The Study and Application of Velocity Transmission in a Queue

变速队伍中速度传递问题的研究及其实际应用

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Abstract

This paper starts with the study of velocity transmission in a queue.

We assume that the velocity of an element in a queue and the distance between the two adjacent elements satisfies a linear relation. Based upon this assumption, models are built to mathematically analyze the influence of the acceleration (or deceleration) of the first object in the sequence to the following objects. The computer program Mathematica® is used to help with the calculation. The final results are presented by a function based on Gamma Function and can also be described by the Velocity-Time graph of every element. Real statistics are used to verify the model.

Secondly, on the basis of the similarities between the flow of vehicles in traffic and the movement of elements in a queue, the model is applied in traffic forecast. Statistics are taken into consideration to modify the model, which contributes to expecting the expansion of traffic congestion, in terms of both velocity and extent.

Finally, by studying the price relation in an industrial chain, the model is also used to analyze the influence that the price fluctuation of an upstream product may bring to the price of downstream products. In this way, the model can be applied in traffic dispatch, fluctuation prediction and many other fields in real life.

Key words： Queue; transmission of velocity; traffic congestion; industrial chain; price fluctuation

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Chapter Ⅰ

Introduction

With its glamorous exhibitions and heart-warming services, the 2010 Shanghai World Expo attracted over 70 million visitors from all over the world. When we were paying a visit, however, it was not the Expo itself that fascinated us most. It was an idea that we came across while lining up in the queue: each move of a queue is actually caused by the move of the first person in the queue. In other words, when the first person accelerates, his or her velocity is passed on to every following person. Thus, given the velocity of the first person and the distances between every two people, we can build a model to mathematically analyze the motion of the queue.

There are many similar situations in our real life where velocity transmission occurs. To quote an example, in a traffic congestion, it is the first car's sudden stop that causes every following car to stop their motion. In these cases, our model can be used to efficiently predict the velocity and distance of the spread of the congestion. Similarities can also be seen in the transmission of price fluctuation in an industrial chain. In this situation, our model contributes to the prediction of the price change in downstream products.

In a word, we believe that the Model of Velocity Transmission in a Queue extends from pure mathematics to the field of application. It indeed is an art of anticipation, as well as a question of balance. Today, when we are all heading for a more efficient society, we sincerely hope that by mathematical methods, we can understand the world in a more sensible way. We are glad to use our own knowledge and efforts to solve the challenges that we are facing. We believe that in the near future, mathematics will play even a more significant role in our joint venture: making this world a better place.

Chapter Ⅱ:

Modeling: Velocity Transmission in a Queue

2.1 Preliminary

2.1.1 Gamma Function

Gamma function (represented by the capital Greek letter **Γ**) is an extension of the factorial function, with its argument shifted down by 1, to real and complex numbers.

- The Gamma function is defined as

$$Gamma[z] = \int_0^\infty t^{z-1} e^{-t} dt$$

- The incomplete gamma function is defined as

$$Gamma[a, z] = \int_z^\infty t^{a-1} e^{-t} dt$$

2.1.2 Notation

n The total number of elements in a queue

a_i The i-th element in a queue

S_i The distance between a_i and a_{i+1}

V_i The velocity of a_i

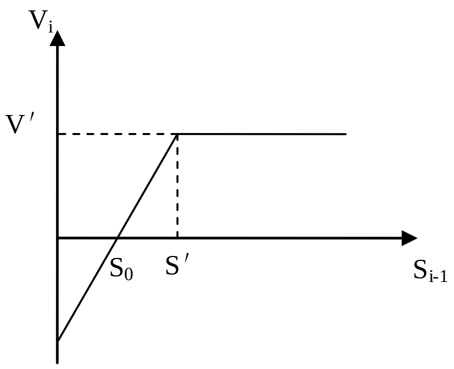
2.2 Mathematical Model and Its Solution

2.2.1 Velocity Transmission during Acceleration

First, let us consider how the velocity transmits in a straight line after the first element starts to move. We assume that there are n elements in a queue. Number the n elements as a_1, a_2, $a_3,\cdots,$ a_n. When the queue remains still, the distance between every two adjacent elements is assumed to be the same (denoted by S_0). When the distance is less than a certain value (denoted by S'), V_i changes linearly with S_{i-1}; When $S_i > S'$, the later element moves at its maximum velocity (denoted by V'), *i.e.*, its velocity remains a fixed value.

We assume that the relation between V and S satisfies

$$V_i = f(S_{i-1}) \ , \ i = 2,3,\cdots,n$$



(Figure 2.1)

To simplify the calculation, we assume that the relation between V_i and S_{i-1} satisfies (as shown in figure 2.1)

$$V_i = f(S_{i-1}) = \begin{cases} k(S_{i-1} - S_0) & \mathrm{S}_{i-1} \in [0, \mathrm{S}') \\ V' & \mathrm{S}_{i-1} \in [\mathrm{S}', +\infty) \end{cases},$$

where k , V', S_0, S' are all positive constants.

When $t = 0$, we have

$$S_1 = S_2 = S_3 = \cdots = S_i = S_0$$

$$V_1 = V' \quad V_2 = V_3 = \cdots = V_i = 0$$

Now let's consider what the distance S_i of every elements in a queue is at time t. For the first element, we have

$$V_1 = V' \quad S_1 = S_0 + V't - \int_0^t V_2 dt$$

(S_1 can be determined by adding the distance that a_1 has traveled to the initial value S_0, then subtracting the distance that a_2 has traveled.)

For the second element a_2, we have

$$V_2 = f(s_1) = k(V't - \int_0^t V_2 dt) \qquad (2.1)$$

(In an infinitesimally small period of time after a_1 starts, V_2 will not exceed V'. Namely, $V_2 = k(S_1 - S_0)$. We first consider how elements move under this circumstance.)

Differentiating (2.1), we obtain,

$$\frac{dV_2}{dt} = k(V' - V_2)$$

$$\Leftrightarrow \frac{1}{V' - V_2} dV_2 = kdt$$

Integrating both sides with respect to t, we have

$$\ln(\frac{1}{V' - V_2}) = kt + C$$

Given that when $t = 0$, $V_2 = 0$, we obtain

$$C = \ln(\frac{1}{V'})$$

$$\therefore \ln(\frac{1}{V' - V_2}) = kt + \ln(\frac{1}{V'})$$

Therefore, V_2 satisfies

$$V_2 = V'(1 - e^{-kt}) \qquad (2.2)$$

(From the expression (2.2), we can see that when the infinitesimally small period of time extends, V_2 will still be less than V'. Only when $t \to +\infty$, $V_2 \to V'$.)

By the definition of distance, we have

$$S_2 = S_0 + \int_0^t V_2 dt - \int_0^t V_3 dt$$

According to the assumption,

$$V_3 = k(S_2 - S_0) = k(\int_0^t V_2 dt - \int_0^t V_3 dt) \qquad (2.3)$$

Substituting (2.2) into (2.3), we have

$$k(V't - \int_0^t V_3 dt) = V'(1 - e^{-kt}) + V_3$$

Therefore, V_3 satisfies

$$V_3 = V'(1 - e^{-kt}(1 + kt))$$

Similarly, we obtain

$$V_4 = V'[1 - e^{-kt}(1 + kt + \frac{1}{2}k^2t^2)]$$

$$V_5 = V'[1 - e^{-kt}(1 + kt + \frac{1}{2}k^2t^2 + \frac{1}{6}k^3t^3)]$$

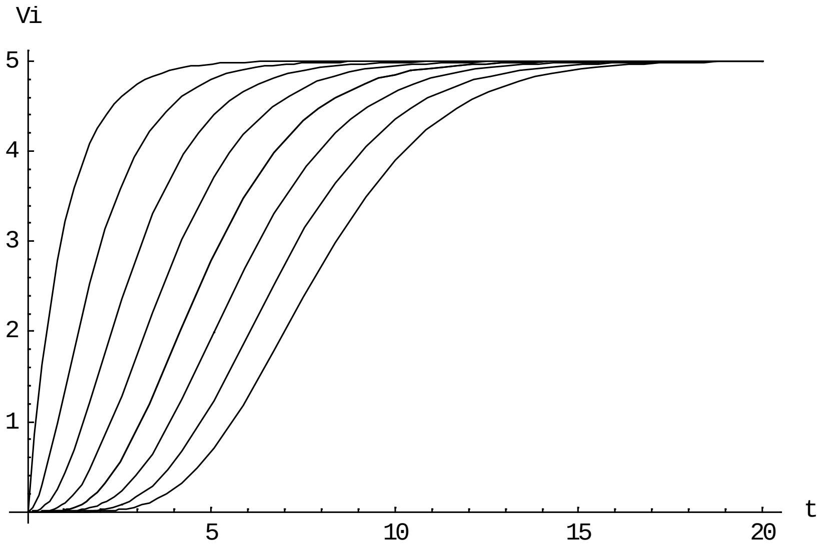
By induction on i,

$$V_i = V'(1 - e^{-kt}\sum_{j=0}^{i-2}\frac{k^j t^j}{j!})$$

Using Gamma function to simplify the equation, we have

$$V_i = V'(1 - \frac{Gamma[i-1, kt]}{Gamma[i-1]}) \qquad (2.4)$$

We use Mathematica® to plot function (2.4):



(Figure 2.2)

2.2.2 Velocity Transmission during Deceleration

Now we discuss how the velocity transmits when the first element suddenly stops.
We assume that from the moment T, the velocity of the first element suddenly becomes 0.
Under this circumstance, we have

$$V_1 = 0 \qquad S_1 = S_0 + V'T - \int_0^T V'(1-e^{-kt})dt - \int_T^t V_2 dt$$

Given that

$$V_2 = f(s_1) = k(V'T - \int_0^T V'(1-e^{-kt})dt - \int_T^t V_2 dt)$$

And that when $t = T$,

$$\int_0^T V'(1-e^{-kt})dt = V'(\frac{-1+e^{-kT}}{k} + T),$$

we have

$$V_2 = V'e^{-kt}(-1+e^{kT})$$

Similarly,

$$V_3 = V'e^{-kt}(-k(t-T) + e^{kT}(1+k(t-T) - Gamma[2,kT]))$$

$$V_4 = V'e^{-kt}(-k(t-T) - \frac{1}{2}k^2(t^2-T^2) + e^{kT}(1+k(t-T) + \frac{1}{2}k^2(t-T)^2 - \frac{1}{2}Gamma[3,kT]))$$

$$V_5 = V'e^{-kt}(-k(t-T) - \frac{1}{2}k^2(t^2 - T^2) - \frac{1}{6}k^3(t^3 - T^3)) +$$
$$e^{kT}(1 + k(t-T) + \frac{1}{2}k^2(t-T)^2 + \frac{1}{6}k^3(t-T)^3 - \frac{1}{6}Gamma[4, kT]))$$

By induction on i, we have

$$V_i = V'e^{-kt}(-\sum_{j=1}^{i-2}\frac{1}{j!}k^j(t^j - T^j) + e^{kT}(\sum_{j=0}^{i-2}\frac{1}{j!}k^j(t-T)^j - \frac{1}{(i-2)!}e^{-kT}\sum_{j=1}^{i-1}\frac{(i-2)!k^{j-1}T^{j-1}}{(j-1)!}))$$

$$\Leftrightarrow V_i = V'e^{-kt}(-\sum_{j=1}^{i-2}\frac{k^j t^j}{j!} + e^{kT}\sum_{j=0}^{i-2}\frac{k^j(t-T)^j}{j!} - 1)$$

Using Gamma function to simplify the equation, we have

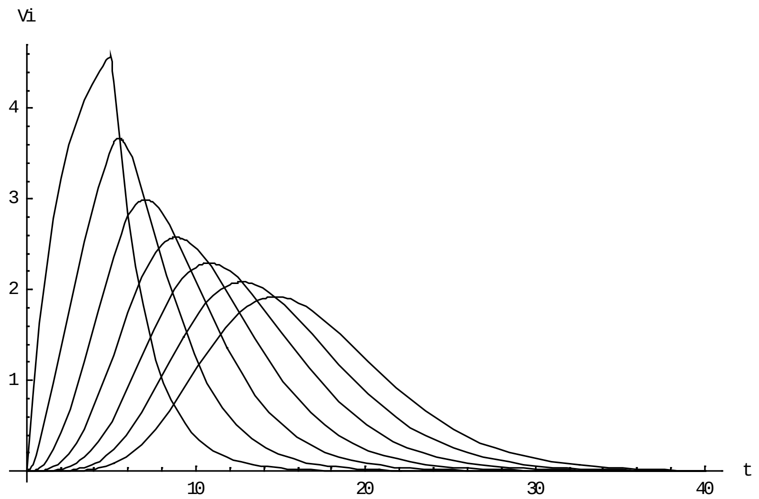
$$V_i = \frac{V'(-Gamma[i-1, kt] + Gamma[i-1, k(t-T)])}{Gamma[i-1]} \qquad (2.5)$$

Therefore, given the equations (2.4), (2.5), V_i can be presented as:

$$V_i = \begin{cases} V'(1 - \dfrac{Gamma[i-1, kt]}{Gamma[i-1]}) & t \in [0, T) \\ \dfrac{V'(-Gamma[i-1, kt] + Gamma[i-1, k(t-T)])}{Gamma[i-1]} & t \in [T, +\infty) \end{cases},$$

where i, k, V' are all positive constants.

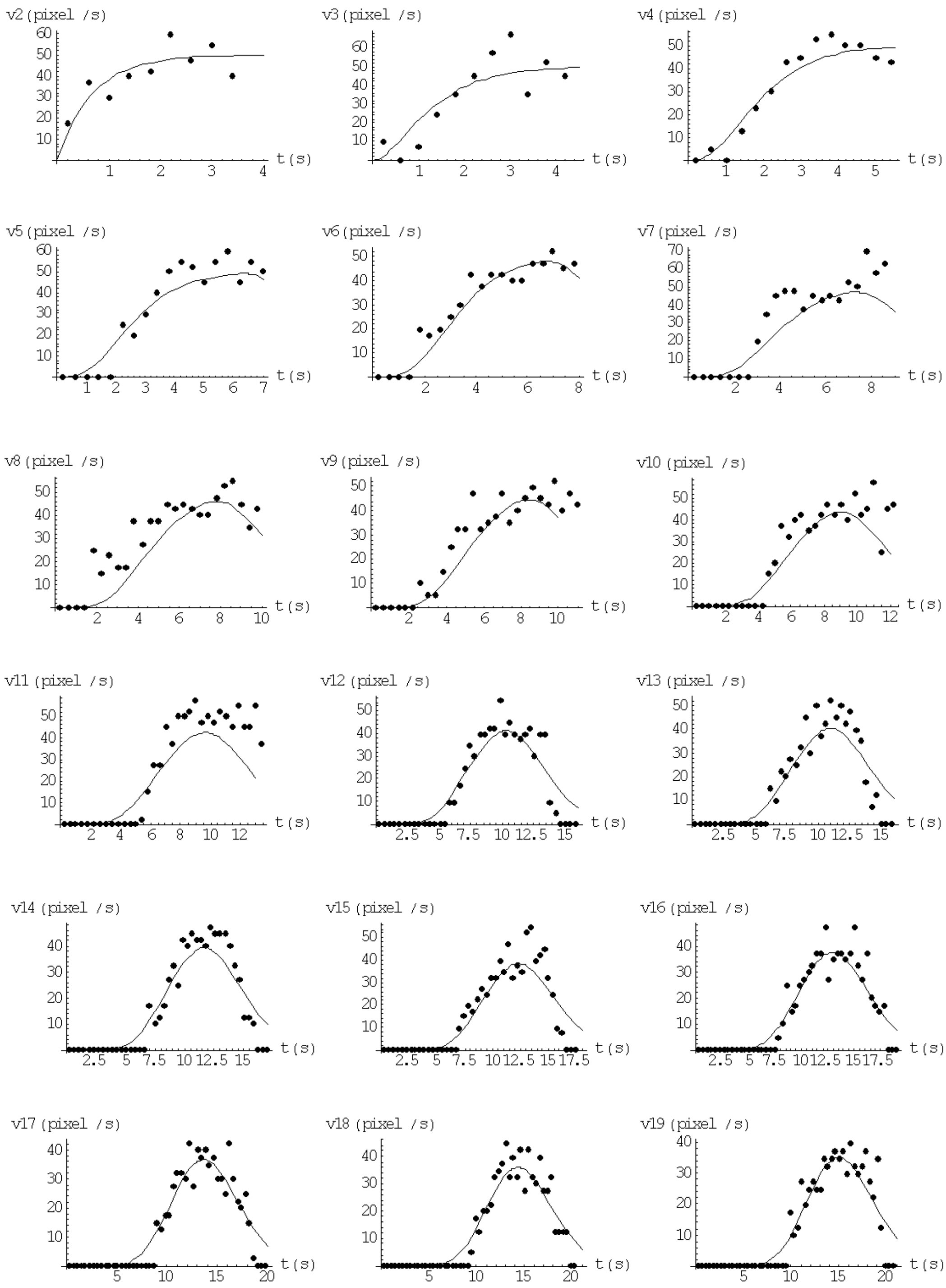
We use Mathematica® to produce the following graph:

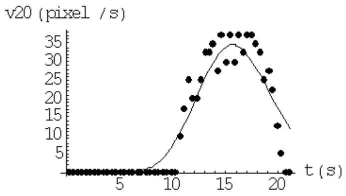


(Figure 2.3)

2.3 Verification

To verify the model, we investigated a real case of Velocity Transmission in a Queue. We took a video at the China Pavilion, 2010 Shanghai World Expo. From the video, we took down the positions at different time t of the first twenty people in the queue. We use the statistics to plot the Velocity-Time graphs of the twenty people. Combining the $V-t$ graphs and the curves of V_i, we have the results below:





(Figure 2.4)

*In the function of V_i, the parameters are $k \to 1.45, V' \to 50, T \to 6$.

From Figure (2.4) we can see that the model is verified. Although when $i \le 3$, velocity transmission may not strictly follow our assumption, for it can be interfered by many other factors, the function in general can well describe the motion of the people in a queue. Thus, we consider the model a good reflection of reality.

2.4 Conclusion

From the discussion above, we can see that the change of velocity of the first element in a queue can be transmitted to all subsequent elements. The longer the distance between a_i and a_0 is, the smaller the range of the change of its velocity appears to be. The delay of the transmission is also illustrated by Figure 2.3, where the function of V_i satisfies

$$V_i = \begin{cases} V'\left(1 - \dfrac{Gamma[i-1, kt]}{Gamma[i-1]}\right) & t \in [0, T) \\ \dfrac{V'(-Gamma[i-1, kt] + Gamma[i-1, k(t-T)])}{Gamma[i-1]} & t \in [T, +\infty) \end{cases}$$

Chapter Ⅲ：Application 1

The Spread of Traffic Congestion

3.1 Study Background

Nowadays, cities have been bothered by traffic congestion. At rush hours, the traffic becomes so heavy that congestion in a comparatively small area, if not directed in time, will soon expand. Thus, traffic jam can cause great trouble for the traffic department. It is also a waste of people’s time. On the basis of the similarities between these circumstances and our model, the Model of Velocity Transmission in a Queue can be used to forecast the spread of traffic congestion. We can ultimately predict how fast and how far the congestion will spread. According to the result, concerned department will be able to take measures in advance.

3.2 Modeling & Calculation

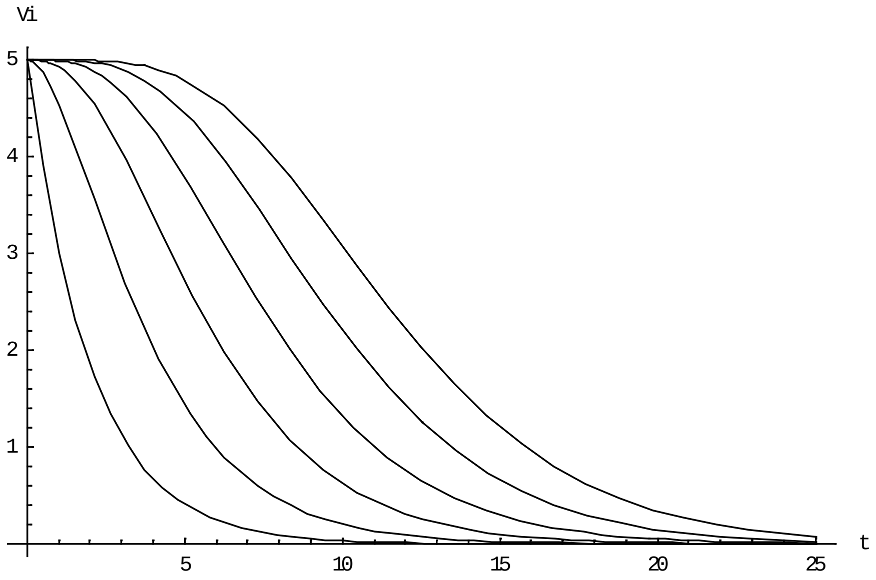
We assume that there are n cars traveling along a straight track. The distances between every two adjacent cars are the same. Now, traffic congestion takes place at the front-end of the queue due to some unpredictable reasons. We further assume that the initial velocity of all cars is a constant V', and that the distance between every two adjacent cars $S_i = S' = \frac{V'}{k} + S_0$. Let the velocity of the first car turn zero at t=0.

According to the model,

$$V_i = V'e^{-kt}\sum_{j=0}^{i-2}\frac{k^j t^j}{j!}$$

$$V_i = \frac{V'(i-1)Gamma[i-1,kt]}{Gamma[i]} \qquad (3.1)$$

We use Mathematica® to obtain the figure below:



(Figure 3.1)

Figure 3.1 presents us with the Velocity-Time graph of every car.
As in the idealized model the velocity of any of the cars will not decrease to zero, we introduce $b = \frac{V_i}{V'}$. When $b \le b_0$, the car is regarded to be in a state of stillness.

To illustrate the main idea of the solution, we take, for example, (In real cases, the constants need to be changed to adapt to different circumstances.)

$$V' = 20(m/s)$$

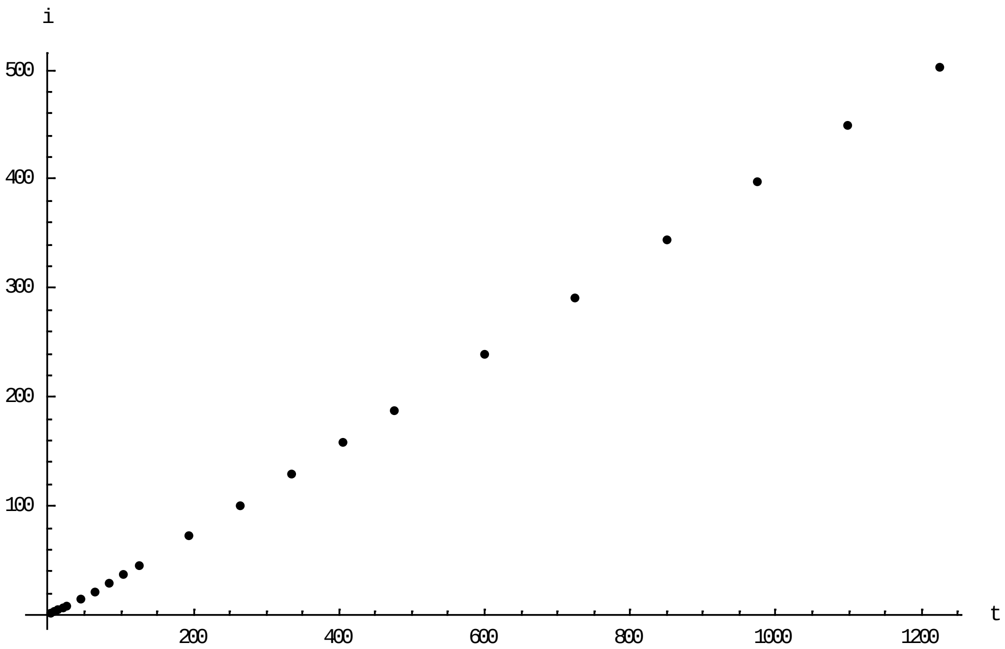
$$k = 0.44(s^{-1})$$

$$S_0 = 5(m)$$

$$S' = 50(m)$$

$$b_0 = 0.05$$

Substituting the constants above into (3.1), we obtain the value of i (representing the serial number of the car that stops at time t). The resulting sequence of i is depicted by Figure 3.2.



(Figure 3.2)

Since

$$V = \frac{dS}{dt} = \frac{di}{dt} = \lim_{dt \to 0} \frac{f[t+dt] - f[t]}{dt}$$

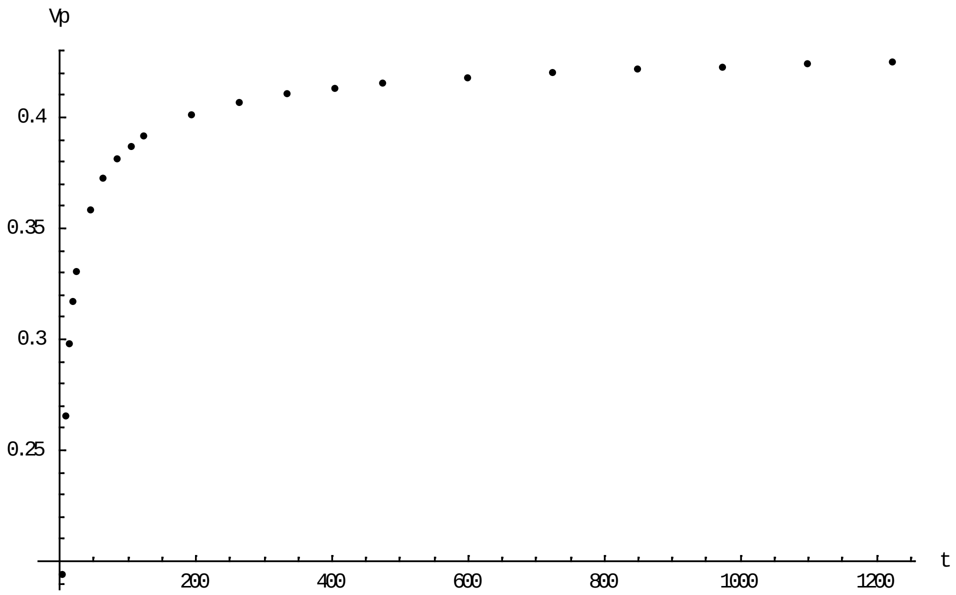
We have

b	t	$f[t]$	$f[t+dt]$	V_p
0.05	0	/		
0.05	5	1.617904	1.618097	0.193265
0.05	10	2.790471	2.790736	0.264746
0.05	15	4.204527	4.204825	0.297544
0.05	20	5.744342	5.744659	0.316969
0.05	25	7.36399	7.364321	0.330148
0.05	45	14.29082	14.29118	0.358375
0.05	65	21.60893	21.6093	0.372164
0.05	85	29.14347	29.14385	0.380715
0.05	105	36.82058	36.82096	0.38668

0.05	125	44.60075	44.60114	0.391143
0.05	195	72.36023	72.36063	0.400902
0.05	265	100.6331	100.6335	0.406468
0.05	335	129.2234	129.2238	0.41018
0.05	405	158.0351	158.0355	0.412882
0.05	475	187.0125	187.0129	0.414961
0.05	600	239.0653	239.0657	0.417723
0.05	725	291.4123	291.4128	0.419735
0.05	850	343.9799	343.9804	0.421285
0.05	975	396.7208	396.7212	0.422526
0.05	1100	449.6024	449.6029	0.423549
0.05	1225	502.6014	502.6018	0.424411

(Table 3.1)

Using the figures in the table, we have



(Figure 3.3)

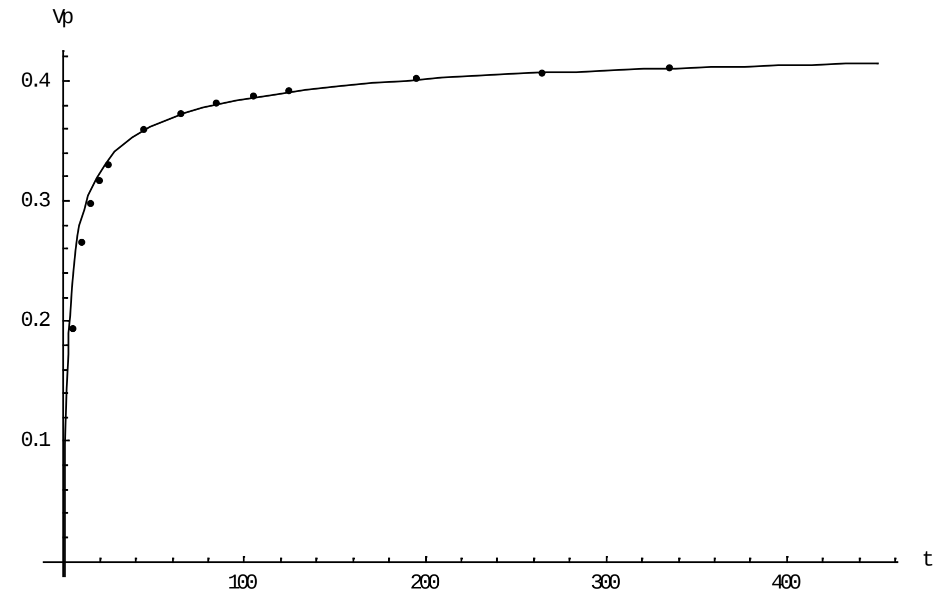
From Figure (3.3), it can be seen that the transmission velocity of the phase is an increasing function of Time. Its slope appears to be decreasing.
We then use the method of Least Square to fit the data.

We find that the function can be best fitted by the equation $V = at^{c}Ln(t) + b$. Using the computer program Mathematica® to solve this equation, we have

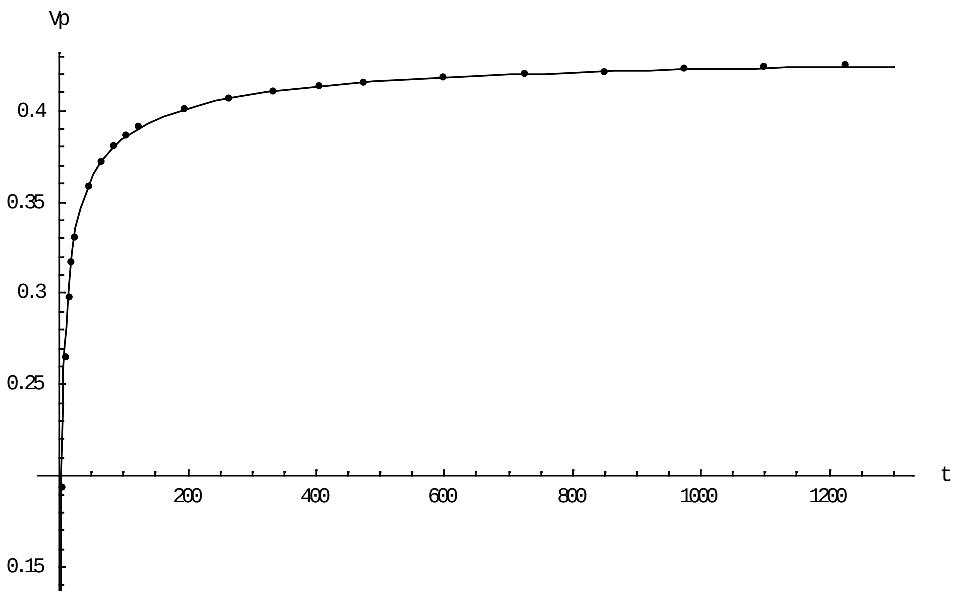
$$a \to 0.11237237, b \to 0.09181098, c \to -0.12348419$$

$$V_p = 0.11237237t^{-0.123484}Ln(t) + 0.09181098 \qquad (3.2)$$

Using computer to plot the function, we have



(Figure 3.4)



(Figure 3.5)

From Figure (3.4) and (3.5), we can see when $t > 50$, the data are well fitted. When $t < 50$, there is a deviation of the function from the statistics.

Then we calculate the distance that traffic congestion (the status at which a car moves

at a relatively low velocity, which is $b \cdot V'$) travels.
Considering the relation between the distance that traffic congestion travels (denoted by D_p) and the distance that a_i travels (denoted by D_i), we have

$$D_p = (i-1)S_0 - D_i$$

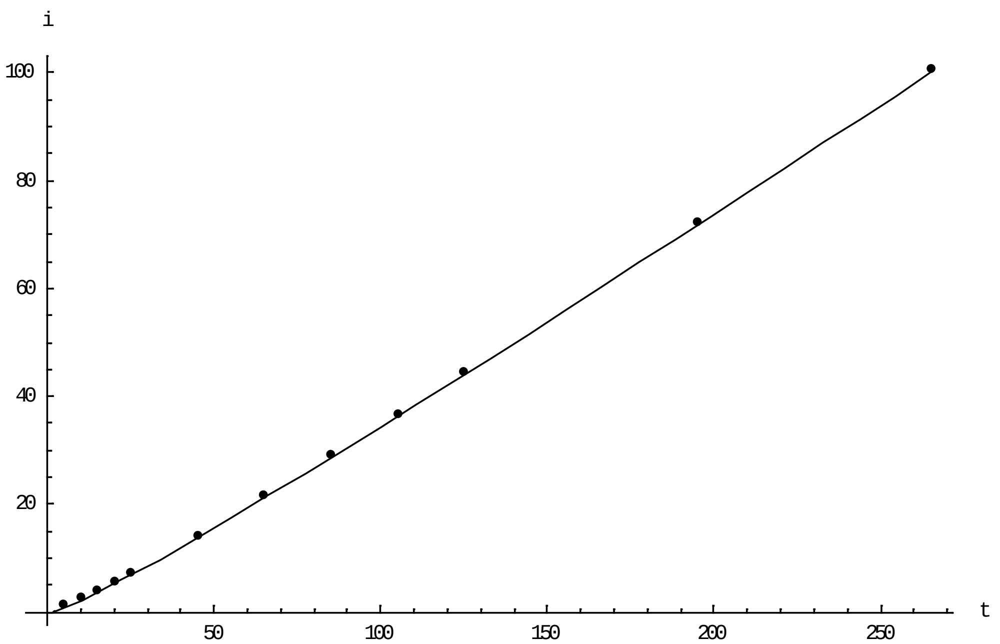
Integrating (3.1), we have

$$D_i = \int_0^t V_i dt = \int_0^t \frac{V'(i-1)Gamma[i-1,kt]}{Gamma[i]} dt$$

$$D_i = \frac{V'(ktGamma[i-1,kt] + Gamma[i,0] - Gamma[i,kt])}{kGamma[i-1]}$$

Similarly, by integrating (3.2), we get

$$i = \int_0^t V dt = 0.0918t + 0.1124t^{0.8765}(-1.302 + 1.141Ln[t])$$



(Figure 3.6)

From Figure (3.6) we note that there is deviation from the fitted curve to the given data points. Thus we come to update the fitting.

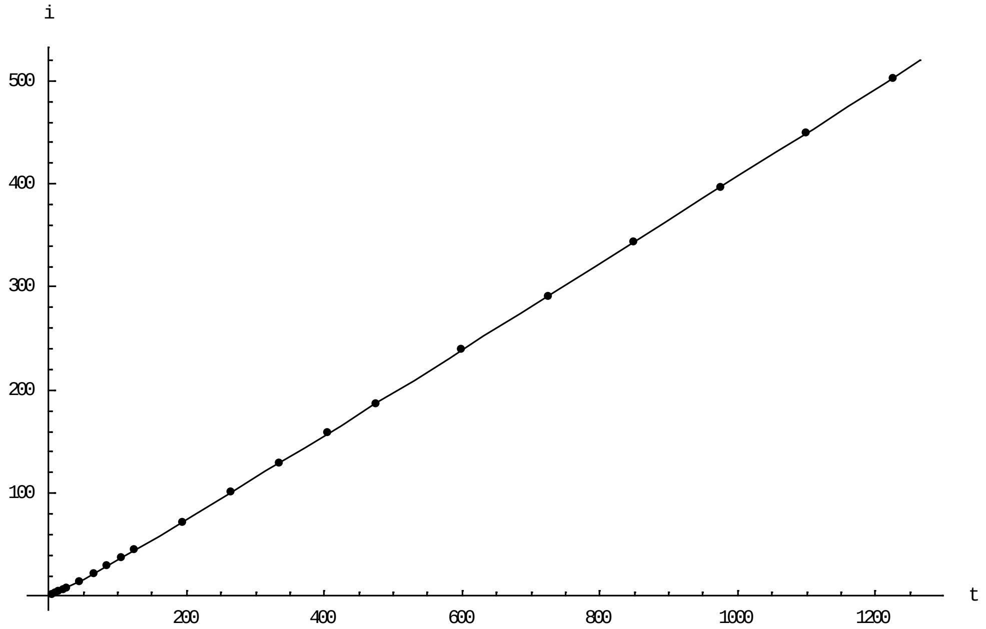
Let $i = at + bt^c(d + eLn[t])$. We find the fitting constants below using the method of Least Square:

a $\rightarrow$ -3.4553978 , b $\rightarrow$ 0.99490992 , c $\rightarrow$ 0.95957738 ,
d $\rightarrow$ 3.5569785 , e $\rightarrow$ 0.22815992

Hence, the function i can be written as:

$$i = -3.455398\,t + 0.99490992\,t^{0.95957738}(3.5569785 + 0.22815992Ln[t]) \qquad (3.3)$$

Having plotted the function in Figure (3.7), we can see that the original statistics are better fitted with the function.



(Figure 3.7)

Substituting i into the formula of D_i, we have

$$D_i = V'(ktGamma[-3.455t + 0.995t^{0.9596}(3.557 + 0.22816Ln[t]) - 1, kt]$$
$$+ Gamma[-3.455t + 0.995t^{0.9596}(3.557 + 0.22816Ln[t]), 0]$$
$$- Gamma[-3.455t + 0.995t^{0.9596}(3.557 + 0.22816Ln[t]), kt])$$
$$/(kGamma[-3.455t + 0.995t^{0.9596}(3.557 + 0.22816Ln[t]) - 1])$$

Therefore, we get,

$$D_p = (-3.455t + 0.995t^{0.9596}(3.557 + 0.22816Ln[t]) - 1)S_0$$
$$- V'(ktGamma[-3.455t + 0.995t^{0.9596}(3.557 + 0.22816Ln[t]) - 1, kt]$$
$$+ Gamma[-3.455t + 0.995t^{0.9596}(3.557 + 0.22816Ln[t]), 0]$$
$$- Gamma[-3.455t + 0.995t^{0.9596}(3.557 + 0.22816Ln[t]), kt])$$
$$/(kGamma[-3.455t + 0.995t^{0.9596}(3.557 + 0.22816Ln[t]) - 1])$$

Substituting the following data:

$$V' = 20(m/s)$$

$$k = 0.44(s^{-1})$$

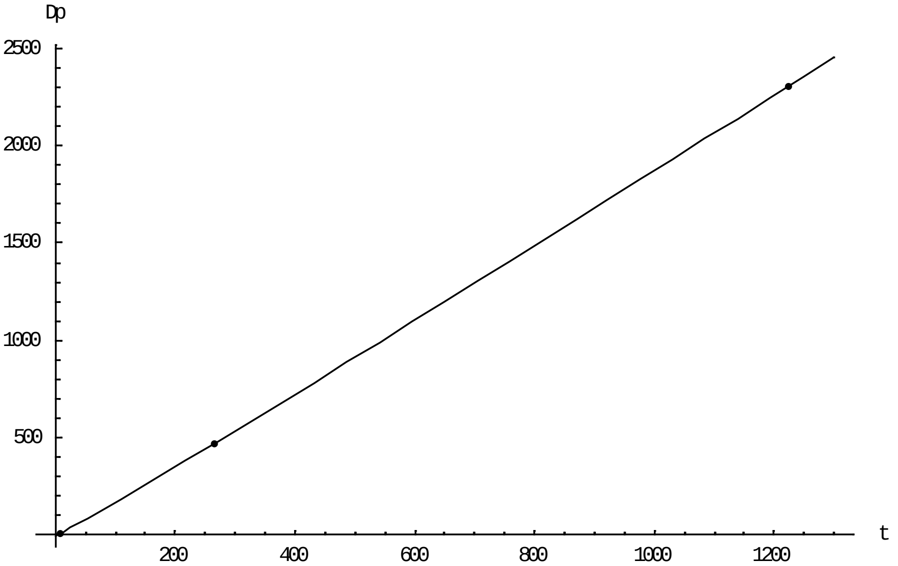
$$S_0 = 5(m)$$

$$S' = 50(m)$$

We have

$$
\begin{aligned}
D_p &= -45.4545 + 176.94t^{0.9596} - 172.77t \\
&- (45.45(0.44t Gamma[\text{-}3.455t + 0.995t^{0.9596}(3.557 + 0.22816 Ln[t]) - 1, 0.44t] \\
&+ Gamma[\text{-}3.455t + 0.995t^{0.9596}(3.557 + 0.22816 Ln[t]), 0] \\
&- Gamma[\text{-}3.455t + 0.995t^{0.9596}(3.557 + 0.22816 Ln[t]), 0.44t])) \\
&/ Gamma[\text{-}3.455t + 0.995t^{0.9596}(3.557 + 0.22816 Ln[t]) - 1] + 11.35t^{0.9596} Ln[t]
\end{aligned} \qquad (3.4)
$$

Now we re-plot the $D_p - t$ graph (Figure 3.8). It can be seen that the fitting is enhanced.



(Figure 3.8)

Conclusion:

From (3.2), (3.3), (3.4), we can obtain V_p (the velocity at which the traffic congestion spreads) at time t and D_p (the spread distance from the original congestion spot).

3.3 Application Analysis

From the analyses above, we can see that the model of Velocity Transmission in a Queue can be applied in traffic problems. In real life, we can obtain the initial velocity and distance between cars by investigation. Once given those statistics, we will be able to substitute them in to the function and then calculate the velocity and extent of the congestion transmission. It will be a convenient and accurate way to predict the congestion. With the help of the model, the traffic department could direct the cars in advance, which would be a remedy for the congested area, and thus reduce the negative impact that traffic congestion may bring.

Chapter Ⅳ：Application 2

The Transmission of Price Fluctuation in an Industrial Chain

4.1 Study Background

The interaction between the upstream and downstream products in an industrial chain is one of the important topics studied in microeconomics. With a certain product's price going up, change will be brought to the cost of the next part in the chain. And this fluctuation is passed on to every following product, which finally to the increase of CPI and other consumption indexes. In real life, the rise of price may be influenced by many factors, such as financial policies, supply-demand relationship *etc*. This fact in some way makes the fluctuation unpredictable. Here, we simplify the situation, assuming that the fluctuation in prices of downstream products is only affected by the upstream ones. By calculation, we hope that we can predict the influence that the price fluctuation of an upstream product may bring to the price of downstream products.

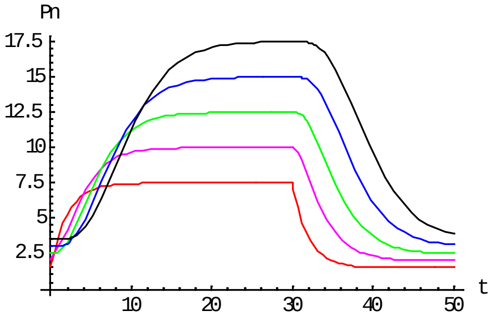
4.2 Modeling and Computations of the model

We assume that when $t=0$, the price of a benchmark product suddenly changes from P_0 to P'. When $t=T$, the price goes back to P_0.
In the primal model in Chapter II, the longer the distance between a_i and a_1 is, the smaller the influence of the acceleration of a_1 appears to be. Similarly in this case, we use n to depict this influence. Therefore, The price of the a-th product in the chain satisfies $P_a = A_a \cdot f(k_a, t)$. k_a can be understood as a parameter that indicates how close P_a is related to P_{a-1}. A_a denotes the ratio of the initial P_a to P_0.

From Chapter Ⅱ, we have

$$P_a = \begin{cases} A_a(P_0 + (P'-P_0)(1-\dfrac{Gamma[n_a-1, k_a t]}{Gamma[n_a-1]})) & t\in[0,T) \\ \\ A_a(P_0 + \dfrac{(P'-P_0)(-Gamma[n_a-1,k_a t]+Gamma[n_a-1,k_a(t-T)])}{Gamma[n_a-1]}) & t\in[T,+\infty) \end{cases}$$

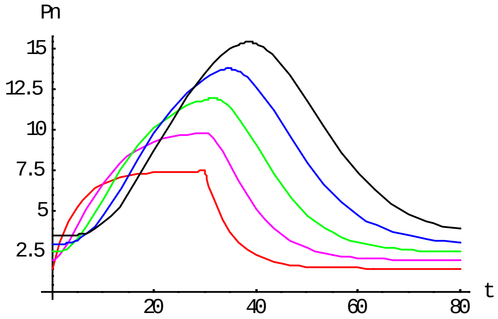
Substituting different parameters into the function above, we have the following graphs:



$P_0 \to 1, P' \to 5, T \to 30$

$k_2 \to 0.5, k_3 \to 0.5, k_4 \to 0.5, k_5 \to 0.5, k_6 \to 0.5$

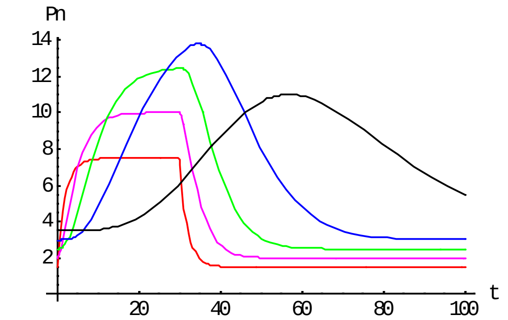
$A_2 \to 1.5, A_3 \to 2.0, A_4 \to 2.5, A_5 \to 3.0, A_6 \to 3.5$



$P_0 \to 1, P' \to 5, T \to 30$

$k_2 \to 0.2, k_3 \to 0.2, k_4 \to 0.2, k_5 \to 0.2, k_6 \to 0.2$

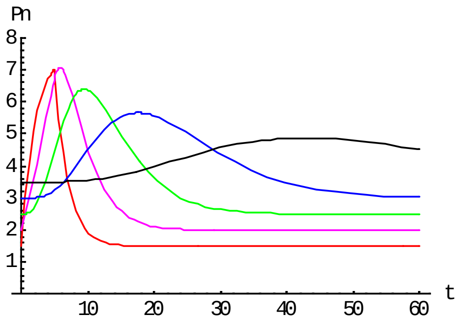
$A_2 \to 1.5, A_3 \to 2.0, A_4 \to 2.5, A_5 \to 3.0, A_6 \to 3.5$



$P_0 \to 1, P' \to 5, T \to 30$

$k_2 \to 0.5, k_3 \to 0.4, k_4 \to 0.3, k_5 \to 0.2, k_6 \to 0.1$

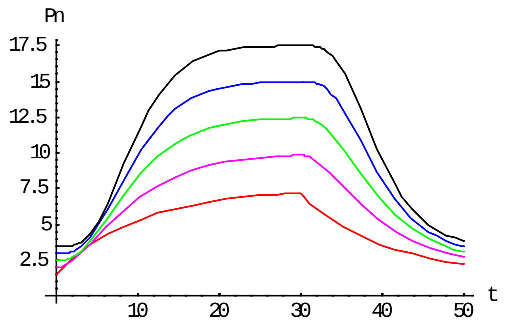
$A_2 \to 1.5, A_3 \to 2.0, A_4 \to 2.5, A_5 \to 3.0, A_6 \to 3.5$



$P_0 \to 1, P' \to 5, T \to 5$

$k_2 \to 0.5, k_3 \to 0.4, k_4 \to 0.3, k_5 \to 0.2, k_6 \to 0.1$

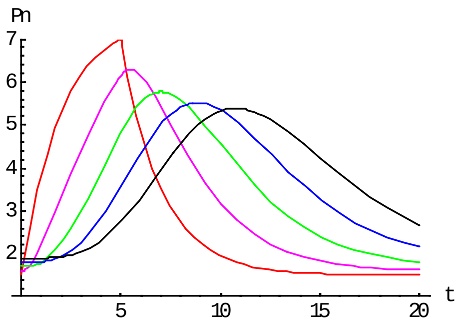
$A_2 \to 1.5, A_3 \to 2.0, A_4 \to 2.5, A_5 \to 3.0, A_6 \to 3.5$



$P_0 \to 1, P' \to 5, T \to 30$

$k_2 \to 0.1, k_3 \to 0.2, k_4 \to 0.3, k_5 \to 0.4, k_6 \to 0.5$

$A_2 \to 1.5, A_3 \to 2.0, A_4 \to 2.5, A_5 \to 3.0, A_6 \to 3.5$



$P_0 \to 1, P' \to 5, T \to 30$

$k_2 \to 0.5, k_3 \to 0.5, k_4 \to 0.5, k_5 \to 0.5, k_6 \to 0.5$

$A_2 \to 1.5, A_3 \to 1.6, A_4 \to 1.7, A_5 \to 1.8, A_6 \to 1.9$

(Figure 4.1)

From the graphs, we can see that price fluctuation in upstream products do bring influences to the whole industrial chain. However, when the parameters are different, the results vary accordingly. In some cases, the fluctuation is magnified in the transmission, which means that the fluctuation becomes more obvious. In other cases, the fluctuation can be digested in the process, making the influence weaker in downstream products.

4.3 Case Analysis

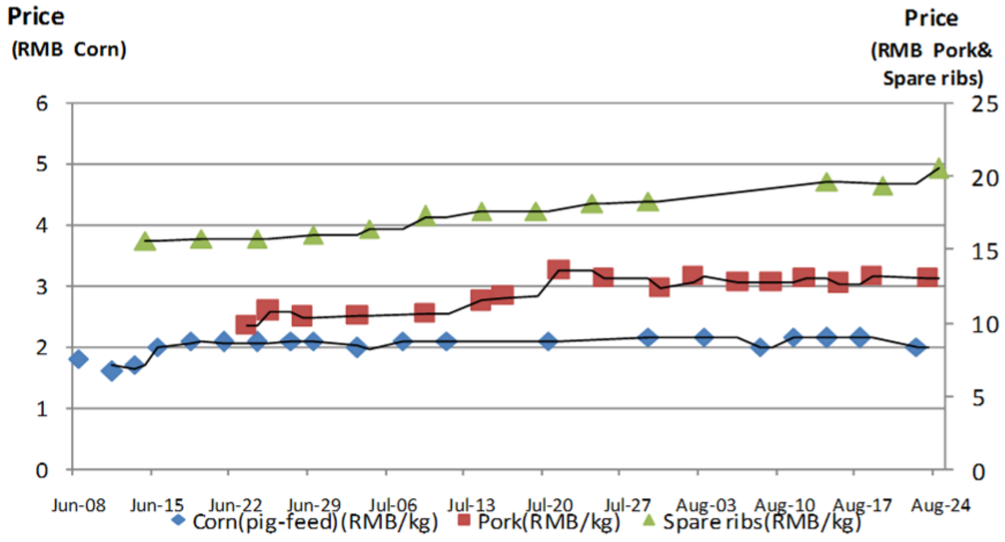
Pork price has always been the indicator of consumption prices. Its fluctuation can reflect the price of pig feed, as well as affecting the price of the industrial chain of pork. Now, we use the industrial chain of corn (pig feed)-pork-spare ribs to analyze the price fluctuation in real life.

The table below shows the prices of corn (pig feed), pork and spare ribs from June 8th, 2010 to August 24th, 2010.

Date	Corn (pig feed) (RMB/kg)	Pork (RMB/kg)	Spare ribs (RMB/kg)	Date	Corn (pig feed) (RMB/kg)	Pork (RMB/kg)	Spare ribs (RMB/kg)
Jun-08	1.80	N/A	N/A	Jul-20	2.10	N/A	N/A
Jun-11	1.60	N/A	N/A	Jul-21	N/A	13.60	N/A
Jun-13	1.70	N/A	N/A	Jul-24	N/A	N/A	18.12
Jun-14	N/A	N/A	15.62	Jul-25	N/A	13.00	N/A
Jun-15	2.00	N/A	N/A	Jul-29	2.16	N/A	18.26
Jun-18	2.10	N/A	N/A	Jul-30	N/A	12.40	N/A
Jun-19	N/A	N/A	15.76	Aug-02	N/A	13.20	N/A
Jun-21	2.08	N/A	N/A	Aug-03	2.16	N/A	N/A
Jun-23	N/A	9.80	N/A	Aug-06	N/A	12.80	N/A
Jun-24	2.08	N/A	15.76	Aug-08	2.00	N/A	N/A
Jun-25	N/A	10.80	N/A	Aug-09	N/A	12.80	N/A
Jun-27	2.10	N/A	N/A	Aug-11	2.16	N/A	N/A
Jun-28	N/A	10.40	N/A	Aug-12	N/A	13.00	N/A
Jun-29	2.10	N/A	16.00	Aug-14	2.15	N/A	19.62
Jul-03	1.98	10.50	N/A	Aug-15	N/A	12.70	N/A
Jul-04	N/A	N/A	16.38	Aug-17	2.15	N/A	N/A
Jul-07	2.10	N/A	N/A	Aug-18	N/A	13.20	N/A
Jul-09	N/A	10.60	17.26	Aug-19	N/A	N/A	19.38
Jul-11	2.10	N/A	N/A	Aug-22	2.00	N/A	N/A
Jul-14	N/A	11.50	17.62	Aug-23	N/A	13.00	N/A
Jul-16	N/A	11.80	N/A	Aug-24	N/A	N/A	20.50
Jul-19	N/A	N/A	17.62				

(Table 4.1)

The data in Table (4.1) can be presented by the graph below:



(Figure 4.2)

Let June 8th be the time $t = 0$. We then try to find the fitting function.
Considering that during this period of time, the prices are monotone increasing, we may only analyze the part of function when $t < T$.
Therefore,

$$P_a = A_a(P_0 + (P' - P_0)(1 - \frac{Gamma[n_a - 1, k_a t]}{Gamma[n_a - 1]}))$$

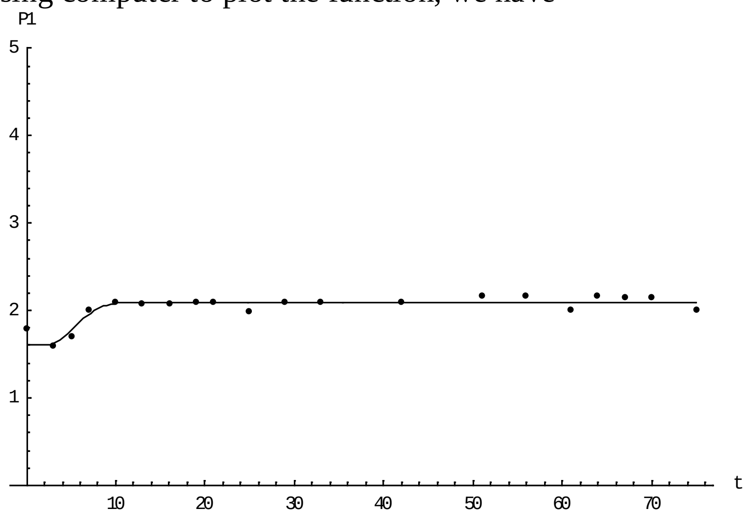
Adjusting the parameters, we have

$$P_0 \to 1.6, P' \to 2.1, A_1 \to 1, n_1 \to 10, k_1 \to 1.5$$

That is,

$$P_1 = 1.6 + (2.1 - 1.6)(1 - \frac{Gamma[10 - 1, 1.5t]}{Gamma[10 - 1]})$$

Using computer to plot the function, we have



(Figure 4.3)

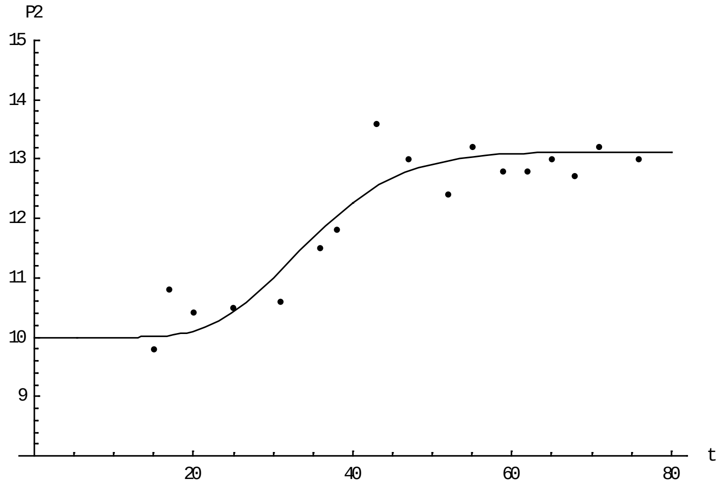
Substituting $P_0 \to 1.6, P' \to 2.1, A_2 = \frac{10}{1.6} = 6.25$ into the equation of P_2, we have

$$P_0 \to 1.6, P' \to 2.1, A_2 \to 6.25, n_2 \to 15, k_2 \to 0.4$$

That is,

$$P_2 = 6.25(1.6 + (2.1 - 1.6)(1 - \frac{Gamma[15-1, 0.4t]}{Gamma[15-1]}))$$

The expression of P_2 can be graphed as Figure (4.4):



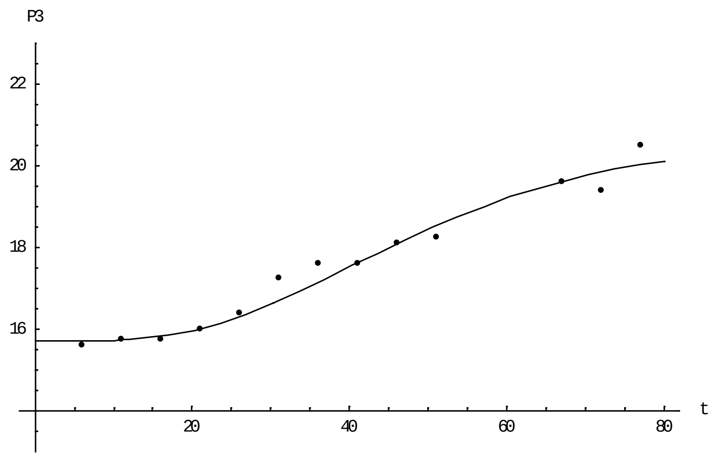
(Figure 4.4)

Substituting $P_0 \to 1.6, P' \to 2.1, A_3 = \frac{15.7}{1.6} = 9.81$ into the equation of P_3, similarly, we have

$$P_0 \to 1.6, P' \to 2.1, A_3 \to 9.81, n_3 \to 6, k_3 \to 0.1$$

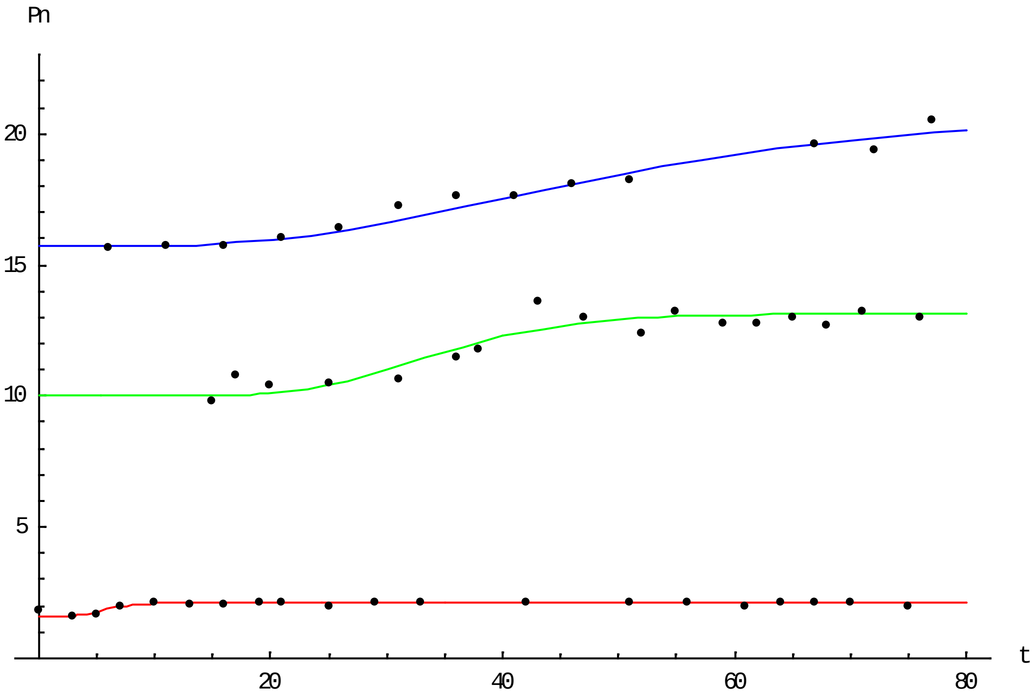
Hence,

$$P_3 = 9.81(1.6 + (2.1 - 1.6)(1 - \frac{Gamma[6-1, 0.1t]}{Gamma[6-1]}))$$



(Figure 4.5)

Combining the three graphs, we have



(Figure 4.6)

4.4 Conclusion

From the graph we can see that our function well fits with the price fluctuation in real life. With the help of the price model, we can analyze the price change of a certain product, and then to anticipate the fluctuation that may appear in the downstream prices. The influences that price fluctuation may bring to an economy are significant. Using the model of Velocity Transmission in a Queue will be of great value for the producers' management. Meanwhile, policy-makers can take it as a reference when trying to exert macro adjustment policies. Consumers may also refer to it for help when making their shopping choices.

Chapter Ⅴ

Retrospect

The study of this project has been a precious experience for us. On the one hand, the study has greatly improved our capacity of mathematical thinking. We have learned how to build up the model on the basis of a practical problem, and more importantly, to use the model in real life. In the process, we used many practical methods to solve the challenges we are faced with. When trying to cope with transcendental equations, for instance, we substituted real statistics into the equation. Also, we have used graphic tools to plot the functions to deal with the data, which has greatly helped with the analysis. We believe that these methods will play a significant role in our future math study as well as in our life.

On the other hand, we have had the opportunity to learn some advanced mathematical knowledge. From our research about Gamma Function, to the solution of transcendental equations, to the use of graphic tools, we have gained a better understanding of the true meaning of mathematics that it is of not only pure beauty, but also practical value. We have come to realize that in an era when progress is being made in every single field, mathematics has never been more connected to our life as it is now. We appreciate the opportunity it offers us with: Equipped with knowledge, shouldered with moral responsibilities, we can make full use of mathematics to serve our community.

Meanwhile, our project does call for further improvement. Firstly, we have simplified the queue as a straight line, which is unlikely in real cases. Secondly, when we are studying the transmission of price fluctuation in an industrial chain, we have neglected the influence of other factors, such as the financial policies and the supply-demand relationship. Partly due to this, our function is still slightly deviated from the real figures.

In the future, we want to keep perfecting this model. For instance, we are considering introducing another variable to indicate the effect caused by other factors. We can also take the differences between the maximum velocities of people into consideration. We believe that the model can be further improved in this way.

At the very end of this report, we want to express our gratitude to our mentors at Hangzhou Foreign Languages School. This report will not be the destination of our study. In the future, our joint efforts will still be a great encouragement to our further study on this project, which we are all looking forward to.

References

[1] The department of mathematics at Tongji University.

Advanced Mathematics [M] ，China Higher Education Press (CHEP)，2007.

[2] Wikipedia - Gamma function. http://en.wikipedia.org/wiki/Gamma_function

[3] Wolfram Mathematica® documentation center.

http://reference.wolfram.com/mathematica

[4] Data sources： http://www.czny.gov.cn

http://www.tczizhu.cn

http://www.pigol.cn

Appendix Ⅰ：The Original Data Used in 2.3

Time (sec.)	Position (pixel) * 1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
0	110	148	162	182	234	256	280	316	311	336
0.4	101	141	158	182	234	256	280	316	311	336
0.8	88	126	158	180	234	256	280	316	311	336
1.2	73	114	155	180	234	256	280	316	311	336
1.6	49	98	145	175	234	256	280	316	311	336
2	36	81	131	166	234	248	280	306	311	336
2.4	18	57	113	154	224	241	280	300	311	336
2.8	N/A	38	90	137	216	233	280	291	307	336
3.2	N/A	16	63	119	204	223	272	284	305	336
3.6	N/A	0	49	98	188	211	258	277	303	336
4	N/A	N/A	28	76	168	194	240	262	297	336
4.4	N/A	N/A	10	56	146	179	221	251	287	336
4.8	N/A	N/A	N/A	36	125	162	202	236	274	330
5.2	N/A	N/A	N/A	18	107	145	187	221	261	322
5.6	N/A	N/A	N/A	1	85	129	169	203	242	307
6	N/A	N/A	N/A	N/A	61	113	152	186	229	294
6.4	N/A	N/A	N/A	N/A	43	94	134	168	215	278
6.8	N/A	N/A	N/A	N/A	21	75	117	151	200	261
7.2	N/A	N/A	N/A	N/A	1	54	96	135	181	247
7.6	N/A	N/A	N/A	N/A	N/A	36	76	119	167	232
8	N/A	N/A	N/A	N/A	N/A	17	48	100	151	215
8.4	N/A	N/A	N/A	N/A	N/A	N/A	25	79	133	196
8.8	N/A	N/A	N/A	N/A	N/A	N/A	0	57	113	179
9.2	N/A	N/A	N/A	N/A	N/A	N/A	N/A	39	95	160
9.6	N/A	N/A	N/A	N/A	N/A	N/A	N/A	25	78	144
10	N/A	N/A	N/A	N/A	N/A	N/A	N/A	8	57	123
10.4	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	41	106
10.8	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	22	88
11.2	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	5	65
11.6	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	55
12	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	37
12.4	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	18

Time (sec.)	Position (pixel) * 11th	12th	13th	14th	15th	16th	17th	18th	19th	20th

0	363	389	413	426	462	475	482	509	526	544
0.4	363	389	413	426	462	475	482	509	526	544
0.8	363	389	413	426	462	475	482	509	526	544
1.2	363	389	413	426	462	475	482	509	526	544
1.6	363	389	413	426	462	475	482	509	526	544
2	363	389	413	426	462	475	482	509	526	544
2.4	363	389	413	426	462	475	482	509	526	544
2.8	363	389	413	426	462	475	482	509	526	544
3.2	363	389	413	426	462	475	482	509	526	544
3.6	363	389	413	426	462	475	482	509	526	544
4	363	389	413	426	462	475	482	509	526	544
4.4	363	389	413	426	462	475	482	509	526	544
4.8	363	389	413	426	462	475	482	509	526	544
5.2	363	389	413	426	462	475	482	509	526	544
5.6	362	389	413	426	462	475	482	509	526	544
6	356	385	413	426	462	475	482	509	526	544
6.4	345	381	407	426	462	475	482	509	526	544
6.8	334	374	403	426	462	475	482	509	526	544
7.2	316	364	394	419	458	475	482	509	526	544
7.6	301	350	386	415	452	475	482	509	526	544
8	281	338	375	410	444	473	482	509	526	544
8.4	261	322	365	403	437	469	482	509	526	544
8.8	240	306	352	392	428	459	482	509	526	544
9.2	217	289	334	379	417	453	476	509	526	544
9.6	198	272	322	369	407	446	471	507	526	544
10	178	250	302	352	394	436	464	500	519	544
10.4	159	234	287	336	381	425	457	495	515	544
10.8	138	216	270	318	365	413	446	487	510	540
11.2	118	200	249	301	351	400	433	479	499	533
11.6	100	185	231	284	332	385	420	470	491	523
12	78	169	211	268	319	370	408	457	481	515
12.4	60	152	194	249	304	351	391	443	470	507
12.8	42	140	175	231	290	340	380	428	460	497
13.2	20	124	159	213	269	326	364	410	450	484
13.6	5	108	145	195	247	311	349	397	436	471
14	N/A	104	138	179	231	296	333	381	423	457
14.4	N/A	102	135	166	214	282	319	368	409	446
14.8	N/A	102	130	155	196	267	304	351	394	431
15.2	N/A	102	130	150	183	248	292	340	380	419
15.6	N/A	102	130	145	173	235	280	323	365	404
16	N/A	102	130	141	169	224	270	310	353	392
16.4	N/A	102	130	141	166	209	253	298	337	377
16.8	N/A	102	130	141	166	201	241	282	324	364
17.2	N/A	102	130	141	166	194	232	271	312	349

17.6	N/A	102	130	141	166	188	224	260	299	334
18	N/A	102	130	141	166	181	214	247	284	320
18.4	N/A	102	130	141	166	181	208	242	273	307
18.8	N/A	102	130	141	166	181	207	237	264	297
19.2	N/A	102	130	141	166	181	207	232	250	286
19.6	N/A	102	130	141	166	181	207	227	245	277
20	N/A	102	130	141	166	181	207	227	245	272
20.4	N/A	102	130	141	166	181	207	227	245	270
20.8	N/A	102	130	141	166	181	207	227	245	270
21.2	N/A	102	130	141	166	181	207	227	245	270

*The position of each person is measured from the video took at China Pavilion.

Appendix Ⅱ: Mathematica® Programs

In[1]:=

```
DSolve[{v'[t] == k*(m - v[t]), v[0] == 0}, v[t], t]
```

Out[1]=

$\{\{v[t] \to e^{-k t} (-1 + e^{k t}) m\}\}$

In[2]:=

```
Simplify[%]
```

Out[2]=

$\{\{v[t] \to m - e^{-k t} m\}\}$

In[3]:=

$DSolve[\{k*v[t] == k*m - m*k*e^{-k t} - v'[t], v[0] == 0\}, v[t], t]$

Out[3]=

$\{\{v[t] \to e^{-k t} m (-1 + e^{k t} - k t)\}\}$

In[4]:=

$DSolve[\{k*v[t] + v'[t] == k*e^{-k t} m (-1 + e^{k t} - k t), v[0] == 0\}, v[t], t]$

Out[4]=

$\{\{v[t] \to \frac{1}{2} e^{-k t} m (-2 + 2 e^{k t} - 2 k t - k^2 t^2)\}\}$

In[5]:=

$DSolve[\{k*v[t] + v'[t] == k*\frac{1}{2} e^{-k t} m (-2 + 2 e^{k t} - 2 k t - k^2 t^2), v[0] == 0\}, v[t], t]$

Out[5]=

$\{\{v[t] \to \frac{1}{6} e^{-k t} m (-6 + 6 e^{k t} - 6 k t - 3 k^2 t^2 - k^3 t^3)\}\}$

In[6]:=

$m*\left(1 - \sum_{i=0}^{n-2} ((k\verb|^|i*t\verb|^|i)/(i!))*e^{-k t}\right)$

Out[6]=

$$m\left(1-\frac{(-1+n)\,\text{Gamma}[-1+n,\,k\,t]}{\text{Gamma}[n]}\right)$$

In[7]:=

$$\textbf{FullSimplify}\left[m\left(1-\frac{(-1+n)\,\text{Gamma}[-1+n,\,k\,t]}{\text{Gamma}[n]}\right)\right]$$

Out[7]=

$$m-\frac{m\,\text{Gamma}[-1+n,\,k\,t]}{\text{Gamma}[-1+n]}$$

In[8]:=

$$\textbf{DSolve}\left[\left\{v'[t]==k*(0-v[t]),\ v[0]==m\left(1-\frac{(-1+2)\,\text{Gamma}[-1+2,\,k\,b]}{\text{Gamma}[2]}\right)\right\},\ v[t],\ t\right]$$

Out[8]=

$$\{\{v[t]\to e^{-b\,k-k\,t}\,(-1+e^{b\,k})\,m\}\}$$

In[9]:=

$$\textbf{DSolve}\left[\left\{v'[t]==k*\left(e^{-b\,k-k\,t}\,(-1+e^{b\,k})\,m-v[t]\right),\ v[0]==m\left(1-\frac{(-1+3)\,\text{Gamma}[-1+3,\,k\,b]}{\text{Gamma}[3]}\right)\right\},\ v[t],\ t\right]$$

Out[9]=

$$\{\{v[t]\to e^{-b\,k-k\,t}\,m\,(e^{b\,k}-k\,t+e^{b\,k}\,k\,t-e^{b\,k}\,\text{Gamma}[2,\,b\,k])\}\}$$

In[10]:=

$$\textbf{DSolve}\left[\left\{v'[t]==k*\left(e^{-b\,k-k\,t}\,m\,\left(e^{b\,k}-k\,t+e^{b\,k}\,k\,t-e^{b\,k}\,\text{Gamma}[2,\,b\,k]\right)-v[t]\right),\ v[0]==m\left(1-\frac{(-1+4)\,\text{Gamma}[-1+4,\,k\,b]}{\text{Gamma}[4]}\right)\right\},\ v[t],\ t\right]$$

Out[10]=

$$\left\{\left\{v[t]\to\frac{1}{2}\,e^{-b\,k-k\,t}\,m\,(2\,e^{b\,k}-2\,k\,t+2\,e^{b\,k}\,k\,t-2\,b\,k^2\,t-k^2\,t^2+e^{b\,k}\,k^2\,t^2-e^{b\,k}\,\text{Gamma}[3,\,b\,k])\right\}\right\}$$

In[11]:=

$$\textbf{DSolve}\left[\left\{v'[t]==k*\left(\frac{1}{2}\,e^{-b\,k-k\,t}\,m\,\left(2\,e^{b\,k}-2\,k\,t+2\,e^{b\,k}\,k\,t-2\,b\,k^2\,t-k^2\,t^2+e^{b\,k}\,k^2\,t^2-e^{b\,k}\,\text{Gamma}[3,\,b\,k]\right)-v[t]\right),\ v[0]==m\left(1-\frac{(-1+5)\,\text{Gamma}[-1+5,\,k\,b]}{\text{Gamma}[5]}\right)\right\},\ v[t],\ t\right]$$

Out[11]=

$$\left\{\left\{v[t] \to \frac{1}{6} e^{-b k-k t} m \left(6 e^{b k}-6 k t+6 e^{b k} k t-6 b k^2 t-3 b^2 k^3 t-3 k^2 t^2+3 e^{b k} k^2 t^2-3 b k^3 t^2-k^3 t^3+e^{b k} k^3 t^3-e^{b k} \text{Gamma}[4, b k]\right)\right\}\right\}$$

In[12]:=

```
FullSimplify[
 {m * e^(-k*t) (-Sum[k^i * (t^i - b^i)/i!, {i, 1, n-2}] + e^(k*b) * (Sum[k^i * (t - b)^i/i!, {i, 0, n-2}] - 1/(n - 2)! Gamma[n - 1, k * b]))}]
```

$$\left\{\frac{m\, (-\text{Gamma}[-1+n, k\, t]+\text{Gamma}[-1+n, k\, (-b+t)])}{\text{Gamma}[-1+n]}\right\}$$

In[13]:=

```
f[t_] := If[0 ≤ t ≤ b, m - m Gamma[-1 + n, k t]/Gamma[-1 + n],
  m (-Gamma[-1 + n, k t] + Gamma[-1 + n, k (-b + t)])/Gamma[-1 + n]]
```

In[14]:=

```
Plot[{f[t] /. {n → 2, m → 5, k → 0.5, b → 5}, f[t] /. {n → 3, m → 5, k → 0.5, b → 5},
  f[t] /. {n → 4, m → 5, k → 0.5, b → 5}, f[t] /. {n → 5, m → 5, k → 0.5, b → 5},
  f[t] /. {n → 6, m → 5, k → 0.5, b → 5}, f[t] /. {n → 7, m → 5, k → 0.5, b → 5},
  f[t] /. {n → 8, m → 5, k → 0.5, b → 5}}, {t, 0, 40}, AxesLabel → {t, Vn}]
```

Out[14]=

- Graphics -

In[15]:=

```
Plot[{m (-1 + n) Gamma[-1 + n, k t]/Gamma[n] /. {m → 5, k → 0.5, n → 2},
  m (-1 + n) Gamma[-1 + n, k t]/Gamma[n] /. {m → 5, k → 0.5, n → 3},
  m (-1 + n) Gamma[-1 + n, k t]/Gamma[n] /. {m → 5, k → 0.5, n → 4},
  m (-1 + n) Gamma[-1 + n, k t]/Gamma[n] /. {m → 5, k → 0.5, n → 5},
  m (-1 + n) Gamma[-1 + n, k t]/Gamma[n] /. {m → 5, k → 0.5, n → 6},
  m (-1 + n) Gamma[-1 + n, k t]/Gamma[n] /. {m → 5, k → 0.5, n → 7}}, {t, 0, 25}, AxesLabel → {t, Vn}]
```

Out[15]=

- Graphics -

In[16]:=

```
pts = {{5, 0.193264861}, {10, 0.264745909}, {15, 0.29754447}, {20, 0.316969089},
  {25, 0.330147927}, {45, 0.358375118}, {65, 0.37216392}, {85, 0.380715453},
  {105, 0.386679602}, {125, 0.391143396}, {195, 0.400901785}, {265, 0.406468223},
  {335, 0.410180352}, {405, 0.412881733}, {475, 0.414960955}, {600, 0.417722877},
  {725, 0.419735016}, {850, 0.4212849}, {975, 0.422526127}, {1100, 0.423549188},
  {1225, 0.424411317}}
```

Out[16]=

```
{{5, 0.193265}, {10, 0.264746}, {15, 0.297544}, {20, 0.316969},
 {25, 0.330148}, {45, 0.358375}, {65, 0.372164}, {85, 0.380715}, {105, 0.38668},
 {125, 0.391143}, {195, 0.400902}, {265, 0.406468}, {335, 0.41018},
 {405, 0.412882}, {475, 0.414961}, {600, 0.417723}, {725, 0.419735},
 {850, 0.421285}, {975, 0.422526}, {1100, 0.423549}, {1225, 0.424411}}
```

```
FindFit[pts, a*Log[x]*x^c + b, {a, b, c}, x]
```

```
{a → 0.167739, b → -0.0159267, c → -0.141343}
```

In[17]:=

```
Show[ListPlot[pts, PlotStyle → PointSize[0.01], AxesLabel → {t, V}],
 Plot[a*Log[x]*x^c + b /. {a → 0.11237237271871775`, b → 0.09181098106195348`,
    c → -0.12348418701155632`}, {x, 0, 1300}]]
```

Out[17]=

```
- Graphics -
```

In[18]:=

```
FindFit[{{5, 1.617903887}, {10, 2.79047123}, {15, 4.20452728}, {20, 5.744342022},
  {25, 7.363990446}, {45, 14.29081861}, {65, 21.60892536}, {85, 29.14346833},
  {105, 36.82057517}, {125, 44.60075318}, {195, 72.36022529}, {265, 100.6331049},
  {335, 129.2234163}, {405, 158.0350738}, {475, 187.0124679}, {600, 239.0652823},
  {725, 291.4123477}, {850, 343.9799432}, {975, 396.7207926}, {1100, 449.6024275},
  {1225, 502.6014086}}, a*x + b*x^c (d + e*Log[x]), {a, b, c, d, e}, x]
```

```
FindFit::cvmit :
 Failed to converge to the requested accuracy or precision within 100 iterations. More…
```

Out[18]=

```
{a → -3.45365, b → 0.994849, c → 0.959569, d → 3.55543, e → 0.228133}
```

In[19]:=

$$\text{FullSimplify}\left[\int_0^t \frac{m\,(-1+n)\,\text{Gamma}[-1+n,\,k\,t]}{\text{Gamma}[n]}\,dt\right]$$

Out[19]=

```
m (k t Gamma[-1 + n, k t] + Gamma[n, 0] - Gamma[n, k t])
--------------------------------------------------------
                  k Gamma[-1 + n]
```

In[20]:=

```
m (k t Gamma[-1 + n, k t] + Gamma[n, 0] - Gamma[n, k t])
-------------------------------------------------------- /.
                  k Gamma[-1 + n]
 n ->
  {a * t + b * t ^c (d + e * Log[t]) /. {a → -3.455397765703369`, b → 0.994909920785172`,
     c → 0.9595773762661212`, d → 3.5569785040642334`, e → 0.22815991816971726`}}
```

Out[20]=

```
{(m (k t Gamma[-1 - 3.4554 t + 0.99491 t^0.959577 (3.55698 + 0.22816 Log[t]), k t] +
      Gamma[-3.4554 t + 0.99491 t^0.959577 (3.55698 + 0.22816 Log[t]), 0] -
      Gamma[-3.4554 t + 0.99491 t^0.959577 (3.55698 + 0.22816 Log[t]), k t])) /
  (k Gamma[-1 - 3.4554 t + 0.99491 t^0.959577 (3.55698 + 0.22816 Log[t])])}
```

In[21]:=

```
FullSimplify[
 (-3.455397765703369` t + 0.994909920785172` t^0.9595773762661212`
      (3.5569785040642334` + 0.22815991816971726` Log[t]) - 1) * 50 -
   (m
      (k t Gamma[-1 - 3.455397765703369` t + 0.994909920785172` t^0.9595773762661212`
            (3.5569785040642334` + 0.22815991816971726` Log[t]), k t] +
        Gamma[-3.455397765703369` t + 0.994909920785172` t^0.9595773762661212`
           (3.5569785040642334` + 0.22815991816971726` Log[t]), 0] -
        Gamma[-3.455397765703369` t + 0.994909920785172` t^0.9595773762661212`
           (3.5569785040642334` + 0.22815991816971726` Log[t]), k t])) /
    (k Gamma[-1.` - 3.455397765703369` t + 0.994909920785172` t^0.9595773762661212`
          (3.5569785040642334` + 0.22815991816971726` Log[t])]) /. {k → 0.44, m → 20}]
```

Out[21]=

```
-50. + 176.944 t^0.959577 - 172.77 t -
  (45.4545 (0.44 t Gamma[-1 - 3.4554 t + 0.99491 t^0.959577 (3.55698 + 0.22816 Log[t]), 0.44 t] +
      Gamma[-3.4554 t + 0.99491 t^0.959577 (3.55698 + 0.22816 Log[t]), 0] -
      Gamma[-3.4554 t + 0.99491 t^0.959577 (3.55698 + 0.22816 Log[t]), 0.44 t])) /
   Gamma[-1. - 3.4554 t + 0.99491 t^0.959577 (3.55698 + 0.22816 Log[t])] +
  11.3499 t^0.959577 Log[t]
```

In[22]:=

```
Show[Plot[-45.45454545454546` + 176.94366008565532` t^0.9595773762661212` -
    172.76988828516846` t -
    (45.45454545454546`
      (0.44` t Gamma[-1 - 3.455397765703369` t +
           0.994909920785172` t^0.9595773762661212`
            (3.5569785040642334` + 0.22815991816971726` Log[t]), 0.44` t] +
        Gamma[-3.455397765703369` t + 0.994909920785172` t^0.9595773762661212`
           (3.5569785040642334` + 0.22815991816971726` Log[t]), 0] -
        Gamma[-3.455397765703369` t + 0.994909920785172` t^0.9595773762661212`
           (3.5569785040642334` + 0.22815991816971726` Log[t]), 0.44` t])) /
     Gamma[-1.` - 3.455397765703369` t +
       0.994909920785172` t^0.9595773762661212`
        (3.5569785040642334` + 0.22815991816971726` Log[t])] +
    11.349928305629236` t^0.9595773762661212` Log[t], {t, 0, 1300}, AxesLabel -> {t, Sp}],
  ListPlot[{{10, 10.734856342759997}, {265, 463.6486080089826},
    {1225, 2302.5604021482977}}, PlotStyle -> PointSize[0.01]]]
```

```
Plot::plnr : -45.4545 + 176.944 t^<<19>> - <<19>> t - (<<18>> (<<1>>))/(Gamma[<<1>>]) + 11.3499 t^0.959577 Log[t]
   is not a machine-size real number at t = 0.0000541666666666664`. More…
```

Out[22]=

```
- Graphics -
```

In[23]:=

```
f[t_] := If[0 ≤ t ≤ b, a (p + (m - p) (1 - Gamma[-1 + n, k t]/Gamma[-1 + n])),
   a (p + ((m - p) (-Gamma[-1 + n, k t] + Gamma[-1 + n, k (-b + t)]))/Gamma[-1 + n])]
```

In[24]:=

```
Plot[{
   f[t] /. {n -> 2, k -> 0.5, m -> 5, p -> 1, b -> 30, a -> 1.5},
   f[t] /. {n -> 3, k -> 0.5, m -> 5, p -> 1, b -> 30, a -> 2},
   f[t] /. {n -> 4, k -> 0.5, m -> 5, p -> 1, b -> 30, a -> 2.5},
   f[t] /. {n -> 5, k -> 0.5, m -> 5, p -> 1, b -> 30, a -> 3},
   f[t] /. {n -> 6, k -> 0.5, m -> 5, p -> 1, b -> 30, a -> 3.5}}, {t, 0, 50},
  PlotStyle -> {Red, Magenta, Green, Blue, Black}, AxesLabel -> {t, Pn},
  PlotRange -> Automatic]
```

Out[24]=

```
- Graphics -
```

In[25]:=

```
f[t_] := a (p + (m - p) (1 - Gamma[-1 + n, k t] / Gamma[-1 + n]))
```

In[26]:=

```
Show[Plot[f[t] /. {n → 10, k → 1.5, m → 2.1, p → 1.6, a → 1}, {t, 0, 75},
  PlotRange → {0, 5}, AxesLabel → {t, P1}],
 ListPlot[{{0, 1.8}, {3, 1.6}, {5, 1.7}, {7, 2}, {10, 2.1}, {13, 2.08}, {16, 2.08},
   {19, 2.1}, {21, 2.1}, {25, 1.98}, {29, 2.1}, {33, 2.1}, {42, 2.1}, {51, 2.16},
   {56, 2.16}, {61, 2}, {64, 2.16}, {67, 2.15}, {70, 2.15}, {75, 2}},
  PlotStyle → PointSize[0.01]]]
```

Out[26]=

```
- Graphics -
```

In[27]:=

```
Show[Plot[f[t] /. {n → 15, k → 0.4, m → 2.1, p → 1.6, a → 6.25}, {t, 0, 80},
  PlotRange → {8, 15}, AxesLabel → {t, P2}],
 ListPlot[{{15, 9.8}, {17, 10.8}, {20, 10.4}, {25, 10.5}, {31, 10.6}, {36, 11.5},
   {38, 11.8}, {43, 13.6}, {47, 13}, {52, 12.4}, {55, 13.2}, {59, 12.8}, {62, 12.8},
   {65, 13}, {68, 12.7}, {71, 13.2}, {76, 13}}, PlotStyle → PointSize[0.01]]]
```

Out[27]=

```
- Graphics -
```

In[28]:=

```
Show[Plot[f[t] /. {n → 6, k → 0.1, m → 2.1, p → 1.6, a → 9.81}, {t, 0, 80},
  PlotRange → {13, 23}, AxesLabel → {t, P3}],
 ListPlot[{{6, 15.62}, {11, 15.76}, {16, 15.76}, {21, 16}, {26, 16.38}, {31, 17.26},
   {36, 17.62}, {41, 17.62}, {46, 18.12}, {51, 18.26}, {67, 19.62}, {72, 19.38},
   {77, 20.5}}, PlotStyle → PointSize[0.01]]]
```

Out[28]=

```
- Graphics -
```