THERE IS A NON-VAN DOUWEN MAD FAMILY

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Abstract

Two functions f and g on $\mathbb{N}$ are almost disjoint if and only if f and g agree on finite many natural numbers, i.e., $\{n \in \mathbb{N} | f(n) = g(n)\}$ is finite. A set $\mathcal{A} \subset \mathbb{N}^{\mathbb{N}}$ is an almost disjoint family if and only if for any two functions f and g in $\mathcal{A}$, f and g are almost disjoint. A maximal almost disjoint (MAD) family is with respect to the almost disjointness in $\mathbb{N}^{\mathbb{N}}$. A Van Douwen MAD family is maximal in $\mathbb{N}^{\mathbb{N}}$, moreover it is also maximal with the respect of infinite partial functions on $\mathbb{N}$.

In [5] D. Raghavan proved the existence of a Van Douwen MAD family, thus he closed a long standing question in set theory. On the other hand, although obviously, Axiom of Choice (or Zorn's Lemma) implies the existence of a MAD family, we do not know whether such a MAD family is Van Douwen or not.

In this short article, the author gives some attempt to construct a MAD family which is not Van Douwen under the Continuum Hypothesis. He hopes the method that is used here can shed some light on the construction of a non-Van Douwen MAD family in ZFC and some related problems, e.g. whether or not there is a closed MAD family in $\mathbb{N}^{\mathbb{N}}$, etc., see [1].

The results in this paper was initially inspired by Yi Zhang's talk in ShenZhen Middle School in October 2010.

摘要

两个$\mathbb{N}^{\mathbb{N}}$中的函数f和g几乎不相交，当且仅当f和g在图像上仅有有限个交点，也就是说集合$\{n \in \mathbb{N} | f(n) = g(n)\}$是可数的。集合$\mathcal{A} \subset \mathbb{N}^{\mathbb{N}}$是几乎不相交集族当且仅当集合$\mathcal{A}$中的函数两两几乎不相交。称一个几乎不相交集族$\mathcal{A}$极大当且仅当对任意在集合$\mathbb{N}^{\mathbb{N}} - \mathcal{A}$中的函数$g$，在中总存在一个函数$f$与$g$几乎不相交。一个Van Douwen极大几乎不相交集族不仅在$\mathbb{N}^{\mathbb{N}}$中极大，也在所有无穷部分函数的集合中极大。

在[5]中D. Raghavan证明了Van Douwen极大几乎不相交集族的存在性，解决了一个存在已久的集合论问题。另一方面，尽管选择公理(或佐恩引理)蕴涵着极大几乎不相交集族的存在性，我们并不知道这样的极大几乎不相交集族是否是Van Douwen极大几乎不相交集族。

在这篇论文中，作者证明了若连续统假设成立，则存在一个非Van Douwen极大几乎不相交集族。作者希望本文中所用的方法能对非Van Douwen极大几乎不相交集族得构造，以及一些相关问题如是否存在一个闭的极大几乎不相交集族，见[1]，起到帮助。

本论文是最初受到张翼教授2010年10月在深圳中学的讲座启发。

1 INTRODUCTION

In this paper, we will make an attempt to shed some light on the structure of a maximal almost disjoint family in $\mathbb{N}^{\mathbb{N}}$. Namely we will try to construct a maximal almost disjoint family which is not Van Douwen [5] under the Continuum Hypothesis. Two functions f and g in $\mathbb{N}^{\mathbb{N}}$ are said to be almost disjoint if they only have finite intersections. So we use the following as our definition of the almost disjointness.

Definition 1.1. Functions f and g in $\mathbb{N}^{\mathbb{N}} = \{f|f : \mathbb{N} \to \mathbb{N}\}$ are said to be almost disjoint if and only if $|f \cap g| < \aleph_0$, i.e. $\{n \in \mathbb{N}|f(n) = g(n)\}$ is finite.

For a set of functions in $\mathbb{N}^{\mathbb{N}}$, if all functions in the family are pairwise almost disjoint, we call such family an almost disjoint family.

Definition 1.2. A family $\mathcal{A}$ in $\mathbb{N}^{\mathbb{N}}$ is called an almost disjoint family if and only if for any two functions f and g in $\mathcal{A}$, $|f \cap g| < \aleph_0$. An almost disjoint family $\mathcal{A}$ in $\mathbb{N}^{\mathbb{N}}$ is called a maximal almost disjoint family or a MAD family, if and only if for any function g in $\mathbb{N}^{\mathbb{N}}$ there is a function f in $\mathcal{A}$ so that f and g are not almost disjoint.

Van Douwen asked whether there is a MAD family of functions $\mathcal{A}$ in $\mathbb{N}^{\mathbb{N}}$ that is also maximal with respect to all infinite partial functions [3]. We call such a family Van Douwen MAD [5].

Definition 1.3. A MAD family $\mathcal{A}$ in $\mathbb{N}^{\mathbb{N}}$ is called a Van Douwen MAD family if and only if for any infinite partial function g, $|f_i \cap g| = \aleph_0$, for all $f_i \in \mathcal{A}$.

Van Douwen's question dates to the 1980s. It occurs as problem 4.2 in A. Miller's problem list [3]. By axiom of choice or by Zorn's lemma, we know that there is a MAD family, e.g. see [6]. But we do not have sufficient details to determine whether the MAD family is Van Douwen or not. In 1999 Y. Zhang [7] got some partial results on this problem. He showed the cardinalities of Van Douwen MAD families are independent of ZFC, etc. D. Raghavan [5] solved this problem in 2010. He showed that there is a Van Douwen MAD family in ZFC. Van Douwen's problem was solved. By solving this problem, Raghavan got the Sacks Prize. We now know that there is a Van Douwen MAD family in ZFC. Naturally we can also ask whether or not all MAD families are Van Douwen. If a MAD family is not Van Douwen, we call the family a non-Van Douwen MAD family.

Definition 1.4. A MAD family $\mathcal{A}$ in $\mathbb{N}^{\mathbb{N}}$ is called non-Van Douwen if and only if there is an infinite partial function g, for any function f in $\mathcal{A}$, f and g are almost disjoint.

In this paper, unless otherwise defined, we will use the standard terminology of set theory, see, e.g. [2]. In Section 2 we will discuss some lemmas which are essential for the process of the construction. We will also define an infinite partial function for the construction of the non-Van Douwen MAD family. Then in Section 3 we will construct a Van Douwen MAD family step by step under the Continuum Hypothesis. We will also show that Continuum Hypothesis implies that there is a dense non-Van Douwen MAD family. However, the existence of non-Van Douwen MAD family in ZCF is still an open question.

2 SOME TECHNICAL LEMMAS

In this section we will discuss some lemmas which are essential for the construction of a non Van Douwen MAD family. We will also list some well-known results as lemmas for later use.

Lemma 2.1. *The biggest prime number does not exist.*

Definition 2.2. Let g be an infinite partial function such that the $dom(g) = \{p, p^p, p^{p^p}, \cdots\}$, in which p is a large enough prime number, and $g(p) = p^p$, $g(g(p)) = p^{g(p)}$, $g(g(g(p))) = p^{g(g(p))}$, $\cdots$, $\underbrace{g(g(\cdots g(p)\cdots))}_{n\ gs} = p^{\overbrace{g(\cdots g(p)\cdots)}^{(n-1)\ gs}}$, $\cdots$.

We will frequently use this function g in later sections.

Lemma 2.3. *Let* $\mathcal{A} = \{\alpha \in ord | 0 \leq \alpha < \aleph_1\}$ *and* $\mathcal{B} = \{\beta \in ord | \aleph_0 \leq \beta < \aleph_1\}$*. There is a bijection between* $\mathcal{A}$ *and* $\mathcal{B}$*.*

Proof. Let f be the bijection from $\mathcal{A}$ to $\mathcal{B}$.

$$f(x) = \begin{cases} \aleph_0 + 2x + 1 & \text{if } 0 \leq x < \aleph_0, \\ \aleph_0 + 2y & \text{if } \aleph_0 \leq x < \aleph_0 + \aleph_0 \text{ and } x = \aleph_0 + y, \\ x & \text{if } \aleph_0 + \aleph_0 \leq x < \aleph_1. \end{cases}$$

So there is a bijection between $\mathcal{A}$ and $\mathcal{B}$.

□

Lemma 2.4. *For a function* f *in* $\mathbb{N}^{\mathbb{N}}$ *and an countable almost disjoint family* $\mathcal{A}$ *in* $\mathbb{N}^{\mathbb{N}}$*, if* $\mathcal{A} \cup \{f\}$ *is also an almost disjoint family, but* f *is not almost disjoint with the infinite partial function* g*, which was defined in Definition 2.2, we can always construct a new function* f' *such that*
(1) $|f' \cap f| = \aleph_0$*, and*
(2) $|f' \cap h| < \aleph_0$*, for* $h \in \mathcal{A}$*, and*
(3) $|f' \cap g| < \aleph_0$*.*

Proof. By Definition 2.2, $dom(g) = \{p, p^p, p^{p^p}, \cdots\}$. Assume that the almost disjoint family

$$\mathcal{A} = \{f_i \in \mathbb{N}^{\mathbb{N}} | 0 \leq i \leq \alpha, \aleph_0 \leq \alpha < \aleph_1\}.$$

Let

$$f'(x) = \begin{cases} 1 + g(x) + f(x) + \sum_{i=0}^{x-1} f_{i-1}(x) & \text{if } x \in dom(g), \\ f(x) & \text{if } x \in \mathbb{N} - dom(g). \end{cases}$$

Since f is almost disjoint with all functions in $\mathcal{A}$, and

$$\text{if } x \in dom(g), f'(x) > f_i(x), \text{ for } 0 \leq i < x,$$

f' is almost disjoint with all functions in $\mathcal{A}$. Also since for $x \in dom(g)$ $f'(x) > g(x)$, f' is almost disjoint with g. □

The following are lemmas that are essential for the construction of a dense non-Van Douwen MAD family.

Lemma 2.5. *There are countably many finite partial functions from* $\mathbb{N}$ *to* $\mathbb{N}$*.*

Lemma 2.6. *A set* $\mathcal{D}$ *is dense in* $\mathbb{N}^{\mathbb{N}}$ *if and only if for any basic open set* $\mathcal{B}_s$*,* $\mathcal{D} \cap \mathcal{B}_s \neq \emptyset$*.*

3 CONSTRUCTING A NON-VAN DOUWEN MAD FAMILY UNDER THE CONTINUUM HYPOTHESIS

In this section, we will prove our main theorem, i.e. we will construct a non-Van Douwen MAD family.

Theorem 3.1. *Continuum Hypothesis implies that there is a non-Van Douwen MAD family.*

Proof. By definition of the non-Van Douwen MAD family, to construct a non-Van Dauwen MAD family, we need to fix an infinite partial function. Let this infinite partial function be the function g that we defined in Definition 2.2. To construct a non-Van Douwen MAD family, we first construct an almost disjoint family $\mathcal{A}_{\aleph_0}$ as following,

$$\mathcal{A}_{\aleph_0} = \{f_i \in \mathbb{N} | f_i(x) = \bar{i}, i \in \mathbb{N}\}.$$

We will extend $\mathcal{A}_{\aleph_0}$ to a non-Van Douwen MAD family. By Lemma 2.3 and CH, we can list all functions in $\mathbb{N}^{\mathbb{N}}$ as follows: $f_{\aleph_0}, f_{\aleph_0+1}, f_{\aleph_0+2}, \cdots, f_\alpha, \cdots$, where $\aleph_0 \leq \alpha < \aleph_1$. At each stage α, we can do as following. For the function f_α, if $\{f_\alpha\} \cup \mathcal{A}_\alpha$ is not an almost disjoint family, let

$$\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha,$$

if $\{f_\alpha\} \cup \mathcal{A}_\alpha$ is an almost disjoint family, and f_α is almost disjoint with g, let

$$\mathcal{A}_{\alpha+1} = \{f_\alpha\} \cup \mathcal{A}_\alpha,$$

if $\{f_\alpha\} \cup \mathcal{A}_\alpha$ is an almost disjoint family, but f_α is not almost disjoint with g, by Lemma 2.4 we construct a new function f'_α such that
(1) $|f'_\alpha \cap f_\alpha| = \aleph_0$, and
(2) $|f'_\alpha \cap h| < \aleph_0$, for $h \in \mathcal{A}_\alpha$, and
(3) $|f'_\alpha \cap g| < \aleph_0$,
and let

$$\mathcal{A}_{\alpha+1} = \{f'_\alpha\} \cup \mathcal{A}_\alpha.$$

For a limit ordinal α, we let

$$\mathcal{A}_\alpha = \bigcup_{\aleph_0 \leq \beta < \alpha} \mathcal{A}_\beta.$$

We start with $\mathcal{A}_{\aleph_0}$ to continue doing this process under the Continuum Hypothesis and we will get a series of almost disjoint families:$\mathcal{A}_{\aleph_0}$, $\mathcal{A}_{\aleph_0+1}$, $\mathcal{A}_{\aleph_0+2}$, $\cdots$, $\mathcal{A}_\alpha$, $\cdots (\alpha < \aleph_1)$. Let

$$\mathcal{A} = \bigcup_{\aleph_0 \leq \alpha < \aleph_1} \mathcal{A}_\alpha.$$

$\mathcal{A}$ is a non-Van Douwen MAD family.

We will first verify that the family is a maximal almost disjoint family.

Suppose that there is a function h in $\mathbb{N}^{\mathbb{N}} - \mathcal{A}$ such that h is almost disjoint with all functions in $\mathcal{A}$. Let us consider h in the following two cases.

a. Let h be almost disjoint with g, which was defined in Definition 2.2.

Since we have considered all functions in $\mathbb{N}^{\mathbb{N}}$, there must be a β such that $h = f_\beta$, $\aleph_0 \leq \beta < \aleph_1$. Now for the family A_β, h is the next function we need to consider. Since h is almost disjoint with all functions in $\mathcal{A}$, h is almost disjoint with all functions in $\mathcal{A}_\beta$. Also h is almost disjoint with g, which was defined in Definition 2.2. As a result based on the rules of the process, we would add h into $\mathcal{A}_\beta$: let $\mathcal{A}_{\beta+1} = \mathcal{A}_\beta \cup \{h\}$. Therefore if there were a such function h, h should have been in $\mathcal{A}$. Thus h should not exist.

b. Let h be not almost disjoint with g, which was defined in Definition 2.2.

Since we have considered all functions in $\mathbb{N}^{\mathbb{N}}$, there must be a β such that $h = f_\beta$, $\aleph_0 \leq \beta < \aleph_1$. For the almost disjoint family A_β, h is the next function we need to consider. We know that h is not almost disjoint with g, which was defined in Definition 2.2. Based on the rules of the process, we would change part of h to construct a new function h' in $\mathbb{N}^{\mathbb{N}}$ such that h' is almost disjoint with all functions in $\mathcal{A}_\beta$ and h' is almost disjoint with g, which was defined in Definition 2.2. Then we added h' into $\mathcal{A}_\beta$ instead of h: let $\mathcal{A}_{\beta+1} = \mathcal{A}_\beta \cup \{h'\}$. We can inferred that $h' \in \mathcal{A}$, but we know that h is not almost disjoint with h'. Therefore such function h could not exist.

Hence such function h in the assumption above does not exist.

Secondly, we will verify that $\mathcal{A}$ is a non-Van Douwen MAD family.

We know that functions in $\mathcal{A}_{\aleph_0} = \{f_i \in \mathbb{N} | f_i(x) = \bar{i}, i \in \mathbb{N}\}$ are almost disjoint with g. Based on the rules of the construction, we know that the functions we added into $\mathcal{A}_{\aleph_0}$ later are all almost disjoint with g. As a result all functions in $\mathcal{A}$ are almost disjoint with g. $\mathcal{A}$ is a non-Van Douwen MAD family.

□

In $\mathbb{N}^{\mathbb{N}}$, we consider its natural topology, e.g. see [4], i.e. the basic open set $\mathcal{B}_s = \{f \in \mathbb{N}^{\mathbb{N}} | f \supset s\}$, where s is a finite partial function. We know that it is an open problem whether there is a closed MAD family in $\mathbb{N}^{\mathbb{N}}$, see e.g. [1]. However, by our construction of non-Van Douwen MAD family above, we can easily prove the following corollaries.

Corollary 3.2. *Continuum Hypothesis implies that there is a dense non-Van Douwen MAD family* $\mathcal{A} \subseteq \mathbb{N}^{\mathbb{N}}$.

Proof. We will use a similar way to construct a dense non-Van Douwen MAD family. By Theorem 2.5, we can list all finite partial function as following: S_0, S_1, S_2, $\cdots$, S_n, $\cdots$, in which $n \in \mathbb{N}$. Let

$$f_n(i) = \begin{cases} s_n(i) & \text{if } i \in dom(s_n), \\ n & \text{if } i \notin dom(s_n), \end{cases}$$

and

$$\mathcal{A}_{\aleph_0} = \{f_i|, i \in \mathbb{N}\}.$$

For any basic open set $\mathcal{B}_s$, there must be a natural number n such that $f_n \in \mathcal{B}_s$. As a result for any basic open set B_s, $\mathcal{A}_{\aleph_0} \cap \mathcal{B}_s \neq \emptyset$. By Lemma 2.6, we know that $\mathcal{A}_{\aleph_0}$ is dense in $\mathbb{N}^{\mathbb{N}}$. Then we list all functions in $\mathbb{N}^{\mathbb{N}}$ as follows: $f_{\aleph_0}, f_{\aleph_0+1}, f_{\aleph_0+2}, \cdots, f_\alpha, \cdots$, where $\aleph_0 \leq \alpha < \aleph_1$, and use similar method to construct $\mathcal{A}_{\aleph_0+1}$, $\mathcal{A}_{\aleph_0+2}$, $\cdots$, $\mathcal{A}_\alpha$, $\cdots$, where α is an ordinal such that $\aleph_0 < \alpha < \aleph_1$. Then similarly we let

$$\mathcal{A} = \bigcup_{\aleph_0 \leq \alpha < \aleph_1} \mathcal{A}_\alpha.$$

Since $\mathcal{A}_{\aleph_0} \subset \mathcal{A}$ is dense in $\mathbb{N}^{\mathbb{N}}$, $\mathcal{A}$ is dense in $\mathbb{N}^{\mathbb{N}}$. Therefore $\mathcal{A}$ is a dense non-Van Douwen MAD family.

□

Corollary 3.3. *There is a dense MAD family* $\mathcal{A} \subset \mathbb{N}^{\mathbb{N}}$

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