Journal of Combinatorial Theory, Series B 144 (2020) 309–358

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

The Kelmans-Seymour conjecture IV: A proof[☆]

Dawei He[1], Yan Wang, Xingxing Yu[2]

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA

ARTICLE INFO

Article history:
Received 21 December 2016
Available online 19 December 2019

Keywords:
K_5-subdivision
Independent paths
Separation
Connectivity
Discharging
Contraction

ABSTRACT

A well known theorem of Kuratowski in 1932 states that a graph is planar if, and only if, it does not contain a subdivision of K_5 or $K_{3,3}$. Wagner proved in 1937 that if a graph other than K_5 does not contain any subdivision of $K_{3,3}$ then it is planar or it admits a cut of size at most 2. Kelmans and, independently, Seymour conjectured in the 1970s that if a graph does not contain any subdivision of K_5 then it is planar or it admits a cut of size at most 4. In this paper, we give a proof of the Kelmans-Seymour conjecture. We also discuss several related results and problems.

1. Introduction

For a graph G, we use TG to denote a subdivision of G, and the vertices in TG that correspond to the vertices of G are said to be its *branch* vertices. Thus, TK_5 denotes a subdivision of K_5, and the vertices in a TK_5 of degree four are its branch vertices. For

☆ Partially supported by NSF grants DMS-1265564 and DMS-1600738.

E-mail addresses: dhe9@math.gatech.edu (D. He), yanwang@gatech.edu (Y. Wang), yu@math.gatech.edu (X. Yu).

[1] This work was started when DH was a student at East China Normal University.

[2] Part of the work was done while XY was a short-term visitor at East China Normal University which was made possible by STCSM grant No. 13dz2260400 awarded to the mathematics department of ECNU.

https://doi.org/10.1016/j.jctb.2019.12.002
0095-8956/

graphs H and K, we say that H contains TK if H contains a subgraph isomorphic to a TK.

The well known result of Kuratowski [18] states that a graph is planar if, and only if, it does not contain TK_5 or $TK_{3,3}$. A simple application of Euler's formula for planar graphs shows that, for $n \geq 3$, if an n-vertex graph has at least $3n - 5$ edges then it must be nonplanar and, hence, contains TK_5 or $TK_{3,3}$. Dirac [5] conjectured that for $n \geq 3$, if an n-vertex graph has at least $3n - 5$ edges then it must contain TK_5. This conjecture was also reported by Erdős and Hajnal [7]. Kézdy and McGuiness [15] showed that a minimal counterexample to Dirac's conjecture must be 5-connected and contains K_4^-, where K_4^- is the graph obtained from the complete graph K_4 by deleting an edge. (However, Kelmans [14] and Seymour (see [22]) knew in the 1970s that a minimal counterexample to Dirac's conjecture must be 5-connected.) After some partial results in [28,30,33,34], Dirac's conjecture was proved by Mader [22], where he also showed that every 5-connected n-vertex graph with at least $3n - 6$ edges contains TK_5 or K_4^-.

Seymour [26] (also see [22,34]) and, independently, Kelmans [14] conjectured that every 5-connected nonplanar graph contains TK_5. Thus, the Kelmans-Seymour conjecture implies Mader's theorem. This conjecture is also related to several interesting problems, which we will discuss in Section 7.

The authors [9–11] produced lemmas needed for resolving this Kelmans-Seymour conjecture, and we are now ready to prove it in this paper.

Theorem 1.1. *Every 5-connected non-planar graph contains TK_5.*

The starting point of our work is the following result of Ma and Yu [20,21]: Every 5-connected nonplanar graph containing K_4^- has a TK_5. This result, combined with the result of Kézdy and McGuiness [15] on minimal counterexamples to Dirac's conjecture, gives an alternative proof of Mader's theorem. Also using this result, Aigner-Horev [1] proved that every 5-connected nonplanar apex graph contains TK_5. A simpler proof of Aigner-Horev's result using discharging argument was obtained by Ma, Thomas, and Yu, and, independently, by Kawarabayashi, see [13].

We now briefly describe the process for proving Theorem 1.1. For a more detailed version, we recommend the reader to read Section 6 first, which should also give motivation to some of the technical lemmas listed in Sections 2, 3, 4 and 5.

Suppose G is a 5-connected non-planar graph not containing K_4^-. We fix a vertex $v \in V(G)$, and let M be a maximal connected subgraph of G such that $v \in V(M)$, G/M (the graph obtained from G by contracting M) is nonplanar, G/M contains no K_4^-, and G/M is 5-connected (i.e., M is contractible). Note that $V(M) = \{v\}$ is possible. Let x denote the vertex of $H := G/M$ resulting from the contraction of M. Then, for each subgraph T of H with $v \in V(T)$ and with $T \cong K_2$ or $T \cong K_3$, H/T is planar, or H/T contains K_4^-, or H/T is not 5-connected. If, for some T, H/T is planar or contains K_4^- then we can find a TK_5 in G using results from [9–11]. Thus, in this paper, our main work is to deal with the final case: for any subgraph T of H with $x \in V(T)$ and with

$T \cong K_2$ or $T \cong K_3$, it follows that H/T is nonplanar, H/T contains no K_4^-, and H/T is not 5-connected. In this case, there exists $S_T \subseteq V(H)$ such that $V(T) \subseteq S_T$, $|S_T| = 5$ or $|S_T| = 6$, and $H - S_T$ is not connected. We will be using such cuts to divide the graph into smaller parts and use them to find a special TK_5 in H. The reason to also include the case $T \cong K_3$ is to avoid the situation when $T \cong K_2$, $|S_T| = 5$, and $H - S_T$ has exactly two components, one of which is trivial. This situation does not cause problems when $T \cong K_3$, as the graph H would then contain K_4^-, and we could use results from [9–11].

We will need a number of results from [9–11], which are given in Section 2. In Section 3, we derive a simplified version of a result on disjoint paths from [39–41], which will be used several times in Section 4. For each subgraph T of H with $v \in V(T)$ and with $T \cong K_2$ or $T \cong K_3$, we will associate to it a quadruple (T, S_T, A, B), where, roughly, $A \cap B = \emptyset$, $H - S_T = A \cup B$, and H has no edge between A and B. (A precise definition of a quadruple is given in Section 4.) In Section 4, we prove some basic properties of quadruples, and take care of two special cases involving quadruples (using disjoint paths results from Section 3). In Section 5, we take care of other cases involving quadruples. We complete the proof of Theorem 1.1 in Section 6, and discuss several related problems in Section 7.

We end this section with some notation and terminology. Let G be a graph. By $S \subseteq G$ we mean that S is a subgraph of G. We may view $S \subseteq V(G)$ as a subgraph of G with vertex set S and no edges. For $S \subseteq G$, we use $G[S]$ to denote the subgraph of G induced by $V(S)$. For any $x \in V(G)$ we use $N_G(x)$ to denote the neighborhood of x in G, and for $S \subseteq G$ let $N_G(S) = \{x \in V(G) - V(S) : N_G(x) \cap V(S) \neq \emptyset\}$. When understood, the reference to G may be dropped. For $S \subseteq E(G)$, $G - S$ denotes the graph obtained from G by deleting all edges in S; and for $K, L \subseteq G$, $K - L$ denotes the graph obtained from K by deleting $V(K \cap L)$ and all edges of K incident with $V(K \cap L)$.

A *separation* in a graph G consists of a pair of subgraphs G_1, G_2 of G, denoted as (G_1, G_2), such that $V(G) = V(G_1) \cup V(G_2)$, $E(G_1) \cup E(G_2) = E(G)$, $E(G_1 \cap G_2) = \emptyset$, $E(G_1) \cup (V(G_1) - V(G_2)) \neq \emptyset$, and $E(G_2) \cup (V(G_2) - V(G_1)) \neq \emptyset$. The *order* of this separation is $|V(G_1) \cap V(G_2)|$, and (G_1, G_2) is said to be a *k-separation* if its order is k. A set $S \subseteq V(G)$ is a *k-cut* (or a *cut* of size k) in G, where k is a positive integer, if $|S| = k$ and G has a separation (G_1, G_2) such that $V(G_1) \cap V(G_2) = S$, $V(G_1 - S) \neq \emptyset$, and $V(G_2 - S) \neq \emptyset$. (Thus, for a separation (G_1, G_2) in a graph G, $V(G_1) \cap V(G_2)$ need not be a cut in G.) If $v \in V(G)$ and $\{v\}$ is a cut of G, then v is said to be a *cut vertex* of G. For $A \subseteq V(G)$ with $G - A \neq \emptyset$ and for a positive integer k, we say that G is *(k, A)-connected* if, for any cut S with $|S| < k$, every component of $G - S$ contains a vertex from A. Thus, if G is a k-connected graph and (G_1, G_2) is a separation in G such that $V(G_2) - V(G_1) \neq \emptyset$, then G_2 is $(k, V(G_1 \cap G_2))$-connected.

Given a path P in a graph and $x, y \in V(P)$, xPy denotes the subpath of P between x and y (inclusive). The *ends* of the path P are the vertices of the minimum degree in P, and all other vertices of P (if any) are its *internal* vertices. A path P with ends u and v (or an u-v path) is also said to be *from u to v* or *between u and v*. A collection of paths is said to be *independent* if no vertex of any path in this collection is an internal vertex of any other path in the collection.

Let G be a graph. Let $K \subseteq G$, $S \subseteq V(G)$, and T a collection of 2-element subsets of $V(K) \cup S$. Then $K + (S \cup T)$ denotes the graph with vertex set $V(K) \cup S$ and edge set $E(K) \cup T$, and if $T = \{\{x, y\}\}$ we write $K + xy$ instead of $K + \{\{x, y\}\}$.

For any positive integer k, let $[k] := \{1, \dots, k\}$ (and let $[0] = \emptyset$). A *3-planar graph* $(G, \mathcal{A})$ consists of a graph G and a set $\mathcal{A} = \{A_1, \dots, A_k\}$ of pairwise disjoint subsets of $V(G)$ (possibly $\mathcal{A} = \emptyset$ when $k = 0$) such that

(a) for distinct $i, j \in [k]$, $N(A_i) \cap A_j = \emptyset$,
(b) for $i \in [k]$, $|N(A_i)| \leq 3$, and
(c) if $p(G, \mathcal{A})$ denotes the graph obtained from G by (for each i) deleting A_i and adding edges joining every pair of distinct vertices in $N(A_i)$ that are not already adjacent in G, then $p(G, \mathcal{A})$ may be drawn in a closed disc D in the plane with no pair of edges crossing such that, for each A_i with $|N(A_i)| = 3$, $N(A_i)$ induces a facial triangle in $p(G, \mathcal{A})$.

If, in addition, $b_1, \dots, b_n$ are vertices of G such that $b_i \notin A_j$ for any $i \in [n]$ and $j \in [k]$ and $b_1, \dots, b_n$ occur on the boundary of the disc D in that cyclic order, then we say that $(G, \mathcal{A}, b_1, \dots, b_n)$ is *3-planar* or, simply, $(G, b_1, \dots, b_n)$ is 3-planar (if there is no need to mention $\mathcal{A}$). If there is no need to specify the order of $b_1, \dots, b_n$ then we simply say that $(G, \mathcal{A}, \{b_1, \dots, b_n\})$ or $(G, \{b_1, \dots, b_n\})$ is 3-planar. When $\mathcal{A} = \emptyset$, we say that $(G, b_1, \dots, b_n)$ or $(G, \{b_1, \dots, b_n\})$ is planar (in this case, G is actually a planar graph).

Note that if $(G, \{b_1, \dots, b_n\})$ is 3-planar and G is $(4, \{b_1, \dots, b_n\})$-connected, then $(G, \{b_1, \dots, b_n\})$ is in fact planar and G has a plane drawing in a closed disc with $b_1, \dots, b_n$ on the boundary of the disk.

2. Previous results

In this section, we list a number of previous results which we will use as lemmas in our proof of Theorem 1.1. We begin with the following result of Ma and Yu [20,21].

Lemma 2.1. *Every* 5*-connected nonplanar graph containing* K_4^- *has a* TK_5*.*

We also need the main result of [10], to take care of the case when the vertex x in $H = G/M$ (see Section 1) is a degree 2 vertex in a K_4^- (which is y_2 in the lemma below).

Lemma 2.2. *Let* G *be a* 5*-connected nonplanar graph and* $\{x_1, x_2, y_1, y_2\} \subseteq V(G)$ *such that* $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ *with* $y_1y_2 \notin E(G)$*. Then one of the following holds:*

(*i*) G *contains a* TK_5 *in which* y_2 *is not a branch vertex.*
(*ii*) $G - y_2$ *contains* K_4^-*.*

(*iii*) *G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{y_2, a_1, a_2, a_3, a_4\}$, and G_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4$ a_1 and the 4-cycle $b_1b_2b_3b_4b_1$ by adding y_2 and the edges y_2b_i for $i \in [4]$.*

(*iv*) *For all distinct $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$, $G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$ contains TK_5.*

To deal with conclusion (*iii*) of Lemma 2.2, we need Proposition 1.3 from [9] in which a plays the role of y_2 in Lemma 2.2.

Lemma 2.3. *Let G be a 5-connected nonplanar graph, (G_1, G_2) a 5-separation in G, $V(G_1 \cap G_2) = \{a, a_1, a_2, a_3, a_4\}$ such that G_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding a and the edges ab_i, $i \in [4]$. Suppose $|V(G_1)| \geq 7$. Then, for any distinct $u_1, u_2 \in N(a) - \{b_1, b_2, b_3\}$, $G - \{av : v \notin \{b_1, b_2, b_3, u_1, u_2\}\}$ contains TK_5.*

Next we list a few results from [9–11]. For convenience, we state their versions from [11]. First, we need Theorem 1.1 in [11] to take care of the case when the vertex x in $H = G/M$ (see Section 1) is a degree 3 vertex in a K_4^- (which is x_1 in the lemma below).

Lemma 2.4. *Let G be a 5-connected nonplanar graph and $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. Then one of the following holds:*

(*i*) *G contains a TK_5 in which x_1 is not a branch vertex.*

(*ii*) *$G - x_1$ contains K_4^-, or G contains a K_4^- in which x_1 is of degree 2.*

(*iii*) *x_2, y_1, y_2 may be chosen so that for any distinct $z_0, z_1 \in N(x_1) - \{x_2, y_1, y_2\}$, $G - \{x_1v : v \notin \{x_2, y_1, y_2, z_0, z_1\}\}$ contains TK_5.*

When applying the next three lemmas, the vertex a will correspond to the vertex x in $H = G/M$ in Section 1. The following result is a direct consequence of Theorem 1.1 in [9], which deals with 5-separations with an apex side.

Lemma 2.5. *Let G be a 5-connected nonplanar graph and let (G_1, G_2) be a 5-separation in G. Suppose $|V(G_i)| \geq 7$ for $i \in [2]$, $a \in V(G_1 \cap G_2)$, and $(G_2 - a, V(G_1 \cap G_2) - \{a\})$ is planar. Then one of the following holds:*

(*i*) *for any $a^* \in V(G_1 - G_2) \cup \{a\}$, G contains a TK_5 in which a^* is not a branch vertex.*

(*ii*) *$G - a$ contains K_4^-, or G contains a K_4^- in which a is of degree 2.*

The next result is Lemma 2.8 in [11], which will be used to take care of 5-cuts containing the vertices of a triangle.

Lemma 2.6. *Let G be a 5-connected graph and (G_1, G_2) be a 5-separation in G. Suppose that $|V(G_i)| \geq 7$ for $i \in [2]$ and $G[V(G_1 \cap G_2)]$ contains a triangle aa_1a_2a. Then one of the following holds:*

- (i) *G contains a TK_5 in which a is not a branch vertex.*
- (ii) *$G - a$ contains K_4^-, or G contains a K_4^- in which a is of degree 2.*
- (iii) *For any distinct $u_1, u_2, u_3 \in N(a) - \{a_1, a_2\}$, $G - \{av : v \notin \{a_1, a_2, u_1, u_2, u_3\}\}$ contains TK_5.*

The following is Lemma 2.9 in [11].

Lemma 2.7. *Let G be a graph, $A \subseteq V(G)$, and $a \in A$ such that $|A| = 6$, $|V(G)| \geq 8$, $(G - a, A - \{a\})$ is planar, and G is $(5, A)$-connected. Then one of the following holds:*

- (i) *$G - a$ contains K_4^-, or G contains a K_4^- in which the degree of a is 2.*
- (ii) *G has a 5-separation (G_1, G_2) such that $a \in V(G_1 \cap G_2)$, $|V(G_2)| \geq 7$, $A \subseteq V(G_1)$, and $(G_2 - a, V(G_1 \cap G_2) - \{a\})$ is planar.*

We need Theorem 1.4 in [9]. This will be used to show that, for a quadruple (T, S_T, A, B) in $H = G/M$ with $x \in V(T)$ (see Section 1), x has a neighbor in A (which corresponds to $G_1 - G_2$ in the statement).

Lemma 2.8. *Let G be a 5-connected graph and $x \in V(G)$, and let (G_1, G_2) be a 6-separation in G such that $x \in V(G_1 \cap G_2)$, $G[V(G_1 \cap G_2)]$ contains a triangle xx_1x_2x, and $|V(G_i)| \geq 7$ for $i \in [2]$. Moreover, assume that (G_1, G_2) is chosen so that, subject to $\{x, x_1, x_2\} \subseteq V(G_1 \cap G_2)$ and $|V(G_i)| \geq 7$ for $i \in [2]$, G_1 is minimal. Let $V(G_1 \cap G_2) = \{x, x_1, x_2, v_1, v_2, v_3\}$. Then $N(x) \cap V(G_1 - G_2) \neq \emptyset$, or one of the following holds:*

- (i) *G contains a TK_5 in which x is not a branch vertex.*
- (ii) *G contains K_4^-.*
- (iii) *There exists $x_3 \in N(x)$ such that for any distinct $y_1, y_2 \in N(x) - \{x_1, x_2, x_3\}$, $G - \{xv : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}$ contains TK_5.*
- (iv) *For some $i \in [2]$ and some $j \in [3]$, $N(x_i) \subseteq V(G_1 - G_2) \cup \{x, x_{3-i}\}$, and any three independent paths in $G_1 - x$ from $\{x_1, x_2\}$ to v_1, v_2, v_3, respectively, with two from x_i and one from x_{3-i}, must contain a path from x_{3-i} to v_j.*

We remark that conclusion (iv) in Lemma 2.8 will be dealt with in Section 4, using a result on disjoint paths from [39–41]. We also need Proposition 4.1 from [9] to deal with the case when H/T is planar (see Section 1) for some $T \subseteq H$ with $x \in V(T)$ and with $T \cong K_2$ or $T \cong K_3$.

Lemma 2.9. *Let G be a 5-connected nonplanar graph, $x \in V(G)$, and $T \subseteq G$, such that $x \in V(T)$, $T \cong K_2$ or $T \cong K_3$, and G/T is 5-connected and planar. Then $G - T$ contains K_4^-.*

We conclude this section with three additional results, first of which is a result of Seymour [25]; equivalent versions are proved in [31,24,27].

Lemma 2.10. *Let G be a graph and let $s_1, s_2, t_1, t_2 \in V(G)$ be distinct. Then either G contains disjoint paths from s_1 to t_1 and from s_2 to t_2, or (G, s_1, s_2, t_1, t_2) is 3-planar.*

The second result is due to Perfect [23].

Lemma 2.11. *Let G be a graph, $u \in V(G)$, and $A \subseteq V(G - u)$. Suppose there exist k independent paths from u to distinct $a_1, \dots, a_k \in A$, respectively, and internally disjoint from A. Then for any $n \geq k$, if there exist n independent paths $P_1, \dots, P_n$ in G from u to n distinct vertices in A and internally disjoint from A then $P_1, \dots, P_n$ may be chosen so that $a_i \in V(P_i)$ for $i \in [k]$.*

The third result is due to Watkins and Mesner [38], which gives a characterization of graphs G with no cycle through three given vertices y_1, y_2, y_3. Roughly, G has 2-cuts separating these three vertices. See Fig. 1 for an illustration.

Lemma 2.12. *Let G be a 2-connected graph and let y_1, y_2, y_3 be three distinct vertices of G. Then G has no cycle containing $\{y_1, y_2, y_3\}$ if, and only if, one of the following holds:*

- (i) *There exists a 2-cut S in G and there exist pairwise disjoint subgraphs D_{y_i} of $G-S$, $i \in [3]$, such that $y_i \in V(D_{y_i})$ and each D_{y_i} is a union of components of $G - S$.*
- (ii) *There exist 2-cuts S_{y_i} in G, $i \in [3]$, and pairwise disjoint subgraphs D_{y_i} of G, such that $y_i \in V(D_{y_i})$, each D_{y_i} is a union of components of $G - S_{y_i}$, there exists $z \in S_{y_1} \cap S_{y_2} \cap S_{y_3}$, and $S_{y_1} - \{z\}, S_{y_2} - \{z\}, S_{y_3} - \{z\}$ are pairwise disjoint.*
- (iii) *There exist pairwise disjoint 2-cuts S_{y_i} in G and pairwise disjoint subgraphs D_{y_i} of $G - S_{y_i}$, $i \in [3]$, such that $y_i \in V(D_{y_i})$, D_{y_i} is a union of components of $G - S_{y_i}$, and $G - V(D_{y_1} \cup D_{y_2} \cup D_{y_3})$ has precisely two components, each containing exactly one vertex from S_{y_i} for $i \in [3]$.*

3. Obstruction to three paths

In order to deal with (iv) of Lemma 2.8, we need a result of the third author [39–41], which characterizes graphs G in which any three disjoint paths from $\{a, b, c\} \subseteq V(G)$ to $\{a', b', c'\} \subseteq V(G)$ must contain a path from b to b'. The objective of this section is to derive a much simpler version of that characterization by imposing extra conditions on

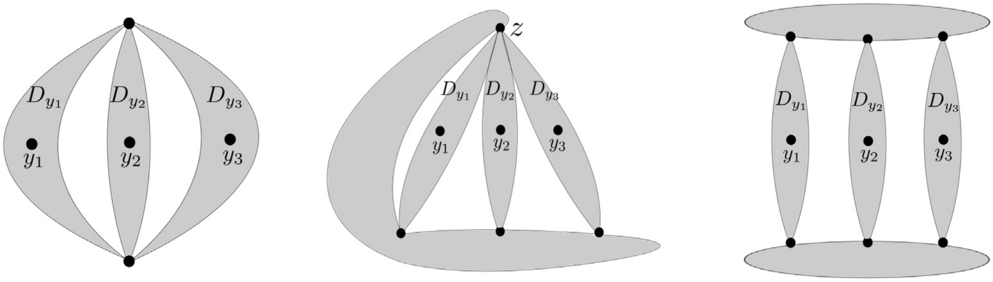

Fig. 1. No cycle containing $\{y_1, y_2, y_3\}$.

G. This result will be used several times in the proofs of Lemmas 4.4 and 4.6. To state the result from [39–41], we need to describe *rungs* and *ladders*.

Let G be a graph, $\{a, b, c\} \subseteq V(G)$, and $\{a', b', c'\} \subseteq V(G)$. (Here, a, b, c are pairwise distinct, and a', b', c' are pairwise distinct.) Suppose $\{a, b, c\} \neq \{a', b', c'\}$, and assume that G has no separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 3$, $\{a, b, c\} \subseteq V(G_1)$, and $\{a', b', c'\} \subseteq V(G_2)$. (So $\{a, b, c\}$ and $\{a', b', c'\}$ are independent sets in G.) We say that $(G, (a, b, c), (a', b', c'))$ is a *rung* if one of the following holds:

(1) $b = b'$ or $\{a, c\} = \{a', c'\}$.
(2) $a = a'$ and $(G - a, c, c', b', b)$ is 3-planar, or $c = c'$ and $(G - c, a, a', b', b)$ is 3-planar.
(3) $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$ and (G, a', b', c', c, b, a) or (G, a', b', c', a, b, c) is 3-planar.
(4) $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, G has a 1-separation (G_1, G_2) such that (i) $\{a, a', b, b'\} \subseteq V(G_1)$, $\{c, c'\} \subseteq V(G_2)$, and (G_1, a, a', b', b) is 3-planar, or (ii) $\{c, c', b, b'\} \subseteq V(G_1)$, $\{a, a'\} \subseteq V(G_2)$, and (G_1, c, c', b', b) is 3-planar.
(5) $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, and G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{z, b\}$ (or $V(G_1 \cap G_2) = \{z, b'\}$), and (i) (G, a, a', b', b) is 3-planar, $\{a, a', b, b'\} \subseteq V(G_1)$, $\{c, c'\} \subseteq V(G_2)$, and (G_2, c, c', z, b) (or (G_2, c, c', b', z)) is 3-planar, or (ii) (G, c, c', b', b) is 3-planar, $\{c, c', b, b'\} \subseteq V(G_1)$, $\{a, a'\} \subseteq V(G_2)$, and (G_2, a, a', z, b) (or (G_2, a, a', b', z)) is 3-planar.
(6) $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, and there are pairwise edge disjoint subgraphs G_a, G_c, M of G such that $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{u, w\}$, $V(G_c \cap M) = \{p, q\}$, $V(G_a \cap G_c) = \emptyset$, and (i) $\{a, a', b'\} \subseteq V(G_a)$, $\{c, c', b\} \subseteq V(G_c)$, and (G_a, a, a', b', w, u) and (G_c, c', c, b, p, q) are 3-planar, or (ii) $\{a, a', b\} \subseteq V(G_a)$, $\{c, c', b'\} \subseteq V(G_c)$, (G_a, b, a, a', w, u), and (G_c, b', c', c, p, q) are 3-planar.
(7) $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, and there are pairwise edge disjoint subgraphs G_a, G_c, M of G such that $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{b, b', w\}$, $V(G_c \cap M) = \{b, b', p\}$, $V(G_a \cap G_c) = \{b, b'\}$, $\{a, a'\} \subseteq V(G_a)$, $\{c, c'\} \subseteq V(G_c)$, and (G_a, a, a', b', w, b) and (G_c, c', c, b, p, b') are 3-planar.

Let L be a graph and let $R_1, \ldots, R_m$ be edge disjoint subgraphs of L such that

(*i*) $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))$ is a rung for each $i \in [m]$,
(*ii*) $V(R_i \cap R_j) = \{x_i, v_i, y_i\} \cap \{x_{j-1}, v_{j-1}, y_{j-1}\}$ for $i, j \in [m]$ with $i < j$,
(*iii*) for any $i, j \in [m] \cup \{0\}$, if $x_i = x_j$ then $x_k = x_i$ for all $i \le k \le j$, if $v_i = v_j$ then $v_k = v_i$ for all $i \le k \le j$, and if $y_i = y_j$ then $y_k = y_i$ for all $i \le k \le j$, and
(*iv*) $L = (\bigcup_{i=1}^{m} R_i) + S$, where S consists of those edges of L each of which has both ends in $\{x_i, v_i, y_i\}$ for some $i \in [m] \cup \{0\}$.

Then $(L, (x_0, v_0, y_0), (x_m, v_m, y_m))$ is a *ladder with rungs* $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))$, $i \in [m]$, or simply, a *ladder along* $v_0 \dots v_m$.

By definition, for any rung $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))$, R_i has three disjoint paths from $\{x_{i-1}, v_{i-1}, y_{i-1}\}$ to $\{x_i, v_i, y_i\}$. So for any ladder $(L, (x_0, v_0, y_0), (x_m, v_m, y_m))$, L has three disjoint paths from $\{x_0, v_0, y_0\}$ to $\{x_m, v_m, y_m\}$.

For a sequence W, the *reduced sequence* of W is the sequence obtained from W by removing all but one consecutive identical elements. For example, the reduced sequence of *aaabcca* is *abca*. We can now state the main result in [41].

Lemma 3.1. *Let G be a graph, $\{a, b, c\} \subseteq V(G)$, and $\{a', b', c'\} \subseteq V(G)$ such that $\{a, b, c\} \neq \{a', b', c'\}$. Assume that G is $(4, \{a, b, c\} \cup \{a', b', c'\})$-connected. Then any three disjoint paths in G from $\{a, b, c\}$ to $\{a', b', c'\}$ must include one from b to b' if, and only if, one of the following statements holds:*

(*i*) *G has a separation (G_1, G_2) of order at most 2 such that $\{a, b, c\} \subseteq V(G_1)$ and $\{a', b', c'\} \subseteq V(G_2)$.*
(*ii*) *$(G, (a, b, c), (a', b', c'))$ is a ladder.*
(*iii*) *G has a separation (J, L) such that $V(J \cap L) = \{w_0, \dots, w_n\}$, $(J, w_0, \dots, w_n)$ is planar, $\{a, b, c\} \cup \{a', b', c'\} \subseteq V(L)$, $(L, (a, b, c), (a', b', c'))$ is a ladder along a sequence $v_0 \dots v_m$, where $v_0 = b$, $v_m = b'$, and $w_0 \dots w_n$ is the reduced sequence of $v_0 \dots v_m$.*

Remark 1. We may remove the assumption that, for any $T \subseteq V(G)$ with $|T| \le 3$, every component of $G - T$ contains some element of $\{a, b, c\} \cup \{a', b', c'\}$. When we do, the conclusion of Lemma 3.1 holds by simply replacing "$(J, w_0, \dots, w_n)$ is planar" in (*iii*) with "$(J, w_0, \dots, w_n)$ is 3-planar".

Remark 2. We may view (*ii*) as a special case of (*iii*) by letting J be a subgraph of L. In the applications of Lemma 3.1 in this paper, we will consider rungs and ladders in a 5-connected graph without TK_5. With such extra conditions, the rungs have much simpler structure, as given in the next three lemmas. See Fig. 2. This first lemma follows from a simple inspection of the definition of rungs.

Lemma 3.2. *Let $(G, (a, b, c), (a', b', c'))$ be a rung. If $\{a, c\} \cap \{a', c'\} = \emptyset$ and a and c have the same set of neighbors in G, then $b = b'$.*

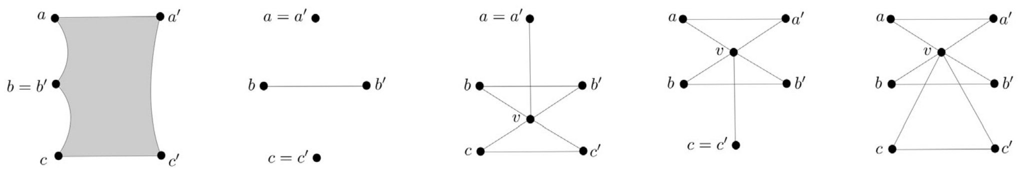

Fig. 2. The simple rungs as in Lemma 3.4.

Lemma 3.3. *Let G be a 5-connected graph and (R, R') a separation in G such that $R' - R \neq \emptyset$, $V(R \cap R') = \{a, b\} \cup \{a', b', c'\}$, $a \neq b$, $\{a, b\} \nsubseteq \{a', b', c'\}$, and a', b', c' are pairwise distinct. Let R^* be obtained from R by adding the new vertex c and joining c to each neighbor of a in R with an edge, and assume $(R^*, (a, b, c), (a', b', c'))$ is a rung. Then $b = b'$, $V(R) = \{a, b, a', c'\}$ and $E(R) = \{aa', ac'\}$.*

Proof. Note that if $(R^*, (a, b, c), (a', b', c'))$ is a rung of type (3)–(7) then a and c must have different sets of neighbors in R^*. For otherwise, by checking each of these five types, we see that R^* would admit a separation (H_1, H_2) such that $|V(H_1 \cap H_2)| \leq 3$, $\{a, b, c\} \subseteq V(H_1)$, and $\{a', b', c'\} \subseteq V(H_2)$.

Hence, since a and c have the same set of neighbors in R^*, $(R^*, (a, b, c), (a', b', c'))$ is of type (1) or (2). Thus, $|V(R \cap R')| = |\{a, b\} \cup \{a', b', c'\}| \leq 4$ and, since G is 5-connected and $R' - R \neq \emptyset$, it follows that $V(R) = \{a, b\} \cup \{a', b', c'\}$.

Suppose $(R^*, (a, b, c), (a', b', c'))$ is of type (2). Then, since $c \neq c'$, we have $a = a'$ and $(R^* - a, c, c', b', b)$ is 3-planar. Hence, $cb' \notin E(G)$ or $c'b \notin E(G)$. Thus, $\{a, b, c'\}$ or $\{a, b', c\}$ would be a cut in R^* separating $\{a, b, c\}$ from $\{a', b', c'\}$, a contradiction.

So $(R^*, (a, b, c), (a', b', c'))$ is of type (1). Then $b = b'$, as $c \notin \{a', c'\}$. Since $\{a, b\} \nsubseteq \{a', b', c'\}$, we have $a \neq a'$. Hence, since R^* has no separation of order at most 3 separating $\{a, b, c\}$ from $\{a', b', c'\}$, we deduce that $E(R) = \{aa', ac'\}$. □

Note that the conclusion of Lemma 3.3 is a special case of (i) of the next lemma.

Lemma 3.4. *Let G be a 5-connected nonplanar graph and (R, R') a separation in G such that $|V(R')| \geq 8$, $V(R \cap R') = \{a, b, c\} \cup \{a', b', c'\}$, $\{a, b, c\} \neq \{a', b', c'\}$, and $(R, (a, b, c), (a', b', c'))$ is a rung. Then for every $x \in V(R' - R)$, G contains TK_5 in which x is not a branch vertex; or G contains K_4^-; or one of the following holds:*

- (i) *$b = b'$.*
- (ii) *$\{a, c\} = \{a', c'\}$, $V(R) = \{a, c, b, b'\}$, and $E(R) = \{bb'\}$.*
- (iii) *$V(R) - (\{a, b, c\} \cup \{a', b', c'\}) = \{v\}$ and $N_G(v) = \{a, b, c\} \cup \{a', b', c'\}$, and either $a = a'$ and $E(R - v) = \{bb', cc'\}$ or $c = c'$ and $E(R - v) = \{bb', aa'\}$.*
- (iv) *$\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, $V(R) - \{a, a', b, b', c, c'\} = \{v\}$, $N_G(v) = \{a, a', b, b', c, c'\}$, and $E(R - v) = \{aa', bb', cc'\}$.*

Proof. By the definition of a rung, R has three disjoint paths from $\{a, b, c\}$ to $\{a', b', c'\}$, with one path from b to b'. So by the symmetry between a and c and the symmetry between a' and c', we may let A, B, C be disjoint paths in R from a, b, c to a', b', c', respectively. First, we consider the case when $\{a, b, c\} \cap \{a', b', c'\} \neq \emptyset$. If $b = b'$ then (i) holds; so we may assume $b \neq b'$. If $a = a'$ and $c = c'$ then, since G is 5-connected, $V(R) = \{a, b, b', c\}$; so $bb' \in E(R)$ (because of the paths A, B, C), and we have (ii). Thus by symmetry between $\{a, a'\}$ and $\{c, c'\}$, we may assume $c \neq c'$. Suppose $a = a'$. Then by the definition of a rung, $R - a$ has no disjoint paths from b, c to c', b', respectively. So by Lemma 2.10, $(R - a, c, c', b', b)$ is 3-planar. Since G is 5-connected, $R - a$ is $(4, \{b, b', c, c'\})$-connected; so $(R - a, c, c', b', b)$ is in fact planar. If $|V(R)| \geq 7$ then G contains TK_5 or K_4^- (by Lemmas 2.5 and 2.2, using the separation (R, R')). If $V(R) = \{a, b, b', c, c'\}$ then, since $(R - a, c, c', b', b)$ is planar, either $\{a, b, c'\}$ or $\{a, b', c\}$ is a 3-cut in R separating $\{a, b, c\}$ from $\{a', b', c'\}$, contradicting the definition of a rung. Thus, we may assume $|V(R)| = 6$ and let $v \in V(R) - \{a, b, b', c, c'\}$. Since G is 5-connected, $N_G(v) = \{a, b, b', c, c'\}$. Therefore, since $(R - a, c, c', b', b)$ is planar, $bc', cb' \notin E(R)$. So $bb', cc' \in E(R)$, as otherwise $\{a, v, c\}$ or $\{a, v, b\}$ would be a 3-cut in R separating $\{a, b, c\}$ from $\{a', b', c'\}$, contradicting the definition of a rung. Hence, (iii) holds.

Thus, we may assume that $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$. We need to deal with (3)–(7) in the definition of a rung. We deal with (4)–(7) in order, and treat (3) last (which is the most complicated case where we use the discharging technique).

Suppose (4) holds for $(R, (a, b, c), (a', b', c'))$. By symmetry, assume that R has a 1-separation (G_1, G_2) such that $\{a, a', b, b'\} \subseteq V(G_1)$, $\{c, c'\} \subseteq V(G_2)$, and (G_1, a, a', b', b) is 3-planar. Let $V(G_1 \cap G_2) = \{v\}$. Note that $v \notin \{a, b, c, a', b', c'\}$; otherwise, $\{a, b, c'\}$ or $\{a', b', c\}$ would be cut in R separating $\{a, b, c\}$ from $\{a', b', c'\}$. Since G is 5-connected, $V(G_2) = \{v, c, c'\}$. Again, since G is 5-connected, G_1 is $(5, \{a, a', b, b', v\})$-connected; so (G_1, a, a', b', b) is planar. Moreover, $vc, vc', cc' \in E(G)$; for otherwise $\{a, b, c'\}$ or $\{a', b', c\}$ or $\{a, b, v\}$ would be a cut in R separating $\{a, b, c\}$ from $\{a', b', c'\}$. If $|V(G_1)| \geq 7$ then the assertion follows from Lemmas 2.5 and 2.2, using the separation $(G_1, G_2 \cup R')$. So we may assume $|V(G_1)| \leq 6$. If $|V(G_1)| = 6$ then let $t \in V(G_1) - \{a, a', b, b', v\}$; now $N_G(t) = \{a, a', b, b', v\}$ and $|(N_G(v) - \{c, c'\}) \cap N_G(t)| \geq 2$ (since G is 5-connected), and hence R (and therefore G) contains K_4^-. So we may assume $V(G_1) = \{a, a', b, b', v\}$. Then $va' \in E(G)$; otherwise $N_G(v) = \{a, b, b', c, c'\}$ and, hence, $a'b \notin E(G)$ (as (G_1, a, a', b', b) is planar), which implies that $\{a, b', c'\}$ is a cut in R separating $\{a, b, c\}$ from $\{a', b', c'\}$, a contradiction. Similarly, $va, vb, vb' \in E(G)$. Then by planarity of (G_1, a, a', b', b), we have $ab', ba' \notin E(G)$. So $aa', bb' \in E(G)$ as $\{b, c, v\}$ and $\{a, v, c\}$ are not 3-cuts in R separating $\{a, b, c\}$ from $\{a', b', c'\}$. Thus we have (iv).

Suppose (5) holds for $(R, (a, b, c), (a', b', c'))$, and assume by symmetry that (R, a, a', b', b) is 3-planar, and R has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{z, b\}$, $\{a, a', b, b'\} \subseteq V(G_1)$, $\{c, c'\} \subseteq V(G_2)$, and (G_2, c, c', z, b) is 3-planar. Since G is 5-connected, $V(G_2) = \{b, z, c, c'\}$. Then $cz, cc' \in E(G)$ as, otherwise, $\{a, b, c'\}$ or $\{a, b, z\}$ would be a 3-cut in R separating $\{a, b, c\}$ from $\{a', b', c'\}$. Hence, since (G_2, b, z, c', c) is planar, $bc' \notin E(G)$. Since (R, a, a', b', b) is 3-planar, (G_1, a, a', b', b) is 3-planar. Thus, the separation $(G_1, G_2 - b)$ shows that $(R, (a, b, c), (a', b', c'))$ is of type (4); so we may assume that (iv) holds by the argument in the previous paragraph.

Now suppose (6) holds for $(R, (a, b, c), (a', b', c'))$, and, by symmetry, assume that there are pairwise edge disjoint subgraphs G_a, G_c, M of R such that $R = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{u, w\}$, $V(G_c \cap M) = \{p, q\}$, $V(G_a \cap G_c) = \emptyset$, $\{a, a', b'\} \subseteq V(G_a)$, $\{c, c', b\} \subseteq V(G_c)$, and (G_a, a, a', b', w, u) and (G_c, c', c, b, p, q) are 3-planar. Since G is 5-connected, $V(M) = \{p, q, u, w\}$. Also since G is 5-connected, G_a is $(5, \{a, a', b', w, u\})$-connected and G_c is $(5, \{c', c, b, p, q\})$-connected; so (G_a, a, a', b', w, u) and (G_c, c', c, b, p, q) are planar. We may assume that $|V(G_c) - \{b, c, c', p, q\}| \leq 1$ and $|V(G_a) - \{a, a', b', u, w\}| \leq 1$, as otherwise the assertion follows from Lemmas 2.5 and 2.2 (using the separation $(G_c, G_a \cup M \cup R')$ or $(G_a, G_c \cup M \cup R')$ in G). If there exists $v \in V(G_c) - \{b, c, c', p, q\}$ then, since G is 5-connected, $N_G(v) = \{b, c, c', p, q\}$ and $|(N_G(p) - \{u, w\}) \cap \{b, c, c', q\}| \geq 2$; so R (and hence G) contains K_4^-. Thus we may assume $V(G_c) = \{b, c, c', p, q\}$. Since G is 5-connected, p and q each have at least five neighbors in $G_c \cup M$. Hence, since (G_c, b, c, c', q, p) is planar, $N_G(p) = \{u, w, b, c, q\}$ and $N_G(q) = \{u, w, c, c', p\}$; so $G[\{p, q, u, w\}]$ (and hence G) contains K_4^-.

Suppose (7) holds for $(R, (a, b, c), (a', b', c'))$. Then there are pairwise edge disjoint subgraphs G_a, G_c, M of R such that $R = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{b, b', w\}$, $V(G_c \cap M) = \{b, b', p\}$, $V(G_a \cap G_c) = \{b, b'\}$, $\{a, a'\} \subseteq V(G_a)$, $\{c, c'\} \subseteq V(G_c)$, and (G_a, a, a', b', w, b) and (G_c, c', c, b, p, b') are 3-planar. Since G is 5-connected, $V(M) = \{b, b', p, w\}$, G_a is $(5, \{a, a', b', w, b\})$-connected, and G_c is $(5, \{c', c, b, p, b'\})$-connected. Hence, (G_a, a, a', b', w, b) and (G_c, c', c, b, p, b') are actually planar. If $|V(G_c)| \geq 7$ then the assertion follows from Lemmas 2.5 and 2.2 (using the separation $(G_c, G_a \cup M \cup R')$ in G). So we may assume $|V(G_c)| \leq 6$. If there exists $q \in V(G_c) - \{b, b', c, c', p\}$ then $N_G(q) = \{b, b', c, c', p\}$ (as G is 5-connected); therefore, since (G_c, c', c, b, p, b') is planar, $N_G(p) \subseteq \{b, b', w, q\}$, a contradiction. Thus $V(G_c) = \{b, b', c, c', p\}$ and, hence, $N_G(p) = \{b, b', c, c', w\}$. Similarly, by considering G_a, we may assume $N_G(w) = \{a, a', b, b', p\}$. Thus $G[\{b, b', p, w\}]$ (and hence G) contains K_4^-.

Finally, assume that (3) holds for $(R, (a, b, c), (a', b', c'))$. So (R, a', b', c', c, b, a) is planar (as G is 5-connected), and we may assume that R is embedded in a closed disc in the plane with no edge crossings such that a, b, c, c', b', a' occur on the boundary of the disc in clockwise order. We apply the discharging method. For convenience, let $A = \{a, b, c, a', b', c'\}$, $F(R)$ denote the set of faces of R, and f_∞ denote the outer face of R (which is incident with all vertices in A). For each $f \in F(R)$, let $d_R(f)$ denote the number of incidences of the edges of R with f, and ∂f denote the set of vertices of R incident with f. For $x \in V(R) \cup F(R)$, let $\sigma(x) = d_R(x) - 4$ be the charge of x.

We claim that R is connected. As G is 5-connected, each component of R must contain a vertex of $\{a, b, c\} \cup \{a', b', c'\}$. Recall that R has disjoint paths A, B, C from a, b, c to a', b', c', respectively. Therefore, if R is not connected then R would have separation (R_1, R_2) such that $V(R_1 \cap R_2) = \{a, b, c'\}$ or $V(R_1 \cap R_2) = \{a, b', c'\}$, $\{a, b, c\} \subseteq V(R_1)$ and $\{a', b', c'\} \subseteq V(R_2)$, a contradiction. So R is connected.

Hence, by Euler's formula, $\sum_{x\in V(R)\cup F(R)}\sigma(x) = -8$. We redistribute charges according to the following rule: For each $v \in V(R) - A$, v sends 1/2 to each $f \in F(R)$ that is incident with v and has $d_R(f) = 3$. Let $\tau(x)$ denote the new charge for all $x \in V(R) \cup F(R)$. Then

$$\sum_{x\in V(R)\cup F(R)} \tau(x) = \sum_{x\in V(R)\cup F(R)} \sigma(x) = -8.$$

Note that we may assume $K_4^- \not\subseteq G$. Thus, each $v \in V(R) - A$ is incident with at most $\lfloor d_R(v)/2 \rfloor$ faces $f \in F(R)$ with $d_R(f) = 3$; so $\tau(v) \geq 0$ (as $d_R(v) \geq 5$). Moreover, for $f \in F(R)$, $\tau(f) \geq 0$ unless $d_R(f) = 3$ and f is incident with at least two vertices in A.

Since R has no separation (R_1, R_2) of order at most 3 such that $\{a, b, c\} \subseteq V(R_1)$ and $\{a', b', c'\} \subseteq V(R_2)$, we see that $\{a, b, c\}$ and $\{a', b', c'\}$ are independent in R. Moreover, since (R, a, a', b', c', c, b) is planar, it follows that $ab', ac', ba', bc', ca', cb' \notin E(R)$, and $d_R(v) \geq 2$ for $v \in A$. (For example, if $ab' \in E(G)$ then $\{a, b', c'\}$ would be a cut in R separating $\{a, b, c\}$ from $\{a', b', c'\}$, and if $d_R(a) = 1$ then $N_R(a) \cup \{b, c\}$ would be a cut in R separating $\{a, b, c\}$ from $\{a', b', c'\}$.) Then $bb' \notin E(R)$; otherwise, since (R, a, a', b', c', c, b) is planar and G is 5-connected, $V(R) = A$ (to avoid 4-cuts $\{a, a', b, b'\}$ and $\{b, b', c, c'\}$), which in turn would force $d_R(v) \leq 1$ for some $v \in A$.

From above, we have $E(R[A]) \subseteq \{aa', cc'\}$ and $d_R(v) \geq 2$ for $v \in A$. Hence, $d_R(f_\infty) \geq 10$, and if $f \in F(R)$ with $d_R(f) = 3$ and $|\partial f \cap A| \geq 2$ then $\partial f \cap A = \{a, a'\}$ or $\partial f \cap A = \{c, c'\}$. Hence,

$$\begin{aligned}\sum_{x\in V(R)\cup F(R)} \tau(x) &\geq \sum_{v\in V(R)} \tau(v) + \sum_{f\in F(R), |\partial f\cap A|\geq 2} \tau(f)\\ &\geq \sum_{v\in A}(d_R(v) - 4) + (d_R(f_\infty) - 4) + \sum_{d_R(f)=3, |\partial f\cap A|\geq 2} (d_R(f) - 4 + 1/2)\\ &\geq (-12) + (10 - 4) + (-1/2) \times 2\\ &= -7,\end{aligned}$$

a contradiction. □

4. Quadruples and special structure

As mentioned in Section 1, we need to deal with 5-connected graphs in which every edge or triangle at a given vertex is contained in a cut of size 5 or 6. Thus, for convenience, we introduce the following concept of a quadruple.

Let G be a graph. For $x \in V(G)$, let $\mathcal{Q}_x$ denote the set of all quadruples (T, S_T, A, B), such that

(1) $T \subseteq G$, $x \in V(T)$, and either $T \cong K_2$ or $T \cong K_3$,

(2) S_T is a cut in G with $V(T) \subseteq S_T$, A is a nonempty union of components of $G - S_T$, and $B = G - A - S_T \neq \emptyset$,
(3) if $T \cong K_3$ then $5 \leq |S_T| \leq 6$, and
(4) if $T \cong K_2$ then $|S_T| = 5$, $|V(A)| \geq 2$, and $|V(B)| \geq 2$.

Note, in particular, that if $T \cong K_3$, then we allow $|V(A)| = 1$ or $|V(B)| = 1$.

The purpose of this section is to derive useful properties of quadruples, in particular, of those (T, S_T, A, B) that minimize $|V(A)|$. We begin with a few simple properties, first of which gives a bound on $|V(A)|$.

Lemma 4.1. *Let G be a 5-connected graph, $x \in V(G)$, and $(T, S_T, A, B) \in \mathcal{Q}_x$. Then G contains K_4^-, or* $\min\{|V(A)|, |V(B)|\} \geq 5$.

Proof. Suppose there exists $(T, S_T, A, B) \in \mathcal{Q}_x$ such that $|V(A)| \leq 4$ or $|V(B)| \leq 4$. We choose such $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum. Then $|V(A)| \leq 4$. Let δ denote the minimum degree of A, and let $u \in V(A)$ such that u has degree δ in A.

We may assume $\delta \geq 1$. For, suppose $\delta = 0$. If $T \cong K_3$ then, since G is 5-connected, $|N_G(u) \cap S_T| \geq 5$; so $G[T + u]$ contains K_4^-. Hence we may assume $T \cong K_2$. Then $|V(A)| \geq 2$ (see (4) above). Indeed, $|V(A)| = 2$ and A consists of two isolated vertices (as otherwise $(T, S_T, A - \{u\}, G[B + u])$ contradicts the choice of (T, S_T, A, B) (that $|V(A)|$ is minimum). Now $G[A \cup T]$ contains K_4^-.

Case 1. $\delta = 1$.

Then $|N_G(u) \cap S_T| \geq 4$. Let v be the unique neighbor of u in A. Since $|V(A)| \leq 4$ and G is 5-connected, $|N_G(v) \cap S_T| \geq 2$. We may assume $|N_G(u) \cap N_G(v) \cap S_T| \leq 1$; for, otherwise, $G[S_T \cup \{u, v\}]$ contains K_4^-.

Suppose $|N_G(v) \cap S_T| \geq 3$ or $N_G(u) \cap N_G(v) \cap S_T = \emptyset$. Then $|S_T| = 6$ and, hence, $T \cong K_3$. Therefore, $|N_G(u) \cap V(T)| \geq 2$ or $|N_G(v) \cap V(T)| \geq 2$; so $G[T + u]$ or $G[T + v]$ contains K_4^-.

Hence, we may assume that $|N_G(v) \cap S_T| \leq 2$ and $|N_G(u) \cap N_G(v) \cap S_T| = 1$. Then, since $|V(A)| \leq 4$ and G is 5-connected, $|N_G(v) \cap S_T| = 2$, $|N_G(v) \cap V(A)| = 3$, and $|V(A)| = 4$. Let $v_1, v_2 \in V(A) - \{u, v\}$, and let $w \in N_G(u) \cap N_G(v) \cap S_T$. Since G is 5-connected, $|N_G(v_i) \cap S_T| \geq 3$ for $i \in [2]$.

We may assume $w \notin V(T)$; for, if $w \in V(T)$ then $|V(T) \cap N_G(u)| \geq 2$ or $|V(T) \cap N_G(v)| \geq 2$, and $G[T + \{u, v\}]$ contains K_4^-. We may also assume $w \notin N_G(v_i)$ for $i \in [2]$, as otherwise $G[\{u, v, w, v_i\}]$ contains K_4^-.

If $v_1v_2 \notin E(G)$ then $|N_G(v_i) \cap S_T| \geq 4$ for $i \in [2]$; so $|N_G(v_i) \cap V(T)| \geq 2$ for $i \in [2]$ (since $w \notin N_G(v_i)$ and $w \notin V(T)$), and hence, $G[T + \{v_1, v_2\}]$ contains K_4^-. So assume $v_1v_2 \in E(G)$. Since G is 5-connected and $w \notin N_G(v_i)$ for $i \in [2]$, there exists $w' \in N_G(v_1) \cap N_G(v_2) \cap S_T$. Now $G[\{v, v_1, v_2, w'\}]$ contains K_4^-.

Case 2. $\delta \geq 2$.

If $|V(A)| = 3$ then $A \cong K_3$ and, since G is 5-connected, $|N_G(a) \cap S_T| \geq 3$ for all $a \in V(A)$; hence, since $|S_T| \leq 6$, $G[V(A) \cup S_T]$ contains K_4^-. So assume $|V(A)| = 4$. We may further assume that A is a cycle as otherwise A contains K_4^-. Hence, $|N_G(a) \cap S_T| \geq 3$ for all $a \in V(A)$. Moreover, we may assume that for any $st \in E(A)$, $|N_G(s) \cap N_G(t) \cap S_T| \leq 1$; for otherwise $G[\{s,t\} \cup S_T]$ contains K_4^-. Let $A = uvwru$.

Suppose $T \cong K_2$. Then for any $st \in E(A)$, $(N_G(s) \cup N_G(t)) - V(A) = S_T$ and $|N_G(s) \cap N_G(t) \cap S_T| = 1$. Let $S_T = \{x_1, x_2, x_3, x_4, x_5\}$ and, without loss of generality, let $N_G(u) \cap A = \{x_1, x_2, x_3\}$ and $N_G(v) \cap A = \{x_3, x_4, x_5\}$. Since $(N_G(w) \cup N_G(r)) - V(A) = S_T$, $wx_3 \in E(G)$ or $rx_3 \in E(G)$. Then $G[\{u, v, w, x_3\}] \cong K_4^-$ or $G[\{r, u, v, x_3\}] \cong K_4^-$.

Now assume $T \cong K_3$. Let $S_T = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ such that $V(T) = \{x_1, x_2, x_3\}$. We may assume $|N_G(a) \cap V(T)| \leq 1$ for each $a \in V(A)$, for, otherwise, $G[T + a]$ contains K_4^-. Hence, we may assume by symmetry that $x_4, x_5 \in N_G(u)$, $x_5, x_6 \in N_G(v)$, and $x_6, x_4 \in N_G(w)$. Note that $N_G(r) \cap \{x_4, x_6\} \neq \emptyset$. If $x_4 \in N_G(r)$ then $G[\{u, w, r, x_4\}] \cong K_4^-$, and if $x_6 \in N_G(r)$ then $G[\{v, w, r, x_6\}] \cong K_4^-$. □

Next, we show that if a graph G has no contractible edge or triangle at some vertex x then every edge of G at x is associated with a quadruple in $\mathcal{Q}_x$.

Lemma 4.2. *Let G be a 5-connected graph and $x \in V(G)$. Suppose for any $T \subseteq G$ with $x \in V(T)$ and with $T \cong K_2$ or $T \cong K_3$, G/T is not 5-connected. Then for any $ax \in E(G)$, there exists $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ such that $\{a, x\} \subseteq V(T')$.*

Proof. Let $T_1 = ax$. By assumption, G/T_1 is not 5-connected. So there exists a 5-cut S_{T_1} in G with $V(T_1) \subseteq S_{T_1}$. We may assume that $G - S_{T_1}$ has a trivial component; for otherwise, let C be a component of $G - S_{T_1}$ and $D = (G - S_{T_1}) - C$. Then $(T_1, S_{T_1}, C, D) \in \mathcal{Q}_x$ is the desired quadruple, with T_1 as T'.

So let $y \in V(G)$ such that y is a component of $G - S_{T_1}$. Since G is 5-connected, $N_G(u) = S_{T_1}$. Let $T_2 := G[T_1 + y] \cong K_3$. By assumption, G/T_2 is not 5-connected. So there exists a cut S_{T_2} in G such that $V(T_2) \subseteq S_{T_2}$ and $|S_{T_2}| \in \{5, 6\}$. Let C be a component of $G - S_{T_2}$ and $D = (G - S_{T_2}) - C$. Then $(T_2, S_{T_2}, C, D) \in \mathcal{Q}_x$ is the desired quadruple, with T_2 as T'. □

We now show that if (T, S_T, A, B) is chosen to minimize $|V(A)|$ then we may assume $T \cong K_3$. Throughout Sections 4, 5, and 6, we use Fig. 3 to illustrate the parts of G divided by two quadruples.

Lemma 4.3. *Let G be a 5-connected graph and $x \in V(G)$. Suppose for any $T \subseteq G$ with $x \in V(T)$ and with $T \cong K_2$ or $T \cong K_3$, G/T is not 5-connected. Then G contains K_4^-, or, for any $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum, $T \cong K_3$.*

Proof. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum, and assume $T \cong K_2$. Then $|S_T| = 5$. Let $a \in N_G(x) \cap V(A)$ (which exists as G is 5-connected). By Lemma 4.2,

	A	S_T	B
C	$A \cap C$	$S_T \cap V(C)$	$B \cap C$
$S_{T'}$	$S_{T'} \cap V(A)$	$S_T \cap S_{T'}$	$S_{T'} \cap V(B)$
D	$A \cap D$	$S_T \cap V(D)$	$B \cap D$

$x \in V(T) \cap V(T') \subseteq S_T \cap S_{T'}$

Fig. 3. Two quadruples (T, S_T, A, B) and $(T', S_{T'}, C, D)$.

there exists $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ such that $\{a, x\} \subseteq V(T')$. Note that $T' \cong K_2$ and $|S_{T'}| = 5$, or $T' \cong K_3$ and $|S_{T'}| \in \{5, 6\}$. We may assume

$$\min\{|V(A)|, |V(B)|\} \geq 5 \text{ and } \min\{|V(C)|, |V(D)|\} \geq 5;$$

for, if not, then G contains K_4^- by Lemma 4.1.

We may assume that if $A \cap C \neq \emptyset$ then $|(S_{T'} \cup S_T) - V(B \cup D)| \geq |S_{T'}| + 1$. For, suppose $A \cap C \neq \emptyset$ and $|(S_{T'} \cup S_T) - V(B \cup D)| \leq |S_{T'}|$. If $|V(A \cap C)| \geq 2$ or $T' \cong K_3$ then $(T', (S_{T'} \cup S_T) - V(B \cup D), A \cap C, G[B \cup D]) \in \mathcal{Q}_x$ and $|V(A \cap C)| \leq |V(A) - \{a\}| < |V(A)|$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. So assume $|V(A \cap C)| = 1$ and $T' \cong K_2$. Then $|S_{T'}| = 5$. Since $|(S_{T'} \cup S_T) - V(B \cup D)| \leq |S_{T'}|$ and G is 5-connected, $|(S_{T'} \cup S_T) - V(B \cup D)| = 5$. Assume for the moment $A \cap D = \emptyset$. Then, since $|S_{T'}| = 5$, $|V(A \cap C)| = 1$, and $x \in S_T \cap S_{T'}$, it follows that $|S_{T'} \cap V(A)| = 4$, $|S_{T'} \cap S_T| = 1$, $|S_{T'} \cap V(B)| = 0$, and $|S_T \cap V(C)| = 0$ (as $|(S_{T'} \cup S_T) - V(B \cup D)| = 5$). Since $|V(C)| \geq 5$, $B \cap C \neq \emptyset$. So $S_T \cap S_{T'}$ is a 1-cut in G, contradicting the assumption that G is 5-connected. Hence, $A \cap D \neq \emptyset$. Suppose for the moment $|(S_{T'} \cup S_T) - V(B \cup C)| = |S_{T'}|$. Then we may assume $|V(A \cap D)| \geq 2$; as otherwise, since G is 5-connected, $G[(A \cap C) \cup (A \cap D) \cup \{a, x\}] \cong K_4^-$. Now $(T', (S_{T'} \cup S_T) - V(B \cup C), A \cap D, G[B \cup C]) \in \mathcal{Q}_x$ and $2 \leq |V(A \cap D)| < |V(A)|$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Hence, $|(S_{T'} \cup S_T) - V(B \cup C)| \geq |S_{T'}| + 1 = 6$; so $|(S_{T'} \cup S_T) - V(A \cup D)| = |S_T| + |S_{T'}| - |(S_{T'} \cup S_T) - V(B \cup C)| \leq 4$. Since G is 5-connected, $B \cap C = \emptyset$. Since $|(S_{T'} \cup S_T) - V(B \cup D)| = 5$, $|S_T \cap V(C)| \leq 3$. Therefore, $|V(C)| \leq 4$, a contradiction.

Similarly, we may assume that if $A \cap D \neq \emptyset$ then $|(S_{T'} \cup S_T) - V(B \cup C)| \geq |S_{T'}| + 1$.

Suppose $A \cap C = A \cap D = \emptyset$. Then, since $|V(A)| \geq 5$, $|S_{T'}| \leq 6$, and $x \in S_T \cap S_{T'}$, it follows that $|S_{T'} \cap V(A)| = |V(A)| = 5$, $|S_T \cap S_{T'}| = 1$, and $|S_{T'} \cap V(B)| = 0$. Since $|S_T| = 5$ and G is 5-connected, we see that $B \cap C = \emptyset$ or $B \cap D = \emptyset$. However, this implies $|V(C)| \leq 4$ or $|V(D)| \leq 4$, a contradiction.

We may thus assume by symmetry that $A \cap C \neq \emptyset$. Then $|(S_{T'} \cup S_T) - V(B \cup D)| \geq |S_{T'}| + 1$. So $|(S_{T'} \cup S_T) - V(A \cup C)| = |S_T| + |S_{T'}| - |(S_{T'} \cup S_T) - V(B \cup D)| \leq 4$. Since G is 5-connected, $B \cap D = \emptyset$. In addition, $A \cap D \neq \emptyset$; as otherwise, $|V(D)| \leq 4$, a contradiction. Therefore, $|(S_{T'} \cup S_T) - V(B \cup C)| \geq |S_{T'}| + 1$. Hence, $|(S_{T'} \cup S_T) - V(A \cup D)| =$

$|S_T| + |S_{T'}| - |(S_{T'} \cup S_T) - V(B \cup C)| \leq 4$. Since G is 5-connected, $B \cap C = \emptyset$. Thus, $|V(B)| \leq |S_{T'} - V(T')| = 3$, a contradiction. $\square$

The next lemma will allow us to assume that if $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum and $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$ then $T \cong K_3$ and $T' \cong K_3$.

Lemma 4.4. *Let G be a 5-connected graph and $x \in V(G)$. Suppose for any $T \subseteq G$ with $x \in V(T)$ and with $T \cong K_2$ or $T \cong K_3$, G/T is not 5-connected. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum and $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$. Suppose $T' \cong K_2$. Then one of the following holds:*

- *(i) G contains a TK_5 in which x is not a branch vertex.*
- *(ii) G contains K_4^-.*
- *(iii) There exist distinct $x_1, x_2, x_3 \in N_G(x)$ such that for any distinct $y_1, y_2 \in N_G(x) - \{x_1, x_2, x_3\}$, $G' := G - \{xv : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}$ contains TK_5.*

Proof. Lemma 4.1, we may assume $\min\{|V(A)|, |V(B)|\} \geq 5$ and $\min\{|V(C)|, |V(D)\} \geq 5$. By Lemma 4.3, we may assume $T \cong K_3$. By Lemma 2.6, we may further assume $|S_T| = 6$. Note the symmetry between C and D, and assume that $V(T) \subseteq S_T - V(D)$. Since $|V(T')| = 2$, $|S_{T'}| = 5$. Since $T' \cap A \neq \emptyset$, $|V(A \cap C)| + |V(A \cap D)| < |V(A)|$.

Suppose $A \cap C \neq \emptyset$. Then $|(S_{T'} \cup S_T) - V(B \cup D)| \geq 7$; otherwise, $(T, (S_{T'} \cup S_T) - V(B \cup D), A \cap C, G[B \cup D]) \in \mathcal{Q}_x$ and $0 < |V(A \cap C)| < |V(A)|$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Hence, $|(S_{T'} \cup S_T) - V(A \cup C)| = |S_T| + |S_{T'}| - |(S_{T'} \cup S_T) - V(B \cup D)| \leq 4$. Since G is 5-connected, $B \cap D = \emptyset$. We may assume $A \cap D \neq \emptyset$; otherwise, $|V(D)| \leq |S_T - V(T)| = 3$, a contradiction. We may also assume $|V(D)| > |V(A)|$; otherwise, $(T', S_{T'}, D, C) \in \mathcal{Q}_x$ and, by Lemma 4.3, G contains K_4^-. Hence, $|V(D) \cap S_T| > |V(A \cap C)| + |V(A) \cap S_{T'}| \geq |V(A) \cap S_{T'}| + 1$. Then, since $|S_T| = 6$ and $V(T) \subseteq S_T - V(D)$, $|V(D) \cap S_T| = 3$ and $|V(A) \cap S_{T'}| = 1$. Hence, $|(S_{T'} \cup S_T) - V(B \cup D)| = |S_T - (V(D) \cap S_T)| + |S_{T'} \cap V(A)| = 4$. However, $(S_{T'} \cup S_T) - V(B \cup D)$ is a cut in G, a contradiction as G is 5-connected.

Now assume $A \cap C = \emptyset$. Then, since $|S_{T'} \cap V(A)| \leq 4$ (as $x \in S_T \cap S_{T'}$ and $|S_{T'}| = 5$), $A \cap D \neq \emptyset$.

Suppose $|(S_{T'} \cup S_T) - V(B \cup C)| = 5$. Then $|V(A \cap D)| = 1$; otherwise, since $|V(C)| \geq 2$ as $|S_{T'}| = 5$, $(T', (S_{T'} \cup S_T) - V(B \cup C), A \cap D, G[B \cup C])$ contradicts the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Hence $|V(A) \cap S_{T'}| = 4$; so $V(B) \cap S_{T'} = V(D) \cap S_T = \emptyset$ as $x \in S_T \cap S_{T'}$. Since G is 5-connected, $B \cap D = \emptyset$. So $|V(D)| = 1$, a contradiction.

Hence, we may assume $|(S_{T'} \cup S_T) - V(B \cup C)| \geq 6$. Then $S_T \cap V(D) \neq \emptyset$ because $|S_{T'}| = 5$. So $B \cap C \neq \emptyset$ (otherwise $|V(C)| \leq 4$ as $V(C) = V(C) \cap S_T$, $|S_T| = 6$, $x \in S_T \cap S_{T'}$, and $S_T \cap V(D) \neq \emptyset$). Hence, since G is 5-connected, $|(S_{T'} \cup S_T) - V(A \cup D)| \geq 5$. Since $|(S_{T'} \cup S_T) - V(A \cup D)| + |(S_{T'} \cup S_T) - V(B \cup C)| = |S_T| + |S_{T'}| = 11$, we have $|(S_{T'} \cup S_T) - V(A \cup D)| = 5$. If $|V(B \cap C)| = 1$ then, since G is 5-connected and $T \subseteq S_T - V(D)$, $G[T \cup (B \cap C)] \cong K_4^-$. If $|V(B \cap C)| \geq 2$ then, since $V(T) \subseteq (S_{T'} \cup S_T) - V(A \cup D)$, the assertion follows from Lemma 2.6. $\square$

The proofs of the remaining two results in this section use Lemmas 3.1, 3.3, and 3.4. The result below will allow us to assume that if $(T, S_T, A, B) \in \mathcal{Q}_x$ is chosen to minimize $|V(A)|$ then $N_G(x) \cap V(A) \neq \emptyset$, which in turn will allow us to choose another quadruple at x.

Lemma 4.5. *Let G be a 5-connected nonplanar graph and $x \in V(G)$. Suppose for any $H \subseteq G$ with $x \in V(H)$ and with $H \cong K_2$ or $H \cong K_3$, G/H is not 5-connected. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ minimizing $|V(A)|$. Then $N(x) \cap V(A) \neq \emptyset$, or one of the following holds:*

- *(i) G contains a TK_5 in which x is not a branch vertex.*
- *(ii) G contains K_4^-.*
- *(iii) There exist distinct $x_1, x_2, x_3 \in N_G(x)$ such that for any distinct $u_1, u_2 \in N_G(x) - \{x_1, x_2, x_3\}$, $G' := G - \{xv : v \notin \{x_1, x_2, x_3, u_1, u_2\}\}$ contains TK_5.*

Proof. Suppose $N_G(x) \cap V(A) = \emptyset$. Then, since G is 5-connected, it follows that $|S_T| = 6$, $T \cong K_3$, and every vertex in $S_T - \{x\}$ has a neighbor in A. Let $V(T) = \{x, x_1, x_2\}$ and $S_T = \{x, x_1, x_2, v_1, v_2, v_3\}$. By Lemma 2.8, we may assume $N_G(x_1) \subseteq V(A) \cup \{x, x_2\}$, and any three independent paths in $G_A := G[A + (S_T - \{x\})] - E(S_T)$ from $\{x_1, x_2\}$ to v_1, v_2, v_3, respectively, with two from x_1 and one from x_2, must include a path from x_2 to v_1.

We wish to apply Lemma 3.1. Let G'_A be obtained from G_A by adding a new vertex x'_1 and joining x'_1 to each vertex in $N_G(x_1) \cap V(A)$ with an edge. Thus, in G'_A, x_1 and x'_1 have the same set of neighbors. Note that $\{x_1, x'_1, x_2\}$ and $\{v_1, v_2, v_3\}$ are independent sets in G'_A.

Claim 1. If (A_1, A_2) is a separation in G'_A such that $|V(A_1 \cap A_2)| \leq 3$, $\{x_1, x'_1, x_2\} \subseteq V(A_1)$, and $\{v_1, v_2, v_3\} \subseteq V(A_2)$, then x_2 has a unique neighbor in A, say x'_2, $V(A_1 \cap A_2) = \{x_1, x'_1, x'_2\}$, and $V(A_1) = \{x_1, x'_1, x_2, x'_2\}$.

To prove Claim 1, let (A_1, A_2) be a separation in G'_A such that $|V(A_1 \cap A_2)| \leq 3$, $\{x_1, x'_1, x_2\} \subseteq V(A_1)$, and $\{v_1, v_2, v_3\} \subseteq V(A_2)$.

We may assume $\{x_1, x'_1\} \not\subseteq V(A_1 \cap A_2)$. For, suppose $\{x_1, x'_1\} \subseteq V(A_1 \cap A_2)$. Then, since $\{x_1, x'_1, x_2\}$ is independent in G'_A, $x_2 \notin V(A_1 \cap A_2)$ and $A_1 - \{x_1, x'_1, x_2\} \neq \emptyset$. Since G is 4-connected, $\{x_1, x_2\} \cup (V(A_1 \cap A_2) - \{x_1, x'_1\})$ is not a 3-cut in G. Hence, $|V(A_1)| = 4$ as $N_G(x_2) \cap V(A) \neq \emptyset$. So x_2 has a unique neighbor in A, say x'_2, and we must have $V(A_1 \cap A_2) = \{x_1, x'_1, x'_2\}$ and $V(A) = \{x_1, x'_1, x_2, x'_2\}$.

Thus, we may assume by symmetry that $x_1 \notin V(A_1 \cap A_2)$. Then (A_1, A_2) may be chosen so that $x'_1 \notin V(A_1 \cap A_2)$ (as x'_1 and x_1 have the same set of neighbors in G'_A). Moreover, $V(A_1) - V(A_2) \subseteq \{x_1, x'_1, x_2\}$; otherwise $S'_T := V(A_1 \cap A_2) \cup V(T)$ is a cut in G with $|S'_T| \leq 6$, and $G - S'_T$ has a component strictly contained in A (as

$V(A_2) \neq \{v_1, v_2, v_3\}$ since $\{v_1, v_2, v_3\}$ is independent in G'_A), contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum.

Hence, $x_1 \in V(A_1) - V(A_2)$. Recall that $N_G(x_1) \subseteq V(A) \cup \{x, x_2\}$. So $V(A_1 \cap A_2) \cup \{x, x_2\}$ is a cut in G. Since G is 5-connected, $V(A_1 \cap A_2) \cup \{x, x_2\}$ is not a 4-cut in G. Hence, $x_2 \in V(A_1) - V(A_2)$ and $|V(A_1 \cap A_2)| = 3$. Since G is 5-connected and $V(A_1) - V(A_2) \subseteq \{x_1, x'_1, x_2\}$, it follows that $N_G(x_1) = \{x, x_2\} \cup V(A_1 \cap A_2)$. Since $N_G(x_2) \cap V(A) \neq \emptyset$ and $N_G(x_2) \cap V(A) \subseteq N_G(x_2) \cap V(A_1 \cap A_2)$, there exists $v \in V(A_1 \cap A_2)$ such that $vx_2 \in E(G)$. Now $G[\{v, x, x_1, x_2\}] \cong K_4^-$ and (ii) holds. □

Since any three disjoint paths in G'_A from $\{x_1, x_2, x'_1\}$ to $\{v_1, v_2, v_3\}$ contain a path from x_2 to v_1, it follows from Claim 1 and Lemma 3.1 that G'_A has a separation (J, L) such that $V(J \cap L) = \{w_0, \ldots, w_n\}$, $(J, w_0, \ldots, w_n)$ is planar, $(L, (x_1, x_2, x'_1), (v_2, v_1, v_3))$ is a ladder along some sequence $b_0 \ldots b_m$, where $b_0 = x_2$, $b_m = v_1$, and $w_0 \ldots w_n$ is the reduced sequence of $b_0 \ldots b_m$. (Note that if (ii) of Lemma 3.1 holds then $(G'_A, (x_1, x_2, x'_1), (v_2, v_1, v_3))$ is a ladder. For convenience, we let $L = G'_A$, let J consist of v_1 and x_2, or let J consist of v_1, x_2, x'_2 if x_2 has a unique neighbor x'_2 in G_A.)

Since L is a ladder, L contains three disjoint paths P_1, P_2, P_3 from x_1, x_2, x'_1, respectively, to $\{v_1, v_2, v_3\}$, with $v_1 \in V(P_2)$. Without loss of generality, we may further assume that $v_2 \in V(P_1)$ and $v_3 \in V(P_3)$. Let $(R_i, (a_{i-1}, b_{i-1}, c_{i-1}), (a_i, b_i, c_i))$, $i \in [m]$, be the rungs in L, with $a_i \in V(P_1)$, $b_i \in V(P_2)$, and $c_i \in V(P_3)$ for $i = 0, \ldots, m$. Since G is 5-connected, $(J, w_0, \ldots, w_n)$ is planar and, by Lemmas 3.3 and 3.4, we may assume that the rungs in L have the simple structures as in Lemma 3.4. For convenience, we view P_3 as a path in G_A from v_3 to x_1.

Claim 2. There exist $t \in V(A)$ and independent paths Q_1, Q_2, Q_3, Q_4, Q_5 in G_A such that Q_1, Q_2, Q_3, Q_4 are from t to x_1, x_2, v_1, v_2, respectively, and Q_5 is from x_1 to v_3; and there exist $t' \in V(A)$ and independent paths $Q'_1, Q'_2, Q'_3, Q'_4, Q'_5$ in G_A such that Q'_1, Q'_2, Q'_3, Q'_4 are from t' to x_1, x_2, v_1, v_3, respectively, and Q'_5 is from x_1 to v_2.

First, we may assume that for $i \in [m]$, $(R_i, (a_{i-1}, b_{i-1}, c_{i-1}), (a_i, b_i, c_i))$ is not of type (iv) as in Lemma 3.4. For, suppose $(R_i, (a_{i-1}, b_{i-1}, c_{i-1}), (a_i, b_i, c_i))$ is of type (iv) for some $i \in [m]$, and let $v \in V(R_i) - (\{a_{i-1}, b_{i-1}, c_{i-1}\} \cup \{a_i, b_i, c_i\})$. Then Claim 2 holds with $v, va_{i-1} \cup a_{i-1}P_1x_1, vb_{i-1} \cup b_{i-1}P_2x_2, vb_i \cup b_iP_2v_1, va_i \cup a_iP_1v_2, P_3$ as $t, Q_1, Q_2, Q_3, Q_4, Q_5$, respectively, and with $v, vc_{i-1} \cup c_{i-1}P_3x_1, vb_{i-1} \cup b_{i-1}P_2x_2$, $vb_i \cup b_iP_2v_1$, $vc_i \cup c_iP_3v_3, P_1$ as $t', Q'_1, Q'_2, Q'_3, Q'_4, Q'_5$, respectively.

We claim that there exists $q \in [m]$ with $x_1b_q \in E(G)$. To see this, let $q \geq 1$ be the smallest integer such that $(R_q, (a_{q-1}, b_{q-1}, c_{q-1}), (a_q, b_q, c_q))$ is not of type (ii) as in Lemma 3.4, which must exist as $x_1 \notin \{v_1, v_2, v_3\}$. Then $a_{q-1} = x_1$ and $c_{q-1} = x'_1$. Since G is 5-connected, $(R_q, (a_{q-1}, b_{q-1}, c_{q-1}), (a_q, b_q, c_q))$ cannot be of type (iii); for, otherwise, $\{a_{q-1}, b_{q-1}, c_{q-1}\} \cup \{a_q, b_q, c_q\}$ would give a 4-cut in G consisting of x_1, b_{q-1}, b_q and one of $\{a_q, c_q\}$. Thus, $(R_q, (a_{q-1}, b_{q-1}, c_{q-1}), (a_q, b_q, c_q))$ must be of type (i) as in Lemma 3.4. Since x_1 and x'_1 have the same set of neighbors in G'_A, $a_q \neq x_1$ and $c_q \neq x'_1$. Since G is 5-connected, $\{x_1, a_q, b_q, c_q\}$ cannot be a cut in G; so $V(R_q) = \{x_1, x'_1, a_q, b_q, c_q\}$. Since

$N_G(x_1) \subseteq V(A) \cup \{x, x_2\}$, $N_G(x_1) \subseteq V(R_q) \cup \{x, x_2\}$. Hence, since G is 5-connected, $N_G(x_1) = \{x, x_2, a_q, b_q, c_q\}$. In particular, $x_1 b_q \in E(G)$.

Recall that L is a ladder with rungs $(R_i, (a_{i-1}, b_{i-1}, c_{i-1}), (a_i, b_i, c_i))$ for $i \in [m]$, $b_0 = x_2$ and $b_m = v_1$, and P_2 is a path through $b_0, b_1, \ldots, b_m$ in order. So we choose b_q such that $x_1 b_q \in E(G)$ and, subject to this, q is maximum. Note that $q < m$ as $x_1 v_1 \notin E(G)$ (since $N_G(x_1) \subseteq V(A) \cup \{x, x_2\}$).

We now show the existence of t and $Q_i, i \in [5]$; the proof of the existence of t' and $Q'_i, i \in [5]$, is symmetric (by exchanging the roles of v_2, P_1 and v_3, P_3).

We may assume that there does not exist r, with $q < r \leq m$, such that L has disjoint paths S, S' from b_r, x_1 to v_2, v_3, respectively, and internally disjoint from $J \cup P_2$. For, suppose such r, S, S' do exist. By Claim 1, $J \cup P_2$ or $(J \cup P_2) - x_2$ is 2-connected. Let P'_2 denote the path between x_2 and v_1 in $J \cup P_2$ such that $P'_2 \cup P_2$ bounds the infinite face of $J \cup P_2$. (Here we assume that $J \cup P_2$ is drawn in a closed disc in the plane with $w_0, \ldots, w_n$ on the boundary of the disc in cyclic order.) Let $t \in V(P'_2)$ such that $x_2 t \in E(P'_2)$. If there exist independent paths L_1, L_2 in $J \cup P_2$ from t to b_q, b_r, respectively, and internally disjoint from P'_2, then $L_1 \cup b_q x_1, t x_2, t P'_2 v_1, L_2 \cup S, S'$ give the desired paths Q_1, Q_2, Q_3, Q_4, Q_5, respectively. Thus we may assume that such L_1, L_2 do not exist. So by Menger's theorem, $J \cup P_2$ has a separation (J_1, J_2) such that $|V(J_1 \cap J_2)| \leq 3$, $t \in V(J_1) - V(J_2)$, and $\{b_q, b_r, v_1, x_2\} \subseteq V(J_2)$. Because of P'_2, $V(J_1 \cap J_2)$ contains x_2 and a vertex $t^* \in V(t P'_2 v_1)$. Note that $V(J_1 \cap J_2) \neq \{t^*, x_2\}$ as otherwise by planarity of $J \cup P_2$, $\{t^*, x_2\}$ would be a cut in G separating t from $B \cup P_1 \cup P_2 \cup P_3$. So let $v \in V(J_1 \cap J_2) - \{t^*, x_2\}$. If $v \notin V(P_2)$ then by planarity of $J \cup P_2$, $\{t^*, v, x_2\}$ is a cut in G separating t from $B \cup P_1 \cup P_3$, a contradiction. So $v \in V(P_2)$. If $v \in V(x_2 P_2 b_q)$ then by planarity of $J \cup P_2$, $\{t^*, v, x_1, x_2\}$ is a cut in G separating t from $B \cup P_1 \cup P_3$, a contradiction. So $v \in V(b_r P_2 v_1)$ and we may assume $v = b_s$ for some s with $r + 1 \leq s \leq m$. Then $V(T) \cup \{a_s, b_s, c_s\}$ is a cut in G separating $\bigcup_{i=1}^{s} R_s$ from $B + t$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum.

Hence, for any $r > q$, it follows from the nonexistence of S, S' above and the maximality of q that $(R_r, (a_{r-1}, b_{r-1}, c_{r-1}), (a_r, b_r, c_r))$ must be of type (i) or (ii) as in Lemma 3.4, and there is no edge in G'_A from $b_q P_2 v_2 - b_q$ to $P_1 - x_1$.

Also notice that, for $r \leq q$ with $b_{r-1} \neq b_q$, because of the edges $x_1 b_q, x'_1 b_q$ in G'_A, $(R_r, (a_{r-1}, b_{r-1}, c_{r-1}), (a_r, b_r, c_r))$ must be of type (ii) as in Lemma 3.4. For $r \leq q$ with $b_{r-1} = b_q$, we see that $V(R_r) = \{x_1, x'_1, a_r, b_q, c_r\}$ to avoid the cut $\{x_1, a_r, b_q, c_r\}$ in G, and we may assume that $b_q a_r \notin E(G)$ (otherwise, $b_q, b_q x_1, b_q P_2 x_2, b_q P_2 v_1, b_q a_r \cup a_r P_1 v_2, P_3$ give the desired $t, Q_1, Q_2, Q_3, Q_4, Q_5$, respectively). Thus, in particular, $x_1 \notin \{a_q, c_q\}$ as x_1 and x'_1 have the same set of neighbors in R_r.

We may assume that for some $j > q$, $\{a_{j-1}, c_{j-1}\} \cap \{a_j, c_j\} = \emptyset$. For, otherwise, for each $j > q$, $(R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))$ is of type (ii) or of type (i) with $|V(R_i)| = 4$ (as G is 5-connected). Hence, since G'_A has no edge from P_2 to $P_1 - x_1$ (as $b_q a_q \notin E(G)$), it follows that $(G_A, x_1, x_2, v_1, v_3, v_2)$ is planar; so the assertion follows from Lemmas 2.5 and 2.2.

Thus, $(R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))$ is of type (i) as in Lemma 3.4.

If $R_j - a_{j-1}$ contains disjoint paths S_1, S_2 from b_j, c_{j-1} to a_j, c_j, respectively, then b_j and the paths $S_1 \cup a_j P_1 v_2, x_1 P_3 c_{j-1} \cup S_2 \cup c_j P_3 v_3$ contradict the nonexistence of b_r, S, S'. So assume that such S_1, S_2 do not exist. Then by Lemma 2.10, $(R_j - a_{j-1}, a_j, c_j, b_j, c_{j-1})$ is 3-planar. Since G is 5-connected, R_j is $(5, \{a_{j-1}, a_j, c_j, b_j, c_{j-1}\})$-connected. So $(R_j - a_{j-1}, a_j, c_j, b_j, c_{j-1})$ is in fact planar. By Lemmas 2.5 and 2.2, we may assume $|V(R_j - a_{j-1})| \leq 5$ as otherwise the desired TK_5 or K_4^- exists in G.

If $|V(R_j - a_{j-1})| = 5$ then there exists $v \in V(R_j) - \{a_{j-1}, a_j, b_j, c_{j-1}, c_j\}$. Since G is 5-connected, $N_G(v) = \{a_{j-1}, a_j, b_j, c_{j-1}, c_j\}$. Since $j > q$, by taking $r = j$ we see that b_j, $b_j v a_j \cup a_j P_1 v_2, P_3$ contradict the nonexistence of b_r, S, S'.

Hence, we may assume $|V(R_j - a_{j-1})| = 4$. Then, since R_j has no cut of size at most 3 separating $\{a_{j-1}, b_{j-1}, c_{j-1}\}$ from $\{a_j, b_j, c_j\}$, we must have $a_{j-1}c_j, a_j c_{j-1} \in E(G)$. Note that there exists $r \geq q$ such that L has a path Z from b_r to $z \in V(x_1 P_1 a_{j-1} - x_1) \cup V(x_1' P_3 c_{j-1} - x_1')$ and internally disjoint from $J \cup P_1 \cup P_2 \cup P_3$; for otherwise, $\{a_j, b_j, c_j, x_1\}$ would be a cut in G.

Suppose $z \in V(x_1 P_1 a_{j-1} - x_1)$. If $r > q$ then $b_r, Z \cup z P_1 v_2, P_3$ contradict the nonexistence of b_r, S, S'. So $r = q$. Then b_q, $b_q x_1$, $b_q P_2 x_2$, $b_q P_2 v_1$, $Z \cup z P_1 v_2$, P_3 give the desired $t, Q_1, Q_2, Q_3, Q_4, Q_5$, respectively.

So assume $z \in V(x_1 P_3 c_{j-1} - x_1)$. If $r > q$ then b_r, $Z \cup z P_3 c_{j-1} \cup c_{j-1} a_j \cup a_j P_1 v_2, x_1 P_1 a_{j-1} \cup a_{j-1} c_j \cup c_j P_3 v_3$ contradict the nonexistence of b_r, S, S'. So $r = q$. Then b_q, $b_q x_1$, $b_q P_2 x_2$, $b_q P_2 v_1$, $Z \cup z P_3 c_{j-1} \cup c_{j-1} a_j \cup a_j P_1 v_2$, $x_1 P_1 a_{j-1} \cup a_{j-1} c_j \cup c_j P_3 v_3$ give the desired t, Q_1, Q_2, Q_3, Q_4, Q_5, respectively. $\square$

Now that we have the paths in Claim 2, we turn to $G_B := G[B + (S_T - \{x_1\})]$. Choose $x_3 \in N_G(x) \cap V(B)$, let $u_1 := x_3$ and let $u_2 \in N_G(x) - \{x_1, x_2, x_3\}$ be arbitrary. Note that $u_2 \in S_T \cup V(B)$ (as $N_G(x) \cap V(A) = \emptyset$). We wish to prove *(iii)* by attempting to find a TK_5 in $G' := G - \{xv : v \notin \{u_1, u_2, x_1, x_2\}\}$. Since G is 5-connected and $N_G(x_1) \cap V(B) = \emptyset$, $G_B - x$ is $(4, \{x_2, v_1, v_2, v_3\})$-connected and, hence, has four independent paths B_1, B_2, B_3, B_4 from u_1 to v_1, v_2, v_3, x_2, respectively. We further choose these paths to be induced in G_B.

Claim 3. We may assume $u_2 \in V(B)$.

For, otherwise, we have $u_2 \in S_T$. If $u_2 = v_1$ then $T \cup Q_1 \cup Q_2 \cup (Q_3 \cup v_1 x) \cup u_1 x \cup B_4 \cup (B_2 \cup Q_4) \cup (B_3 \cup Q_5)$ is a TK_5 in G' with branch vertices t, u_1, x, x_1, x_2. If $u_2 = v_2$ then $T \cup Q_1 \cup Q_2 \cup (Q_4 \cup v_2 x) \cup u_1 x \cup B_4 \cup (B_1 \cup Q_3) \cup (B_3 \cup Q_5)$ is a TK_5 in G' with branch vertices t, u_1, x, x_1, x_2. Now assume $u_2 = v_3$. Then $T \cup Q_1' \cup Q_2' \cup (Q_4' \cup v_3 x) \cup u_1 x \cup B_4 \cup (B_1 \cup Q_3') \cup (B_2 \cup Q_5')$ is a TK_5 in G' with branch vertices t', u_1, x, x_1, x_2. $\square$

Let P be a path in $G_B - x$ from u_2 to some $w_2 \in V(B_1 \cup B_2 \cup B_3 \cup B_4) - \{u_1\}$ and internally disjoint from $B_1 \cup B_2 \cup B_3 \cup B_4$.

Claim 4. We may assume that $w_2 \in V(B_4)$ for every choice of P.

For, if $w_2 \in V(B_1)$ then $T \cup Q_1 \cup Q_2 \cup (Q_3 \cup v_1 B_1 w_2 \cup P \cup u_2 x) \cup u_1 x \cup B_4 \cup (B_2 \cup Q_4) \cup (B_3 \cup Q_5)$ is a TK_5 in G' with branch vertices t, u_1, x, x_1, x_2. If $w_2 \in V(B_2)$ then $T \cup Q_1 \cup Q_2 \cup (Q_4 \cup v_2 B_2 w_2 \cup P \cup u_2 x) \cup u_1 x \cup B_4 \cup (B_1 \cup Q_3) \cup (B_3 \cup Q_5)$ is a TK_5 in G' with branch vertices t, u_1, x, x_1, x_2. If $w_2 \in V(B_3)$ then $T \cup Q'_1 \cup Q'_2 \cup (Q'_4 \cup v_3B_3w_2 \cup P \cup u_2x) \cup u_1x \cup B_4 \cup (B_1 \cup Q'_3) \cup (B_2 \cup Q'_5)$ is a TK_5 in G' with branch vertices t', u_1, x, x_1, x_2. $\square$

Let U_2 denote the $(B_1 \cup B_2 \cup B_3)$-bridge of $G_B - x$ containing $B_4 + u_2$. That is, U_2 is the subgraph of $G_B - x$ induced by the edges in the component of $(G_B - x) - (B_1 \cup B_2 \cup B_3)$ containing $B_4 + u_2$ and the edges from that component to $B_1 \cup B_2 \cup B_3$.

Claim 5. We may assume that $N_G((U_2 - x_2) - (B_1 \cup B_2 \cup B_3)) \subseteq V(B_1) \cup \{x, x_2\}$.

For, otherwise, there exists $w \in N_G((U_2 - x_2) - (B_1 \cup B_2 \cup B_3))$ such that $w \notin V(B_1) \cup \{x, x_2\}$. By symmetry, we may assume $w \in V(B_2 - u_1)$ and choose w so that wB_2v_2 is minimal.

Then U_2 has a path X between x_2 to w and internally disjoint from $B_1 \cup B_2 \cup B_3$, and a path from u_2 to some $u'_2 \in V(X)$ and internally disjoint from $X \cup B_1 \cup B_2 \cup B_3$. (Note that $u'_2 = u_2$ is possible.) Since G is 5-connected, U_2 is $(4, (V(U_2) \cap V(B_1 \cup B_2 \cup B_3)) \cup \{u_2, x_2\})$-connected. Hence, U_2 has four independent paths from u'_2 to four distinct vertices in $(V(U_2) \cap V(B_1 \cup B_2 \cup B_3)) \cup \{u_2, x_2\}$ and internally disjoint from $B_1 \cup B_2 \cup B_3$. Thus, by Lemma 2.11, U_2 contains independent paths L_1, L_2, L_3, L_4 from u'_2 to u_2, x_2, w, w', respectively, and internally disjoint from $B_1 \cup B_2 \cup B_3$, where $w' \in V(B_1 \cup B_2 \cup B_3)$.

If $w' \in V(B_2)$ then by the minimality of wB_2v_2, $w' \in V(wB_2u_1 - w)$; so $T \cup (L_1 \cup u_2x) \cup L_2 \cup (L_3 \cup wB_2v_2 \cup P_1) \cup (u_1B_2w' \cup L_4) \cup u_1x \cup (B_1 \cup P_2) \cup (B_3 \cup P_3)$ is a TK_5 in G' with branch vertices u_1, u'_2, x, x_1, x_2. (Recall that we view P_3 as a path in G_A from x_1 to v_3.)

If $w' \in V(B_1 - u_1)$ then (using Claim 2) we see that $T \cup Q'_1 \cup Q'_2 \cup (Q'_4 \cup B_3 \cup u_1x) \cup (L_1 \cup u_2x) \cup L_2 \cup (L_3 \cup wB_2v_2 \cup Q'_5) \cup (L_4 \cup w'B_1v_1 \cup Q'_3)$ is a TK_5 in G' with branch vertices t', u'_2, x, x_1, x_2.

If $w' \in V(B_3 - u_1)$ then (using Claim 2) we see that $T \cup Q_1 \cup Q_2 \cup (Q_3 \cup B_1 \cup u_1x) \cup (L_1 \cup u_2x) \cup L_2 \cup (L_3 \cup wB_2v_2 \cup Q_4) \cup (L_4 \cup w'B_3v_3 \cup Q_5)$ is a TK_5 in G' with branch vertices t, u'_2, x, x_1, x_2. $\square$

Now let $z \in N_G((U_2 - x_2) - (B_1 \cup B_2 \cup B_3))$ with $z \in V(B_1)$ such that zB_1v_1 is minimal. Since G is 5-connected, $\{u_1, x_2, z\}$ cannot be a cut in $G_B - x$ (in particular, $|V(zB_1u_1)| \geq 3$). So $G_B - x$ has a path Y from some $y \in V(zB_1u_1) - \{u_1, z\}$ to some $y' \in V(B_2 \cup B_3) - \{u_1\}$ and internally disjoint from $U_2 \cup B_1 \cup B_2 \cup B_3$.

Claim 6. We may assume that $G[(U_2 - B_1) + z]$ has no independent paths from u_2 to x_2, z, respectively.

For, suppose $G[(U_2 - B_1) + z]$ (and hence $G[U_2 \cup zB_1u_1]$) has independent paths from u_2 to x_2, z, respectively. Since G is 5-connected, it follows from Claim 5 and the choice of z that $G[U_2 \cup zB_1u_1]$ is $(4, V(zB_1u_1) \cup \{x_2\})$-connected. So by Lemma 2.11, $G[U_2 \cup zB_1u_1]$ has independent paths L_1, L_2, L_3, L_4 from u_2 to distinct vertices x_2, z, z_1, z_2, respectively, and internally disjoint from B_1, where u_1, z_2, z_1, z occur on B_1 in the order listed. Possibly, $u_1 = z_2$.

If $y' \in V(B_2 - u_1)$ then (using Claim 2) we see that $T \cup Q'_1 \cup Q'_2 \cup (Q'_4 \cup B_3 \cup u_1x) \cup u_2x \cup L_1 \cup (L_2 \cup zB_1v_1 \cup Q'_3) \cup (L_3 \cup z_1B_1y \cup Y \cup y'B_2v_2 \cup Q'_5)$ is a TK_5 in G' with branch vertices t', u_2, x, x_1, x_2.

If $y' \in V(B_3 - u_1)$ then (using Claim 2) we see that $T \cup Q_1 \cup Q_2 \cup (Q_4 \cup B_2 \cup u_1x) \cup u_2x \cup L_1 \cup (L_2 \cup zB_1v_1 \cup Q_3) \cup (L_3 \cup z_1B_1y \cup Y \cup y'B_3v_3 \cup Q_5)$ is a TK_5 in G' with branch vertices t, u_2, x, x_1, x_2. $\square$

By Claim 6, $G[(U_2 - B_1) + z]$ has a 1-separation (U_{21}, U_{22}) such that $u_2 \in V(U_{21}) - V(U_{22})$ and $\{x_2, z\} \subseteq V(U_{22})$. We choose this separation so that U_{22} is minimal. Let u'_2 denote the unique vertex in $V(U_{21} \cap U_{22})$. By the definition of z, we see that $u'_2 \notin \{x_2, z\}$. Also, $u'_2 \in V(B_4)$ as otherwise by Claim 4, $\{x, u'_2\}$ would be a cut in G. By the minimality of U_{22}, we see that U_{22} has independent paths L_1, L_2 from u'_2 to x_2, z, respectively.

Claim 7. We may assume that u'_2 has exactly two neighbors in U_{22}.

By the minimality of U_{22}, $|N_G(u'_2) \cap V(U_{22})| \geq 2$. Suppose $|N_G(u'_2) \cap V(U_{22})| \geq 3$. Let L be a path in U_{21} from u_2 to u'_2.

We claim that $G[U_{22} \cup zB_1u_1] - u_1$ has three independent paths from u'_2 to three distinct vertices in $V(zB_1u_1 - u_1) \cup \{x_2\}$. Otherwise, $G[U_{22} \cup zB_1u_1] - u_1$ has a separation (Y_1, Y_2) such that $|V(Y_1 \cap Y_2)| \leq 2$, $u'_2 \in V(Y_1) - V(Y_2)$, and $V(zB_1u_1 - u_1) \cup \{x_2\} \subseteq V(Y_2)$. Since $|N_G(u'_2) \cap V(U_{22})| \geq 3$, we see that $V(Y_1 \cap Y_2) \cup \{x, u_1, u'_2\}$ is cut of G and, hence, of order 5 (as G is 5-connected). Let $V(Y_1 \cap Y_2) = \{t_1, t_2\}$ and let T_1, T_2 be disjoint paths in Y_2 from t_1, t_2 to z, x_2, respectively. Thus G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{t_1, t_2, x, u_1, u'_2\}$, $G_A \cup Y_2 \subseteq G_1$, and $Y_1 \subseteq G_2$. Clearly, $|V(G_1)| \geq 7$. We may assume $|V(G_2)| \geq 7$; for otherwise, $|V(G_2)| = 6$ and the vertex in $V(G_2 - G_1)$ together with u'_2, t_1, t_2 induces a subgraph of G containing K_4^- ((*ii*) holds). Thus, we may further assume that $(G_2 - x, V(G_1 \cap G_2) - \{x\})$ is not planar; as otherwise the assertion of this lemma follows from Lemmas 2.5 and 2.2. So by Lemma 2.10, $G_2 - x$ has disjoint paths S_1, S_2 from t_1, t_2 to u'_2, u_1, respectively. Then $L \cup S_1 \cup T_1$ is a path in $G_B - x$ from u_2 to B_1 and disjoint from the $(T_2 \cup S_2) \cup B_1 \cup B_2 \cup B_3$; so we have a contradiction to Claim 4 by replacing B_4 with $T_2 \cup S_2$.

So by Lemma 2.11, $G[U_{22} \cup zB_1u_1] - u_1$ has independent paths L'_1, L'_2, L'_3 from u'_2 to x_2, z, z_1, respectively, and internally disjoint from B_1, where $z_1 \in V(zBu_1) - \{u_1, z\})$.

If $y' \in V(B_2 - u_1)$ then $T \cup Q'_1 \cup Q'_2 \cup (Q'_4 \cup B_3 \cup u_1x) \cup (L \cup u_2x) \cup L'_1 \cup (L'_2 \cup zB_1v_1 \cup Q'_3) \cup (L'_3 \cup z_1B_1y \cup Y \cup y'B_2v_2 \cup Q'_5)$ is a TK_5 in G' with branch vertices t', u'_2, x, x_1, x_2.

If $y' \in V(B_3 - u_1)$ then $T \cup Q_1 \cup Q_2 \cup (Q_4 \cup B_2 \cup u_1x) \cup (L \cup u_2x) \cup L'_1 \cup (L'_2 \cup zB_1v_1 \cup Q_3) \cup (L'_3 \cup z_1B_1y \cup Y \cup y'B_3v_3 \cup Q_5)$ is a TK_5 in G' with branch vertices t, u'_2, x, x_1, x_2. $\square$

Since G is 5-connected, it follows from Claim 7 that u'_2 has at least two neighbors in $U_{21} \cup (zB_1u_1 - z)$. Moreover, $u'_2B_4u_1 \neq u'_2u_1$ as, otherwise, it follows from Claim 4 that $\{u'_2, u_1, x\}$ would be a cut in G. Hence, since B_4 is induced, $u'_2u_1 \notin E(G)$.

Then $G[U_{21} \cup zB_1u_1] - \{z, u_1\}$ has two independent paths from u'_2 to two vertices in $(zB_1u_1 - \{z, u_1\}) \cup \{u_2\}$. For, otherwise, $G[U_{21} \cup (zB_1u_1 - z)]$ has a separation (U', U'') such that $|V(U' \cap U'')| \leq 2$, $u_1 \in V(U' \cap U'')$, $u'_2 \in V(U' - U'')$, and $(zB_1u_1 - z) + u_2 \subseteq U''$.

Since $u_2'u_1 \notin E(G)$ and u_2' has at least two neighbors in $U_{21} \cup (zB_1u_1 - z)$, it follows from Claims 4 and 5 that $V(U' \cap U'') \cup \{u_2', x\}$ is a cut in G of size at most 4, a contradiction.

Hence, by Lemma 2.11, $G[U_{21} \cup zB_1u_1] - \{z, u_1\}$ has independent paths L_3, L_4 from u_2' to z_1, u_2, respectively, and internally disjoint from B_1, where $z_1 \in V(zB_1u_1) - \{z, u_1\}$.

If $y' \in V(B_2 - u_1)$ then $T \cup Q_1' \cup Q_2' \cup (Q_4' \cup B_3 \cup u_1x) \cup (L_4 \cup u_2x) \cup L_1 \cup (L_2 \cup zB_1v_1 \cup Q_3') \cup (L_3 \cup z_1B_1y \cup Y \cup y'B_2v_2 \cup Q_5')$ is a TK_5 in G' with branch vertices t', u_2', x, x_1, x_2.

If $y' \in V(B_3 - u_1)$ then $T \cup Q_1 \cup Q_2 \cup (Q_4 \cup B_2 \cup u_1x) \cup (L_4 \cup u_2x) \cup L_1 \cup (L_2 \cup zB_1v_1 \cup Q_3) \cup (L_3 \cup z_1B_1y \cup Y \cup y'B_3v_3 \cup Q_5)$ is a TK_5 in G' with branch vertices t, u_2', x, x_1, x_2. $\square$

We conclude this section with another technical lemma, which deals with a special case that occurs in the proof of Lemma 5.5. It is included in this section because its proof also makes use of Lemmas 3.1, 3.3, and 3.4.

Lemma 4.6. *Let G be a 5-connected nonplanar graph and $x \in V(G)$. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ such that $|V(A)|$ is minimum, and suppose there exists $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ such that $T' \cong K_3$, $T' \cap A \neq \emptyset$, $V(A \cap C) = S_T \cap V(C) = V(B \cap D) = V(B) \cap S_{T'} = \emptyset$, $|V(A) \cap S_{T'}| = |V(D) \cap S_T| = |V(D \cap T)| = 1$, $A \cap D \neq \emptyset$, and $|S_T \cap S_{T'}| = 5$. Suppose that for any $H \subseteq G$ with $x \in V(H)$ and with $H \cong K_2$ or $H \cong K_3$, G/H is not 5-connected, and that for any $(H, S_H, A_H, B_H) \in \mathcal{Q}_x$, we have $|V(H \cap A)| \leq 1$, and $H \cong K_3$ when $H \cap A \neq \emptyset$. Then one of the following holds:*

- (i) *G has a TK_5 in which x is not a branch vertex.*
- (ii) *G contains K_4^-.*
- (iii) *There exist $x_1, x_2, x_3 \in N_G(x)$ such that, for any distinct $y_1, y_2 \in N_G(x) - \{x_1, x_2, x_3\}$, $G' := G - \{xv : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}$ contains TK_5.*

Proof. Note that $|S_T| = |S_T \cap S_{T'}| + |V(D \cap T)| = 6$. So $T \cong K_3$. Let $V(T) = \{x, w, x_1\}$ and $V(T') = \{x, a, b\}$ such that $V(A) \cap S_{T'} = \{a\}$ and $V(D) \cap S_T = \{w\}$, and let $S_T \cap S_{T'} = \{x, x_1, b, z_1, z_2\}$. Then $|V(D)| = |V(A)| = |V(A \cap D)| + 1$. Moreover,

(1) $|N_G(s) \cap V(A)| \geq 2$ for $s \in \{b, z_1, z_2\}$,

for, otherwise, $(T, (S_T - \{s\}) \cup (N_G(s) \cap V(A)), A - N_G(s), G[B + s]) \in \mathcal{Q}_x$ and $|A - N_G(s)| \geq 2$ (by Lemma 4.1), contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. We may assume that

(2) G has no edge from $T - x$ to $T' - x$,

as otherwise $G[T \cup T']$ contains K_4^- and (ii) holds. We may also assume that

(3) $N_G(x_1) \cap V(D) \neq \{w\}$ and $N_G(w) \cap V(A) \neq \emptyset$,

for, otherwise, if $N_G(x_1) \cap V(D) = \{w\}$ then let $S := S_T - \{x_1\}$ and $B' := G[B + x_1]$, and if $N_G(w) \cap V(A) = \emptyset$ then let $S := S_T - \{w\}$ and $B' := G[B + w]$; now $|S| = 5$ and $(xw, S, A, B') \in \mathcal{Q}_x$ or $(xx_1, S, A, B') \in \mathcal{Q}_x$ and, hence, (ii) follows from Lemma 4.3. We may further assume that

(4) for any $x' \in N_G(x) \cap V(A \cap D)$, $xx'z_1x$ or $xx'z_2x$ is a triangle in G.

For, let $x' \in N_G(x) \cap V(A \cap D)$. By Lemma 4.2, we may assume that there exists $H \subseteq G$ with $x, x' \in V(H)$ and with $H \cong K_2$ or $H \cong K_3$. By the assumption of this lemma, $H \cong K_3$ and $V(H) \cap S_T \neq \{x\}$. If $V(H) \cap \{b, w, x_1\} \neq \emptyset$ then $H \cup T$ or $H \cup T'$ contains K_4^-, and (ii) holds. So we may assume $V(H) \cap \{z_1, z_2\} \neq \emptyset$ and, hence, $xx'z_1x$ or $xx'z_2x$ is a triangle in G.

We may assume that

(5) $|N_G(x) \cap V(A \cap D)| \leq 2$.

For, otherwise, by (4), there exist $i \in [2]$ and distinct $x', x'' \in N_G(x) \cap V(A \cap D) \cap N_G(z_i)$. So $G[\{x, x', x'', z_i\}]$ contains K_4^-, and (ii) holds.

We now distinguish two cases.

Case 1. $z_i \notin N_G(x)$ for $i \in [2]$.

Then by (4), $N_G(x) \cap V(A \cap D) = \emptyset$. We prove that (iii) holds with $x_2 = w$ and $x_3 = b$. Let $y_1, y_2 \in N_G(x) - \{x_1, x_2, x_3\}$. Since G is 5-connected and $z_1, z_2 \notin N_G(x)$, we may assume $y_1 \in V(B \cap C)$. Then $G_B := G[B + \{b, x_1, z_1, z_2\}]$ has independent paths Y_1, Y_2, Y_3, Y_4 from y_1 to z_1, z_2, x_1, b, respectively.

We may assume that $wz_i \notin E(G)$ for $i \in [2]$. For, suppose, by symmetry, $wz_1 \in E(G)$. If $G[A + \{b, w, x_1\}]$ has independent paths Q_1, Q_2 from b to x_1, w, respectively, then $T \cup bx \cup Q_1 \cup Q_2 \cup y_1x \cup (Y_1 \cup z_1w) \cup Y_3 \cup Y_4$ is a TK_5 in G' with branch vertices b, w, x, x_1, y_1. So we may assume that such Q_1, Q_2 do not exist. Then $G[A + \{b, w, x_1\}]$ has a cut vertex v separating b from $\{w, x_1\}$. Let K denote the component of $G[A + \{b, w, x_1\}] - v$ containing b. Since $|N_G(b) \cap V(A)| \geq 2$ (by (1)), $|V(K)| \geq 2$. Now $\{b, v, x, z_1, z_2\}$ is a cut in G, and G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b, v, x, z_1, z_2\}$, $|V(G_1)| \geq 6$ and $\{a, b\} \subseteq V(G_1)$, and $B + \{w, x_1\} \subseteq G_2$. By the choice of (T, S_T, A, B) with $|V(A)|$ minimum, it follows that $|V(G_1)| = 6$. Let $u \in V(G_1) - V(G_2)$; then $V(G_1 \cap G_2) \subseteq N_G(u)$ (since G is 5-connected). If $u = a$ then $bv \in E(G)$ (since $|N_G(b) \cap V(A)| \geq 2$); now $G[\{a, b, v, x\}]$ contains K_4^-, and (ii) holds. So assume $u \neq a$. Then $v = a$ and $G[\{b, u, v, x\}]$ contains K_4^-; so (ii) holds.

We may also assume that $G_A := G[A + \{b, w, x_1, z_1, z_2\}]$ does not contain three independent paths, with one from x_1 to b, one from b to w, and one from w to z_i for some $i \in [2]$. For, otherwise, such three paths and $T \cup bx \cup y_1x \cup Y_i \cup Y_3 \cup Y_4$ form a TK_5 in G' with branch vertices b, w, x, x_1, y_1.

We wish to apply Lemma 3.1. Let G'_A be the graph obtained from G_A by identifying z_1 and z_2 as z', and duplicating w, b with w', b', respectively (adding edges from w' to all vertices in $N_{G_A}(w)$, and from b' to all vertices in $N_{G_A}(b)$). Then any three disjoint paths in G'_A from $\{w, x_1, w'\}$ to $\{b, z', b'\}$, if exist, must contain a path from x_1 to z'.

Suppose G'_A has a separation (A_1, A_2) such that $|V(A_1 \cap A_2)| \leq 2$, $\{w, x_1, w'\} \subseteq V(A_1)$, and $\{b, z', b'\} \subseteq V(A_2)$. Since w and w' have the same set of neighbors in G'_A, we may assume $\{w, w'\} \subseteq V(A_1 \cap A_2)$ or $\{w, w'\} \cap V(A_1 \cap A_2) = \emptyset$. If $\{w, w'\} \subseteq V(A_1 \cap A_2)$ then $V(A_1) = \{x_1\} \cup V(A_1 \cap A_2)$ as $\{x, x_1, w\}$ cannot be a cut in G; hence, $N_G(x_1) \cap V(D) = \{w\}$, contradicting (3). So $\{w, w'\} \cap V(A_1 \cap A_2) = \emptyset$. Suppose $\{b, b', z'\} \cap V(A_1 \cap A_2) = \emptyset$. Then, since $wz_i \notin E(G)$ for $i \in [2]$, $V(A_1 \cap A_2) \cup \{x_1, x\}$ is a cut in G separating w from $B + \{b, z_1, z_2\}$, contradicting the fact that G is 5-connected. So $\{b, b', z'\} \cap V(A_1 \cap A_2) \neq \emptyset$. Note that $\{b, b'\} \not\subseteq V(A_1 \cap A_2)$; as otherwise $\{b, x, x_1\}$ would be a cut in G separating w from $B + \{z_1, z_2\}$. Thus, we may assume that $b, b' \notin V(A_1 \cap A_2)$ as b and b' have the same set of neighbors in G'_A. Hence, $z' \in V(A_1 \cap A_2)$. Now $S := \{x, x_1, z_1, z_2\} \cup (V(A_1 \cap A_2) - \{z'\})$ is a cut in G separating w from $B + b$. Since G is 5-connected, $x_1 \notin V(A_1 \cap A_2)$. If $|V(A_1 - x_1 - A_2)| \geq 2$ then $(xx_1, S, A_1 - x_1 - A_2, G - S - A_1) \in \mathcal{Q}_x$ which contradicts the choice of (T, S_T, A, B) with $|V(A)|$ minimum. So $V(A_1 - x_1 - A_2) = \{w\}$. Since G is 5-connected, $wz_i \in E(G)$ for $i \in [2]$, a contradiction.

Hence, by Lemma 3.1, G'_A has a separation (J, L) such that $V(J \cap L) = \{w_0, \ldots, w_n\}$, $(J, w_0, \ldots, w_n)$ is planar (since G is 5-connected), $(L, (w, x_1, w'), (b, z', b'))$ is a ladder along a sequence $b_0 \ldots b_m$, where $b_0 = x_1$, $b_m = z'$, and $w_0 \ldots w_n$ is the reduced sequence of $b_0 \ldots b_m$. Moreover, we may assume that L has disjoint induced paths P_1, P_2, P_3 from w, x_1, w' to b, z', b', respectively, and J is a connected plane graph with $b_0, b_1, \ldots, b_m$ occurring on the outer walk of J in cyclic order. (For convenience, when (*ii*) of Lemma 3.1 holds, we let J consist of $w_0, \ldots, w_n$ only.) Note that by Lemmas 3.3 and 3.4, each rung of $(L, (w, x_1, w'), (b, z', b'))$ is of type (*i*)–(*iv*) as in Lemma 3.4, with possible exceptions of those rungs containing a or z'. Let $(R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))$, $j \in [m]$, be the rungs in $(L, (w, x_1, w'), (b, z', b'))$ such that $a_j \in V(P_1)$ and $c_j \in V(P_3)$ for $j = 0, 1, \ldots, m$.

We now show that there exists $t \in N_{G'_A}(w)$ such that $t \in V(P_2) - \{x_1, z'\}$. For, suppose such t does not exist. Choose the largest j such that $\{w, w'\} \subseteq V(R_j)$. We claim that $z' \notin V(R_j)$. For, otherwise, $z' = b_j$. Then $b_{j-1} \neq b_j$; as, otherwise, $\{x_1, z_1, z_2\}$ would be a cut in G (since t does not exist and $N_G(x_1) \cap V(D) \neq \{w\}$). Since w, w' have the same set of neighbors in G'_A and $wb \notin E(G)$ (by (2)), we see that $(R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))$ cannot be a rung of type (2)–(7) (see the definition of rungs in front of Lemma 3.1) and, hence, must be a rung of type (1), with $\{w, w'\} = \{a_{j-1}, c_{j-1}\} = \{a_j, b_j\}$. This, however, contradicts the maximality of j. Thus, we can instead choose the largest j such that $\{w, w'\} \subseteq V(R_j)$ and $(R_j, (a_{j-1}, b_{j-1}, c_{j-1}),$ $(a_j, b_j, c_j))$ is not of type (*ii*) in Lemma 3.4, which exists as $w \neq b$. By the above claim we know $z' \notin V(R_j)$. Since G is 5-connected and because w and w' have the same set of neighbors in G'_A, $(R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))$ cannot be of type (*iii*) as in Lemma 3.4. Moreover, $(R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))$ is not of type (iv) as in Lemma 3.4, as otherwise G contains K_4^- (obtained from $R_j - \{b_{j-1}, b_j\}$ after identifying w with w'). So $(R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))$ is of type (i) as in Lemma 3.4. Now $V(R_j) = \{a_j, b_j, c_j, w, w'\}$, as otherwise $\{a_j, b_j, c_j, w\}$ would be a cut in G. Then $wb_j \in E(G)$; for otherwise, $N_G(w) \subseteq \{a_j, c_j, x, x_1\}$, a contradiction as G is 5-connected. Hence $t := b_j$ is as desired.

Without loss of generality, we may assume that the edge of P_2 incident with z' corresponds to an edge of G incident with z_1. We view P_3 as a path in G_A from b to w.

Then $G_A - V(P_1 \cup P_3) - z_2$ has independent paths from t to x_1, z_1, respectively. Recall that $N_G(x) \cap V(A \cap D) = \emptyset$. So G_A is $(5, V(P_1 \cup P_3) \cup \{a, x_1, z_1, z_2\})$-connected. Hence, by Lemma 2.11, G_A has five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from t and internally disjoint from $V(P_1 \cup P_3) \cup \{a, z_2\}$, with Q_1, Q_2, Q_3 ending at x_1, w, z_1, respectively, and Q_4, Q_5 ending at distinct vertices in $(V(P_1 \cup P_3) - \{w\}) \cup \{a, z_2\}$. By symmetry between P_1 and P_3, we may assume that Q_4 ends in $V(P_3 - w) \cup \{a\}$. Then $Q_4 \cup (P_3 - w) \cup ba$ contains a path Q_4^* from t to b. Let $G_B = G[B + \{b, x, x_1, z_1, z_2\}]$. Since $N_G(w) \cap V(B) = \emptyset$, G_B is $(5, \{b, x, x_1, z_1, z_2\})$-connected.

If $G_B - x$ contains disjoint paths S_1, S_2 from z_1, b to y_1, x_1, respectively, then $T \cup bx \cup P_1 \cup S_2 \cup Q_1 \cup Q_2 \cup (Q_3 \cup S_1 \cup y_1x) \cup Q_4^*$ is a TK_5 in G' with branch vertices b, t, w, x, x_1. Hence, we may assume such S_1, S_2 do not exist. Then by Lemma 2.10, there exists a collection $\mathcal{D}$ of subsets of $V(G_B - x) - \{z_1, b, y_1, x_1\}$ such that $(G_B - x, \mathcal{D}, z_1, b, y_1, x_1)$ is 3-planar. We choose such $\mathcal{D}$ that each $D \in \mathcal{D}$ is minimal.

If $(G_B - x, \{b, x_1, z_1, z_2\})$ is planar then the assertion of the lemma follows from Lemmas 2.5, with the cut $\{b, x, x_1, z_1, z_2\}$ giving the required 5-separation (G_1, G_2).

So we may assume that $(G_B - x, \{b, x_1, z_1, z_2\})$ is not planar. Then either $\mathcal{D} = \emptyset$ and z_2 does not belong to the facial walk of $G_B - x$ containing $\{b, x_1, y_1, z_1\}$, or $\mathcal{D} = \{D\}$ for some $D \subseteq V(G_B - x) - \{b, x_1, y_1, z_1\}$ and $z_2 \in D$ (as G is 5-connected).

In the following three paragraphs, we show that we may assume that $G_B - x$ has disjoint paths S_1', S_2' from z_2, b to y_1, x_1, respectively, and if b has degree at least two in $G_B - x$ then $G_B - x$ has independent paths Y, Y_2', Y_3', Y_4', with Y from b to z_1 and Y_2', Y_3', Y_4' from y_1 to z_2, x_1, b, respectively.

First, suppose $\mathcal{D} = \emptyset$ and z_2 does not belong to the facial walk F of $G_B - x$ containing $\{b, x_1, y_1, z_1\}$ (for every planar drawing of $G_B - x$ with b, x_1, y_1, z_1 incident with a common face). Let S_2' denote the path in $F - y_1$ from b to x_1, let Y be the path in $F - y_1$ from b to z_1, and let Y_3', Y_4' denote the paths in F from y_1 to x_1, b, respectively, neither of which contains S_2'. Note that Y_3' is internally disjoint from Y and Y_4', and if b has at least two neighbors in $G_B - x$ then Y and Y_4' are also internally disjoint. We claim that $G_B - x$ contains a path Y_2' from y_1 to z_2 and internally disjoint $Y \cup Y_3' \cup Y_4'$; for otherwise, $G_B - x$ has a cut T with $|T| \le 2$ such that $T \subseteq V(Y \cup Y_3' \cup Y_4')$ and T separates z_2 from y_1, but then $T \cup \{b, x\}$ is a cut in G, a contradiction. Next, we may assume $G_B - x - S_2'$ contains no path from z_2 to y_1 since such a path gives the desired S_1'. Then there exist $z_1', z_2' \in V(S_2')$ and a separation (H_1, H_2) in $G_B - x$ such that $V(H_1 \cap H_2) = \{z_1', z_2'\}$, $z_2 \in V(H_1 - H_2)$, and $y_1 \in V(H_2 - H_1)$. Hence, $z_1 \in V(H_1 - H_2)$ as otherwise $|V(H_1)| = 3$ (since G is

5-connected) and, hence, $(G_B - x, \{b, x, z_1, z_2\})$ is planar, a contradiction. Now G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b, x, x_1, z_1', z_2'\}$, $H_1 \cup A \subseteq G_1$, $H_2 \subseteq G_2$, and $(G_2 - x, \{b, x_1, z_1', z_2'\})$ is planar. If $|V(G_2)| \geq 7$ then the assertion of the lemma follows from Lemmas 2.5 and 2.2. So $|V(G_2)| = 6$. Hence, $G[\{b, x, x_1, y_1\}] \cong K_4^-$.

Now suppose that $\mathcal{D} = \{D\}$ for some $D \subseteq V(G_B - x) - \{b, x_1, y_1, z_1\}$ and $z_2 \in D$. Note that $|N_G(D)| = 3$; for, otherwise, $V(D) = \{z_2\}$ (as G is 5-connected) and $(G_B - x, \{b, x_1, y_1, z_1\})$ is planar, a contradiction. Moreover, for any partition T_1, T_2 of $N_G(D) \cup \{z_2\}$ with $|T_1| = |T_2| = 2$, $D' := G[D \cup N_G(D)]$ has two disjoint paths, one between the vertices in T_1 and the other between the vertices in T_2; for, otherwise, $(D', T_1 \cup T_2)$ is planar by the minimality of D and, hence, $(G_B - x, \{z_1, b, y_1, x_1\})$ is planar, contradicting the choice of $\mathcal{D}$.

Let H be obtained from $G_B - x$ by deleting $D - z_2$ and adding a complete graph on $N_G(D) \cup \{z_2\}$. Then (H, b, z_1, x_1, y_1) is planar and z_2 does not belong to the facial walk of H containing $\{b, x_1, y_1, z_1\}$. Hence, the same argument for the case $\mathcal{D} = \emptyset$ above applied to H (instead of $G_B - x$) shows the existence of the desired paths. (Note that if the paths use the K_4 on $N_G(D) \cup \{z_2\}$, we can always replace them with disjoint paths in D'.)

Next, we may assume that G_A contains a path Z from z_2 to some $z_2' \in V(P_1 \cup P_3) - \{b, b'\}$ and internally disjoint from $P_1 \cup P_3 \cup ((J - z') \cup P_2 + \{z_1, z_2\})$. To see this, consider the component F of $G_A' - ((J - z') \cup P_2) - \{b, b'\}$ containing z_2. If F contains $(P_1 - b) \cup (P_3 - b')$ then the desired path Z exists. So all neighbors of F in G_A are contained in $(J - z') \cup P_2 \cup \{b\}$. We claim that there exists $v \in V(tXx_1 - x_1)$ such that $F' := G[(J - z') \cup P_2 \cup F + w]$ has three independent paths from v to x_1, z_2, w, respectively. If $F' - w$ has independent paths from t to x_1, z_2, respectively, then these two paths and tw show that $v := t$ works. So assume that there exists $v \in V(tP_2x_1 - x_1)$ and a separation (A_1, A_2) in $F' - w$ such that $V(A_1 \cap A_2) = \{v\}$, $t \in V(A_1 - A_2)$, and $F + x_1 \subseteq A_2$. Choose v and (A_1, A_2) to minimize A_2. Then, since $|N_G(z_2) \cap V(A)| \geq 2$ (by (1)), A_2 has independent paths from v to x_1, z_2, respectively. Now these two paths and vP_2tw give the desired paths in F'. Since $N_G(x) \cap V(A \cap D) = \emptyset$, G_A is $(5, V(P_1 \cup P_3) \cup \{a, x_1, z_1, z_2\})$-connected. Hence, by Lemma 2.11, G_A has five independent paths $Q_1', Q_2', Q_3', Q_4', Q_5'$ from t to $x_1, w, z_2, (V(P_1 \cup P_3) - \{w\}) \cup \{a, z_1\}$, respectively, with only t in common, and internally disjoint from $P_1 \cup P_3$. By symmetry between P_1 and P_3, we may assume that Q_4' ends in $V(P_3 - w) \cup \{a\}$. So $Q_4' \cup (P_3 - w) \cup ba$ contains a path Q_4^+ from t to b. Now $T \cup bx \cup P_1 \cup S_2' \cup Q_1' \cup Q_2' \cup (Q_3' \cup S_1' \cup y_1x) \cup Q_4^+$ is TK_5 in G' with branch vertices b, t, w, x, x_1.

We may further assume that b has only one neighbor in $G_B - x$ and, in particular, $bz_i \notin E(G)$ for $i \in [2]$. For, otherwise, $G_B - x$ has the paths Y', Y_2', Y_3', Y_4'. By symmetry between P_1 and P_3, we may assume that $z_2' \in V(P_3 - b')$. Then $T \cup bx \cup P_1 \cup (Y \cup P_2) \cup y_1x \cup (Y_2' \cup Z \cup z_2'P_3w) \cup Y_3' \cup Y_4'$ is a TK_5 in G' with branch vertices b, w, x, x_1, y_1.

Thus, since G is 5-connected and $bw \notin E(G)$ (by (2)) and since P_1, P_3 are induced paths in L, b has a neighbor $u \in V(A) - V(P_1 \cup P_3)$. Let $(R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))$

be the rung containing $\{b, b', u\}$. Since b and b' have the same set of neighbors in G'_A, $a_{j-1} = b$ if, and only if, $c_{j-1} = b'$. Moreover, we must have $b_j = z'$ because of the path Z.

We claim that $b_{j-1} = z'$ and, hence, $a_{j-1} \neq b$ and $c_{j-1} \neq b'$. For suppose $b_{j-1} \neq z'$. Since $b_{j-1} \neq b_j$ and since b and b' have the same set of neighbors in G'_A, we must have $a_{j-1} = b$ and $c_{j-1} = b'$. This, however, contradicts the existence of the path Z.

We now show that we may further choose Z so that Z is internally disjoint R_j as well. To see this, let F denote the component of $G'_A - ((J - z') \cup P_3) - (R_j - \{a_{j-1}, c_{j-1}\})$ containing z_2. If F contains $(P_1 \cup P_3) - (R_j - \{a_{j-1}, c_{j-1}\})$ then clearly Z may be chosen to be internally disjoint from R_j as well. So F is disjoint from $(P_1 \cup P_3) - (R_j - \{a_{j-1}, c_{j-1}\})$. Then F has a neighbor in $(J \cup P_2) - \{x_1, z'\}$; for otherwise, $\{a_{j-1}, c_{j-1}, z_1\} \cup V(T)$ is a cut in G separating $(J - z') \cup P_2 \cup wP_1a_{j-1} \cup w'P_3c_{j-1}$ from $B \cup F \cup R_j$, contradicting the choice of T (that A is minimal). We claim that there exists $v \in V(tP_2x_1 - x_1)$ such that $F' := G[(J - z') \cup P_2 \cup F + w]$ has three independent paths from v to w, x_1, z_2, respectively. Note that we can let $v := t$ if $F' - w$ has independent paths from t to x_1, z_2, respectively. So assume such paths do not exist in $F' - w$. Then there exist $v \in V(tP_2x_1)$ and a separation (A_1, A_2) in $F' - w$ such that $V(A_1 \cap A_2) = \{v\}$, $t \in V(A_1 - v)$, and $F + x_1 \subseteq A_2$. Choose v and (A_1, A_2) so that A_2 is minimal. Then since F has a neighbor in $J - \{x_1, z'\}$, A_2 has independent paths from v to x_1, z_2, respectively, which together with vP_2tw gives the desired paths in F'. Since $N_G(x) \cap V(A \cap D) = \emptyset$, G_A is $(5, V(P_1 \cup P_3) \cup \{a, x_1, z_1, z_2\})$-connected. Hence, by Lemma 2.11, G_A has five independent paths $Q'_1, Q'_2, Q'_3, Q'_4, Q'_5$ from v to $x_1, w, z_2, (V(P_1 \cup P_3) - \{w\}) \cup \{a, z_1\}$, respectively, with only v in common, and internally disjoint from $P_1 \cup P_3$. By symmetry between P_1 and P_3, we may assume that Q'_4 ends in $V(P_3 - w) \cup \{a\}$. So $Q'_4 \cup (P_3 - w) \cup ba$ contains a path Q_4^+ from t to b. Now $T \cup bx \cup P_1 \cup S'_2 \cup Q'_1 \cup Q'_2 \cup (Q'_3 \cup S'_1 \cup y_1x) \cup Q_4^+$ is TK_5 in G' with branch vertices b, v, w, x, x_1.

Finally, we show that $G[R_j - \{b', z'\} + z_1]$ has independent paths P'_1, P'_2, S from b to a_{j-1}, c_{j-1}, z_1, respectively. For, otherwise, $G[R_j - \{b', z'\} + \{z_1, z_2\}]$ has a 3-separation (A_1, A_2) such that $z_2 \in V(A_1 \cap A_2)$, $b \in V(A_1 - A_2)$, and $\{a_{j-1}, c_{j-1}, z_1\} \subseteq V(A_2)$. Now G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b, x\} \cup V(A_1 \cap A_2)$, $bP_1a_{j-1} \cup bP_3c_{j-1} \cup \{u\} \subseteq G_1$, and $B \cup (J - z') \cup A_2 \subseteq G_2$. Note that $u \in V(G_1 - G_2)$. By the choice of T (that A is minimal), we have $|V(G_1)| = 6$ and, hence, $N_G(u) = \{b, x\} \cup V(A_1 \cap A_2)$. Now $G[\{a_{j-1}, c_{j-1}, b, u\}]$ contains K_4^-.

By symmetry between P_1 and P_3, we may assume that $z'_2 \in V(P_3 - b')$. Now $T \cup bx \cup (P'_1 \cup a_{j-1}P_1w) \cup (S \cup z_1P_2x_1) \cup y_1x \cup (Y_2 \cup Z \cup z'_2P_3w) \cup Y_3 \cup Y_4$ is TK_5 in G' with branch vertices b, w, x, x_1, y_1.

Case 2. $N_G(x) \cap \{z_1, z_2\} \neq \emptyset$.

Without loss of generality, we may assume $xz_1 \in E(G)$. We may further assume z_1 is not adjacent to any of $\{a, b, w, x_1\}$; for otherwise, $G[T + z_1]$ or $G[T' + z_1]$ contains K_4^-, and (ii) holds. We wish to prove (iii), with $x_2 = b$ and $x_3 = z_1$. Let $y_1, y_2 \in N_G(x) - \{b, x_1, z_1\}$ be distinct.

Subcase 2.1. For some $i \in [2]$, $y_i \in V(B) \cup \{z_2\}$.

Without loss of generality, assume $y_1 \in V(B) \cup \{z_2\}$ and, whenever possible, let $y_1 \in V(B)$. Let $G_B := G[B + \{b, x_1, z_1, z_2\}]$. When $y_1 \in V(B)$ let $t = y_1$ and let Y_1, Y_2, Y_3, Y_4, Y_5 be independent paths in G_B from t to z_1, y_1, b, x_1, z_2, respectively. When $y_1 = z_2$ let $t \in V(B)$ be arbitrary and let Y_1, Y_2, Y_3, Y_4 be independent paths in G_B from t to z_1, y_1, b, x_1, respectively. Let $G_A = G[A + \{b, w, x_1, z_1\}]$.

We may assume that there is no cycle in G_A containing $\{b, x_1, z_1\}$. For, such a cycle and $xb \cup xx_1 \cup xz_1 \cup Y_1 \cup (Y_2 \cup y_1 x) \cup Y_3 \cup Y_4$ is a TK_5 in G' with branch vertices b, t, x, x_1, z_1.

We may also assume that G_A is 2-connected. To see this, we first assume $N_G(x_1) \cap N_G(w) = \{x\}$; for otherwise, letting $u \in (N_G(x_1) \cap N_G(w)) - \{x\}$ we see that $G[T + u]$ contains K_4^- and (ii) holds. Therefore, since $N_G(w) \cap V(A) \neq \emptyset$ and $N_G(x_1) \cap V(A) \neq \emptyset$ (by (3)), it suffices to show that $G[A + \{b, z_1\}]$ is 2-connected. So assume for a contradiction that there exists a separation (A_1, A_2) in $G[A + \{b, z_1\}]$ such that $|V(A_1 \cap A_2)| \leq 1$. Without loss of generality, let $|\{b, z_1\} \cap V(A_1)| \leq 1$. Then $V(A_1) \nsubseteq V(A_2) \cup \{b, z_1\}$ as $|N_G(s) \cap V(A)| \geq 2$ for $s \in \{b, z_1\}$ (by (1)). Hence, $V(T) \cup (\{b, z_1\} \cap V(A_1)) \cup V(A_1 \cap A_2) \cup \{z_2\}$ is a cut in G of size at most 6 which separates A_1 from the rest of G, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum.

Then, since G_A has no cycle containing $\{b, x_1, z_1\}$, it follows that (i), or (ii), or (iii) of Lemma 2.12 holds for G_A and $\{b, x_1, z_1\}$. So for each $u \in \{b, x_1, z_1\}$, G_A has a 2-cut S_u separating u from $\{b, x_1, z_1\} - \{u\}$, and let D_u denote a union of components of $G_A - S_u$ such that $u \in V(D_u)$ for $u \in \{b, x_1, z_1\}$ and D_b, D_{x_1}, D_{z_1} are pairwise disjoint. We choose S_u and D_u, $u \in \{b, x_1, z_1\}$, to maximize $D_b \cup D_{x_1} \cup D_{z_1}$. Note that, since $wx_1 \in E(G)$, we have $w \notin V(D_b \cup D_{z_1})$.

We claim that for $u \in \{b, x_1, z_1\}$, $V(D_u) = \{u\}$. For, otherwise, $S := S_u \cup \{u, x, z_2\}$ is a cut in G separating $D_u - u$ from the rest of G. If $|V(D_u)| \geq 3$ then $(ux, S, D_u - u, G - S - D_u) \in \mathcal{Q}_x$, contradicting the choice of (T, S_T, A, B) with $|V(A)|$ minimum. So let $V(D_u) = \{u, u'\}$ and let $S_u = \{s_u, t_u\}$. Since G is 5-connected, $N_G(u') = \{s_u, t_u, u, x, z_2\}$. Since $|N_G(u) \cap V(A + w)| \geq 2$ (by (1) and (3)), we may assume that $us_u \in E(G)$. Then $G[\{s_u, u, u', x\}]$ contains K_4^-, and (ii) holds.

For $u \in \{b, x_1, z_1\}$, let $S_u = \{s_u, t_u\}$. Since G_A is 2-connected, $\{us_u, ut_u\} \subseteq E(G)$. Note $a \in \{s_b, t_b\}$; so we may assume $s_b t_b \notin E(G)$ because otherwise $G[\{x, b, s_b, t_b\}]$ contains K_4^-, and (ii) holds. Similarly, $w \in \{s_{x_1}, t_{x_1}\}$ and we may assume $s_{x_1} t_{x_1} \notin E(G)$. If (i) of Lemma 2.12 occurs then $ax_1 \in E(G)$, contradicting (2). If (iii) of Lemma 2.12 occurs then let R_1, R_2 be the components of $G_A - V(D_b \cup D_{x_1} \cup D_{z_1})$ and assume without loss of generality that $s_u \in V(R_1)$ and $t_u \in V(R_2)$ for $u \in \{b, x_1, z_1\}$. By symmetry, assume $w \notin V(R_1)$. Hence, $(xb, \{x, b, x_1, s_{z_1}, z_2\}, R_1 - s_{z_1}, G - R_1 - \{x, b, x_1, z_2\}]) \in \mathcal{Q}_x$ with $2 \leq |V(R_1 - s_{z_1})| < |V(A)|$, contradicting the choice of (T, S_T, A, B).

So we may assume that (ii) of Lemma 2.12 holds. Without loss of generality, let $z = s_b = s_{x_1} = s_{z_1}$. By (2), $z \neq a$ and $z \neq w$. So $a = t_b$ and $w = t_{x_1}$. Thus, we may assume $xz \notin E(G)$ as, otherwise, $G[T + z]$ contains K_4^- and (ii) holds. Moreover, we may assume $za, zw \notin E(G)$ as otherwise $G[T' + z]$ or $G[T + z]$ contains K_4^- and (ii)

holds. Hence, since G is 5-connected, $R := G_A - \{b, x_1, z_1\}$ is connected. Thus, by (1) and by the maximality of $D_b \cup D_{x_1} \cup D_{z_1}$, $G[R + z_2]$ is 2-connected as G is 5-connected.

We claim that there exist distinct $t_1, t_2 \in \{a, w, t_{z_1}\}$ such that $G[R + z_2]$ contains disjoint paths P_1, P_2 from z, t_1 to z_2, t_2, respectively. For, suppose $\{a, w\}$ cannot serve as $\{t_1, t_2\}$. Then, by Lemma 2.10, $(G[R + z_2], a, z_2, w, z)$ is 3-planar. Thus, since $G[R + z_2]$ is 2-connected, $G[R + z_2]$ has a cycle, say C, through a, z, w, z_2 in cyclic order. Let C_a, C_w denote the paths in C between z and z_2 that contain a, w, respectively. Since G_A is 2-connected, $G[R + z_2]$ has a path P from t_{z_1} to a vertex $t \in V(C)$ such that P is internally disjoint from C. If $t \in V(C_a)$ then $C \cup P$ has disjoint paths from z, a to z_2, t_{z_1}, respectively; and if $t \in V(C_w)$ then $C \cup P$ has disjoint paths from z, w to z_2, t_{z_1}, respectively.

Suppose $z_2 \neq y_1$. Recall the definition of t and the paths Y_1, Y_2, Y_3, Y_4, Y_5. If $\{t_1, t_2\} = \{a, w\}$ then $bxx_1zb \cup xz_1z \cup (x_1w \cup P_2 \cup ab) \cup (Y_2 \cup y_1x) \cup (Y_5 \cup P_1) \cup Y_3 \cup Y_4$ is a TK_5 in G' with branch vertices b, t, x, x_1, z. If $\{t_1, t_2\} = \{a, t_{z_1}\}$ then $bxz_1zb \cup xx_1z \cup (z_1t_{z_1} \cup P_2 \cup ab) \cup Y_1 \cup (Y_2 \cup y_1x) \cup Y_3 \cup (Y_5 \cup P_1)$ is a TK_5 in G' with branch vertices b, t, x, z, z_1. If $\{t_1, t_2\} = \{w, t_{z_1}\}$ then $x_1xz_1zx_1 \cup xbz \cup (x_1w \cup P_2 \cup t_{z_1}z_1) \cup Y_1 \cup (Y_2 \cup y_1x) \cup Y_4 \cup (Y_5 \cup P_1)$ is a TK_5 in G' with branch vertices t, x, x_1, z, z_1.

So assume $z_2 = y_1$. Then $y_2 \neq z_2$; and hence, by the choice of y_1, we have $y_2 \in V(A) \cup \{w\}$. If $R - z$ has independent paths S_1, S_2, S_3 from y_2 to a, w, t_{z_1}, respectively, then $xbzx_1x \cup y_2x \cup (S_1 \cup ab) \cup (S_2 \cup wx_1) \cup Y_3 \cup Y_4 \cup (Y_1 \cup z_1t_{z_1} \cup S_3) \cup (Y_2 \cup z_2x)$ is a TK_5 in G' with branch vertices b, t, x, x_1, y_2. So assume such S_1, S_2, S_3 do not exist. This implies that $y_2 \notin \{a, w\}$ by the maximality of $D_a \cup D_{x_1} \cup D_{z_1}$. Then $R - z$ has a separation (A_1, A_2) such that $|V(A_1 \cap A_2)| \le 2$, $y_2 \in V(A_1 - A_2)$, and $\{a, w, t_{z_1}\} \subseteq V(A_2)$. Thus $S := \{x, z, z_2\} \cup V(A_1 \cap A_2)$ is a cut in G separating y_2 from $B \cup A_2 \cup \{b, x_1, z_1, z\}$. Since G is 5-connected, $|S| = 5$. By the choice of (T, S_T, A, B) (with $|V(A)|$ minimum), $V(A_1 - A_2) = \{y_2\}$. Therefore, since G is 5-connected, $N_G(y_2) = S$; in particular, $y_2z \in E(G)$. By the maximality of $D_b \cup D_{x_1} \cup D_{z_1}$, $R_2 - \{y_2, z\}$ has a path Q from a to w. Then $bxx_1zb \cup (ba \cup Q \cup wx_1) \cup zy_2x \cup (Y_1 \cup z_1z) \cup (Y_2 \cup z_2x) \cup Y_3 \cup Y_4$ is a TK_5 in G' with branch vertices b, t, x, x_1, z.

Subcase 2.2. $y_1, y_2 \in V(A) \cup \{w\}$.

First, we show that we may assume $y_1 = w$. To see this, we explore the symmetry betwee a and w by noting that $|V(D)| = |V(A)|$ and the symmetry between (T, S_T, A, B) and $(T', S_{T'}, D, C)$. Thus, if $a \in \{y_1, y_2\}$ then we could exchange the roles of a and w and the roles of (T, S_T, A, B) and $(T', S_{T'}, D, C)$. Hence, we may assume $a \notin \{y_1, y_2\}$. Then by (4), for each $i \in [2]$ there exists $j_i \in [2]$ such that $T_i := xy_iz_{j_i}x$ is a triangle in G. If $j_1 = j_2$ then $G[\{x, y_1, y_2, z_{j_1}\}]$ contains K_4^-. So assume $j_1 \neq j_2$, say $j_1 = 1$ and $j_2 = 2$. Now $S_{T_1} = V(T_1) \cup \{b, x_1, z_1, z_2\}$ is a cut in G separating $(A - \{y_1\}) \cup \{w\}$ from B. Note the symmetry between T_1, S_{T_1} and T, S_T, and we may choose T_1, S_{T_1} as T, S_T, respectively; so that the assumption of this lemma still holds. Hence, we may assume $y_1 = w$ (as y_1 now plays the role of w and z_1 now plays the role of x_1).

Let $t \in V(B)$, and let L_1, L_2, L_3, L_4 be independent paths in $G_B = G[B + \{b, x_1, z_1, z_2\}]$ from t to z_1, z_2, b, x_1, respectively. Let $G_A := G[A + \{b, w, x_1, z_2\}]$. Note that, by the same argument as in Subcase 2.1 (with z_2 in place of z_1), we may assume that G_A is 2-connected.

We may assume that G_A does not contain independent paths from z_2, w, b to w, b, x_1, respectively; for otherwise, these paths and $T \cup bx \cup (L_1 \cup z_1x) \cup (L_2 \cup z_2w) \cup L_3 \cup L_4$ form a TK_5 in G with branch vertices b, t, w, x, x_1.

We claim that $wz_2 \notin E(G)$. For, suppose $wz_2 \in E(G)$. Then $G_A - z_2$ has no independent paths from b to w, x_1, respectively; as, otherwise, such paths and wz_2 give independent paths in G_A from z_2, w, b to w, b, x_1, respectively. So G_A has a 2-separation (A_1, A_2) such that $z_2 \in V(A_1 \cap A_2)$, $b \in V(A_1 - A_2)$, and $\{w, x_1\} \subseteq V(A_2)$. If $V(A_1 - A_2) \not\subseteq \{a, b\}$ then $V(T') \cup V(A_1 \cap A_2) \cup \{z_1\}$ is a cut in G separating $A_1 - A_2 - \{a, b\}$ from $B \cup A_2$, contradicting the choice of T (that A is minimum). So $V(A_1 - A_2) \subseteq \{a, b\}$. Thus, by (1), $a \notin V(A_1 \cap A_2)$. Hence, $N_G(a) = V(A_1 \cap A_2) \cup \{b, x\}$, a contradiction as G is 5-connected.

Recall that $wz_1 \notin E(G)$ (see beginning of Case 2) and $wa, wb \notin E(G)$ (by (2)). Therefore, since G is 5-connected, it follows that

$$|N_G(w) \cap V(A \cap D)| \geq 3.$$

Let G'_A be the graph obtained from G_A by duplicating w, b with w', b', respectively, and adding all edges from w' to $N_{G_A}(w)$, and from b' to $N_{G_A}(b)$. Then any three disjoint paths in G'_A from $\{b, b', z_2\}$ to $\{w, w', x_1\}$ must have a path from z_2 to x_1, and we wish to apply Lemma 3.1.

First, we note that G'_A has no cut of size at most 2 separating $\{x_1, w, w'\}$ from $\{b, b', z_2\}$. For, otherwise, G'_A has a separation (A_1, A_2) such that $|V(A_1 \cap A_2)| \leq 2$, $\{x_1, w, w'\} \subseteq V(A_1)$, and $\{b, b', z_2\} \subseteq V(A_2)$. Note that $V(A_1 \cap A_2) \neq \{w, w'\}$ as, otherwise, w would be a cut vertex in G_A. Further, $\{w, w'\} \cap V(A_1 \cap A_2) = \emptyset$; for, otherwise, since w and w' have the same set of neighbors in G'_A, it follows from (3) that $V(A_1 \cap A_2) - \{w, w'\}$ would be a cut in G_A of size at most one. On the other hand, $V(A_1 - A_2) \subseteq \{x_1, w\}$; as, otherwise $(T, V(T) \cup \{z_1\} \cup V(A_1 \cap A_2), (A_1 - A_2) - w', G - (T \cup A_1)) \in \mathcal{Q}_x$ and $1 \leq |V((A_1 - A_2) - \{w'\})| < |V(A)|$, contradicting the choice of (T, S_T, A, B). However, this implies $|N_G(w) \cap V(A \cap D)| \leq |V(A_1 \cap A_2)| \leq 2$, a contradiction.

Hence by Lemma 3.1 (and Remark 1 following Lemma 3.1), G'_A has a separation (J, L) such that $V(J \cap L) = \{w_0, \ldots, w_n\}$, $(J, w_0, \ldots, w_n)$ is 3-planar, $(L, (b, z_2, b'), (w, x_1, w'))$ is a ladder along some sequence $b_0 \ldots b_m$, where $b_0 = z_2$, $b_m = x_1$, and $w_0 \ldots w_n$ is the reduced sequence of $b_0 \ldots b_m$. Let P_1, P_2, P_3 be three disjoint paths in L from b, z_2, b' to w, x_1, w', respectively, and assume that they are induced in G'_A. (For convenience, when (ii) of Lemma 3.1 holds, we let $L = G'_A$ and J consist of $w_0, \ldots, w_n$ only.) Let $(R_i, (a_{i-1}, b_{i-1}, c_{i-1}), (a_i, b_i, c_i))$, $i \in [m]$, be the rungs in L with $a_i \in V(P_1)$ and $c_i \in V(P_3)$ for $i = 0, 1, \ldots, m$. (We caution that here we may not use Lemma 3.4 as its condition on separation (R, R') is not satisfied because x and z_1 can have neighbors inside R_i.)

We show that there exists $u \in V(P_2)-\{x_1,z_2\}$ such that $G[G_A+\{x,z_1\}]$ has five independent paths Q_1,Q_2,Q_3,Q_4,Q_5 from u to distinct vertices x_1,w,z_2,u_1,u_2, respectively, with $u_1,u_2 \in V(P_1-w)\cup V(P_3-\{b',w'\})\cup\{x,z_1\}$, and internally disjoint from $P_1\cup(P_3-\{b',w'\})$. Since $|N_G(w)\cap V(A\cap D)|\geq 3$ and P_1,P_3 are induced paths in G'_A, there exists $w^* \in (N_G(w)\cap V(A))-V(P_1\cup P_3)$ such that $w^* \notin \{x_1,z_2\}$. If $w^* \in V(P_2)$ then let $u=w^*$ and we see that there exist independent paths in $G_A-(V(P_1-w)\cup V(P_3-\{b',w'\}))$ from u to x_1,w,z_2, respectively; so the paths $Q_1,\dots,Q_5$ exist by Lemma 2.11 as $G[G_A+\{x,z_1\}]$ is $(5,V(P_1-w)\cup V(P_3-\{b',w'\})\cup\{x,z_1\})$-connected. Now suppose $w^* \notin V(P_2)$. Let $(R_i,(a_{i-1},b_{i-1},c_{i-1}),(w,b_i,w'))$ be the rung in L containing $\{w,w',w^*\}$. Since w and w' have the same set of neighbors in G'_A, $w=a_{i-1}$ iff $w'=c_{i-1}$. If $w=a_{i-1}$ and $w'=c_{i-1}$ then $S^*_T := V(T)\cup\{b_{i-1},b_i,z_1\}$ is a cut in G of size at most 6, and $G-S^*_T$ has a component of size smaller than $|V(A)|$, contradicting the choice of (T,S_T,A,B). So $w\neq a_{i-1}$ and $w'\neq c_{i-1}$. Therefore, $b_{i-1}=b_i$ by Lemma 3.2. Then $b_{i-1}\neq x_1$; for, otherwise, $T_1 := V(T)\cup\{a_{i-1},c_{i-1},z_1\}$ is a cut in G such that one component of $G-T_1$ is contained in $R_i-\{a_{i-1},c_{i-1},b_{i-1}\}$, contradicting the choice of T. Suppose R_i has a separation (R',R'') such that $|V(R'\cap R'')|\leq 2$, $w\in V(R'-R'')$, and $\{a_{i-1},c_{i-1},b_{i-1}\}\subseteq V(R'')$. Then we may assume $w'\in V(R'-R'')$ as w and w' have the same set of neighbors in G'_A. Therefore, since $|N_G(w)\cap V(A\cap D)|\geq 3$, $S^*_T := V(T)\cup V(R'\cap R'')\cup\{z_1\}$ is a cut in G of size at most 6, and $G-S^*_T$ has a component of size smaller than $|V(A)|$, contradicting the choice of (T,S_T,A,B). Thus we may assume, by Lemma 2.11, R_i contains three independent paths from w to a_{i-1},c_{i-1},b_{i-1}, respectively. Again since w and w' have the same set of neighbors in G'_A, the parts of P_1,P_3 inside R can be modified so that the three paths in R_i correspond to $wP_1a_{i-1}, w'P_3c_{i-1}$, and a path from w to b_{i-1} and internally disjoint from $P_1\cup P_2\cup P_3$. Thus, there exist independent paths in $G_A-(V(P_1-w)\cup V(P_3-\{b',w'\}))$ from $u:=b_{i-1}$ to x_1,w,z_2, respectively. Note that $b_{i-1}\neq z_2$ as, otherwise, $\{x,b,w,z_1,z_2\}$ is a cut in G separating $P_1\cup P_3+a$ from $(J-z')\cup B$, contradicting minimality of A. Now the paths $Q_1,\dots,Q_5$ exist by Lemma 2.11, as $G[G_A+\{x,z_1\}]$ is $(5,V(P_1-w)\cup V(P_3-\{b',w'\})\cup\{x,z_1\})$-connected.

We may assume $\{u_1,u_2\}=\{x,z_1\}$ for any choice of $Q_1,\dots,Q_5$. For, otherwise, we may assume by symmetry that $u_1\in V(P_1-w)$. If G_B-x has disjoint paths B_1,B_2 from z_1,b to z_2,x_1, respectively, then $T\cup bx\cup P_3\cup B_2\cup Q_1\cup Q_2\cup(Q_3\cup B_1\cup z_1x)\cup(Q_4\cup u_1P_1b)$ is a TK_5 in G with branch vertices b,u,w,x,x_1. (Here we view P_3 as a path in G by identifying b',w' with b,w, respectively.) So we may assume that such B_1,B_2 do not exist. Then, since G_B-x is $(4,\{b,x_1,z_1,z_2\})$-connected (as G is 5-connected), it follows from Lemma 2.10 that (G_B-x,z_1,b,z_2,x_1) is planar; so the assertion of the lemma follows from Lemma 2.5.

We may also assume $|N_G(b)\cap V(B)|\leq 1$. For, suppose $|N_G(b)\cap V(B)|\geq 2$. Then, $G[B+\{b,x_1,z_2\}]$ contains independent paths B_1,B_2 from b to x_1,z_2, respectively; for, otherwise, $G[B+\{b,x_1,z_2\}]$ has a cut vertex t separating b from $\{x_1,z_2\}$ and, hence, $\{b,t,x,z_1\}$ is a cut in G, a contradiction. Hence, $T\cup bx\cup P_3\cup B_1\cup Q_1\cup Q_2\cup(Q_3\cup B_2)\cup(Q_4\cup z_1x)$ is a TK_5 in G with branch vertices b,u,w,x,x_1, where we view P_3 as a path in G' by identifying b',w' with b,w, respectively.

Then $|N_G(b) \cap V(A + z_2)| \geq 3$ as $bz_1 \notin E(G)$ (see the beginning of Case 2). Let $b^* \in (N_G(b) \cap V(A + z_2)) - V(P_1 \cup P_3)$. Let $(R_j, (b, b_{j-1}, b'), (a_j, b_j, c_j))$ be the rung in L containing $\{b, b', b^*\}$. Since b and b' have the same set of neighbors in G'_A, $b = a_j$ iff $b' = c_j$.

If $b^* \in V(P_2)$ let $z = b^*$ and let $P = bz$ which is internally disjoint from $P_1 \cup P_2 \cup P_3$.

Now suppose $b^* \notin V(P_2)$. If $b = a_j$ and $b' = c_j$ then $S^*_T := V(T') \cup \{b_{j-1}, b_j, z_1\}$ is a cut in G of size 6 (otherwise, $a = b^*$ and $b^*z_1 \in E(G)$) and $G - S^*_T$ has a component of size smaller than $|V(A)|$, contradicting the choice of (T, S_T, A, B). So $b \neq a_j$ and $b' \neq c_j$. Hence, $b_{j-1} = b_j$ by Lemma 3.2. We claim that $P_1 \cap R_j$ and $P_3 \cap R_j$ may be modified so that G_A contains a path P from b to b_j and internally disjoint from $P_1 \cup P_2 \cup (P_3 - \{b', w'\})$. If R_j contains three independent paths from b to a_j, c_j, b_j, then $P_1 \cap R_j, P_3 \cap R_j$ can be modified so that the three paths in R_j correspond to $bP_1a_j, b'P_3c_j$, and a path P from b to $z := b_j$ and internally disjoint from $P_1 \cup P_2 \cup (P_3 - \{b', w'\})$. So assume that such three paths in R_j do not exist. Then R_j has a separation (A_1, A_2) with $|V(A_1 \cap A_2)| \leq 2$, $V(A_1 \cap A_2) \subseteq V(P_1 \cup P_3)$, $b \in V(A_1 - A_2)$ and $\{a_j, c_j, b_j\} \subseteq V(A_2)$. Since b' is a copy of b in G'_A, we may assume $b' \in V(A_1 - A_2)$. Now $V(A_1 \cap A_2) \cup \{x, b, z_1\}$ is a cut in G; so $V(A_1) = V(A_1 \cap A_2) \cup \{b, b', b^*\}$ by the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Then $b^*x, b^*z_1 \in E(G)$ (as G is 5-connected); so $G[\{x, b^*, b, z_1\}]$ contains K_4^-, and (ii) holds.

Suppose $R_i \neq R_j$. Since G is 5-connected, $G[B + \{b, x_1\}]$ has a path B_1 from b to x_1. Since $\{u_1, u_2\} = \{x, z_1\}$ for any choice of $Q_1, \ldots, Q_5$, we see that $Q_1, \ldots, Q_5$ are all internally disjoint from $P \cup P_1 \cup P_3$. Thus, we can modify Q_1, Q_3 so that $Q_1 \cup Q_3 \subseteq J \cup P_2$. Hence, because $(J \cup P_2, w_0, \ldots, w_n)$ is 3-planar, we may assume that $z \in V(Q_3)$. By symmetry between Q_4 and Q_5, we may assume $u_1 = z_1$. Then $T \cup bx \cup P_3 \cup B_1 \cup Q_1 \cup Q_2 \cup (uQ_3z \cup P) \cup (Q_4 \cup z_1x)$ is a TK_5 in G' with branch vertices b, u, w, x, x_1, where we view P_3 as a path in G by identifying b', w' with b, w, respectively.

So $R_i = R_j$. Then $a_{i-1} = b$ and $c_{i-1} = b'$. Recall $bw \notin E(G)$ (by (2)). Then $b_{i-1} = b_i$ by Lemma 3.2. Hence, $\{b, b_i, w, x, z_1\}$ is a cut in G separating $P_1 \cup (P_3 - \{b', w'\})$ from $B \cup J$. Since $bw \notin E(G)$, $|V(P_1 - \{b, w\}) \cup V(P_3 - \{b', w'\})| \geq 2$. This contradicts the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. $\square$

5. Interactions between quadruples

In this section, we explore the structure of G by considering a quadruple (T, S_T, A, B) with $|V(A)|$ minimum and a quadruple $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$. The lemma below allows us to assume that if $T \cap C = \emptyset$ then $A \cap C = \emptyset$.

Lemma 5.1. *Let G be a 5-connected nonplanar graph and $x \in V(G)$. Suppose for any $H \subseteq G$ with $x \in V(H)$ and with $H \cong K_2$ or $H \cong K_3$, G/H is not 5-connected. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum, and $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$. Suppose $T \cap C = \emptyset$. Then $A \cap C = \emptyset$, or one of the following holds:*

(*i*) *G contains a* TK_5 *in which* x *is not a branch vertex.*
(*ii*) *G contains* K_4^-.
(*iii*) *There exist* $x_1, x_2, x_3 \in N_G(x)$ *such that for any* $y_1, y_2 \in N_G(x) - \{x_1, x_2, x_3\}$, $G - \{xv : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}$ *contains* TK_5.

Proof. We may assume $T \cong K_3$ (by Lemma 4.3) and $T' \cong K_3$ (by Lemma 4.4). Suppose $A \cap C \neq \emptyset$.

Then $|(S_T \cup S_{T'}) - V(B \cup D)| \geq 7$; otherwise $(T', (S_{T'} \cup S_T) - V(B \cup D), A \cap C, G[B \cup D]) \in \mathcal{Q}_x$ and $1 \leq |V(A \cap C)| \leq |V(A) - V(T' \cap A)| < |V(A)|$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Hence $|(S_T \cup S_{T'}) - V(A \cup C)| \leq 5$, as $|S_T| = |S_{T'}| = 6$. Since $T \cap C = \emptyset$, $V(T) \subseteq (S_T \cup S_{T'}) - V(A \cup C)$.

Suppose $|V(B \cap D)| \geq 2$. Then G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = (S_T \cup S_{T'}) - V(A \cup C)$ and $|V(G_i)| \geq 7$. Note that $G[V(G_1 \cap G_2)]$ contains the triangle T. So the assertion of this lemma follows from Lemma 2.6.

Hence, we may assume $|V(B \cap D)| \leq 1$. Therefore, by the minimality of $|V(A)|$, $|S_T \cap V(D)| \geq |S_{T'} \cap V(A)|$. But this implies that $|S_T| \geq |(S_T \cup S_{T'}) - V(B \cup D)| \geq 7$, a contradiction. □

We need a lemma on paths in $G[A + S_T]$ to deal with a special case when $A \cap C = \emptyset$ for quadruples $(T, S_T, A, B), (T', S_{T'}, C, D) \in \mathcal{Q}_x$.

Lemma 5.2. *Let* G *be a 5-connected nonplanar graph and* $x \in V(G)$, *and suppose for any* $H \subseteq G$ *with* $x \in V(H)$ *and with* $H \cong K_2$ *or* $H \cong K_3$, G/H *is not 5-connected. Let* $(T, S_T, A, B) \in \mathcal{Q}_x$ *with* $|V(A)|$ *minimum and* $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ *with* $T' \cap A \neq \emptyset$. *Let* $V(T) = \{x, x_1, x_2\}$ *and* $V(T') = \{x, a, b\}$ *with* $a \in V(A)$. *Suppose* $A \cap C = \emptyset$, $|S_T| = 6 = |S_{T'}|$, $V(T) \subseteq S_T - V(C)$, $|(S_T \cup S_{T'}) - V(B \cup C)| = 7$, *and* $(S_T \cup S_{T'}) - V(B \cup C \cup T \cup T') = \{x_3, x_4\}$. *Then* G *contains* K_4^-, *or the following statements hold:*

(*i*) $N_G(b) \cap V(A - a) \neq \emptyset$ *and if* $t \in N_G(b) \cap V(A - a)$ *then* $G[(A - a) + \{b, x_1, x_2, x_3, x_4\}]$ *has independent paths from* t *to* b, x_1, x_2, x_3, x_4, *respectively, and*
(*ii*) *if* $b \in S_T$ *then* $G[A + \{b, x_1, x_2\}]$ *has independent paths from* b *to* x_1, x_2, *respectively.*

Proof. First, we may assume $A - a \neq \emptyset$; for, otherwise, G contains K_4^- by Lemma 4.1. Next, $N_G(b) \cap V(A - a) \neq \emptyset$; otherwise, $(T, (S_T \cup S_{T'}) - V(B \cup C) - \{b\}, A - a, G[B \cup C + b]) \in \mathcal{Q}_x$ contradicts the choice of (T, S_T, A, B) that $|V(A)|$ is minimum.

To complete the proof of (*i*), let $t \in N_G(b) \cap V(A - a)$. If $G[(A - a) + \{x_1, x_2, x_3, x_4\}] - b$ has four independent paths from t to x_1, x_2, x_3, x_4, respectively, then these four paths and tb give the desired five paths. So we may assume that such four paths do not exist. Then $G[(A - a) + \{x_1, x_2, x_3, x_4\}] - b$ has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 3$, $t \in V(G_1 - G_2)$ and $\{x_1, x_2, x_3, x_4\} \subseteq V(G_2)$. Hence, $(T', V(T') \cup V(G_1 \cap G_2), G_1 - G_2, G - T' - G_1) \in \mathcal{Q}_x$ and $1 \leq |V(G_1 - G_2)| \leq |V(A - a)| < |V(A)|$, contradicting the choice of (T, S_T, A, B).

To prove (ii), let $b \in S_T$ and assume that the two paths for (ii) do not exist. Note that if $b \in V(T)$ then $T \cup T'$ contains K_4^-. So we may assume $b \notin V(T)$. Then, $G[A+\{b, x_1, x_2\}]$ has a separation (G_1, G_2) such that $|V(G_1) \cap V(G_2)| \leq 1$, $b \in V(G_1) - V(G_2)$ and $\{x_1, x_2\} \subseteq V(G_2)$. Since $N_G(b) \cap V(A-a) \neq \emptyset$ and $|V(G_1) \cap V(G_2)| \leq 1$, $|V(G_1 - G_2)| \geq 2$. Let $S_{bx} = (S_T - \{x_1, x_2\}) \cup V(G_1 \cap G_2)$ (which is a cut in G), and let $F = G_1 - S_{bx}$. Then $|V(F)| \geq 1$ as $|V(G_1 - G_2)| \geq 2$. We may assume x_1 and x_2 have no common neighbor other than x, as otherwise G contains K_4^-. So $|V(G_2 \cap A)| \geq 2$ as x_1, x_2 each have a neighbor in A. Thus, $|V(F)| < |V(A)|$. If $|V(F)| \geq 2$ then $(bx, S_{bx}, F, G - S_{bx} - F) \in \mathcal{Q}_x$ with $2 \leq |V(F)| < |V(A)|$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. So assume $|V(F)| = 1$ and let $v \in V(F)$. Since G is 5-connected, v is adjacent to all vertices in S_{bx}. If $v \neq a$ then $V(G_1 \cap G_2) = \{a\}$; so $G[\{a, b, v, x\}]$ contains K_4^-. Now assume $v = a$. Let $w \in V(G_1 \cap G_2)$. Since $N_G(b) \cap V(A-a) \neq \emptyset$, we have $bw \in E(G)$. So $G[\{a, b, w, x\}]$ contains K_4^-. □

In the next two lemmas, we consider the case when quadruples (T, S_T, A, B) and $(T', S_{T'}, C, D)$ may be chosen so that $|V(T' \cap A)| = 2$.

Lemma 5.3. *Let G be a 5-connected nonplanar graph and $x \in V(G)$. Suppose for any $H \subseteq G$ with $x \in V(H)$ and with $H \cong K_2$ or $H \cong K_3$, G/H is not 5-connected. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum. Suppose there exists $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ such that $T' \cong K_3$ and $|V(T' \cap A)| = 2$. Then one of the following holds:*

- (i) *G contains a TK_5 in which x is not a branch vertex.*
- (ii) *G contains K_4^-.*
- (iii) *There exist $x_1, x_2, x_3 \in N_G(x)$ such that for any $y_1, y_2 \in N_G(x) - \{x_1, x_2, x_3\}$, $G - \{xv : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}$ contains TK_5.*
- (iv) *$|S_T \cap S_{T'}| = 1$, $|S_{T'} \cap V(B)| = 2$, and either $|S_T \cap V(C)| = 2$ and $T \cap C = \emptyset$ or $|S_T \cap V(D)| = 2$ and $T \cap D = \emptyset$.*

Proof. We may assume $T \cong K_3$ (by Lemma 4.3). We may also assume that $|S_T| = |S_{T'}| = 6$; for, otherwise, (i) or (ii) or (iii) follows from Lemma 2.6. We may further assume $|V(A)| \geq 5$; as otherwise, by Lemma 4.1, G contains K_4^- and (ii) holds.

Let $T' = \{a, b, x\}$ with $a, b \in V(A)$. By symmetry, assume $T \cap C = \emptyset$. Then, by Lemma 5.1, we may assume $A \cap C = \emptyset$. Now $B \cap C \neq \emptyset$; for, otherwise, $|V(C)| = |S_T \cap V(C)| \leq 3 < |V(A)|$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Hence, $S_T \cap V(C) \neq \emptyset$ as $S_{T'} - \{a, b\}$ is not a cut in G. Moreover, $A \cap D \neq \emptyset$. For, otherwise, $|V(A) \cap S_{T'}| = 5$ and, hence, $|S_{T'} \cap S_T| = 1$ and $|S_{T'} \cap V(B)| = 0$; so $(S_T \cup S_{T'}) - V(A \cup D)$ is a cut in G of size at most 4 and separating $B \cap C$ from $A \cup D$, a contradiction.

We claim that $|(S_T \cup S_{T'}) - V(B \cup C)| = 7$ and $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$. First, note that $|(S_T \cup S_{T'}) - V(B \cup C)| \geq 7$; otherwise, $(T', (S_T \cup S_{T'}) - V(B \cup C), A \cap D, G[B \cup C]) \in \mathcal{Q}_x$ and $1 \leq |V(A \cap D)| \leq |V(A-a)| < |V(A)|$, contradicting the choice of (T, S_T, A, B) that $|V(A)$ is minimum. Also note that $|(S_T \cup S_{T'}) - V(A \cup D)| \geq 5$, since $(S_T \cup S_{T'}) - V(A \cup D)$ is a cut in G (as $B \cap C \neq \emptyset$) and G is 5-connected. Thus the claim follows from the fact that $|(S_T \cup S_{T'}) - V(B \cup C)| + |(S_T \cup S_{T'}) - V(A \cup D)| = |S_T| + |S_{T'}| = 12$.

We may assume that $|S_T \cap V(C)| \neq 1$ or $|S_{T'} \cap V(A)| \neq 2$. For, otherwise, let $S_T \cap V(C) = \{c\}$ and $S_{T'} \cap V(A) = \{a, b\}$. If $a, b \in N_G(c)$ then $G[T' + c]$ contains K_4^- and (ii) holds. So by the symmetry between a and b, we may assume that $ca \notin E(G)$. Then $(T, (S_T - c) \cup \{b\}, A - b, G[B + c]) \in \mathcal{Q}_x$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum.

We may also assume $T \cap D \neq \emptyset$; for, otherwise, since $A \cap D \neq \emptyset$, (i) or (ii) or (iii) follows from Lemma 5.1. Therefore, $S_T \cap V(D) \neq \emptyset$. Note that $1 \leq |S_T \cap S_{T'}| \leq 4$, and we distinguish four cases according to $|S_T \cap S_{T'}|$.

Suppose $|S_T \cap S_{T'}| = 4$. Then $S_{T'} \cap V(B) = \emptyset$ and $|S_T \cap V(C)| = |S_T \cap V(D)| = 1$. Therefore, by the minimality of $|V(A)|$, $B \cap D \neq \emptyset$. Hence, $S_T - V(C)$ is a 5-cut in G and $V(T) \subseteq S_T - V(C)$. By the choice of (T, S_T, A, B) that $|V(A)|$ is minimum, $|V(B \cap D)| \geq 5$. Now (i) or (ii) or (iii) follows from Lemma 2.6.

Consider $|S_T \cap S_{T'}| = 3$. Suppose for the moment $S_{T'} \cap V(B) = \emptyset$. Then $|S_T \cap V(C)| = 2$ as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$. So $B \cap D = \emptyset$ as otherwise $S_T - V(C)$ would be a 4-cut in G. However, this implies $|V(D)| < |V(A)|$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. So $S_{T'} \cap V(B) \neq \emptyset$. Therefore, since $|S_{T'}| = 6$, we have $|S_{T'} \cap V(B)| = 1$ and $S_{T'} \cap V(A) = \{a, b\}$. Since $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$, $|S_T \cap V(C)| = 1$. This is a contradiction, as we have $|S_T \cap V(C)| \neq 1$ or $|S_{T'} \cap V(A)| \neq 2$.

Now let $|S_T \cap S_{T'}| = 2$. First, assume $|S_T \cap V(C)| = 1$. Then $|S_{T'} \cap V(B)| = 2$ (as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$) and, hence, $|S_{T'} \cap V(A)| = 2$ (as $|S_{T'}| = 6$), a contradiction. So we may assume that $|S_T \cap V(C)| \geq 2$, which implies $|S_{T'} \cap V(B)| \leq 1$ as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$. Hence, since $|S_T| = |S_{T'}| = 6$, $|S_{T'} \cap V(A)| \geq 3$ and $|S_T \cap V(D)| \leq 2$. Therefore, by the minimality of $|V(A)|$, $B \cap D \neq \emptyset$. Thus $(S_T \cap S_{T'}) - V(A \cup C)$ is a 5-cut in G and contains $V(T)$. So $|V(B \cap D)| \geq 5$ by the minimality of $|V(A)|$. Now (i) or (ii) or (iii) follows from Lemma 2.6.

Finally, assume $|S_T \cap S_{T'}| = 1$. If $|S_{T'} \cap V(B)| = 2$ then $|S_T \cap V(C)| = 2$ (as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$); so (iv) holds. If $|S_{T'} \cap V(B)| = 3$ then $|S_T \cap V(C)| = 1$ (since $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$) and $S_{T'} \cap V(A) = \{a, b\}$ (as $|S_{T'}| = 6$), a contradiction. Hence, we may assume $|S_{T'} \cap V(B)| \leq 1$. Then $|S_T \cap V(C)| \geq 3$ (since $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$), $|S_{T'} \cap V(A)| \geq 4$, and $|(S_T \cup S_{T'}) - V(A \cup C)| \leq 4$. Hence, since G is 5-connected, $B \cap D = \emptyset$; so $|V(D)| < |V(A)|$. However, this shows that $(T', S_{T'}, D, C)$ contradicts the choice of (T, S_T, A, B). $\square$

Next, we take care of the case when (iv) of Lemma 5.3 holds.

Lemma 5.4. *Let G be a 5-connected nonplanar graph and $x \in V(G)$, and suppose for any $H \subseteq G$ with $x \in V(H)$ and with $H \cong K_2$ or $H \cong K_3$, G/H is not 5-connected. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum, and $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$.*

Suppose $T \cap C = \emptyset$, $S_T \cap S_{T'} = \{x\}$, and $|S_T \cap V(C)| = |S_{T'} \cap V(B)| = 2$. Then one of the following holds:

(i) G contains a TK_5 in which x is not a branch vertex.
(ii) G contains K_4^-.
(iii) There exist $x_1, x_2, x_3 \in N_G(x)$ such that, for any $y_1, y_2 \in N_G(x) - \{x_1, x_2, x_3\}$, $G' := G - \{xv : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}$ contains TK_5.

Proof. We may assume $T \cong K_3$ (by Lemma 4.3) and $T' \cong K_3$ (by Lemma 4.4). By Lemma 4.1, we may assume $|V(A)| \geq 5$. We may further assume that $|S_T| = |S_{T'}| = 6$; for, otherwise, the assertion follows from Lemma 2.6.

Let $V(T) = \{x, x_1, x_2\}$, $V(T') = \{x, a, b\}$, $S_T \cap V(C) = \{p_1, p_2\}$, $S_{T'} \cap V(A) = \{a, b, q\}$, and $S_T \cap V(D) = \{x_1, x_2, w\}$. Since $T \cap C = \emptyset$, we may assume by Lemma 5.1 that $A \cap C = \emptyset$. Then $B \cap C \neq \emptyset$ as $|V(C)| \geq |V(A)| \geq 5$.

We may assume $N_G(p_1) \cap V(A) = \{a, q\}$ and $N_G(p_2) \cap V(A) = \{b, q\}$. To see this, for $i \in [2]$, let $S_i := (S_T - \{p_i\}) \cup (N_G(p_i) \cap \{a, b, q\})$ which is a cut in G and containing $V(T)$. If $N_G(p_i) \cap \{a, b, q\} = \emptyset$ then $|S_i| = 5$ and the assertion of this lemma follows from Lemma 2.6. If $|N_G(p_i) \cap \{a, b, q\}| = 1$ then $(T, S_i, A - (N_G(p_i) \cap \{a, b, q\}), S_i, G[B + p_i]) \in \mathcal{Q}_x$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Hence, we may assume that $|N_G(p_i) \cap \{a, b, q\}| \geq 2$ for $i \in [2]$. We may also assume $\{a, b\} \not\subseteq N_G(p_i)$ for $i \in [2]$; as, otherwise, $G[T' + p_i]$ contains K_4^- and (ii) holds. Moreover, $N_G(p_1) \cap \{a, b, q\} \neq N_G(p_2) \cap \{a, b, q\}$, as otherwise, $S := (S_T - \{p_1, p_2\}) \cup (N_G(p_1) \cap \{a, b, q\})$ is a cut in G containing $V(T)$; so $(T, S, A - (N_G(p_1) \cap \{a, b, q\}), G[B + \{p_1, p_2\}]) \in \mathcal{Q}_x$, contradicting the choice of (T, S_T, A, B) with $|V(A)|$ minimum. Hence, we may assume by symmetry that $N_G(p_1) \cap V(A) = \{a, q\}$ and $N_G(p_2) \cap V(A) = \{b, q\}$.

Note that $N_G(x_i) \cap V(B) \neq \emptyset$ for $i \in [2]$; for, otherwise, $S := V(T') \cup \{q, x_{3-i}, w\}$ is a cut in G, and $(T', S, G[(A \cap D) + x_i], G[B + \{p_1, p_2\}]) \in \mathcal{Q}_x$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Moreover, we may assume $N_G(w) \cap V(B) \neq \emptyset$; as otherwise, $S_T - \{w\}$ is a 5-cut in G and $V(T) \subseteq S_T - \{w\}$, and the assertion of this lemma follows from Lemma 2.6.

We wish to prove (iii) with $x_3 = b$. Let $y_1, y_2 \in N_G(x) - \{x_1, x_2, x_3\}$ be distinct. Choose $v \in \{y_1, y_2\} - \{a\}$. We may assume $v \notin \{p_1, p_2\}$, as otherwise $G[T' + v]$ contains K_4^- and (ii) holds. By Lemma 5.2, we may choose $t \in N_G(b) \cap V(A - a)$ such that $G[(A - a) + \{b, q, x_1, x_2, w\}]$ has independent paths P_1, P_2, P_3, P_4, P_5 from t to b, x_1, x_2, w, q respectively. We distinguish four cases according to the location of v.

Case 1. $v \in V(B)$.

Let W be the component of B containing v. First, suppose $N_G(x_i) \cap W \neq \emptyset$ for $i \in [2]$. Then there exists $v^* \in V(W)$ such that $G[W + \{x_1, x_2\}]$ has three independent paths from v^* to v, x_1, x_2, respectively. Hence by Lemma 2.11, $G[W + (S_T - \{x\})]$ (which is $(4, S_T - \{x\})$-connected) has independent paths Q_1, Q_2, Q_3, Q_4 from v^* to v, x_1, x_2, u, respectively, and internally disjoint from S_T, where $u \in S_T - \{x, x_1, x_2\}$. If $u = w$ then

$T \cup (P_1 \cup bx) \cup P_2 \cup P_3 \cup (Q_1 \cup vx) \cup Q_2 \cup Q_3 \cup (Q_4 \cup P_4)$ is a TK_5 in G' with branch vertices t, v^*, x, x_1, x_2. If $u = p_i$ for some $i \in [2]$ then $T \cup (P_1 \cup bx) \cup P_2 \cup P_3 \cup (Q_1 \cup vx) \cup Q_2 \cup Q_3 \cup (Q_4 \cup p_i q \cup P_5)$ is a TK_5 in G' with branch vertices t, v^*, x, x_1, x_2.

Thus, we may assume that $N_G(x_1) \cap W = \emptyset$. Since G is 5-connected, $G[W + (S_T - \{x_1\})]$ is $(5, S_T - \{x_1\})$-connected; so it has independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from v to x, x_2, w, p_1, p_2, respectively. Clearly, we may assume that $Q_1 = vx$. Since $N_G(x_1) \cap V(B) \neq \emptyset$, let W' be a component of B with $N_G(x_1) \cap V(W') \neq \emptyset$. Since G is 5-connected, there exists $i \in [2]$ such that $N_G(p_i) \cap V(W') \neq \emptyset$. Hence, $G[W' + \{x_1, p_i\}]$ has a path R from x_1 to p_i, and, by symmetry, assume R is from x_1 to p_1. Now $T \cup (P_1 \cup bx) \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup (Q_3 \cup P_4) \cup (Q_4 \cup R)$ is a TK_5 in G' with branch vertices t, v, x, x_1, x_2.

Case 2. $v \in V(A \cap D)$.

First, we show that $G[(A \cap D) + \{q, w, x, x_1, x_2\}]$ has independent paths $P_1', P_2', P_3', P_4', P_5'$ from v to q, x, x_1, x_2, w, respectively (and we may assume that $P_2' = vx$). This is clear if $G[(A \cap D) + \{q, w, x_1, x_2\}]$ has independent paths from v to q, x_1, x_2, w, respectively. So we may assume that $G[(A \cap D) + \{q, w, x_1, x_2\}]$ has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 3$, $v \in V(G_1 - G_2)$, and $\{q, w, x_1, x_2\} \subseteq V(G_2)$. Then $S := V(T') \cup V(G_1 \cap G_2)$ is a cut in G, and $(T', S, G_1 - G_2, G - S - G_1) \in \mathcal{Q}_x$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum.

Suppose B has a component W such that $N_G(x_i) \cap W \neq \emptyset$ for $i \in [2]$. Then there exists $z \in V(W)$ such that $G[W + \{x_1, x_2\}]$ has independent paths from z to x_1, x_2, respectively. Hence by Lemma 2.11, $G[W + (S_T - \{x\})]$ has four independent paths Q_1, Q_2, Q_3, Q_4 from z to x_1, x_2, u_1, u_2, respectively, and internally disjoint from S_T, where $u_1, u_2 \in \{w, p_1, p_2\}$ are distinct. If $\{u_1, u_2\} = \{w, p_1\}$ then we may assume $u_1 = w$ and $u_2 = p_1$; now $T \cup P_2' \cup P_3' \cup P_4' \cup Q_1 \cup Q_2 \cup (Q_3 \cup P_5') \cup (Q_4 \cup p_1 abx)$ is a TK_5 in G' with branch vertices v, x, x_1, x_2, z. If $\{u_1, u_2\} = \{w, p_2\}$ then we may assume $u_1 = w$ and $u_2 = p_2$; now $T \cup P_2' \cup P_3' \cup P_4' \cup Q_1 \cup Q_2 \cup (Q_3 \cup P_5') \cup (Q_4 \cup p_2 bx)$ is a TK_5 in G' with branch vertices v, x, x_1, x_2, z. So assume $\{u_1, u_2\} = \{p_1, p_2\}$. We may further assume $u_i = p_i$ for $i \in [2]$. Then $T \cup P_2' \cup P_3' \cup P_4' \cup Q_1 \cup Q_2 \cup (Q_3 \cup p_1 q \cup P_1') \cup (Q_4 \cup p_2 bx)$ is a TK_5 in G' with branch vertices v, x, x_1, x_2, z.

Hence, we may assume that no component of B contains neighbors of both x_1 and x_2. Since G is 5-connected, we may assume by symmetry that Z is a component of B such that $N_G(x_1) \cap V(Z) = \emptyset$ and $N_G(x_2) \cap V(Z) \neq \emptyset$. Again, since G is 5-connected, $G[Z + (S_T - \{x_1\})]$ has five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from some $z \in V(Z)$ to x_2, w, p_1, p_2, x, respectively. Since $N_G(x_1) \cap V(B) \neq \emptyset$, let Z' be a component of B with $N_G(x_1) \cap Z' \neq \emptyset$. Then $N_G(x_2) \cap V(Z') = \emptyset$. So $G[Z' + \{x_1, p_1\}]$ contains a path R from x_1 to p_1. Now $T \cup P_2' \cup P_3' \cup P_4' \cup (Q_4 \cup p_2 bx) \cup Q_1 \cup (Q_3 \cup R) \cup (Q_2 \cup P_5')$ is a TK_5 in G' with branch vertices v, x, x_1, x_2, z.

Case 3. $v = q$.

Suppose B has a component Z such that $\{w, x_1, x_2\} \subseteq N_G(Z)$. Then there exists $z \in V(Z)$ such that $G[Z + \{w, x_1, x_2\}]$ has independent paths from z to w, x_1, x_2, respectively. By Lemma 2.11, $G[Z + (S_T - \{x\})]$ has independent paths Q_1, Q_2, Q_3, Q_4 from z to

x_1, x_2, w, u, respectively, and internally disjoint from S_T, where $u \in \{p_1, p_2\}$. Let $S = Q_4 \cup p_1abx$ if $u = p_1$, and let $S = Q_4 \cup p_2bx$ if $u = p_2$. Then $T \cup Q_1 \cup Q_2 \cup S \cup (P_4 \cup Q_3) \cup P_2 \cup P_3 \cup (P_5 \cup qx)$ is a TK_5 in G' with branch vertices t, x, x_1, x_2, z.

So we may assume that no component of B is adjacent to all of x_1, x_2 and w. Since $N_G(w) \cap V(B) \neq \emptyset$, there exists a component Z of B such that $N_G(w) \cap V(Z) \neq \emptyset$. Since G is 5-connected, we may assume by symmetry that $N_G(x_2) \cap V(Z) \neq \emptyset$. Then $N_G(x_1) \cap V(Z) = \emptyset$. Since G is 5-connected, $G[Z + (S_T - \{x_1\})]$ has independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from some $z \in V(Z)$ to x_2, w, p_1, p_2, x, respectively. Since $N_G(x_1) \cap V(B) \neq \emptyset$, there exists some component Z' of B with $N_G(x_1) \cap V(Z') \neq \emptyset$. Hence, $N_G(x_2) \cap V(Z') = \emptyset$ or $N_G(w) \cap V(Z') = \emptyset$; so $G[Z' + \{x_1, p_1\}]$ contains a path R from x_1 to p_1. Now $T \cup Q_1 \cup (Q_3 \cup R) \cup (Q_4 \cup p_2bx) \cup (P_4 \cup Q_2) \cup P_2 \cup P_3 \cup (P_5 \cup qx)$ is a TK_5 in G' with branch vertices t, x, x_1, x_2, z.

Case 4. $v = w$.

Suppose B has a component Z such that $\{w, x_1, x_2\} \subseteq N_G(Z)$. Then there exists $z \in V(Z)$ such that $G[Z + \{w, x_1, x_2\}]$ has three independent paths from z to w, x_1, x_2, respectively. Hence, by Lemma 2.11, $G[Z + (S_T - \{x\})]$ has independent paths Q_1, Q_2, Q_3, Q_4 from z to x_1, x_2, w, u, respectively, and internally disjoint from S_T, where $u = p_i$ for some $i \in [2]$. Then $T \cup Q_1 \cup Q_2 \cup (Q_3 \cup wx) \cup (P_1 \cup bx) \cup P_2 \cup P_3 \cup (P_5 \cup qp_i \cup Q_4)$ is a TK_5 in G' with branch vertices t, x, x_1, x_2, z.

Hence, we may assume that no component of B is adjacent to all of w, x_1, x_2. Since $N_G(w) \cap V(B) \neq \emptyset$, B has a component Z such that $N_G(w) \cap V(Z) \neq \emptyset$. Since G is 5-connected, we may assume by symmetry that $N_G(x_2) \cap V(Z) \neq \emptyset$. Then $N_G(x_1) \cap V(Z) = \emptyset$. Since G is 5-connected, $G[Z + (S_T - \{x_1\})]$ has five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from z to x_2, w, p_1, p_2, x, respectively. Since $N_G(x_1) \cap V(B) \neq \emptyset$, B has a component Z' such that $N_G(x_1) \cap V(Z') \neq \emptyset$. Then $N_G(x_2) \cap V(Z') = \emptyset$ or $N_G(w) \cap V(Z') = \emptyset$; so $G[Z' + \{x_1, p_1\}]$ contains a path R from x_1 to p_1. Now $T \cup Q_1 \cup (Q_2 \cup wx) \cup (Q_3 \cup R) \cup (P_1 \cup bx) \cup P_2 \cup P_3 \cup (P_5 \cup qp_2 \cup Q_4)$ is a TK_5 in G' with branch vertices t, x, x_1, x_2, z. $\square$

We end this section with the following lemma which deals with another special case when $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum, $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$, and $A \cap C = \emptyset$.

Lemma 5.5. *Let G be a 5-connected nonplanar graph and $x \in V(G)$ such that for any $H \subseteq G$ with $x \in V(H)$ and with $H \cong K_2$ or $H \cong K_3$, G/H is not 5-connected. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum, and $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$. Suppose $A \cap C = \emptyset$, $|S_T| = 6$, $|S_{T'}| = 6$, $V(T') \cap S_T = \{x, b\}$, $V(T' \cap A) = S_{T'} \cap V(A) = \{a\}$ and $V(C) \cap S_T = \emptyset$. Then, one of the following holds:*

(*i*) *G contains a TK_5 in which x is not a branch vertex.*
(*ii*) *G contains K_4^-.*

(iii) There exist distinct $x_1, x_2 \in N_G(x)$ such that for any distinct $y_1, y_2 \in N_G(x) - \{b, x_1, x_2\}$, $G' := G - \{xv : v \notin \{x_1, x_2, b, y_1, y_2\}\}$ contains TK_5.

Proof. By assumption, $V(T') = \{a, b, x\}$ with $a \in V(A)$ and $b, x \in S_T \cap S_{T'}$. Let $V(T) = \{x, x_1, x_2\}$ and $S_T = \{b, x, x_1, x_2, x_3, x_4\}$. We wish to prove (iii); so let $y_1, y_2 \in N_G(x) - \{b, x_1, x_2\}$ be distinct. Let $v \in \{y_1, y_2\} - \{a\}$.

We may assume by Lemma 4.1 that $B \cap C \neq \emptyset$ as $S_{T'}$ is a cut in G and $S_T \cap V(C) = \emptyset$. So $|(S_T \cup S_{T'}) - V(A \cup D)| \geq 5$ (as $(S_T \cup S_{T'}) - V(A \cup D)$ is a cut in G). Moreover, we may assume $A \cap D \neq \emptyset$ by Lemma 4.1. So $|(S_T \cup S_{T'}) - V(B \cup C)| \geq 7$; for otherwise $(T, (S_T \cup S_{T'}) - V(B \cup C), A \cap D, G[B \cup C])$ contradicts the choice of (T, S_T, A, B) that $|V(A)|$ is minimum. Since $|S_T| = |S_{T'}| = 6$, we have

$$|(S_T \cup S_{T'}) - V(A \cup D)| = 5 \text{ and } |(S_T \cup S_{T'}) - V(B \cup C)| = 7.$$

We may assume that $N_G(x_i) \cap V(B) \neq \emptyset$ for $i \in [2]$. For, suppose this is not true and by symmetry assume $N_G(x_1) \cap V(B) = \emptyset$. Let $S = (S_T - \{x_1\}) \cup \{a\}$, $C' = B$, and $D' = G[(A - a) + x_1]$. Then $(T', S, C', D') \in \mathcal{Q}_x$. We now apply Lemma 4.6 to (T, S_T, A, B) and (T', S, C', D'). Note that $|S \cap S_T| = 5$, $V(A \cap C') = S_T \cap V(C') = S \cap V(B) = V(B \cap D') = \emptyset$, and $|S \cap V(A)| = |S_T \cap V(D')| = |V(T \cap D')| = 1$. To verify the other condition in Lemma 4.6, let $(H, S_H, C_H, D_H) \in \mathcal{Q}_x$. By Lemma 4.4, we may assume that $H \cong K_3$ when $H \cap A \neq \emptyset$. By Lemmas 5.3 and 5.4, we may assume that $|V(H \cap A)| \leq 1$. Therefore, the assertion of this lemma follows from Lemma 4.6. Hence, we may assume $N_G(x_i) \cap B \neq \emptyset$ for $i \in [2]$.

We may also assume that for any component W of B, $N_G(b) \cap W \neq \emptyset$; for, otherwise, $S_T - \{b\}$ is a 5-cut in G, and the assertion of this lemma follows from Lemma 2.6. We consider three cases according to the location of v.

Case 1. $v \in V(B)$.

Let B_v be the component of B containing v. First, suppose $N_G(x_i) \cap V(B_v) \neq \emptyset$ for $i \in [2]$. Then $G[B_v + \{x_1, x_2\}]$ has independent paths from some $v^* \in V(B_v)$ to v, x_1, x_2, respectively. Thus, by Lemma 2.11, $G[B_v + (S_T - \{x\})]$ has independent paths P_1, P_2, P_3, P_4 from v^* to v, x_1, x_2, u, respectively, and internally disjoint from S_T, where $u \in \{b, x_3, x_4\}$. Suppose $u = b$. By Lemma 5.2, we may assume that $G[A + \{b, x_1, x_2\}]$ contains independent paths R_1, R_2 from b to x_1, x_2, respectively. Then $T \cup R_1 \cup R_2 \cup bx \cup (P_1 \cup vx) \cup P_2 \cup P_3 \cup P_4$ is a TK_5 in G' with branch vertices b, v^*, x, x_1, x_2. So we may assume by symmetry that $u = x_3$. By Lemma 5.2 again, we may choose $t \in N_G(b) \cap V(A - a)$ and let Q_1, Q_2, Q_3, Q_4, Q_5 be independent paths in $G[(A - a) + \{b, x_1, x_2, x_3, x_4\}]$ from t to b, x_1, x_2, x_3, x_4, respectively. Then, $T \cup (Q_1 \cup bx) \cup Q_2 \cup Q_3 \cup (P_1 \cup vx) \cup P_2 \cup P_3 \cup (P_4 \cup Q_4)$ is a TK_5 in G' with branch vertices t, v^*, x, x_1, x_2.

Therefore, we may assume by symmetry that $N_G(x_1) \cap V(B_v) = \emptyset$. Since G is 5-connected, $G[B_v + (S_T - \{x_1\})]$ has independent paths P_1, P_2, P_3, P_4, P_5 from v to x, b, x_2, x_3, x_4, respectively, and we may assume that $P_1 = vx$. Since $N_G(x_1) \cap V(B) \neq \emptyset$, B has a component B_{x_1} such that $N_G(x_1) \cap V(B_{x_1}) \neq \emptyset$. Again, since G is 5-connected,

$N_G(x_j) \cap V(B_{x_1}) \neq \emptyset$ for some $j \in \{3,4\}$, and we may assume $j = 3$. Then $G[B_{x_1} + \{x_1, x_3\}]$ contains a path Q from x_1 to x_3. Let $t \in N_G(b) \cap V(A - a)$. By Lemma 5.2, we may assume that $G[(A-a) + \{b, x_1, x_2, x_3, x_4\}]$ has independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from t to b, x_1, x_2, x_3, x_4, respectively. Then $T \cup (Q_1 \cup bx) \cup Q_2 \cup Q_3 \cup (P_5 \cup Q_5) \cup (P_4 \cup Q) \cup P_1 \cup P_3$ is a TK_5 in G' with branch vertices t, v, x, x_1, x_2.

Case 2. $v \in V(A \cap D)$.

We claim that $G[(A-a) + \{x, x_1, x_2, x_3, x_4\}]$ has independent paths P_1, P_2, P_3, P_4, P_5 from v to x, x_1, x_2, x_3, x_4, respectively (and we may assume $P_1 = vx$). This is clear if $G[(A-a) + \{x_1, x_2, x_3, x_4\}]$ has independent paths from v to x_1, x_2, x_3, x_4, respectively; so we may assume such paths do not exist. Then there exists a separation (G_1, G_2) in $G[(A-a) + \{x_1, x_2, x_3, x_4\}]$ such that $|V(G_1 \cap G_2)| \leq 3$, $v \in V(G_1 - G_2)$, and $\{x_1, x_2, x_3, x_4\} \subseteq V(G_2)$. Let $S := V(G_1 \cap G_2) \cup V(T')$, which is a cut in G of size at most 6. Since G is 5-connected, $|V(G_1 \cap G_2)| \geq 2$. Then, $(T', S, G_1 - G_2, (G-S) - G_1) \in \mathcal{Q}_x$ and $1 \leq |V(G_1 - G_2)| \leq |V(A-a)| < |V(A)|$, contradicting the choice of (T, S_T, A, B) that $|V(A)|$ is minimum.

Suppose that B has a component W such that $N_G(x_i) \cap V(W) \neq \emptyset$ for $i \in [2]$. Then there exists $w \in V(W)$ such that $G[W + b]$ has independent paths from w to x_1, x_2, b, respectively. By Lemma 2.11, $G[B + S_T]$ has independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from w to x_1, x_2, b, u_1, u_2, respectively, and internally disjoint from S_T, where $u_1, u_2 \in \{x, x_3, x_4\}$ are distinct. By symmetry, we may assume $u_1 = x_3$. Then $T \cup P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup (Q_3 \cup bx) \cup (Q_4 \cup P_4)$ is a TK_5 in G' with branch vertices v, w, x, x_1, x_2.

Hence, we may assume that no component of B is adjacent to both x_1 and x_2. Let W be a component of B such that $N_G(x_2) \cap V(W) \neq \emptyset$. Then $N_G(x_1) \cap V(W) = \emptyset$. Since G is 5-connected, $G[W + (S_T - \{x_1\})]$ has independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from some $w \in V(W)$ to b, x_2, x_3, x_4, x, respectively. Since $N_G(x_1) \cap V(B) \neq \emptyset$, B has a component B_x such that $N_G(x_1) \cap V(B_x) \neq \emptyset$. Then $N_G(x_2) \cap V(B_x) = \emptyset$. Again, since G is 5-connected, $G[B_x + \{x_1, x_3\}]$ contains a path R from x_1 to x_3. Now $T \cup P_1 \cup P_2 \cup P_3 \cup (Q_1 \cup bx) \cup Q_2 \cup (Q_3 \cup R) \cup (Q_4 \cup P_5)$ is a TK_5 in G' with branch vertices v, w, x, x_1, x_2.

Case 3. $v \in S_T$.

We may assume that $v = x_3$. By Lemma 5.2, we may assume $t \in N_G(b) \cap V(A-a)$ and $G[(A-a) + \{b, x_1, x_2, x_3, x_4\}]$ has independent paths P_1, P_2, P_3, P_4, P_5 from t to b, x_1, x_2, x_3, x_4, respectively, with $P_1 = tb$. Also by Lemma 5.2, we may assume that $G[A + \{b, x_1, x_2\}]$ has independent paths Q_1, Q_2 from b to x_1, x_2, respectively.

Suppose B has a component W such that $\{x_1, x_2\} \subseteq N_G(W)$. Then there exists $w \in V(W)$ such that $G[W + \{b, x_1, x_2\}]$ has independent paths from w to b, x_1, x_2, respectively. So by Lemma 2.11, $G[B + S_T]$ has independent paths R_1, R_2, R_3, R_4, R_5 from w to x_1, x_2, b, u_1, u_2, respectively, and internally disjoint from S_T, where $u_1, u_2 \in \{x, x_3, x_4\}$ are distinct. Assume by symmetry that $u_1 \in \{x_3, x_4\}$. If $u_1 = x_3$, then $T \cup bx \cup Q_1 \cup Q_2 \cup R_1 \cup R_2 \cup R_3 \cup (R_4 \cup x_3x)$ is a TK_5 in G' with branch vertices

b, w, x, x_1, x_2. If $u_1 = x_4$, then $T \cup (P_4 \cup x_3x) \cup P_2 \cup P_3 \cup R_1 \cup R_2 \cup (R_3 \cup bx) \cup (R_4 \cup P_5)$ is a TK_5 in G' with branch vertices t, w, x, x_1, x_2.

Thus, we may assume that no component of B is adjacent to both x_1 and x_2. Since G is 5-connected, we may assume by symmetry that W is a component of B such that $N_G(x_2) \cap V(W) \neq \emptyset$ and $N_G(x_1) \cap V(W) = \emptyset$. Let $w \in V(W)$. Since G is 5-connected, $G[W + (S_T - \{x_1\})]$ has independent paths R_1, R_2, R_3, R_4, R_5 from w to x, x_2, x_3, x_4, b, respectively. Since $N_G(x_1) \cap B \neq \emptyset$, B has a component B_x such that $N_G(x_1) \cap V(B_x) \neq \emptyset$. Then $N_G(x_2) \cap V(B_x) = \emptyset$. Since G is 5-connected, $G[B_x + \{x_1, x_4\}]$ contains a path R from x_1 to x_4. Now $T \cup bx \cup Q_1 \cup Q_2 \cup R_2 \cup (R_3 \cup x_3x) \cup R_5 \cup (R_4 \cup R)$ is a TK_5 in G' with branch vertices b, w, x, x_1, x_2. $\square$

6. Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1, using the lemmas we have proved so far. Let G be a 5-connected nonplanar graph. We proceed to find a TK_5 in G. By Lemma 2.1, we may assume that

(1) G contains no K_4^-.

Let M denote a maximal connected subgraph of G such that

$$H := G/M \text{ is 5-connected and nonplanar, and contains no } K_4^-.$$

Note that $|V(M)| = 1$ (i.e., $H = G$) is possible. Let x denote the vertex of H resulting from the contraction of M. Then, for any $T \subseteq H$ with $x \in V(T)$ and with $T \cong K_2$ or $T \cong K_3$, one of the following holds:

$$H/T \text{ contains } K_4^-, \text{ or } H/T \text{ is planar, or } H/T \text{ is not 5-connected.}$$

For convenience, we will use x_T to denote the vertex of H/T resulting from the contraction of T (for any such T). We may assume that

(2) for any $T \subseteq H$ with $x \in V(T)$ and with $T \cong K_2$ or $T \cong K_3$, if F is a TK_5 in H/T then x_T is a branch vertex of F.

For, suppose that F is a TK_5 in H/T in which x_T is not a branch vertex. If $x_T \notin V(F)$ then F is also TK_5 in G. So assume $x_T \in V(T)$. Let $u, v \in V(F)$ such that $x_Tu, x_Tv \in E(F)$. Since M is connected, $G[M + \{u, v\}]$ has a path P from u to v. Thus, $(F - x_T) \cup P$ is a TK_5 in G. So we may assume (2).

Suppose there exists $T \subseteq V(H)$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, such that H/T is 5-connected and planar. Then by Lemma 2.9, $H - T$ contains K_4^-, contradicting (1). So

(3) for any $T \subseteq H$ with $x \in V(T)$ and with $T \cong K_2$ or $T \cong K_3$, if H/T is 5-connected then H/T is nonplanar.

We now show that

(4) if $T \subseteq H$ with $x \in V(T)$ and with $T \cong K_2$ or $T \cong K_3$ and if there exist $x_1, x_2, x_3 \in N_{H/T}(x_T)$ such that $H/T - \{x_T v : v \notin \{u_1, u_2, x_1, x_2, x_3\}\}$ contains TK_5 for every choice of distinct $u_1, u_2 \in N_{H/T}(x_T) - \{x_1, x_2, x_3\}$, then G contains TK_5.

To prove (4), let $A = N_G(M \cup T) = N_{H/T}(x_T)$. Consider the subgraph $G[(M \cup T) + A]$. Since $M \cup T$ is connected, there is a vertex $v \in V(M \cup T)$ such that $G[(M \cup T) + \{x_1, x_2, x_3\}]$ has independent paths from v to x_1, x_2, x_3, respectively. Since G is 5-connected, $G[(M \cup T) + A]$ is $(5, A)$-connected; so it has five independent paths from v to A with only v in common and internally disjoint from A. Hence, by Lemma 2.11, there exist distinct $u_1, u_2 \in A - \{x_1, x_2, x_3\}$ such that $G[(M \cup T) + A]$ has five independent paths P_1, P_2, P_3, P_4, P_5 from v to x_1, x_2, x_3, u_1, u_2, respectively, and internally disjoint from A. Now suppose F is a TK_5 in $H/T - \{x_T v : v \notin \{x_1, x_2, x_3, u_1, u_2\}\}$. If $x_T \notin V(F)$ then F is also a TK_5 in G. So we may assume $x_T \in V(F)$. Since $F \subseteq H/T - \{x_T v : v \notin \{x_1, x_2, x_3, u_1, u_2\}\}$, each edge of F incident with x_T is one of $\{x_T v : v \in \{x_1, x_2, x_3, u_1, u_2\}\}$. Hence, $F - x_T$ and the (two or four) paths among P_1, P_2, P_3, P_4, P_5 corresponding to the edges of F at x_T form a TK_5 in G. So we may assume (4).

We have two cases by (3): For some $T \subseteq H$ with $x \in V(T)$ and with $T \cong K_2$ or $T \cong K_3$, H/T is 5-connected and nonplanar but, due to maximality of M, contains K_4^-; or for every $T \subseteq H$ with $x \in V(T)$ and with $T \cong K_2$ or $T \cong K_3$, H/T is not 5-connected.

Case 1. There exists $T \subseteq H$ with $x \in V(T)$ and with $T \cong K_2$ or $T \cong K_3$ such that H/T is 5-connected and nonplanar, and H/T contains K_4^-.

Let $K \subseteq H/T$ such that $K \cong K_4^-$, and let $V(K) = \{x_1, x_2, y_1, y_2\}$ with $y_1 y_2 \notin E(H)$. By (1), $x_T \in V(K)$.

Subcase 1.1. x_T has degree 2 in K.

Then we may assume that the notation is chosen so that $x_T = y_2$. By Lemma 2.2, one of the following holds:

(i) H/T contains a TK_5 in which x_T is not a branch vertex.
(ii) $H/T - x_T$ contains K_4^-.
(iii) H/T has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x_T, a_1, a_2, a_3, a_4\}$, and G_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4 a_1$ and the 4-cycle $b_1 b_2 b_3 b_4 b_1$ by adding x_T and the edges $x_T b_i$ for $i \in [4]$.
(iv) For any distinct $w_1, w_2, w_3 \in N_{H/T}(x_T) - \{x_1, x_2\}$, $H/T - \{x_T v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$ contains TK_5.

Note that (i) does not occur because of (2), and (ii) does not occur because of (1).

Now suppose (iii) occurs. First, assume $|V(G_1)| \geq 7$. Then by Lemma 2.3, for any distinct $u_1, u_2 \in N(x_T) - \{b_1, b_2, b_3\}$, $H/T - \{x_T v : v \notin \{b_1, b_2, b_3, u_1, u_2\}\}$ contains TK_5. Hence, by (4) (with x_i as b_i for $i \in [3]$), G contains TK_5. So we may assume that $|V(G_1)| = 6$, and let $v \in V(G_1 - G_2)$. By (1), $a_i a_{i+1} \notin E(G)$ for $i \in [4]$, where $a_5 = a_1$. Hence, since G is 5-connected, $a_1 a_3, a_2 a_4 \in E(G)$. Now $(H - x_T) - \{a_1 v, a_1 b_4, a_4 v, a_4 b_4\}$ is a TK_5 with branch vertices a_2, a_3, b_1, b_2, b_3, contradicting (2).

Finally, suppose (iv) holds. Then, by (4) (with w_1, w_2, w_3 as x_3, u_1, u_2, respectively), we see that G contains TK_5.

Subcase 1.2. x_T has degree 3 in K.

Then we may assume that the notation is chosen so that $x_T = x_1$. By Lemma 2.4, one of the following holds:

- (i) H/T contains a TK_5 in which x_T is not a branch vertex.
- (ii) $H/T - x_T$ contains K_4^-, or H/T contains a K_4^- in which x_T is of degree 2.
- (iii) x_2, y_1, y_2 may be chosen so that for any distinct $z_0, z_1 \in N_{H/T}(x_T) - \{x_2, y_1, y_2\}$, $H/T - \{x_T v : v \notin \{z_0, z_1, x_2, y_1, y_2\}\}$ contains TK_5.

By (2), (i) does not occur. If (ii) holds then, by (1), H/T contains K_4^- in which x_T is of degree 2; and we are back in Subcase 1.1. If (iii) holds then G contains TK_5 by (4).

Case 2. H/T is not 5-connected for every $T \subseteq H$ with $x \in V(T)$ and with $T \cong K_2$ or $T \cong K_3$.

Let $\mathcal{Q}_x$ denote the set of all quadruples (T, S_T, A, B), such that

- $T \subseteq V(H)$, $x \in V(T)$, and either $T \cong K_2$ or $T \cong K_3$,
- S_T is a cut in H with $V(T) \subseteq S_T$, A is a nonempty union of components of $H - S_T$, and $B = H - S_T - A \neq \emptyset$,
- if $T \cong K_3$ then $5 \leq |S_T| \leq 6$, and
- if $T \cong K_2$ then $|S_T| = 5$, $|V(A)| \geq 2$, and $|V(B)| \geq 2$.

We choose a quadruple (T, S_T, A, B) from $\mathcal{Q}_x$ such that $|V(A)|$ is minimum. By Lemma 4.3, $T \cong K_3$ (as $K_4^- \not\subseteq H$). By Lemma 4.5 and by (2) and (4), we may assume $N_G(x) \cap V(A) \neq \emptyset$. So let $a \in V(A)$ such that $ax \in E(H)$. Then by Lemma 4.2, there exists $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ such that $\{a, x\} \subseteq V(T')$, and either $T' \cong K_2$ or $T' \cong K_3$. Again since $K_4^- \not\subseteq H$, $T' \cong K_3$ by Lemma 4.4 and by (2) and (4).

Note that $T \cap C = \emptyset$ or $T \cap D = \emptyset$. We may assume, without loss of generality, that $T \cap C = \emptyset$. So $|V(C) \cap S_T| \leq 3$. Hence, by Lemma 5.1 and by (2) and (4), $A \cap C = \emptyset$ (since $K_4^- \not\subseteq H$). We may assume $B \cap C \neq \emptyset$; for otherwise, $|V(A)| \leq |V(C)| = |V(C) \cap S_T| \leq 3$ and, by Lemma 4.1, H contains K_4^-, a contradiction.

We may assume that $|V(T') \cap S_T| = 2$ for any choice of $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$; otherwise, by Lemmas 5.3 and 5.4, we derive a contradiction to (2), or (4), or the fact $K_4^- \not\subseteq H$. Hence, since $K_4^- \not\subseteq H$, we have $A \cap D \neq \emptyset$ by Lemma 4.1.

Note that $|S_T| = |S_{T'}| = 6$; for otherwise, by Lemma 2.6, we derive a contradiction to (2), or (4), or the fact $K_4^- \nsubseteq H$. We claim that

$$|(S_T \cup S_{T'}) - V(B \cup C)| = 7 \text{ and } |(S_T \cup S_{T'}) - V(A \cup D)| = 5.$$

First, note that $|(S_T \cup S_{T'}) - V(B \cup C)| \geq 7$; otherwise, $(T', (S_T \cup S_{T'}) - V(B \cup C), A \cap D, G[B \cup C]) \in \mathcal{Q}_x$ and $1 \leq |V(A \cap D)| < |V(A)|$, contradicting the choice of (T, S_T, A, B) with $|V(A)|$ minimum. Since H is 5-connected and $B \cap C \neq \emptyset$, $|(S_T \cup S_{T'}) - V(A \cup D)| \geq 5$. So the claim follows from the fact that $|(S_T \cup S_{T'}) - V(B \cup C)| + |(S_T \cup S_{T'}) - V(A \cup D) = |S_T| + |S_{T'}| = 12$.

If $S_T \cap V(C) = \emptyset$ for some choice $(T', S_{T'}, C, D)$ then $|S_{T'} \cap V(A)| = 1$ as $|S_{T'}| = 6$ and $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$; so by Lemma 5.5, we derive a contradiction to (2), or (4), or the fact $K_4^- \nsubseteq H$.

Hence, we may assume that

$$S_T \cap V(C) \neq \emptyset$$

for any choice of $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$. Then $2 \leq |S_T \cap S_{T'}| \leq 4$ as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$.

Suppose $|S_T \cap S_{T'}| = 4$. Then $|S_{T'} \cap V(B)| = 0$ and $|S_T \cap V(C)| = 1$, as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$. Since $|S_T| = |S_{T'}| = 6$, $|S_T \cap V(D)| = 1$ and $|S_{T'} \cap V(A)| = 2$. Hence, $B \cap D \neq \emptyset$ (since $|V(D)| \geq V(A)|$). So $S_T - V(C)$ is a 5-cut in H and $V(T) \subseteq S_T - V(C)$. Note $|V(B \cap D)| \geq 2$; for otherwise, since H is 5-connected, $H[T \cup (B \cap D)]$ contains K_4^-, a contradiction. Hence, by Lemma 2.6, we derive a contradiction to (2), or (4), or the fact $K_4^- \nsubseteq H$.

Now assume $|S_T \cap S_{T'}| = 3$. Then, $|S_{T'} \cap V(B)| \leq 1$ as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$ and $|S_T \cap V(C)| > 0$. Suppose $|S_{T'} \cap V(B)| = 0$. Then $|S_{T'} \cap V(A)| = 3$ as $|S_{T'}| = 6$. So $|S_T \cap V(D)| = 1$ since $|(S_T \cup S_{T'}) - V(B \cup C)| = 7$. Thus, since H is 5-connected, $B \cap D = \emptyset$. However, this implies that $|V(D)| < |V(A)|$, a contradiction. So $|S_{T'} \cap V(B)| = 1$. Then $|S_{T'} \cap V(A)| = 2$ as $|S_{T'}| = 6$, and $|S_T \cap V(C)| = 1$ as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$. Let $q \in S_{T'} \cap V(A - T')$, $S' := (S_{T'} - \{q\}) \cup (S_T \cap V(C))$, $C' := B \cap C$, and $D' = G[D + q]$. Then $(T', S', C', D') \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$ and $T \cap C' = \emptyset$. However, $S_T \cap V(C') = \emptyset$, a contradiction.

Finally, assume $|S_T \cap S_{T'}| = 2$. Suppose $|S_T \cap V(C)| \geq 2$. Then $|S_{T'} \cap V(B)| \leq 1$ (as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$), and $|S_{T'} \cap V(A)| \geq 3$ (as $|S_{T'}| = 6$). So $B \cap D \neq \emptyset$ as $|V(D)| \geq |V(A)|$. Hence, $(S_T \cup S_{T'}) - V(A \cup C)$ is a 5-cut in H and contains $V(T)$. If $|V(B \cap D)| = 1$ then, since H is 5-connected, $H[T \cup (B \cap D)]$ contains K_4^-, a contradiction. So $|V(B \cap D)| \geq 2$. Then, by Lemma 2.6, we derive a contradiction to (2), or (4), or the fact $K_4^- \nsubseteq H$. Therefore, we may assume $|S_T \cap V(C)| = 1$. Hence, $|S_T \cap V(D)| = 3$ (as $|S_T| = 6$), $|S_{T'} \cap V(B)| = 2$ (as $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$), and $|S_{T'} \cap V(A)| = 2$ (as $|S_{T'}| = 6$). Let $q \in S_{T'} \cap V(A - T')$, $S' := (S_{T'} - \{q\}) \cup (S_T \cap V(C))$, $C' := B \cap C$, and $D' = G[D + q]$. Then $(T', S', C', D') \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$ and $T \cap C' = \emptyset$. However, $S_T \cap V(C') = \emptyset$, a contradiction. $\square$

7. Concluding remarks

We have shown that every 5-connected nonplanar graph contains TK_5. Thus, if a graph contains no TK_5 then it is planar, or admits a cut of size at most 4. This is a step towards a more useful structural description of the class of graphs containing no TK_5. There is a nice result for graphs containing no $TK_{3,3}$ due to Wagner [37]: Every such graph is planar, or is a K_5, or admits a cut of size at most 2.

Mader [22] conjectured that every simple graph with minimum degree at least 5 and no K_4^- contains TK_5, and he also asked the following.

Question 7.1. Does every simple graph on $n \geq 4$ vertices with more than $12(n-2)/5$ edges contain K_4^-, $K_{2,3}$, or TK_5?

In a recent paper [13], it is shown that an affirmative answer to Question 7.1 implies the Kelmans-Seymour conjecture. As an independent approach to resolve the Kelmans-Seymour conjecture, Kawarabayashi, Ma, and Yu considered a contractible cycle in a 5-connected nonplanar graph containing no K_4^- or $K_{2,3}$, and then use such a cycle to find a TK_5 by applying augmenting path arguments. This plan (if successful), combined with the results in [21,13], would give an alternative solution to the Kelmans-Seymour conjecture.

One of the motivations for us to work on the Kelmans-Seymour conjecture was the following conjecture of Hajós (see e.g., [35]) which, if true, would generalize the Four Color Theorem.

Conjecture 7.2. *Graphs containing no TK_5 are 4-colorable.*

It is known that Conjecture 7.2 holds for graphs with large girth (see Kühn and Osthus [17]). Let G be a possible counterexample to Conjecture 7.2 with $|V(G)|$ minimum. Then our result on the Kelmans-Seymour conjecture implies that G has connectivity at most 4. By a standard coloring argument, it is easy to show that G must be 3-connected. It is shown in [42] that G must be 4-connected. It is further shown in [29] that for every 4-cut T of G, $G - T$ has exactly two components. The work in [42,29] suggests that G should be “close” to being 5-connected.

Hajós actually made a more general conjecture in the 1950s: For any positive integer k, every graph containing no TK_{k+1} is k-colorable. This is easy to verify for $k \leq 3$ (see [4]), and disproved in [2] for $k \geq 6$. However, it remains open for $k = 4$ (Conjecture 7.2) and $k = 5$. Thomassen [35] pointed out connections between Hajós' conjecture and Ramsey numbers, maximum cuts, and perfect graphs. We refer the reader to [35] for other work and references related to Hajós' conjecture and topological minors.

In fact, Erdős and Fajtlowicz [6] showed that the above general Hajós' conjecture for $k \geq 6$ fails for almost all graphs. Let $H(n) := \max\{\chi(G)/\sigma(G) : G \text{ is a graph with } |V(G)| = n\}$, where $\chi(G)$ denotes the chromatic number of G and

$\sigma(G)$ denotes the largest t such that G contains TK_t. Erdős and Fajtlowicz [6] showed that $H(n) = \Omega(\sqrt{n}/\log n)$, and conjectured that $H(n) = \Theta(\sqrt{n}/\log n)$. This conjecture was verified by Fox, Lee, and Sudakov [8], by studying $\sigma(G)$ in terms of independence number $\alpha(G)$. The following conjecture of Fox, Lee, and Sudakov [8] is interesting.

Conjecture 7.3. *There is a constant $c > 0$ such that every graph G with $\chi(G) = k$ satisfies $\sigma(G) \geq c\sqrt{k \log k}$.*

A key idea in [20,21,9–11] for finding TK_5 in graphs containing K_4^- is to find a nonseparating path in a graph that avoids two given vertices. Let G be a 5-connected nonplanar graph and $x_1, x_2, y_1, y_2 \in V(G)$ such that $\{x_1, x_2, y_1, y_2\}$ induces a K_4^- in which x_1, x_2 are of degree 3. We used an induced path X in G between x_1 and x_2 such that $G - X$ is 2-connected and $\{y_1, y_2\} \not\subseteq V(X)$, and in certain cases we need X to contain a special edge at x_1 (for example, in Section 6, $x_1 = x$ is the special vertex representing the contraction of M). If we could find such X that $G - X$ is 3-connected then our proofs would have been much simpler. This is related to the following conjecture of Lovász [19].

Conjecture 7.4. *There exists an integer valued function $f(k)$ such that for any $f(k)$-connected graph G and for any $A \subseteq V(G)$ with $|A| = 2$, there exist vertex disjoint subgraphs G_1, G_2 of G such that $V(G_1) \cup V(G_2) = V(G)$, G_1 is a path between the vertices in A, and G_2 is k-connected.*

A classical result of Tutte [36] implies $f(1) = 3$. That $f(2) = 5$ was proved by Kriesell [16] and, independently, by Chen, Gould and Yu [3]. Despite much effort from the research community, Conjecture 7.4 remains open for $k \geq 3$. Variations of Conjecture 7.4 for $k = 2$ are used in [20,21,9–11] to resolve the Kelmans-Seymour conjecture.

An edge version of Conjecture 7.4 was conjectured by Kriesell and proved by Kawarabayashi *et al.* [12]. Thomassen [32] conjectured a statement that is more general than Conjecture 7.4 by allowing $|A| \geq 2$ and requiring that $A \subseteq V(G_1)$ and G_1 be k-connected.

Acknowledgment

We thank the referees for carefully reading the manuscript and providing valuable comments that improved the readability of this paper. In particular, we are greatly indebted to the referee who checked every single detail and corrected a number of subtle omissions.

References

[1] E. Aigner-Horev, Subdivisions in apex graphs, Abh. Math. Semin. Univ. Hamb. 82 (1) (April 2012) 83–113.

[2] P.A. Catlin, Hajós' graph coloring conjecture: variations and counterexamples, J. Comb. Theory, Ser. B 26 (1979) 268–274.
[3] G. Chen, R. Gould, X. Yu, Graph connectivity after path removal, Combinatorica 23 (2003) 185–203.
[4] G.A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, J. Lond. Math. Soc., Ser. B 27 (1952) 85–92.
[5] G.A. Dirac, Homomorphism theorems for graphs, Math. Ann. 153 (1964) 69–80.
[6] P. Erdős, S. Fajtlowicz, On the conjecture of Hajós, Combinatorica 1 (1981) 141–143.
[7] P. Erdős, H. Hajnal, On complete topological subgraphs of certain graphs, Ann. Univ. Sci. Budapest, Sect. Math. 7 (1964) 143–149.
[8] J. Fox, C. Lee, B. Sudakov, Chromatic number, clique subdivisions, and the conjecture of Hajós, Combinatorica 33 (2003) 181–197.
[9] D. He, Y. Wang, X. Yu, The Kelmans-Seymour conjecture I, special separations, submitted for publication, arXiv:1511.05020.
[10] D. He, Y. Wang, X. Yu, The Kelmans-Seymour conjecture II, 2-vertices in K_4^-, submitted for publication, arXiv:1602.07557.
[11] D. He, Y. Wang, X. Yu, The Kelmans-Seymour conjecture III, 3-vertices in K_4^-, submitted for publication, arXiv:1609.05747.
[12] K. Kawarabayashi, O. Lee, P. Wollan, B. Reed, A weaker version of Lovász' path removal conjecture, J. Comb. Theory, Ser. B 98 (2008) 972–979.
[13] K. Kawarabayashi, J. Ma, X. Yu, K_5-subdivisions in graphs containing $K_{2,3}$, J. Comb. Theory, Ser. B 113 (2015) 18–67.
[14] A.K. Kelmans, Every minimal counterexample to the Dirac conjecture is 5-connected, Lectures to the Moscow Seminar on Discrete Mathematics (1979).
[15] A.E. Kézdy, P.J. McGuiness, Do $3n-5$ edges suffice for a subdivision of K_5?, J. Graph Theory 15 (1991) 389–406.
[16] M. Kriesell, Induced paths in 5-connected graphs, J. Graph Theory (2001) 52–58.
[17] D. Kühn, D. Osthus, Topological minors in graphs of large girth, J. Comb. Theory, Ser. B 86 (2002) 364–380.
[18] K. Kuratowski, Sur le probleme des courbes gauches en topologie, Fundam. Math. 15 (1930) 271–283 (in French).
[19] L. Lovász, Problems, in: M. Fiedler (Ed.), Recent Advances in Graph Theory, Academia, Prague, 1975.
[20] J. Ma, X. Yu, Independent paths and K_5-subdivisions, J. Comb. Theory, Ser. B 100 (2010) 600–616.
[21] J. Ma, X. Yu, K_5-subdivisions in graphs containing K_4^-, J. Comb. Theory, Ser. B 103 (2013) 713–732.
[22] W. Mader, $3n-5$ edges do force a subdivision of K_5, Combinatorica 18 (1998) 569–595.
[23] H. Perfect, Applications of Menger's graph theorem, J. Math. Anal. Appl. 22 (1968) 96–111.
[24] N. Robertson, K. Chakravarti, Covering three edges with a bond in a nonseparable graph, in: Deza, Rosenberg (Eds.), Ann. Discrete Math. (1979) 247.
[25] P.D. Seymour, Disjoint paths in graphs, Discrete Math. 29 (1980) 293–309.
[26] P.D. Seymour, private communication.
[27] Y. Shiloach, A polynomial solution to the undirected two paths problem, J. Assoc. Comput. Mach. 27 (1980) 445–456.
[28] Z. Skupień, On the locally Hamiltonian graphs and Kuratowski's theorem, Rocz. Pol. Tow. Mat., 3 Mat. Stosow. 11 (1968) 255–268.
[29] Y. Sun, X. Yu, On a coloring conjecture of Hajós, Graphs Comb. (2015), https://doi.org/10.1007/s00373-015-1539-0.
[30] C. Thomassen, Some homeomorphism properties of graphs, Math. Nachr. 64 (1974) 119–133.
[31] C. Thomassen, 2-Linked graphs, Eur. J. Comb. 1 (1980) 371–378.
[32] C. Thomassen, Graph decomposition with applications to subdivisions and path systems modulo k, J. Graph Theory 7 (1983) 261–271.
[33] C. Thomassen, K_5-subdivisions in graphs, Comb. Probab. Comput. 5 (1996) 179–189.
[34] C. Thomassen, Dirac's conjecture on K_5-subdivisions, Discrete Math. 165/166 (1997) 607–608.
[35] C. Thomassen, Some remarks on Hajós' conjecture, J. Comb. Theory, Ser. B 93 (2005) 95–105.
[36] W.T. Tutte, How to draw a graph, Proc. Lond. Math. Soc. 13 (1963) 743–768.
[37] K. Wagner, Über eine erweiterung eines satzes von Kuratowski, Dtsch. Math. 2 (1937) 280–285.
[38] M.E. Watkins, D.M. Mesner, Cycles and connectivity in graphs, Can. J. Math. 19 (1967) 1319–1328.
[39] X. Yu, Disjoint paths in graphs I, 3-planar graphs and basic obstructions, Ann. Comb. 7 (2003) 89–103.

[40] X. Yu, Disjoint paths in graphs II, a special case, Ann. Comb. 7 (2003) 105–126.
[41] X. Yu, Disjoint paths in graphs III, characterization, Ann. Comb. 7 (2003) 229–246.
[42] X. Yu, F. Zickfeld, Reducing Hajós' coloring conjecture, J. Comb. Theory, Ser. B 96 (2006) 482–492.