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The Kelmans-Seymour conjecture I: Special separations ☆

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A R T I C L E I N F O

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Seymour and, independently, Kelmans conjectured in the 1970s that every 5-connected nonplanar graph contains a subdivision of K_5. This conjecture was proved by Ma and Yu for graphs containing K_4^-, and an important step in their proof is to deal with a 5-separation in the graph with a planar side. In order to establish the Kelmans-Seymour conjecture for all graphs, we need to consider 5-separations and 6-separations with less restrictive structures. The goal of this paper is to deal with special 5-separations and 6-separations, including those with an apex side. Results will be used in subsequent papers to prove the Kelmans-Seymour conjecture.

☆ This work started when DH was a student at ECNU and XY was visiting ECNU.

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1. Introduction

Let K be a graph; we use TK to denote a subdivision of K and call the vertices of the TK corresponding to the vertices of K its *branch* vertices. Kuratowski's theorem states that a graph is planar if, and only if, it contains neither $TK_{3,3}$ nor TK_5. Graphs containing no $TK_{3,3}$ are subgraphs of those graphs constructed from planar graphs and copies of K_5 by pasting them along cliques of size at most two. The structure of graphs containing no TK_5 is not well understood. Kelmans [4] and, independently, Seymour [10] conjectured that 5-connected nonplanar graphs must contain TK_5. Thus, if the Kelmans-Seymour conjecture is true then graphs containing no TK_5 is planar or admits a cut of size at most 4. Note that the requirement on connectivity is best possible (see, for example, $K_{4,4}$).

Ma and Yu [5,6] proved the Kelmans-Seymour conjecture for graphs containing K_4^-, and Kawarabayashi, Ma and Yu [3] proved the Kelmans-Seymour conjecture for graphs containing $K_{2,3}$. We refer the reader to [5,6,3] for problems and results (as well as references) related to the Kelmans-Seymour conjecture.

It turns out that K_4^- is the right intermediate structure for studying the Kelmans-Seymour structure. By a result of Kawarabayashi [2], any 5-connected graph containing no K_4^- has an edge e that is *contractible* (i.e., G/e is also 5-connected). Thus, our strategy for proving the Kelmans-Seymour conjecture is to keep contracting edges incident with a special vertex to produce a smaller 5-connected graph. To avoid trivial components associated with 5-cuts or 6-cuts, we also contract triangles (but we give preference to edges).

We now give a more detailed description of our strategy. For a graph G and a connected subgraph M of G (respectively, an edge e of G), we use G/M (respectively, G/e) to denote the graph obtained from G by contracting M (respectively, e). Let G be a 5-connected nonplanar graph containing no K_4^-. Then G contains an edge e such that G/e is 5-connected. If G/e is planar, we can apply a discharging argument. So assume G/e is not planar. Let M be a maximal connected subgraph of G such that G/M is 5-connected and nonplanar. Let z denote the vertex of G/M representing the contraction of M, and let $H = G/M$. Then one of the following holds.

(a) H contains a subgraph K such that $K \cong K_4^-$ and z has degree 2 in K.
(b) H contains a subgraph K such that $K \cong K_4^-$ and z has degree 3 in K.
(c) H does not contain K_4^-, and there exists $T \subseteq H$, with $z \in V(T)$ and either $T \cong K_2$ or $T \cong K_3$, such that H/T is 5-connected and planar.
(d) H does not contain K_4^-, and for any $T \subseteq H$ with $z \in V(T)$ and either $T \cong K_2$ or $T \cong K_3$, H/T is not 5-connected.

Note that by $T \subseteq H$ we mean that T is a subgraph of H. We plan a series of four papers to establish the Kelmans-Seymour conjecture. The purpose of this paper is to prove several results about 5-separations and 6-separations, which will be used in subsequent papers to reduce (c) and (d) to (a) or (b). Note that 5-separations and 6-separations arise naturally when (d) occurs. In the second paper, we will handle (a). The third paper takes care of (b). In the final paper, we will deal with (c) and (d). In our arguments throughout this series, we frequently encounter the case when $H - z$ contains K_4^-; so the exclusion of K_4^- (in [5,6]) is very useful.

One of the main steps in the proofs in [5,6] is to deal with 5-connected nonplanar graphs that admit a 5-separation with a planar side. A *separation* in a graph G consists of a pair of subgraphs G_1, G_2 of G, denoted as (G_1, G_2), such that $E(G_1) \cup E(G_2) = E(G)$, $E(G_1) \cap E(G_2) = \emptyset$, and neither G_1 nor G_2 is a subgraph of the other. The *order* of this separation is $|V(G_1) \cap V(G_2)|$, and (G_1, G_2) is said to be a *k-separation* if its order is k. Let G be a graph and $A \subseteq V(G)$. We say that (G, A) is *planar* if G has a plane representation in which the vertices in A are incident with a common face. Ma and Yu [5] proved that if G has a 5-separation (G_1, G_2) such that $|V(G_i)| \geq 7$ (for $i = 1, 2$) and $(G_2, V(G_1) \cap V(G_2))$ is planar then G contains a TK_5 with all branch vertices contained in G_2. In order to establish the Kelmans-Seymour conjecture for graphs containing no K_4^-, we need to study 5-separations and 6-separations with less restrictive structures.

Let G be a graph. For two subgraphs G_1, G_2 of G, we use $G_1 \cup G_2$ and $G_1 \cap G_2$ to denote their union and intersection, respectively. For $a \in V(G)$, we use $G - a$ to denote the subgraph obtained from G by deleting a and all edges of G incident with a. For a positive integer n, we let $[n] = \{1, \dots, n\}$. We often represent a path (or cycle) by a sequence of vertices. The following result deals with one type of 5-separations.

Theorem 1.1. *Let G be a 5-connected nonplanar graph and let (G_1, G_2) be a 5-separation in G. Suppose $|V(G_i)| \geq 7$ for $i \in [2]$, $a \in V(G_1 \cap G_2)$, and $(G_2 - a, V(G_1 \cap G_2) - \{a\})$ is planar. Then one of the following holds.*

- *(i) For any $a^* \in V(G_1 - G_2) \cup \{a\}$, G contains a TK_5 in which a^* is not a branch vertex.*
- *(ii) $G - a$ contains K_4^-.*
- *(iii) G has a 5-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{a, a_1, a_2, a_3, a_4\}$ and G_2' is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding a and the edges ab_i for $i \in [4]$.*

Let G be a graph. For $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, and let $G - S = G[V(G) - S]$. For $H \subseteq G$, we write $G[H]$ instead of $G[V(H)]$. For $S \subseteq E(G)$, $G - S$ denotes the graph obtained from G by deleting the edges in S. Another type of 5-separations considered in this paper are those (G_1, G_2) with the property that $G[V(G_1 \cap G_2)]$ contains a triangle.

Theorem 1.2. *Let G be a 5-connected graph and (G_1, G_2) be a 5-separation in G. Suppose that $|V(G_i)| \geq 7$ for $i \in [2]$ and $G[V(G_1 \cap G_2)]$ contains a triangle aa_1a_2a. Then one of the following holds.*

(*i*) *G contains a* TK_5 *in which a is not a branch vertex.*
(*ii*) $G - a$ *contains* K_4^-.
(*iii*) *G has a* 5*-separation* (G_1', G_2') *such that* $V(G_1' \cap G_2') = \{a, a_1, a_2, a_3, a_4\}$ *and* G_2' *is the graph obtained from the edge-disjoint union of the* 8*-cycle* $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ *and the* 4*-cycle* $b_1b_2b_3b_4b_1$ *by adding a and the edges* ab_i *for* $i \in [4]$.
(*iv*) *For any distinct* $u_1, u_2, u_3 \in N(a) - \{a_1, a_2\}$, $G - \{av : v \notin \{a_1, a_2, u_1, u_2, u_3\}\}$ *contains* TK_5.

In the applications of Theorems 1.1 and 1.2, the vertex a will represent the special vertex resulting from the contraction of a connected subgraph formed by a sequence of edges and triangles. (Allowing the contraction of triangles will ensure that $|V(G_i)| \geq 7$ in the applications of Theorems 1.1 and 1.2.) So (*i*) of Theorems 1.1 and 1.2 gives a TK_5 in the original graph, and (*ii*) of Theorems 1.1 and 1.2 allows us to apply the result of Ma and Yu [6] to get a TK_5 in the original graph. The TK_5 given in (*iv*) of Theorem 1.2 can be used to derive a TK_5 in the original graph. When (*iii*) of Theorems 1.1 and 1.2 occurs, we will use Proposition 1.3 below, whose proof is included in this section (as it is short).

Let G be a graph. Recall that we use $H \subseteq G$ to mean that H is a subgraph of G. When $K \subseteq G$ and $L \subseteq G$, we let $K - L = K - V(K \cap L)$. For $S \subseteq V(G)$, we may view S as a subgraph of G with vertex set S and edge set $\emptyset$. For $H \subseteq G$, $N_G(H)$ denotes the neighborhood of H (not including the vertices in $V(H)$). For any $x \in V(G)$, we use $N_G(x)$ to denote the neighborhood of x in G. When understood, the reference to G may be dropped. We may view paths as sequences of vertices. The *ends* of a path P are the vertices of the minimum degree in P, and all other vertices of P (if any) are its *internal* vertices. A collection of paths are said to be *independent* if no vertex of any path in this collection is an internal vertex of any other path in the collection.

Proposition 1.3. *Let G be a 5-connected nonplanar graph,* (G_1, G_2) *a 5-separation in G,* $V(G_1 \cap G_2) = \{a, a_1, a_2, a_3, a_4\}$ *such that* G_2 *is the graph obtained from the edge-disjoint union of the* 8*-cycle* $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ *and the* 4*-cycle* $b_1b_2b_3b_4b_1$ *by adding a and the edges* ab_i *for* $i \in [4]$. *Suppose* $|V(G_1)| \geq 7$. *Then, for any* $u_1, u_2 \in N(a) - \{b_1, b_2, b_3\}$, $G - \{av : v \notin \{b_1, b_2, b_3, u_1, u_2\}\}$ *contains* TK_5.

Proof. By symmetry between u_1 and u_2, we may assume that $u_1 \neq b_4$. For convenience, let $G' := G - \{av : v \notin \{b_1, b_2, b_3, u_1, u_2\}\}$.

First, suppose $u_1 \in V(G_1 - G_2)$. Since G is 5-connected, $G_1 - a$ has independent paths Q_1, Q_2, Q_3, Q_4 from u_1 to a_1, a_2, a_3, a_4, respectively. Then $G[\{a, b_1, b_2, b_3, u_1\}] \cup (Q_1 \cup a_1b_1) \cup (Q_2 \cup a_2b_2) \cup (Q_3 \cup a_3b_3) \cup b_1b_4b_3$ is a TK_5 in G' with branch vertices a, b_1, b_2, b_3, u_1.

Now suppose $u_1 \in \{a_2, a_3\}$. By symmetry, we may assume $u_1 = a_2$. Note that $G_1 - a$ contains a path R from a_2 to a_3; for otherwise, a_2 and a_3 are in different components of $G_1 - a$ and, since $|V(G_1)| \geq 6$, $\{a, a_1, a_2, a_4\}$ or $\{a, a_1, a_3, a_4\}$ would be a cut in G, a contradiction. Then $G[\{a, b_1, b_2, b_3, u_1\}] \cup (R \cup a_3b_3) \cup b_1b_4b_3$ is a TK_5 in G' with branch vertices a, b_1, b_2, b_3, u_1.

Finally, assume $u_1 \in \{a_1, a_4\}$. By symmetry, we may assume $u_1 = a_1$. If $G_1 - a$ has independent paths R_1, R_2 from a_1 to a_2, a_3, respectively, then $G[\{a, b_1, b_2, b_3, u_1\}] \cup (R_1 \cup a_2b_2) \cup (R_2 \cup a_3b_3) \cup b_1b_4b_3$ is a TK_5 in G' with branch vertices a, b_1, b_2, b_3, u_1. So we may assume that such R_1, R_2 do not exist. Then there exists a separation (K, L) in $G_1 - a$ such that $|V(K \cap L)| \leq 1$, $a_1 \in V(K - L)$, and $\{a_2, a_3\} \subseteq V(L)$. We claim that $V(K) \subseteq \{a_1, a_4\} \cup V(K \cap L)$; for, otherwise, $\{a, a_1, a_4\} \cup V(K \cap L)$ would a cut in G, contradicting the assumption that G is 5-connected. If $a_4 \in V(L)$ then $N(a) \subseteq \{a, b_1, b_4\} \cup V(K \cap L)$, contradiction. So $a_4 \notin V(L)$. Then $V(L) = \{a_2, a_3\} \cup V(K \cap L)$; as otherwise $\{a, a_2, a_3\} \cup V(K \cap L)$ would be a cut in G of size at most 4, a contradiction. But this implies $|V(G_1)| \leq 6$, a contradiction. □

The next result deals with a special type of 6-separations, whose proof makes use of Theorems 1.1 and 1.2.

Theorem 1.4. *Let G be a 5-connected graph, let (G_1, G_2) be a 6-separation in G, and let $x, x_1, x_2 \in V(G_1 \cap G_2)$ such that xx_1x_2x is a triangle in G and $|V(G_i)| \geq 7$ for $i \in [2]$. Moreover, assume that (G_1, G_2) is chosen so that, subject to $\{x, x_1, x_2\} \subseteq V(G_1 \cap G_2)$ and $|V(G_i)| \geq 7$ for $i \in [2]$, G_1 is minimal. Let $V(G_1 \cap G_2) = \{x, x_1, x_2, v_1, v_2, v_3\}$. Then $N(x) \cap V(G_1 - G_2) \neq \emptyset$, or one of the following holds.*

- *(i) G contains a TK_5 in which x is not a branch vertex.*
- *(ii) G contains K_4^-.*
- *(iii) There exists $x_3 \in N(x)$ such that for any distinct $y_1, y_2 \in N(x) - \{x_1, x_2, x_3\}$, $G - \{xv : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}$ contains TK_5.*
- *(iv) There exist $i \in [2]$ and $j \in [3]$ such that $N(x_i) \subseteq V(G_1 - G_2) \cup \{x, x_{3-i}\}$, and any three independent paths in $G_1 - x$ from $\{x_1, x_2\}$ to v_1, v_2, v_3, respectively, with two from x_i and one from x_{3-i}, must contain a path from x_{3-i} to v_j.*

Note that in (*ii*) of Theorem 1.4, we ask that G contain K_4^- instead of the stronger statement "$G - a$ contains K_4^-" as in (ii) of Theorems 1.1 and 1.2. Thus, (*iii*) of Theorems 1.1 and 1.2 does not occur in Theorem 1.4 as it gives (*ii*) of Theorem 1.4.

This paper is organized as follows. In Section 2, we use the discharging technique to prove two lemmas about K_4^- in apex graphs. (A graph is apex if it has a vertex whose removal results in a planar graph.) In Section 3, we collect a number of known results and prove Theorem 1.1. In Section 4, we prove a result about apex graphs (from which one can see how (c) might be taken care of). In Section 5, we prove Theorem 1.2. In Section 6, we prove Theorem 1.4, using Theorems 1.1 and 1.2.

We end this section with additional notation and terminology. Let G be a graph. Let $K \subseteq G$, $S \subseteq V(G)$, and T a collection of 2-element subsets of $V(K) \cup S$; then $K + (S \cup T)$

denotes the graph with vertex set $V(K) \cup S$ and edge set $E(K) \cup T$. If $T = \{\{x, y\}\}$ and $x, y \in V(K)$, we write $K + xy$ instead of $K + \{\{x, y\}\}$.

A set $S \subseteq V(G)$ is a *k-cut* (or a *cut* of size k) in a graph G, where k is a positive integer, if $|S| = k$ and G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = S$ and $V(G_i - S) \neq \emptyset$ for $i \in [2]$. If $v \in V(G)$ and $\{v\}$ is a cut of G, then v is said to be a *cut vertex* of G.

Given a path P in a graph and $x, y \in V(P)$, xPy denotes the subpath of P between x and y (inclusive). A path P with ends u and v (or an u-v path) is also said to be *from u to v* or *between u and v*.

A *plane* graph is a graph drawn in the plane with no edge-crossings. The unbounded face of a plane graph is usually called its *outer face*. The *outer walk* of a plane graph G is the subgraph of G consisting of those vertices and edges of G incident with the outer face of G. (An *outer cycle* is simply an outer walk that is a cycle.) It is well known that if G is a 2-connected plane graph then every facial boundary of G is a cycle. Let D be a cycle in a plane graph. Given $x, y \in V(D)$, if $x \neq y$ then xDy denotes the subpath of D between x and y (inclusive) in clockwise order; and if $x = y$ then xDy is simply the trivial path with the single vertex $x = y$.

2. Discharging and K_4^-

In this section, we prove results about K_4^- in certain apex graphs, using the discharging technique. First, we give a simple lemma on discharging. For a plane graph G, let $F(G)$ denote the set of all faces of G and, for each $f \in F(G)$, let $d(f)$ denote the number of edges of G incident with f (with each cut edge counted twice). For each $v \in V(G)$, we use $d_G(v)$ (or $d(v)$ when G is understood) to denote the degree of v in G.

Lemma 2.1. *Let G be a connected plane graph and let $\sigma : V(G) \cup F(G) \longrightarrow \mathbb{Z}$, the set of integers, such that $\sigma(t) = 4 - d(t)$ for all $t \in V(G) \cup F(G)$ (which is called the number of units of charge of t). Let τ be obtained from σ as follows: For each $f \in F(G)$ with $d(f) = 3$, choose two vertices incident with f and send charge $1/2$ from f to each of these two vertices. Then*

$$\sum_{v \in V(G)} \sigma(v) + \sum_{f \in F(G)} \sigma(f) = 8,$$

and if $K_4^- \nsubseteq G$ then, for $v \in V(G)$,

$$\tau(v) \leq \begin{cases} 4 - 3k, & \text{if } d(v) = 4k; \\ 3 - 3k, & \text{if } d(v) = 4k + 1; \\ 5/2 - 3k, & \text{if } d(v) = 4k + 2; \\ 3/2 - 3k, & \text{if } d(v) = 4k + 3. \end{cases}$$

Proof. By Euler's formula, we have

$$\sum_{v\in V(G)} \sigma(v) + \sum_{f\in F(G)} \sigma(f) = 8.$$

Now suppose $K_4^- \not\subseteq G$. Then for each $v \in V(G)$, v is contained in at most $\lfloor d(v)/2\rfloor$ facial triangles. So $\tau(v) \le \sigma(v) + \lfloor d(v)/2\rfloor/2 = 4 - d(v) + \lfloor d(v)/2\rfloor/2$. By considering $d(v)$ modulo 4, we get the desired bounds on $\tau(v)$. □

To state the remaining results in this section, we need a concept on connectivity. Let G be a graph and $A \subseteq V(G)$, and let k be a positive integer. We say that G is (k, A)-*connected* if, for any cut T of G with $|T| < k$, each component of $G - T$ contains a vertex in A. Thus, every vertex of G not in A has degree at least k in G. Recall that (G, A) is planar if G has a plane representation in which the vertices in A are incident with a common face.

Lemma 2.2. *Let G be a connected graph and $A \subseteq V(G)$ such that $|A| = 5$, $|V(G)| \ge 7$, G is $(5, A)$-connected, and (G, A) is planar. Then, for any $a \in A$, $G - a$ contains K_4^-.*

Proof. Since G is $(5, A)$-connected, each component of $G - A$ must contain a neighbor of every vertex in A; thus $G - E(G[A])$ is connected. Clearly, $G - E(G[A])$ is $(5, A)$-connected, and $(G - E(G[A]), A)$ is planar. Therefore, it suffices to prove the lemma for the case when A is an independent set in G. So we assume that A is an independent set in G.

First, we show that $G - a$ is connected for any $a \in A$. For, if not, then we may let C_1, C_2 be two components of $G - a$ such that $V(C_1) \not\subseteq A$. Since G is $(5, A)$-connected, $V(C_i) \cap A \ne \emptyset$ for $i \in [2]$. Now $S := (A \cap V(C_1)) \cup \{a\}$ is a cut in G such that $|S| \le 4$ and $G - S$ has a component contained in $C_1 - A$, contradicting the assumption that G is $(5, A)$-connected.

Take a plane representation of G such that the vertices in A are incident with the outer face of G. Let $\sigma : V(G - a) \cup F(G - a) \longrightarrow \mathbb{Z}$ such that $\sigma(t) = 4 - d_{G-a}(t)$ for all $t \in V(G - a) \cup F(G - a)$. Then by Lemma 2.1, the total charge is

$$\sum_{t\in V(G-a)\cup F(G-a)} \sigma(t) = 8.$$

Note that for any $t \in V(G - a) \cup F(G - a)$, if $\sigma(t) > 0$ then $t \in A$, or $t \in F(G - a)$ and $d_{G-a}(t) = 3$ (in which case, $\sigma(t) = 1$). For each $f \in F(G - a)$ with $d_{G-a}(f) = 3$, choose two vertices of $G - A$ incident with f (which exists as A is independent), and send a charge $1/2$ from f to each of these two vertices. Let τ denote the resulting charge function.

Then $\tau(f) \le 0$ for all $f \in F(G-a)$. Suppose $K_4^- \not\subseteq G-a$. Then, for each $v \in V(G-a)$, $\tau(v)$ has the upper bound in Lemma 2.1. So $\tau(v) \le 0$ when $v \notin N(a) \cup A$, $\tau(v) \le 1$ for $v \in N(a)$ (as A is independent), and $\tau(v) = \sigma(v) \le 3$ for $v \in A - \{a\}$.

Let f_∞ denote the outer face of $G-a$. Since A is independent in G, $d_{G-a}(f_\infty) \geq |N(a)|+7$; so $\tau(f_\infty) = \sigma(f_\infty) \leq 4-(|N(a)|+7) = -3-|N(a)|$. If at least two vertices in $A-\{a\}$ each have degree 2 or more in $G-a$, then $\sum_{v \in A-\{a\}} \tau(v) \leq 2+2+3+3 = 10$; so

$$\tau(f_\infty) + \sum_{v \in A-\{a\}} \tau(v) \leq -3-|N(a)|+10 = 7-|N(a)|.$$

Now assume that at most one vertex in $A-\{a\}$ has degree 2 or more in $G-a$. Since G is $(5,A)$-connected, the vertices in $A-a$ with degree 1 in $G-a$ cannot share a neighbor in $G-a$. Hence, since A is independent, $d_{G-a}(f_\infty) \geq |N(a)|+9$. Since $\sum_{v \in A-\{a\}} \tau(v) \leq 3+3+3+3 = 12$,

$$\tau(f_\infty) + \sum_{v \in A-\{a\}} \tau(v) \leq 4-(|N(a)|+9)+12 = 7-|N(a)|.$$

Therefore, since

$$\sum_{t \in V(G-a) \cup F(G-a)} \tau(t) \leq \tau(f_\infty) + \sum_{v \in A-\{a\}} \tau(v) + \sum_{v \in N(a)} \tau(v),$$

we have

$$\sum_{t \in V(G-a) \cup F(G-a)} \tau(t) \leq (7-|N(a)|)+|N(a)| = 7 < 8 = \sum_{t \in V(G-a) \cup F(G-a)} \sigma(t),$$

a contradiction. □

The next result will not be used in this paper, but will be used in subsequent papers in the series; we include it here as its proof also uses discharging.

Proposition 2.3. *Let G be a graph, $A \subseteq V(G)$, and $a \in A$ such that $|A| = 6$, $|V(G)| \geq 8$, $(G-a, A-\{a\})$ is planar, and G is $(5,A)$-connected. Then one of the following holds.*

(i) $G-a$ contains K_4^-, or G contains a subgraph $K \cong K_4^-$ such that $a \in V(K)$ and the degree of a in K is 2.
(ii) G has a 5-separation (G_1, G_2) such that $a \in V(G_1 \cap G_2)$, $A \subseteq V(G_1)$, $|V(G_2)| \geq 7$, and $(G_2-a, V(G_1 \cap G_2)-\{a\})$ is planar.

Proof. We may assume that

(1) G has no 5-separation or 6-separation (G', G'') such that $a \in V(G' \cap G'')$, $A \subseteq V(G')$, $|V(G'')| \geq 8$, and $(G''-a, V(G' \cap G'')-\{a\})$ is planar.

For, suppose such a separation (G', G'') does exist in G. Then G'' is $(5, V(G' \cap G''))$-connected. If $|V(G' \cap G'')| = 5$ then (ii) holds. If $|V(G' \cap G'')| = 6$ we may work with G'' and $V(G' \cap G'')$ instead of G and A.

Note that A is independent in G. For, if there exist $a_1, a_2 \in A$ such that $a_1a_2 \in E(G)$ then let $G'' = G - a_1a_2$ and G' consists of A and the edge a_1a_2. Now (G', G'') is a separation in G, which contradicts (1).

(2) $|V(G - a)| \geq 8$ and each vertex in $A - \{a\}$ has at least two neighbors in $G - A$.

First, suppose $|V(G - a)| = 7$. Let b_1, b_2 denote the two vertices in $V(G) - A$. Since G is $(5, A)$-connected, $|N(b_1) \cap N(b_2) \cap A| \geq 3$. But this contradicts the assumption that $(G - a, A - \{a\})$ is planar.

So $|V(G - a)| \geq 8$. Then by (1), each vertex in $A - \{a\}$ has at least two neighbors in $G - A$, completing the proof of (2).

Moreover,

(3) $G - A$ is connected.

For, otherwise, let C_1, C_2 be two components of $G - A$. Since G is $(5, A)$-connected, $|N(C_i) \cap (A - \{a\})| \geq 4$ for $i \in [2]$. Hence, $|N(C_1) \cap N(C_2) \cap (A - \{a\})| \geq 3$, contradicting the assumption that $(G - a, A - \{a\})$ is planar. This proves (3).

We now apply a discharging argument to $G - a$ by taking a plane representation of $G - a$ in which the vertices in $A - \{a\}$ are incident with its outer face. Let $\sigma : V(G - a) \cup F(G - a) \longrightarrow \mathbb{Z}$ such that $\sigma(t) = 4 - d_{G-a}(t)$ for all $t \in V(G - a) \cup F(G - a)$. By (2) and (3), $G - a$ is connected. So by Lemma 2.1, the total charge is

$$\sum_{t \in V(G-a) \cup F(G-a)} \sigma(t) = 8.$$

Note that for any $t \in V(G - a) \cup F(G - a)$, if $\sigma(t) > 0$ then $t \in A - \{a\}$, or $t \in F(G - a)$ and $d_{G-a}(t) = 3$ (in which case, $\sigma(t) = 1$). For each $f \in F(G - a)$ with $d_{G-a}(f) = 3$, we may assume that f is incident with two vertices in $V(G - a) - N(a)$; for, otherwise, (i) holds. So for each $f \in F(G - a)$ with $d_{G-a}(f) = 3$, we choose two vertices from $V(G - a) - N(a)$ incident with f, and send a charge $1/2$ from f to each of these two vertices. Let τ denote the resulting charge function. Then $\tau(f) \leq 0$ for all $f \in F(G - a)$. We may assume that $K_4^- \not\subseteq G - a$, as otherwise (i) holds. Then for each $v \in V(G - a)$, $\tau(v)$ has the upper bound in Lemma 2.1. Thus, $\tau(v) \leq 0$ if $v \notin A$ (since vertices in $N(a)$ do not receive charge), and $\tau(v) \leq 5/2$ if $v \in A - \{a\}$ (by (2)).

Denote by f_∞ the outer face of $G - a$. Since A is independent in G, $\tau(f_\infty) = \sigma(f_\infty) \leq -6$. Therefore, $\tau(f_\infty) + \sum_{v \in A - \{a\}} \tau(v) \leq -6 + 25/2 = 13/2$. Hence, the total charge is

$$\sum_{t\in V(G-a)\cup F(G-a)} \tau(t) \leq \tau(f_\infty) + \sum_{v\in A-\{a\}} \tau(v) \leq 13/2 < 8 = \sum_{t\in V(G-a)\cup F(G-a)} \sigma(t).$$

This is a contradiction. □

3. Apex separations

In this section, we prove Theorem 1.1. For convenience, we introduce the following terminology. Let G be a graph and $A \subseteq V(G)$. We say that (G, A) is *plane* if G is drawn in the plane with no edge crossings such that the vertices in A are incident with the outer face. Moreover, for $a_1, \ldots, a_k \in V(G)$, we say $(G, a_1, \ldots, a_k)$ is *plane* (respectively, *planar*) if G is drawn (respectively, has a drawing) in a closed disc in the plane with no edge crossings such that $a_1, \ldots, a_k$ occur on the boundary of the disc in this cyclic order (clockwise or counterclockwise).

We also need a few known results. The first result is a consequence of a more general result of Seymour [9] (with equivalent versions proved in [1,12,11]).

Lemma 3.1. *Let G be a graph and let $s_1, s_2, t_1, t_2 \in V(G)$ be distinct such that G is $(4, \{s_1, s_2, t_1, t_2\})$-connected. Then either G contains disjoint paths from s_1 to t_1 and from s_2 to t_2, or (G, s_1, s_2, t_1, t_2) is planar.*

The next lemma is Theorem 4.3 in [5], where it is used to prove that if a 5-connected nonplanar graph has a 5-separation with one side planar and nontrivial then it contains TK_5.

Lemma 3.2. *Let G be a graph and let $a_1, a_2, a_3, a_4, a_5 \in V(G)$ be distinct such that $(G, a_1, a_2, a_3, a_4, a_5)$ is plane. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and suppose G is $(5, A)$-connected and $|V(G)| \geq 7$. Then there exist $w \in V(G) - A$, a cycle C_w in $(G - A) - w$, and four paths P_1, P_2, P_3, P_4 from w to A such that*

(i) $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 4$, and $|V(P_i \cap C_w)| = 1$ for $i \in [4]$, and
(ii) there exist $1 \leq i \neq j \leq 4$ such that a_1 is an end of P_i and a_5 is an end of P_j.

We need two results from [3], which may be viewed as apex versions of Lemma 3.2. They are used in [3] to deal with 5-separations with an apex side. Lemma 3.3 is Corollary 2.11 in [3], and Lemma 3.4 is Corollary 2.12 in [3].

Lemma 3.3. *Let G be a connected graph with $|V(G)| \geq 7$, $A \subseteq V(G)$ with $|A| = 5$, and $a \in A$, such that G is $(5, A)$-connected, $(G - a, A - \{a\})$ is plane, and G has no 5-separation (G_1, G_2) with $A \subseteq G_1$ and $|V(G_2)| \geq 7$. Let $w \in N(a)$ such that w is not incident with the outer face of $G - a$. Then*

(i) the vertices of $G - a$ cofacial with w induce a cycle C_w in $G - a$, and

(*ii*) *$G - a$ contains paths P_1, P_2, P_3 from w to $A - \{a\}$ such that $V(P_i \cap P_j) = \{w\}$ for $1 \le i < j \le 3$, and $|V(P_i \cap C_w)| = |V(P_i) \cap A| = 1$ for $i \in [3]$.*

Lemma 3.4. *Let G be a connected graph with $|V(G)| \ge 7$, $A \subseteq V(G)$ with $|A| = 5$, and $a \in A$, such that $K_4^- \nsubseteq G - a$, G is $(5, A)$-connected, $(G - a, (A - \{a\}) \cup N(a))$ is plane, and G has no 5-separation (G_1, G_2) with $A \subseteq V(G_1)$ and $|V(G_2)| \ge 7$. Then $G - a$ is 2-connected. Moreover, either G is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding a and the edges ab_i for $i \in [4]$, with $A = \{a, a_1, a_2, a_3, a_4\}$, or there exists $w \in V(G) - A$ such that*

(*i*) *the vertices of $G-a$ cofacial with w induce a cycle C_w in $G-a$ such that $C_w \cap D = \emptyset$, where D denotes the outer cycle of $G - a$,*
(*ii*) *there exist paths P_1, P_2, P_3, P_4 in G from w to A such that $V(P_i \cap P_j) = \{w\}$ for $1 \le i < j \le 4$, and $|V(P_i \cap C_w)| = |V(P_i) \cap A| = 1$ for $i \in [4]$, and*
(*iii*) *either $a \notin \bigcup_{i=1}^4 V(P_i)$, or $a \in \bigcup_{i=1}^4 V(P_i)$ and we may write $A - \{a\} = \{a_1, a_2, a_3, a_4\}$ such that $a \in V(P_1)$, $a_i \in V(P_i)$ for $2 \le i \le 4$, and $a_1, a_2, a_3, V(P_1 \cap D), a_4$ occur on D in a cyclic order.*

Proof of Theorem 1.1. Let $a^* \in V(G_1 - G_2) \cup \{a\}$. We choose the separation (G_1, G_2) so that G_2 is minimal. Then we may assume that G_2 has no 5-separation (G_2', G_2'') such that $|V(G_2' \cap G_2'')| \le 5$, $V(G_1 \cap G_2) \subseteq V(G_2')$, and $|V(G_2'')| \ge 7$. For, suppose such (G_2', G_2'') does exist. Then $a \notin V(G_2' \cap G_2'')$; otherwise, $(G_1 \cup G_2', G_2'')$ contradicts the choice of (G_1, G_2) (that G_2 is minimal). Hence, $(G_2'', V(G_2' \cap G_2''))$ is planar; so (*ii*) holds by Lemma 2.2.

Let $A := V(G_1 \cap G_2) = \{a, a_1, a_2, a_3, a_4\}$ such that $(G_2 - a, a_1, a_2, a_3, a_4)$ is plane. Let D denote the outer walk of $G_2 - a$; so a_1, a_2, a_3, a_4 occur on D in clockwise order. We may assume that neither (G_1, A) nor (G_2, A) is planar; else (*ii*) holds by Lemma 2.2.

Suppose there exists some $w \in N(a) \cap V(G_2 - D)$. Then by Lemma 3.3, the vertices of $G_2 - a$ cofacial with w induce a cycle C_w in $G_2 - a$, and $G_2 - a$ contains paths P_1, P_2, P_3 from w to $A - \{a\}$ such that $V(P_i \cap P_j) = \{w\}$ for $1 \le i < j \le 3$, and $|V(P_i \cap C_w)| = |V(P_i) \cap A| = 1$ for $i \in [3]$. Without loss of generality, we may assume that $V(P_i) \cap A = \{a_i\}$ for $i \in [3]$. Let $y \in V(G_1 - A)$ with $y \ne a^*$. Since G is 5-connected, there exist independent paths Q_1, Q_2, Q_3, Q_4 in $G_1 - a_4$ from y to a_1, a_2, a_3, a, respectively. Then $C_w \cup P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3 \cup (Q_4 \cup wa)$ is a TK_5 in G in which a^* is not a branch vertex. So (*i*) holds.

Thus, we may assume that $N(a) \cap V(G_2) \subseteq V(D)$. By the minimality of G_2, A is independent in G_2. Hence $(G_2 - a, (A - a) \cup (N(a) \cap V(G_2)))$ is planar. Moreover, we may assume $|N(a) \cap V(G_2 - A)| \ge 2$; for, otherwise, (G_2, A) is planar and, hence, (*ii*) holds by Lemma 2.2.

Suppose (*ii*) and (*iii*) of Theorem 1.1 both fail. Then by Lemma 3.4, $G_2 - a$ is 2-connected (so D is a cycle) and there exists $w \in V(G_2) - A$ such that

(1) the vertices of G_2-a cofacial with w induce a cycle C_w in G_2-a such that $C_w \cap D = \emptyset$,
(2) there exist paths P_1, P_2, P_3, P_4 in G_2 from w to A such that $V(P_i \cap P_j) = \{w\}$ for $1 \le i < j \le 4$, and $|V(P_i \cap C_w)| = |V(P_i) \cap A| = 1$ for $i \in [4]$, and
(3) either $a \notin \bigcup_{i=1}^{4} V(P_i)$, or $a \in \bigcup_{i=1}^{4} V(P_i)$ and we may relabel a_1, a_2, a_3, a_4 (if necessary) such that $a \in V(P_1)$, $a_i \in V(P_i)$ for $2 \le i \le 4$, and $a_1, a_2, a_3, V(P_1 \cap D), a_4$ occur on D in a cyclic order.

For convenience, let $L = C_w \cup P_1 \cup P_2 \cup P_3 \cup P_4$ and assume, without loss of generality, that a_1, a_2, a_3, a_4 occur on D in clockwise order. By the planarity of $G_2 - a$, we may assume that $P_i \cap D$ is a path for $i \in [4]$.

Case 1. $a \notin \bigcup_{i=1}^{4} V(P_i)$.

Without loss of generality, we may assume that $a_i \in V(P_i)$ for $i \in [4]$. If $G_1 - a$ has disjoint paths S_1, S_2 from a_1, a_2 to a_3, a_4, respectively, then $L \cup S_1 \cup S_2$ is a TK_5 in G in which a^* is not a branch vertex; so (i) holds. Hence, we may assume that such S_1, S_2 do not exist. Then, since G_1 is $(5, A)$-connected, it follows from Lemma 3.1 that $(G_1 - a, a_4, a_3, a_2, a_1)$ is plane and, hence, $G_1 - A$ is connected.

Subcase 1.1. $G_1 - A$ is not 2-connected.

If $|V(G_1-A)| = 2$ then let C_1, C_2 denote the two disjoint 1-vertex subgraphs of G_1-A, and otherwise let (C_1, C_2) denote a 1-separation in $G_1 - A$ with $V(C_i - C_{3-i}) \neq \emptyset$ for $i \in [2]$. Since G is 5-connected and $(G_1-a, a_4, a_3, a_2, a_1)$ is plane, $N(a) \cap V(C_i - C_{3-i}) \neq \emptyset$ and $C_i - C_{3-i}$ is connected, for $i \in [2]$. Without loss of generality, we may assume that $\{a, a_1, a_2, a_3\} \subseteq N(C_1)$ and $\{a, a_1, a_3, a_4\} \subseteq N(C_2)$.

Suppose $N(a) \cap V(G_2 - A) \subseteq V(P_1 \cup P_3)$. If $N(a) \cap V(G_2 - A) \subseteq V(P_i)$ for some $i \in \{1, 3\}$ then (G_2, A) is planar; so (ii) holds by Lemma 2.2. Thus, we may assume that there exist $a' \in N(a) \cap V(P_1) - V(P_3)$ and $a'' \in N(a) \cap V(P_3) - V(P_1)$. Let L' be obtained from L by replacing P_1, P_3 with wP_1a', wP_3a'', respectively, and let R be a path in $G_1 - \{a, a_1, a_3\}$ from a_2 to a_4. Then $L' \cup a'aa'' \cup R$ is a TK_5 in G in which a^* is not a branch vertex. So (i) holds.

Hence, we may assume that there exists $a' \in N(a) \cap V(G_2 - A)$ such that $a' \notin V(P_1 \cup P_3)$. Recall that $N(a) \cap V(G_2 - A) \subseteq V(D)$. Then $D - V(P_1 \cup P_3)$ has a path Q which is either from a' to a_2 (and disjoint from P_4) or from a' to a_4 (and disjoint from P_2).

First, assume that Q is from a' to a_2. Let P_2' be the path in $P_2 \cup Q$ between w and a', and L' be obtained from L by replacing P_2 with P_2'. Let R_1 be a path in $G_1[(C_1 - C_2) + \{a_1, a_3\}]$ from a_1 to a_3, and R_2 be a path in $G_1[(C_2 - C_1) + \{a, a_4\}]$ from a to a_4. Now $L' \cup R_1 \cup (R_2 \cup aa')$ is a TK_5 in G in which a^* is not a branch vertex, and (i) holds.

So we may assume that Q is from a' to a_4. Then let P_4' be the path in $P_4 \cup Q$ between w and a', and L' be obtained from L by replacing P_4 with P_4'. Let R_1 be a path in $G_1[(C_1 - C_2) + \{a, a_2\}]$ from a to a_2, and R_2 be a path in $G_1[(C_2 - C_1) + \{a_1, a_3\}]$ from

a_1 to a_3. Now $L' \cup (R_1 \cup aa') \cup R_2$ is a TK_5 in G in which a^* is not a branch vertex, and (i) holds.

Subcase 1.2. $G_1 - A$ is 2-connected.

Let D' denote the outer cycle of $G_1 - A$, and let $a_i', a_i'' \in N(a_i) \cap V(D')$ such that a_1', a_1'', a_2', a_2'', a_3', a_3'', a_4', a_4'' occur on D' in counterclockwise order, with $a_i'' D' a_i'$ maximal for $i \in [4]$. Note that a_i', a_i'' exist as G is 5-connected and $(G_1 - a, a_4, a_3, a_2, a_1)$ is plane, and note that $a_i' = a_i''$ is possible. Recall that $|N(a) \cap V(D)| \geq 2$. Let $a', a'' \in N(a) \cap V(D)$ be distinct.

Suppose a', a'' may be chosen such that D contains disjoint paths U, W from a', a'' to a_1, a_3 (or a_2, a_4), respectively, and disjoint from $P_2 \cup P_4$ (or $P_1 \cup P_3$). By symmetry, we may assume that U, W are paths in D from a', a'' to a_1, a_3, respectively, and disjoint from $P_2 \cup P_4$. Let L' be obtained from L by replacing P_1 with the path in $P_1 \cup U$ from w to a' and replacing P_3 with the path in $P_3 \cup W$ from w to a''. Let R be a path in $G_1 - \{a, a_1, a_3\}$ from a_2 to a_4. Now $L' \cup a'aa'' \cup R$ is a TK_5 in G in which a^* is not a branch vertex, and (i) holds.

So we may assume that no such a', a'' can be chosen. Then, without loss of generality, we may assume that D has disjoint paths U, W from a', a'' to a_3, a_4, respectively, and disjoint from $P_1 \cup P_2$.

Suppose a has a neighbor on $a_2'' D' a_4' - \{a_4', a_2''\}$ or $a_3'' D' a_1' - \{a_1', a_3''\}$. By symmetry, we may assume the former. Then $G_1 - a_3$ has disjoint paths R_1, R_2 from a, a_2 to a_1, a_4, respectively. We modify L to obtain L' by replacing P_3 with the path in $P_3 \cup U$ from w to a'. Then $L' \cup (a'a \cup R_1) \cup R_2$ is a TK_5 in G in which a^* is not a branch vertex, and (i) holds.

Thus, we may assume that $N(a) \cap V(D') \subseteq V(a_4' D' a_3'')$. Since (G_1, A) is not planar, a has a neighbor in $V(G_1 - A) - V(D')$, say b. Since G is 5-connected, G_1 has no 2-cut $\{c_1, c_2\} \in V(a_4' D' a_2'')$ separating b from a_1. So $(G_1 - \{a, a_2, a_3, a_4\}) - a_4' D' a_2''$ has a path R from b to a_1. Let L' be obtained from L by replacing P_3 with the path in $P_3 \cup U$ from w to a'. Then $L' \cup (R \cup baa') \cup (a_4 a_4' \cup a_4' D' a_2'' \cup a_2'' a_2)$ is a TK_5 in G in which a^* is not a branch vertex, and (i) holds.

Case 2. $a \in \bigcup_{i=1}^{4} V(P_i)$.

Recall the notation in (3). If $G_1 - a_1$ contains disjoint paths S_1, S_2 from a, a_3 to a_2, a_4, respectively, then $L \cup S_1 \cup S_2$ is a TK_5 in G in which a^* is not a branch vertex (so (i) holds). Hence, we may assume that such S_1, S_2 do not exist. Then, since G is 5-connected, it follows from Lemma 3.1 that $(G_1 - a_1, a, a_3, a_2, a_4)$ is plane and, hence, $G_1 - A$ is connected. We may assume that a, a_3, a_2, a_4 occur on the outer walk of $G_1 - a_1$ in clockwise order.

Recall that $P_i \cap D$ is a path for $i \in [4]$ (see the sentence before Case 1). So let $P_1 \cap D = a' P_1 a''$ and $P_i \cap D = a_i' P_i a_i$ for $2 \leq i \leq 4$. Moreover, $a \in V(P_1)$ by (3) (since we are in Case 2); so we may assume that a' is the only neighbor of a on $a' P_1 a''$ (otherwise we can shorten P_1). Let $b, c \in V(D)$ such that $a_2 \in V(bDc)$ and $bDc \cap (P_3 \cup P_4) = \emptyset$ and, subject to this, bDc is maximal.

Suppose there exists $v \in N(a) \cap V(bDc)$. Let P_2' be the path in $P_2 \cup bDc$ from w to v, and let L' be obtained from L by replacing P_2 with P_2'. Let Q be a path in $G_1 - \{a, a_1, a_2\}$ from a_3 to a_4. Then $L' \cup vaa' \cup Q$ is a TK_5 in G in which a^* is not a branch vertex, and (i) holds.

So we may assume $N(a) \cap V(bDc) = \emptyset$. However, since (G_2, A) is not planar, a must have a neighbor in $V(D) - V(a_3Da_4)$. Hence, by the maximality of bDc, one of the following holds:

(a) a_3, a_4, a_4' occur on D in clockwise order and there exists $v \in N(a) \cap V(a_4Da_4' - a_4)$; or

(b) a_3', a_3, a_4 occur on D in clockwise order and there exists $v \in N(a) \cap V(a_3'Da_3 - a_3)$.

We consider two cases according to whether or not $G_1 - A$ is 2-connected.

Subcase 2.1. $G_1 - A$ is 2-connected.

Let D' denote the outer cycle of $G_1 - A$ and g_2, g_3, g, g_4 be neighbors of a_2, a_3, a, a_4 on D', respectively. We may choose g_2, g_3, g, g_4 so that they are pairwise distinct and occur on D' in counterclockwise order.

If (a) holds, let $R_1 = a_2g_2 \cup g_2D'g_4 \cup g_4a_4$, $R_2 = ag \cup gD'g_3 \cup g_3a_3$, and L_a be obtained from L by replacing P_1 with $wP_1a'' \cup a''Da_4$ and replacing P_4 with $wP_4a_4' \cup vDa_4' \cup va$. Then $L_a \cup R_1 \cup R_2$ is a TK_5 in G in which a^* is not a branch vertex, and (i) holds.

If (b) holds, then let $R_1 = a_3g_3 \cup g_3D'g_2 \cup g_2a_2$, $R_2 = a_4g_4 \cup g_4D'g \cup ga$, and L_b be obtained from L by replacing P_1 with $wP_1a'' \cup a_3Da''$ and replacing P_3 with $wP_3a_3' \cup a_3'Dv \cup va$. Then $L_b \cup R_1 \cup R_2$ is a TK_5 in G in which a^* is not a branch vertex, and (i) holds.

Subcase 2.2. $G_1 - A$ is not 2-connected.

If $|V(G_1 - A)| = 2$ then let C_1, C_2 denote the two disjoint 1-vertex subgraphs of $G_1 - A$. If $|V(G_1 - A)| \geq 3$ then let (C_1, C_2) be a 1-separation of $G_1 - A$. Since G_1 is $(5, A)$-connected and $(G_1 - a_1, a, a_3, a_2, a_4)$ is planar, we may assume without loss of generality that $\{a_1, a, a_3, a_4\} \subseteq N(C_1)$ and $\{a_1, a_2, a_3, a_4\} \subseteq N(C_2)$, or $\{a_1, a_2, a_3, a\} \subseteq N(C_1)$ and $\{a_1, a_2, a_4, a\} \subseteq N(C_2)$. Moreover, $C_i - C_{3-i}$ is connected for $i \in [2]$.

Suppose $\{a_1, a, a_3, a_4\} \subseteq N(C_1)$ and $\{a_1, a_2, a_3, a_4\} \subseteq N(C_2)$. Let R_1 be a path in $G[(C_1 - C_2) + \{a, a_1\}] \cup a_1Da_2 \cup P_2$ from w to a, and L' be obtained from L by replacing P_2 with R_1. Let R_2 be a path in $G_1[(C_2 - C_1) + \{a_3, a_4\}]$ from a_3 to a_4. Then $L' \cup R_2$ is a TK_5 in G in which a^* is not a branch vertex, and (i) holds.

Thus, we may assume that $\{a_1, a_2, a_3, a\} \subseteq N(C_1)$ and $\{a_1, a_2, a_4, a\} \subseteq N(C_2)$.

If (a) holds, let R_1 be a path in $G_1[(C_1 - C_2) + \{a, a_3\}]$ from a to a_3, R_2 be a path in $G_1[(C_2 - C_1) + \{a_2, a_4\}]$ from a_2 to a_4, and L_a be obtained from L by replacing P_1 with $wP_1a'' \cup a''Da_4$ and replacing P_4 with $wP_4a_4' \cup vDa_4' \cup va$. Then $L_a \cup R_1 \cup R_2$ is a TK_5 in G in which a^* is not a branch vertex, and (i) holds.

So assume that (b) holds. Let R_1 be a path in $G[(C_1 - C_2) + \{a_3, a_2\}]$ from a_3 to a_2, R_2 be a path in $G[(C_2 - C_1) + \{a, a_4\}]$ from a to a_4, and L_b be obtained from L by replacing P_1 with $wP_1a'' \cup a_3Da''$ and replacing P_3 with $wP_3a'_3 \cup a'_3Dv \cup va$. Then $L_b \cup R_1 \cup R_2$ is a TK_5 in G in which a^* is not a branch vertex, and (i) holds. □

4. Apex graphs

In this section, we consider apex graphs. The purpose is to give an indication how case (c) in Section 1 will be taken care of later (by combining with later results on K_4^-). First, we give a consequence of Lemma 2.2.

Proposition 4.1. *Let G be a 5-connected nonplanar graph, $a \in V(G)$, and $T \subseteq G$ with $T \cong K_2$ or $T \cong K_3$, such that $a \in V(T)$ and G/T is 5-connected and planar. Then $G - V(T)$ contains K_4^-.*

Proof. Let z denote the vertex of G/T representing the contraction of T. Take a plane representation of G/T. If some facial cycle of G/T has length at least 5 then we choose a set A of five vertices from this facial cycle, with $z \in A$. Then G/T is $(5, A)$-connected and $(G/T, A)$ is planar. Since G/T is 5-connected and planar, $|V(G/T)| \geq 7$. Hence, by Lemma 2.2, $G/T - z$ contains K_4^-. So $G - V(T)$ contains K_4^-.

So assume that every facial cycle of G/T containing z has length at most 4. Let F denote a facial cycle of G/T containing z, and let e be an edge of F incident with z. We subdivide e once or twice so that the new cycle F' obtained from F has length 5, and we use H to denote the new plane graph (resulted from G/T). Now let $A = V(F')$. Note that H is connected and $(5, A)$-connected, and (H, A) is planar. Hence, by Lemma 2.2, $H - z$ contains K_4^-. Now a K_4^- in $H - z$ is also contained in $G/T - z$, as the vertices in H but not in G/T cannot be contained in the K_4^-. Hence, $G - V(T)$ contains K_4^-. □

Next we give a result which strengthens Corollary 2.9 in [3].

Proposition 4.2. *Let G be a 5-connected nonplanar graph and $a \in V(G)$ such that $G - a$ is planar. Then one of the following holds.*

- *(i) G contains a TK_5 in which a is not a branch vertex.*
- *(ii) $G - a$ contains K_4^-.*
- *(iii) G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a, a_1, a_2, a_3, a_4\}$ and G_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding a and the edges ab_i for $i \in [4]$.*

Proof. We may assume that $G - a$ is embedded in the plane with no edge crossings. We may further assume that $K_4^- \nsubseteq G - a$; as otherwise (ii) holds. Let $S := \{v \in V(G - a) : d_{G-a}(v) = 4\}$.

We now show $|S| \geq 8$ by applying to $G - a$ the same discharging argument used in Section 2. Let $\sigma : V(G-a) \cup F(G-a) \longrightarrow \mathbb{Z}$ such that $\sigma(t) = 4 - d_{G-a}(t)$ for all $t \in V(G-a) \cup F(G-a)$. Then by Lemma 2.1, the total charge is

$$\sum_{t \in V(G-a) \cup F(G-a)} \sigma(t) = 8.$$

Note that for any $t \in V(G-a) \cup F(G-a)$, if $\sigma(t) > 0$ then $t \in F(G-a)$, $d_{G-a}(t) = 3$, and $\sigma(t) = 1$ (i.e., t is a triangle in $G - a$). For each $f \in F(G-a)$ with $d_{G-a}(f) = 3$, pick two of its incident vertices, and send a charge $1/2$ from f to each of these two vertices. Let τ denote the resulting charge function. Then $\tau(f) \leq 0$ for all $f \in F(G-a)$. Since we assume $K_4^- \not\subseteq G$, for $v \in V(G-a)$, $\tau(v)$ satisfies the upper bound in Lemma 2.1. Hence, $\tau(v) \leq 0$ for $v \in V(G-a) - S$, and $\tau(v) \leq 1$ for $v \in S$. Therefore, $|S| \geq 8$ as

$$\sum_{t \in V(G-a) \cup F(G-a)} \tau(t) = \sum_{t \in V(G-a) \cup F(G-a)} \sigma(t).$$

Let $b \in S$ be arbitrary, let $N(b) = \{a, b_1, b_2, b_3, b_4\}$, and let D_b denote the facial cycle of $(G-a)-b$ containing $\{b_1, b_2, b_3, b_4\}$. Note that $G - b$ is $(5, \{a, b_1, b_2, b_3, b_4\})$-connected and $((G-b)-a, \{b_1, b_2, b_3, b_4\})$ is planar.

Suppose there exists $w \in N(a) - V(D_b)$. Let $G' = (G-b) - E(G[\{a, b_1, b_2, b_3, b_4\}])$. We wish to apply Lemma 3.3 to G' and $\{a, b_1, b_2, b_3, b_4\}$. Since $|S| \geq 8$, $|V(G')| \geq 7$. Suppose G' has a 5-separation (G_1', G_2') with $\{a, b_1, b_2, b_3, b_4\} \subseteq V(G_1')$ and $|V(G_2')| \geq 7$. Then G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = V(G_1' \cap G_2')$, $V(G_1) = V(G_1') \cup \{b\}$ and $G_2 = G_2'$. Note that $|V(G_1')| \geq 6$ as $\{a, b_1, b_2, b_3, b_4\}$ is independent in G'. So $|V(G_1)| \geq 7$. If $a \in V(G_1 \cap G_2)$ then we may apply Theorem 1.1 to G to conclude that (i) or (ii) or (iii) holds. If $a \notin V(G_1 \cap G_2)$ then $(G_2, V(G_1 \cap G_2))$ is planar; so (ii) holds by Lemma 2.2. Hence, we may assume that no such 5-separation (G_1', G_2') exists in G'. Then by Lemma 3.3, the vertices of $G' - a$ cofacial with w induce a cycle C_w (in $G' - a$) and $G' - a$ contains three paths P_1, P_2, P_3 from w to $\{b_1, b_2, b_3, b_4\}$ such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 3$, and $|V(P_i \cap C_w)| = |V(P_i) \cap \{b_1, b_2, b_3, b_4\}| = 1$ for $i \in [3]$. Without loss of generality, assume that $b_i \in V(P_i)$ for $i \in [3]$. Now $C_w \cup baw \cup (P_1 \cup b_1 b) \cup (P_2 \cup b_2 b) \cup (P_3 \cup b_3 b)$ is a TK_5 in G in which a is not a branch vertex, and (i) holds.

So we may assume that $N(a) - \{s\} \subseteq V(D_s)$ for all $s \in S$. Without loss of generality, we may assume that b_1, b_2, b_3, b_4 occur on D_b in clockwise order.

Since $|S| \geq 8$, we may further assume that there exists $v \in S \cap V(b_1 D_b b_2 - \{b_1, b_2\})$. Let $N(v) = \{a, v_1, v_2, v_3, v_4\}$ with $v_1 \in V(b_1 D_b v)$ and $v_2 \in V(v D_b b_2)$. Let D_v be the facial cycle of $(G-a)-v$ containing $\{v_1, v_2, v_3, v_4\}$. Since $v \in S$, we have $N(a) - \{v\} \subseteq V(D_v)$. Thus, $N(a) - \{b, v\} \subseteq V(D_v \cap D_b) = V(b_1 D_b b_2 - v)$ (since G is 5-connected). However, this implies that G is planar, a contradiction. □

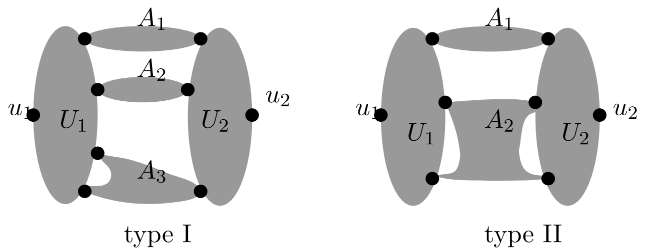

Fig. 1. Obstructions of type I and type II.

5. Topological H and 5-separations

In this section, we prove Theorem 1.2. We fix the notation H to represent the tree on six vertices, two of which are adjacent and of degree 3. We need a result on TH in the proof of Theorem 1.2 (and also in the proof of Theorem 1.4 in Section 6).

Let G be a graph and $u_1, u_2, a_1, a_2, a_3, a_4$ be distinct vertices of G. We say that a TH in G is *rooted* at $u_1, u_2, \{a_1, a_2, a_3, a_4\}$ if in the TH, u_1, u_2 are of degree 3 and a_1, a_2, a_3, a_4 are of degree 1. For convenience, we use *quadruple* to denote (G, u_1, u_2, A) where u_1, u_2 are distinct vertices of the graph G, $A \subseteq V(G) - \{u_1, u_2\}$, and $|A| = 4$. We say that (G, u_1, u_2, A) is *feasible* if G has a TH rooted at u_1, u_2, A.

In [7], infeasible quadruples are characterized in terms of obstructions. Here we include the descriptions and figures of obstructions given in [7]. A quadruple (G, u_1, u_2, A) is an *obstruction* if G has subgraphs U_1, U_2 (called *sides*) and $A_1, \dots, A_k$ (called *middle parts*), where $2 \le k \le 4$, such that

- $V(G) = V(U_1) \cup V(U_2) \cup A_{[k]}$, where $A_{[k]} = \bigcup_{i \in [k]} V(A_i)$,
- $V(A_i), i \in [k]$, are pairwise disjoint,
- $E(G - A)$ is the disjoint union of $E(U_1 - A), E(U_2 - A)$, and $E(A_i - A)$ for $i \in [k]$,
- $V(U_1 \cap U_2) \subseteq A \subseteq A_{[k]}$, $u_1 \in V(U_1) - A_{[k]}$, and $u_2 \in V(U_2) - A_{[k]}$,
- for any $i \in [k], V(A_i) \cap A \neq \emptyset$ and $|V(A_i) \cap A| \le |V(A_i) \cap V(U_1 \cup U_2)| \le |V(A_i) \cap A| + 1$,
- if $|V(A_i)| \ge 2$, then $V(A_i) \cap V(U_1 \cup U_2) \cap A = \emptyset$ and $N(V(A_i) \cap A) \subseteq V(A_i)$.

Note that $V(A_i) \cap V(U_1 \cup U_2) \cap A \neq \emptyset$ if, and only if, $|V(A_i)| = 1$, in which case $V(A_i) \subseteq A \cap V(U_1 \cap U_2)$ and there is no restriction on $N(A_i)$.

An obstruction (G, u_1, u_2, A) is said to be of *type I* if $k = 3$, $|V(A_i) \cap A| = 1$ for $i \in [2]$, $|V(A_3) \cap A| = 2$, $|V(U_i \cap A_j)| = 1$ for $(i, j) \neq (1, 3)$, and $|V(U_1 \cap A_3)| = 2$.

An obstruction (G, u_1, u_2, A) is said to be of *type II* if $k = 2$, $|V(A_1) \cap A| = 1$, $|V(A_2) \cap A| = 3$, $|V(U_i \cap A_1)| = 1$ for $i \in [2]$, and $|V(U_i \cap A_2)| = 2$ for $i \in [2]$ (Fig. 1).

An obstruction (G, u_1, u_2, A) is said to be of *type III* if $k = 2$, $|V(A_i) \cap A| = 2$ for $i \in [2]$, $|V(U_1 \cap A_1)| = |V(U_2 \cap A_2)| = 1$, and $|V(U_1 \cap A_2)| = |V(U_2 \cap A_1)| = 2$.

An obstruction (G, u_1, u_2, A) is said to be of *type IV* if $k = 4$ and, for $i \in [4]$ and $j \in [2]$, $|V(A_i) \cap A| = |V(U_j \cap A_i)| = 1$ (Fig. 2).

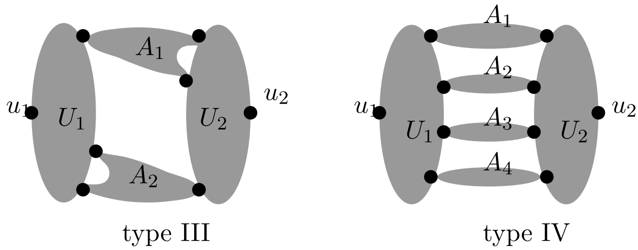

Fig. 2. Obstructions of types III and IV.

The following result is proved by Ma, Xie and Yu in [7], which gives a characterization of feasible quadruples.

Lemma 5.1. *Let (G, u_1, u_2, A) be a quadruple. Then one of the following holds.*

- *(i)* *(G, u_1, u_2, A) is feasible.*
- *(ii)* *G has a separation (K, L) such that $|V(K \cap L)| \leq 2$ and for some $i \in [2]$, $u_i \in V(K - L)$ and $A \cup \{u_{3-i}\} \subseteq V(L)$.*
- *(iii)* *G has a separation (K, L) such that $|V(K \cap L)| \leq 4$, $u_1, u_2 \in V(K - L)$, and $A \subseteq V(L)$.*
- *(iv)* *(G, u_1, u_2, A) is an obstruction of type I, or type II, or type III, or type IV.*

It is easy to verify that if G is $(4, A)$-connected and has no separation as in (iii) of Lemma 5.1, then (G, u_1, u_2, A) is not feasible if, and only if, it is an obstruction of type IV. Moreover, in the case when G is $(4, A)$-connected and (G, u_1, u_2, A) is an obstruction of type IV, it is straightforward to verify that for any $i \in [2]$ and $a \in A$, G contains four independent paths: one from u_i to a, and three from u_{3-i} to the three vertices in $A - \{a\}$, respectively.

Proof of Theorem 1.2. Let G be a 5-connected graph and (G_1, G_2) be a 5-separation in G such that $|V(G_i)| \geq 7$ for $i \in [2]$. Let $V(G_1 \cap G_2) = \{a, a_1, a_2, a_3, a_4\}$ such that $G[\{a, a_1, a_2\}] \cong K_3$. We wish to prove (iv); so let $u_1, u_2, u_3 \in N(a) - \{a_1, a_2\}$ be pairwise distinct and $G' := G - \{av : v \notin \{a_1, a_2, u_1, u_2, u_3\}\}$. Without loss of generality, we may assume that $u_1 \in V(G_2) - V(G_1)$.

Claim 1. We may assume $u_2, u_3 \in V(G_2) - V(G_1)$.

Since G is 5-connected, $G_2 - a$ has independent paths P_1, P_2, P_3, P_4 from u_1 to a_1, a_2, a_3, a_4, respectively.

First, suppose $u_2 \notin V(G_2)$; the case when $u_3 \notin V(G_2)$ can be treated in the same way. Since G is 5-connected, $G_1 - a$ has independent paths Q_1, Q_2, Q_3, Q_4 from u_2 to a_1, a_2, a_3, a_4, respectively. Then $G[\{a, a_1, a_2\}] \cup u_1a \cup P_1 \cup P_2 \cup u_2a \cup Q_1 \cup Q_2 \cup (P_3 \cup Q_3)$ is a TK_5 in G' with branch vertices a, a_1, a_2, u_1, u_2.

Next, assume that $u_2 \in \{a_3, a_4\}$; the case when $u_3 \in \{a_3, a_4\}$ is the same. Without loss of generality, we may assume that $u_2 = a_3$. Let $b \in V(G_1) - V(G_2)$. Since G

is 5-connected, $G_1 - a$ has independent paths Q_1, Q_2, Q_3, Q_4 from b to a_1, a_2, a_3, a_4, respectively. Now $G[\{a, a_1, a_2\}] \cup u_1a \cup P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup (Q_3 \cup a_3a) \cup (P_4 \cup Q_4)$ is a TK_5 in G' with branch vertices a, a_1, a_2, b, u_1. Thus we have Claim 1.

Before proceeding with the proof, we introduce additional notation. If there is a 4-separation (K, L) in $G_2 - a$ such that $u_1, u_2 \in V(K - L)$ and $A := \{a_1, a_2, a_3, a_4\} \subseteq V(L)$, we choose such a separation that K is minimal. If such a 4-separation does not exist, we let $K = G_2$, $V(L) = A$ and $E(L) = \emptyset$. Let $V(K \cap L) = \{y_1, y_2, y_3, y_4\}$. Since G is 5-connected, we may assume that L contains disjoint paths Y_1, Y_2, Y_3, Y_4 from a_1, a_2, a_3, a_4 to y_1, y_2, y_3, y_4, respectively (by relabeling if necessary).

Claim 2. We may assume that $(K, u_1, u_2, \{y_1, y_2, y_3, y_4\})$ is not feasible.

For, suppose K has a TH rooted at $u_1, u_2, \{y_1, y_2, y_3, y_4\}$, say T. First, assume that there exists $i \in [2]$ such that $T - u_{3-i}$ has independent paths from u_i to y_1, y_2, respectively. If $G_1 - a$ has two disjoint paths P, Q from $\{a_3, a_4\}$ to $\{a_1, a_2\}$ then $G[\{a, a_1, a_2, u_1, u_2\}] \cup T \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup P \cup Q$ is a TK_5 in G' with branch vertices a, a_1, a_2, u_1, u_2. So assume such paths P, Q do not exist in $G_1 - a$. Then G_1 has a separation (G_1', G_2') such that $a \in V(G_1' \cap G_2')$, $|V(G_1' \cap G_2')| \leq 2$, $\{a_1, a_2\} \subseteq V(G_1')$, and $\{a_3, a_4\} \subseteq V(G_2')$. Since G is 5-connected, $|V(G_i')| = 4$ for $i \in [2]$. This forces $|V(G_1)| = 6$, a contradiction.

Thus, we may assume without loss of generality that $T - u_2$ has independent paths from u_1 to y_1, y_4, respectively. If $G_1 - a$ has disjoint paths P_1, P_2 from a_3, a_4 to a_1, a_2, respectively, then $G[\{a, a_1, a_2, u_1, u_2\}] \cup T \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup P_1 \cup P_2$ is a TK_5 in G', with branch vertices a, a_1, a_2, u_1, u_2. Hence, we may assume that such P_1, P_2 do not exist. Then by Lemma 3.1, $(G_1 - a, \{a_1, a_2, a_3, a_4\})$ is planar. Since $|V(G_i)| \geq 7$ for $i \in [2]$, it follows from Theorem 1.1 that (i), or (ii), or (iii) holds, completing the proof of Claim 2.

Since G is 5-connected, it follows from the choice of (K, L), Claim 2 and Lemma 5.1 that $(K, u_1, u_2, \{y_1, y_2, y_3, y_4\})$ is an obstruction of type IV. Thus, K has four independent paths R_1, R_2, R_3, R_4, with R_1, R_2, R_3 from u_1 to y_1, y_2, y_3, respectively, and R_4 from u_2 to y_4. Let $b \in V(G_1) - V(G_2)$. Since G is 5-connected, $G_1 - a$ contains independent paths Q_1, Q_2, Q_3, Q_4 from b to a_1, a_2, a_3, a_4, respectively. Therefore, $G[\{a, a_1, a_2, u_1\}] \cup (R_1 \cup Y_1) \cup (R_2 \cup Y_2) \cup (R_3 \cup Y_3 \cup Q_3) \cup Q_1 \cup Q_2 \cup (au_2 \cup R_4 \cup Y_4 \cup Q_4)$ is a TK_5 in G' with branch vertices a, a_1, a_2, b, u_1. Hence, (iv) holds. $\square$

6. 6-Separations

In this section, we prove Theorem 1.4. We need the following result of Perfect [8].

Lemma 6.1. *Let G be a graph, $u \in V(G)$, and $A \subseteq V(G - u)$. Suppose there exist k independent paths from u to distinct $a_1, \ldots, a_k \in A$, respectively, and otherwise disjoint from A. Then for any $n \geq k$, if there exist n independent paths $P_1, \ldots, P_n$ in G from u*

to n distinct vertices in A and otherwise disjoint from A then $P_1, \ldots, P_n$ may be chosen so that $a_i \in V(P_i)$ for $i \in [k]$.

We also need the next lemma, which is due to Watkins and Mesner [13].

Lemma 6.2. *Let R be a 2-connected graph and let $y_1, y_2, y_3 \in V(R)$ be distinct. Then there is no cycle through y_1, y_2, y_3 in R if, and only if, one of the following statements holds.*

- *(i) There exists a 2-cut S in R and there exist pairwise disjoint subgraphs $D_{y_i} \subseteq R-S$, $i \in [3]$, such that $y_i \in V(D_{y_i})$ and each D_{y_i} is a union of components of $R - S$.*
- *(ii) There exist 2-cuts S_{y_i} of R and pairwise disjoint subgraphs D_{y_i} of R, $i \in [3]$, such that $y_i \in V(D_{y_i})$, each D_{y_i} is a union of components of $R - S_{y_i}$, $S_{y_1} \cap S_{y_2} \cap S_{y_3} = \{z\}$, and $S_{y_1} - \{z\}, S_{y_2} - \{z\}, S_{y_3} - \{z\}$ are pairwise disjoint.*
- *(iii) There exist pairwise disjoint 2-cuts S_{y_i} in R and pairwise disjoint subgraphs D_{y_i} of R such that $y_i \in V(D_{y_i})$, D_{y_i} is a union of components of $R-S_{y_i}$, and $R-V(D_{y_1} \cup D_{y_2} \cup D_{y_3})$ has precisely two components, each containing exactly one vertex from S_{y_i}.*

Proof of Theorem 1.4. Suppose $N(x) \cap V(G_1 - G_2) = \emptyset$. We proceed to show that one of (i)–(iv) holds. We may assume that

(1) $|V(G_i)| \geq 8$ for $i \in [2]$;

for, otherwise, G contains K_4^- (as G is 5-connected) and, hence, (ii) holds. Since G is 5-connected,

(2) $N(x_i) \cap V(G_1 - G_2) \neq \emptyset$ for $i \in [2]$, and $N(\{x_1, x_2\}) \cap V(G_2 - G_1) \neq \emptyset$.

Moreover, by the minimality of G_1 and by (1), we have

(3) $|N(v_i) \cap V(G_1 - G_2)| \geq 2$ for $i \in [3]$.

We may assume that

(4) $N(v_i) \cap V(B) \neq \emptyset$ for $i \in [3]$ and for any component B of $G_2 - G_1$.

For, suppose B is a component of $G_2 - G_1$ such that $N(v_i) \cap V(B) = \emptyset$ for some $i \in [3]$. If $|V(B)| = 1$ then G contains K_4^- (as G is 5-connected) and (ii) holds. So assume $|V(B)| \geq 2$. Let B' be obtained from B by adding $\{x, x_1, x_2, v_1, v_2, v_3\} - \{v_i\}$ and all edges of G from B to $\{x, x_1, x_2, v_1, v_2, v_3\} - \{v_i\}$. Now we may apply Theorem 1.2 to the separation $(G - B, B')$. If (i) or (ii) of Theorem 1.2 holds then we have (i) or (ii).

If (iii) of Theorem 1.2 holds then G contains K_4^-; so (ii) holds. Now assume (iv) of Theorem 1.2 holds. Then clearly, (iii) holds. So we may assume (4).

Let $a \in V(G_1 - G_2)$. Since G is 5-connected and $N(x) \cap V(G_1 - G_2) = \emptyset$, $G_1 - x$ has independent paths P_1, P_2, P_3, P_4, P_5 from a to x_1, x_2, v_1, v_2, v_3, respectively.

If $N(x) = \{x_1, x_2, v_1, v_2, v_3\}$, then let $b \in V(G_2 - G_1)$. Since G is 5-connected, $G_2 - x$ has independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from b to x_1, x_2, v_1, v_2, v_3, respectively. Now $G[\{x, x_1, x_2\}] \cup P_1 \cup P_2 \cup (P_3 \cup v_1x) \cup Q_1 \cup Q_2 \cup (Q_4 \cup v_2x) \cup (P_5 \cup Q_5)$ is TK_5 in G' with branch vertices a, b, x, x_1, x_2. Thus (iii) holds with $x_3 = v_1$, and $\{y_1, y_2\} = \{v_2, v_3\}$ (as $d(x) = 5$ in this case). So we may assume that

(5) $|N(x) \cap V(G_2 - G_1)| \geq 1$, and let $x_3 \in N(x) \cap V(G_2 - G_1)$.

We wish to prove that (iii) holds; so let $y_1, y_2 \in N(x) - \{x_1, x_2, x_3\}$ and $G' := G - \{xv : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}$.

Case 1. $N(x_i) \cap V(G_2 - G_1) \neq \emptyset$ for $i \in [2]$.

Claim 1. We may assume $\{y_1, y_2\} \nsubseteq \{v_1, v_2, v_3\}$.

For, suppose $\{y_1, y_2\} \subseteq \{v_1, v_2, v_3\}$. Without loss of generality, let $y_i = v_i$ for $i \in [2]$. Let X denote the component of $G_2 - G_1$ containing x_3.

First, suppose $N(x_i) \cap V(X) \neq \emptyset$ for $i \in [2]$. Then $G[X + \{x_1, x_2\}] - x_1x_2$ contains a path P from x_1 to x_2, and a path from x_3 to some vertex $b \in V(P) - \{x_1, x_2\}$ (where $b = x_3$ if $x_3 \in V(P)$) and internally disjoint from P; so $G[X + \{x_1, x_2\}]$ has independent paths from b to x_1, x_2, x_3, respectively. By Lemma 6.1, $G[X + \{x_1, x_2, v_1, v_2, v_3\}]$ has independent paths Q_1, Q_2, Q_3, Q_4 from b to x_1, x_2, x_3, v_j (for some $j \in [3]$), respectively, and internally disjoint from $\{v_1, v_2, v_3\}$. By symmetry between $v_1 = y_1$ and $v_2 = y_2$, we may assume $j \in \{2, 3\}$. Let $Q = Q_4 \cup P_4$ if $j = 2$, and $Q = Q_4 \cup P_5$ if $j = 3$. Now $G[\{x, x_1, x_2\}] \cup P_1 \cup P_2 \cup (P_3 \cup v_1x) \cup Q \cup Q_1 \cup Q_2 \cup (Q_3 \cup x_3x)$ is a TK_5 in G' with branch vertices a, b, x, x_1, x_2.

Hence, we may assume by symmetry that $N(x_1) \cap V(X) = \emptyset$. Since G is 5-connected, $G[X + \{x_2, v_1, v_2, v_3\}]$ has independent paths Q_1, Q_2, Q_3, Q_4 from x_3 to x_2, v_1, v_2, v_3, respectively. Since $N(x_1) \cap V(G_2 - G_1) \neq \emptyset$, $G_2 - G_1$ has a component X' such that $N(x_1) \cap V(X') \neq \emptyset$. Then by (4), $G[X' + \{x_1, v_3\}]$ contains a path Q from x_1 to v_3. So $G[\{x, x_1, x_2\}] \cup P_1 \cup P_2 \cup (P_3 \cup v_1x) \cup (P_4 \cup Q_3) \cup Q_1 \cup (Q_4 \cup Q) \cup x_3x$ is a TK_5 in G' with branch vertices a, x, x_1, x_2, x_3.

By Claim 1, we may let $y_1 \in V(G_2 - G_1)$. For convenience, let $u_1 = x_3$ and $u_2 = y_1$.

Claim 2. We may assume that for any $i \in [3]$, there do not exist $w_1, w_2 \in V(G_2 - G_1)$ and two disjoint paths W_1, W_2 in $G_2 - G_1$ from w_1, w_2 to u_1, u_2, respectively, such that $(G_2 - x) - ((W_1 - w_1) \cup (W_2 - w_2))$ has a TH rooted at $w_1, w_2, \{x_1, x_2\} \cup (\{v_1, v_2, v_3\} - \{v_i\})$.

Suppose otherwise and, without loss of generality, assume that there exist $w_1, w_2 \in V(G_2 - G_1)$, paths W_1, W_2 in $G_2 - G_1$, and a TH in $(G_2 - x) - ((W_1 - w_1) \cup (W_2 - w_2))$ rooted at $w_1, w_2, \{x_1, x_2, v_1, v_2\}$. Let K denote the union of W_1, W_2 and that TH.

We may assume that $(G_1 - x) - v_3$ contains disjoint paths X_1, X_2 from x_1, x_2 to v_1, v_2, respectively, and disjoint paths X_1', X_2' from x_1, x_2 to v_2, v_1, respectively. For, otherwise, since $(G_1 - x) - v_3$ is $(4, \{x_1, x_2, v_1, v_2\})$-connected, $((G_1 - x) - v_3, \{x_1, x_2, v_1, v_2\})$ is planar by Lemma 3.1. Hence by (1), (i) or (ii) follows from Theorem 1.1 (as (iii) of Theorem 1.1 implies that G contains K_4^-).

Now $G[\{x, x_1, x_2, u_1, u_2\}] \cup K \cup X_1 \cup X_2$ or $G[\{x, x_1, x_2, u_1, u_2\}] \cup K \cup X_1' \cup X_2'$ is a TK_5 in G' with branch vertices w_1, w_2, x, x_1, x_2. This completes the proof of Claim 2.

We now set up some notation. Let $G_2' = (G_2 - x) + \{v_3v_2, v_3v_1\}$. If there exists a separation (K, L) in G_2' such that $|V(K \cap L)| \leq 4$, $u_1, u_2 \in V(K - L)$, and $\{v_1, v_2, x_1, x_2\} \subseteq V(L)$, we choose such (K, L) that $v_3 \in V(L)$ whenever possible and, subject to this, K is minimal. If such a separation does not exist we let $K = G_2'$, $V(L) = \{v_1, v_2, x_1, x_2\}$, and $E(L) = \emptyset$. Note that if $L \not\subseteq K$ then $\{v_1, v_2, x_1, x_2\} \neq V(K \cap L)$; for, otherwise, $K = G_2'$ by (4), a contradiction. Since $v_3v_1, v_3v_2 \in E(G_2')$, if $v_3 \notin V(L)$ then $v_1, v_2 \in V(K \cap L)$. Let $V(K \cap L) = \{s_1, s_2, s_3, s_4\}$.

Claim 3. If $v_3 \notin V(L)$ then L has disjoint paths S_1, S_2, S_3, S_4 from s_1, s_2, s_3, s_4, respectively, to $\{v_1, v_2, x_1, x_2\}$; and if $v_3 \in V(L)$ then L has disjoint paths S_1, S_2, S_3, S_4 from s_1, s_2, s_3, s_4, respectively, to $\{v_1, v_2, x_1, x_2\}$, or $\{v_1, v_3, x_1, x_2\}$, or $\{v_2, v_3, x_1, x_2\}$, and internally disjoint from $\{v_1, v_2, v_3\}$.

Suppose L does not contain four disjoint paths from $\{s_1, s_2, s_3, s_4\}$ to $\{v_1, v_2, x_1, x_2\}$. Then L has a separation (L_1, L_2) such that $|V(L_1 \cap L_2)| \leq 3$, $\{v_1, v_2, x_1, x_2\} \subseteq V(L_1)$ and $\{s_1, s_2, s_3, s_4\} \subseteq V(L_2)$. Note $v_3 \notin V(L_1)$, or else $\{x\} \cup V(L_1 \cap L_2)$ would be a cut in G of size at most 4. Moreover, $|V(L_1 \cap L_2)| = 3$, as otherwise $\{x, v_3\} \cup V(L_1 \cap L_2)$ would be a cut in G of size at most 4. Hence, since $v_3v_1, v_3v_2 \in E(G_2')$, we have $v_1, v_2 \in V(L_1 \cap L_2)$.

First, assume $v_3 \notin V(L_2)$. Then $v_3 \in V(K - L)$; but then $V(L_1 \cap L_2) \cup \{v_3\}$ is a cut in G of size 4 and separating $\{u_1, u_2\}$ from $\{x_1, x_2, v_1, v_2, v_3\}$, contradicting the choice of (K, L) (that $v_3 \in V(L)$ whenever possible).

So $v_3 \in V(L_2)$. Let $V(L_1 \cap L_2) = \{v_1, v_2, t\}$. Since G is 5-connected, L_2 has four disjoint paths S_1', S_2', S_3', S_4' from s_1, s_2, s_3, s_4, respectively, to $\{t, v_1, v_2, v_3\}$. If $L_1 - v_2$ has disjoint paths X_1, X_2 from $\{t, v_1\}$ to $\{x_1, x_2\}$, then S_1', S_2', S_3', S_4' and X_1, X_2 form four disjoint paths in L from $\{s_1, s_2, s_3, s_4\}$ to $\{v_2, v_3, x_1, x_2\}$ which give the desired S_1, S_2, S_3, S_4 after appropriate labeling. So assume that such X_1, X_2 do not exist. Then L_1 has a separation (L_{11}, L_{12}) such that $v_2 \in V(L_{11} \cap L_{12})$, $|V(L_{11} \cap L_{12})| \leq 2$, $\{x_1, x_2\} \subseteq V(L_{11})$ and $\{t, v_1\} \subseteq V(L_{12})$. Since $N(x_i) \cap V(G_2 - G_1) \neq \emptyset$ for $i \in [2]$ (as we are in Case 1), $V(L_{11}) \neq \{v_2, x_1, x_2\}$. If $|V(L_{11} - v_2)| = 3$ then $G[V(L_{11} - v_2) \cup \{x\}]$ contains K_4^-, and (ii) holds. So we may assume $|V(L_{11} - v_2)| \geq 4$. Then $V(L_{11} \cap L_{12}) \cup \{x, x_1, x_2\}$ is a 5-cut in G. Hence, G has a 5-separation (H_1, H_2) such that $V(H_1 \cap H_2) = V(L_{11} \cap L_{12}) \cup \{x, x_1, x_2\}$, $L_{11} \subseteq H_1$, $L_{12} \cup L_2 \cup K \cup G_1 \subseteq H_2$. Clearly, $|V(H_2)| \geq 7$ and $|V(H_1)| \geq 6$. If $|V(H_1)| = 6$ then G contains K_4^- (as G is 5-connected); so (ii) holds. If $|V(H_1)| \geq 7$ then we may apply Theorem 1.2. Since the conclusions of Theorem 1.4 follows from those of Theorem 1.2, we have Claim 3.

Note that a TH in K rooted at $u_1, u_2, \{s_1, s_2, s_3, s_4\}$ and the paths in Claim 3 would give a contradiction to Claim 2. Hence, $(K, u_1, u_2, \{s_1, s_2, s_3, s_4\})$ is not feasible. So by the minimality of K and the 5-connectedness of G, it follows from Lemma 5.1 that $(K, u_1, u_2, \{s_1, s_2, s_3, s_4\})$ is an obstruction of type I, or type II, or type III, or type IV. Let $U_1, U_2, A_i, A_{[k]}$ be defined as in Section 5 with $\{s_1, s_2, s_3, s_4\}$ as A.

Claim 4. We may assume $(K, u_1, u_2, \{s_1, s_2, s_3, s_4\})$ is an obstruction of type IV.

If $(K, u_1, u_2, \{s_1, s_2, s_3, s_4\})$ is an obstruction of type II or type III, then by symmetry we may assume that $v_3 \notin V(U_2) - A_{[k]}$; now $V(U_2 \cap A_{[k]}) \cup \{x\}$ is a 4-cut in G, a contradiction.

So assume $(K, u_1, u_2, \{s_1, s_2, s_3, s_4\})$ is an obstruction of type I. Then $v_3 \in V(U_2) - A_{[k]}$, as otherwise $V(U_2 \cap A_{[k]}) \cup \{x\}$ would be a 4-cut in G. Since $v_3v_1, v_3v_2 \in E(G_2')$, we may assume $V(A_1) = \{v_1\}$ and $V(A_2) = \{v_2\}$. Without loss of generality, assume $s_1 = v_1$ and $s_2 = v_2$. Let $V(A_3 \cap U_2) = \{t_2\}$ and $V(A_3 \cap U_1) = \{t_{11}, t_{12}\}$.

We may assume that A_3 contains two disjoint paths X_1, X_2 from $\{s_3, s_4\}$ to $\{t_{11}, t_{12}, t_2\}$. For, otherwise, A_3 has a separation (A_{31}, A_{32}) such that $|V(A_{31} \cap A_{32})| \le 1$, $\{s_3, s_4\} \subseteq V(A_{31})$, and $\{t_{11}, t_{12}, t_2\} \subseteq V(A_{32})$. Then $V(A_{31} \cap A_{32}) \cup \{v_1, v_2, x\}$ is a cut in G of size at most 4, a contradiction.

Since G is 5-connected, U_1 has independent paths $Q_1^1, Q_2^1, Q_3^1, Q_4^1$ from u_1 to v_1, v_2, t_{11}, t_{12}, respectively, and U_2 has four independent paths $Q_1^2, Q_2^2, Q_3^2, Q_4^2$ from u_2 to v_1, v_2, v_3, t_2, respectively.

If both X_1, X_2 end in $\{t_{11}, t_{12}\}$ then $(Q_1^1 \cup Q_1^2) \cup (Q_3^1 \cup Q_4^1 \cup X_1 \cup X_2) \cup Q_2^2 \cup Q_3^2 \cup S_3 \cup S_4$ is a TH in $G_2 - x$ rooted at $u_1, u_2, \{x_1, x_2, v_2, v_3\}$, contradicting Claim 2. So assume $t_{11} \in V(X_1)$ and $t_2 \in V(X_2)$. Then $(Q_1^1 \cup Q_1^2) \cup Q_2^1 \cup (Q_3^1 \cup X_1) \cup Q_3^2 \cup (Q_4^2 \cup X_2) \cup S_3 \cup S_4$ is a TH in $G_2 - x$ rooted at $u_1, u_2, \{x_1, x_2, v_2, v_3\}$, contradicting Claim 2 and completing the proof of Claim 4.

By Claim 4, we may assume $s_i \in V(A_i)$ for $i \in [4]$. Let $V(A_i \cap U_1) = \{r_i\}$ and $V(A_i \cap U_2) = \{t_i\}$ for $i \in [4]$.

Subcase 1.1. $v_3 \in V(K - L)$.

Then $v_1, v_2 \in \{s_1, s_2, s_3, s_4\}$, and without loss of generality, assume $v_1 = s_3$ and $v_2 = s_4$. Since $v_3v_1, v_3v_2 \in E(G_2')$, we may assume by symmetry that $v_3 \in V(A_3 \cup U_2)$. Then $V(A_4) = \{v_2\}$ and $r_4 = s_4 = t_4 = v_2$. Since G is 5-connected, U_1 has independent paths R_1, R_2, R_3, R_4 from u_1 to r_1, r_2, r_3, r_4, respectively. Let X_1 be a path in A_1 from r_1 to s_1, and X_2 be a path in A_2 from t_2 to s_2.

Suppose $v_3 \in V(A_3)$. Then $V(A_3) \ne \{v_1\}$. We may assume that A_3 has disjoint paths R, R' from r_3, t_3, respectively, to $\{v_1, v_3\}$. For otherwise, A_3 has a separation (A_{31}, A_{32}) such that $|V(A_{31} \cap A_{32})| \le 1$, $\{r_3, t_3\} \subseteq V(A_{31})$ and $\{v_1, v_3\} \subseteq V(A_{32})$. Then $V(A_{31} \cap A_{32}) \cup \{s_1, s_2, s_4\}$ is a cut in K separating $\{u_1, u_2\}$ from $\{s_1, s_2, s_3, s_4\}$, contradicting the minimality of K. Since G is 5-connected, U_2 has independent paths T_1, T_2, T_3, T_4 from u_2 to t_1, t_2, t_3, t_4, respectively. Then $(R_4 \cup T_4) \cup (R_1 \cup X_1) \cup (R_3 \cup R) \cup (T_2 \cup X_2) \cup (T_3 \cup R')$ is a TH in K rooted at $u_1, u_2, \{s_1, s_2, v_1, v_3\}$. This TH and the paths S_1, S_2 in Claim 3 (from

s_1, s_2 to $\{x_1, x_2\}$) form a TH in $G_2 - x$ rooted at $u_1, u_2, \{x_1, x_2, v_1, v_3\}$, contradicting Claim 2.

Thus, $v_3 \in V(U_2 - A_3)$. Then $V(A_3) = \{v_1\}$ and $r_3 = t_3 = v_1 = s_3$. Suppose $U_2 - \{t_1, t_2, t_3, t_4\}$ contains a path from u_2 to v_3. Since G is 5-connected, U_2 has four independent paths from u_2 to $\{v_3, t_1, t_2, t_3, t_4\}$ with only u_2 in common. So by Lemma 6.1, U_2 contains three independent paths Q_1, Q_2, Q_3 from u_2 to t_p, t_q, v_3, respectively, such that $p \in \{1, 2\}$, $q \in \{3, 4\}$, and disjoint from $\{t_1, t_2, t_3, t_4\} - \{t_p, t_q\}$. Without loss of generality, we may assume $p = 2$ and $q = 3$. Then $(R_3 \cup Q_2) \cup Q_3 \cup (Q_1 \cup X_2) \cup (R_1 \cup X_1) \cup R_4$ is a TH in K rooted at $u_1, u_2, \{s_1, s_2, v_2, v_3\}$. This TH and the paths S_1, S_2 in Claim 3 (from s_1, s_2 to $\{x_1, x_2\}$) form a TH in $G_2 - x$ rooted at $u_1, u_2, \{x_1, x_2, v_2, v_3\}$, contradicting Claim 2.

Hence, $U_2 - \{t_1, t_2, t_3, t_4\}$ contains no path from u_2 to v_3. Then U_2 has a separation (U_{21}, U_{22}) such that $V(U_{21} \cap U_{22}) = \{t_1, t_2, t_3, t_4\}$, $v_3 \in V(U_{21})$ and $u_2 \in V(U_{22})$. By the minimality of K, $K - \{s_1, s_2, s_3, s_4\}$ must have a path from v_3 to u_2. Thus we may assume that $t_1 \neq s_1$ and $G[V(U_{21})] - \{t_2, t_3, t_4\}$ contains a path Q from v_3 to t_1. Let Q_1, Q_2, Q_3, Q_4 be independent paths in U_{22} from u_2 to t_1, t_2, t_3, t_4, respectively. Then $(Q_3 \cup R_3) \cup (Q_1 \cup Q) \cup (Q_2 \cup X_2) \cup (R_1 \cup X_1) \cup R_4$ is a TH in K rooted at $u_1, u_2, \{s_1, s_2, v_2, v_3\}$. This TH and the paths S_1, S_2 in Claim 3 (from s_1, s_2 to $\{x_1, x_2\}$) give a TH in $G_2 - x$ rooted at $u_1, u_2, \{x_1, x_2, v_2, v_3\}$, contradicting Claim 2.

Subcase 1.2. $v_3 \notin V(K - L)$.

Since G is 5-connected, $|V(A_i)| = 1$ or $|V(A_i)| = 3$ for $i \in [4]$; as otherwise $\{r_i, s_i, t_i, x\}$ would be a cut in G of size 4. Moreover, by the choice of (K, L) (the minimality of K), if $|V(A_i)| = 3$ then $r_i s_i t_i$ is a path in G. Thus, since G is 5-connected, K contains eight independent paths R_1, R_2, R_3, R_4 from u_1 to s_1, s_2, s_3, s_4, respectively, and Q_1, Q_2, Q_3, Q_4 from u_2 to s_1, s_2, s_3, s_4, respectively. Let S_1, S_2, S_3, S_4 be the paths in Claim 3 with $s_i \in S_i$ for $i \in [4]$.

Suppose for some $i \in [4]$ and $j \in [2]$, s_i has at least two neighbors in U_j. Without loss of generality, assume $i = j = 1$. Then, $r_1 = s_1 = t_1$. Since G is 5-connected, $U_1 - \{r_2, r_3, r_4\}$ has a path from s_1 to u_1. Moreover, U_1 has two independent paths from s_1 to $\{u_1, r_2, r_3, r_4\}$ with only s_1 in common. For, otherwise, U_1 has a separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| \leq 1$, $s_1 \in V(U_{11} - U_{12})$, and $\{u_1, r_2, r_3, r_4\} \subseteq V(U_{12})$. Now $|V(U_{11})| \geq 3$, as s_1 has at least two neighbors in U_1. Hence, $V(U_{11} \cap U_{12}) \cup \{x, s_1\}$ is a cut in G of size at most 3, a contradiction. So by Lemma 6.1, U_1 has two independent paths R'_1, R'_2 from s_1 to u_1, r_p, respectively, with $p \in \{2, 3, 4\}$, and internally disjoint from $\{r_2, r_3, r_4\}$. Without loss of generality, we may assume $p = 2$. Now $Q_1 \cup (Q_3 \cup S_3) \cup (Q_4 \cup S_4) \cup S_1 \cup (R'_2 \cup r_2 s_2 \cup S_2)$ form a TH in $(G_2 - x) - (R'_1 - s_1)$ rooted at $s_1, u_2, \{x_1, x_2, v_1, v_2\}$, or $s_1, u_2, \{x_1, x_2, v_1, v_3\}$, or $s_1, u_2, \{x_1, x_2, v_2, v_3\}$. However, this contradicts Claim 2.

Thus, we may assume that no such i, j exist. Then each s_i has at least two neighbors in $L - \{s_1, s_2, s_3, s_4\}$, unless $s_i \in \{x_1, x_2, v_1, v_2, v_3\}$.

Suppose there exists some $s_i \notin \{x_1, x_2, v_1, v_2, v_3\}$, say $i = 1$. Let t be the end of S_1 other than s_1. By the similar argument as above, L has two independent paths from

s_1 to $\{t, v_1, v_2, v_3\} \cup V(S_2 \cup S_3 \cup S_4)$. Then by Lemma 6.1, there exist two independent paths S_1', R from s_1 to t, r, respectively, with $r \in \{v_1, v_2, v_3\} \cup V(S_2 \cup S_3 \cup S_4)$ and internally disjoint from $\{v_1, v_2, v_3\} \cup V(S_2 \cup S_3 \cup S_4)$. If $r \in \{v_1, v_2, v_3\} - V(S_2 \cup S_3 \cup S_4)$ then $S_1' \cup R \cup Q_1 \cup (Q_3 \cup S_3) \cup (Q_4 \cup S_4)$ is a TH in $(G_2 - x) - (R_1 - s_1)$ rooted at $s_1, u_2, \{v_1, v_2, x_1, x_2\}$, or $s_1, u_2, \{v_1, v_3, x_1, x_2\}$, or $s_1, u_2, \{v_2, v_3, x_1, x_2\}$, contradicting Claim 2. Hence, without loss of generality, we may assume $r \in V(S_2)$ and let r' be the end of S_2 other than s_2. Now $S_1' \cup (R \cup rS_2r'))) \cup Q_1 \cup (Q_3 \cup S_3) \cup (Q_4 \cup S_4)$ is a TH in $(G_2 - x) - (R_1 - s_1)$ rooted at $s_1, u_2, \{v_1, v_2, x_1, x_2\}$, or $s_1, u_2, \{v_1, v_3, x_1, x_2\}$, or $s_1, u_2, \{v_2, v_3, x_1, x_2\}$. This contradicts Claim 2.

Hence, $\{s_1, s_2, s_3, s_4\} \subseteq \{x_1, x_2, v_1, v_2, v_3\}$. By (4) and by symmetry between x_1 and x_2, we may assume that $s_i = v_i$ for $i \in [3]$ and $s_4 = x_2$. Since $N(x_1) \cap V(G_2 - G_1) \neq \emptyset$ (as we are in Case 1), $N(x_1) \cap V(L - K) \neq \emptyset$. Thus, by (4), $G[L - K + \{x_1, v_1\}]$ has a path T from v_1 to x_1. If $(G_1 - x) - \{v_1, v_2\}$ has two independent paths T_1, T_2 from v_3 to x_1, x_2, respectively, then $G[\{x, x_1, x_2, u_1\}] \cup T_1 \cup T_2 \cup (Q_3 \cup u_2x) \cup (R_1 \cup T) \cup R_3 \cup R_4$ is a TK_5 in G' with branch vertices u_1, v_3, x, x_1, x_2. So assume such T_1, T_2 do not exist. Then $G_1 - x$ has a separation (G_1', G_1'') such that $|V(G_1' \cap G_1'')| \leq 3$, $\{v_1, v_2\} \subseteq V(G_1' \cap G_1'')$, $v_3 \in V(G_1')$ and $\{x_1, x_2\} \subseteq V(G_1'')$. Since $|N(v_3) \cap V(G_1 - G_2)| \geq 2$ by (3), $V(G_1' \cap G_1'') \cup \{v_3\}$ is a cut in G of size at most 4, a contradiction.

Case 2. $N(x_i) \cap V(G_2 - G_1) = \emptyset$ for some $i \in [2]$.

Without loss of generality, we may assume that $N(x_1) \cap V(G_2 - G_1) = \emptyset$.

Claim 1. We may assume $\{y_1, y_2\} \not\subseteq \{v_1, v_2, v_3\}$.

For, otherwise, we may assume $y_i = v_i$ for $i \in [2]$. Since G is 5-connected, $G_2 - \{x, x_1\}$ contains independent paths Q_1, Q_2, Q_3, Q_4 from x_3 to v_1, v_2, v_3, x_2, respectively. If $G_1 - \{x, x_2\}$ has a cycle C containing $\{x_1, v_1, v_2\}$, then $G[\{v_1, v_2, x, x_1, x_3\}] \cup C \cup (Q_4 \cup x_2x_1) \cup Q_1 \cup Q_2$ is a TK_5 in G' with branch vertices v_1, v_2, x, x_1, x_3. So we may assume that such a cycle C does not exist. We wish to apply Lemma 6.2.

First, we show that $G_1 - \{x, x_2\}$ is 2-connected. Note that $G_1 - G_2$ is connected by the minimality of G_1. Suppose $G_1 - \{x, x_2\}$ has a cut vertex, say v. Then $v \in V(G_1 - G_2)$, and $G_1 - \{x, x_2\}$ has a separation (G_{11}, G_{12}) such that $V(G_{11} \cap G_{12}) = \{v\}$, $x_1 \in V(G_{11})$ and $V(G_{12}) \cap \{v_1, v_2, v_3\} \neq \emptyset$. But this implies that $V(G_{12}) - \{v_1, v_2, v_3\} = \{v\}$ or $\{v_1, v_2, v_3\} \subseteq V(G_{12})$ (as G is 5-connected and $N(x) \cap V(G_1 - G_2) = \emptyset$). If $V(G_{12}) - \{v_1, v_2, v_3\} = \{v\}$, then any vertex in $V(G_{12}) \cap \{v_1, v_2, v_3\}$ has at most one neighbor in $G_1 - G_2$, contradicting (3). So $\{v_1, v_2, v_3\} \subseteq V(G_{12})$. This implies $V(G_{11}) = \{x_1, v\}$ (as G is 5-connected). But now $N(x_1) \subseteq \{x, x_2, v\}$, which is a contradiction.

So (i) or (ii) or (iii) of Lemma 6.2 holds and, in either case, $G_1 - \{x, x_2\}$ has a 2-cut $\{z_1, z_2\}$ separating x_1 from $\{v_1, v_2\}$. Let D_{x_1} denote the component of $(G_1 - \{x, x_2\}) - \{z_1, z_2\}$ containing x_1. Then $|V(D_{x_1}) - \{x_1, v_3\}| \leq 1$; otherwise $S := \{x, x_1, x_2, z_1, z_2, v_3\}$ is a 6-cut in G such that $G - S$ has a component strictly contained in $G_1 - G_2$, contradicting the choice of (G_1, G_2) (that G_1 is minimal). Suppose $|V(D_{x_1}) - \{x_1, v_3\}| = 1$, and let $v \in V(D_{x_1}) - \{x_1, v_3\}$. Since G is 5-connected and $N(x) \cap V(G_1 - G_2) = \emptyset$, $N(v) = \{v_3, x_1, x_2, z_1, z_2\}$. So $G[\{v, x, x_1, x_2\}] \cong K_4^-$, and (ii) holds. Therefore, we may assume $V(D_{x_1}) \subseteq \{x_1, v_3\}$. Since $N(x_1) \cap V(G_2 - G_1) = \emptyset$, $N(x_1) = \{x, x_2, z_1, z_2, v_3\}$. Hence, $v_3 \in V(D_{x_1})$. Since $|N(v_3) \cap V(G_1 - G_2)| \geq 2$, we have $z_1, z_2 \in N(v_3)$. Therefore, $G[\{x_1, v_3, z_1, z_2\}]$ contains K_4^-, and (ii) holds.

By Claim 1, let $y_1 \in V(G_2 - G_1)$. For convenience, let $u_1 := x_3$ and $u_2 := y_1$.

Claim 2. We may assume that there exist $w_1, w_2 \in V(G_2 - G_1)$ and two disjoint paths W_1, W_2 in $G_2 - G_1$ from w_1, w_2 to u_1, u_2, respectively, such that $(G_2 - \{x, x_1\}) - ((W_1 - w_1) \cup (W_2 - w_2))$ has a TH rooted at $w_1, w_2, \{x_2, v_1, v_2, v_3\}$.

First, we set up some notation. If $G_2 - \{x, x_1\}$ has a separation (K, L) such that $|V(K \cap L)| \leq 4$, $u_1, u_2 \in V(K - L)$ and $\{x_2, v_1, v_2, v_3\} \subseteq V(L)$ we select (K, L) so that K is minimal; in this case, since G is 5-connected, $|V(K \cap L)| = 4$, and we let $V(K \cap L) = \{s_1, s_2, s_3, s_4\}$. If such a separation (K, L) does not exist in $G_2 - \{x, x_1\}$, we let $K = G_2 - \{x, x_1\}$, $V(L) = \{s_1, s_2, s_3, s_4\} = \{x_2, v_1, v_2, v_3\}$, and $E(L) = \emptyset$.

Since G is 5-connected, L has four disjoint paths S_1, S_2, S_3, S_4 from s_1, s_2, s_3, s_4, respectively, to $\{x_2, v_1, v_2, v_3\}$. Thus, we may assume that K has no TH rooted at $u_1, u_2, \{s_1, s_2, s_3, s_4\}$; otherwise such TH and S_1, S_2, S_3, S_4 form a TH in $G_2 - \{x, x_1\}$ rooted at $u_1, u_2, \{x_2, v_1, v_2, v_3\}$ and, hence, Claim 2 holds by letting, for $i \in [2]$, $u_i = w_i$ and W_i be the trivial path consisting of w_i only.

Since G is 5-connected and by the choice of (K, L) and Lemma 5.1, $(K, u_1, u_2, \{s_1, s_2, s_3, s_4\})$ is an obstruction of type IV. (The detailed proof is the same as for Claim 4 in Case 1.) As before, we use the notation U_1, U_2 and A_i from Section 5, with $s_i \in V(A_i)$ for $i \in [4]$ and with $\{s_1, s_2, s_3, s_4\}$ as A. For $i \in [4]$, let $V(A_i \cap U_1) = \{r_i\}$ and $V(A_i \cap U_2) = \{t_i\}$. Since G is 5-connected, $|V(A_i)| = 1$ or $|V(A_i)| = 3$, and $r_i s_i t_i$ is a path in A_i; so K has eight independent paths $P_1^i, P_2^i, P_3^i, P_4^i$ from u_i to s_1, s_2, s_3, s_4, respectively.

Suppose $\{s_1, s_2, s_3, s_4\} = \{x_2, v_1, v_2, v_3\}$. Without loss of generality, let $s_i = v_i$ for $i \in [3]$, and $s_4 = x_2$. If $G[G_1 - \{x, x_2\}]$ has three independent paths P_1, P_2, P_3 from x_1 to v_1, v_2, v_3, respectively, then $G[\{u_1, u_2, x, x_1, x_2\}] \cup P_4^1 \cup P_4^2 \cup (P_3^1 \cup P_3^2) \cup (P_1 \cup P_1^1) \cup (P_2 \cup P_2^2)$ is a TK_5 in G' with branch vertices u_1, u_2, x, x_1, x_2. So assume that such paths do not exist. Then $G[G_1 - x]$ has a separation (G_1', G_1'') such that $x_2 \in V(G_1' \cap G_1'')$, $|V(G_1' \cap G_1'')| \leq 3$, $x_1 \in V(G_1')$ and $\{v_1, v_2, v_3\} \subseteq V(G_1'')$. Since $N(x_1) \cap V(G_2 - G_1) = \emptyset$, $V(G_1' \cap G_1'') \cup \{x_1\}$ is a cut in G of size at most 4, a contradiction.

Thus, we may assume without loss of generality that $s_1 \notin \{x_2, v_1, v_2, v_3\}$. Since G is 5-connected, s_1 has at least two neighbors in L, or at least two neighbors in U_i for some $i \in [2]$.

First, assume that s_1 has at least two neighbors in some U_i, say U_1 (by symmetry). Then $V(A_1) = \{s_1\}$ and $s_1 \in V(U_1 \cap U_2)$. We claim that U_1 has two independent paths from s_1 to two distinct vertices in $\{u_1, r_2, r_3, r_4\}$. For, otherwise, U_1 has a separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| \leq 1$, $s_1 \in V(U_{11})$ and $\{u_1, r_2, r_3, r_4\} \subseteq V(U_{12})$. Since s_1 has at least two neighbors in U_1, we see that $|V(U_{11})| \geq 3$. So $V(U_{11} \cap U_{12}) \cup \{x, s_1\}$ is cut in G of size at most 3, a contradiction. Since $U_1 - \{r_2, r_3, r_4\}$ has a path from s_1 to u_1, it follows from Lemma 6.1 that U_1 contains two independent paths Q_1, Q_2 from s_1 to u_1, r_i, respectively, for some $i \in \{2, 3, 4\}$, and disjoint from $\{r_2, r_3, r_4\} - \{r_i\}$. Without loss of generality, we may assume that Q_2 ends at r_2. Now $P_1^2 \cup (P_3^2 \cup S_3) \cup (P_4^2 \cup S_4) \cup (Q_2 \cup r_2s_2 \cup S_2) \cup S_1$ is a TH in $(G_2 - \{x, x_1\}) - (Q_1 - s_1)$ rooted at $s_1, u_2, \{x_2, v_1, v_2, v_3\}$. Hence Claim 2 holds with $w_1 = s_1$, $W_1 = Q_1$, $w_2 = u_2$, and $W_2 = w_2$.

So we may assume that s_1 has at least two neighbors in L. Let $s \in \{x_2, v_1, v_2, v_3\}$ be the end of S_1 other than s_1. Since s_1 has at least two neighbors in L and G is 5-connected, L has two independent paths S_1', S_2' from s_1 to $\{s\} \cup V(S_2 \cup S_3 \cup S_4)$ and internally disjoint from $\{s\} \cup S_2 \cup S_3 \cup S_4$. Thus by Lemma 6.1 (and the existence of S_1), L has independent paths S_1', S_2' from s_1 to s, s', respectively, and internally disjoint from $S_2 \cup S_3 \cup S_4$, with $s' \in V(S_2 \cup S_3 \cup S_4)$. Without loss of generality, we may further assume that S_2' ends at $s' \in V(S_2)$ and let t be the end of S_2 in $\{x_2, v_1, v_2, v_3\}$. Now $S_1' \cup (s_1S_2's' \cup s'S_2t \cup P_1^2 \cup (P_3^2 \cup S_3) \cup (P_4^2 \cup S_4)$ is a TH in $(G_2 - \{x, x_1\}) - (P_1^1 - s_1)$ rooted at $s_1, u_2, \{x_2, v_1, v_2, v_3\}$. Hence, Claim 2 holds with $w_1 = s_1$, $W_1 = P_1^1$, $w_2 = u_2$, and $W_2 = w_2$. This completes the proof of Claim 2.

By Claim 2, let Z denote the union of W_1, W_2 and a TH in $(G_2 - \{x, x_1\}) - ((W_1 - w_1) \cup (W_2 - w_2))$ rooted at $w_1, w_2, \{x_2, v_1, v_2, v_3\}$. Without loss of generality, we may assume that $Z - w_2$ has independent paths from w_1 to u_1, x_2, v_1, respectively, and $Z - w_1$ has independent paths from w_2 to u_2, v_2, v_3, respectively.

It remains to show that (iv) holds for $i = j = 1$ if we cannot find the desired TK_5 for (iii). Suppose $G_1 - x$ contains three independent paths, with one from x_2 to v_3 and two from x_1 to v_1, v_2, respectively, or with one from x_2 to v_2 and two from x_1 to v_1, v_3, respectively. It is easy to verify that these paths and Z form a TK_5 in G' with branch vertices w_1, w_2, x, x_1, x_2. So we may assume that any three independent paths from $\{x_1, x_2\}$ to v_1, v_2, v_3, respectively, with two from x_1 and one from x_2, must contain a path from x_2 to v_1. Hence (iv) holds. □

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