SPECIAL LAGRANGIAN SUBMANIFOLDS OF LOG CALABI–YAU MANIFOLDS

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To S.-T. Yau with admiration on the occasion of his 70th birthday

Abstract

We study the existence of special Lagrangian submanifolds of log Calabi–Yau manifolds equipped with the complete Ricci-flat Kähler metric constructed by Tian and Yau. We prove that if X is a Tian–Yau manifold and if the compact Calabi–Yau manifold at infinity admits a single special Lagrangian, then X admits infinitely many disjoint special Lagrangians. In complex dimension 2, we prove that if Y is a del Pezzo surface or a rational elliptic surface and $D \in |-K_Y|$ is a smooth divisor with $D^2 = d$, then $X = Y \setminus D$ admits a special Lagrangian torus fibration, as conjectured by Strominger–Yau–Zaslow and Auroux. In fact, we show that X admits twin special Lagrangian fibrations, confirming a prediction of Leung and Yau. In the special case that Y is a rational elliptic surface or $Y = \mathbb{P}^2$, we identify the singular fibers for generic data, thereby confirming two conjectures of Auroux. Finally, we prove that after a hyper-Kähler rotation, X can be compactified to the complement of a Kodaira type I_d fiber appearing as a singular fiber in a rational elliptic surface $\check{\pi} : \check{Y} \to \mathbb{P}^1$.

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1. Introduction

Mirror symmetry arose from physics as a mysterious duality between Hodge numbers of certain Calabi–Yau 3-folds Y, $\check{Y}$ (see, e.g., [18], [22]). Over the past thirty years, mirror symmetry has attracted intense interest from mathematicians. In 1994, Kontsevich [57] proposed that mirror symmetry could be explained as a certain duality between categories: the derived category of coherent sheaves $D^b\,\mathrm{Coh}(Y)$ on the one hand and the Fukaya category $\mathrm{Fuk}(\check{Y})$ on the other. This proposal has come to be known as *homological mirror symmetry*. Strominger, Yau, and Zaslow [82] proposed a geometric mechanism for mirror symmetry based on the prediction that, in certain limits, Calabi–Yau manifolds admit fibrations by special Lagrangian tori. Mirror symmetry is then obtained by fiberwise T-duality and the categories $D^b\,\mathrm{Coh}(Y)$, $\mathrm{Fuk}(\check{Y})$ are related by a real Fourier–Mukai transform (see [61]).

A fundamental difficulty in making progress on the Strominger–Yau–Zaslow (SYZ) proposal has been the dearth of special Lagrangian submanifolds in Calabi–Yau manifolds. Indeed, it is unknown whether a general Calabi–Yau manifold admits even a single special Lagrangian submanifold. In fact, the only examples of Calabi–Yau manifolds which are known to admit SYZ fibrations are essentially trivial: complex tori and hyper-Kähler manifolds with holomorphic fibrations by complex tori, where SYZ fibrations can be produced by a hyper-Kähler rotation. Nevertheless, these examples can be used to give nontrivial evidence for the SYZ picture. For example, when Y is an elliptically fibered $K3$ surface with twenty-four I_1 singular fibers, Gross and Wilson [44] used the hyper-Kähler rotation trick, together with a careful analysis of the Calabi–Yau metrics, to confirm the SYZ picture. We refer the reader to [42], [43], [71], [72], [88]–[90], [94] and references therein for related work.

In the case of noncompact manifolds, only slightly more is known. In special cases, Goldstein [35] and Gross [39] have constructed special Lagrangian fibrations using group actions. However, it is important to emphasize here that the symplectic form is *not* the Ricci-flat symplectic form.

Since Kontsevich's original proposal, mirror symmetry has been extended beyond the setting of Calabi–Yau manifolds thanks to the work Batryev [11], [12], Kontsevich [58], Givental [32]–[34], Hori–Vafa [52] and many others. When $-K_Y \to Y$ is effective, the mirror can no longer be compact and instead is expected to be a Landau–Ginzburg model $(\check{Y}, W)$ consisting of a noncompact Kähler manifold $\check{Y}$ and a holomorphic function W called the *superpotential*.

A beautiful proposal of Auroux [2] suggests that when $-K_Y$ is effective, the mirror of Y should be constructed by applying SYZ mirror symmetry to the complement $Y \setminus D$, where $D \in |-K_Y|$ is an anticanonical divisor; such pairs (Y, D) are usually referred to as *log Calabi–Yau manifolds*. Note that SYZ mirror symmetry makes sense since $Y \setminus D$ carries a nonvanishing holomorphic volume form with a simple pole along D. In particular, by the SYZ proposal, $Y \setminus D$ should admit a special Lagrangian torus fibration and the mirror $(\check{Y}, W)$ should be constructed from $Y \setminus D$ by T-duality along the fibers. Furthermore, the superpotential W is generated by the Floer theory of $Y \setminus D$. Auroux's proposal is in part inspired by Seidel's address to the International Congress of Mathematicians (see [75]), in which he explained how the Fukaya category of the complement of a hyperplane divisor in a projective Calabi–Yau manifold could be effectively understood.

We note that when Y is a projective surface and D is a singular nodal curve, Gross, Hacking, and Keel [40] constructed an algebraic mirror of $Y \setminus D$ using tropical techniques, along the lines of the Gross–Siebert program. We refer the reader to [41] and the references therein for an introduction to this active area of research.

In this paper, motivated by Auroux's work, we study the SYZ proposal for log Calabi–Yau manifolds (Y, D). Our main interest will be the existence of special Lagrangian submanifolds in $X = Y \setminus D$. We will consider the following two cases:

(I) Y is a Fano variety and $D \in |-K_Y|$ is a smooth divisor,

(II) Y admits a fibration $\pi : Y \to B$ onto a smooth algebraic curve with connected fibers and $D \in |-K_X|$ is a smooth fiber of π.

For example, the second case can be achieved by blowing up the basepoints of a pencil of anticanonical divisors in a Fano manifold. In each of these cases, it is a fundamental result of Tian and Yau [86], [87] that $X = Y \setminus D$ admits a complete Ricci-flat metric making $(X, \omega_{\mathrm{TY}})$ a complete noncompact Calabi–Yau manifold. Our first main theorem is the following.

THEOREM 1.1

Suppose that (Y, D) is a log Calabi–Yau manifold of type I or II. Suppose that the Calabi–Yau manifold D admits a smooth, immersed special Lagrangian $L \subset D$. Then $X = Y \setminus D$ equipped with the Tian–Yau metric admits a countable infinity of disjoint, immersed special Lagrangian submanifolds $\{\tilde{L}_i\}_{i\in\mathbb{N}}$ with topology $L \times S^1$ and having the following property: for each i, there is a sequence $j_i \to \infty$ such that $d(\tilde{L}_i, \tilde{L}_{j_i}) \to \infty$ as $j_i \to \infty$.

Remark 1.2

The final property is meant to emphasize that the countable infinity of special Lagrangians are not small deformations of a single special Lagrangian.

One way to view this result is as a lifting result for special Lagrangian submanifolds from Calabi–Yau manifolds of dimension $n - 1$ to Calabi–Yau manifolds of dimension n. For instance, there are several examples of projective Calabi–Yau manifolds admitting special Lagrangian submanifolds (see [17], [49], [55]), and it is well known that elliptically fibered $K3$ surfaces can be hyper-Kähler rotated to produce special Lagrangian fibrations, and in some examples hyper-Kähler rotation remains algebraic. By Theorem 1.1, we obtain noncompact Calabi–Yau 3-folds admitting many distinct special Lagrangian submanifolds. Conceivably, one could glue two such Calabi–Yau manifolds along infinity to obtain a compact Calabi–Yau manifold with finitely many distinct special Lagrangian submanifolds. If one were exceedingly lucky, this construction could be repeated to obtain special Lagrangian submanifolds of Calabi–Yau manifolds in higher and higher dimensions. We note that Talbot [83] has obtained gluing results of this type for noncompact, asymptotically cylindrical special Lagrangians in asymptotically cylindrical Calabi–Yau manifolds.

One case in which the existence of special Lagrangians in D is trivial is when Y has dimension 2, so that D is a torus. In this case, we obtain the following result which confirms the SYZ conjecture in this case.

THEOREM 1.3
Suppose that Y is a del Pezzo surface or a rational elliptic surface and that $D \in |-K_Y|$ is smooth. Then $X = Y \setminus D$ equipped with the Tian–Yau metric admits a special Lagrangian fibration $\pi : X \to \mathbb{R}^2$ with a section. Furthermore, near ∞, the fibers of π are contained in a neighborhood of D and the smooth fibers are topologically S^1-bundles over special Lagrangian submanifolds of D.

Theorem 1.3 resolves a conjecture of Auroux [4, Conjecture 5.1] in complex dimension 2. To our knowledge, this theorem produces the first examples of SYZ fibrations in the literature which are neither trivial, nor obtained by hyper-Kähler rotating from an existing holomorphic torus fibration (as in the case of $K3$ surfaces). As an application of this result we obtain the following.

COROLLARY 1.4 (Auroux [3, Conjecture 2.9])
Let D be a smooth cubic in $\mathbb{P}^2$. Then $X = \mathbb{P}^2 \setminus D$ admits a special Lagrangian fibration with respect to the Tian–Yau metric $\pi : X \to \mathbb{R}^2$. Furthermore, the fibration π has three singular fibers, each of which is a nodal special Lagrangian sphere, that is, of Kodaira type I_1.

Corollary 1.4 resolves a conjecture of Auroux [3, Conjecture 2.9]. Secondly (in the type II case), we obtain the following corollary, which resolves another conjecture of Auroux.

COROLLARY 1.5 (Auroux [3, Conjecture 2.10])
Let Y be a rational elliptic surface, and let $D \in |-K_Y|$ be a smooth divisor. Then

for any choice of Kähler class $[\omega]$ on Y, $X = Y \setminus D$ admits a special Lagrangian fibration $\pi : X \to \mathbb{R}^2$ with respect to the Tian–Yau metric. For generic $(Y, [\omega], D)$, this fibration has twelve singular fibers, each of which is a nodal special Lagrangian sphere.

Finally, we apply our results to mirror symmetry for del Pezzo surfaces and rational elliptic surfaces. Before stating our result, let us explain the context. At a homological level, mirror symmetry for del Pezzo surfaces and rational elliptic surfaces is quite well understood. In this setting, Auroux, Katzarkov, and Orlov [6] proved one direction of the mirror correspondence, namely, showing that the derived category of coherent sheaves on a del Pezzo surface Y is equivalent to the derived category of vanishing cycles of a certain elliptic fibration

$$W : \check{X} \to \mathbb{C}.$$

W here is the superpotential of the Landau–Ginzburg mirror of the del Pezzo surface Y. One of the key ideas in their work is that there is an elliptic fibration (in fact, a rational elliptic surface)

$$\overline{W} : \check{Y} \to \mathbb{P}^1$$

and that $\check{Y} \setminus \overline{W}^{-1}(\infty)$ is the fiberwise compactification of $W : \check{X} \to \mathbb{C}$. In fact, if Y_k is the del Pezzo surface obtained by blowing up $\mathbb{P}^2$ at $9-k$ points, then $\overline{W}^{-1}(\infty)$ is a singular fiber of the elliptic fibration consisting of k rational curves. This correspondence is constructed by hand, exploiting the relative flexibility of the symplectic category. Subsequently, Lunts and Przyjalkowski [64] showed that this construction gives mirror symmetry at the level of Hodge numbers, following a proposal by Katzarkov, Kontsevich, and Pantev [54]. Using a different approach motivated by the Doran–Harder–Thompson conjecture in [25], Doran and Thompson [26] showed that the mirror correspondence between del Pezzo complements and rational elliptic surfaces holds true at a lattice-theoretical level.

On a more historical note, it was originally thought that mirror symmetry for Calabi–Yau surfaces (and hyper-Kähler manifolds more generally) could be obtained by hyper-Kähler rotation (see [10], [16], [53]). It is now understood that this is not the case in general. Nevertheless, we prove the following result.

THEOREM 1.6
Let Y be a del Pezzo surface or rational elliptic surface, and let $D \in |-K_Y|$ be a smooth anticanonical divisor with $D^2 = d$. Let $X = Y \setminus D$, and equip X with the Ricci-flat Tian–Yau metric g_{TY}. Denote this complete noncompact Calabi–Yau manifold by (X, g_{TY}, J). Then for any choice of homology class $[\gamma] \in H_1(D, \mathbb{Z})$ repre-

sented by a special Lagrangian, (X, g_{TY}, J) *admits a special Lagrangian torus fibration* $\pi : X \to \mathbb{R}^2$ *with fibers topologically* $S^1 \times \gamma$*. We can perform a hyper-Kähler rotation to a complex structure* I *so that the fibration* $\pi : (X, g_{\mathrm{TY}}, I) \to \mathbb{C}$ *is holomorphic, with generic fiber an elliptic curve. Furthermore, we have* $(X, I) = \check{Y} \setminus \check{D}$*, where* $\check{\pi} : \check{Y} \to \mathbb{P}^1$ *is a rational elliptic surface and* $\check{D}$ *is a singular fiber of* $\check{\pi}$ *of Kodaira type* I_d*.*

It is important to remark that we do not know if the manifold $\check{Y} \setminus \check{D}$ is mirror in the sense of SYZ to X; the correct mirror obtained by torus duality may have a different complex structure. Nevertheless, it is in the correct family, as suggested by the results of [6], [26], [54], and [64].

Finally, we remark that Theorem 1.6 in fact produces many inequivalent special Lagrangian fibrations on X. Given the elliptic curve D, we can choose two distinct special Lagrangians γ, γ' intersecting at one point and generating $H_1(D, \mathbb{Z})$. Every such choice gives rise to a special Lagrangian fibration. Since the mirror of X is a rational elliptic surface, the existence of such *twin* special Lagrangian fibrations confirms a prediction of Leung and Yau [59], [60].

This article is structured as follows. In order to prove Theorem 1.1, we use the explicit form of the Tian–Yau metrics near infinity to construct *approximate* special Lagrangians in the asymptotic geometry. This construction proceeds in two steps. Fix a point $x_0 \in X$, and let $d(x_0, \cdot)$ be the distance to x_0 with respect to the Tian–Yau metric. In the first step, we construct explicit special Lagrangians L_R in the model geometry to which the Tian–Yau metrics converge. We find explicit bounds for the geometry of these special Lagrangians in terms of the parameter R, which roughly measures $d(x_0, L_R)$. Next, we transfer the special Lagrangians L_R to *approximate* special Lagrangians L'_R in the Tian–Yau manifolds, while maintaining precise control of the geometry of L'_R in terms of $d(x_0, L'_R)$.

The second step is to run the Lagrangian mean curvature flow (LMCF) in order to deform L'_R to a genuine special Lagrangian. In the type II case, the Tian–Yau metrics are asymptotically cylindrical and the geometry of the approximate Lagrangians L'_R as well as the Tian–Yau metric is uniformly controlled near infinity. This allows us to appeal to a theorem of Li [62], which in the current setting implies that for R sufficiently large, the LMCF starting at L'_R converges smoothly and exponentially fast to a special Lagrangian. In the type I case, the situation is substantially more difficult, as the geometry of L'_R as well as the Tian–Yau metric degenerates at infinity. Nevertheless, by exploiting the precise control of the geometry achieved in the construction of L'_R, we prove that the LMCF converges smoothly and exponentially fast to a special Lagrangian submanifold (see Theorem 4.23). These results occupy Sections 2, 3, and 4.

Next, we focus on the surface case. Using the deformation theory of special Lagrangians in [66], together with the theory of J-holomorphic curves and a hyper-Kähler rotation trick, we show that the existence of two disjoint immersed special Lagrangians representing the same primitive homology class infers the existence of a special Lagrangian fibration. Combining this result with Theorem 1.1, we obtain Theorem 1.3. Finally, in Section 6 we refine our results when Y is a del Pezzo surface or a rational elliptic surface. We prove Theorem 1.6 as well as identify the (generic) singular fibers of the special Lagrangian fibrations, as predicted by Auroux's conjectures (see Corollaries 1.4 and 1.5).

2. Perturbation of Lagrangians

In this section we collect together a few formulas for the variation of geometric quantities on a Lagrangian or, more generally, a submanifold M, under variations in the Riemannian metric. The primary application of these formulas will be in controlling the following perturbation problem. Suppose that $(X_{\text{mod}}, \omega_{\text{mod}}, J_{\text{mod}}, \Omega_{\text{mod}}, g_{\text{mod}})$ is a Calabi–Yau manifold (perhaps not complete), and suppose that $(X, \omega, J, \Omega, g)$ is a complete noncompact Calabi–Yau manifold with one end. Fix some point $p \in X$, and suppose that, for a large number $R < \infty$, there is a diffeomorphism Φ such that

$$X_{\text{mod}} \xrightarrow{\Phi} \{x \in X : d(p,x) > R\} \subset X$$

with the property that $\Phi^*\omega - \omega_{\text{mod}} = d\beta$ for some 1-form β. Suppose that $M_{\text{mod}} \subset X_{\text{mod}}$ is a special Lagrangian. The goal is to perturb M_{mod} to a submanifold M which is Lagrangian with respect to $\Phi^*\omega$, while maintaining control of the Riemannian geometry of M.

The natural way to accomplish this is via Moser's trick. Define a time-dependent family of symplectic forms $\omega_t = (1-t)\omega_{\text{mod}} + t\Phi^*\omega$ for $t \in [0,1]$, and define the time-dependent vector field V_t on X_{mod} via

$$i_{V_t}\omega_t = -\beta.$$

Let F_t be the time-dependent diffeomorphism generated by the flow of V_t, that is, defined by $\frac{dF_t}{dt} = V_t \circ F_t$ and $F_0 = \text{id}$. Then

$$\frac{d}{dt}F_t^*\omega_t := F_t^*\Big(\frac{d}{dt}\omega_t + \mathcal{L}_{V_t}\omega_t\Big).$$

Applying Cartan's formula and using that ω_t is closed gives

$$\frac{d}{dt}F_t^*\omega_t := F_t^*\big(\Phi^*\omega - \omega_{\text{mod}} + d(i_{V_t}\omega_t)\big) = F_t^*\big(d\beta + d(i_{V_t}\omega_t)\big) = 0.$$

From this we conclude that $F_t^*\omega_t = \omega_{\text{mod}}$. Setting $t = 1$ gives $F_1^*\Phi^*\omega = \omega_{\text{mod}}$. Thus, for any Lagrangian M_{mod} with respect to ω_{mod}, the image $F_1(M_{\text{mod}})$ will be a smooth

Lagrangian with respect to $\Phi^*\omega$. We can now transplant M_{mod} to a Lagrangian in X by taking $M := \Phi(F_1(M_{\text{mod}}))$.

To keep track of the Riemannian geometry throughout this process, we need to perturb the metrics. Since the flow F_t may not map J_{mod} to Φ^*J, the Riemannian structure is not naturally inherited from the flow of symplectic forms. Instead, we will consider the one-parameter family of metrics $\tilde{g}_t = (1-t)g_{\text{mod}} + t\Phi^*g$ for $t \in [0,1]$. Note that the geometry of $F_t(L_{\text{mod}})$ as a submanifold of $(X_{\text{mod}}, \tilde{g}_t)$ is just the same as the geometry of L_{mod} as a submanifold of X_{mod} equipped with the one-parameter metric $g_t = F_t^*\tilde{g}_t$, for $t \in [0,1]$. In particular, we are essentially reduced to understanding how various geometric quantities vary under changes in the metric. We begin with a simple lemma.

LEMMA 2.1

Let X be a Riemannian manifold. For $t \in [0,1]$, consider time-dependent Riemannian metrics g_t, and let V_t be a time-dependent vector field. Let F_t be the diffeomorphism of X defined by $\frac{\partial}{\partial t}F_t(p) = V_t(F_t(p))$, with $F_0(p) = p$. Then we have

$$\frac{\partial}{\partial t}F_t^*g_t = F_t^*\Big(\mathcal{L}_{V_t}g_t + \frac{\partial}{\partial t}g_t\Big).$$

In particular, we have

$$\Big|\frac{\partial}{\partial t}F_t^*g_t\Big|_{F_t^*g_t} \le 2|\nabla^t V_t|_{g_t} + \Big|\frac{\partial}{\partial t}g_t\Big|_{g_t},$$

where ∇^t denotes the covariant derivative with respect to g_t.

Proof

The formula for $\frac{\partial}{\partial t}F_t^*g_t$ is a straightforward computation. The estimate follows from the formula $\mathcal{L}_{V_t}g_t = 2g_t(\nabla_\cdot^t V_t, \cdot)$ and the observation that, for any tensor T, we have $|F_t^*T|_{F_t^*g_t} = |T|_{g_t}$. □

Next, we consider the variation of the second fundamental form of a submanifold $M \subset X$.

LEMMA 2.2

Let $M^k \subset X^{n+k}$ be a submanifold, and let g_t be a family of Riemannian metrics on X for $t \in (-\varepsilon, \varepsilon)$. Let A_t denote the second fundamental form of M in X. Then we have

$$\Big|\frac{\partial}{\partial t}|A_t|^2_{g_t}\Big| \le 10\big(|\nabla^t\partial_t g_t|_{g_t}|A_t|_{g_t} + |\partial_t g_t|_{g_t}|A_t|^2_{g_t}\big),$$

$$\Big|\frac{\partial}{\partial t}|H_t|^2_{g_t}\Big| \le 10\big(|\nabla^t\partial_t g_t|_{g_t}|H_t|_{g_t} + |\partial_t g_t|_{g_t}|H_t|^2_{g_t}\big).$$

Proof

The lemma follows immediately from the variational formula for the second fundamental form. We refer the reader to the appendix and Lemma A.1 for a complete proof. Let $\overline{X} = X \times (-\varepsilon, \varepsilon)$, and let $\bar{g} = g(t) + dt^2$. If $\overline{\nabla}$ denotes the covariant derivative of $\bar{g}$ on $\overline{X}$, then Lemma A.1 gives

$$\left|\frac{\partial}{\partial t}|A|^2_{g_t}\right| = 2\left|\langle A, \overline{\nabla}_{\partial_t} A\rangle\right| \leq 10\big(|A|_{g_t}|\nabla^t \partial_t g|_{g_t} + |A|^2_{g_t}|\partial_t g|_{g_t}\big)$$

and, similarly,

$$\left|\frac{\partial}{\partial t}|H|^2_{g_t}\right| \leq 10\big(|H|_{g_t}|\nabla^t \partial_t g|_{g_t} + |H|^2_{g_t}|\partial_t g|_{g_t}\big),$$

which is the desired result. □

Finally, we examine how the first nonzero eigenfunction of the Laplacian changes under a change in the metric.

LEMMA 2.3

Let M be a compact Riemannian manifold, and let g_t be a smooth family of Riemannian metrics for $t \in (-\varepsilon, \varepsilon)$. Let $\lambda_1(t)$ denote the first nonzero eigenvalue of the Laplacian on (M, g_t). Then we have

$$e^{-3\mu(t)}\lambda_1(0) \leq \lambda_1(t) \leq e^{3\mu(t)}\lambda_1(0),$$

where $\mu(t) = \int_0^t \sup_M |\partial_s g_s|_{g_s}\, ds$.

Proof

We define the eigenvalues of the Laplacian by $\Delta f + \lambda f = 0$. Recall that the first nonzero eigenvalue of the Laplacian on M is given by the Rayleigh quotient

$$\lambda_1(t) = \inf \frac{\int_M |\nabla^t f_t|^2_{g_t}\, d\mathrm{Vol}_t}{\int_M f_t^2\, d\mathrm{Vol}_t},$$

where the infimum is taken over all smooth functions with $\int_M f\, d\mathrm{Vol}_t = 0$. Given a function f such that $\int_M f\, d\mathrm{Vol}_0 = 0$, define

$$f_t = f - \frac{1}{\mathrm{Vol}(M, g_t)} \int_M f\, d\mathrm{Vol}_t,$$

and note that $df_t = df$ for all t. Then

$$\frac{\partial}{\partial t}\int_M |\nabla^t f_t|^2_{g_t}\, d\mathrm{Vol}_t = \int_M \Big(\frac{1}{2}(g_t^{ij}\partial_t g_{ij}) \cdot |\nabla^t f_t|^2_{g_t} - \langle \partial_t g, df_t \otimes df_t\rangle_{g_t}\Big)\, d\mathrm{Vol}_t$$

and so

$$\Big|\frac{\partial}{\partial t}\int_M |\nabla^t f|^2_{g_t}\, d\mathrm{Vol}_t\Big| \leq 2\sup_M |\partial_t g|_{g_t}\Big(\int_M |\nabla^t f_t|^2_{g_t}\, d\mathrm{Vol}_t\Big).$$

Similarly, we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_M f_t^2\, d\mathrm{Vol}_t &= \frac{1}{2}\int_M f_t^2(g_t^{ij}\partial_t g_{ij})\, d\mathrm{Vol}_t + 2\frac{\partial}{\partial t} f_t \int_M f_t\, d\mathrm{Vol}_t \\ &= \frac{1}{2}\int_M f_t^2(g_t^{ij}\partial_t g_{ij})\, d\mathrm{Vol}_t,\end{aligned}$$

where we used that $\frac{\partial}{\partial t} f_t$ is constant on M and $\int_M f_t\, d\mathrm{Vol}_t = 0$. Therefore,

$$\Big|\frac{\partial}{\partial t}\int_M f_t^2\, d\mathrm{Vol}_t\Big| \leq \sup_M |\partial_t g_t|_{g_t}\int_M f_t^2\, d\mathrm{Vol}_t.$$

Define

$$\mu(t) = \int_0^t \sup_M |\partial_s g_s|_{g_s}\, ds,$$

and denote the quotient by $F(t) := \frac{\int_M |\nabla^t f_t|^2_{g_t}\, d\mathrm{Vol}_t}{\int_M f_t^2\, d\mathrm{Vol}_t}$. The above inequalities imply that the function $e^{-3\mu(t)}F(t)$ is nonincreasing in time, while $e^{3\mu(t)}F(t)$ is nondecreasing. Thus, we conclude that

$$e^{-3\mu(t)}F(0) \leq F(t) \leq e^{3\mu(t)}F(0),$$

from which the result follows. □

3. Special Lagrangians in asymptotically cylindrical Calabi–Yau manifolds

We first turn to the case of asymptotically cylindrical Calabi–Yau manifolds and prove the existence of infinitely many special Lagrangian submanifolds assuming the existence of one special Lagrangian in the asymptotic cross section. This case is much simpler than what we prove in the subsequent section, although the basic idea is similar.

Definition 3.1

A complete Riemannian manifold (X, g) is called *asymptotically cylindrical* (ACyl) if there exists a compact subset $V \subset X$, a closed Riemannian manifold (Y, h), and a diffeomorphism $\Phi : \mathbb{R}^+ \times Y \to X \setminus V$ satisfying

$$\big|\nabla^k_{g_\infty}(\Phi^* g - g_\infty)\big|_{g_\infty} = O(e^{-\delta\ell}) \tag{3.1}$$

for some $\delta > 0$ and all $k \in \mathbb{N}$. The limiting metric is given by $g_\infty = d\ell^2 \oplus h$, where ℓ is the coordinate on $\mathbb{R}^+$.

Assume that $(X, \omega, J, \Omega, g)$ is a simply connected, irreducible ACyl Calabi–Yau manifold and thus Ricci-flat. By the Cheeger–Gromoll splitting theorem, unless X is a product cylinder, it can only have a single cylindrical end.

3.1. The model cylindrical geometry

We discuss the geometry of the limiting cylindrical end and construct a one-parameter family of special Lagrangians in this model space. Let $n = \dim_{\mathbb{C}} X$, and first assume that $n > 2$. In this case, Theorem B of [47] applies.

THEOREM 3.2 (Haskins, Hein, and Nordström [47])
Let X be a simply connected, irreducible ACyl Calabi–Yau manifold with $n > 2$. There exists a compact Calabi–Yau manifold D with a Kähler isometry ι of finite order m such that the cross section Y of X can be written as $Y = (S^1 \times D)/\iota$, where ι acts on the product via $\iota(\theta, x) = (\theta + \frac{2\pi}{m}, \iota(x))$. Moreover, ι preserves the holomorphic volume form Ω_D on D.

To construct the model cylinder, first consider $\tilde{X}_\infty := \mathbb{R}^+ \times S^1 \times D$ with the product complex structure J_∞ and Hermitian metric $g_\infty = d\ell^2 + d\theta^2 + g_D$. Here g_D is a Ricci-flat metric on D, $\theta \in S^1$ and $J_\infty(\frac{\partial}{\partial \ell}) = \frac{\partial}{\partial \theta}$. The associated Kähler form and holomorphic $(n, 0)$-form are given by

$$\omega_\infty = d\ell \wedge d\theta + \omega_D \qquad \text{and} \qquad \Omega_\infty = (d\ell + i\, d\theta) \wedge \Omega_D,$$

respectively. As in Theorem 3.2, the action of ι on D extends to $S^1 \times D$, and, furthermore, ω_∞ and Ω_∞ are fixed under this action. Thus both forms descend to the smooth Kähler manifold

$$X_\infty := \mathbb{R}^+ \times (S^1 \times D)/\iota,$$

which serves as the cylindrical model for the end of X.

We now construct our one-parameter family of special Lagrangians in X_∞. Assume that D admits a special Lagrangian submanifold $N \subset D$ which does not contain any fixed points of ι. Because ι is an isometry which preserves Ω_D, $\iota^k(N)$ is a special Lagrangian for $1 \le k \le m - 1$. For any $\rho \in \mathbb{R}^+$, consider the following union, which consists of m submanifolds with boundary sitting inside $\tilde{X}_\infty$:

$$\tilde{M}_\rho := \bigcup_{k=1}^{m} \{\rho\} \times \iota^{k-1}\Big(\Big[0, \frac{2\pi}{m}\Big] \times N\Big).$$

$\tilde{M}_\rho$ descends to a smooth submanifold of X_∞, which we denote by M_ρ. The following lemma is immediately clear from our construction.

LEMMA 3.3
M_ρ is isometric to $S^1 \times N$ with the product metric, where the S^1 factor has length $\frac{2\pi}{m}$. In particular, if N is a torus, then so is M_ρ.

Let $\{\frac{\partial}{\partial x^i}\}_{i=1}^{n-1}$ form a basis for TN. At any point $p \in M_\rho$ the tangent space T_pM_ρ is spanned by the vectors $\{\frac{\partial}{\partial\theta}, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n-1}}\}$. Because we assumed that $N \subset D$ is Lagrangian, it is clear that $f|_{M_\rho} = 0$ is as well. Furthermore,

$$\Omega_\infty|_{M_\rho} = i\, d\theta\Big(\frac{\partial}{\partial\theta}\Big)\Omega_D\Big(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n-1}}\Big) = i\,\Omega_D|_N,$$

which is constant if $N \subset D$ is in addition a special Lagrangian. Additionally, the induced metric on M_ρ is given by

$$g_{M_\rho} = d\theta^2 + g_D|_N.$$

In particular, we note that this metric is independent of ρ. This gives the following lemma.

LEMMA 3.4
Assume that $N \subset D$ is a special Lagrangian which does not contain fixed points of ι. Then M_ρ is a special Lagrangian submanifold of $(X_\infty, \omega_\infty, \Omega_\infty)$ for all ρ. Furthermore, the induced metric on M_ρ is independent of $\rho \in \mathbb{R}^+$ and thus the geometric quantities associated to M_ρ, including the second fundamental form and the first eigenvalue of the Laplacian, depend only on N and are independent of ρ.

3.2. Perturbation

Given our one-parameter family of special Lagrangians in the model space, we demonstrate how to perturb this family into a family of Lagrangians with respect to the ACyl metric. By assumption, our Calabi–Yau manifold $(X, \omega, J, \Omega, g)$ comes equipped with a cylindrical diffeomorphism $\Phi : X_\infty \to X \setminus V$ for a given compact subset $V \subset X$. Pulling back via Φ allows us to work on the half-cylinder X_∞, where in addition to (3.1) we have the following decay for all $k \geq 0$:

$$\big|\nabla^k_{g_\infty}(\Phi^*\omega - \omega_\infty)\big|_{g_\infty} = O(e^{-\delta\ell}), \tag{3.2}$$

$$\big|\nabla^k_{g_\infty}(\Phi^*J - J_\infty)\big|_{g_\infty} = O(e^{-\delta\ell}), \tag{3.3}$$

$$\big|\nabla^k_{g_\infty}(\Phi^*\Omega - \Omega_\infty)\big|_{g_\infty} = O(e^{-\delta\ell}). \tag{3.4}$$

We will apply the perturbation results from Section 2. First, we demonstrate that the two Kähler forms $\Phi^*\omega$ and ω_∞, are cohomologous, using an argument from [47].

Denote the difference in Kähler forms by $\eta := \omega_\infty - \Phi^*\omega$, and decompose this 2-form as $\eta = d\ell \wedge \eta_1 + \eta_2$. Since $\Phi^*\omega$ and ω_∞ are closed, $d\eta = 0$ and so

$$d\eta_2 = d\ell \wedge \tilde{d}\eta_1,$$

where $\tilde{d}$ denotes the differential on the cross section $(S^1 \times D)/\iota$ only. Since the right-hand side above contains $d\ell$, the exterior derivative of η_2 must be of the form $d\eta_2 = d\ell \wedge \frac{\partial}{\partial \ell}\eta_2$, which implies that $\tilde{d}\eta_1 = \frac{\partial}{\partial \ell}\eta_2$.

Define a 1-form τ on X_∞ by integrating η_1 in the ℓ direction:

$$\tau(\ell, p) = -\int_\ell^\infty \eta_1(s, p)\, ds,$$

which is finite by (3.2). Taking the exterior derivative gives

$$d\tau = d\ell \wedge \eta_1 - \int_\ell^\infty \tilde{d}\eta_1\, ds = d\ell \wedge \eta_1 - \int_\ell^\infty \frac{\partial}{\partial \ell}\eta_2\, ds = d\ell \wedge \eta_1 + \eta_2 = \eta.$$

Thus $d\tau = \omega_\infty - \Phi^*\omega$ and τ is determined by the initial data.

Next, we employ the standard Moser trick from Section 2, setting ω_{mod} as ω_∞. This gives a family of diffeomorphisms F_t of X_∞ satisfying $F_t^*\omega_t = \omega_\infty$. Setting $t = 1$ gives $F_1^*\Phi^*\omega = \omega_\infty$. Thus, for any Lagrangian M_ρ with respect to ω_∞, the image $F_1(M_\rho)$ will be Lagrangian with respect to $\Phi^*\omega$. In addition to being a Lagrangian, we have the following control of the geometry of $\hat{M}_\rho := F_1(M_\rho)$.

PROPOSITION 3.5

For any $\rho > 0$, the submanifold $\hat{M}_\rho := F_1(M_\rho) \subset X_\infty$ is Lagrangian with respect to $\Phi^*\omega$ with vanishing Maslov class. Furthermore, for ρ sufficiently large, there are uniform constants $C > 2$ and $\delta_0 > 0$, depending on N, (3.1), (3.2), (3.3), and (3.4), so that

(1) *the coordinate ℓ on $\mathbb{R}^+$ restricted to $\hat{M}_\rho$ satisfies*

$$\rho - C < \ell|_{\hat{M}_\rho} < \rho + C,$$

(2) *the second fundamental form satisfies*

$$|A|^2 \le C,$$

(3) *the mean curvature satisfies*

$$|H|^2 \le Ce^{-\delta_0\rho},$$

(4) *the volume satisfies*

$$(1-C^{-1})\mathrm{Vol}_{g_D}(N) \le \mathrm{Vol}_{\Phi^* g}(\hat{M}_\rho) \le (1+C^{-1})\mathrm{Vol}_{g_D}(N),$$

(5) *the first positive eigenvalue* $\lambda_1(\hat{M}_\rho)$ *satisfies*

$$C^{-1}\lambda_1(N) \le \lambda_1(\hat{M}_\rho) \le C\lambda_1(N).$$

Proof
We have already seen that $\hat{M}_\rho$ is Lagrangian with respect to $\Phi^*\omega$. Furthermore, because $\hat{M}_\rho$ is homotopic to M_ρ via a one-parameter family of diffeomorphisms, the Maslov class of $\hat{M}_\rho$ vanishes.

Since V_t is uniquely determined by τ, it depends only on $\Phi^*\omega - \omega_\infty$. Thus, by (3.2), we see that

$$|\nabla^k_{g_\infty} V_t|_{g_\infty} = O(e^{-\delta\ell}).$$

Furthermore, given the Riemannian metrics $g_t = (1-t)g_\infty + t\Phi^* g$, (3.1) implies that

$$\frac{1}{2}g_\infty \le g_t \le 2g_\infty, \qquad |\nabla^k_{g_\infty} g_t|_{g_\infty} = O(e^{-\delta\ell}).$$

Putting these together gives

$$|\nabla^k_{g_t} V_t|_{g_t} = O(e^{-\delta\ell}). \tag{3.5}$$

Now, by definition we have the restriction $\ell|_{M_\rho} = \rho$. Our control of V_t bounds how far the points in M_ρ can move by the diffeomorphism F_1, proving (1).

To bound the remaining quantities, we rely on our analysis from Section 2. First we see that control of $(F_t(M_\rho), g_t)$ follows from control of the geometry of $M_\rho \subset X_\infty$ with respect to the Riemannian metrics $\tilde{g}_t := F_t^* g_t$. The above bound (3.5), together with Lemma 2.1, gives

$$\begin{aligned} |\partial_t \tilde{g}_t|_{\tilde{g}_t} &\le C_0 e^{-\delta\ell}, \\ |\nabla_{\tilde{g}_t}\partial_t \tilde{g}_t|_{\tilde{g}_t} &\le C_0 e^{-\delta\ell}. \end{aligned} \tag{3.6}$$

Turning to the second fundamental form and the mean curvature of M_ρ with respect to $\tilde{g}_t$, we consider the ODE

$$\frac{df}{dt} = c(f^{\frac{1}{2}} + f), \qquad f(0) \ge 0, \tag{3.7}$$

whose solution is $f(t) = (-1 + [1 + f(0)^{\frac{1}{2}}]e^{\frac{ct}{2}})^2$. By Lemma 2.2, equation (3.6), and part (1) of the proposition, both $|H|^2(t)$ and $|A|^2(t)$ are subsolutions of (3.7) with constant $c = C'e^{-\delta'\rho}$, where $C', \delta' > 0$ are uniform constants. For ρ sufficiently large depending only on C', δ', we obtain

$$|A|^2(t) \le 100(C')^2 e^{-2\delta'\rho} + 4|A|^2(0),$$
$$|H|^2(t) \le 100(C')^2 e^{-2\delta'\rho} + 4|H|^2(0).$$

Lemma 3.4 describes the geometry of M_ρ with the metric induced by $\tilde{g}_0$, from which we see that $|A|(0)$ is controlled by the supremum of the second fundamental form of the compact Lagrangian $N \subset D$. This establishes (2). Also, since M_ρ is minimal with respect to $\tilde{g}_0$, we have $|H|(0) = 0$, establishing (3).

Estimate (4) follows immediately from (3.6), while (5) follows from (3.6) and Lemma 2.3, again using our understanding of the geometry of M_ρ for $t = 0$ given by Lemma 3.4. This completes the proof. □

3.3. *The mean curvature flow*

We now evolve the perturbed Lagrangian $\hat{M}_\rho$ by mean curvature flow. We apply the following result of Li [62], based on work of Chen and Li [19] in the Kähler–Ricci flow.

THEOREM 3.6 (Li [62, Theorem 1.2])
Let (X, g) be a complete Kähler–Einstein manifold with scalar curvature $\overline{R} \ge 0$, and let M be a compact Lagrangian submanifold smoothly immersed in X with vanishing Maslov class. For any V_0, Λ_0, and $\delta_0 > 0$, there exists an $\varepsilon > 0$, depending on V_0, Λ_0, δ_0, $\overline{R}$, a lower bound for the injectivity radius of X, and an upper bound for $\sum_{k=0}^{5} |\nabla^k \mathrm{Rm}|_g$, so that if

$$\lambda_1 \ge \frac{\overline{R}}{2n} + \delta_0, \qquad \mathrm{Vol}(M) \le V_0, \qquad |A| \le \Lambda_0, \qquad \int_M |H|^2 \le \varepsilon,$$

then the Lagrangian mean curvature flow with initial data M will converge exponentially fast to a minimal Lagrangian submanifold in X.

To apply the above result to the Lagrangian $\hat{M}_\rho$, note that the constants δ_0, V_0, and Λ_0 can be specified by Proposition 3.5. Asymptotic decay (3.1) gives the desired control of background Riemannian curvature tensor and also allows us to bound the injectivity radius of $(X_\infty, \Phi^* g)$ below by the injectivity radius of the cross section $(S^1 \times D)/\iota$. All of these quantities are independent of ρ; thus, using Proposition 3.5(3), for any $\varepsilon > 0$ we can choose ρ large enough so that

$$|H|^2 \le C e^{-\delta_0 \rho} < \varepsilon.$$

The hypothesis of Theorem 3.6 now applies and as a result the mean curvature flow beginning with $\hat{M}_\rho$ converges to a special Lagrangian submanifold, which we denote by $\mathcal{M}_\rho$.

We now show that there is a countable family of special Lagrangians. Let C be the fixed constant from Proposition 3.5. Fix ρ so that $\hat{M}_\rho$ converges to $\mathcal{M}_\rho$ along the mean curvature flow. Now, Proposition 3.5(1) allows us to choose $\rho_1 > \rho + 2C + 1$ large enough so that the corresponding perturbed Lagrangians $\hat{M}_\rho$ and $\hat{M}_{\rho_1}$ are at least distance 1 apart. To see that they stay distinct along the flow, we only need to observe that the mean curvature vector decays exponentially along the flow. Specifically, Lemma 5.2 in [62] gives

$$\big|H(t)\big| \leq C e^{\frac{-\delta_0 \rho}{n+2}} e^{\frac{-\beta_0}{2n+4} t},$$

which controls how far each Lagrangian can travel. Thus, for ρ large enough, we can construct a sequence $\rho_{i+1} = \rho_i + 2(C+1)$, so that the corresponding limiting special Lagrangians $\mathcal{M}_{\rho_i}$ are distinct.

We conclude this section with a note about the case $n = 2$. Recall that Theorem 3.2 stipulates $n > 2$, but this is only required to obtain a compactification (see [47, Remark 1.6]). However, in our setting we are not working on an arbitrary ACyl Calabi–Yau manifold. Rather, we consider Y a compact Kähler surface which admits a holomorphic fibration and let $X = Y \setminus D$ be the complement of a smooth fiber in $|-K_Y|$. The complete metric ω constructed by Tian and Yau [86] (see also [47], [48]) on X is ACyl with cross section $\mathbb{T}^3 = S^1 \times \mathbb{T}^2$, equipped with a flat metric h of the form

$$h = \gamma\, d\theta^2 + g_{\varepsilon,\tau_0}.$$

Here $\theta \in S^1$, γ is a fixed length, and g_{ε,τ_0} is the unique flat metric of area ε and modulus τ_0 on $\mathbb{T}^2$. Our above construction of a special Lagrangian can now be carried out, with the model Lagrangian M_ρ given by the product of the S^1 factor with any line of rational slope in $\mathbb{T}^2$. Thus we have the following.

COROLLARY 3.7

Let Y be a rational elliptic surface, and let $D \in |-K_Y|$ be a smooth divisor. Then for any closed loop $[\gamma] \in H_1(D, \mathbb{Z})$, the noncompact Calabi–Yau manifold $Y \setminus D$ admits infinitely many special Lagrangian submanifolds which are topologically $[\gamma] \times S^1$.

4. Special Lagrangians in Tian–Yau spaces of type I

In this section, we prove the existence of infinitely many special Lagrangian submanifolds in Tian–Yau spaces, under the assumption that the Calabi–Yau manifold at infinity admits one smooth special Lagrangian. We begin by reviewing the model geometry for the Tian–Yau spaces.

4.1. *The Calabi model geometry*

Suppose that D is a projective Calabi–Yau manifold of dimension $n-1$, with $K_D = \mathcal{O}_D$, and let $\pi : L \to D$ be an ample line bundle. By Yau's theorem in [92], we can find a Hermitian metric h on L, unique up to scaling, so that $\omega_D = -\sqrt{-1}\partial\overline{\partial}\log(h)$ defines a Ricci-flat Kähler metric. Consider the open n-dimensional complex manifold

$$\mathcal{C} = \{\xi \in L : 0 < |\xi|_h < 1\}.$$

The space $\mathcal{C}$ has a natural, nonvanishing holomorphic $(n,0)$-form induced in the following way. Fix local holomorphic coordinates $(z_1,\dots,z_{n-1})$ on D and a local trivialization ξ of L. We get coordinates $(z_1,\dots,z_{n-1},w)$ on L by

$$(z_1,\dots,z_{n-1},w) \mapsto (z_1,\dots,z_{n-1},w\xi).$$

Let Ω_D be a holomorphic volume form on D. We fix a scale for Ω_D by requiring that

$$\frac{1}{2}\int_D (\sqrt{-1})^{(n-1)^2}\Omega_D \wedge \overline{\Omega}_D = \int_D \big(2\pi c_1(L)\big)^{n-1}. \tag{4.1}$$

The volume form Ω_D can be locally written as $f(z)\,dz_1 \wedge \cdots \wedge dz_{n-1}$. Then

$$\Omega_{\mathcal{C}} = \frac{f(z)}{w}\,dz_1 \wedge \cdots \wedge dz_{n-1} \wedge dw$$

is a local, nonvanishing holomorphic $(n,0)$-form on $\mathcal{C}$. It is easy to see that this expression is independent of the choice of trivialization of L, and hence $\Omega_{\mathcal{C}}$ glues to a trivialization of $K_{\mathcal{C}}$. We will assume that Ω_D is normalized such that

$$\omega_D^{n-1} = \frac{1}{2}(\sqrt{-1})^{(n-1)^2}\Omega_D \wedge \bar{\Omega}_D. \tag{4.2}$$

Define a function on $\mathcal{C}$ by

$$\mathcal{C} \ni \xi \longmapsto \frac{n}{n+1}\big(-\log|\xi|_h^2\big)^{\frac{n+1}{n}},$$

and let

$$\omega_{\mathcal{C}} = \sqrt{-1}\partial\overline{\partial}\frac{n}{n+1}\big(-\log|\xi|_h^2\big)^{\frac{n+1}{n}}.$$

By direct computation one can verify that $\omega_{\mathcal{C}}$ defines a Kähler metric on $\mathcal{C}$, which is complete as $|\xi|_h \to 0$, but incomplete as $|\xi|_h \to 1$. Fix a point $p \in D$, and let

$(z_1, \dots, z_{n-1})$ be coordinates centered at p. Choose a trivialization of L so that $h(p) = 1$, $dh(p) = 0$, and write $h = e^{-\varphi}$ for a locally defined function φ. If we write $w = re^{\sqrt{-1}\theta}$, then the Kähler form and the Riemannian metric are given by

$$\begin{aligned}
\omega_{\mathcal{C}} &= \sqrt{-1}\big(-\log\big(|w|^2 e^{-\varphi}\big)\big)^{\frac{1}{n}-1}\frac{1}{n}\Big(\frac{dw}{w} + \partial\varphi\Big) \wedge \Big(\frac{d\bar{w}}{\bar{w}} + \bar{\partial}\varphi\Big) \\
&\quad + \big(-\log\big(|w|^2 e^{-\varphi}\big)\big)^{\frac{1}{n}}\pi^*\omega_D, \\
g_{\mathcal{C}} &= \big(-\log(r^2 e^{-\varphi})\big)^{\frac{1}{n}-1}\frac{1}{n}\Big(\Big(\frac{dr}{r} - \frac{1}{2}\,d\varphi\Big)^2 + \Big(d\theta + \frac{1}{2}J\,d\varphi\Big)^2\Big) \\
&\quad + \big(-\log(r^2 e^{-\varphi})\big)^{\frac{1}{n}}\pi^* g_D.
\end{aligned} \tag{4.3}$$

At any point $(0, \dots, 0, re^{\sqrt{-1}\theta})$ this reduces to

$$\begin{aligned}
\omega_C &= \sqrt{-1}\frac{(-\log(|w|^2))^{\frac{1}{n}-1}}{n}\frac{dw \wedge d\bar{w}}{|w|^2} + \big(-\log\big(|w|^2\big)\big)^{\frac{1}{n}}\pi^*\omega_D, \\
g_C &= \big(-\log(r^2)\big)^{\frac{1}{n}-1}\frac{1}{n}\Big(\frac{dr^2}{r^2} + d\theta^2\Big) + \big(-\log(r^2)\big)^{\frac{1}{n}}\pi^* g_D.
\end{aligned} \tag{4.4}$$

Completeness as $r \to 0$ easily follows from this formula. Furthermore, from (4.2), we have

$$\omega_{\mathcal{C}}^n = \sqrt{-1}\frac{dw \wedge d\bar{w}}{|w|^2} \wedge \pi^*\omega_D^{n-1} = \frac{1}{2}(\sqrt{-1})^{n^2}\Omega_{\mathcal{C}} \wedge \overline{\Omega_{\mathcal{C}}},$$

so $\omega_{\mathcal{C}}$ is Ricci-flat. Let us introduce the following terminology.

Definition 4.1
We define the *scale function* on $\mathcal{C}$ to be

$$\ell_0 = \big(-\log|\xi|_h^2\big)^{\frac{1}{2n}}.$$

Remark 4.2
Note that the scale function satisfies

$$|\nabla_{g_{\mathcal{C}}}\ell_0^{n+1}|^2 = \frac{(n+1)^2}{4n}.$$

Furthermore, if $p \in \mathcal{C}$ is a fixed point, then there is a constant C_1 so that

$$C_1^{-1}\ell_0^{n+1} \le d_{\mathcal{C}}(p, x) \le C_1\ell_0^{n+1}.$$

By direct computation we have the following.

PROPOSITION 4.3
Let $g_{\mathcal{C}}$ be the Riemannian metric on $\mathcal{C}$ induced by $\omega_{\mathcal{C}}$, and suppose that $n \geq 3$. Then for all $k \in \mathbb{N} \cup \{0\}$ there is a constant C_k so that

$$\left|\nabla^k \mathrm{Rm}(g_{\mathcal{C}})\right|_{g_{\mathcal{C}}} \leq C_k \ell_0^{-(k+2)}.$$

If $n = 2$, then we have

$$\left|\nabla^k \mathrm{Rm}(g_{\mathcal{C}})\right|_{g_{\mathcal{C}}} \leq C_k \ell_0^{-(k+6)}.$$

The injectivity radius satisfies

$$C_\iota^{-1} \ell_0^{1-n} \leq \operatorname{inj} g_{\mathcal{C}} \leq C_\iota \ell_0^{1-n}$$

for a uniform constant C_ι.

Having now understood the Riemannian geometry of $\mathcal{C}$, we move on to the study of the special Lagrangian submanifolds of $\mathcal{C}$. To this end, suppose that $N \subset D$ is a special Lagrangian submanifold of D. Fix $\varepsilon > 0$, and consider the S^1-bundle over D,

$$\mathcal{C}_\varepsilon = \left\{\xi \in \mathcal{C} : |\xi|_h^2 = \varepsilon\right\} \xrightarrow{\pi} D.$$

For each $\varepsilon \in (0, 1)$, we define a smooth, real codimension n submanifold of $\mathcal{C}$ by

$$M_\varepsilon = \pi^{-1}(N) \cap \mathcal{C}_\varepsilon.$$

In other words, M_ε is the manifold obtained by restricting the circle bundle $\mathcal{C}_\varepsilon$ to N. First, we describe the topology of M_ε.

LEMMA 4.4
The manifold M_ε is topologically $S^1 \times N$. In particular, if $N = (S^1)^{n-1}$ is a torus, then so is M_ε.

Proof
Since N is Lagrangian, we have $\omega|_N = 0$. On the other hand, ω is the first Chern class of $L \to N$. Since M_ε is the circle bundle in L, it follows that the Euler class of the circle bundle $M_\varepsilon \to N$ vanishes, and hence M_ε is topologically a trivial S^1-bundle over N. □

Our primary case of interest will be when N is a torus; hence the above lemma ensures that in this case M_ε will also be a torus.

Next, we claim that M_ε is Lagrangian. This can be achieved by a pointwise calculation, using the coordinate expression for $\omega_{\mathcal{C}}$ in (4.4). Fix a point $p \in N$. As before, let ξ be a local trivialization of L so that $h(p)=1$, $dh(p)=0$, and write $h=e^{-\varphi}$. Let w the corresponding local coordinate on L, and write $w=re^{\sqrt{-1}\theta}$, where $\theta\in S^1$. Locally we have

$$M_\varepsilon=\{(p,w):p\in N,|w|^2e^{-\varphi(p)}=\varepsilon\}.$$

Therefore, $\frac{dw}{w}=\frac{dr}{r}+\sqrt{-1}\,d\theta$ and we have

$$\frac{dw}{w}\Big|_{M_\varepsilon}=\frac{1}{2}d\varphi+\sqrt{-1}\,d\theta.$$

By our choice of trivialization using (4.4), we have $d\varphi(p)=0$ and hence

$$\frac{dw\wedge d\bar{w}}{|w|^2}\Big|_{M_\varepsilon}=0,\qquad \Omega_{\mathcal{C}}|_{M_\varepsilon}=\sqrt{-1}\pi^*\Omega_D\wedge d\theta|_{M_\varepsilon}.$$

It follows from (4.4) that M_ε is Lagrangian. Furthermore,

$$e^{-\sqrt{-1}\frac{\pi}{2}}\Omega_{\mathcal{C}}|_{M_\varepsilon}=\pi^*\operatorname{Im}\Omega_D|_N\wedge d\theta=0. \tag{4.5}$$

If necessary, rotate Ω_D by a unit complex number so that $\operatorname{Im}(\Omega_D|_N)=0$ and $\operatorname{Re}(\Omega_D|_N)=d\operatorname{Vol}_N$. Then we have $\operatorname{Im}(e^{-\sqrt{-1}\frac{\pi}{2}}\Omega_C|_{M_\varepsilon})=0$. Summarizing, we have the following.

LEMMA 4.5
If $N\subset D$ is special Lagrangian, then for each $\varepsilon>0$ the manifold

$$M_\varepsilon=\{\xi\in L:\pi(\xi)\in N,|\xi|_h^2=\varepsilon\}$$

is a special Lagrangian submanifold of $(\mathcal{C},\omega_{\mathcal{C}})$ which is topologically $N\times S^1$.

As described in the introduction and already executed in Section 3, we will transplant these special Lagrangians into the Tian–Yau spaces and run the Lagrangian mean curvature flow in order to produce special Lagrangians. In order to prove the convergence of the mean curvature flow, we will need to understand the Riemannian geometry of M_ε in some detail. This is what we take up next. First, we compute the volume of M_ε with respect to the induced metric.

LEMMA 4.6
The volume of $M_\varepsilon\subset(\mathcal{C},g_{\mathcal{C}})$ is independent of ε and given by

$$\operatorname{Vol}(M_\varepsilon,g_{\mathcal{C}})=\frac{2\pi}{\sqrt{n}}\operatorname{Vol}(N,g_D).$$

Proof
One can directly compute the volume form of the induced metric on M_ε from the formula of $\omega_{\mathcal{C}}$ (4.3) to find that

$$d\,\mathrm{Vol}_{M_\varepsilon} = \frac{1}{\sqrt{n}}\pi^* d\,\mathrm{Vol}_N \wedge d\theta.$$

The lemma follows by integration over M_ε. ☐

Next, we examine the bottom of the spectrum of the Laplacian on M_ε. Recall that the first nonzero eigenvalue of the Laplacian on a compact, boundaryless Riemannian manifold (M, g) is characterized by the Rayleigh quotient

$$\lambda_1(M, g) = \inf \frac{\int_M |\nabla f|^2}{\int_M f^2},$$

where the infimum is taken over all real-valued L^2 functions on M with $\int_M f = 0$. Suppose that $f : N \to \mathbb{R}$ and $\int_N f = 0$. By the formula for the volume form of M_ε, we obtain $\int_{M_\varepsilon} \pi^* f = 0$ and

$$\int_{M_\varepsilon} (\pi^* f)^2 = 2\pi \int_N f^2.$$

Furthermore, by a local computation using (4.4) we have

$$|\nabla \pi^* f|^2_{g_{\mathcal{C}}} = \big(-\log(\varepsilon)\big)^{-1/n} \pi^* |\nabla f|^2_{g_D}.$$

It follows immediately from the Rayleigh quotient formula that

$$\lambda_1(M_\varepsilon) \leq \frac{\lambda_1(N)}{(-\log(\varepsilon))^{1/n}}. \tag{4.6}$$

We claim that we in fact have equality in (4.6), provided that ε is sufficiently small. This essentially follows from work of Bérard-Bergery and Bourguignon [13] (see also [14]), but we will give a simple explicit proof for the reader's convenience.

LEMMA 4.7
The first nonzero eigenvalue of the Laplacian on M_ε satisfies

$$\lambda_1(M_\varepsilon) = \frac{\lambda_1(N)}{(-\log(\varepsilon))^{1/n}},$$

provided that ε is sufficiently small, depending only on $\lambda_1(N)$, n.

Proof
Fix a point $p \in N$, and choose normal coordinates $(x_1, \dots, x_{n-1})$ centered at p. As before, we use $(x_1, \dots, x_{n-1}, \theta)$ as coordinates on M_ε, choosing the trivialization of L so that

$$g_{\mathcal{C}}|_{M_\varepsilon} = \big(-\log(\varepsilon)\big)^{\frac{1}{n}-1}\frac{1}{n}\,d\theta^2 + \big(-\log(\varepsilon)\big)^{\frac{1}{n}}\pi^* g_D|_N \tag{4.7}$$

at any point $q \in \pi^{-1}(p)$. It is easy to see that the circle $\pi^{-1}(p)$ is a geodesic, and so the Laplacian takes the form

$$\begin{aligned}\Delta_{M_\varepsilon} f &= g^{\alpha\beta}\nabla_\alpha\nabla_\beta f - \nabla_{\nabla_{\partial_\alpha}\partial_\beta} f\\ &= n\big(-\log(\varepsilon)\big)^{1-\frac{1}{n}}\nabla_\theta\nabla_\theta f + \Delta_H f,\end{aligned}$$

where

$$\Delta_H f = \big(-\log(\varepsilon)\big)^{-\frac{1}{n}}\sum_{1\le i,j\le n-1}(g_D)^{ij}\nabla_i\nabla_j f - \big(-\log(\varepsilon)\big)^{-\frac{1}{n}}(g_D)^{ij}\Gamma_{ij}^{\theta}\frac{\partial f}{\partial\theta}.$$

From this formula it is clear that if f is an eigenfunction on N with eigenvalue λ, then $\pi^* f$ is an eigenfunction on M_ε with eigenvalue $(-\log(\varepsilon))^{-1/n}\lambda$. In particular, $\frac{\lambda_1(N)}{(-\log(\varepsilon))^{1/n}}$ is an eigenvalue of Δ_{M_ε}.

Now suppose that f is an eigenfunction of Δ_{M_ε} with eigenvalue μ. Define (local) smooth functions $a_k(x)$ by

$$a_k(x) = \frac{1}{2\pi}\int_{S^1} f(x,\theta)e^{-\sqrt{-1}k\theta}\,d\theta,$$

and let $f_k(x,\theta) = a_k(x)e^{\sqrt{-1}k\theta}$. Then by standard results in Fourier analysis we have

$$f = \sum_{k\in\mathbb{Z}} f_k(x,\theta),$$

and this series converges locally smoothly. In fact, $f_k(x,\theta)$ are globally defined smooth functions corresponding to the decomposition of $L^2(M_\varepsilon)$ into weight spaces induced by the natural isometric $U(1)$-action on M_ε. Since the $U(1)$-action is Killing, for each $\ell \in \mathbb{Z}$ we have

$$\begin{aligned}\mu f_\ell &= \Delta_{M_\varepsilon} f_\ell\\ &= 2n\big(-\log(\varepsilon)\big)^{1-\frac{1}{n}}(-\ell^2)f_\ell + \Delta_H f_\ell.\end{aligned}$$

Therefore, we have

$$\Delta_H f_\ell = \left(\mu + 2n\left(-\log(\varepsilon)\right)^{1-\frac{1}{2}}\ell^2\right) f_\ell. \tag{4.8}$$

On the other hand, for any complex-valued function h we have

$$\begin{aligned}\int_{M_\varepsilon} \Delta_H h\overline{h} &= \int_{M_\varepsilon} \Delta h\overline{h} - 2n\left(-\log(\varepsilon)\right)^{1-\frac{1}{n}} \nabla_\theta \nabla_\theta h\overline{h} \\ &= -\int_{M_\varepsilon} |\nabla h|^2 + \int_{M_\varepsilon} |\nabla_\theta h|^2 \leq 0. \end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9), we conclude that if $f_\ell \neq 0$, then

$$\left(\mu + 2n\left(-\log(\varepsilon)\right)^{1-\frac{1}{n}}\ell^2\right) \leq 0.$$

If ε is sufficiently small, then

$$-2n\left(-\log(\varepsilon)\right)^{1-\frac{1}{n}} \ll \frac{\lambda_1(N)}{(-\log(\varepsilon))^{1/n}},$$

and so to be a competitor for $\lambda_1(M_\varepsilon)$ we must have $f_\ell = 0$ for all $\ell \neq 0$. But in this case it is clear that $f = \pi^* \tilde{f}$ for some eigenfunction $\tilde{f}$ of the Laplacian on N. The lemma follows immediately. ☐

Next, we will compute the second fundamental form of $M_\varepsilon \subset (\mathcal{C}, g_{\mathcal{C}})$. To this end, choose real coordinates $(x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1})$ for D centered at a point $p \in N$ so that $(x_1, \dots, x_{n-1})$ are coordinates on N and at p we have $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\}$ orthonormal and

$$J_D(p)\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}.$$

Since g_D is Hermitian, $\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{n-1}}\}$ form an orthonormal basis for $T_pN^\perp \subset T_pD$. As before, we choose a trivialization of L so that $h(p) = 1$ and $dh(p) = 0$ and write $h = e^{-\varphi}$. Let w be the induced coordinate on L, and write $w = re^{\sqrt{-1}\theta}$. Define a new coordinate u by

$$u = \log(r) - \frac{1}{2}\varphi.$$

Then $(x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}, u, \theta)$ form local coordinates for any point near $(0, u, \theta)$. In these coordinates the metric is given by

$$g_{\mathcal{C}} = (-2u)^{\frac{1}{n}-1}\frac{1}{n}\left(du^2 + \left(d\theta + \frac{1}{2}J\,d\varphi\right)^2\right) + (-2u)^{\frac{1}{n}}\pi^* g_D,$$

and this simplifies at $(0, u, \theta)$ since $d\varphi = 0$, so, in particular, the metric is block diagonal there. Fix a point $q \in M_\varepsilon \subset \{u = \frac{1}{2}\log(\varepsilon)\}$ with $\pi(q) = p$. To compute the second fundamental form, we note that

$$Y_k = \big(-\log(\varepsilon)\big)^{-\frac{1}{2n}} \frac{\partial}{\partial y_i}, \qquad U = \sqrt{n}\big(-\log(\varepsilon)\big)^{n-1} 2n \frac{\partial}{\partial u}$$

form an orthonormal basis for $T_q M_\varepsilon$. It suffices to compute $\langle \nabla_X Y, Z\rangle$, where X, Y run over $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial \theta}\}$ and Z runs over $\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{n-1}}, \frac{\partial}{\partial u}\}$. In other words, we need to compute some of the Christoffel symbols of $g_{\mathcal{C}}$ at q. We begin by computing $\Gamma^\alpha_{x_i x_j}$, where $\alpha = y_k, u$. We have

$$\Gamma^u_{x_i x_j} = \frac{1}{2} n \big(-\log(\varepsilon)\big)^{1-\frac{1}{n}} (-\partial_u g_{x_i x_j}).$$

Now $g_{x_i x_j} = (-2u)^{\frac{1}{n}} (g_D)_{x_i x_j} + (-2u)^{\frac{1}{n}-1} O((d\varphi)^2)$ so that

$$\Gamma^u_{x_i x_j} = \delta_{ij}, \qquad \langle \nabla_{\partial_{x_i}} \partial_{x_j}, U\rangle = \frac{(-\log(\varepsilon))^{\frac{1-n}{2n}}}{\sqrt{n}} \delta_{ij}.$$

Next, since $d\varphi = 0$ at q, it is straightforward to show that $\Gamma^{y_k}_{x_i x_j} = (\Gamma_D)^{y_k}_{x_i x_j}$, where the Γ_D are the Christoffel symbols at $p \in D$. Thus,

$$\langle \nabla_{\partial_{x_i}} \partial_{x_j}, Y_k\rangle = \big(-\log(\varepsilon)\big)^{\frac{1}{2n}} \langle \nabla^D_{\partial_{x_i}} \partial_{x_j}, \partial_{y_k}\rangle_{g_D},$$

which is a rescaling of the second fundamental form of $N \subset (D, g_D)$. Next, consider $\nabla_{\partial_\theta} \partial_\theta$. We begin by computing

$$\Gamma^u_{\theta\theta} = -\frac{1}{2} \partial_u \log\Big(\frac{(-2u)^{\frac{1}{n}-1}}{n}\Big) = \frac{n-1}{2nu}.$$

Therefore,

$$\langle \nabla_{\partial_\theta} \partial_\theta, U\rangle = \frac{-(n-1)}{n\sqrt{n}} (-2u)^{\frac{1-3n}{2n}}.$$

One easily checks that $\langle \nabla_{\partial_\theta} \partial_\theta, Y_k\rangle = 0$ and so it only remains to compute the contribution from $\nabla_{\partial_\theta} \partial_{x_i}$. We have

$$\Gamma^u_{\theta x_i} = \frac{n}{2} (-2u)^{1-\frac{1}{n}} \Big(-\partial_u (J\, d\varphi)\Big(\frac{\partial}{\partial x_i}\Big)\Big) = 0,$$

since $J\, d\varphi$ is independent of u. It therefore suffices to compute

$$\Gamma^{y_k}_{\theta x_i} = \frac{1}{2} \big(-\log(\varepsilon)\big)^{-\frac{1}{n}} (\partial_{x_i} g_{y_k \theta} + \partial_\theta g_{x_i y_k} - \partial_{y_k} g_{x_i \theta}).$$

Now $\partial_\theta g_{x_i y_k} = 0$ since the metric is θ-independent. On the other hand,

$$\partial_{x_i} g_{y_k \theta} = (-2u)^{\frac{1}{n}-1} \frac{1}{2n} \frac{\partial}{\partial x_i} \Big[(J\, d\varphi)\Big(\frac{\partial}{\partial y_k}\Big)\Big].$$

Since $p = \pi(q)$ satisfies $d\varphi(p) = 0$, we have $J\, d\varphi(\partial_{y_k}) = -\partial_{x_k}\varphi + O(x^2)$, and so we obtain

$$\partial_{x_i} g_{y_k\theta} = -(-2u)^{\frac{1}{n}-1}\frac{1}{2n}\frac{\partial^2\varphi}{\partial x_i x_k}.$$

Similarly, we have $\partial_{y_k} g_{x_i\theta} = (-2u)^{\frac{1}{n}-1}\frac{1}{2n}\frac{\partial^2\varphi}{\partial y_i y_k}$. Therefore,

$$\Gamma^{y_k}_{\theta x_i} = \frac{-1}{n}\big(-\log(\varepsilon)\big)^{-1}\frac{1}{4}\Big(\frac{\partial^2\varphi}{\partial x_i x_k} + \frac{\partial^2\varphi}{\partial y_i y_k}\Big).$$

On the other hand, we have

$$g_{i\bar{j}} = \frac{\partial^2\varphi}{\partial z_i\,\partial\bar{z}_j} = \frac{1}{4}\Big(\frac{\partial^2\varphi}{\partial x_i\,\partial x_j} + \frac{\partial^2\varphi}{\partial y_i\,\partial y_j}\Big) + \frac{\sqrt{-1}}{4}\Big(\frac{\partial^2\varphi}{\partial x_i\,\partial y_j} - \frac{\partial^2}{\partial y_i\,\partial x_j}\Big),$$

and so, by our choice of coordinates, we have

$$\Gamma^{y_k}_{\theta x_i} = \frac{-1}{n}\big(-\log(\varepsilon)\big)^{-1}\delta_{ik}.$$

Therefore,

$$\langle\nabla_{\partial_\theta}\partial_{x_i}, Y_k\rangle = \frac{-1}{n}\big(-\log(\varepsilon)\big)^{\frac{1}{2n}-1}\delta_{ik}.$$

Taking the norm of the second fundamental form, we obtain the following.

LEMMA 4.8
The special Lagrangian submanifold $M_\varepsilon \subset (\mathcal{C}, g_{\mathcal{C}})$ satisfies

$$|A|^2_g \le C(N,n)\big(-\log(\varepsilon)\big)^{-\frac{1}{n}}$$

for a constant $C(N,n)$ depending only on the dimension and the second fundamental form of $N \subset (D, g_D)$.

Definition 4.9
Let (M,g) be a Riemannian manifold of dimension n, and let $\kappa, r_0 > 0$. We say that (M,g) is κ-non-collapsing at scale r_0 if for every $0 < r < r_0$ and for every $p \in M$, we have

$$\mathrm{Vol}\big(B(p,r)\big) \ge \kappa r^n,$$

where $\mathrm{Vol}(B(p,r))$ denotes the volume with respect to g of the geodesic ball of radius r about p in (M,g).

Finally, we have the following noncollapsing result.

LEMMA 4.10
Assume that N is κ_N-non-collapsing at scale r_N. Define constants $\kappa, r_\varepsilon > 0$ by

$$\kappa = \frac{\kappa_N}{2^{n-1}}, \qquad r_\varepsilon = \frac{2\pi}{\sqrt{n}}\big(-\log(\varepsilon)\big)^{\frac{1-n}{2n}}.$$

For ε sufficiently small, depending only on (N, g_D), M_ε is κ-non-collapsing at scale r_ε.

Proof
Fix a point $p \in M_\varepsilon$. Choose ε sufficiently small so that

$$\frac{2\pi}{\sqrt{n}}\big(-\log(\varepsilon)\big)^{-\frac{1}{2}} < r_N.$$

Let d_N denote the distance function on (N, g_D). For $r < \frac{2\pi}{\sqrt{n}}$, consider the set $Q = \{q \in N : d_N(\pi(p), q) < \frac{r}{2}(-\log\varepsilon)^{-1/2}\}$. For each point $q \in Q$ with $\pi(p) \neq q$ we can choose a unit speed geodesic $\gamma(t)$ in (N, g_D) connecting $\pi(p)$ to q and having length $d_N(\pi(p), q) < \frac{r}{2}(-\log\varepsilon)^{-1/2}$. We now take the horizontal lift of this curve through p. Recall that the metric on M_ε is given in local coordinates by

$$g_{\mathcal{C}}|_{M_\varepsilon} = \big(-\log(\varepsilon)\big)^{\frac{1}{n}-1}\frac{1}{n}\Big(d\theta + \frac{1}{2}J\,d\varphi\Big)^2 + \big(-\log(\varepsilon)\big)^{\frac{1}{n}}\pi^* g_D|_N$$

and the tangent space of the fibers $\pi : M_\varepsilon \to N$ is spanned by $\frac{\partial}{\partial\theta}$. Thus, the horizontal lift $\bar{\gamma}$ is a lift of γ to M_ε such that

$$\iota_{\dot{\bar{\gamma}}}\Big(d\theta + \frac{1}{2}J\,d\varphi\Big) = 0.$$

In particular, if $\bar{\gamma}$ denotes the horizontal lift of γ to M_ε passing through p, then $\bar{\gamma}$ connects p to a point $\bar{\gamma}(1) \in \pi^{-1}(q)$ and $\bar{\gamma}$ satisfies

$$|\dot{\bar{\gamma}}|^2 = \big(-\log(\varepsilon)\big)^{\frac{1}{n}}|\dot{\gamma}|^2_{g_D}.$$

Thus, $\bar{\gamma}$ has length in M_ε given by

$$\mathrm{Length}_{(M_\varepsilon, g_{\mathcal{C}})}(\bar{\gamma}) = \big(-\log(\varepsilon)\big)^{\frac{1}{2n}} d_N\big(\pi(p), q\big) < \frac{r}{2}\big(-\log(\varepsilon)\big)^{\frac{1-n}{2n}}.$$

Consider the ball $B := B(p, r(-\log(\varepsilon))^{\frac{1-n}{2n}}) \subset M_\varepsilon$. Choose coordinates $(x_1, \dots, x_{n-1}, \theta)$ centered at $\bar{\gamma}(1)$ so that the metric takes the form (4.7) at any point in the fiber containing $\overline{\gamma}(1)$. If q' is any point in the fiber containing $\overline{\gamma}(1)$ with

$|\theta(q')| < \frac{\sqrt{n}r}{2}$, then we claim that $q' \in B$. To see this, connect q' to p by concatenating the curve $\overline{\gamma}$ with the curve in the fiber from $\overline{\gamma}(1)$ to q'. The length of this curve is at most $r(-\log(\varepsilon))^{\frac{1-n}{2n}}$. Recall that, by assumption, $\frac{\sqrt{n}r}{2} < \pi$. Since this holds for every point $q \in Q$ and the volume form of M_ε is given by $\frac{1}{\sqrt{n}}\, d\theta \wedge \pi^* \, d\mathrm{Vol}_N$, we have

$$\int_B d\,\mathrm{Vol}_{M_\varepsilon} \geq r \int_Q d\,\mathrm{Vol}_N > r\kappa_N \Big(\frac{r}{2}\Big)^{n-1} \big(-\log(\varepsilon)\big)^{\frac{1-n}{2}}$$
$$= \frac{\kappa_N}{2^{n-1}} \Big(r\big(-\log(\varepsilon)\big)^{\frac{1-n}{2n}}\Big)^n,$$

which is the desired result. □

We now essentially understand the geometry of M_ε as a subset of the model space $(\mathcal{C}, g_{\mathcal{C}})$.

4.2. Transplantation and perturbation

The next step in the construction is to transplant the special Lagrangians in the Calabi model into the Tian–Yau spaces to produce approximate special Lagrangians using the calculations in Section 2. To begin, we recall the identification of the end of the Tian–Yau spaces with the Calabi model; our discussion follows closely that of [50]. Thus, we fix a Fano Kähler manifold X and assume that $D = \{\sigma = 0\}$ is a smooth anticanonical divisor in the linear system $|-K_X|$. Denote by $L = -K_X|_D$ the normal bundle of D in Y. We fix a holomorphic volume form on D satisfying the normalization (4.1). We view $\frac{1}{\sigma} = \Omega_X$ as a holomorphic $(n, 0)$-form on X with its normalization chosen so that the residue of Ω_X on D is Ω_D. Let h_D be the unique up-to-scale positively curved metric on $-K_X|_D \to D$ such that

$$-\sqrt{-1}\partial\overline{\partial}\log h_D = \omega_D$$

is Ricci-flat on D. Let h_X be a positively curved Hermitian metric on $-K_X$ extending h_D, and put

$$\omega_X = \sqrt{-1}\partial\overline{\partial}\frac{n}{n+1}\big(-\log|\sigma|^2_{h_X}\big)^{\frac{n+1}{n}}.$$

After possibly scaling h_X by a sufficiently small positive constant, we can assume that ω_X defines a smooth positive Kähler metric on $X \setminus D$ and evidently ω_X is asymptotic to the Calabi model. The following theorem is due to Tian and Yau [86] with the exponential decay estimates due to Hein [48].

THEOREM 4.11
There exists a function $\varphi : X \setminus D \to \mathbb{R}$ *such that* $\omega_{\mathrm{TY}} := \omega_X + \sqrt{-1}\partial\bar{\partial}\varphi$ *is a complete Ricci-flat Kähler metric on* $X \setminus D$ *solving the Monge–Ampère equation*

$$\omega_{\mathrm{TY}}^n = \frac{(\sqrt{-1})^{n^2}}{2}\Omega_X \wedge \overline{\Omega}_X. \tag{4.10}$$

Furthermore, there is a constant $\delta_0 = \delta_0(M, D)$ *such that*

$$|\nabla_{g_X}^k \varphi|_{g_X} = O(e^{-\delta_0 \ell_0^{n+1}}),$$

where ℓ_0 *is the scale function of Definition 4.1*

Fix a smooth Kähler metric g on X. Using g, we can identify L with $(T^{1,0}D)^\perp$ as C^∞ complex line bundles. Using the g-exponential map, we get a diffeomorphism Φ between a neighborhood of the zero section in L and a neighborhood of D in X. Under this identification, we have the following estimates (see [48], [50]).

PROPOSITION 4.12
There is a diffeomorphism $\Phi : \mathcal{C} \setminus K' \to X \setminus K$, *where* $K \subset X$ *is compact, and* $K' := \{|\xi|_h \geq \frac{1}{2}\}$ *such that the following estimates hold uniformly for all large enough values of* ℓ_0*:*

(1) $|\nabla_{g_C}^k(\Phi^* J_X - J_C)|_{g_C} = O(e^{-(\frac{1}{2}-\varepsilon)\ell_0^{2n}})$ *for all* $k \geq 0, \varepsilon > 0$,

(2) $|\nabla_{g_C}^k(\Phi^* \Omega_X - \Omega_C)|_{g_C} = O(e^{-(\frac{1}{2}-\varepsilon)\ell_0^{2n}})$ *for all* $k \geq 0, \varepsilon > 0$,

(3) $|\nabla_{g_C}^k(\Phi^* \omega_X - \omega_C)|_{g_C} = O(e^{-(\frac{1}{2}-\varepsilon)\ell_0^{2n}}))$ *for all* $k \geq 0, \varepsilon > 0$,

(4) $|\nabla_{g_C}^k(\Phi^*(-\log|\sigma|_{h_X}^2)^{\frac{n+1}{n}} - (-\log|\xi|_{h_D}^2)^{\frac{n+1}{n}})|_{g_C} = O(e^{-(\frac{1}{2}-\varepsilon)\ell_0^{2n}})$ *for all* $k \geq 0, \varepsilon > 0$,

(5) *there is a number* $\underline{\delta} > 0$ *such that for all* $k \geq 0$, *we have*

$$\left|\nabla_{g_C}^k(\Phi^* \omega_{\mathrm{TY}} - \omega_C)\right|_{g_C} = O(e^{-\underline{\delta}\ell_0^{2n}}),$$

$$\left|\nabla_{g_C}^k(\Phi^* g_{\mathrm{TY}} - g_C)\right|_{g_C} = O(e^{-\underline{\delta}\ell_0^{2n}}).$$

We will (somewhat abusively) use ℓ_0 to denote the scale function pulled back by Φ^{-1} to X. Note that by Proposition 4.12, there is a uniform constant C such that

$$C^{-1}\big(-\log|\sigma|_{h_X}^2\big)^{\frac{1}{2n}} \leq \ell_0 \leq C\big(-\log|\sigma|_{h_X}^2\big)^{\frac{1}{2n}}.$$

Furthermore, Proposition 4.12 together with Proposition 4.3 can be equivalently reformulated as the following.

COROLLARY 4.13
For all $k \geq 0$ and $n \geq 3$, we have

$$\left|\nabla^k_{g_{\mathrm{TY}}}\mathrm{Rm}(g_{\mathrm{TY}})\right|_{g_{\mathrm{TY}}} = O(\ell_0^{-(2+k)}).$$

When $n = 2$, then for all $k \geq 0$ we have

$$\left|\nabla^k_{g_{\mathrm{TY}}}\mathrm{Rm}(g_{\mathrm{TY}})\right|_{g_{\mathrm{TY}}} = O(\ell_0^{-(6+k)}).$$

The injectivity radius satisfies

$$C_\iota^{-1}\ell_0^{1-n} \leq \mathrm{inj}\, g_{\mathrm{TY}} \leq C_\iota \ell_0^{1-n}$$

for a uniform constant C_ι.

Note that

$$\begin{aligned}
&\Phi^*\omega_{\mathrm{TY}} - \omega_C \\
&\quad = d\Big(\Phi^*\Big(\sqrt{-1}\,\overline{\partial}_X \frac{n}{n+1}\big(-\log|\sigma|^2_{h_X}\big)^{\frac{n+1}{n}} + \overline{\partial}_X\varphi\Big) \\
&\qquad - \overline{\partial}_C \frac{n}{n+1}\big(-\log|\xi|^2_h\big)^{\frac{n+1}{n}}\Big) \\
&\quad = \frac{-n}{2(n+1)}\, d\Big((\Phi^* J_X)\, d\Phi^*\Big[\big(-\log|\sigma|^2_{h_X}\big)^{\frac{n+1}{n}} + \frac{n+1}{n}\varphi\Big] \\
&\qquad - J_C\, d\big(-\log|\xi|^2_h\big)^{\frac{n+1}{n}}\Big).
\end{aligned}$$

Thus, by parts (1) and (4) of Proposition 4.12, together with Theorem 4.11, we get a 1-form β satisfying

$$\Phi^*\omega_{\mathrm{TY}} - \omega_C = d\beta, \qquad \sum_{\ell=0}^{\ell} |\nabla^\ell_{g_C}\beta|_{g_C} = O(e^{-(\frac{1}{2}-\varepsilon)\ell_0^{2n}}),$$

for all $k \geq 0$ and $\varepsilon > 0$. Define symplectic forms $\omega_t = (1-t)\omega_C + t\,\Phi^*\omega_{\mathrm{TY}}$ for $t \in [0,1]$ and a time-dependent vector field V_t by

$$\iota_{V_t}\omega_t = -\beta. \tag{4.11}$$

It follows from the estimates in Proposition 4.12 that, for all $k \geq 0$,

$$\sum_{\ell=0}^{k} |\nabla^\ell_{g_C} V_t|_{g_C} = O(e^{-\underline{\delta}\ell_0^{2n}}). \tag{4.12}$$

Define Riemannian metrics $g_t = (1-t)g_C + t\Phi^* g_{\mathrm{TY}}$. Proposition 4.12 implies that for all $k \geq 0$, we have

$$\frac{1}{2} g_C \leq g_t \leq 2 g_C, \qquad |\nabla^k_{g_C} g_t|_{g_C} = O(e^{-\underline{\delta}\ell_0^{2n}}).$$

We obtain the following.

COROLLARY 4.14

Let V_t, g_t be as above. Then there is a number $\underline{\delta} > 0$ so that, for all $t \in [0,1]$ and all $k \geq 0$, we have

$$\sum_{\ell=0}^{k} |\nabla^\ell_{g_t} V_t|_{g_t} = O(e^{-\underline{\delta}\ell_0^{2n}}).$$

We can now apply the analysis of Section 2 to conclude the following.

PROPOSITION 4.15

Suppose that $N \subset (D, \omega_D)$ is a special Lagrangian submanifold. Then for all $K \gg 0$ sufficiently large, there exists a Lagrangian submanifold $M_K \subset (X, \omega_{\mathrm{TY}}, g_{\mathrm{TY}}, J_{\mathrm{TY}}, \Omega_{\mathrm{TY}})$, which is topologically $N \times S^1$ and has vanishing Maslov class. Furthermore, there are uniform constants $C > 2$, $\delta' > 0$, depending only on N and the estimates in Proposition 4.12, such that

(1) *the function ℓ_0 satisfies*

$$C^{-1}K < \ell_0|_{M_K} < CK,$$

(2) *the second fundamental form satisfies*

$$|A|^2_{g_{\mathrm{TY}}} \leq CK^{-2},$$

(3) *the mean curvature satisfies*

$$|H|^2_{g_{\mathrm{TY}}} \leq Ce^{-\delta' K^{2n}},$$

(4) *the volume satisfies*

$$(1-C^{-1})\mathrm{Vol}_{g_D}(N) \leq \mathrm{Vol}_{g_{\mathrm{TY}}}(M_K) \leq (1+C^{-1})\mathrm{Vol}_{g_D}(N),$$

(5) *the first positive eigenvalue $\lambda_1(M_K)$ of (M_K, g_{TY}) satisfies*

$$C^{-1}\lambda_1(N)\ell_0^{-2} \leq \lambda_1(M_K) \leq C\lambda_1(N)\ell_0^{-2},$$

(6) *(M_K, g_{TY}) is κ_0-non-collapsing on scale r_K, where*

$$\kappa_0 = C^{-1}, \qquad r_K = C^{-1}K^{1-n},$$

where C, κ_N are uniform constants depending only on N and the estimates in Proposition 4.12.

Proof
The proof is the culmination of the estimates in Section 4.1 together with the arguments in Section 2. Given K large, define $\varepsilon > 0$ by $K = (-\log(\varepsilon))^{\frac{1}{2n}}$. Let M_ε be the special Lagrangian in $(C, \omega_C, J_C, \Omega_C)$ constructed in Section 4.1. By the calculations in Section 4.1, the above estimates hold, with constants depending only on n, N for M_ε. We now follow the arguments in Section 2 to transplant and perturb M_ε to a Lagrangian in the Tian–Yau space. To this end, let F_t be the time t flow of the vector field V_t defined in (4.11), and let $g_t = (1-t)g_C + t\Phi^* g_{\mathrm{TY}}$ as above. By definition, we have that $F_t^*\omega_t = \omega_C$, so $F_t(M_\varepsilon)$ is Lagrangian with Maslov class 0 with respect to ω_t. It follows that

$$M_K := \Phi\big(F_1(M_\varepsilon)\big) \subset (X, \omega_{\mathrm{TY}})$$

is Lagrangian with vanishing Maslov class.

To control the geometry, we begin by estimating the function $\ell_0|_{F_t(M_\varepsilon)}$. By definition, we have $\ell_0|_{F_0(M_\varepsilon)} = K$. By Remark 4.2 and the estimates in Proposition 4.12, we have

$$|\nabla_{g_t}\ell_0^{n+1}|_{g_t}^2 \le \frac{(n+1)^2}{4n} + Ce^{-\underline{\delta}\ell_0^{2n}} \le (n+1)^2,$$

provided that K is sufficiently large. Therefore,

$$\left|\frac{\partial}{\partial t}\ell_0^{n+1}\right| = \left|\langle \nabla_{g_t}\ell_0^{n+1}, V_t\rangle_{g_t}\right| \le (n+1)e^{-\underline{\delta}\ell_0^{2n}}.$$

Thus, if K is sufficiently large, then we will have

$$\frac{K}{2} \le \ell_0(M_K) \le 2K.$$

It remains to control the geometry of $(F_t(M_\varepsilon), g_t)$. By the discussion in Section 2, it suffices to control the geometry of $M_\varepsilon \subset C$ with respect to the Riemannian metrics $\tilde{g}_t := F_t^* g_t$. By Corollary 4.14 together with Lemma 2.1 we have that

$$\begin{aligned} |\partial_t \tilde{g}_t|_{\tilde{g}_t} &\le C_0 e^{-\underline{\delta}\ell_0^{2n}}, \\ |\nabla_{\tilde{g}_t}\partial_t \tilde{g}_t|_{\tilde{g}_t} &\le C_0 e^{-\underline{\delta}\ell_0^{2n}}. \end{aligned} \tag{4.13}$$

Let us first consider the second fundamental form and the mean curvature of M_ε with respect to $\tilde{g}_t$. Consider the ODE

$$\frac{df}{dt} = c(f^{\frac{1}{2}} + f), \qquad f(0) \geq 0, \tag{4.14}$$

whose solution is $f(t) = (-1 + [1 + f(0)^{\frac{1}{2}}]e^{\frac{ct}{2}})^2$. By Lemma 2.2, equation (4.13), and Proposition 4.15(1), both $|H|^2(t)$ and $|A|^2(t)$ are subsolutions of (4.14) with constant $c = C'e^{-\delta' K^{2n}}$, where $C', \delta' > 0$ are uniform constants. For K sufficiently large depending only on C', δ', we obtain

$$|A|^2(t) \leq 100(C')^2 e^{-2\delta' K^{2n}} + 4|A|^2(0),$$
$$|H|^2(t) \leq 100(C')^2 e^{-2\delta' K^{2n}} + 4|H|^2(0).$$

Since M_ε is minimal with respect to $\tilde{g}_0$, we have $|H|(0) = 0$, while by Lemma 4.8 we have $|A|(0) \leq C_1 K^{-2}$ for a constant C_1 depending only on n and N. This establishes (2).

Estimate (4) follows immediately from (4.13), while (5) follows from (4.13) and Lemma 2.3. Finally, estimate (6) follows from Lemma 2.3, since (4.13) together with (1) implies that for K sufficiently large, depending only on N, n and the constants in Proposition 4.12,

$$\frac{1}{2}\tilde{g}_t \leq \tilde{g}_0 \leq 2\tilde{g}_t$$

for all $t \in [0, 1]$. Combining these calculations with the discussion at the beginning of Section 2, we obtain the result. □

4.3. The mean curvature flow

The only remaining task is to prove that the almost special Lagrangian manifold M_K constructed in Proposition 4.15 can be perturbed to a special Lagrangian. To do this, we will show that for K sufficiently large the Lagrangian mean curvature flow starting from M_K converges to a special Lagrangian. Furthermore, by controlling the scale function along the flow, we will show that we can construct infinitely many distinct special Lagrangians.

In Section 3, we appealed to a theorem of Li [62]; see Theorem 3.6. However, it is clear that this result does not apply in the Tian–Yau spaces, since the injectivity radius is not bounded below. More crucially, however, in order to make the mean curvature of the initial manifold M_K very small, we may have to take K very large. In turn, by the estimates in Proposition 4.15, this causes $\lambda_1(M_K)$ and the noncollapsing scale to become *even smaller*.

We therefore need an effective version of Li's result tailored to our situation which exploits the fact that the mean curvature of M_K decays exponentially in K, while the quantities r_K, $\lambda_1(M_K)$ decay only polynomially.

Before beginning the proof, let us fix some notation. In what follows, we will use unbarred quantities g, ∇, Rm, ∇Rm, and so forth, to denote quantities associated with the manifold (X, g_{TY}). The corresponding barred quantities $\overline{g}$, $\overline{\nabla}$, $\overline{\mathrm{Rm}}$, $\overline{\nabla}\,\overline{\mathrm{Rm}}$ will denote quantities computed on M_t with respect to the induced metric.

Let us briefly explain the idea of the proof. The key result is the following.

LEMMA 4.16
Let M_t be compact Lagrangian submanifolds of vanishing Maslov class moving by the LMCF. Then the mean curvature satisfies

$$\frac{\partial}{\partial t}\int_{M_t} |H|_g^2 \, d\mathrm{Vol}_g \leq -2\big(\lambda_1(M_t) - \sup_{M_t} |A|_g |H|_g\big) \int_{M_t} |H|_g^2 \, d\mathrm{Vol}_g,$$

where g denotes the metric induced by the Tian–Yau metric on X and $\lambda_1(M_t)$ is the first positive eigenvalue of the Laplacian on $(M_t, g|_{M_t})$. In particular, if $\lambda_1(M_t) > \varepsilon$ and $\varepsilon > 2\sup_{M_t} |A|_g |H|_g$ on some interval $[0, T]$, then we have

$$\int_{M_t} |H|_g^2 \, d\mathrm{Vol}_g \leq e^{-\varepsilon t} \int_{M_0} |H|_g^2 \, d\mathrm{Vol}_g$$

on $[0, T]$.

Proof
The proof is straightforward. To ease notation, we will suppress the metric g, with the understanding that it is the metric induced by g_{TY}. A standard computation (see [80]) shows that the mean curvature 1-form satisfies

$$\frac{\partial}{\partial t} H_j = \nabla_j \nabla^i H_i$$

along the flow. Combining this formula with the evolution for the metric $\bar{g}$ yields

$$\frac{\partial}{\partial t}\int_{M_t} |H|^2 \, d\mathrm{Vol} \leq 2\int_{M_t} \big(\overline{g}^{j\ell} H_\ell \nabla_j \nabla^i H_i + 2|A||H|^3 - |H|^4\big) \, d\mathrm{Vol}.$$

Integration by parts on the first term yields

$$\frac{\partial}{\partial t}\int_{M_t} |H|^2 \, d\mathrm{Vol} \leq -2\int_{M_t} |\nabla^i H_i|^2 + 2\sup_{M_t} |A||H| \int_{M_t} |H|^2 \, d\mathrm{Vol}.$$

Now, by the Maslov class 0 assumption there is a function $\theta(t)$ so that $H_j = \nabla_j \theta(t)$. In particular, we have

$$\int_{M_t} |\nabla^i H_i|^2 = \int_{M_t} |\Delta_{\overline{g}} \theta|^2.$$

Write $\theta = \sum_i f_i$, where $f_i = \alpha_i \psi_i$ for $\alpha_i \in \mathbb{R}$ and ψ_i is an orthonormal basis of L^2 consisting of eigenfunctions of $\Delta_{\bar{g}}$; we say that ψ_i has eigenvalue λ_i if $\Delta\psi_i + \lambda_i \psi_i = 0$. Then we have (suppressing the volume form)

$$\begin{aligned}\int_{M_t} |\Delta_{\bar{g}}\theta|^2 &= \sum_i \lambda_i^2 \int_{M_t} f_i^2 \geq \lambda_1 \sum_i \lambda_i \int_{M_t} f_i^2 \\ &= -\lambda_1 \int_{M_t} \theta \Delta_{\bar{g}} \theta = \lambda_1 \int_{M_t} |\nabla\theta|^2.\end{aligned}$$

As a consequence, we have

$$\int_{M_t} |\nabla^i H_i|^2 \, d\mathrm{Vol} \geq \lambda_1 \int_{M_t} |H|^2 \, d\mathrm{Vol},$$

and the lemma follows immediately. □

The general idea of the proof is that if $\lambda_1(M_t)$ is large compared to $|A|$, $|H|$ and bounded from below on some time interval $[0, T]$, then the previous lemma implies the exponential decay of the L^2-norm of the mean curvature. This implies pointwise exponential decay for $|H|^2$, provided that M_t is noncollapsing and $|\nabla H|^2$ is controlled. The exponential decay of the mean curvature strongly controls the geometry of the flow on $[0, T]$ and yields exponential decay on an even larger interval. To ease the presentation, we make the following definition.

Definition 4.17

A Maslov class 0 Lagrangian submanifold $M \subset (X, \omega_{\mathrm{TY}}, g_{\mathrm{TY}})$ has (C, K, δ')-bounded geometry if

(1) the function ℓ_0 satisfies

$$C^{-1}K < \ell_0|_M < CK,$$

(2) the second fundamental form satisfies

$$|A|^2 \leq CK^{-2},$$

(3) the mean curvature satisfies

$$|H|^2 \leq Ce^{-\delta' K^{2n}},$$

(4) the volume satisfies

$$C^{-1} \leq \mathrm{Vol}(M) \leq C,$$

(5) the first positive eigenvalue $\lambda_1(M)$ satisfies

$$C^{-1}K^{-2} \leq \lambda_1(M_K) \leq CK^{-2},$$

(6) (M, g_{TY}) is κ_0-non-collapsing on scale r_0, where

$$\kappa_0 \geq C^{-1}, \qquad r_0 \geq C^{-1}K^{1-n}.$$

As a first step, we show that control of $|A|^2$, $|H|^2$, and ℓ_0 at time $t = 0$ implies control on a suitably large time interval.

LEMMA 4.18
Suppose that M_0 is a Maslov class 0 Lagrangian in $(X, \omega_{\mathrm{TY}})$ with (C, K, δ')-bounded geometry. Let M_t be a solution of the mean curvature flow starting at M_0. Then for all $\delta \in (0, 10)$ there is a constant $\alpha = \alpha(C, \delta) > 0$ so that M_t has $((1+\delta)C, K, \delta')$-bounded geometry for $t \in [0, \alpha K^2)$.

Proof
Define three times $T_S, T_A, T_H > 0$ by

$$T_S := \sup\Big\{s > 0 : \frac{1}{(1+\delta)}C^{-1}K \leq \ell_0|_{M_t} < (1+\delta)CK \text{ for all } t \in [0, s)\Big\},$$
$$T_A := \sup\big\{s > 0 : |A|^2(t) < (1+\delta)CK^{-2} \text{ for all } t \in [0, s)\big\},$$
$$T_H := \sup\big\{s > 0 : |H|^2(t) < (1+\delta)Ce^{-\delta' K^{2n}} \text{ for all } t \in [0, s)\big\}.$$

We first estimate T_A. Recall the evolution equation for the norm of the second fundamental form (see [80]):

$$\frac{\partial}{\partial t}|A| \leq \Delta|A| + 8|A|^3 + 20|\mathrm{Rm}||A| + 4|\nabla \mathrm{Rm}|.$$

By Corollary 4.13, there is a uniform constant D so that on $[0, T_S)$ we have

$$\sup_{t\in[0,T_S)} \sup_{M_t} |\mathrm{Rm}| \leq D\big((1+\delta)CK^{-2}\big), \qquad \sup_{t\in[0,T_S)} \sup_{M_t} |\nabla \mathrm{Rm}| \leq D\big((1+\delta)CK^{-3}\big).$$

Combining this with the definition of T_A, we conclude that on the interval $[0, \min\{T_S, T_A\})$ there holds

$$\frac{\partial}{\partial t}|A| \leq \Delta|A| + 10^3(C^{3/2}K^{-3} + DC^{3/2}K^{-3} + DCK^{-3}).$$

By the comparison principle, there is a constant $c_A > 0$, depending only on C, D, δ so that

$$T_A \geq \min\{c_A K^2, T_S\}.$$

We estimate T_H in a similar way. Recall that along the LMCF, $|H|$ satisfies the inequality

$$\frac{\partial}{\partial t}|H| \leq \Delta|H| + 2|A|^2|H| + |\mathrm{Rm}||H|.$$

Arguing as above, we have that, as long as $0 < t < \min\{T_S, c_A K^2\}$, there holds

$$\frac{\partial}{\partial t}|H| \leq \Delta|H| + 2|A|^2|H| + |\mathrm{Rm}||H| \leq \Delta|H| + (1+\delta)CK^{-2}(1+D)|H|,$$

and so by the comparison principle,

$$\left|H(t)\right|^2 \leq \left|H(0)\right|^2 e^{2(1+\delta)CK^{-2}(1+D)t} \leq Ce^{-\delta' K^{2n}} e^{100CK^{-2}(1+D)t}.$$

As a result, there is a constant c_H depending only on C, D, δ so that $T_H \geq \min\{c_H K^2, c_A K^2, T_S\}$.

On the other hand, it is easy to see that $T_S > T_H$. Arguing as in the proof of Proposition 4.15, by the equivalence of $\Phi^* g_{\mathrm{TY}}$, g_C near infinity, we have

$$\left|\frac{\partial}{\partial t}\ell_0^{n+1}(M_t)\right| \leq |\nabla \ell_0^{n+1}||H| \leq (n+1)Ce^{-\delta' K^{2n}}$$

as long as $t \leq T_H$. In particular, we can certainly choose a constant c_S depending only on n, δ, δ', C so that if $t < \min\{T_H, c_S K^2\}$, then we have

$$\frac{1}{(1+\delta)}C^{-1}K \leq \ell_0|_{M_t} \leq (1+\delta)CK,$$

and so $T_S > \min\{T_H, c_S K^2\}$. Combining these estimates, we conclude that there is a constant α' such that

$$\min\{T_S, T_H, T_A\} \geq \alpha' K^2.$$

It remains only to prove that $\lambda_1(M_t)$ and the noncollapsing scale are under control. This is straightforward. A standard computation shows that the induced metric $\overline{g}_t$ on M_t satisfies

$$\frac{d\overline{g}_t}{dt} = -2\langle H(t), A(t)\rangle_{g_t}.$$

Define

$$\mu(t) = 2\int_0^t \big(\sup_{M_t}|H|\big) \cdot \big(\sup_{M_t}|A|\big)\, dt.$$

Then by Lemma 2.3, we have

$$\lambda_1(M_0)e^{-3\mu(t)} \le \lambda_1(M_t) \le \lambda_1(M_0)e^{3\mu(t)}$$

and trivially $e^{-\mu(t)}\overline{g}(0) \le \overline{g}(t) \le e^{\mu(t)}\overline{g}(0)$. If M_0 is κ_0-non-collapsing on scale r_K, then M_t will be $\kappa_0 e^{-(n+1)\mu(t)}$-non-collapsing on scale r_K. Now, if $t < \alpha K^2$ for $\alpha \le \alpha'$, then we have

$$\mu(t) \le t(1+\delta)^2 C^2 K^{-2} e^{-\delta' K^{2n}} \le \alpha(1+\delta)^2 C^2 e^{-\delta' K^{2n}} \le \frac{1}{3(n+1)} \log(1+\delta),$$

provided that α is taken sufficiently small depending only on C, δ. The lemma follows. □

We extract the bounds for λ_1 and the noncollapsing constant in the following elementary corollary, which follows from [62, Lemma 3.4] and the computations in Section 2.

COROLLARY 4.19
Suppose that M_t evolves by LMCF and that M_0 is κ_0-non-collapsing at scale r_0. Denote by $\lambda_1(M_t)$ the first positive eigenvalue of the Laplacian on M_t, and define

$$\mu(t) = 2\int_0^t \big(\sup_{M_t} |H|\big) \cdot \big(\sup_{M_t} |A|\big)\, dt.$$

Then M_t is $\kappa_0 e^{-(n+1)\mu(t)}$-non-collapsing at scale r_0 and satisfies

$$\lambda_1(M_0)e^{-3\mu(t)} \le \lambda_1(M_t) \le \lambda_1(M_0)e^{3\mu(t)}.$$

Once we have control of the second fundamental form, we get control of all higher derivatives along the flow by the smoothing estimates for the mean curvature flow. In our setting, we can state these estimates succinctly as follows.

LEMMA 4.20
Suppose that M_t has (C, K, δ')-bounded geometry for all $t \in [0, \alpha K^2)$. Then for all $\ell \ge 0$, there is a constant $C(\ell)$ depending only on C, α and the constants in Corollary 4.13 so that, for all $t \in [0, \alpha K^2)$, we have

$$|\nabla^\ell A|^2 \le C(\ell)\frac{K^{-2}}{t^\ell}.$$

This result is well known, but since we have not been able to find a reference in the literature with the dependence we need, we have included a proof in the appendix.

With the smoothing estimates we can turn integral estimates of geometric quantities into pointwise estimates by the following simple lemma (see, e.g., [62, Lemma 3.5]).

LEMMA 4.21
Suppose that a Riemannian manifold (M,g) is κ_0-non-collapsing on scale r_0, and suppose that S is a tensor with

$$\int_M |S|^2 \leq \varepsilon, \qquad |\nabla S| \leq C.$$

If $\varepsilon < r_0^{n+2}$, then

$$\sup_M |S| \leq \Big(\frac{1}{\sqrt{\kappa_0}} + C\Big)\varepsilon^{\frac{1}{n+2}}.$$

The next step is to show that exponential decay of the mean curvature, together with a bound on the second fundamental form, implies *improved* estimates on the second fundamental form.

LEMMA 4.22
Suppose that M_0 has (C,K,δ')-bounded geometry, and let M_t be the solution of the LMCF with initial data M_0. Suppose that, on some interval $[0,T]$, M_t has $(4C,K,\frac{\delta'}{n+2})$-bounded geometry and, furthermore, that there is a constant $a>0$ so that

$$\big|H(t)\big|^2 < e^{-\frac{\delta'}{n+2}K^{2n}-at}.$$

Then for K sufficiently large depending only on C, n, δ', we have

$$\big|A(t)\big|^2 \leq 2CK^{-2} + \frac{1}{a}e^{-\frac{\delta'}{n+2}K^{2n}}$$

for all $t \in [0,T]$.

Proof
By Lemma 4.18, we can assume that $T > \alpha K^2$ for α depending only on C. If K is sufficiently large, depending only on C, then we may assume that $T > 1$ and that

$$\big|A(t)\big|^2 < 2CK^{-2} \quad \text{for } t \in [0,1]. \tag{4.15}$$

By the smoothing estimates in Lemma 4.20, for all $\ell \geq 0$ and $t \in [1,T)$ we have

$$|\nabla^\ell A|^2 \leq C(\ell)K^{-2} \tag{4.16}$$

for a constant $C(\ell)$ depending only on ℓ and C.

The second fundamental form satisfies the following inequality along the flow (see [80]):

$$\frac{\partial}{\partial t}|A|^2 \leq 100\big(|A||\nabla^2 H| + |A|^3|H| + |\mathrm{Rm}||H|\big). \tag{4.17}$$

Since the mean curvature decays exponentially, the only problematic term is $|\nabla^2 H|$. On the other hand, using (4.16) and integrating by parts gives

$$\int |\nabla^2 H|^2 \leq \int |H||\nabla^4 H| \leq \sqrt{C(4)}K^{-1}e^{-\frac{\delta'}{2(n+2)}K^{2n}-\frac{a}{2}t}$$
$$\leq e^{-\frac{\delta'}{2(n+2)}K^{2n}-\frac{a}{2}t}$$

for $t > 1$, provided that K is sufficiently large depending only on C. Since M_t has $(4C, K, \frac{\delta'}{n+2})$-bounded geometry, we can apply Lemma 4.21 to conclude that, for all $t \in [1, T)$, there holds

$$|\nabla^2 H| \leq \big(2\sqrt{C} + \sqrt{C(3)}K^{-1}\big)e^{-\frac{\delta'}{2(n+2)^2}K^{2n}-\frac{a}{2(n+2)}t},$$

provided that we choose K sufficiently large, depending only on δ', C, n so that

$$e^{-\frac{\delta'}{2(n+2)}K^{2n}-\frac{a}{2}t} \leq \frac{K^{(1-n)(n+2)}}{(4C)^{n+2}}.$$

Plugging this estimate into (4.17), we conclude that, for K sufficiently large, depending only on δ', C, n and the constants in Corollary 4.13, we have

$$\frac{\partial}{\partial t}|A|^2(t) \leq e^{-\frac{\delta'}{2(n+2)^2}K^{2n}-\frac{a}{2(n+2)}t}.$$

Combining this estimate with (4.15) and integrating in time yields

$$\big|A(t)\big|^2 \leq 2CK^{-2} + \frac{2(n+2)}{a}e^{-\frac{\delta'}{2(n+2)^2}K^{2n}} \quad \text{for all } t \in [0, T]. \qquad \square$$

We can now prove the main theorem.

THEOREM 4.23
Fix constants $C > 1$ and $\delta' > 0$. There is a constant $K_0 > 0$ such that, for all $K \geq K_0$, if M_K is a Lagrangian submanifold of $(X, \omega_{\mathrm{TY}})$ with Maslov class 0 and (C, K, δ')-bounded geometry, then the LMCF starting at M_K converges smoothly and exponentially fast to a special Lagrangian M_∞ with $(4C, K, \frac{\delta'}{n+2})$-bounded geometry.

Proof
Let M_t be the solution of the LMCF with initial data $M_0 = M_K$. First we show that,

for K sufficiently large, as long as M_t has $(4C, K, \frac{\delta'}{n+2})$-bounded geometry, the mean curvature decays exponentially.

Suppose that M_t has $(4C, K, \frac{\delta'}{n+2})$-bounded geometry on $[0, T]$. By Lemma 4.18, we can assume that $T > 1$ and M_t has $(2C, K, \delta')$-bounded geometry on $[0, 1]$, provided that K is large enough depending only on C. For all $t \in [0, T]$, we have

$$\begin{aligned}\lambda_1(M_t) - \sup_{M_t} |A||H| &\geq (4C)^{-1}K^{-2} - 4CK^{-1}e^{-\frac{\delta'}{2(n+2)}K^2}\\ &> (8C)^{-1}K^{-2} =: aK^{-2},\end{aligned}$$

provided that K is sufficiently large depending on C, δ', n. Therefore, by Lemma 4.16 we have

$$\int_{M_t} |H|^2 \leq e^{-aK^{-2}t} \int_{M_0} |H|^2 \leq C^2 e^{-\delta' K^{2n} - aK^{-2}t},$$

since M_0 has (C, K, δ')-bounded geometry. We now use Lemma 4.21 to turn this estimate into a pointwise bound. By the smoothing estimates in Lemma 4.20, we have

$$|\nabla A|^2 \leq C(1)K^{-2}$$

for $t \in [1, T]$ for a constant $C(1)$ depending only on $C, (X, g_{\mathrm{TY}})$. Therefore, Lemma 4.21 yields the estimate

$$|H|(t) \leq \big(2\sqrt{C} + \sqrt{C(1)}K^{-1}\big)e^{-\frac{a}{n+2}K^{-2}t} C^{\frac{2}{n+2}} e^{-\frac{\delta'}{n+2}K^{2n}}$$

as long as K is chosen sufficiently large, depending only on C, δ', so that

$$e^{-aK^{-2}t}C^2 e^{-\delta' K^{2n}} < (4C)^{-(n+2)} K^{-(n-1)(n+2)}.$$

Increasing K if necessary, we can assume that

$$|H|^2(t) \leq e^{-\frac{\delta'}{n+2}K^{2n} - \frac{2a}{n+2}K^{-2}t} \tag{4.18}$$

for all $t \in [1, T]$. In particular, we have shown that $|H|^2(t)$ decays exponentially as long as M_t has $(4C, K, \frac{\delta'}{n+2})$-bounded geometry and K is chosen sufficiently large depending on n, δ', C and (X, g_{TY}).

Next, we claim that this exponential decay implies that M_t has $(4C, K, \frac{\delta'}{n+2})$-bounded geometry for all time. Define

$$T_{\max} = \sup\Big\{T : M_t \text{ has } \Big(4C, K, \frac{\delta'}{n+2}\Big)\text{-bounded geometry } \forall t \in [0, T)\Big\}.$$

First, for $t \in [0, T_{\max})$ we estimate

$$\begin{aligned}\mu(t) &= 2\int_0^t \big(\sup_{M_t}|H|\big)\cdot\big(\sup_{M_t}|A|\big)\,dt \\ &\le 8Ce^{-\frac{\delta'}{2(n+2)}K^{2n}}K^{-1} + 2\sqrt{C}K^{-1}\int_1^T e^{-\frac{\delta'}{n+2}K^{2n}-\frac{2a}{n+2}K^{-2}t}\,dt \\ &< \frac{1}{3(n+1)}\log(2),\end{aligned}$$

provided that K is sufficiently large depending only on n, C, δ'. In particular, by Corollary 4.19, on M_t we have

$$\kappa_0(M_t) > \frac{1}{2}C^{-1}, \qquad r_0(M_t) = r_0, \qquad \lambda_1(M_t) \ge \frac{1}{2}C^{-1}K^{-2}.$$

Similarly, we have

$$\frac{\partial}{\partial t}\mathrm{Vol}(M_t) = -\int_{M_t}|H_t|^2 \ge -4Ce^{-\frac{\delta'}{n+2}K^{2n}-\frac{2a}{n+2}K^{-2}t}.$$

Thus, if K is sufficiently large depending only on n, C, δ', then

$$C \ge \mathrm{Vol}(M_0) \ge \mathrm{Vol}(M_t) \ge \frac{1}{2}\mathrm{Vol}(M_0) \ge \frac{1}{2C}.$$

To control ℓ_0, we argue as in the proof of Proposition 4.15. Thanks to Remark 4.2 and Proposition 4.12, for K sufficiently large depending only on (X, g_{TY}) we have $|\nabla \ell_0^{n+1}|^2_{g_{\mathrm{TY}}} \le (n+1)^2$. Therefore,

$$\Big|\frac{\partial}{\partial t}\ell_0^{n+1}\Big| \le (n+1)\big|H(t)\big| \le (n+1)e^{-\frac{\delta'}{2(n+2)}K^{2n}-\frac{a}{n+2}K^{-2}t}.$$

Choosing K sufficiently large depending only on δ', n, C, we can ensure that

$$\frac{1}{2}\ell_0|_{M_0} \le \ell_0|_{M_t} \le 2\ell_0|_{M_0}.$$

Finally, we apply Lemma 4.22 to conclude that for K sufficiently large depending on C, δ', n, we have

$$\big|A(t)\big|^2 \le 2CK^{-2} + (n+2)^2 8CK^2 e^{-\frac{\delta'}{2(n+2)^2}K^{2n}} \quad \text{for all } t \in [0, T].$$

Increasing K if necessary, we obtain $|A(t)| < 3CK^{-2}$. It follows that M_t has $(3C, K, \frac{\delta'}{n+2})$-bounded geometry on $[0, T_{\max}]$. By Lemma 4.18, it follows that $T_{\max} = +\infty$ and M_t has $(4C, K, \frac{\delta'}{n+2})$-bounded geometry for all time. The estimate (4.18) holds for all time, and hence M_t converges smoothly and exponentially fast to a special Lagrangian with $(4C, K, \frac{\delta'}{n+2})$-bounded geometry. □

As an immediate consequence, we obtain Theorem 1.1 for Tian–Yau spaces.

Proof of Theorem 1.1 for Tian–Yau spaces
By Proposition 4.15, there are constants $C > 1$, $\delta' > 0$, $K_0 \gg 1$ depending only on $N, (X, \omega_{\mathrm{TY}})$ such that, for $K \geq K_0$, $(X, \omega_{\mathrm{TY}})$ admits Lagrangians M_K with (C, K, δ')-bounded geometry and zero Maslov class. By Theorem 4.23, after possibly increasing K_0 depending only on C, n, δ', we can assume that the LMCF starting at M_K converges to a special Lagrangian submanifold $M_{K,\infty}$ with $(4C, K, \frac{\delta'}{n+1})$-bounded geometry. Define a sequence K_i, starting with K_0, having $K_i = 100C^2 K_{i-1}$. Then we have

$$4CK_i < (4C)^{-1} K_{i+1},$$

and so, using the scale function ℓ_0, we see that $M_{K_i,\infty}$ is disjoint from $M_{K_j,\infty}$ for all $i \neq j$. □

Remark 4.24
Note that the proof, and in particular the exponential decay of the mean curvature, shows that for any $\varepsilon, C, \delta' > 0$, there is a constant $K(\varepsilon, C, \delta')$ with the following effect. If M_0 has (C, K, δ')-bounded geometry for $K \geq K(\varepsilon, C, \delta')$, then the Lagrangian mean curvature flow starting from M_0 converges to an immersed special Lagrangian $M_\infty \subset (X, \omega_{\mathrm{TY}}, g_{\mathrm{TY}})$ and $M_\infty \subset B(M_0, \varepsilon)$. In particular, if N, N' are two special Lagrangians in D, then for ℓ_0 sufficiently large, the LMCF starting from the models M_0, M_0' constructed in Proposition 4.15 will converge to disjoint special Lagrangians. Clearly, the same result holds for the constructions in Section 3.

5. Special Lagrangian fibrations in dimension 2

In this section, we prove that, under fairly general assumptions in complex dimension 2, the existence of a single special Lagrangian torus with primitive homology class and zero self-intersection in a Calabi–Yau surface with controlled geometry implies the existence of a global special Lagrangian torus fibration. The three main tools we use are the deformation theory of special Lagrangians, hyper-Kähler rotation, and the moduli and compactness theory of holomorphic curves.

Recall that a hyper-Kähler manifold is a Riemannian manifold (X, g) equipped with a triple of parallel, orthogonal, integrable complex structures (I, J, K) satisfying the quaternion relations

$$I^2 = J^2 = K^2 = IJK = -1.$$

This data yields an S^2 worth of complex structures given by $\{(aI + bJ + cK) : a^2 + b^2 + c^2 = 1\}$ on (X, g) compatible with the Riemannian structure, inducing distinct Kähler structures on (X,g). Equivalently (see [24]), in real dimension 4, a hyper-Kähler structure on the oriented manifold $(X, d\mathrm{Vol}_0)$ is a triple of closed 2-forms $(\omega_1,\omega_2,\omega_3)$ satisfying, for every $1 \le i \le j \le 3$,

$$\frac{1}{2}\omega_i \wedge \omega_j = Q_{ij}\, d\mathrm{Vol}_0,$$
$$\frac{1}{2}\omega_i \wedge \omega_j = \frac{1}{6}\delta_{ij}(\omega_1^2 + \omega_2^2 + \omega_3^2)$$

for Q_{ij} a positive definite matrix. The hyper-Kähler triple $(\omega_1,\omega_2,\omega_3)$ induces a Riemannian metric g such that each ω_j is self-dual with respect to g. Such a metric g is called a *hyper-Kähler metric*. Each form ω_i is symplectic and induces an integrable complex structure J_i such that $\Omega_i = \omega_j + \sqrt{-1}\omega_k$, $i \ne j \ne k$ is a holomorphic 2-form.

In the present setting, we have a Calabi–Yau manifold (X,g,J,ω,Ω) of complex dimension 2 with Kähler form satisfying

$$\omega^2 = \frac{1}{2}\Omega \wedge \bar{\Omega}.$$

Direct calculation shows that $(\mathrm{Re}(\Omega),\omega,\mathrm{Im}(\Omega))$ is a hyper-Kähler triple, with associated complex structures (I,J,K) satisfying the quaternion relations. Finally, associated to the hyper-Kähler structure is the twistor space $\mathcal{X}$, a smooth complex manifold diffeomorphic to $X \times \mathbb{P}^1$, but with complex structure on the fiber over $\zeta \in \mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$ given by

$$J_\zeta = \frac{\sqrt{-1}(-\zeta + \bar{\zeta})I - (\zeta + \bar{\zeta})K + (1 - |\zeta|^2)J}{1 + |\zeta|^2}.$$

In particular, $\mathcal{X}$ has a nontrivial holomorphic fibration. The holomorphic volume form on a fiber (X,g,J_ζ) for $\zeta \in \mathbb{P}^1$ is given by

$$\Omega_\zeta = \Omega + 2\zeta\sqrt{-1}\omega - \zeta^2\overline{\Omega}. \tag{5.1}$$

A crucial point for us is the observation that if $L \subset (X,g,J,\omega,\Omega)$ is special Lagrangian with $\omega|_L = \mathrm{Im}(\Omega)|_L = 0$ and $\mathrm{Re}(\Omega)|_L = d\mathrm{Vol}_g$, then Wirtinger's inequality implies that L is a *holomorphic* subvariety of the Calabi–Yau manifold $(X,g,I,\mathrm{Re}(\Omega),\omega - \sqrt{-1}\,\mathrm{Im}(\Omega))$. We will denote this Calabi–Yau manifold by (X,g,I).

We begin with the following lemma.

LEMMA 5.1
Suppose that $L \subset (X,g,J,\omega_J)$ is a (possibly immersed) special Lagrangian torus

and that $[L]^2 = 0$. *Then* L *is embedded and there exists a neighborhood* U *of* L *and a fibration* $\pi : U \to B$ *to a complex manifold* B *such that the fibers of* π *are special Lagrangian tori.*

Proof
McLean's deformation theory for (possibly immersed) special Lagrangian tori (see [66]; see also [51]) implies that for each nonzero harmonic 1-form representing a class in $H^1(L)$, we obtain a nontrivial deformation L' of L. By hyper-Kähler rotating, we obtain holomorphic curves $C \neq C'$ with $[C] = [C'] = [L]$. By assumption, $0 = [L]^2 = [C] \cdot [C'] = [L].[L']$, so $C \cap C' = \emptyset$, and hence L, L' are disjoint. Furthermore, since $C^2 = 0$ and the canonical bundle of (X, I) is trivial, the adjunction formula (see [9, p. 69]) implies that any immersed torus fiber π is, in fact, a smooth embedded torus. It follows that the deformations of L are all disjoint smooth, embedded Lagrangian tori, and hence we obtain an open set U containing L such that $\pi : U \to B$, where B is an open neighborhood of $0 \in H^1(L)$, is a fibration whose fibers are embedded Lagrangian tori. That the base of the fibration admits a natural complex structure is due to Hitchin [51]. □

The remainder of this section is devoted to proving that this local fibration extends to a global fibration. The basic idea is to prove that the set of points which lie on a (possibly singular) special Lagrangian L' deformable to L is both open and closed. We will make heavy use of the theory of holomorphic curves. Since our manifold is not compact, we need a result to ensure that our holomorphic curves cannot escape to infinity. We begin by noting the following lemma, which is likely well known to experts in the field.

LEMMA 5.2
Suppose that (X, g, J) *is a complete Kähler manifold. For* $p \in X$, *let* $\mathrm{inj}(p)$ *denote the injectivity radius, and let*

$$K(p) = \sup_{B(p,\mathrm{inj}(p))} |\mathrm{Rm}|.$$

There is a universal constant $C_1 > 0$ *with the following effect. For any* $r < \min\{\mathrm{inj}(x), C_1 K(x)^{-1/2}\}$, *if* $u : \Sigma \to X$ *is a* J*-holomorphic curve with* $x \in u(\Sigma)$ *and* $u(\partial\Sigma) \subset \partial B(x, r)$, *then*

$$\mathrm{Area}\big(u(\Sigma) \cap B(x, r)\big) \geq \frac{\pi}{4} r^2.$$

Proof
This is a standard fact in symplectic geometry, which is based on the fact that holo-

morphic curves are absolutely area minimizing in their homology class; we refer the reader to [79] for a transparent proof. However, since the dependence on the geometry is not explicit there, we sketch the details. First, using the Rauch theorems, one can easily show that there is a universal constant $R > 0$ so that, in normal coordinates centered at x, we have

$$\frac{1}{2} g_{\mathrm{Euc}} \leq g \leq 2 g_{\mathrm{Euc}}$$

on any ball of radius $r < \min\{\mathrm{inj}(x), C_1 K(x)^{-1/2}\}$. In particular, it suffices to estimate the area of $u(\Sigma) \cap B(x,r)$ with respect to the Euclidean metric. We now apply a comparison argument that goes back to Blaschke [15, p. 176]. Since $u(\partial\Sigma) \subset \partial B(x,r)$, we can choose a point $x_0 \in u(\partial\Sigma)$. Taking the cone of $u(\partial\Sigma)$ over x_0 yields a surface D developable onto a disk. One can then apply the isoperimetric inequality in the plane for nonsimple curves (see [7]) to conclude that—in the Euclidean space—the area and length satisfy

$$4\pi \,\mathrm{Area}_{\mathrm{Euc}}(D) \leq \mathrm{Length}^2_{\mathrm{Euc}}(\partial D) = \mathrm{Length}^2_{\mathrm{Euc}} u(\partial\Sigma).$$

(We refer the reader to [69] for a nice discussion of this argument.) We can now apply [79, Proposition 4.3.1]. □

PROPOSITION 5.3

Let (X, g) be a complete Kähler manifold. Fix a point $x_0 \in X$, and let $r(x) = d(x_0, x)$. Suppose that

(1) *the sectional curvature of (X, g) is bounded by a constant C_2,*

(2) *there is a nonincreasing function $f : [0,\infty) \to \mathbb{R}_{>0}$ such that $\int_0^{+\infty} f(s)\,ds = +\infty$ and*

$$\mathrm{inj}(x) \geq f\big(r(x)\big).$$

Let K be a compact set in X. If Σ is a connected holomorphic curve with $\Sigma \cap K \neq \emptyset$, $\partial\Sigma \subset K$, and $\mathrm{Area}(\Sigma) \leq A$, then there is a constant $e = e(X, K, A) > 0$, independent of Σ, so that $\Sigma \subset B_e(K)$.

Proof

Fix x_0 and a constant A as in the statement of the proposition. We may as well assume that $K = B(x_0, R_0)$ for some $R_0 \in \mathbb{N}$. For $m \geq R_0$, let

$$\Sigma_m = \Sigma \cap \big(B(x_0, m+1) \setminus B(x_0, m)\big),$$

and note that each of these sets is either connected or empty. We will show that there is a constant $N \in \mathbb{N}$, $N \geq R_0 + 1$ independent of Σ and depending only on X, A with the following property: if $\mathrm{Area}(\Sigma) \leq A$, then $\Sigma_m = \emptyset$ for $m \geq N$.

For the sake of contradiction, let us suppose that $\Sigma_N \neq \emptyset$. Cover Σ by balls of radius $\delta(x) = 5^{-1}\min\{\mathrm{inj}(x), \frac{1}{2}, C_1 C_2^{-1/2}\}$, where C_1 is the constant appearing in Lemma 5.2. By the Vitali covering lemma, we can extract a countable collection of points x_j such that $B(x_j, \delta(x_j))$ are disjoint and

$$\Sigma \subset \bigcup_{j=0}^{\infty} B\big(x_j, 5\delta(x_j)\big).$$

Define

$$N_m = \{j \in \mathbb{N} : x_j \in \Sigma_m\}, \qquad n_m = \# N_m.$$

Since $\delta(x_j) < 1$ for each $j \in N_m$, we have $B(x_j, \delta(x_j)) \cap K = \emptyset$ and hence, since $\partial\Sigma \subset K$, $B(x_j, \delta(x_j))$ is disjoint from $\partial\Sigma$. We can therefore apply Lemma 5.2, in combination with the fact that the balls $B(x_j, \delta(x_j))$ are disjoint, to obtain

$$\begin{aligned} \mathrm{Area}(\Sigma) &\geq \sum_{R_0 < m} \sum_{j \in N_m} \mathrm{Vol}\big(B\big(x_j, \delta(x_j)\big) \cap \Sigma\big) \\ &\geq \frac{\pi}{4} \sum_{R_0 < m} \sum_{j \in N_m} \delta(x_j)^2. \end{aligned} \tag{5.2}$$

Next, we claim that if $m' > m + 1 > R_0$ and $n_{m'} \neq 0 \neq n_m$, then for every $m < m'' < m'$, we have $n_{m''} \neq 0$. This follows easily from connectedness, since if $\Sigma_{m'} \neq \emptyset$ and $\Sigma_m \neq \emptyset$, then the same is true for every $m'' \in [m, m']$. On the other hand, since $\delta(x_j) < \frac{1}{10}$, it is easy to see that

$$\Sigma_{m''} \not\subset \bigcup_{j \in N_m \cup N_{m'}} B\big(x_j, 5\delta(x_j)\big).$$

Concretely, no point in $\Sigma \cap B(x_0, m'' + \frac{1}{2})$ can be contained in the set on the right-hand side. But, since $B(x_j, 5\delta(x_j))$ cover Σ, we conclude that $N_{m''} \neq \emptyset$. Now, since we assumed that $\Sigma_N \neq \emptyset$, we obtain

$$\mathrm{Area}(\Sigma) \geq \frac{\pi}{4} \sum_{R_0 < m < N} \min_{j \in N_m} \delta(x_j)^2.$$

On the other hand, for each $j \in N_m$ we have

$$\delta(x_j) \geq 5^{-1} \min\Big\{ f(m+1), \frac{1}{2}, C_1 C_2^{-1/2} \Big\}.$$

Thanks to the assumption that $\int_0^\infty f(s)\, ds = \infty$, we conclude that there is a constant $N' \in \mathbb{N}$, depending only on C_1, C_2, f, so that if $N \geq N'$, then

$$\text{Area}(\Sigma) \geq A + 1,$$

a contradiction. □

Remark 5.4
The integrability of the complex structure is not needed in the proof of Proposition 5.3. The result holds even if J is an almost complex structure and the symplectic form ω is "uniformly" J-tame (see [79]). Y. Groman has pointed out to us that he independently obtained a similar result in [37, Theorem 4.10].

We can now state the main theorem of this section, whose proof and consequences will occupy the remainder of this section.

THEOREM 5.5
Let (X, g) be a complete hyper-Kähler surface. Fix a point $x_0 \in X$, and let $r(x) = d(x_0, x)$. Suppose that
(1) *the sectional curvature of (X, g) is bounded,*
(2) *there is a nonincreasing function $f : [0, \infty) \to \mathbb{R}_{>0}$ such that $\int_0^{+\infty} f(s)\, ds = +\infty$ and*

$$\text{inj}(x) \geq f\big(r(x)\big),$$

(3) *X has finite Euler characteristic: $\chi(X) < +\infty$.*

Assume that there exists a (possibly immersed) special Lagrangian torus L with $[L] \in H_2(X, \mathbb{Z})$ primitive and $[L]^2 = 0$. Then
(1) *X admits a special Lagrangian fibration with L as one of the fibers,*
(2) *there are at most $\chi(X)$ singular fibers, each classified by Kodaira and no fiber is multiple,*
(3) *L is a smooth embedded torus.*

Remark 5.6
The assumption that $[L]$ is primitive in $H_2(X, \mathbb{Z})$ is not fundamental and can be weakened. However, since it holds in all cases we have considered and streamlines parts of the argument, we have included it for convenience.

Let us briefly recall the current state of affairs. We have a complete noncompact Calabi–Yau surface $(X, g, J, \omega_J, \Omega_J)$ with bounded curvature, which we can think of as a Tian–Yau space of either type I or type II, though our results apply in a rather general setting. In addition, $(X, g, J, \omega_J, \Omega_J)$ contains a special Lagrangian torus L, with $[L]^2 = 0$. By Lemma 5.1, this special Lagrangian generates a local fibration. We now hyper-Kähler rotate to a complex structure I so that $(X, g, I, \omega_I, \Omega_I)$ is again a

Calabi–Yau surface, but now L becomes a holomorphic submanifold of genus 1 and L has a neighborhood admitting a holomorphic genus 1 fibration. We are going to consider the moduli space of such submanifolds.

Let Σ denote a smooth surface of genus 1, and consider the space

$$\mathcal{M}\big([L], I\big) = \big\{u : (\Sigma, j) \to (X, g, I) : u \text{ is holomorphic, } u_*[\Sigma] = [L]\big\},$$

the moduli space of parameterized I-holomorphic maps from Σ into (X, g, I, ω_I) having image homologous to our fixed elliptic curve L. Note that we are not fixing the complex structure j on Σ, which we allow to vary over the moduli space. We let $\mathcal{M}_1$ denote the connected component of $\mathcal{M}$ containing the fixed holomorphic curve L and let $X_1 \subset X$ be the set of points lying on $u(\Sigma)$ for some $u \in \mathcal{M}_1$. By Lemma 5.1, X_1 is open and, thanks to Hitchin [51], $\mathcal{M}_1$ has a canonical complex structure. The goal of the remainder of this section will be to prove that $X \setminus X_1$ consists of finitely many singular elliptic curves, each classified by Kodaira. For simplicity, denote by $X_2 = \partial X_1$, and note that X_2 is closed. As a first step, we observe the following.

LEMMA 5.7
If $p \in \partial X_1$, then there is an I-holomorphic cusp curve $u : \bigcup_\alpha \Sigma_\alpha \to (X, g, I)$ with $p \in \bigcup_\alpha u(\Sigma_\alpha)$. Furthermore, $u \notin \mathcal{M}_1$.

Proof
This follows immediately from compactness theory for holomorphic curves. If $u_i : \Sigma \to (X, g, I)$ is a sequence of holomorphic maps with points $p_i \in u_i(\Sigma)$ and $p_i \to p$, then by Proposition 5.3, the holomorphic curves $u_i(\Sigma)$ all lie in a fixed compact set. Since every curve in $\mathcal{M}_1$ has the same volume and hence energy, the standard Gromov–Sacks–Uhlenbeck compactness theory (see [38], [65], [73], [93]) implies that u_i converges to a cusp curve (or stable curve) $u : \bigcup_\alpha \Sigma_\alpha \to (X, g, I)$. Here $\bigcup_\alpha \Sigma_\alpha$ is some tree of Riemann surfaces; its precise structure is irrelevant for our current considerations. If $\bigcup_\alpha \Sigma_\alpha = \Sigma$ irreducible, then $u : \Sigma \to (X, g, I)$ and by the deformation theory of holomorphic Lagrangians, Lemma 5.1, we obtain that p is in the interior of X_1, a contradiction. □

Let $\mathcal{M}_2$ denote the space parameterizing I-holomorphic cusp curves, or stable maps, appearing as limits of holomorphic curves in X_1. Recall from [93] that a cusp curve in (X, I) is a disjoint union $\bigcup_\alpha \Sigma_\alpha$ of finitely many Riemann surfaces Σ_α, together with an identification of a finite number of points (called "nodes") and a holomorphic curve $u : \bigcup_\alpha \Sigma_\alpha \to X$, compatible with the identification. Any such holomorphic curve has connected image. Recall that a holomorphic curve $u : \Sigma \to X$ is called *multiply covered* if $u = \hat{u} \circ \pi$, where $\pi : \Sigma \to \Sigma'$ is a holomorphic branched cover of degree larger than 1. The curve u is called *simple* if it is not multiply covered. If u is a multiply covered holomorphic map from $\mathbb{P}^1$, then by the Riemann–Hurwitz formula, $\Sigma' = \mathbb{P}^1$ also.

Given a holomorphic cusp curve $C = u(\Sigma)$, we will denote by $C^{(k)}$ the irreducible components of C, n_k their multiplicities, and denote by

$$u_k : \Sigma_k \to C^{(k)}$$

the associated simple holomorphic curve.

LEMMA 5.8
Suppose that $x \in X_2$. Then there is a number $N(x) \in \mathbb{N}$, depending only on x, such that any component of a cusp curve containing x and corresponding to a point $u \in \mathcal{M}_2$ has number of components (counted with multiplicity) bounded by $N(x)$.

Proof
Suppose that $u_i \in \mathcal{M}_1$ is a sequence of holomorphic maps Gromov–Sacks–Uhlenbeck converging to a stable map u such that $x \in \mathrm{Im}(u)$. Write the image of u as $C = \sum_k n_k C^{(k)}$ for $C^{(k)}$ reduced irreducible holomorphic curves. Notice that $\mathrm{Vol}_g(u_i(\Sigma_i)) = \int_\Sigma u_i^* \omega_I = \mathrm{Vol}_g(L)$ is fixed since the u_i's are homotopic to each other. By Proposition 5.3, all the curves $u_i(\Sigma)$ fall in a fixed compact set $K \subset X$ and hence $C \subset K$. From the convergence we have

$$\mathrm{Vol}_g(L) = \int_C \omega_I = \sum_k n_k \int_{(C_k)_{\mathrm{reg}}} \omega_I .$$

On the other hand, since the sectional curvature and injectivity radius are bounded below, $\int_{(C_k)_{\mathrm{reg}}} \omega_I \geq \hbar > 0$ for some constant $\hbar$ depending on x (see [79, Proposition 4.3.1]). The lemma follows. □

Since L moves in a local fibration, the general fiber is disjoint from any singular curve C obtained as a limit of curves in $\mathcal{M}_1$. Thus, if we write $[C] = \sum_k n_k [C^{(k)}]$, then $0 = [L] \cdot [C^{(k)}] = [C] \cdot [C^{(k)}]$ (see also [9, Proposition III.8.2]). Thus, we obtain the following.

LEMMA 5.9
Suppose that $C = \sum_{k=1}^m n_k C^{(k)}$ is a singular holomorphic curve obtained as a Gromov–Sacks–Uhlenbeck limit of holomorphic curves in $\mathcal{M}_1$. Let Q denote the negative intersection form on the components $[C^{(k)}]$ with components $q_{ij} = -[C^{(i)}] \cdot [C^{(j)}]$. Then Q is positive semidefinite and the annihilator of Q is 1-dimensional and spanned by $[C] = \sum_k n_k [C^{(k)}]$. In particular, if there is a component $[C^{(\ell)}]$ such that $[C^{(\ell)}]^2 = 0$, then $C = n_\ell C^{(\ell)}$ has only one component.

Proof
Consider the vector space over $\mathbb{Q}$ spanned by the classes $[C^{(k)}]$ with the quadratic form defined as above.

Then we have $q_{pk} \leq 0$ for all $p \neq kl$. Furthermore, since C is connected, there is no partition of $\{1, \dots, m\}$ into nonempty disjoint sets P, K so that $q_{pk} = 0$ for all $p \in P$, $k \in K$. Finally, since

$$[C] = \sum_{k=1}^{m} n_k [C^{(k)}]$$

for $n_k > 0$ and $[C]^2 = 0$, we can apply [9, Lemma I.2.10] to conclude that $Q \geq 0$ and the annihilator of Q is spanned by $[C]$. The lemma follows. □

We can now classify the singular fibers appearing in $\mathcal{M}_2$.

PROPOSITION 5.10
Suppose that C is a singular holomorphic curve obtained as a Gromov–Sacks–Uhlenbeck limit of holomorphic curves in $\mathcal{M}_1$. Write $C = \sum_{k=1}^{m} n_k C^{(k)}$, with C_k reduced and irreducible. Then the components $C^{(k)}$ satisfy $[C^{(k)}]^2 = 0, -2$, and

(1) *if $[C^{(k)}]^2 = 0$, for some k, then C has one component and is a singular fiber of Kodaira type I_1 or II,*
(2) *if $[C^{(k)}]^2 = -2$, for all k, then C is a singular fiber of Kodaira type III, I_n, IV, I_0^*, I_n^*, IV^*, III^*, or II^*.*

Proof
Suppose that we have a sequence of I-holomorphic curves $u_k : \Sigma \to X$ converging in the sense of Gromov–Sacks–Uhlenbeck to a cusp curve $u : \bigcup_k \Sigma_k \to X$. As discussed above, we let

$$u_k : \Sigma_k \to X$$

be the simple holomorphic curves, $[C^{(k)}] = (u_k)_*[\Sigma_k]$ (which may be zero if u_k is constant) and write

$$[L] = \sum_{k} n_k [C^{(k)}]$$

for positive integers n_k. First, thanks to the Gromov–Sacks–Uhlenbeck compactness theory (see [38], [65], [73], [93]), only Riemann surfaces with genus 1 or 0 appear in the limit.

Suppose that the map $u_\alpha : \Sigma_\alpha \to X$ is nonconstant. By the Riemann–Hurwitz formula, u_α is either simple or factors through a branched covering $\pi : \Sigma_\alpha \to \hat{\Sigma}_\alpha$;

in order to lighten notation we will denote by $v_\alpha : \hat{\Sigma}_\alpha \to X$ the simple holomorphic map. Keep in mind that $\hat{\Sigma}_\alpha$ can be either the sphere or the torus, with the latter case occurring if u_α factors through an unramified cover of the torus (thanks to the Riemann–Hurwitz formula). Consider the curve $C^{(\alpha)} = v_\alpha(\hat{\Sigma}_\alpha)$, and let

$$\nu : \tilde{C}_\alpha \to C^{(\alpha)}$$

be the normalization. Since $\hat{\Sigma}_\alpha$ has genus 0 or 1, it follows easily from the universal property of the normalization and Riemann–Hurwitz that $\tilde{C}_\alpha$ has genus 0 or 1. Furthermore, if $\hat{\Sigma}_\alpha$ has genus 0, then so does $\tilde{C}_\alpha$. We now appeal to the adjunction formula (see [9, p. 69]), which gives

$$\text{genus}(\tilde{C}_\alpha) + \delta = 1 + \frac{1}{2}\big(K_X + [C^{(\alpha)}]\big) \cdot [C^{(\alpha)}], \tag{5.3}$$

where the δ invariant is given by

$$\delta = \sum_{x \in C_\alpha} \dim_{\mathbb{C}}(\nu_* \mathcal{O}_{\tilde{C}_\alpha} / \mathcal{O}_{C_\alpha}).$$

Recall that X has $K_X = \mathcal{O}_X$ and $[C^{(\alpha)}]^2 \le 0$ by Lemma 5.9.

Let us first treat the case that $\tilde{C}_\alpha$ has genus 1 for some α. In this case, we must have that $[C^{(\alpha)}]^2 = 0$, $\delta = 0$, and $\hat{\Sigma}_\alpha$ has genus 1. In particular, by Lemma 5.9, C has only one component and u factors through an unramified cover of the torus. But since $[L] = (u)_*[\bigcup_\alpha \Sigma_\alpha]$ is primitive in $H_2(X, \mathbb{Z})$, we must have that $\bigcup_\alpha \Sigma_\alpha$ is a single Riemann surface of genus 1. Since $\delta = 0$, ν is an isomorphism and C is smooth. An application of Riemann–Hurwitz implies that $\hat{\Sigma}_1$ is biholomorphic to $\tilde{C}$ and hence u_1 is an embedding. Thus $C = C^{(1)}$ is a smooth elliptic curve, which yields a contradiction.

We may therefore assume that $\tilde{C}_\alpha$ has genus 0 for all α. Equation (5.3) becomes

$$\delta = 1 + \frac{1}{2}[C^{(\alpha)}]^2,$$

and so either $[C^{(\alpha)}]^2 = 0$ or -2. If $[C^{(\alpha)}]^2 = 0$, then by Lemma 5.9, $C = n_\alpha C^{(\alpha)}$ has only one component, which has $\delta = 1$. By an exercise in algebraic geometry (see, e.g., [30, Chapter 1, Exercise 4]), $C^{(\alpha)}$ has either a single ordinary double point or a single cusp and hence is (a positive integer multiple of) a fiber of Kodaira type I_1 or II.

It remains to consider the case when $[C^{(\alpha)}]^2 = -2$ for all α. In this case, each $C^{(\alpha)}$ is a smooth rational curve by the adjunction formula and C must have more than one component. Furthermore, by Lemma 5.9, for any $\alpha \ne \beta$ we have

$$0 \ge \big([C^{(\alpha)}] + [C^{(\beta)}]\big)^2 = -4 + 2[C^{(\alpha)}] \cdot [C^{(\beta)}]$$

and so $[C^{(\alpha)}]\cdot[C^{(\beta)}]\le 2$, and by Lemma 5.9 equality is achieved if and only if $[C]=n([C^{(\alpha)}]+[C^{(\beta)}])$. In particular, C is a multiple of a fiber of Kodaira type I_2 or III. We are reduced to considering the case when $0\le[C^{(\alpha)}]\cdot[C^{(\beta)}]\le 1$ for all $\alpha\neq\beta$. We can now directly apply [9, Lemma 2.12] to conclude that the intersection matrix is of type $\tilde{A}_n$, $\tilde{D}_n$, or $\tilde{E}_k$ for $k=6,7,8$. By inspection, these yield singular fibers of type I_n (or type IV if $n=3$), or of type I_0^*, I_n^*, IV^*, III^*, II^*.

It only remains to rule out the case of multiple fibers, but this follows from the assumption that $[C]=[L]$ is primitive in $H_2(X,\mathbb{Z})$. □

Remark 5.11
Note that every singular fiber appearing in the Kodaira classification result Proposition 5.10 contains a component with multiplicity 1.

Next, we will show that, in fact, $X_1\cup X_2=X$. The main technical issue is to prove that the set of points lying on singular elliptic curves is "discrete" in an appropriate sense. Geometrically, we will prove that, for any singular elliptic curve C lying in X_2, there is an $\varepsilon>0$ such that the ε-neighborhood $B(C,\varepsilon)\subset(X,g)$ contains no other singular elliptic curve in X_2. We begin by proving this statement when all the singular curves under consideration have multiple components.

LEMMA 5.12
Suppose that $C\subset X_2$ has m irreducible components for some $m\ge 2$. Then there exists $\varepsilon>0$ such that $B(C,\varepsilon)$ does not contain any singular curve $C'\neq C$ homologous to $[L]$ with more than one component.

Proof
Suppose not. Write $C=\sum_{k=1}^m n_k C^{(k)}$. For every $\ell\in\mathbb{N}$, there is a singular elliptic curve C_ℓ in $B(C,\ell^{-1})$. By Lemma 5.8, we can assume that each curve C_ℓ has $m\ge 2$ components with multiplicity n'_k. That is, we can write

$$C_\ell=\sum_{k=1}^m n'_k C_\ell^{(k)},$$

for $C_m^{(k)}$ smooth, irreducible rational curves, thanks to Proposition 5.10. It follows from the Mayer–Vietoris theorem that, for all ℓ sufficiently large, $H_2(B(C,\ell^{-1}),\mathbb{Z})$ is generated (over $\mathbb{Z}$) by $[C^{(k)}]$ for $1\le k\le m$. Thus, we can assume that $[C_\ell^{(k)}]=[C^{(k)}]$ for all m sufficiently large. On the other hand, since $C_\ell\neq C$, we must have that, for some k and some ℓ sufficiently large, $C_\ell^{(k)}\neq C^{(k)}$. Thus,

$$[C^{(k)}]^2 = [C^{(k)}] \cdot [C_\ell^{(k)}] \geq 0.$$

But by Lemma 5.9 we have $[C^{(k)}]^2 = -2$, a contradiction. □

The next step is to rule out the accumulation of singular curves with only one component at a singular curve with only one component. By Proposition 5.10, singular curves with only one component correspond to reduced and irreducible divisors obtained as holomorphic images of $\mathbb{P}^1$ with either nodal or cuspidal singularities. So, suppose that we have a simple holomorphic curve

$$u : \mathbb{P}^1 \to C \subset (X, g, I, \omega_I)$$

such that $[C] = [L]$. Suppose that, for all $\ell \geq 0$, $B(C, \ell^{-1})$ contains a reduced, irreducible rational curve $C_\ell \subset X_2$, corresponding to a simple holomorphic map

$$u_\ell : \mathbb{P}^1 \to C_\ell \subset (X, g, I, \omega_I),$$

with $[C_\ell] = [L]$. We will address the nodal and cuspidal cases separately, but first we prove a general lemma.

LEMMA 5.13

Suppose that $u : \mathbb{P}^1 \to C \subset (X, g, I, \omega_I)$ is a simple holomorphic curve with $[C] = [L]$ and that C is reduced and irreducible. Suppose that for every $\ell \in \mathbb{N}$ there is a simple holomorphic curve $u_\ell : \mathbb{P}^1 \to B(C, \ell^{-1})$ such that $(u_\ell)_*[\mathbb{P}^1] = [L]$. Then u_ℓ converges to u in the sense of Gromov–Sacks–Uhlenbeck.

Proof

By Proposition 5.3, up to taking a subsequence, u_ℓ Gromov–Sacks–Uhlenbeck converges to a cusp curve, or stable map $u' : \bigcup_\alpha \Sigma_\alpha \to (X, g, I, \omega_I)$, with each $\Sigma_\alpha = \mathbb{P}^1$, whose image is contained in C (see [38], [65], [93]). Since C is irreducible, the image of u' must be C. Furthermore, since $(u_\ell)_*[\mathbb{P}^1] = [C_\ell] = [C]$, we have that $(u')_*[\bigcup_\alpha \Sigma_\alpha] = [C]$. It follows that u' is constant on all but one component of $\bigcup_\alpha \Sigma_\alpha$. Forgetting the constant components of the map, we obtain $u' : \mathbb{P}^1 \to C$ and since $(u')_*[\mathbb{P}^1] = [C]$, the map u' is simple. Therefore, $u' = u \circ \tau$ for some $\tau \in \mathrm{PSL}(2, \mathbb{C})$. Therefore, u_ℓ converges in the sense of Gromov–Sacks–Uhlenbeck to u. □

PROPOSITION 5.14

Suppose that $C \subset X_2$ is a reduced and irreducible rational curve with a nodal singularity obtained as a Gromov–Sacks–Uhlenbeck limit of I-holomorphic curves in $\mathcal{M}_1$. Then there exists an $\varepsilon > 0$ such that $B(C, \varepsilon)$ does not contain any irreducible singular curve in $\mathcal{M}_2$ distinct from C.

Proof
We begin with a simple calculation. By assumption, the rational curve $u : \mathbb{P}^1 \to X$ is nodal and hence du is injective. Therefore, we have an injection of holomorphic vector bundles on $\mathbb{P}^1$ by

$$0 \to T\mathbb{P}^1 \to u^*TX.$$

By Grothendieck's theorem (see [45]) and the fact that $c_1(u^*TX) = 0$, u^*TX splits as a direct sum $u^*TX = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$ for some $a \geq 0$. Since $T\mathbb{P}^1 = \mathcal{O}_{\mathbb{P}^1}(2)$, there is a nowhere vanishing section of $\mathcal{O}_{\mathbb{P}^1}(a-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-a-2)$. This immediately implies $a = 2$ and hence

$$u^*TX = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2).$$

We now consider the twistor space of X, which we denote by $\mathcal{X}$. By the fiber exact sequence, we have

$$0 \to TX \to T\mathcal{X}|_X \to \mathcal{O}_X \to 0.$$

Composing the holomorphic map u with the inclusion, we obtain

$$0 \to \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \to u^*T\mathcal{X}|_X \to \mathcal{O}_{\mathbb{P}^1} \to 0.$$

We need the following lemma.

LEMMA 5.15
In the above notation, we have

$$u^*T\mathcal{X}|_X = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Let us assume the lemma for now and finish the proof. Let $\iota : X \hookrightarrow \mathcal{X}$ be the inclusion of X into the twistor space by the complex structure I. Combining Lemma 5.15 with [65, Lemma 3.3.1] and [65, Theorem 3.1.6], we conclude that the moduli space of parameterized, simple holomorphic rational curves in the twistor space homologous to $\iota_*[C]$ is a smooth manifold with Gromov–Sacks–Uhlenbeck topology. Moreover, it has real dimension 6 by the Riemann–Roch theorem (see also [65, Theorem 3.1.6]). On the other hand, the 3-complex-dimensional group $\mathrm{PSL}(2,\mathbb{C})$ acts on the holomorphic rational curves by reparameterization. It follows that there is an open neighborhood U of u in the Gromov–Sacks–Uhlenbeck topology such that, if $u' \in U$, then u' is obtained from u by precomposing with a Möbius transformation of $\mathbb{P}^1$; in particular, $u(\mathbb{P}^1) = u'(\mathbb{P}^1)$. Since any I-holomorphic rational curve into X homologous to $[C]$ induces a holomorphic rational curve into $\mathcal{X}$ homologous to $\iota_*[C]$, we deduce that the same result holds true for X.

Assume for the sake of contradiction that there are rational curves $u_\ell : \mathbb{P}^1 \to (X, g, I)$ in $\mathcal{M}_2$ such that $C_\ell = u_\ell(\mathbb{P}^1) \subset B_{\ell^{-1}}(C)$, but $C_\ell \neq C$. By Lemma 5.13, the rational curves u_ℓ Gromov–Sacks–Uhlenbeck converge to u. In particular, for ℓ sufficiently large, $u_\ell = u \circ \tau$ for some $\tau \in \mathrm{PSL}(2, \mathbb{C})$. But this implies that $u_\ell(\mathbb{P}^1) = u(\mathbb{P}^1)$, a contradiction. □

It only remains to prove Lemma 5.15.

Proof of Lemma 5.15
Consider the exact sequence of vector bundles

$$0 \to TX \hookrightarrow T\mathcal{X}|_X \to \mathcal{O}_X \to 0. \tag{5.4}$$

Restricting to C and pulling back gives the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \to u^* T\mathcal{X}|_X \to \mathcal{O}_{\mathbb{P}^1} \to 0.$$

An easy computation shows that $\dim_{\mathbb{C}} \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) = 1$, and so it suffices to show that the exact sequence is not split. Taking the long exact sequence in cohomology, this question is reduced to understanding the connecting homomorphism

$$\delta : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \to H^1(\mathbb{P}^1, u^*TX).$$

In particular, it is easy to see that if δ is not the zero map, then the exact sequence cannot be split.

On the other hand, since the twistor family $\mathcal{X} \to \mathbb{P}^1$ is a nontrivial deformation of complex structures, the Kodaira–Spencer map

$$\delta : H^0(X, \mathcal{O}_X) \to H^1(X, TX)$$

of the long exact sequence associated with (5.4) is nontrivial. By a well-known computation (see, e.g., [85, Lemma 7.2]), the contraction of the image of the Kodaira–Spencer map (viewed as a (TX, I)-valued $(0,1)$-form) with the holomorphic 2-form is the $(1,1)$ component of $\frac{d}{d\zeta}|_{\zeta=\sqrt{-1}} \Omega_\zeta$. By (5.1), we have

$$\frac{d}{d\zeta}\Big|_{\zeta=\sqrt{-1}} \Omega_\zeta = 2\sqrt{-1}\omega - 2\sqrt{-1}\,\overline{\Omega} = 2\sqrt{-1}\big(\omega + \sqrt{-1}\mathrm{Im}(\Omega)\big) - 2\sqrt{-1}\mathrm{Rm}(\Omega).$$

Since $(\omega + \sqrt{-1}\,\mathrm{Im}(\Omega))$ is holomorphic on X_I, we are reduced to considering $-2\sqrt{-1}\mathrm{Re}(\Omega)$.

It suffices to show that the restriction of the deformation of complex structures to the nodal rational curve C is nontrivial. In other words, it suffices to show that

$$u^*\mathrm{Re}(\Omega) \in H^1(\mathbb{P}^1, u^*\Lambda^{1,1}T^*X_I)$$

is nontrivial. But this is clear, since $u^*\mathrm{Re}(\Omega)$ is a Kähler metric on $\mathbb{P}^1$ and hence cannot be in the image of $\overline{\partial} : u^*\Lambda^{1,0}T^*X_I \to u^*\Lambda^{1,1}T^*X_I$, for otherwise we would have

$$0 < \int_C \mathrm{Re}(\Omega) = \int_{\mathbb{P}^1} \overline{\partial}(u^*\beta) = 0. \qquad \square$$

We next rule out the accumulation of irreducible rational curves at a cuspidal rational curve.

PROPOSITION 5.16
Suppose that $C \subset X_2$ is a reduced and irreducible rational curve with cuspidal singularities obtained as a Gromov–Sacks–Uhlenbeck limit of I-holomorphic curves in $\mathcal{M}_1$. Then there exists an $\varepsilon > 0$ such that $B(C,\varepsilon)$ does not contain any rational curves in X_2 homologous to $[C] = [L]$ and distinct from C.

Proof
The idea is to show that if there exist irreducible rational curves $C_\ell \neq C$ in X_1 contained in $B_{\ell-1}(C)$, then in fact C deforms in a real 2-dimensional family of holomorphic curves sweeping out a neighborhood of the generic point of C. In particular, there will be rational curves homologous to $[C]$ intersecting the smooth elliptic curves homologous to $[C]$ in X_1, which is impossible since $[C]^2 = 0$.

In order to do this, we must first study the deformation theory of the cuspidal rational curve C. Recall (see [65, Chapter 2]) that if u is an I-holomorphic curve (for some almost complex structure I) with $u_*[\mathbb{P}^1] = C$, then we can deform u for λ sufficiently small by

$$u_\lambda = \exp_u(\lambda\xi),$$

where ξ is any smooth (or more generally $W^{k,p}$) section of $u^*TX \to \mathbb{P}^1$. Note that we are considering *parameterized* I-holomorphic curves. Under this identification, one obtains the linearized I-holomorphic curve operator

$$D_{(u,I)} : W^{k,p}(\mathbb{P}^1, u^*TX) \to W^{k-1,p}(\mathbb{P}^1, \Lambda^{0,1}T^*\mathbb{P}^1 \otimes u^*TX), \tag{5.5}$$

where k, p are chosen sufficiently large. The moduli space of I-holomorphic curves with $u_*[\mathbb{P}^1] = [C]$ will be a smooth manifold near (u, I), provided that $D_{(u,I)}$ is surjective.

We begin by showing that, in the present case, $D_{(u,I)}$ is in fact *not* surjective by applying (the proof of) [65, Lemma 3.3.1]. By assumption, $u : \mathbb{P}^1 \to (X, g, I)$

is a cuspidal rational curve and so du has a simple zero at some point $p \in \mathbb{P}^1$. In particular, du induces an injective map

$$du : T\mathbb{P}^1 \otimes \mathcal{O}_{\mathbb{P}^1}(1) \to u^*TX.$$

Since u^*TX has degree 0, Grothendieck's theorem implies that $u^*TX = \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$ and Serre duality yields

$$H^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)\big) = H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-5)\big) \oplus H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)\big).$$

By [65, Lemma 3.3.1], $D_{(u,I)}$ has a 4-real-dimensional cokernel. By [65, Proposition 3.1.11], the index of $D_{(u,I)}$ is 4 and hence the kernel is 8-dimensional.

In order to obtain a smooth moduli space, we must expand the set of almost complex structures under consideration. By a standard argument, we can construct a smooth family $\pi : \mathcal{I} \to D \subset \mathbb{R}^4$ of (not necessarily integrable) almost complex structures compatible with the given symplectic form $\omega_0 := \omega_{I_0}$ over a disk $D \subset \mathbb{R}^4$ with $I_0 := \pi^{-1}(0) = I$ such that the expanded moduli space

$$\mathcal{M}\big([C], \mathbb{P}^1; \mathcal{I}\big) = \big\{(u, I_t) : t \in D, u : \mathbb{P}^1 \to (X, \omega_0, I_t) \text{ is } I_t\text{-holomorphic}\big\}$$

becomes a smooth manifold in a neighborhood of (u, I_0). More precisely, the extended linearized operator

$$\hat{D}_{(u,I_t)}(\xi, Y) = D_{(u,I_t)}\xi + \frac{1}{2}Y(u) \circ du \circ j_{\mathbb{P}^1} \tag{5.6}$$

regarded as a map

$$\hat{D}_{(u,I_t)} : W^{k,p}(\mathbb{P}^1, u^*TX) \times T_{I_t}\mathcal{I} \to W^{k-1,p}(\mathbb{P}^1, \Lambda^{0,1}T^*\mathbb{P}^1 \otimes u^*TX)$$

is surjective. Since this operator is homotopic to the operator $(\xi, Y) \mapsto D_u\xi$, one can easily show that $\mathcal{M}([C], \mathbb{P}^1; \mathcal{I})$ has dimension 8, with tangent space at (u, I_t) given by the kernel of (5.6). The 6-real-dimensional group $\mathrm{PSL}(2, \mathbb{C})$ acts on $\mathcal{M}([C], \mathbb{P}^1; \mathcal{I})$ by reparameterization. Taking the quotient, we obtain a smooth manifold

$$\widetilde{\mathcal{M}}\big([C], \mathbb{P}^1; \mathcal{I}\big) := \mathcal{M}\big([C], \mathbb{P}^1; \mathcal{I}\big)/\mathrm{PSL}(2, \mathbb{C})$$

of real dimension 2 consisting of *unparameterized* holomorphic maps.

Combining the assumption with Lemma 5.13, $\widetilde{\mathcal{M}}([C], \mathbb{P}^1; \mathcal{I})$ contains a sequence of disjoint I_0-holomorphic curves converging to the I_0-holomorphic curve u. Thus, there is a nonzero smooth section w of $u^*TX \to \mathbb{P}^1$ satisfying

$$\hat{D}_{(u,I_0)}(w, 0) = D_{(u,I_0)}w = 0$$

and giving rise to a nontrivial deformation of u. On the other hand, by [65, Remark 3.2.6], the operator $D_u = \hat{D}_{(u,I)}$ is complex linear since $I = I_0$ is integrable. Hence, $(Iw, 0) \neq (w, 0)$ is also in the kernel of $\hat{D}_{(u,I)}$. We claim that Iw gives rise to a nontrivial deformation. To see this, observe that if Iw gives rise to a trivial deformation, then $Iw = du(V_\tau)$, where V_τ is the vector field induced by a 1-parameter subgroup of $\mathrm{PSL}(2,\mathbb{C})$. But by the I-holomorphic curve equation,

$$w = -I(Iw) = -I\, du(V_\tau) = -du(j_{\mathbb{P}^1} V_\tau)$$

and hence w gives rise to a trivial deformation, a contradiction.

Fix a point $p \in C_{\mathrm{reg}}$, where $w(p) \neq 0$. Let z be a local coordinate on C_{reg} near p identifying a neighborhood of p with $B_1 \subset \mathbb{C}$. For $\varepsilon \ll 1$, consider the map

$$B_1 \times \left\{(s,t) \in \mathbb{R}^2 : |s| < \varepsilon, |t| < \varepsilon\right\} \mapsto u_{s,t}(z), \tag{5.7}$$

where $u_{s,t}$ is the deformation of u generated by $sw + tIw$. Let $C_{s,t} = u_{s,t}(\mathbb{P}^1)$. Since p is a regular point of C and $w(p) \neq 0$, (5.7) is an immersion in a neighborhood of the origin. Furthermore, since the deformation is nontrivial and $[C] \cdot [C] = 0$, $C_{s,t}$ is disjoint from $C_{s',t'}$ for $(s,t) \neq (s',t')$. The map (5.7) is therefore an embedding. In particular, deformations of C sweep out a neighborhood of p.

On the other hand, since $C \subset X_2$, for any $\varepsilon > 0$ there are smooth elliptic curves homologous to $[C]$ and intersecting $B_\varepsilon(p)$. It follows that there is an $I = I_0$ holomorphic rational curve $\tilde{C}$ homologous to C such that $\tilde{C}$ intersects X_1 nontrivially. Therefore, we can choose a smooth elliptic curve $\hat{C} \subset X_1$ homologous to C such that

$$0 < \tilde{C} \cdot \hat{C} = [C]^2 = 0,$$

a contradiction. □

Remark 5.17
The proof of Proposition 5.16 could be used to prove Proposition 5.14 as well. The main advantage of the argument given to prove Proposition 5.14 is that the extraneous family of complex structures is constructed explicitly using the twistor space construction.

Finally, we only need to rule out the accumulation of singular rational curves with one component at singular curves with several components and vice versa. Note that if C is an irreducible (nodal or cuspidal) rational curve, then it is a simple consequence of Mayer–Vietoris that, for some $\varepsilon > 0$, $H_2(B(C,\varepsilon),\mathbb{Z}) = \mathbb{Z}[C]$. In particular, singular curves $\sum_k n_k C^{(k)}$ with more than one component cannot accumulate at singular curves with only one component. The converse will be a corollary of the following proposition.

PROPOSITION 5.18

The set X_1 consisting of points lying on smooth elliptic curves is open, dense, and path-connected in X. In particular, $X = X_1 \cup X_2$.

Proof

Choose a smooth elliptic curve C passing through a point $p \in X_1$. Let $q \in X$, and choose $R > 0$ so that $q \in B_R(p)$. We can assume that $q \notin X_2$; otherwise, we are finished. Consider $\overline{B_{2R+e}}(p)$, where e is the constant from Proposition 5.3 for $K = B_{2R}(p)$ and $A = [\omega] \cdot [C]$. We claim that, by Lemma 5.12 and Propositions 5.14 and 5.16, the set $X_2 \cap \overline{B_{2R}}(p)$ is a finite union of sets $C_i \cap \overline{B_{2R}}(p)$, where $C_i \subset \overline{B_{2R+e}}(p)$ are singular holomorphic curves. Suppose that this is not the case. By Proposition 5.3, any connected holomorphic curve intersecting $\overline{B_{2R}(p)}$ is contained in $\overline{B_{2R+e}}(p)$. Thus, for the sake of contradiction, we can assume that

$$X_2 \cap \overline{B_{2R}}(p) = \bigcup_{\alpha \in \mathcal{A}} C_\alpha \cap \overline{B_{2R}}(p), \tag{5.8}$$

where the C_α's are singular holomorphic curves in $\overline{B_{2R+e}}(p)$ and $\mathcal{A}$ is an infinite index set. There is an infinite subset $\mathcal{A}' \subset \mathcal{A}$ such that all the curves $C_{\alpha'}, \alpha' \in \mathcal{A}'$ are either all reduced, irreducible rational curves or all curves with $m \geq 2$ irreducible components. Now, since $\overline{B_{2R+e}}(p)$ is compact, the Hausdorff distance is compact, and so there is a sequence of curves $C_{\alpha'_i}, i \in \mathbb{N}$, converging in the Hausdorff distance to $C_{\alpha'_\infty}$ for some $\alpha'_\infty \in \mathcal{A}'$. If $\mathcal{A}'$ consists only of irreducible rational curves, then this contradicts Propositions 5.14 and 5.16, while if $\mathcal{A}'$ consists of reducible curves, then this contradicts Lemma 5.12. Therefore, the index set $\mathcal{A}$ in (5.8) is finite. Since each C_α has only finitely many components by Lemma 5.8, each with Hausdorff dimension 2, $X_2 \cap B_{2R}(p)$ has Hausdorff dimension $2 < 3 = 4 - 1$. Therefore, the complement $B_{2R}(p) \setminus X_2$ is path-connected (see, e.g., [76]). Hence we can find a smooth curve $\gamma(t)$ such that $\gamma(0) = p$, $\gamma(1) = q$ and $\gamma(t) \notin X_2$ for any $t \in [0,1]$. The set $I := \{t \in [0,1] : \gamma(t) \in X_1\}$ is nonempty and open by Lemma 5.1. Since $\gamma(t) \notin X_2$ for any $t \in [0,1]$, it follows that A is closed and hence $q \in X_1$ as desired. □

The following lemma is straightforward, but we state it for completeness.

LEMMA 5.19

Let Y be a del Pezzo surface or a rational elliptic surface, and let $D \in |-K_Y|$ be a smooth divisor with $D^2 = d$. Then $X = Y \setminus D$ has $\chi(X) = 9 - d$.

Proof

This follows immediately from the fact that, topologically, Y is obtained by blowing up $\mathbb{P}^2$ at $9 - d$ points. Thus, $\chi(Y) = 12$. Therefore, $\chi(X) = 12 - d$. □

Finally, we prove the following lemma, which in combination with the previous results establishes Theorem 5.5.

LEMMA 5.20
There is a complex manifold B of complex dimension 1 *such that* (X, g, I, ω_I) *admits an elliptic fibration* $\pi : X \to B$, *the number of singular fibers bounded by* $\chi(X) < +\infty$. *Furthermore, this fibration is minimal in the sense that no fiber contains a rational curve with self-intersection* (-1).

Proof
First, we show that X contains only finitely many singular fibers. Notice that one can compute the Euler characteristic of X from the Mayer–Vietoris sequence. Since X is a torus fibration, we have

$$+\infty > \chi(X) = \sum_{C:\,\text{singular fibers}} \chi(C).$$

On the other hand, by Proposition 5.10 the singular fibers are classified by Kodaira's list and each singular fiber C has $\chi(C) \geq 1$ (see, e.g., [68]). Therefore, there can only be a finite number of singular fibers. In particular, this rules out the accumulation of singular rational curves with only one component at a singular curve with several components.

Define a fibration $\pi : X \to B$ by sending $x \to [x]$, where we say that $x \sim y$ if x, y lie on the same connected holomorphic curve homologous to $[L]$. Note that by Proposition 5.18, this equivalence relation is well defined on all of X. Let $B_1 = \pi(X_1)$, and recall that, by a result of Hitchin [51], B_1 has a natural complex structure making $\pi : X_1 \to B_1$ a holomorphic fibration whose fibers are smooth genus 1 curves.

For a torus fiber C, choose a point $p \in C$. Then the normal exponential map $\nu : T_pC^{\perp} \to X$ defines a local section of π which is smooth with respect to the smooth structure on B_1. We extend this structure to all of B in the following way. If C is a singular curve in X_2, then choose a point $p \in C_{\text{reg}}$ lying in a component with multiplicity 1; this is possible by Proposition 5.10 and Remark 5.11. Since the singular fibers of π are isolated, there is a small ball $B(p,\varepsilon)$ such that $B(p,\varepsilon) \setminus C$ consists only of points lying on smooth torus fibers. Furthermore, since p lies on a component with multiplicity 1, the normal exponential map from p defines a local section of π intersecting each smooth fiber at one point and hence induces local coordinates in a neighborhood of $\pi(p) \in B$. Since the disk has a unique smooth structure, the resulting smooth structure on B is well defined and independent of any choices.

It only remains to prove that the holomorphic structure on B_1 extends to all of B. Choose a point $b \in B \setminus B_1$, and let $D \ni b$ be a disk with local coordinates (x_1, x_2) centered at b and such that $D^* := D \setminus \{b\} \subset B_1$. By Hitchin's result in [51], D^*

has a complex structure. By a result from complex analysis (see, e.g., [84, Corollary 1.2.7a]), D^* is biholomorphic to either the punctured disk $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$, $\mathbb{C}^*$, or an annulus $\{z \in \mathbb{C} : 1 < |z| < R\}$ for some $R > 1$. We claim that in fact D^* is biholomorphic to Δ^*. Take $p \in C_{\mathrm{reg}}$ a smooth point in a component of multiplicity 1; note that such a component always exists (see Proposition 5.10 and Remark 5.11). We can find holomorphic coordinates (z_1, z_2) on an open ball $B(p, \varepsilon) \subset X$ so that $\{z_2 = 0\} = C \cap B(p, \varepsilon)$ and $p = \{z_1 = z_2 = 0\}$. We claim that for $N \in \mathbb{N}$ sufficiently large, the set $\{(0, z_2) : |z_2| < N^{-1}\varepsilon\}$ will intersect each fiber of π at exactly one point. Suppose that this is not the case. Then for all $i \in \mathbb{N}$, there exist smooth torus fibers $C^i \neq C$, Hausdorff converging to C as $i \to \infty$ and having the following property: for each $i \in \mathbb{N}$, there exists points $p_1^i, p_2^i \in C^i \cap B(p, \varepsilon)$, with $p_1^i \neq p_2^i$ such that $z_1(p_1^i) = z_1(p_2^i) = 0$ and $|z_2(p_1^i)| < i^{-1}\varepsilon$, $|z_2(p_2^i)| < i^{-1}\varepsilon$. Clearly $p_1^i, p_2^i \to p$ as $i \to \infty$.

If there is a subsequence $i_k \to +\infty$ such that $C^{i_k} \cap B(p, \varepsilon)$ has two or more connected components, then necessarily the component of C containing p has multiplicity at least 2. But this contradicts the fact that p is in a component of C with multiplicity 1. Thus, we can assume that for i sufficiently large $C^i \cap B(p, \varepsilon)$ has only one connected component. Thanks to the connectedness and the fact that $z_1(p_j^i) = 0$ for $j = 1, 2$, the mean value theorem gives the existence of a point $q^i \in C^i \cap B(p, \varepsilon)$ such that $dz_1 = 0$ at q^i, but this contradicts the fact that (z_1, z_2) are coordinates.

Thus, after possibly shrinking D, the map $z_2 \mapsto \pi(0, z_2)$ gives a biholomorphic map from a punctured disk to D^*; in particular, D inherits a complex structure extending the one on $D^* \cong \Delta^* \subset \mathbb{C}$. Thus, we have a holomorphic map $\pi : \pi^{-1}(D^*) \to D^* \cong \Delta^* \subset \mathbb{C}$. By the Riemann extension theorem, π extends to a holomorphic map $\pi : \pi^{-1}(D) \to \Delta$, and hence $\pi : X \to B$ is a holomorphic fibration. By the adjunction formula, we conclude that the smooth fibers of π are tori and that no smooth rational curve with self-intersection (-1) can occur in any fiber (this also follows from Proposition 5.10). Thus, $\pi : (X, I) \to B$ is a minimal holomorphic torus fibration. □

Finally, we can apply Theorem 5.5 in conjunction with Theorem 1.1 to prove Theorem 1.3. In order to apply Theorem 5.5, we need to first check that the conditions apply. We begin with the following result.

LEMMA 5.21
Let Y be a compact Kähler surface, and let $D \in |-K_Y|$ be a smooth anticanonical divisor. Let N be a tubular neighborhood of D, and let $X = Y \setminus D$. Assume that $[L] \in H_2(N, \mathbb{Z})$ is primitive, that $H_2(Y, \mathbb{Z})$ is torsion-free, and that $H_1(Y, \mathbb{Z}) = 0$. Then $[L] \in H_2(X, \mathbb{Z})$ is primitive.

Proof
This is a direct consequence of the Mayer–Vietoris sequence. Since $H_1(Y) = 0$, Poincaré duality implies that $H_3(Y) = 0$, and, by Mayer–Vietoris,

$$0 = H_3(Y) \to H_2(N \setminus D) \xrightarrow{\alpha} H_2(X) \oplus H_2(D) \xrightarrow{\beta} H_2(Y)$$
$$[L] \longrightarrow \big(\alpha\big([L]\big), 0\big).$$

If $\alpha([L])$ is not primitive in X, then we can write $\alpha([L]) = m[L']$ for some primitive homology class $[L'] \in H_2(X)$ not in the image of α. But then $\beta([L'])$ is a nonzero class in $H_2(Y)$ with $m\beta([L']) = 0$, contradicting the assumption that $H_2(Y)$ has no torsion. □

THEOREM 5.22
Let Y be a del Pezzo surface or a rational elliptic surface, and let $D \in |-K_Y|$ be a smooth anticanonical divisor. Then $X = Y \setminus D$ admits a special Lagrangian fibration $\pi : X \to \mathbb{R}^2$ with at most finitely many singular fibers. Furthermore, after hyper-Kähler rotation with respect to the Tian–Yau metric, the fibration $\pi : X \to \mathbb{C}$ is holomorphic.

Proof
Let Y be a del Pezzo surface or a rational elliptic surface. Then, topologically, Y is obtained by blowing up $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or the second Hirzebruch surface $\mathbb{F}_2$. It follows that $H_1(Y, \mathbb{Z}) = 0$, $H_2(Y, \mathbb{Z})$ is torsion-free and Y has finite Euler characteristic by Lemma 5.19.

Since the divisor D is a flat torus, D contains infinitely many smooth special Lagrangian circles. By Theorem 1.1, we obtain infinitely many disjoint, possibly embedded, special Lagrangian tori in X with $\mathrm{Im}(\Omega)|_L = 0$ (after possibly rotating Ω) within a fixed homology class $[L]$ which is primitive in a tubular neighborhood of D. Therefore, by Lemma 5.21, $[L]$ is primitive in $H_2(X, \mathbb{Z})$, and since $[L]$ can be represented by disjoint embedded special Lagrangians, $[L]^2 = 0$.

Now, X equipped with the Tian–Yau, or asymptotically cylindrical Calabi–Yau metric, satisfies the assumptions of Theorem 5.5; hence we obtain a special Lagrangian torus fibration $\pi : (X, g, J, \omega_J) \to B$ with finitely many singular fibers, each classified by Kodaira and having no multiple fibers. We only need to prove that $B = \mathbb{R}^2$. After hyper-Kähler rotating, we have a holomorphic fibration $\pi : (X, g, I, \omega_I) \to B$, and B is a noncompact Riemann surface by Hitchin's result in [51]. We need the following lemma.

LEMMA 5.23
The manifold X has torsion first homology group.

Proof
Consider the long exact sequence of relative homologies

$$H_2(X) \to H_2(Y) \to H_2(Y,X) \to H_1(X) \to H_1(Y) = 0.$$

The last equality can be seen from the fact that any del Pezzo surface Y is either a blowup of $\mathbb{P}^2$ in points or $\mathbb{P}^1 \times \mathbb{P}^1$. Under the duality $H_2(Y,X) \cong H^2(D) \cong \mathbb{Z}$, the second map is given by

$$\begin{aligned} H_2(Y) &\to H^2(D), \\ [C] &\mapsto \big([D] \mapsto [C] \cdot [D]\big). \end{aligned}$$

This is surjective after tensoring with $\mathbb{R}$ because D is ample. □

Let B_s be the image of the singular fibers, let $X_s = \pi^{-1}(B_s)$, and recall that by Lemma 5.20, B_s is a finite set. Given any closed curve γ in B, after possibly a small homotopy, we can assume that γ avoids B_s. By lifting γ along the smooth fibration $\pi : X \setminus X_s \to B \setminus B_s$, we obtain a surjection $\pi_1(X) \to \pi_1(B)$. On the other hand, it is a classical result (see, e.g., [1, Theorem 44A]) that the fundamental group of a noncompact Riemann surface is a free group. In particular, if $H_1(X)$ is torsion, then B must be simply connected. By the uniformization theorem, B is biholomorphic to either $\mathbb{C}$ or the unit disk. If B is biholomorphic to a disk, then pulling back the bounded holomorphic function z along the holomorphic fibration π, we obtain (after taking real and imaginary parts) a bounded harmonic function on (X,g). But (X,g) is complete and Ricci-flat, and a well-known result of Yau [91, Corollary 1] says that no such function can exist. Thus B is biholomorphic to $\mathbb{C}$. □

Finally, we prove that, for Tian–Yau surfaces of type I or II, the special Lagrangian fibration in a neighborhood of ∞ is obtained from the Lagrangian mean curvature flow of the model fibration, in a very precise sense. To fix notation, let L_0 be a Lagrangian in the model fibration, and denote by $F_t(L_0)$ the solution of the Lagrangian mean curvature flow starting from L_0 at time $t \in [0,\infty)$. Sections 3 and 4 show that in either the type I or type II cases we can fix a compact set $K_0 \subset X$ such that $X \setminus K_0$ is diffeomorphic to the corresponding model geometry and such that if $L_0 \subset X \setminus K_0$ is a Lagrangian obtained from a special Lagrangian in the model geometry, then Lagrangian mean curvature flow starting at L_0 converges smoothly and exponentially fast to a special Lagrangian, which we denote by $F_\infty(L_0)$.

PROPOSITION 5.24
In the above setting, there are compact sets $K_0 \subset K_1 \subset K_2 \subset X$ with the following property. Let $\tilde{L} \subset X \setminus K_1$ be a fiber of the special Lagrangian fibration of $(X, \omega_{\mathrm{TY}})$.

Then there exists a unique Lagrangian $L \subset (X, \omega_{\mathrm{TY}})$ contained in $X \setminus K_0$ which is a fiber of the model Lagrangian fibration such that

$$\tilde{L} = F_\infty(L).$$

Furthermore, the mean curvature flow $F_t(\cdot), t \in [0, \infty]$ induces a continuous family of continuous maps $F_t : X \setminus K_1 \to X \setminus K_0$ such that F_∞ is injective and $F_\infty(X \setminus K_1) \supset X \setminus K_2$.

The proposition will be the result of the following two lemmas.

LEMMA 5.25
Let L_1, L_2 be two disjoint model Lagrangian submanifolds contained in $X \setminus K_0$, and denote by $\tilde{L}_i = F_\infty(L_i)$ the limits of the LMCF for $i = 1, 2$. Then $\tilde{L}_1$ and $\tilde{L}_2$ are disjoint.

Proof
By our choice of K_0 above, the Lagrangian mean curvature flow starting from any model Lagrangian contained in $X \setminus K_0$ converges smoothly and exponentially fast to a special Lagrangian torus $\tilde{L}$. Let L_1, L_2 be two such model Lagrangians in $X \setminus K_0$, and let $\tilde{L}_1$, $\tilde{L}_2$ be the corresponding special Lagrangians obtained as limits of the LMCF.

We hyper-Kähler rotate so that $\tilde{L}_1$, $\tilde{L}_2$ are holomorphic. Since $0 = [\tilde{L}_1] \cdot [\tilde{L}_2]$, we see that $\tilde{L}_1$, $\tilde{L}_2$ are either disjoint or equal. We only need to rule out the case $\tilde{L}_1 = \tilde{L}_2$. We do this by using a Floer homology-theoretic argument. Recall that LMCF preserves the Hamiltonian isotopy class (see [80]). Since L_1, L_2, $\tilde{L}_1$, $\tilde{L}_2$ are all special Lagrangians (though with respect to different holomorphic volume forms), they are all spin and have Maslov index 0. Since X has complex dimension 2, a standard index calculation shows that the moduli space of holomorphic disks with boundary on any of L_1, L_2, $\tilde{L}_1$, $\tilde{L}_2$ has virtual dimension -1. Thus, by a generic small perturbation of almost complex structures, we can assume that they do not bound any pseudo-holomorphic disks and hence are all unobstructed Lagrangians (see [81] for an even stronger result). In particular, the Floer homology between any pair of L_1, L_2, $\tilde{L}_1$, $\tilde{L}_2$ is well defined (see, e.g., [5]). If $\tilde{L}_1 = \tilde{L}_2$ coincide, then the standard argument of Floer [29] yields

$$H^*(\tilde{L}_1) \cong HF^*(\tilde{L}_1, \tilde{L}_1) = HF(\tilde{L}_1, \tilde{L}_2) \cong HF(L_1, L_2) = 0.$$

But since $\tilde{L}_1$ is a torus, this is absurd. Therefore, $\tilde{L}_1$, $\tilde{L}_2$ are disjoint and the result follows. □

Next we have the following.

LEMMA 5.26

There exist compact sets $K_1 \subset K_2$ *with* $K_0 \subset K_1 \subset X$ *and having the following property. Suppose that* $\tilde{L}_i = F_\infty(L_i)$ *are special Lagrangians in* $X \setminus K_2$ *converging in the Hausdorff topology to* $\tilde{L}_\infty \subset X \setminus K_2$*. Then the sequence* $\{L_i\}_{i \in \mathbb{N}}$ *of model Lagrangian submanifolds is contained in* $X \setminus K_1$ *and converges in the Hausdorff topology to a model Lagrangian* $L_\infty \subset X \setminus K_1$ *with* $F_\infty(L_\infty) = \tilde{L}_\infty$*.*

Proof

By Theorem 4.23 (or Theorem 3.6 in the type II case), the LMCF starting from any model Lagrangian L in $X \setminus K_0$ converges to a special Lagrangian $\tilde{L}$ which is contained in a ball of radius ε around L, where $\varepsilon = \varepsilon(K_0)$ depends only on K_0. Thus, we can choose $K_0 \subset K_1 \subset K_2$ such that

(i) the LMCF starting from any model Lagrangian in $X \setminus K_1$ converges to a special Lagrangian in $X \setminus K_0$,

(ii) if L is a model Lagrangian such that the LMCF starting from L converges to $\tilde{L} = F_\infty(L) \subset X \setminus K_2$, then $L \subset X \setminus K_1$.

In fact, we may as well just take

$$K_i = \overline{B\big(K_0, 100^i \varepsilon(K_0)\big)}$$

for $i = 1, 2$.

Let $\tilde{L}_i \subset X \backslash K_2$ be as in the statement of the lemma. Since $\tilde{L}_i \subset X \setminus K_2$ for all i (including $i = \infty$) and the sequence $\{\tilde{L}_i\}$ converges in the Hausdorff topology, we get that the L_i's are contained in a compact subset of $X \setminus K_1$. From the explicit description of the model fibration, we can pass to a subsequence (not relabeled) converging in the Hausdorff topology (and even smoothly) to a limit model Lagrangian $L_\infty \subset X \setminus K_1$. It suffices to show that $F_\infty(L_\infty) = \tilde{L}_\infty$; this follows from the continuous dependence of the LMCF on initial conditions, together with the exponential decay of the mean curvature established in Theorem 4.23.

First, since $L_i \subset X \setminus K_0$ for all i (including $i = \infty$), the proof of Theorem 4.23 (see (4.18)) shows that (after possibly enlarging K_0) we can assume that the mean curvature along the LMCF satisfies

$$\big|H(t)\big|^2 \le e^{-ct}$$

on $[1, \infty]$ for a uniform constant $c > 0$. Thus, for any $\varepsilon > 0$ we can choose T_ε large so that

$$\int_{T_\varepsilon}^{\infty} \big|H(t)\big|\, dt < \varepsilon.$$

Since the L_i's converge smoothly to L_∞, the continuous dependence of the mean curvature flow on initial data shows that we can choose N large so that if $i \ge N$, then

$$d\big(F_t(L_i), F_t(L_\infty)\big) < \varepsilon$$

for all $t \in [0, T_\varepsilon]$. Combining these estimates, we see that

$$d\big(F_t(L_i), F_t(L_\infty)\big) < 3\varepsilon$$

for all $t \in [0, \infty]$, provided that $i \geq N$. Since the flows $F_t(L_i)$ converge smoothly to $\tilde{L}_i$, we get $d(\tilde{L}_i, F_\infty(L_\infty)) < 3\varepsilon$. Now since ε was arbitrary and the $\tilde{L}_i$'s converge to $\tilde{L}_\infty$, we conclude that $F_\infty(L_\infty) = \tilde{L}_\infty$ as desired. Furthermore, since $\tilde{L}_\infty$ is a smooth fiber of a smooth torus fibration, the convergence $\tilde{L}_i \to \tilde{L}_\infty$ is smooth. □

Proof of Proposition 5.24

Let $K_0 \subset K_1 \subset K_2$ be as in Lemmas 5.25 and 5.26. For $t \in [0, \infty]$, let $F_t : X \setminus K_2 \to X \setminus K_1$ be the map sending a point x to its time t flow under the LMCF. By (the proof of) Lemma 5.26, F_t is continuous for all $t \in [0, \infty]$. By Lemma 5.25, F_∞ is injective, and by Lemma 5.26, $F_\infty(X \setminus K_1) \cap X \setminus K_2$ is relatively closed in $X \setminus K_2$. By invariance of domain, $F_\infty(X \setminus K_1) \cap X \setminus K_2$ is relatively open in $X \setminus K_2$. Since K_2 is compact and X has only one end, $X \setminus K_2$ is connected, and so $F_\infty(X \setminus K_1) \cap X \setminus K_2 = X \setminus K_2$ as claimed. □

6. Applications to mirror symmetry

In this section, we apply the results from Section 5, together with the classification of compact complex surfaces, to prove Corollaries 1.4 and 1.5 and Theorem 1.6. This will be obtained by compactifying the elliptic fibrations obtained by hyper-Kähler rotating our special Lagrangian fibrations. We begin with the following lemma, which says that there is a section in a neighborhood of ∞. The reader may wish to compare with [21, Proposition 5.3.1].

LEMMA 6.1

Let Y be a del Pezzo surface or a rational elliptic surface, and let $D \in |-K_Y|$ be a smooth anticanonical divisor. Let $\pi : (X, g, J) \to \mathbb{R}^2$ be the special Lagrangian fibration whose existence in guaranteed by Theorem 5.22. Then, after hyper-Kähler rotation, the genus 1 fibration $\pi : (X, g, I) \to \mathbb{C}$ admits a local holomorphic section in a neighborhood of ∞.

Proof

By Theorem 5.22, after hyper-Kähler rotation, we have an elliptic fibration $\pi : (X, g, I) \to \mathbb{C}$ with no singular fibers in a neighborhood of infinity. Let Δ^* be a punctured disk neighborhood of ∞, and let $X^* = \pi^{-1}(\Delta^*)$. Since the fibers of π are smooth elliptic curves, the map π is flat (see [28, p. 158]). In particular, the direct image sheaves $R^i\pi_*\mathcal{O}_X$ are locally free. The fiber of $R^i\pi_*\mathcal{O}_X$ over

$b \in \Delta^*$ is, by definition, $H^i(\pi^{-1}(b), \mathcal{O}_{\pi^{-1}(b)})$. When $i = 1$, Serre duality implies that $H^1(\pi^{-1}(b), \mathcal{O}_{\pi^{-1}(b)}) = H^0(\pi^{-1}(b), \mathcal{O}_{\pi^{-1}(b)}) = \mathbb{C}$ and so $R^1\pi_*\mathcal{O}_X$ is a line bundle. Thanks to the fact that Δ^* is Stein, Cartan's theorems A and B imply that $H^1(\Delta^*, R^p\pi_*\mathcal{O}_X) = 0$. By a theorem of Grauert [36, Satz 7] and Röhrl [70], $R^1\pi_*\mathcal{O}_X$ is the trivial line bundle.

Let $X^\#$ denote the sheaf of holomorphic sections of $\pi : X^* \to \Delta^*$. Since $\pi : X^* \to \Delta^*$ is a smooth fibration without multiple fibers, there is an exact sequence of commutative groups (see, e.g., [9, Chapter V, Section 9])

$$0 \to R^1\pi_*\mathbb{Z} \to R^1\pi_*\mathcal{O}_{X^*} \to X^\# \to 0.$$

Taking the long exact sequence in cohomology yields

$$0 \to H^0(R^1\pi_*\mathbb{Z}) \to H^0(R^1\pi_*\mathcal{O}_{X^*}) \to H^0(X^\#) \to H^1(R^1\pi_*\mathbb{Z}) \to 0.$$

Since $H^0(\Delta^*, R^1\pi_*\mathcal{O}_{X^*})$ is the sheaf of global sections of a trivial bundle over Δ^*, it is uncountable. On the other hand, $H^i(\Delta^*, R^1\pi_*\mathbb{Z}), i = 0, 1$ is a lattice and hence countable. Therefore, $H^0(\Delta^*, X^\#)$ is infinite-dimensional and hence we obtain a section. □

We now have a elliptic fibration over a punctured disk with a section. In order to extend this elliptic fibration over 0, we need to control the monodromy of the fibration. We have the following lemma.

LEMMA 6.2
Let Y be a del Pezzo surface or a rational elliptic surface, and let $D \in |-K_Y|$ be a smooth anticanonical divisor with $D^2 = d$. Let $\pi : (X, g, J) \to \mathbb{R}^2$ be the special Lagrangian fibration whose existence is guaranteed by Theorem 5.22. Then around ∞ the torus fibration has monodromy

$$m_{\infty,d} := \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}.$$

Proof
Let γ be a simple closed loop in the base of the special Lagrangian fibration $\pi : X \to \mathbb{R}^2$ circling ∞ with positive orientation. By Proposition 5.3, we can choose γ so that the torus bundle $\pi^{-1}(\gamma)$ is contained in $X \setminus K_2$, where K_2 is the compact set from Proposition 5.24. By Proposition 5.24, we can find a loop $\hat{\gamma}$ in the base of the model fibration such that the model torus fibration over $\hat{\gamma}$ is carried by LMCF to $\pi^{-1}(\gamma)$. Thus $\pi^{-1}(\gamma)$ has the same homotopy type as the model torus fibration over $\hat{\gamma}$ and hence has the same monodromy. But the monodromy of the model fibration in the Calabi model, or asymptotically cylindrical model, is $m_{\infty,d}$ (see [74]). □

We now obtain the following corollary, which follows from Kodaira's classification of singularities of elliptic fibrations in [56] (see also [9]) and the theory of stable reduction.

COROLLARY 6.3
There is a compact complex surface $\check{Y}$ equipped with a relatively minimal elliptic fibration $\check{\pi} : \check{Y} \to \mathbb{P}^1$ without multiple fibers such that $\check{Y} \setminus \check{\pi}^{-1}(\infty) \cong (X, I)$.

Proof
As usual, (X, I) denotes the hyper-Kähler rotation, so we have a fibration $\pi : (X, I) \to \mathbb{C}$. Identify a neighborhood of ∞ with the punctured disk $\Delta^* \subset \mathbb{C}$; let $\pi^* : X^* \to \Delta^*$ be the induced elliptic fibration. By Lemma 6.1, $X^* \to \Delta^*$ has a section, and hence we get a map $f^* : \Delta^* \to \mathcal{M}_{1,1}$, the moduli space of elliptic curves with a marked point. By Lemma 6.2, the fibration $X^* \to \Delta^*$ has monodromy $m_{\infty,d}$ and so, by [46, Proposition 5.9] (and its proof), f extends to a holomorphic map $f : \Delta \to \overline{\mathcal{M}}_{1,1}$. Thus, we can identify $X^* \to \Delta^*$ with the universal family away from the central fiber. We can therefore fill in the fiber over $0 \in \Delta$ and obtain a holomorphic family

$$f : X \to \Delta$$

extending X^* and having reduced central fiber. By taking a minimal resolution and blowing down any (-1) curves contained in the fiber, we obtain a relatively minimal family of elliptic curves $\bar{\pi} : W \to \Delta$ agreeing with the fibration over Δ^*. Since W is isomorphic to X away from the fiber over 0, the fibration has monodromy corresponding to a fiber of type I_d. By Kodaira's classification (see [9], [56]), this implies that the central fiber is of type I_d.

Now, since $W \setminus \bar{\pi}^{-1}(0)$ is isomorphic to X^* we can glue W to (X, I) along X^* to obtain a compact complex surface $\check{Y}$ with a relatively minimal fibration $\check{\pi} : \check{Y} \to \mathbb{P}^1$. □

We are now in a position to prove the following result.

THEOREM 6.4
Let Y_d be a del Pezzo surface or rational elliptic surface, and let $D \in |-K_{Y_d}|$ be a smooth divisor with $D^2 = d$. Let $X_d = Y_d \setminus D$, equip X_d with the Tian–Yau metric g_{TY}, and let $\pi : (X_d, g_{\mathrm{TY}}, J) \to \mathbb{R}^2$ be the special Lagrangian torus fibration of Theorem 5.22. Then after hyper-Kähler rotating to a complex structure I so that $\pi : (X_d, g_{\mathrm{TY}}, I) \to \mathbb{C}$ is a holomorphic elliptic fibration, the following holds: there is a rational elliptic surface $\check{\pi} : \check{Y} \to \mathbb{P}^1$ with a singular fiber of Kodaira type I_d so that (X_d, I) is biholomorphic to $\check{Y} \setminus I_d$.

Proof
Let $\check{\pi} : \check{Y} \to \mathbb{P}^1$ be the surface constructed in Lemma 6.3. Then $\check{Y}$ admits a genus 1 fibration with an I_d fiber over $\infty \in \mathbb{P}^1$. It follows from Lemma 5.19 that $\chi(\check{Y}) = 12$. From the Mayer–Vietoris sequence and Lemma 5.23, we have $b_1(\check{Y}) = 0$. Since $\check{Y} \to \mathbb{P}^1$ is a genus 1 fibration without multiple fibers, the canonical bundle formula (see, e.g., [30, Chapter 7, Theorem 15] or [9, Chapter V, Theorem 12.1]) gives

$$K_{\check{Y}} = \pi^*\big(K^1_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}^1}(k)\big) \tag{6.1}$$

for some $k \geq 0$. Thus $c_1(\check{Y})^2 = 0$. Furthermore, applying [30, Chapter 7, Corollary 17] we conclude that, since $\check{\pi} : \check{Y} \to \mathbb{P}^1$ has no multiple singular fibers, we must have $k > 0$ in (6.1). We can now appeal to the classification of compact complex surfaces.

To begin, assume that $\check{Y}$ is minimal. Since $c_1(\check{Y})^2 = 0$, $b_1(\check{Y}) = 0$, by the Enriques–Kodaira classification (see, e.g., [9, Chapter VI, Table 10]), $\check{Y}$ must be an Enriques surface, a $K3$ surface, or a minimal properly elliptic surface. Since $\chi(\check{Y}) = 12$, $\check{Y}$ is not a $K3$ surface. If $\check{Y}$ is an Enriques surface, then [9, Chapter VIII, Lemma 17.1] gives that $\pi : \check{Y} \to \mathbb{P}^1$ has two multiple fibers. But by construction the fibration $\check{\pi}$ has no multiple fibers. Thus, $\check{Y}$ is not an Enriques surface. It only remains to rule out the possibility that $\check{Y}$ is a minimal properly elliptic surface. We apply Noether's formula in combination with $K^2_{\check{Y}} = 0$, $\chi(\check{Y}) = 12$ to obtain

$$\chi(\mathcal{O}_{\check{Y}}) = \frac{1}{12}\big(K^2_{\check{Y}} + \chi(\check{Y})\big) = 1.$$

By definition, a properly elliptic surface has Kodaira dimension 1, which implies that in equation (6.1) we must have $k \geq 3$. In particular, we have $h^0(\check{Y}, K_{\check{Y}}) > 0$. In combination with Serre duality and $b_1(\check{Y}) = 0$, we obtain

$$1 = \chi(\mathcal{O}_{\check{Y}}) = h^0(\check{Y}, \mathcal{O}_{\check{Y}}) - h^1(\check{Y}, \mathcal{O}_{\check{Y}}) + h^2(\check{Y}, \mathcal{O}_{\check{Y}}) = 1 - 0 + h^0(\check{Y}, K_{\check{Y}}) > 1,$$

a contradiction.

It follows that $\check{Y}$ is not minimal. Let C be a rational curve in Y with $C^2 = -1$. Since the genus 1 fibration is relatively minimal, C must intersect the generic fiber of $\check{\pi} : \check{Y} \to \mathbb{P}^1$ positively; in particular, C is a multisection of the fibration. Let F be a generic fiber of π. Then we have $(C + F)^2 = -1 + 2C \cdot F + F^2 = 2C \cdot F - 1 > 0$. Thus, by [9, Chapter IV, Theorem 5.2], $\check{Y}$ is projective. By the canonical bundle formula (6.1) (and the remarks following it), we have

$$K_{\check{Y}} = \pi^*\big(\mathcal{O}_{\mathbb{P}^1}(k')\big)$$

for some $k' \geq -1$. If $k' \geq 0$, then $\check{Y}$ has $\mathrm{Kod}(\check{Y}) \geq 0$ and so $K_{\check{Y}}$ is effective. But by the adjunction formula, $K_{\check{Y}} \cdot C = -1$, which is a contradiction. Thus, we have

$k' = -1$, and hence $h^1(\check{Y}, \mathcal{O}_{\check{Y}}) = 0$, $h^0(\check{Y}, K^2_{\check{Y}}) = 0$, and C intersects the generic fiber at one point. Therefore, by Castelnuovo's rationality criterion (see [9, Chapter VI, Theorem 2.1]), we conclude that $\check{Y}$ is rational. Thus, $\check{Y}$ is a rational elliptic surface and $C : \mathbb{P}^1 \to \check{Y}$ is a section. □

COROLLARY 6.5
Let Y_d be a del Pezzo surface or rational elliptic surface, and let $D \in |-K_{Y_d}|$ be a smooth divisor with $D^2 = d$. Let $X_d = Y_d \setminus D$, equip X_d with the Tian–Yau metric g_{TY}, and let $\pi : (X_d, g_{\mathrm{TY}}, J) \to \mathbb{R}^2$ be the special Lagrangian torus fibration of Theorem 5.22. Then π admits a global section.

Note that Corollary 6.5 together with Theorem 5.22 establishes Theorem 1.3, modulo the statement that, near infinity, the special Lagrangians are S^1-bundles over special Lagrangians in the divisor at ∞. But this statement is an immediate consequence of Proposition 5.24 and the fact that the model special Lagrangians are S^1-bundles over special Lagrangians in the divisor at ∞.

The next result says that, at least in the special case of $\mathbb{P}^2$, we can identify the rational elliptic surface obtained by hyper-Kähler rotation. Together with Theorem 5.22, this result establishes Corollary 1.4 and a conjecture of Auroux [3, Conjecture 2.9].

PROPOSITION 6.6
Let $D \in |-K_{\mathbb{P}^2}|$ be a smooth cubic, and let $\check{\pi} : \check{Y} \to \mathbb{P}^1$ be the rational elliptic surface obtained via Theorem 6.4. Then $\check{\pi}^{-1}(\infty)$ is a singular fiber of type I_9 and $\check{\pi} : \check{Y} \setminus \pi^{-1}(\infty) \to \mathbb{C}$ has exactly three singular fibers of type I_1.

Proof
Recall that $\check{\pi} : \check{Y} \setminus \pi^{-1}(\infty)$ is an elliptic fibration with no multiple fibers. Since $\chi(\check{Y}) = 12$ and $\check{Y}$ has a singular fiber of type I_9 over $\infty \in \mathbb{P}^1$ with monodromy at ∞ given by

$$m_\infty := \begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix},$$

monodromy considerations (see [9, Chapter V, Table 6]) imply that $\check{Y} \setminus \check{\pi}^{-1}(\infty)$ must have more than one singular fiber. Thus, there are only three possible configurations for the singular fibers in $\check{Y} \setminus \check{\pi}^{-1}(\infty)$; they are $\{I_1, I_1, I_1\}$, $\{I_1, II\}$, and $\{I_1, I_2\}$. If the configuration is $\{I_1, I_2\}$, then there are $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$ so that m_∞ is conjugate in $\mathrm{SL}(2, \mathbb{Z})$ to

$$\begin{pmatrix} 1-ab & a^2 \\ -b^2 & 1+ab \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Since $\mathrm{Tr}(m_\infty) = 2$, we find that $b = 0$ and hence $a = \pm 1$, which implies that m_∞ is conjugate to

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix},$$

which is absurd. If instead the configuration is $\{I_1, II\}$, then we conclude that m_∞ is conjugate to

$$\begin{pmatrix} 1-ab & a^2 \\ -b^2 & 1+ab \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Again, since $\mathrm{Tr}(m_\infty) = 2$, we obtain that a, b solve $a^2 + b^2 - ab + 1 = 0$, which has no real solutions. The result follows. □

In the same vein, we have the following lemma, which together with Theorem 5.22 proves Corollary 1.5 and a conjecture of Auroux [3, Conjecture 2.10].

LEMMA 6.7
Consider the moduli space $\mathcal{Y}$ consisting of 4-tuples $(Y, D, [\omega], \Omega)$, where Y is a rational elliptic surface, $D \in |-K_Y|$ is a smooth divisor, $[\omega] \in H^2(Y, \mathbb{R})$ is a Kähler class, and Ω is a meromorphic 2-form on $X = Y \setminus D$ with simple pole along D. For a generic 4-tuple $(Y, D, [\omega], \Omega) \in \mathcal{Y}$, the special Lagrangian fibration of X with respect to any asymptotically cylindrical metric in $[\omega]|_{Y \setminus D}$ produced by Theorem 5.22 has twelve singular fibers, each of which is a nodal special Lagrangian sphere.

Proof
Recall that a generic rational elliptic surface has twelve singular fibers of type I_1. The idea is to show that for generic choice of data $(Y, D, [\omega], \Omega)$, the hyper-Kähler rotation along the special Lagrangian fibration produced in Theorem 5.22 will produce a generic rational elliptic surface. Fix a rational elliptic surface $\pi : Y \to \mathbb{P}^1$, a smooth divisor $D \in |-K_Y|$, and a class $[\gamma] \in H_1(D, \mathbb{Z})$, and let M_γ be the model Lagrangian induced by γ. Let Ω be the holomorphic volume form on $X := Y \setminus D$ with a simple pole along D and normalized such that $\int_{M_\gamma} \Omega = 1$. Let ω be the asymptotically cylindrical Tian–Yau metric such that $2\omega^2 = \Omega \wedge \bar{\Omega}$. By Theorem 5.22, there exists a special Lagrangian fibration $\pi : X \to \mathbb{R}^2$. Denote by $\check{X}$ the complex surface obtained from hyper-Kähler rotation with the Kähler form and holomorphic volume form

$$\begin{aligned} \check{\Omega} &= \omega - \sqrt{-1}\,\mathrm{Im}\,\Omega, \\ \check{\omega} &= \mathrm{Re}\,\Omega. \end{aligned} \tag{6.2}$$

Note that these choices are compatible with our normalizations, so that after this rotation the fibration $\pi : X \to \mathbb{R}^2$ becomes an elliptic fibration. By Theorem 6.4, there is a rational elliptic surface $\check{Y}$ and a smooth divisor $\check{D} \in |-K_{\check{Y}}|$, so that $X = \check{Y} \setminus \check{D}$. By direct computation in the cylindrical model together with the estimates (3.2), (3.3), and (3.4), one can check that $\check{\Omega}$ has a simple pole along $\check{D}$.

By the Torelli theorem for pairs of rational surfaces with smooth anti-canonical divisors (see [67]), a deformation (Y', D') of the pair (Y, D) is determined by the cohomology class of the meromorphic 2-form Ω' in

$$H^2(Y' \setminus D', \mathbb{C}) \cong H^2(Y \setminus D, \mathbb{C})$$

up to $\mathbb{C}^*$ scaling, but we have fixed the scaling by the $\int_{M_\gamma} \Omega = 1$ (see also [31, Proposition 3.12] to go from markings to periods). Since all the rational elliptic surfaces are deformation equivalent and generic fibers are smooth, all such pairs are in the same deformation family. Assume that Y is a generic rational elliptic surface. Then for a generic small deformation Y' of Y, the cohomology class $[\omega] \in H^2(Y, \mathbb{R})$ is transported by the Gauss–Manin connection to a Kähler form $[\omega'] \in H^2(Y', \mathbb{R})$ (see [23, Theorem 0.9]) (since $H^{2,0}(Y) = 0$). In particular, a small deformation of the Kähler class $[\omega']|_{X'} \in H^2(X', \mathbb{R})$ within the subspace $\mathrm{Im}(H^2(Y', \mathbb{R}) \to H^2(X', \mathbb{R}))$ is Kähler. We now apply the discussion in the preceding paragraph to Y', obtaining a hyper-Kähler rotation as in (6.2). Denote by $(\check{Y}', \check{D}')$ the pair of a rational elliptic surface and a smooth anticanonical divisor so obtained. Let $\check{X}' = \check{Y}' \setminus \check{D}'$. By the Torelli theorem (see [67]), any small deformation of $(\check{Y}, \check{D})$ can be achieved by a suitable choice of (Y', D') and its Kähler class $[\omega']$. The theorem follows from the fact that the generic rational elliptic surface $\check{Y}'$ has twelve singular fibers. □

Remark 6.8
More generally, the above lemma shows that the special Lagrangian fibrations constructed by Theorem 5.22 on the complement of a smooth divisor in a rational elliptic surface can have all possible singularities in Kodaira's list, since this is true of elliptic fibrations on rational elliptic surfaces.

Remark 6.9
In [50, Remark 2.5], Hein, Sun, Viaclovsky, and Zhang note that for special choices of elliptic curves D and homology classes $[\gamma] \in H_1(D, \mathbb{Z})$, hyper-Kähler rotating the Calabi model along the model special Lagrangian fibration over $[\gamma]$ produces the semiflat ansatz in a neighborhood of a type I_d fiber. The semiflat model was used

by Hein [48] to construct Ricci-flat metrics on complements of I_d fibers in rational elliptic surfaces. The authors suggest that this could be used to identify the metrics by global hyper-Kähler rotation, which could lead to a completely different proof of Theorem 1.6 in these special cases.

Appendix. Some analysis lemmas

In this appendix, we record, with proofs, several results which were needed for the analysis in Sections 2, 3, and 4. These results are surely well known. However, since we have not been able to find references containing exactly the statements we need with proofs, we include complete proofs here for the reader's convenience. The first result is a variational formula for the second fundamental form of a submanifold under variations of the metric; see [63] for a related formula in codimension 1.

LEMMA A.1
Let $M^k \subset X^{n+k}$ be a smooth submanifold of a Riemannian manifold (X, g_0). Suppose that $g(t)$ is a smooth variation of Riemannian metrics for $t \in (-\varepsilon, \varepsilon)$. Consider the product manifold $\overline{X} := X \times (-\varepsilon, \varepsilon)$ equipped with the Riemannian metric $\bar{g} = dt^2 + g(t)$. Let $A(t)$ denote the second fundamental form of $M \subset (X, g(t))$, and let $\overline{\nabla}$ denote the covariant derivative of $\bar{g}$. For $p \in M \times \{0\}$, let $(x_1, \dots, x_k)$ be local coordinates on M centered at p which are normal for $g(0)$. Let $\{E_1, \dots, E_n\}$ be a local orthonormal frame for $(TM)^\perp \subset (X, g(0))$. Then we have

$$\overline{\nabla}_t A_{ij}(0) = \sum_{\alpha=1}^{n}\Big((\nabla^0_i \partial_t g_{\alpha j} + \nabla^0_j \partial_t g_{\alpha i} - \nabla^0_\alpha \partial_t g_{ij}) + \frac{1}{2}\sum_{\beta=1}^{n} \partial_t g_{\alpha\beta} A^{\beta}_{ij}(0)\Big) E_\alpha(0)$$
$$- \sum_{\alpha=1}^{n}\sum_{\ell=1}^{k} \frac{1}{2}\partial_t g_{\alpha\ell} A^{\alpha}_{ij}(0)\partial_{x_\ell},$$

where ∇^0 denotes the covariant derivative on $(X, g(0))$.

Proof
As in the statement of the lemma, let $\overline{X} = X \times (-\varepsilon, \varepsilon)$, and equip $\overline{X}$ with the metric $\overline{g} := dt^2 + g(t)$. For $p \in M$, let $(x_1, \dots, x_k)$ be local coordinates on M centered at p which are normal for $g(0) = g_0$. Let $\{E_1, \dots, E_n\}$ be a local orthonormal frame for $(TM)^\perp$ with respect to $g(0)$. Extend $\{E_\alpha\}_{1\le\alpha\le n}$ smoothly in time to a local $g(t)$-orthonormal frame of $TM^\perp \subset (X, g_t)$. Choose local functions $(y_1, \dots, y_n)$ vanishing at p so that $\partial_{y_i} = E_i(0)$ holds at p. Then $(x_1, \dots, x_k, y_1, \dots, y_n)$ form local coordinates for X and g_0 is the identity at p. The second fundamental form is

$$A_{ij}(t) = \sum_{\alpha=1}^{n} \big\langle \nabla^t_{\partial_{x_i}} \partial_{x_j}, E_\alpha(t) \big\rangle_{g_t} E_\alpha(t).$$

Let $\overline{\nabla}$ denote the covariant derivative of $\overline{g}$. Then we have

$$\begin{aligned}\overline{\nabla_t} A_{ij}(t) &= \sum_{\alpha=1}^{n} \big\langle \overline{\nabla}_{\partial_t} \nabla^t_{\partial_{x_i}} \partial_{x_j}, E_\alpha(t) \big\rangle_{g_t} E_\alpha(t) \\ &\quad + \sum_{\alpha=1}^{n} \big\langle \nabla^t_{\partial_{x_i}} \partial_{x_j}, \overline{\nabla}_{\partial_t} E_\alpha(t) \big\rangle_{g_t} E_\alpha(t) \\ &\quad + \sum_{\alpha=1}^{n} \big\langle \nabla^t_{\partial_{x_i}} \partial_{x_j}, E_\alpha(t) \big\rangle_{g_t} \overline{\nabla}_{\partial_t} E_\alpha(t).\end{aligned}$$

At $t = 0$, we have $\nabla^0_{\partial_{x_i}} \partial_{x_j} = (\nabla^0_{\partial_{x_i}} \partial_{x_j})^\perp$ and so

$$\begin{aligned}\big\langle \nabla^0_{\partial_{x_i}} \partial_{x_j}, \overline{\nabla}_{\partial_t} E_\alpha(0) \big\rangle_{g_0} &= \big\langle \nabla^0_{\partial_{x_i}} \partial_{x_j}, (\overline{\nabla}_{\partial_t} E_\alpha)^\perp(0) \big\rangle_{g_0} \\ &= \sum_{\beta} \big\langle \nabla^0_{\partial_{x_i}} \partial_{x_j}, E_\beta(0) \big\rangle_{g_0} \big\langle E_\beta(0), \overline{\nabla}_{\partial_t} E_\alpha(0) \big\rangle_{g_0}.\end{aligned}$$

Since $E_\beta(t)$ is orthogonal to TM with respect to $g(t)$, we can also write

$$\begin{aligned}\overline{\nabla}_{\partial_t} E_\alpha(0) &= \big\langle E_\beta, \overline{\nabla}_{\partial_t} E_\alpha(0) \big\rangle_{g_0} E_\beta + \sum_{\ell=1}^{k} \big\langle \partial_{x_\ell}, \overline{\nabla}_{\partial_t} E_\alpha(0) \big\rangle \partial_{x_\ell} \\ &= \big\langle E_\beta, \overline{\nabla}_{\partial_t} E_\alpha(0) \big\rangle_{g_0} E_\beta - \sum_{\ell=1}^{k} \big\langle \overline{\nabla}_{\partial_t} \partial_{x_\ell}, E_\alpha(0) \big\rangle \partial_{x_\ell}.\end{aligned}$$

Putting these formulas together, we obtain

$$\begin{aligned}\overline{\nabla_t} A_{ij}(t) &= \sum_{\alpha=1}^{n} \big\langle \overline{\nabla}_{\partial_t} \nabla^t_{\partial_{x_i}} \partial_{x_j} |_{t=0}, E_\alpha(0) \big\rangle_{g_t} E_\alpha(0) \\ &\quad + \sum_{1 \le \alpha, \beta \le n} \big\langle \nabla^0_{\partial_{x_i}} \partial_{x_j}, E_\beta(0) \big\rangle_{g_0} \big\langle E_\beta(0), \overline{\nabla}_{\partial_t} E_\alpha(0) \big\rangle_{g_0} E_\alpha(0) \\ &\quad + \sum_{1 \le \alpha, \beta \le n} \big\langle \nabla^0_{\partial_{x_i}} \partial_{x_j}, E_\alpha(0) \big\rangle_{g_0} \big\langle E_\beta, \overline{\nabla}_{\partial_t} E_\alpha(0) \big\rangle_{g_0} E_\beta(0) \\ &\quad - \sum_{\alpha=1}^{n} \sum_{\ell=1}^{k} \big\langle \nabla^0_{\partial_{x_i}} \partial_{x_j}, E_\alpha(0) \big\rangle_{g_0} \big\langle \overline{\nabla}_{\partial_t} \partial_{x_\ell}, E_\alpha(0) \big\rangle \partial_{x_\ell}.\end{aligned}$$

Swapping α, β in the second line and using that $\overline{\partial_t}\langle E_\alpha, E_\beta\rangle_{g_t} = 0$, the second and third lines cancel and we obtain

$$\begin{aligned}\overline{\nabla_t} A_{ij}(t) = &\sum_{\alpha=1}^{n} \big\langle \overline{\nabla}_{\partial_t} \nabla^t_{\partial_{x_i}} \partial_{x_j}|_{t=0}, E_\alpha(0)\big\rangle_{g_0} E_\alpha(0)\\ &- \sum_{\alpha=1}^{n}\sum_{\ell=1}^{k} \big\langle \nabla^0_{\partial_{x_i}} \partial_{x_j}, E_\alpha(0)\big\rangle_{g_0} \big\langle \overline{\nabla}_{\partial_t} \partial_{x_\ell}, E_\alpha(0)\big\rangle \partial_{x_\ell}.\end{aligned}$$

We now compute

$$\begin{aligned}\big\langle \overline{\nabla}_{\partial_t} \nabla^t_{\partial_{x_i}} \partial_{x_j}|_{t=0}, E_\alpha(0)\big\rangle_{g_0} &= \frac{\partial}{\partial t}\Gamma^\alpha_{ij} + \sum_{\ell=1}^{k} \Gamma^\ell_{ij} \langle \overline{\nabla}_{\partial_t} \partial_{x_\ell}, E_\alpha\rangle_{g_0}\\ &\quad + \sum_{\beta=1}^{n} \Gamma^\beta_{ij} \big\langle \overline{\nabla}_{\partial_t} \partial_{y_\beta}, E_\alpha(0)\big\rangle_{g_0}\\ &= \frac{\partial}{\partial t}\Gamma^\alpha_{ij} + \sum_{\beta=1}^{n} A^\beta_{ij}(0)\overline{\Gamma}^\alpha_{t\beta},\end{aligned}$$

where $\overline{\Gamma}$ denotes the Christoffel symbols of $\overline{g}$ in coordinates (x_ℓ, y_α, t) and we have used that $\Gamma^\ell_{ij}(0)$ vanish at p. By straightforward calculation, we have

$$\overline{\Gamma}^\alpha_{t\ell} = \frac{1}{2}\partial_t g_{\alpha\ell}, \qquad \overline{\Gamma}^\alpha_{t\beta} = \frac{1}{2}\partial_t g_{\alpha\beta}.$$

Therefore,

$$\begin{aligned}\overline{\nabla_t} A_{ij}(t) = &\sum_{\alpha=1}^{n} \Big(\frac{\partial}{\partial t}\Gamma^\alpha_{ij} + \frac{1}{2}\sum_{\beta=1}^{n} \partial_t g_{\alpha\beta} A^\beta_{ij}\Big) E_\alpha(0)\\ &- \sum_{\alpha=1}^{n}\sum_{\ell=1}^{k} \frac{1}{2}\partial_t g_{\alpha\ell} A^\alpha_{ij}(0)\partial_{x_\ell}.\end{aligned}$$

By the well-known formula

$$\frac{\partial}{\partial t}\Gamma^\alpha_{ij} = (\nabla^0_i \partial_t g_{\alpha j} + \nabla^0_j \partial_t g_{\alpha i} - \nabla^0_\alpha \partial_t g_{ij}),$$

we obtain

$$\overline{\nabla}_t A_{ij}(t) = \sum_{\alpha=1}^{n} \Big((\nabla_i^0 \partial_t g_{\alpha j} + \nabla_j^0 \partial_t g_{\alpha i} - \nabla_\alpha^0 \partial_t g_{ij}) + \frac{1}{2} \sum_{\beta=1}^{n} \partial_t g_{\alpha\beta} A_{ij}^{\beta}(0) \Big) E_\alpha(0)$$
$$- \sum_{\alpha=1}^{n} \sum_{\ell=1}^{k} \frac{1}{2} \partial_t g_{\alpha\ell} A_{ij}^{\alpha}(0) \partial_{x_\ell},$$

which is the desired result. □

Our next result concerns smoothing estimates to the mean curvature flow. Such estimates are essentially standard in the theory. However, we have been unable to a find reference for these estimates in a scale-invariant form in a nonflat background. We refer the reader to [27, Chapter 3] for a proof when the background geometry is Euclidean and [80, Theorem 1.2] for similar estimates, but which are not manifestly scale-invariant.

PROPOSITION A.2
Suppose that $M_0^k \subset (X^{n+k}, g)$ *is a compact submanifold, and let* M_t *be the mean curvature flow starting at* M_0*. Suppose that there is a constant* $K > 0$ *so that, for all* $t \in [0, \frac{\alpha}{K})$*, we have the following estimates:*

(i) *the second fundamental form* A_t *of* M_t *satisfies*

$$|A_t|^2 \leq K,$$

(ii) *for all* $0 \leq \ell \leq m+1$*, the curvature tensor* Rm *of* (X, g) *satisfies*

$$\sup_{M_t} |\nabla^\ell \mathrm{Rm}|^2 \leq K^{2+\ell}.$$

Then there exists a constant $C > 0$ *depending only on* α, n, k, m *so that*

$$|\nabla^m A|^2 \leq \frac{CK}{t^m}.$$

Proof
The proof is based on Shi's well-known estimates for the Ricci flow in [77] and [78] (see also [8], [20, Chapter 7]). We will only prove the case $m = 1$, the remaining cases being essentially identical and following from an easy induction argument. For tensors S, T, we write $S * T$ for various contractions using the metric and multiplication by dimensional constants; the precise form will be irrelevant for our considerations. We begin by recalling the well-known evolution equations for the second fundamental form along the mean curvature flow. We have

$$\frac{\partial}{\partial t}A = \Delta A + A * A * A + A * \mathrm{Rm} + \nabla \mathrm{Rm},$$
$$\frac{\partial}{\partial t}\nabla A = \nabla \frac{\partial}{\partial t}A + A * A * \nabla A.$$

Furthermore, we have

$$\frac{\partial}{\partial t}A = \Delta A + A * A * A + A * \mathrm{Rm} + \nabla \mathrm{Rm}.$$

Therefore,

$$\begin{aligned}\nabla \frac{\partial}{\partial t}A &= \nabla \Delta A + \nabla A * A * A + \nabla A * \mathrm{Rm} + A * \nabla \mathrm{Rm} + \nabla\nabla \mathrm{Rm} \\ &= \Delta \nabla A + (\nabla A) * A * A + \nabla A * \mathrm{Rm} + A * \nabla \mathrm{Rm} + \nabla\nabla \mathrm{Rm},\end{aligned}$$

where we recall that everything is taken up to dimensional constants. Then we have

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla A|^2 = 2\langle \nabla A, \Delta \nabla A\rangle + (\nabla A)^{*2} * A * A + \nabla A^{*2} * \mathrm{Rm} \\ + \nabla A * A * \nabla \mathrm{Rm} + \nabla A * \nabla\nabla \mathrm{Rm}.\end{aligned}$$

Thus,

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big)|\nabla A|^2 \le -2|\nabla\nabla A|^2 + C\big(|\nabla A|^2|A|^2 + |\nabla A|^2|\mathrm{Rm}| \\ + |\nabla A||A||\nabla \mathrm{Rm}| + |\nabla A||\nabla\nabla \mathrm{Rm}|\big)\end{aligned}$$

for a dimensional constant C. We also have

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)|A|^2 \le -2|\nabla A|^2 + C\big(|A|^4 + |A|^2|\mathrm{Rm}| + |A|\nabla \mathrm{Rm}|\big).$$

Now, as before, assume that, on the interval $t \in [0, \frac{\alpha}{K})$ we have

$$|A|^2 \le K, \qquad |\mathrm{Rm}| \le K, \qquad |\nabla \mathrm{Rm}| \le K^{3/2}, \qquad |\nabla\nabla \mathrm{Rm}| \le K^2.$$

Consider the quantity $F = t|\nabla A|^2 + \beta|A|^2$. Then we have

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big)F \le |\nabla A|^2 \\ &+ Ct\big(|\nabla A|^2|A|^2 + |\nabla A|^2|\mathrm{Rm}| + |\nabla A||A||\nabla \mathrm{Rm}| + |\nabla A||\nabla\nabla \mathrm{Rm}|\big) \\ &- 2\beta|\nabla A|^2 + C\beta\big(|A|^4 + |A|^2|\mathrm{Rm}| + |A|\nabla \mathrm{Rm}|\big).\end{aligned}$$

Now, we want to estimate the $|\nabla A|^2$ term. Write

$$
\begin{aligned}
|\nabla A||\nabla\nabla \mathrm{Rm}| &= \big(|\nabla A||\nabla\nabla \mathrm{Rm}|^{\alpha}\big)\big(|\nabla\nabla \mathrm{Rm}|^{1-\alpha}\big) \\
&\le |\nabla A|^2|\nabla\nabla \mathrm{Rm}|^{2\alpha}) + |\nabla\nabla \mathrm{Rm}|^{2(1-\alpha)}.
\end{aligned}
$$

By considering the scaling of each term, we are led to take $2(1-\alpha)\times 4 = 6$, or, in other words, $\alpha = \frac{1}{4}$. Then we get

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t} - \Delta\Big)F &\le (1 + CtK - 2\beta)|\nabla A|^2 + CtK^3 \\
&\quad + C\beta\big(|A|^4 + |A|^2|\mathrm{Rm}| + |A||\nabla \mathrm{Rm}|\big).
\end{aligned}
$$

Choosing β large depending only on α, C, we obtain

$$
\Big(\frac{\partial}{\partial t} - \Delta\Big)F \le C'\beta K^2
$$

for a uniform constant C'. It follows that $F - C'\beta K^2 t \le F(0) \le \beta K$ and so

$$
t|\nabla A|^2 \le \beta K + C'\beta K^2 t \le C'' K,
$$

which is the desired estimate. □

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