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Uniform bounds on harmonic Beltrami differentials and Weil–Petersson curvatures

By *Martin Bridgeman* at Chestnut Hill and *Yunhui Wu* at Beijing

Abstract. In this article we show that for every finite area hyperbolic surface X of type (g, n) and any harmonic Beltrami differential μ on X, then the magnitude of μ at any point of small injectivity radius is uniform bounded from above by the ratio of the Weil–Petersson norm of μ over the square root of the systole of X up to a uniform positive constant multiplication. We apply the uniform bound above to show that the Weil–Petersson Ricci curvature, restricted at any hyperbolic surface of short systole in the moduli space, is uniformly bounded from below by the negative reciprocal of the systole up to a uniform positive constant multiplication. As an application, we show that the average total Weil–Petersson scalar curvature over the moduli space is uniformly comparable to $-g$ as the genus g goes to infinity.

1. Introduction

In this paper, we derive uniform bounds on the curvature of the Weil–Petersson metric on $\mathcal{M}_g^n$ the moduli space of conformal structures on the surface of genus g with n punctures where $3g + n \geqslant 5$. We write $\mathcal{M}_g$ for $\mathcal{M}_g^0$ for simplicity. These bounds depend on new uniform bounds for the norm of harmonic Beltrami differentials in terms of injectivity radius.

Let $X \in \mathcal{M}_g^n$. Recall that the *systole* $\ell_{\mathrm{sys}}(X)$ of X is shortest length of closed geodesics in the hyperbolic surface X and for $z \in X$, the *injectivity radius* $\mathrm{inj}(z)$ is the maximum radius of an embedded ball centered at z. We denote the Margulis constant in dimension two by

$$\varepsilon_2 = \sinh^{-1}(1).$$

By the Collar Lemma, for $r(z) \leqslant \varepsilon_2$, then z is either contained in a collar $\mathcal{C}_\gamma$ about a closed geodesic γ or z is in a neighborhood $\mathcal{C}_c$ about a cusp c. The tangent space $T_X\mathcal{M}_g^n$ of $\mathcal{M}_g^n$ at X can be identified with the space of harmonic Beltrami differentials on X. Let $\mu \in T_X\mathcal{M}_g^n$. We denote by $\|\mu\|_{\mathrm{WP}}$ the Weil–Petersson norm of μ, which is also the L^2-norm of μ on X. One consequence of our analysis is the following proposition.

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Proposition 1.1. *Let $X \in \mathcal{M}_g^n$ with $\ell_{\mathrm{sys}}(X) \leqslant 2\varepsilon_2$. Then for any $\mu \in T_X \mathcal{M}_g^n$ a harmonic Beltrami differential and $z \in X$ with injectivity radius* $\mathrm{inj}(z) \leqslant \varepsilon_2$

$$|\mu(z)|^2 \leqslant \frac{\|\mu\|_{\mathrm{WP}}^2}{\mathrm{inj}(z)} \leqslant 2\frac{\|\mu\|_{\mathrm{WP}}^2}{\ell_{\mathrm{sys}}(X)}.$$

Remark 1.2. In [16, Corollary 11] or [17, Lemma 11], Wolpert proved a similar bound when $\ell_{\mathrm{sys}}(X)$ is smaller than a positive constant depending on g and n. Our approach is similar to Wolpert's, but using a detailed analysis of the thin parts, we are able to obtain the above uniform bounds independent of g and n. Actually we will prove certain more precise uniform bounds which are Proposition 3.3 and Lemma 3.4. One may see Section 3 for more details.

Using Proposition 1.1, we derive uniform lower bounds on Weil–Petersson curvatures. More precisely, we prove:

Theorem 1.3. *For any $X \in \mathcal{M}_g^n$ with $\ell_{\mathrm{sys}}(X) \leqslant 2\varepsilon_2$, then*

(1) *for any $\mu \in T_X \mathcal{M}_g^n$ with $\|\mu\|_{\mathrm{WP}} = 1$, the Weil–Petersson Ricci curvature satisfies that*

$$\mathrm{Ric}^{\mathrm{WP}}(\mu) \geqslant -\frac{4}{\ell_{\mathrm{sys}}(X)}.$$

(2) *The Weil–Petersson scalar curvature at X satisfies that*

$$\mathrm{Sca}^{\mathrm{WP}}(X) \geqslant -\frac{4}{\ell_{\mathrm{sys}}(X)} \cdot (3g - 3 + n).$$

Remark 1.4. In [9] Teo showed that for any $X \in \mathcal{M}_g$,

(1) $\mathrm{Ric}^{\mathrm{WP}} \geqslant -2C(\frac{\ell_{\mathrm{sys}}(X)}{2})^2$.
(2) $\mathrm{Sca}^{\mathrm{WP}}(X) \geqslant -(6g-6)C(\frac{\ell_{\mathrm{sys}}(X)}{2})^2$.

Here the function $C(\cdot)$ is given by (3.1). As the systole $\ell_{\mathrm{sys}}(X)$ of X tends to zero,

$$C\left(\frac{\ell_{\mathrm{sys}}(X)}{2}\right)^2 = \frac{4}{\pi \ell_{\mathrm{sys}}(X)^2} + o\left(\frac{1}{\ell_{\mathrm{sys}}(X)^2}\right).$$

Also $C(\frac{\ell_{\mathrm{sys}}(X)}{2})^2$ tends to $\frac{3}{4\pi}$ as $\ell_{\mathrm{sys}}(X)$ goes to infinity. Compared to Teo's result, we obtain a better growth rate as $\ell_{\mathrm{sys}}(X) \to 0$. Actually this growth rate $\frac{-1}{\ell_{\mathrm{sys}}(X)}$ is optimal: Wolpert in [16, Theorem 15] or [16, Corollary 16] computed the Weil–Petersson holomorphic sectional curvature along the gradient of certain geodesic length function and showed that it behaves as $\frac{-3}{\pi \ell_\alpha} + O(\ell_\alpha)$ as $\ell_\alpha \to 0$, where $\alpha \subset X$ is a nontrivial loop. Part (1) of Teo's results above in particular implies that the Weil–Petersson sectional curvature, restricted on any ε-thick part of the moduli space, is uniformly bounded from below by a negative constant only depending on ε. This was first obtained by Huang in [4]. One may also see [12] for more general statements.

Remark 1.5. The assumption $\ell_{\mathrm{sys}}(X) \leqslant 2\varepsilon_2$ in Theorem 1.3 can *not* be removed. One may see this in the following two different ways. (1) Tromba [10] and Wolpert [13] showed that for all $X \in \mathcal{M}_g$,

$$\mathrm{Sca}^{\mathrm{WP}}(X) \leqslant \frac{-3}{4\pi} \cdot (3g - 2).$$

In particular for large enough g, the uniform lower bound for scalar curvature in Theorem 1.3 does not hold for Buser–Sarnak surface $\mathcal{X}_g$ (see [2]) whose injectivity radius grows like $\ln(g)$ as $g \to \infty$. Similarly for (2). It was shown in [12, Theorem 1.1] that if $\ell_{\mathrm{sys}}(X)$ is large enough, then

$$\min_{\operatorname{span}\{\mu,v\}\subset T_X\mathcal{M}_g} K^{\mathrm{WP}}(\mu, v) \leqslant -C < 0,$$

where $C > 0$ is a uniform constant independent of g. In particular, the uniform lower bound for Ricci curvature in Theorem 1.3 does not hold for Buser–Sarnak surface $\mathcal{X}_g$ in [2] for large enough g.

Let $X \in \mathcal{M}_g^n$ with $\ell_{\mathrm{sys}}(X) \leqslant 2\varepsilon_2$, and let $P(X) \subset T_X\mathcal{M}_g^n$ be the linear subspace generated by the gradient of short closed geodesic length functions and $P(X)^{\perp}$ its perpendicular. One may see (3.7) and (3.8) for the precise definitions. Our next result says that the Weil–Petersson curvature along any plane in $T_X\mathcal{M}_g$ containing a $\mu \in P(X)^{\perp}$ is uniformly bounded from below. More precisely:

Theorem 1.6. *Let $X \in \mathcal{M}_g^n$ with $\ell_{\mathrm{sys}}(X) \leqslant 2\varepsilon_2$. Then for any $\mu \neq 0 \in P(X)^{\perp}$ and $v \in T_X\mathcal{M}_g^n$, the Weil–Petersson sectional curvature $K^{\mathrm{WP}}(\mu, v)$ along the plane spanned by μ and v satisfies that*

$$K^{\mathrm{WP}}(\mu, v) \geqslant -4.$$

It would be *interesting* to find upper bounds for $K^{\mathrm{WP}}(\mu, v)$ in terms of certain measurements of μ and v.

Recall that the boundary $\partial\mathcal{M}_g$ of $\mathcal{M}_g$ consists of nodal surfaces. As X goes to $\partial\mathcal{M}_g$, the Weil–Petersson scalar curvature $\mathrm{Sca}^{\mathrm{WP}}(X)$ always blows up to $-\infty$ since the Weil–Petersson sectional curvature at X along certain direction goes to $-\infty$ (e.g., see [8] or [16, Corollary 16]). It was not known whether the total scalar curvature $\int_{\mathcal{M}_g} \mathrm{Sca}^{\mathrm{WP}}(X)\,dX$ is finite. We will show it is truly finite. Moreover, combining Theorem 1.3 and a result of Mirzakhani in [7], we determine the asymptotic behavior of $\int_{\mathcal{M}_g} \mathrm{Sca}^{\mathrm{WP}}(X)\,dX$ as $g \to \infty$. More precisely, we prove:

Theorem 1.7. *As $g \to \infty$,*

$$\frac{\int_{\mathcal{M}_g} \mathrm{Sca}^{\mathrm{WP}}(X)\,dX}{\mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}_g)} \asymp -g.$$

Notation. In this paper, we say two functions $f_1(g) \asymp f_2(g)$ if there exists a universal constant $C \geqslant 1$, independent of g, such that

$$\frac{f_2(g)}{C} \leqslant f_1(g) \leqslant C f_2(g).$$

Plan of the paper. Section 2 provides some necessary background and the basic properties on Teichmüller theory and the Weil–Petersson metric. Refined results of Proposition 1.1 are proved in Section 3. We prove several results on uniform lower bounds for Weil–Petersson curvatures including Theorem 1.3 and 1.6. Theorem 1.7 is proved in Section 5.

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2. Preliminaries

In this section, we set our notation and review the relevant background material on Teichmüller space and Weil–Petersson curvature.

2.1. Teichmüller space. We denote by S_g^n an oriented surface of genus g with n punctures where $3g+n \geqslant 5$. Then the Uniformization Theorem implies that the surface S_g^n admits hyperbolic metrics of constant curvature -1. We let $\mathcal{T}_g^n$ be the Teichmüller space of surfaces of genus g with n punctures, which we consider as the equivalence classes under the action of the group $\mathrm{Diff}_0(S_g^n)$ of diffeomorphisms isotopic to the identity of the space of hyperbolic surfaces $X = (S_g^n, \sigma(z)|dz|^2)$. The tangent space $T_X\mathcal{T}_g^n$ at a point $X = (S_g^n, \sigma(z)|dz|^2)$ is identified with the space of finite area *harmonic Beltrami differentials* on X, i.e. forms on X expressible as $\mu = \frac{\overline{\psi}}{\sigma}$, where $\psi \in Q(X)$ is a holomorphic quadratic differential on X. Let $z = x+iy$ and $dA = \sigma(z)dxdy$ be the volume form. The *Weil–Petersson metric* is the Hermitian metric on $\mathcal{T}_g$ arising from the *Petersson scalar product*

$$\langle \varphi, \psi \rangle = \int_X \frac{\varphi \cdot \overline{\psi}}{\sigma^2}\, dA$$

via duality. We will concern ourselves primarily with its Riemannian part g_{WP}. Throughout this paper we denote by $\mathrm{Teich}(S_g^n)$ the Teichmüller space endowed with the Weil–Petersson metric. By definition it is easy to see that the mapping class group $\mathrm{Mod}_g^n := \mathrm{Diff}^+(S_g^n)/\mathrm{Diff}^0(S_g^n)$ acts on $\mathrm{Teich}(S_g^n)$ as isometries. Thus, the Weil–Petersson metric descends to a metric, also called the Weil–Petersson metric, on the moduli space of Riemann surfaces $\mathcal{M}_g^n$ which is defined as $\mathcal{T}_g^n/\mathrm{Mod}_g^n$. Throughout this paper we also denote by $\mathcal{M}_g^n$ the moduli space endowed with the Weil–Petersson metric and write $\mathcal{M}_g = \mathcal{M}_g^0$ for simplicity. One may refer to [15] for recent developments on Weil–Petersson geometry.

2.2. Weil–Petersson curvatures. The Weil–Petersson metric is Kähler. The curvature tensor of the Weil–Petersson metric is given as follows. Let μ_i, μ_j be two elements in the tangent space $T_X\mathcal{M}_g^n$ at X, so that the metric tensor written in local coordinates is

$$g_{i\overline{j}} = \int_X \mu_i \cdot \overline{\mu_j}\, dA.$$

For the inverse of $(g_{i\overline{j}})$, we use the convention

$$g^{i\overline{j}} g_{k\overline{j}} = \delta_{ik}.$$

Then the curvature tensor is given by

$$R_{i\overline{j}k\overline{l}} = \frac{\partial^2}{\partial t^k \partial \overline{t^l}} g_{i\overline{j}} - g^{s\overline{t}} \frac{\partial}{\partial t^k} g_{i\overline{t}} \frac{\partial}{\partial \overline{t^l}} g_{s\overline{j}}.$$

We now describe the curvature formula of Tromba [10] and Wolpert [13] which gives the curvature in terms of the Beltrami–Laplace operator Δ. It has been applied to study various curvature properties of the Weil–Petersson metric. Tromba [10] and Wolpert [13] showed that $\mathcal{M}_g^n$ has negative sectional curvature. In [8] Schumacher showed that $\mathcal{M}_g^n$ has strongly negative curvature in the sense of Siu. Liu, Sun and Yau in [6] showed that $\mathcal{M}_g^n$ has dual Nakano negative curvature, which says that the complex curvature operator on the dual tangent bundle is positive in some sense. The second named author of this article in [18] showed that the $\mathcal{M}_g^n$ has non-positive definite Riemannian curvature operator. One can also see [3, 4, 9, 14, 16, 19] for other aspects of the curvature of $\mathcal{M}_g^n$.

Set

$$D = -2(\Delta - 2)^{-1},$$

where Δ is the Beltrami–Laplace operator on $X = (S, \sigma|dz|^2) \in \mathcal{M}_g^n$. The operator D is positive and self-adjoint.

Theorem 2.1 (Tromba [10], Wolpert [13]). *The curvature tensor satisfies*

$$R_{i\bar{j}k\bar{l}} = \int_X D(\mu_i \mu_{\bar{j}}) \cdot (\mu_k \mu_{\bar{l}})\, dA + \int_X D(\mu_i \mu_{\bar{l}}) \cdot (\mu_k \mu_{\bar{j}})\, dA.$$

2.2.1. Weil–Petersson holomorphic sectional curvatures. Recall that a holomorphic sectional curvature is a sectional curvature along a holomorphic line. Let $\mu \in T_X \mathcal{M}_g^n$ be a harmonic Beltrami differential. By Theorem 2.1 the holomorphic sectional curvature $\mathrm{HolK}^{\mathrm{WP}}(\mu)$ along the holomorphic line spanned by μ is

$$\mathrm{HolK}^{\mathrm{WP}}(\mu) = \frac{-2 \cdot \int_X D(|\mu|^2) \cdot (|\mu|^2)\, dA}{\|\mu\|_{\mathrm{WP}}^4}.$$

Assume that $\|\mu\|_{\mathrm{WP}} = 1$. From [12, Proposition 2.7], which relies on an estimation of Wolf in [11], we know that

$$-2 \int_X |\mu|^4\, dA \leqslant \mathrm{HolK}^{\mathrm{WP}}(\mu) \leqslant -\frac{2}{3} \int_X |\mu|^4\, dA.$$

2.2.2. Weil–Petersson sectional curvatures. We now describe a lower bound on sectional curvatures which follows from [13]. We let $\mu_i, \mu_j \in T_X \mathcal{M}_g^n$ be two orthogonal tangent vectors with $\|\mu_i\|_{\mathrm{WP}} = \|\mu_j\|_{\mathrm{WP}} = 1$. We let $K^{\mathrm{WP}}(\mu_i, \mu_j)$ be the Weil–Petersson sectional curvature of the plane spanned by the real vectors corresponding to μ_i and μ_j. In [13, Theorem 4.5], Wolpert makes the following observations. Wolpert shows that

$$\int_X D(|\mu_i\|\mu_j|)|\mu_i||\mu_j|\, dA \leqslant \int_X D(|\mu_i|^2)|\mu_j|^2\, dA,$$

$$\left|\int_X D(\mu_i \mu_{\bar{j}})\mu_i \mu_{\bar{j}}\, dA\right| \leqslant \int_X D(|\mu_i||\mu_j|)|\mu_i||\mu_j|\, dA,$$

$$\left|\int_X D(\mu_i \mu_{\bar{j}})\mu_{\bar{i}} \mu_j\, dA\right| \leqslant \int_X D(|\mu_i||\mu_j|)|\mu_i||\mu_j|\, dA.$$

Therefore as sectional curvature is given by

$$K^{\mathrm{WP}}(\mu_i, \mu_j) = \mathrm{Re} \int_X D(\mu_i \mu_{\bar{j}})\mu_i \mu_{\bar{j}}\, dA - \frac{1}{2} \int_X D(\mu_i \mu_{\bar{j}})\mu_{\bar{i}} \mu_j\, dA - \frac{1}{2} \int_X D(|\mu_i|^2)|\mu_j|^2\, dA$$

putting these equations together gives

$$K^{\mathrm{WP}}(\mu_i, \mu_j) \geqslant -2 \int_X D(|\mu_i|^2)|\mu_j|^2\, dA. \tag{2.1}$$

2.2.3. Weil–Petersson Ricci curvatures.. Let $\{\mu_i\}_{i=1}^{3g-3+n}$ be a holomorphic orthonormal basis of $T_X\mathcal{M}_g^n$. Then the Ricci curvature $\mathrm{Ric}^{\mathrm{WP}}(\mu_i)$ of $\mathcal{M}_g^n$ at X in the direction μ_i is given by

$$\begin{aligned}\mathrm{Ric}^{\mathrm{WP}}(\mu_i) &= -\sum_{j=1}^{3g-3+n} R_{i\bar{j}j\bar{i}}\\ &= -\sum_{j=1}^{3g-3+n}\left(\int_X D(\mu_i\mu_{\bar{j}})\cdot(\mu_j\mu_{\bar{i}})\,dA + \int_X D(|\mu_i|^2)\cdot(|\mu_j|^2)\,dA\right).\end{aligned}$$

Since $\int_X D(f)\cdot\overline{f}\,dA \geqslant 0$ for any function f on X, by applying the argument in the proof of (2.1) we have

$$-2 \leqslant \frac{\mathrm{Ric}^{\mathrm{WP}}(\mu_i)}{\sum_{j=1}^{3g-3+n}\int_X D(|\mu_i|^2)\cdot(|\mu_j|^2)\,dA} \leqslant -1. \tag{2.2}$$

2.2.4. Weil–Petersson scalar curvature. The scalar curvature $\mathrm{Sca}^{\mathrm{WP}}(X)$ at $X \in \mathcal{M}_g^n$ is the trace of the Ricci tensor. We can express the scalar curvature as

$$\begin{aligned}\mathrm{Sca}^{\mathrm{WP}}(X) = -\sum_{i=1}^{3g-3+n}\sum_{j=1}^{3g-3+n}\bigg(&\int_X D(\mu_i\mu_{\bar{j}})\cdot(\mu_j\mu_{\bar{i}})\,dA\\ &+\int_X D(|\mu_i|^2)\cdot(|\mu_j|^2)\,dA\bigg).\end{aligned}$$

It is known from [12, Proposition 2.5] that $-\mathrm{Sca}^{\mathrm{WP}}(X)$ is uniformly comparable to the quantity $\|\sum_{i=1}^{3g-3+n}|\mu_i|^2\|_{\mathrm{WP}}^2$. More precisely,

$$-2\int_X\left(\sum_{i=1}^{3g-3+n}|\mu_i|^2\right)^2 dA \leqslant \mathrm{Sca}^{\mathrm{WP}}(X) \leqslant -\frac{1}{3}\int_X\left(\sum_{i=1}^{3g-3+n}|\mu_i|^2\right)^2 dA.$$

3. Bounding the pointwise norm by the L^2 norm

In this section we will bound the pointwise norm of a harmonic Beltrami differential $\mu = \frac{\overline{\phi}}{\sigma}$ in terms of its Weil–Petersson norm and the injectivity radius function. Our results will improve on prior work of Teo [9] and Wolpert [16], giving the optimal asymptotics of Wolpert with its uniformity of Teo. As in Wolpert [16, Proposition 7], our approach will be to first decompose ϕ in the thin part of the surface into the leading and non-leading parts of its Laurent expansion. Then by a detailed analysis, we describe the leading term and give an explicit exponentially decaying upper bound on the non-leading term.

Given $X \in \mathcal{M}_g^n$ a hyperbolic surface of finite volume, for $z \in X$ we will let $r(z) = \mathrm{inj}(z)$ be the injectivity radius at z. We will refer several times to the a function $C(r)$ introduced by Teo in [9] which is given by

$$C(r) = \left(\frac{4\pi}{3}\left(1-\mathrm{sech}^6\left(\frac{r}{2}\right)\right)\right)^{-\frac{1}{2}} = \left(\frac{4\pi}{3}\left(1-\left(\frac{4e^r}{(1+e^r)^2}\right)^3\right)\right)^{-\frac{1}{2}}. \tag{3.1}$$

It follows that $C(r)$ is decreasing with respect to r and as r tends to zero we have

$$C(r) = \frac{1}{\sqrt{\pi r}} + O(1).$$

Furthermore, $C(r)$ tends to $\sqrt{\frac{3}{4\pi}}$ as r tends to infinity.

Let $X = (S_g^n, \sigma(z)|dz|^2) \in \mathcal{M}_g^n$ and $\phi \in Q(X)$, where $Q(X)$ is the space of holomorphic quadratic differentials on X. We set

$$\|\phi(z)\| := \frac{|\phi(z)|}{\sigma(z)} \quad \text{for all } z \in X,$$

and

$$\|\phi\|_2 := \left(\int_X \|\phi(z)\|^2 \cdot \sigma(z)|dz|^2 \right)^{\frac{1}{2}}.$$

We have the following result of Teo.

Lemma 3.1 (Teo, [9, Proposition 3.1]). *Let $\phi \in Q(X)$ be a holomorphic quadratic differential on a hyperbolic surface $X \in \mathcal{M}_g^n$, and let $r : X \to \mathbb{R}_+$ be the injectivity radius function. Then*

$$\|\phi(z)\| \leqslant C(r(z)) \cdot \|\phi\|_2 = \frac{\|\phi\|_2}{\sqrt{\pi} \cdot r(z)}(1 + o(r(z))),$$

where the constant $C(\cdot)$ is given by (3.1).

In [16], Wolpert gave the following asymptotically optimal bound. One may also see [17, Lemma 11] for a similar result.

Lemma 3.2 (Wolpert, [16, Corollary 11]). *Let S be a surface of genus g with n punctures, and let $X \in \mathcal{M}_g^n$ be any hyperbolic surface. Then for any $\varepsilon > 0$ there exists a $\delta(\varepsilon, S) > 0$ such that if $\ell_{\mathrm{sys}}(X) \leqslant \delta(\varepsilon, S)$, then for any $\phi \in Q(X)$ and $z \in X$*

$$\|\phi(z)\| \leqslant (1 + \varepsilon)\sqrt{\frac{2}{\pi}} \frac{\|\phi\|_2}{\sqrt{\ell_{\mathrm{sys}}(X)}}.$$

We will now derive a uniform bound that gives the asymptotics of Wolpert's bound above.

3.1. Collar neighborhoods. We let $\phi \in Q(X)$ be a holomorphic quadratic differential on a Riemann surface $X \in \mathcal{M}_g^n$ and let γ be a simple closed geodesic of length L in X. We lift ϕ to $\widetilde{\phi}$ on the annulus

$$A = \left\{ z : e^{-\frac{\pi^2}{L}} < |z| < e^{\frac{\pi^2}{L}} \right\}.$$

Then $\widetilde{\phi}(z) = \frac{f(z)}{z^2} dz^2$. where f is holomorphic on A. Therefore we have the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

We define

$$f_-(z) = \sum_{n<0} a_n z^n, \quad f_0(z) = a_0, \quad f_+(z) = \sum_{n>0} a_n z^n.$$

We therefore have the decomposition

$$\widetilde{\phi}(z) = (f_-(z) + f_0(z) + f_+(z))\frac{dz^2}{z^2} = \phi_-(z) + \phi_0(z) + \phi_+(z).$$

Let $\gamma \subset X \in \mathcal{M}_g^n$ be a closed geodesic of length $L \leqslant 2\varepsilon_2$. By the Collar Lemma (see [1, Chapter 4]) there is an embedded collar $\mathcal{C}_\gamma$ of γ in X as follows:

$$\mathcal{C}_\gamma := \left\{ z \in X : d(z,\gamma) \leqslant \operatorname{arcsinh}\left(\frac{1}{\sinh(\frac{L}{2})}\right)\right\}.$$

We set

$$\|\phi|_{\mathcal{C}_\gamma}\|_2 := \left(\int_{\mathcal{C}_\gamma} \|\phi(z)\|^2 \cdot \sigma(z)\,|dz|^2\right)^{\frac{1}{2}}.$$

As $\mathcal{C}_\gamma$ embeds in A, we have that the injectivity radius function r on A coincides with the injectivity radius function on $\mathcal{C}_\gamma \subseteq X$. Also if $z \in A$ has distance $d(z,\gamma)$ from the core closed geodesic, then

$$\sinh(r(z)) = \sinh\left(\frac{L}{2}\right)\cosh(d(z,\gamma)).$$

Therefore it follows that

$$\mathcal{C}_\gamma = \left\{ z \in A : r(z) \leqslant \sinh^{-1}\left(\cosh\left(\frac{L}{2}\right)\right)\right\}.$$

For $0 < t \leqslant \sinh^{-1}(\cosh(\frac{L}{2}))$ we then define

$$C_t = \{z \in A : r(z) \leqslant t\}.$$

In part of the following proposition we will need to restrict to a sub-collar of the standard collar $\mathcal{C}_\gamma$. For this we define the constant

$$\overline{\varepsilon}_2 = \frac{\log(3)}{2} = \sinh^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

We prove the following:

Proposition 3.3. *Let $\phi \in Q(X)$ and $\mathcal{C}_\gamma$ be the collar about a closed geodesic γ of length $L \leqslant 2\varepsilon_2$. Then:*

(1) *For any $z \in \mathcal{C}_\gamma$,*

$$\|\phi_0(z)\| \leqslant \frac{1}{\sqrt{Lc_0(L)}}\frac{\sinh^2(\frac{L}{2})}{\sinh^2(r(z))}\|\phi|_{\mathcal{C}_\gamma}\|_2,$$

where

$$c_0(L) = \cos^{-1}\left(\tanh\left(\frac{L}{2}\right)\right) + \frac{1}{2}\sin\left(2\cos^{-1}\left(\tanh\left(\frac{L}{2}\right)\right)\right) = \frac{\pi}{2} - \frac{L^3}{12} + O(L^5).$$

(2) *On C_t, $\|\phi_\pm(z)\|$ attains its maximum on ∂C_t.*

(3) *For $z \in \mathcal{C}_\gamma$ in the sub-collar $C_{\overline{\varepsilon}_2} = \{z \in A : r(z) \leqslant \overline{\varepsilon}_2\}$,*

$$\|\phi_\pm(z)\| \leqslant F(r(z))\|\phi|_{\mathcal{C}_\gamma}\|_2,$$

where

$$F(r(z)) = \frac{e^{\pi\sqrt{3}}C(\overline{\varepsilon}_2)e^{-\frac{\pi}{\sinh(r(z))}}}{3\sinh^2(r(z))} \leqslant C(\overline{\varepsilon}_2).$$

(4) *For* $z \in \mathcal{C}_\gamma$ *in the sub-collar* $C_{\overline{\varepsilon}_2}$ *with* $r(z) \leqslant \overline{\varepsilon}_2$

$$\|\phi(z)\| \leqslant G(r(z))\|\phi|_{\mathcal{C}_\gamma}\|_2,$$

where

$$G(r) = \frac{1}{\sqrt{2rc_0(2r)}} + \frac{2e^{\pi\sqrt{3}}C(\overline{\varepsilon}_2)e^{-\frac{\pi}{\sinh(r)}}}{3\sinh^2(r)} = \frac{1}{\sqrt{\pi r}}\left(1 + \frac{2r^3}{3\pi} + O(r^5)\right).$$

(5) *For* $z \in \mathcal{C}_\gamma$ *with* $r(z) \leqslant \varepsilon_2$ *then*

$$\|\phi(z)\| \leqslant \frac{\|\phi\|_2}{\sqrt{r(z)}}.$$

Proof. Let $S = \{z = x + iy : |y| < \frac{\pi}{2}\}$ be the strip, then the hyperbolic metric on S is $\rho_S(z) = \frac{|dz|}{\cos(y)}$. By the Collar Lemma [1, Theorem 4.1.6] the injectivity radius function on S satisfies

$$\sinh(r(z)) = \frac{\sinh(\frac{L}{2})}{\cos(y)}. \tag{3.2}$$

We have the $\mathbb{Z}$ cover $\pi : S \to A$ given by $\pi(z) = e^{\frac{2\pi i z}{L}}$. Therefore the hyperbolic metric on A is given by

$$\rho(z) = \frac{L}{2\pi}\frac{1}{|z|\cos\left(\frac{L}{2\pi}\log|z|\right)}.$$

It follows that $\mathcal{C}_\gamma$ lifts to the strip $\mathcal{S}_\gamma = \{w = x + iy : |y| < h(L)\}$, where

$$h(L) = \cos^{-1}\left(\tanh\left(\frac{L}{2}\right)\right).$$

Therefore $\mathcal{C}_\gamma = \{z \in A : e^{-s(L)} < |z| < e^{s(L)}\}$, where

$$s(L) = 2\pi \cdot \frac{h(L)}{L}.$$

We first show that ϕ_-, ϕ_0, ϕ_+ are all orthogonal on $\mathcal{C}_\gamma$. We have

$$\begin{aligned}
\|\phi|_{\mathcal{C}_\gamma}\|_2^2 &= \int_{\mathcal{C}_\gamma} \frac{|\phi(z)|^2}{\rho^2(z)} = \sum_{n,m}\int_{e^{-s(L)}}^{e^{s(L)}}\int_0^{2\pi} \frac{a_n\overline{a}_m z^n \overline{z}^m}{|z|^4\rho^2(r)} r\,dr\,d\theta \\
&= \sum_{n,m}\left(\int_{e^{-s(L)}}^{e^{s(L)}} \frac{a_n\overline{a}_m r^{n+m-3}}{\rho^2(r)}\,dr\right)\left(\int_0^{2\pi} e^{i(n-m)\theta}\,d\theta\right) \\
&= 2\pi\sum_n \int_{e^{-s(L)}}^{e^{s(L)}} \frac{|a_n|^2 r^{2n-3}}{\rho^2(r)}\,dr.
\end{aligned}$$

Therefore

$$\|\phi|_{\mathcal{C}_\gamma}\|_2^2 = \|\phi_-|_{\mathcal{C}_\gamma}\|_2^2 + \|\phi_0|_{\mathcal{C}_\gamma}\|_2^2 + \|\phi_+|_{\mathcal{C}_\gamma}\|_2^2. \tag{3.3}$$

This gives the bound

$$\|\phi|_{\mathcal{C}_\gamma}\|_2^2 \geqslant \|\phi_0|_{\mathcal{C}_\gamma}\|_2^2 = 2\pi|a_0|^2\frac{4\pi^2}{L^2}\int_{e^{-s(L)}}^{e^{s(L)}} \frac{\cos^2\left(\frac{L}{2\pi}\log r\right)}{r}\,dr.$$

We let $t = \frac{L}{2\pi}\log r$ giving $dt = \frac{L}{2\pi r}dr$ and

$$\|\phi_0|_{\mathcal{C}_\gamma}\|_2^2 = |a_0|^2\frac{16\pi^4}{L^3}\int_{-h(L)}^{h(L)}\cos^2(t)\,dt.$$

We define

$$c_0(L) = \int_{-h(L)}^{h(L)}\cos^2(t)\,dt = h(L) + \frac{1}{2}\sin(2h(L)) = \frac{\pi}{2} - \frac{L^3}{12} + O(L^5).$$

Then

$$\|\phi_0|_{\mathcal{C}_\gamma}\|_2^2 = |a_0|^2\frac{16\pi^4}{L^3}c_0(L).$$

For $z \in \mathcal{C}_\gamma$, we have

$$\|\phi_0(z)\| = \frac{4\pi^2|a_0|}{L^2}\cos^2\left(\frac{L}{2\pi}\log|z|\right) = \frac{1}{\sqrt{Lc_0(L)}}\frac{\sinh^2(\frac{L}{2})}{\sinh^2(r(z))}\|\phi_0|_{\mathcal{C}_\gamma}\|_2,$$

where in the last equality we apply the following version of formula (3.2)

$$\cos\left(\frac{L}{2\pi}\log|z|\right) = \frac{\sinh(\frac{L}{2})}{\sinh(r(z))}.$$

Thus

$$\|\phi_0(z)\| \leqslant \frac{1}{\sqrt{Lc_0(L)}}\frac{\sinh^2(\frac{L}{2})}{\sinh^2(r(z))}\|\phi|_{\mathcal{C}_\gamma}\|_2$$

giving (1).

We consider

$$\phi_+(z) = \frac{f_+(z)dz^2}{z^2}.$$

We have that $f_+(z)$ is holomorphic on the disk $D_+ = \{z : |z| < e^{\frac{\pi^2}{L}}\}$. Furthermore, the function $\frac{f_+(z)}{z}$ extends holomorphically to that disk. By the maximum principle the maximum modulus of $\frac{f_+(z)}{z}$ on $B(s) = \{z : |z| \leqslant s\}$ is on the boundary. Therefore the maximum modulus of $\frac{f_+(z)}{z}$ on $B(s)$ is at some $z_s \in \partial B(s)$ with $M_s = \frac{|f(z_s)|}{|z_s|}$. We have for $z \in B(s)$,

$$\|\phi_+(z)\| = \frac{|f_+(z)|}{|z|^2}\cdot\frac{4\pi^2}{L^2}|z|^2\cos^2\left(\frac{L}{2\pi}\log|z|\right) \leqslant M_s\frac{4\pi^2}{L^2}|z|\cos^2\left(\frac{L}{2\pi}\log|z|\right).$$

Recall that

$$\|\phi_+(z_s)\| = M_s\frac{4\pi^2}{L^2}s\cos^2\left(\frac{L}{2\pi}\log s\right).$$

Therefore

$$\|\phi_+(z)\| \leqslant \frac{\|\phi_+(z_s)\|}{s\cos^2(\frac{L}{2\pi}\log s)}\cdot\left(|z|\cos^2\left(\frac{L}{2\pi}\log|z|\right)\right). \tag{3.4}$$

We observe that $x\cos^2(\frac{L}{2\pi}\log x)$ is monotonically increasing on $[1, e^{s(L)}]$. To see this, we consider equivalently the function $u(t) = e^{\frac{2\pi t}{L}}\cos^2(t)$ on $[-h(L), h(L)]$. Differentiating it, we get

$$u'(t) = 2e^{\frac{2\pi t}{L}}\cos(t)\left(\frac{\pi}{L}\cos(t) - \sin(t)\right).$$

Thus u is monotonic for $\tan(t) \leqslant \frac{\pi}{L}$. As $t \leqslant h(L) = \cos^{-1}(\tanh(\frac{L}{2}))$, we have

$$\tan(t) \leqslant \tan(h(L)) = \frac{1}{\sinh(\frac{L}{2})} \leqslant \frac{2}{L} \leqslant \frac{\pi}{L}.$$

Thus u is monotonic on $[1, \frac{L}{2\pi} \cdot s(L)]$. Therefore $\|\phi_+(z)\|$ has maximum modulus in C_t on the boundary. Similarly one may prove that $\|\phi_-(z)\|$ has maximum modulus in C_t on the boundary by using $\frac{1}{z}$ as a variable. This proves (2).

To prove (3) we use Teo's bound from Lemma 3.1. By Teo

$$\|\phi_+(z_s)\| \leqslant C(r(z_s)) \cdot \|\phi_+|_{B(z_s, r(z_s))}\|_2,$$

where $B(z,r)$ is the hyperbolic ball about z of radius r. We choose z_s in the collar such that $B(z_s, r(z_s)) \subseteq \mathcal{C}_\gamma$. By the Collar Lemma [1, Theorem 4.1.6], a point of injectivity radius r is a distance d from the boundary of the collar, where

$$\sinh(r) = \cosh\left(\frac{L}{2}\right)\cosh d - \sinh d.$$

We note that solving $d = r$ gives

$$r = \tanh^{-1}\left(\frac{\cosh(\frac{L}{2})}{2}\right) \geqslant \tanh^{-1}\left(\frac{1}{2}\right).$$

Therefore we choose z_s such that $r(z_s) = \tanh^{-1}(\frac{1}{2}) = \overline{\varepsilon}_2$. Then by Lemma 3.1 and (3.3)

$$\|\phi_+(z_s)\| \leqslant C(\overline{\varepsilon}_2) \cdot \|\phi_+|_{\mathcal{C}_\gamma}\|_2 \leqslant C(\overline{\varepsilon}_2) \cdot \|\phi|_{\mathcal{C}_\gamma}\|_2.$$

This together with (3.4) implies that

$$\|\phi_+(z)\| \leqslant \frac{C(\overline{\varepsilon}_2)}{s\cos^2(\frac{L}{2\pi}\log s)} \cdot \|\phi|_{\mathcal{C}_\gamma}\|_2\left(|z|\cos^2\left(\frac{L}{2\pi}\log|z|\right)\right).$$

Recall that (3.2) gives

$$\cos\left(\frac{L}{2\pi}\log|z|\right) = \frac{\sinh(\frac{L}{2})}{\sinh(r(z))}.$$

Therefore

$$|z| = e^{\pm\frac{2\pi}{L}\left(\cos^{-1}\left(\frac{\sinh(\frac{L}{2})}{\sinh(r(z))}\right)\right)}$$

where the sign depends on which side of the core closed geodesic you are on. We rewrite the bound in terms of injectivity radius. Recall that $s > 1$. Then for $|z| \geqslant 1$, i.e.,

$$|z| = e^{\frac{2\pi}{L}\left(\cos^{-1}\left(\frac{\sinh(\frac{L}{2})}{\sinh(r(z))}\right)\right)},$$

we have

$$\|\phi_+(z)\| \leqslant \frac{C(\overline{\varepsilon}_2)\sinh^2(\overline{\varepsilon}_2)e^{\frac{2\pi}{L}\left(\cos^{-1}\left(\frac{\sinh(\frac{L}{2})}{\sinh(r(z))}\right)-\cos^{-1}\left(\frac{\sinh(\frac{L}{2})}{\sinh(\overline{\varepsilon}_2)}\right)\right)}}{\sinh^2(r(z))} \cdot \|\phi|_{\mathcal{C}_\gamma}\|_2.$$

Note that $\sinh(\overline{\varepsilon}_2) = \frac{1}{\sqrt{3}}$. Also for $0 < x < y < \pi$ then $x - y \leqslant \cos(y) - \cos(x)$ giving

$$\|\phi_+(z)\|^2 \leqslant \frac{C(\overline{\varepsilon}_2)e^{-\frac{2\pi\sinh(\frac{L}{2})}{L}\left(\frac{1}{\sinh(r(z))}-\sqrt{3}\right)}}{3\sinh^2(r(z))} \cdot \|\phi|_{\mathcal{C}_\gamma}\|_2.$$

As $\sinh(x) \geqslant x$ we have for $|z| \geqslant 1$,

$$\|\phi_+(z)\| \leqslant \frac{C(\bar{\varepsilon}_2)e^{-\pi\left(\frac{1}{\sinh(r(z))}-\sqrt{3}\right)}}{3\sinh^2(r(z))} \cdot \|\phi|_{\mathcal{C}_\gamma}\|_2 = F(r(z)) \cdot \|\phi|_{\mathcal{C}_\gamma}\|_2.$$

We note that $r(z) = r(\frac{1}{z})$. Also by the above, the maximum of $\|\phi_+(z)\|$ on $\{z : \frac{1}{c} \leqslant |z| \leqslant c\}$ is on the boundary $|z| = c$, where $1 < c \leqslant e^{s(L)}$. Therefore for $|z| = \frac{1}{c} < 1$ we have

$$\|\phi_+(z)\| \leqslant \max_{|w|=1/c} \|\phi_+(w)\| \leqslant \max_{|w|=c} \|\phi_+(w)\| \leqslant F(r(z)) \cdot \|\phi|_{\mathcal{C}_\gamma}\|_2.$$

Thus for $r(z) \leqslant \bar{\varepsilon}_2$,

$$\|\phi_+(z)\| \leqslant \left(\frac{C(\bar{\varepsilon}_2)e^{\pi\sqrt{3}}e^{-\frac{\pi}{\sinh(r(z))}}}{3\sinh^2(r(z))}\right)\|\phi|_{\mathcal{C}_\gamma}\|_2 \leqslant C(\bar{\varepsilon}_2)\|\phi|_{\mathcal{C}_\gamma}\|_2,$$

where in the last inequality we apply that $\frac{1}{\sinh^2(r(z))}e^{-\frac{\pi}{\sinh(r(z))}}$ is increasing. Similar as in the proof of part (2) if we consider $\frac{1}{z}$ as a variable, one may also get the same bound for $\|\phi_-(z)\|$. This proves (3).

For proving (4), we combine the bounds above using

$$\|\phi(z)\| \leqslant \|\phi_-(z)\| + \|\phi_0(z)\| + \|\phi_+(z)\|.$$

First observe that both $\frac{\sinh(\frac{L}{2})}{\sqrt{L}}$ and $\frac{\sinh(\frac{L}{2})}{\sqrt{c_0(L)}}$ are increasing. Since $2r(z) \geqslant L$, for any $z \in \mathcal{C}_\gamma$ we have

$$\|\phi_0(z)\| \leqslant \frac{1}{\sqrt{2r(z)c_0(2r(z))}}\|\phi|_{\mathcal{C}_\gamma}\|_2.$$

Therefore for $z \in \mathcal{C}_\gamma$ with $r(z) \leqslant \bar{\varepsilon}_2$,

$$\|\phi(z)\| \leqslant G(r(z))\|\phi|_{\mathcal{C}_\gamma}\|_2,$$

where

$$G(r) = \frac{1}{\sqrt{2rc_0(2r)}} + \frac{2e^{\pi\sqrt{3}}C(\bar{\varepsilon}_2)e^{-\frac{\pi}{\sinh(r)}}}{3\sinh^2(r)}.$$

This proves (4).

To prove (5) we combine the above bound for $r(z) \leqslant \bar{\varepsilon}_2$ with Teo's bound for $r(z) \leqslant \varepsilon_2$. If $r(z) \geqslant \bar{\varepsilon}_2$, by Lemma 3.1 we have that

$$\|\phi(z)\| \leqslant C(r(z)) \cdot \|\phi\|_2 \leqslant \sqrt{r(z)}C(r(z)) \cdot \frac{\|\phi\|_2}{\sqrt{r(z)}}.$$

As $C(x)\sqrt{x}$ is monotonically decreasing with $C(\bar{\varepsilon}_2) \cdot \sqrt{\bar{\varepsilon}_2} = 0.8091$, we have

$$\|\phi(z)\| \leqslant \sqrt{\bar{\varepsilon}_2}C(\bar{\varepsilon}_2) \cdot \frac{\|\phi\|_2}{\sqrt{r(z)}} = 0.8091\frac{\|\phi\|_2}{\sqrt{r(z)}}.$$

We now consider $r(z) \leqslant \bar{\varepsilon}_2$. We have that $H(r) = G(r) \cdot \sqrt{r}$ is monotonically decreasing. Therefore part (4) above together with Lemma 3.1 imply that

$$\|\phi(z)\| \leqslant \min\bigl(H(r(z)), \sqrt{r(z)}C(r(z))\bigr) \cdot \frac{\|\phi\|_2}{\sqrt{r(z)}}.$$

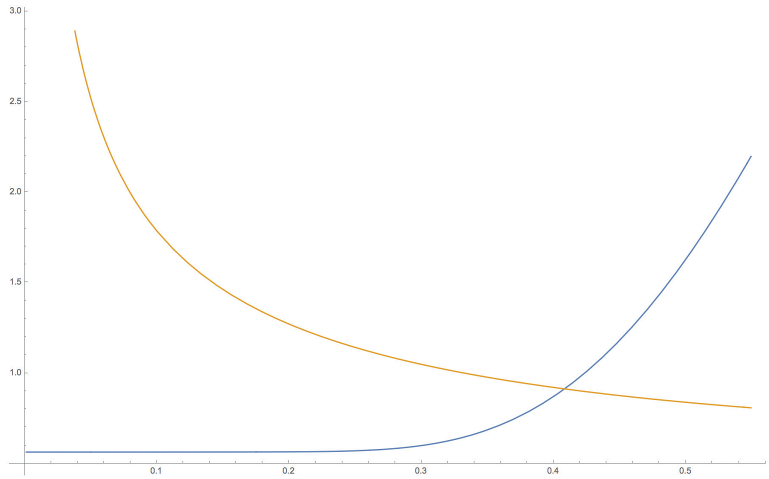

Figure 1. Plot of $H(r)$ and $\sqrt{r}C(r)$ on $(0, \bar{\varepsilon}_2]$.

Considering $m(r) = \min(H(r), \sqrt{r}C(r))$ on $(0, \bar{\varepsilon}_2]$, we have by computation that

$$m(r) \leqslant m_0 = 0.9137$$

(see Figure 1). Therefore for $r(z) \leqslant \varepsilon_2$,

$$\|\phi(z)\| \leqslant \max\{0.8091, 0.9137\} \cdot \frac{\|\phi\|_2}{\sqrt{r(z)}} \leqslant \frac{\|\phi\|_2}{\sqrt{r(z)}}, \tag{3.5}$$

which completes the proof. □

3.2. Cusp neighborhoods. We consider the cusp neighborhoods of $X \in \mathcal{M}_g^n$. Then each cusp c gives a cover $\pi : \Delta^* \to X$, where $\Delta^* = \{z : 0 < |z| < 1\}$. The hyperbolic metric on Δ^* is

$$\rho(z) = -\frac{1}{|z| \log |z|}.$$

By the Collar Lemma (see [1, Chapter 4]), c has a collar $\mathcal{C}_c$ which lifts to the annulus $A_c = \{z : 0 < |z| < e^{-\pi}\}$ with π injective on A_c. Furthermore, as $\mathcal{C}_c$ is embedded, it follows that the injectivity radius function r on X lifts to the injectivity radius function on A_c with $A_c = \{z \in A : r(z) < \varepsilon_2\}$. We have:

Lemma 3.4. *Let $X \in \mathcal{M}_g^n$ and $\phi \in Q(X)$. If $z \in \mathcal{C}_c$, then*

$$\|\phi(z)\| \leqslant K(r(z))\|\phi\|_2 \leqslant C(\varepsilon_2)\|\phi\|_2,$$

where

$$K(r) = \left(\frac{C(\varepsilon_2)e^{\pi} e^{-\frac{\pi}{\sinh(r)}}}{\sinh^2(r)}\right)$$

and $C(\varepsilon_2) = 0.7439$.

Proof. As before we have $\phi = \phi_- + \phi_0 + \phi_+$. We have the hyperbolic metric on A_c is

$$\rho(z) = -\frac{1}{|z| \log |z|}.$$

The lemma is trivially true if $\|\phi\|_2 = \infty$. Therefore we consider $\|\phi\|_2 < \infty$. It follows that $\phi_0 = \phi_- = 0$. We now bound $\|\phi(z)\|$ as above. If

$$\phi(z) = \frac{f(z)dz^2}{z^2},$$

then $\frac{f(z)}{z}$ extends to $B(s) = \{z : |z| < s\}$ and has maximum modulus at z_s with $|z_s| = s$. Therefore

$$\|\phi(z)\| \leqslant \frac{\|\phi(z_s)\|}{s(\log|s|)^2} \cdot |z|(\log|z|)^2.$$

It can easily be checked that $|z|(\log|z|)^2$ is monotonic on A_c. By the Collar Lemma, the injectivity radius on A_c satisfies $\sinh(r(z)) = -\frac{\pi}{\log|z|}$. Therefore by letting $s = e^{-\pi}$ (the maximal cusp) and using Lemma 3.1, we obtain that for $r(z) \leqslant \varepsilon_2$,

$$\|\phi(z)\| \leqslant \left(\frac{C(\varepsilon_2)e^{\pi}e^{-\frac{\pi}{\sinh(r(z))}}}{\sinh^2(r(z))}\right)\|\phi\|_2 = K(r(z)) \cdot \|\phi\|_2.$$

The function $\frac{e^{-\frac{\pi}{\sinh(r)}}}{\sinh^2(r)}$ is monotonically increasing on $[0, \varepsilon_2]$. Recall that $\sinh(\varepsilon_2) = 1$. So we have

$$\|\phi(z)\| \leqslant C(\varepsilon_2)\|\phi\|_2,$$

which completes the proof. □

3.3. Uniform upper bounds for $\|\phi\|$. In this subsection we discuss several applications of Proposition 3.3 and Lemma 3.4. The first one is to show Proposition 1.1.

Proof of Proposition 1.1. Let $z \in X$ with $\mathrm{inj}(z) \leqslant \varepsilon_2$. Then z is in either a collar or a cusp. If z is in a collar, the claim follows by part (5) of Proposition 3.3. If z is in a cusp, the claim follows by Lemma 3.4. □

We define $\ell^+_{\mathrm{sys}}(X) = \min(2\varepsilon_2, \ell_{\mathrm{sys}}(X))$. Then we have:

Corollary 3.5. *Let $X \in \mathcal{M}_g^n$ and $\phi \in Q(X)$. Then*

$$\|\phi\|_\infty \leqslant \sqrt{\frac{2}{\ell^+_{\mathrm{sys}}(X)}}\|\phi\|_2.$$

Proof. If $r(z) \geqslant \varepsilon_2$ or z is in a cusp neighborhood, then as $\ell^+_{\mathrm{sys}}(X) \leqslant 2\varepsilon_2$, it follows by Lemma 3.1 and Lemma 3.4 that

$$\|\phi(z)\| \leqslant C(\varepsilon_2)\|\phi\|_2 \leqslant \sqrt{2\varepsilon_2}C(\varepsilon_2)\frac{\|\phi\|_2}{\sqrt{\ell^+_{\mathrm{sys}}(X)}}.$$

We have $\sqrt{2\varepsilon_2} \cdot C(\varepsilon_2) = 0.9877 < \sqrt{2}$. So the claim follows for these two cases.

If z is in a collar neighborhood with $r(z) \leqslant \varepsilon_2$, it follows by (3.5) that

$$\|\phi(z)\| \leqslant m_0\frac{\|\phi\|_2}{\sqrt{r(z)}} \leqslant \sqrt{2} \cdot m_0 \cdot \frac{\|\phi\|_2}{\sqrt{\ell_{\mathrm{sys}}(X)}}. \tag{3.6}$$

The claim also follows as $m_0 < 0.9137$. □

Remark 3.6. We note that we can use Proposition 3.3 to give a bound for Wolpert's Lemma 3.2 which is independent of topology. We let $H(r) = G(r) \cdot \sqrt{r}$. Then $H(r)$ is monotonically increasing with

$$\lim_{r\to 0} H(r) = \frac{1}{\sqrt{\pi}}.$$

We note for from part (4) of Proposition 3.3 that for $r(z) \leqslant \overline{\varepsilon}_2$,

$$\|\phi(z)\| \leqslant \frac{\|\phi\|_2}{\sqrt{r(z)}}.$$

Thus for $\frac{\pi}{2} \cdot \ell_{\mathrm{sys}}(X) \leqslant r(z) \leqslant \overline{\varepsilon}_2$ we have

$$\|\phi(z)\| \leqslant \frac{\|\phi\|_2}{\sqrt{r(z)}} \leqslant \sqrt{\frac{2}{\pi}} \frac{\|\phi\|_2}{\sqrt{\ell_{\mathrm{sys}}(X)}}.$$

We choose δ_1 such that

$$\delta_1 = \frac{2}{\pi} H^{-1}\left(\frac{1+\varepsilon}{\sqrt{\pi}}\right).$$

Then it follows by part (4) of Proposition 3.3 that for $\ell_{\mathrm{sys}}(X) < \delta_1$ and

$$r(z) \leqslant \min\left\{\frac{\pi}{2} \cdot \ell_{\mathrm{sys}}(X), \overline{\varepsilon}_2\right\} \leqslant \min\left\{H^{-1}\left(\frac{1+\varepsilon}{\sqrt{\pi}}\right), \overline{\varepsilon}_2\right\},$$

we have

$$\|\phi(z)\| \leqslant (1+\varepsilon)\sqrt{\frac{2}{\pi}} \frac{\|\phi\|_2}{\sqrt{\ell_{\mathrm{sys}}(X)}}.$$

Now for $r(z) \geqslant \overline{\varepsilon}_2$ as $C(\overline{\varepsilon}_2) = 1.09 < 2$

$$\|\phi(z)\| \leqslant C(\overline{\varepsilon}_2)\|\phi\|_2 \leqslant 2\|\phi\|_2.$$

Thus for $\ell_{\mathrm{sys}}(X) < \frac{1}{2\pi}$ and $r(z) \geqslant \overline{\varepsilon}_2$ we have

$$\|\phi(z)\| \leqslant \sqrt{\frac{2}{\pi}} \frac{\|\phi\|_2}{\sqrt{\ell_{\mathrm{sys}}(X)}}.$$

We therefore choose $\delta = \min(\delta_1, \frac{1}{2\pi})$ to get the following result.

Theorem 3.7. *Let $X \in \mathcal{M}_g^n$ be any hyperbolic surface. Then for any $\varepsilon > 0$ there exists a constant $\delta(\varepsilon) > 0$ only depending on ε such that if $\ell_{\mathrm{sys}}(X) \leqslant \delta(\varepsilon)$, then for any $\phi \in Q(X)$ and $z \in X$,*

$$\|\phi(z)\| \leqslant (1+\varepsilon)\sqrt{\frac{2}{\pi}} \frac{\|\phi\|_2}{\sqrt{\ell_{\mathrm{sys}}(X)}}.$$

We note by the expansion of G we have for ε small,

$$\delta(\varepsilon) = \frac{2}{\pi} H^{-1}\left(\frac{1+\varepsilon}{\sqrt{\pi}}\right) \simeq \left(\frac{12\varepsilon}{\pi^2}\right)^{\frac{1}{3}}.$$

3.4. Fixing the length of short curves. Let $X \in \mathcal{M}_g^n$ and for α a closed curve, we let l_α be the geodesic length function on $\mathcal{M}_g^n$. Then we let $dL_\alpha \in T^*(M_g^n)$ be the complex one-form such that $\operatorname{Re} dL_\alpha = dl_\alpha$. We define

$$P(X) \subseteq T_X^*(\mathcal{M}_g) = \operatorname{span}\{(dL_\alpha)_X : l_\alpha(X) \leqslant \varepsilon_2\} \tag{3.7}$$

and

$$P(X)^\perp = \{\mu : \langle \phi, \mu \rangle = 0 \text{ for all } \phi \in P(X)\} \subseteq T_X(\mathcal{M}_g^n). \tag{3.8}$$

The plane $P(X)^\perp$ is the set of directions that fix the length of short curves. We have the following immediate consequence of Proposition 3.3.

Lemma 3.8. *Let $\mu \in P(X)^\perp$. Then*

$$\|\mu(z)\| \leqslant \sqrt{2} \cdot \|\mu\|_2.$$

Furthermore, for $r(z) \leqslant \overline{\varepsilon}_2$,

$$\|\mu(z)\| \leqslant 2 \cdot F(r(z)) \cdot \|\mu\|_2,$$

where $F(r(z))$ is defined in Proposition 3.3.

Proof. Let $\mu = \frac{\overline{\phi}}{\rho^2} \in P(X)^\perp$. Recall that $C(\overline{\varepsilon}_2) = 1.0917$. If $r(z) \geqslant \overline{\varepsilon}_2$, then by Lemma 3.1

$$\|\mu(z)\| \leqslant C(\overline{\varepsilon}_2) \cdot \|\mu\|_2 \leqslant \sqrt{2}\|\mu\|_2.$$

Similarly if z is in a cusp neighborhood, then

$$\|\mu(z)\| \leqslant K(r(z)) \leqslant C(\varepsilon_2) \cdot \|\mu\|_2 \leqslant C(\overline{\varepsilon}_2)\|\mu\|_2.$$

Now we consider the remaining case. That is, $r(z) \leqslant \overline{\varepsilon}_2$ and $z \in \mathcal{C}_\alpha$, where $\alpha \subset X$ is a closed geodesic with $l_\alpha(X) \leqslant 2\overline{\varepsilon}_2$. We lift ϕ to $\hat{\phi}$ on the annulus A and have as before

$$\hat{\phi}(z) = \phi_- + \phi_0 + \phi_+$$

with $\phi_0(z) = a\frac{dz^2}{z^2}$ for $a \in \mathbb{C}$. By the Gardiner formula [5] we have

$$\begin{aligned} 0 = \langle dL_\alpha, \mu \rangle &= \frac{2}{\pi} \int_A \frac{\overline{\hat{\phi}(z)}}{\rho(z)^2} \frac{dz^2}{z^2} \\ &= \frac{2}{\pi} \int_A \frac{\overline{\phi_0(z)}}{\rho(z)^2} \frac{dz^2}{z^2} = \frac{2a}{\pi} \int_A \frac{dx\,dy}{r^4 \rho^2(r)} = a l_\alpha(X). \end{aligned}$$

Therefore $a = 0$ and $\phi_0 = 0$ (see also [11, Proposition 8.5]). Then it follows from part (3) of Proposition 3.3 that

$$\|\phi(z)\| \leqslant \|\phi_-(z)\| + \|\phi_+(z)\| \leqslant 2 \cdot F(r(z))\|\phi\|_2, \tag{3.9}$$

where

$$F(r) = \frac{e^{\pi\sqrt{3}} C(\overline{\varepsilon}_2) e^{-\frac{\pi}{\sinh(r)}}}{3\sinh^2(r)}.$$

Together with Lemma 3.1, by letting $m'(r) = \min(2F(r), C(r))$ we have

$$\|\phi(z)\| \leqslant m'(r(z))\|\phi\|_2.$$

On $(0, \overline{\varepsilon}_2]$ by computation we have $m'(r) \leqslant 1.2333$ (see Figure 2).

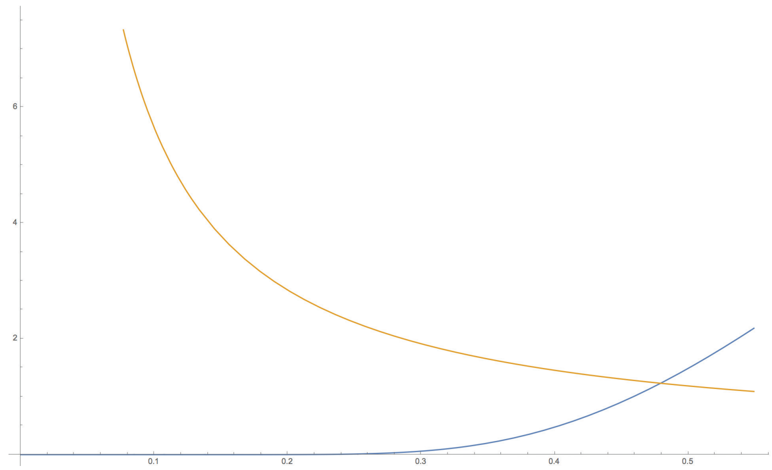

Figure 2. Plot of $2F(r)$ and $C(r)$ on $(0, \overline{\varepsilon}_2]$.

Therefore

$$\|\phi(z)\| \leqslant \sqrt{2} \cdot \|\phi\|_2$$

and proving the first inequality.

We note that $K(r) \leqslant 2F(r)$ on $(0, \overline{\varepsilon}_2]$, where $K(r)$ is defined in Lemma 3.4. Then it follows by Lemma 3.4 and (3.9) for all $r(z) \leqslant \overline{\varepsilon}_2$

$$\|\mu(z)\| \leqslant 2 \cdot F(r(z)) \cdot \|\mu\|_2,$$

which completes the proof. □

Remark 3.9. For $\mu \in P(X)^{\perp}$, the quantity $\mathrm{Comp}(\mu)$ defined in [17, Definition 10] of Wolpert is equal to 1. For this case, [17, Lemma 11] says that $\|\mu(z)\| \leqslant c'' \cdot \|\mu\|_2$, where c'' is a positive constant. The constant in Lemma 3.8 is uniform and explicit. We are grateful to Scott Wolpert for noticing us this reference.

4. Uniform lower bounds for Weil–Petersson curvatures

The following bounds is essentially due to Teo [9]. As we need a slightly modified version, we give the following version due to Ken Bromberg.

Proposition 4.1. *Fix $z \in X$ and let $U \subset T_X \mathcal{M}_g^n$ be a subspace and $K_z > 0$ a constant such that for all harmonic Beltrami differentials $\mu \in U$ we have*

$$\|\mu(z)\| \leq K_z \|\mu\|_2.$$

Then if $\mu_1, \dots, \mu_k$ is an orthonormal family in U, we have

$$\sum_{i=1}^{k} \|\mu_i(z)\|^2 \leq K_z^2.$$

Proof. Pick constants $c_1, \dots, c_k$ such that $|c_i| = \|\mu_i(z)\|$ and the directions of maximal and minimal stretch of the Beltrami differentials $c_i \mu_i$ all agree at z.[1)] We then let

$$\mu_z = \sum_{i=1}^{k} c_i \mu_i$$

and observe that our conditions on the directions of maximal and minimal stretch give that

$$\|\mu_z(z)\| = \sum_{i=1}^{k} \|c_i \mu_i(z)\| = \sum_{i=1}^{k} \|\mu_i(z)\|^2.$$

As the μ_i are orthonormal, we also have

$$\|\mu_z\|^2 = \sum_{i=1}^{k} |c_i|^2 = \sum_{i=1}^{k} \|\mu_i(z)\|^2.$$

As μ_z is a linear combination of harmonic Beltrami differentials, it is also a harmonic Beltrami differential so

$$\|\mu_z(z)\| \leq K_z \|\mu_z\|$$

and therefore

$$\|\mu_z(z)\|^2 \leq K_z^2 \|\mu_z\|^2 = K_z^2 \|\mu_z(z)\|.$$

Dividing by $\|\mu_z(z)\| = \sum_{i=1}^{k} \|\mu_i(z)\|^2$ gives the result. □

In this section we prove Theorem 1.3. Before proving it, we provide a uniform upper bound for any holomorphic orthonormal frame at $X \in \mathcal{M}_g^n$.

First we make a thick-thin decomposition of $X \in \mathcal{M}_g^n$ into three pieces as follows. Let ε_2 be the Margulis constant as in previous sections. We set

$$\begin{aligned} X_1 &:= \{q \in X : \mathrm{inj}(q) \geqslant \varepsilon_2\}, \\ X_2 &:= \{q \in \text{cusps} : \mathrm{inj}(q) < \varepsilon_2\}, \\ X_3 &:= \{q \in \text{collars} : \mathrm{inj}(q) < \varepsilon_2\}. \end{aligned}$$

So $X = \bigcup_{i=1}^{3} X_i$. We note that the set X_2 and X_3 may be empty. Actually Buser and Sarnak [2] showed that

$$\sup_{X \in \mathcal{M}_g} \mathrm{inj}(X) \asymp \ln(g)$$

for all $g \geqslant 2$.

Let $\{\mu_i\}_{i=1}^{3g-3+n}$ be a holomorphic orthonormal basis of $T_X \mathcal{M}_g^n$. Our aim is to bound $\sum_{i=1}^{3g-3+n} |\mu_i|^2(z)$ from above.

First we restrict the discussion on X_1. In this case, Teo's formula [9, equation (3.12)], which extends to the punctured case by Proposition 4.1, gives

$$\sup_{z \in X_1} \sum_{i=1}^{3g-3+n} |\mu_i|^2(z) \leqslant C(\varepsilon_2)^2 = 0.5533. \tag{4.1}$$

This bound is an easy application of Lemma 3.1 and Proposition 4.1.

[1)] For example if we choose a chart near z, in the chart the μ_i are realized by functions and we can let $c_i = \overline{\mu_i(z)}$. Then, in this chart, the directions of maximal and minimal stretch at z of each $c_i \mu_i$ are the real and imaginary axis.

Next we consider the case on X_2. Recall that Lemma 3.4 says that for any $x \in X_2$, $\|\phi(z)\| \leqslant C(\varepsilon_2)\|\phi\|_2$. Therefore it follows by Proposition 4.1 that

$$\sup_{z \in X_2} \sum_{i=1}^{3g-3+n} |\mu_i|^2(z) \leqslant C(\varepsilon_2)^2 = 0.5533 \tag{4.2}$$

Now we deal with the case on X_3. Considering (3.6), we let

$$K_0 = 2 \times (0.9137)^2 = 1.6697.$$

Then by Proposition 4.1 we have

$$\sup_{z \in X_3} \sum_{i=1}^{3g-3+n} |\mu_i|^2(z) \leqslant \frac{K_0}{\ell_{\mathrm{sys}}(X)} = \frac{1.6697}{\ell_{\mathrm{sys}}(X)}. \tag{4.3}$$

On the thick part of the moduli space $\mathcal{M}_g^n$, the Weil–Petersson curvature has been well studied in [4, 9, 12]. Now we study the Weil–Petersson curvatures on Riemann surfaces with short systoles. Our first result in this section is as follows.

Theorem 4.2 (= Theorem 1.3). *For any $X \in \mathcal{M}_g^n$ with $\ell_{\mathrm{sys}}(X) \leqslant 2\varepsilon_2$, then:*

(1) *For any $\mu \in T_X\mathcal{M}_g^n$ with $\|\mu\|_{\mathrm{WP}} = 1$, the Ricci curvature satisfies*

$$\mathrm{Ric}^{\mathrm{WP}}(\mu) \geqslant -\frac{4}{\ell_{\mathrm{sys}}(X)}.$$

(2) *The scalar curvature at X satisfies*

$$\mathrm{Sca}^{\mathrm{WP}}(X) \geqslant -\frac{4}{\ell_{\mathrm{sys}}(X)} \cdot (3g-3+n).$$

Proof. We first show part (1). Let $\mu \in T_X\mathcal{M}_g^n$ with $\|\mu\|_{\mathrm{WP}} = 1$ and one may choose a holomorphic orthonormal basis $\{\mu_i\}_{i=1}^{3g-3+n}$ of $T_X\mathcal{M}_g$ such that $\mu = \mu_1$. Now we split the lower bound in (2.2) into three parts. Since X_1, X_2 and X_3 are mutually disjoint,

$$\begin{aligned}
\mathrm{Ric}^{\mathrm{WP}}(\mu) &\geqslant -2 \sum_{j=1}^{3g-3+n} \int_X D(|\mu|^2) \cdot (|\mu_j|^2)\, dA \\
&= -2 \int_{X_1} D(|\mu|^2) \cdot \left(\sum_{j=1}^{3g-3+n} |\mu_j|^2 \right) dA \\
&\quad - 2 \int_{X_2} D(|\mu|^2) \cdot \left(\sum_{j=1}^{3g-3+n} |\mu_j|^2 \right) dA \\
&\quad - 2 \int_{X_3} D(|\mu|^2) \cdot \left(\sum_{j=1}^{3g-3+n} |\mu_j|^2 \right) dA.
\end{aligned}$$

Since D is a positive operator (see [13]), $D(|\mu|^2) \geqslant 0$. Then it follows by (4.1), (4.2) and (4.3) that

$$\begin{aligned}\mathrm{Ric}^{\mathrm{WP}}(\mu) &\geqslant -2\cdot C(\varepsilon_2)^2\cdot\int_{X_1\bigcup X_2} D(|\mu|^2)\,dA - 2\cdot\frac{K_0}{\ell_{\mathrm{sys}}(X)}\cdot\int_{X_3} D(|\mu|^2)\,dA\\ &\geqslant -\frac{3.3394}{\ell_{\mathrm{sys}}(X)}\int_X D(|\mu|^2)\,dA,\end{aligned}$$

where in the last inequality we note that $2K_0 = 4\times(0.9137^2) = 3.3394$ and $C(\varepsilon_2) = 0.7438$. Recall that the operator D is self-adjoint and $D(1) = 1$. So

$$\int_X D(|\mu|^2)\,dA = \int_X |\mu|^2\cdot D(1)\,dA = \|\mu\|^2_{\mathrm{WP}} = 1.$$

Therefore

$$\mathrm{Ric}^{\mathrm{WP}}(\mu) \geqslant -\frac{3.3394}{\ell_{\mathrm{sys}}(X)} \geqslant -\frac{4}{\ell_{\mathrm{sys}}(X)}. \tag{4.4}$$

Part (2) follows by part (1) as

$$\mathrm{Sca}^{\mathrm{WP}}(X) = \sum_{i=1}^{3g-3+n}\mathrm{Ric}^{\mathrm{WP}}(\mu_i) \geqslant -\frac{4}{\ell_{\mathrm{sys}}(X)}\cdot(3g-3+n).$$

The proof is complete. □

Remark 4.3. For $\mathcal{M}_g = \mathcal{M}_g^0$, the lower bound in part (2) of Theorem 4.2 can be extended to $-\frac{11}{\ell_{\mathrm{sys}}(X)}\cdot(g-1)$ because (4.4) implies that

$$\mathrm{Sca}^{\mathrm{WP}}(X) = \sum_{i=1}^{3g-3}\mathrm{Ric}^{\mathrm{WP}}(\mu_i) \geqslant -\frac{-3\times 3.3394}{\ell_{\mathrm{sys}}(X)}\cdot(g-1) \geqslant -\frac{11}{\ell_{\mathrm{sys}}(X)}\cdot(g-1).$$

Since the Weil–Petersson sectional curvature is negative [10, 13], we have that for any $X\in\mathcal{M}_g^n$ and $\mu, v\in T_X\mathcal{M}_g^n$,

$$\max\{\mathrm{Ric}^{\mathrm{WP}}(\mu), \mathrm{Ric}^{\mathrm{WP}}(v)\} < K^{\mathrm{WP}}(\mu, v).$$

The following result is a direct consequence of Theorem 4.4.

Theorem 4.4. *For any $X\in\mathcal{M}_g^n$ with $\ell_{\mathrm{sys}}(X)\leqslant 2\varepsilon_2$, then for any $\mu, v\in T_X\mathcal{M}_g^n$, the Weil–Petersson sectional curvature satisfies that*

$$K^{\mathrm{WP}}(\mu, v) \geqslant -\frac{4}{\ell_{\mathrm{sys}}(X)}. \tag{4.5}$$

Remark 4.5. Huang in [3] showed that $K^{\mathrm{WP}}(\mu, v) \geqslant -\frac{c(g)}{\ell_{\mathrm{sys}}(X)}$ on $\mathcal{M}_g$, where $c(g) > 0$ is a constant depending on g.

Remark 4.6. The upper bound $2\varepsilon_2$ for $\ell_{\mathrm{sys}}(X)$ in Theorem 4.4 may not be optimal. However, the upper bound for $\ell_{\mathrm{sys}}(X)$ can not be removed: actually it was shown in [12, Theorem 1.1] that if $\ell_{\mathrm{sys}}(X)$ is large enough, then

$$\min_{\mathrm{span}\{\mu,v\}\subset T_X\mathcal{M}_g} K^{\mathrm{WP}}(\mu, v) \leqslant -C < 0,$$

where $C > 0$ is a uniform constant independent of g. In particular, (4.5) does not hold for Buser–Sarnak surface $\mathcal{X}_g$ in [2] whose injectivity radius grows like $\ln(g)$ as $g \to \infty$.

We close this subsection by proving Theorem 1.6.

Theorem 4.7 (= Theorem 1.6). *For any $X \in \mathcal{M}_g^n$ with $\ell_{\mathrm{sys}}(X) \leqslant 2\varepsilon_2$, then for any $\mu \neq 0 \in P(X)^{\perp}$ and $v \in T_X\mathcal{M}_g^n$, the Weil–Petersson sectional curvature $K^{\mathrm{WP}}(\mu, v)$ along then plane spanned by μ and v satisfies that*

$$K^{\mathrm{WP}}(\mu, v) \geqslant -4.$$

Proof. Since $\mu \in P(X)^{\perp}$, by Lemma 3.8 we have

$$\sup_{z \in X} |\mu|(z) \leqslant \sqrt{2}\|\mu\|_{\mathrm{WP}}.$$

By taking a rescaling one may assume $\|\mu\|_{\mathrm{WP}} = 1$. We normalize v such that $\|v\|_{\mathrm{WP}} = 1$. Then it follows by (2.1) that

$$\begin{aligned} K^{\mathrm{WP}}(\mu, v) &\geqslant -2\int_X D(|v|^2)|\mu|^2 \, dA \\ &\geqslant -4\int_X D(|v|^2)\cdot 1 \, dA \\ &= -4\int_X |v|^2 \, dA = -4, \end{aligned}$$

which completes the proof. □

5. Total scalar curvature for large genus

It is known [8,16] that the Weil–Petersson scalar curvature always tends to negative infinity as the surface goes to the boundary of the moduli space. In this section we focus on $\mathcal{M}_g$ and study the total Weil–Petersson scalar curvature $\int_{\mathcal{M}_g} \mathrm{Sca}^{\mathrm{WP}}(X)\, dX$ over the moduli space $\mathcal{M}_g$, where dX is the Weil–Petersson measure induced by the Weil–Petersson metric on $\mathcal{M}_g$.

For any $\varepsilon > 0$, the ε-thick part $\mathcal{M}_g^{\geqslant\varepsilon}$ is the subset defined as

$$\mathcal{M}_g^{\geqslant\varepsilon} := \{X \in \mathcal{M}_g : \ell_{\mathrm{sys}}(X) \geqslant \varepsilon\}.$$

The complement $\mathcal{M}_g^{<\varepsilon} := \mathcal{M}_g \setminus \mathcal{M}_g^{\geqslant\varepsilon}$ is called the ε-thin part of the moduli space. We first recall the following result of Mirzakhani which we will apply.

Theorem 5.1 (Mirzakhani, [7, Corollary 4.3]). *As $g \to \infty$,*

$$\int_{\mathcal{M}_g} \frac{1}{\ell_{\mathrm{sys}}(X)}\, dX \asymp \mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}_g).$$

Now we are ready to state our result in this section.

Theorem 5.2 (= Theorem 1.7). *As $g \to \infty$,*

$$\frac{\int_{\mathcal{M}_g} \mathrm{Sca}^{\mathrm{WP}}(X)\, dX}{\mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}_g)} \asymp -g.$$

Proof. First by Wolpert [13] or Tromba [10] we know that for all $X \in \mathcal{M}_g$,

$$\mathrm{Sca}^{\mathrm{WP}}(X) \leqslant \frac{-3}{4\pi} \cdot (3g-2).$$

Thus,

$$\frac{\int_{\mathcal{M}_g} \mathrm{Sca}^{\mathrm{WP}}(X)\, dX}{\mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}_g)} \leqslant -C_1 \cdot g,$$

where $C_1 > 0$ is a uniform constant independent of g.

Next we prove the other direction. That is to show that

$$\int_{\mathcal{M}_g} \mathrm{Sca}^{\mathrm{WP}}(X)\, dX \geqslant -C_1' \cdot g \cdot \mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}_g), \tag{5.1}$$

where $C_1' > 0$ is a uniform constant independent of g. We split the total scalar curvature into two parts. More precisely, we let $\varepsilon_2 = \sinh^{-1}(1) > 0$,

$$\int_{\mathcal{M}_g} \mathrm{Sca}^{\mathrm{WP}}(X)\, dX = \int_{\mathcal{M}_g^{\geqslant \varepsilon_2}} \mathrm{Sca}^{\mathrm{WP}}(X)\, dX + \int_{\mathcal{M}_g^{<\varepsilon_2}} \mathrm{Sca}^{\mathrm{WP}}(X)\, dX. \tag{5.2}$$

On $\mathcal{M}_g^{\geqslant \varepsilon_2}$ it follows by Lemma 3.1 of Teo that

$$\mathrm{Sca}^{\mathrm{WP}}(X) \geqslant -(6g-6) \cdot C^2(\varepsilon_2).$$

Thus, we have

$$\begin{aligned} \int_{\mathcal{M}_g^{\geqslant \varepsilon_2}} \mathrm{Sca}^{\mathrm{WP}}(X)\, dX &\geqslant -(6g-6) \cdot C^2(\varepsilon_2) \cdot \mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}_g^{\geqslant \varepsilon_2}) \\ &\geqslant -(6g-6) \cdot C^2(\varepsilon_2) \cdot \mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}_g) \\ &\geqslant -C_2 \cdot g \cdot \mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}_g), \end{aligned} \tag{5.3}$$

where $C_2 > 0$ is a uniform constant independent of g.

On $\mathcal{M}_g^{<\varepsilon_2}$ it follows by Theorem 4.2 that

$$\mathrm{Sca}^{\mathrm{WP}}(X) \geqslant -\frac{11}{\ell_{\mathrm{sys}}(X)} \cdot (g-1).$$

Thus, we have

$$\begin{aligned} \int_{\mathcal{M}_g^{<\varepsilon_2}} \mathrm{Sca}^{\mathrm{WP}}(X)\, dX &\geqslant -11(g-1) \cdot \int_{\mathcal{M}_g^{<\varepsilon_2}} \frac{1}{\ell_{\mathrm{sys}}(X)} dX \\ &\geqslant -11(g-1) \cdot \int_{\mathcal{M}_g} \frac{1}{\ell_{\mathrm{sys}}(X)} dX. \end{aligned}$$

By Theorem 5.1 of Mirzakhani we have

$$\int_{\mathcal{M}_g^{<\varepsilon_2}} \mathrm{Sca}^{\mathrm{WP}}(X)\, dX \geqslant -C_3 \cdot g \cdot \mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}_g), \tag{5.4}$$

where $C_3 > 0$ is a uniform constant independent of g.

Then the claim (5.1) follows by (5.2), (5.3) and (5.4). □

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Martin Bridgeman, Boston College, Chestnut Hill, Ma 02467, USA
e-mail: bridgem@bc.edu

Yunhui Wu, Tsinghua University, Haidian District, Beijing 100084, P. R. China
e-mail: yunhui_wu@mail.tsinghua.edu.cn

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