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Distance and angles between Teichmüller geodesics ☆

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We show that the angles between Teichmüller geodesic rays issuing from a common point, defined by using the law of cosines, do not always exist. The proof uses an estimation for the Teichmüller distance on finite dimensional Teichmüller spaces. As a consequence, the Teichmüller space equipped with the Teichmüller metric is not a CAT(k) space for any $k \in \mathbb{R}$. We also discuss some necessary conditions for the existence of angle between the Teichmüller geodesics.

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1. Introduction

Let $S = S_{g,n}$ be a connected oriented surface of genus g with n punctures such that $3g-3+n > 1$. The Teichmüller space $\mathcal{T}(S)$ is the space of equivalence classes of marked conformal structures of analytically finite type on S. It is well known that $\mathcal{T}(S)$ is a complex manifold of dimension $3g-3+n$. The Teichmüller metric is a Finsler metric on $\mathcal{T}(S)$ which is equal to the Kobayashi metric [22].

It is a classical result that the Teichmüller space with the Teichmüller metric is not a metric space of non-positive curvature. This was first proved by Masur [15] by constructing two Teichmüller geodesic rays originating at the same point which are not divergent at infinity. Masur and Wolf [17] proved that the Teichmüller space with the Teichmüller metric is not Gromov hyperbolic. Moreover, the Teichmüller metric is known to exhibit features of positive curvature near the thin regions of Teichmüller space [18].

In this paper, we show that the Teichmüller space with the Teichmüller metric is not a metric space of "curvature bounded from above".

Theorem 1.1. *The Teichmüller space with the Teichmüller metric is not a* $\mathrm{CAT}(k)$ *space for any* $k \in \mathbb{R}$.

The proof uses an estimation for the Teichmüller distance function (Theorem 3.1) to show that the angles between Teichmüller geodesic rays issuing from a common point, defined by using the law of cosines, do not always exist.

Theorem 1.2. *For any* $X \in \mathcal{T}(S)$*, there exist two Teichmüller geodesic rays* $r_i, i = 1, 2$*, issuing from* X *such that the angle between* r_1 *and* r_2 *at* X *does not exist.*

The idea is that, since the Teichmüller metric is Finsler but not Riemannian when $3g-3+n > 1$, the analogous notion of Riemannian angle does not adapt to the Teichmüller metric. We define the angle (if it exists) between two Teichmüller geodesic rays by using the law of cosines for approximated geodesic triangles. The existence of angles for all pairs of Teichmüller geodesic rays (this is true under the $\mathrm{CAT}(k)$ assumption) would imply that the Teichmüller norm arises from an inner product, which is a contradiction.

Since any Teichmüller disk is isometric to the Poincaré disk, the angle between two geodesic rays on the same Teichmüller disk exists. Beyond that, we do not know any other explicit example about the existence of angle. In §4, we discuss some necessary conditions for the existence of angle. Our study may be related to locally holomorphic rigidity for Teichmüller spaces.

Conjecture 1.3. *The angle between two Teichmüller geodesic rays (issuing from a common point) exists if and only if they lie on the same Teichmüller disk.*

Remark 1.4. There are many important results on the metric geometry of the Teichmüller metric, especially recent works of Farb and Masur on the asymptotic cone of the moduli space [8], and the results of Rafi [19] on lack of hyperbolicity. We recommend the survey [16] for a general reference.

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2. Preliminaries

2.1. Teichmüller space

We refer to [11] and [13] for basic knowledge on Teichmüller theory.

A *marked Riemann surface* (X, f) is a conformal structure X of analytically finite type on S, equipped with an orientation-preserving homeomorphism $f : S \to X$. The *Teichmüller space of* S, denoted by $\mathcal{T}(S)$, is the set of equivalence classes of marked Riemann surfaces, where (X_1, f_1) and (X_2, f_2) are equivalent if there is a conformal map $g : X_1 \to X_2$ homotopic to $f_2 \circ f_1^{-1}$. In this paper, we shall denote the equivalence class of a marked Riemann surface (X, f) by X for the sake of simplicity, if no confusion arises.

A measurable $(-1, 1)$-form $\mu = \mu(z)\frac{d\bar{z}}{dz}$ on X such that

$$\|\mu\|_\infty = \operatorname*{ess\,sup}_{z\in X} |\mu(z)| < \infty$$

is called a *Beltrami differential*. Let $B(X)$ be the space of Beltrami differentials on X.

Given $X \in \mathcal{T}(S)$ and $\mu \in B(X)$ with $\|\mu\|_\infty < 1$, the solution of the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z}$$

gives rise to a quasiconformal deformation of X. Denote the solution by f^μ (the solution is unique if we consider the equation on the universal cover $\mathbb{H}^2$ and require that f fixes $0, 1, \infty \in \partial\mathbb{H}^2$). Let $X_\mu = f^\mu(X)$. Two Beltrami differentials μ and ν are *Teichmüller equivalent*, denoted by $\mu \sim \nu$, if (X_μ, f^μ) is Teichmüller equivalent to (X_ν, f^ν). We denote the equivalence class of μ by $[\mu]$. Note that the equivalence classes of Beltrami differentials parametrize the Teichmüller space $\mathcal{T}(X) \cong \mathcal{T}(S)$.

The *maximal dilatation* of a quasiconformal mapping f with Beltrami differential μ, denoted by $K(f)$, is given by

$$K(f) = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}.$$

For $X_1, X_2 \in \mathcal{T}(S)$, the *Teichmüller distance between* X_1 *and* X_2 is defined by

$$d(X_1, X_2) = \frac{1}{2} \log \inf_{g \simeq f_2 \circ f_1^{-1}} K(g),$$

where the infimum is taken over all quasiconformal maps $g : X_1 \to X_2$ homotopic to $f_2 \circ f_1^{-1}$.

A *holomorphic quadratic differential* φ on X is a $(2,0)$-form locally of the form $\varphi = \varphi(z)dz^2$, where $\varphi(z)$ is holomorphic. A holomorphic quadratic differential φ is *integrable* if

$$\|\varphi\| = \iint_X |\varphi(z)| dxdy < \infty.$$

Let $Q(X)$ be the space of all integrable holomorphic quadratic differentials on X. Denote the unit sphere of $Q(X)$ by $Q^1(X)$.

A Beltrami differential μ on X is said to be *extremal* in $[\mu]$ if

$$\|\mu\|_\infty = \inf_{\mu' \in [\mu]} \|\mu'\|_\infty.$$

For any extremal $\mu \in B(X)$ with $\|\mu\|_\infty < 1$, there exist unique $0 < k < 1$ and $\varphi_\mu \in Q^1(X)$ satisfying

$$\mu = k \frac{|\varphi_\mu|}{\varphi_\mu}.$$

For any $X, Y \in \mathcal{T}(S)$, there exists a unique extremal quasiconformal map, called the *Teichmüller map*, in the homotopy class between X and Y. The Beltrami differential of the Teichmüller map is of the form $k\frac{|\varphi|}{\varphi}$, for some $\varphi \in Q^1(X)$ and $0 < k < 1$. The Teichmüller distance $d(X, Y)$ is equal to

$$\frac{1}{2} \log \frac{1+k}{1-k}.$$

2.2. Finsler structure of the Teichmüller metric

Let $\mathcal{M}$ be a smooth manifold. Denote by $\mathrm{T}\mathcal{M}$ the tangent bundle of $\mathcal{M}$. A *Finsler structure* on $\mathcal{M}$ is defined by a continuous map

$$F : \mathrm{T}\mathcal{M} \to \mathbb{R}$$

such that the restriction of F on each $\mathrm{T}_p\mathcal{M}, p \in \mathcal{M}$ (the tangent space to $\mathcal{M}$ at p) is a norm in the weak sense, that is,

- $F(p, v) \geq 0$, $\forall\, v \in \mathrm{T}_p\mathcal{M} \setminus \{0\}$, and $F(p, v) > 0$ if $v \neq 0$;
- $F(p, kv) = kF(p, v)$, $\forall\, k > 0$;
- $F(p, v + w) \leq F(p, v) + F(p, w)$, $\forall\, v, w \in \mathrm{T}_p\mathcal{M}$.

If $\gamma : [0, 1] \to \mathcal{M}$ is a C^1 curve then its Finsler length is defined by

$$L(\gamma) = \int_0^1 F(\gamma(t), \gamma'(t))dt.$$

For any two points $p, q \in \mathcal{M}$ the *Finsler distance* is defined by

$$d_F(p, q) = \inf_{\gamma} L(\gamma)$$

where the infimum is taken over all piecewise C^1 curve joining p to q.

Now we can describe the Finsler structure of the Teichmüller space. There is a natural pairing between $B(X)$ and $Q(X)$ defined by

$$(\mu, \varphi) = \iint_X \mu(z)\varphi(z)dxdy,$$

where $\mu \in B(X)$ and $\varphi \in Q(X)$.

A Beltrami differential $\mu \in B(X)$ is called *infinitesimally trivial* if

$$\iint_X \mu(z)\varphi(z)dxdy = 0$$

for any $\varphi \in Q(X)$. Denote the set of infinitesimal trivial Beltrami differentials on X by $N(X)$. It is known that $\mu \in N(X)$ if and only if $f^{t\mu}$ (for t small) represents a trivial deformation of X. With the above notations, we can identify the (holomorphic) tangent space of $\mathcal{T}(S)$ at X as $B(X)/N(X)$. The cotangent space, dual to $B(X)/N(X)$, is $Q(X)$.

We let

$$\langle \mu, \varphi \rangle = \mathrm{Re} \iint_X \mu(z)\varphi(z)dxdy.$$

For $\mu \in B(X)$, we define the infinitesimal norm

$$\|\mu\|_T = \sup_{\varphi \in Q^1(X)} \langle \mu, \varphi \rangle.$$

It follows that the infinitesimal Teichmüller norm $\|\cdot\|_T$ is just the quotient norm on $B(X)/N(X)$, that is,

$$\|\mu\|_T = \inf ||\mu - \nu||_\infty,$$

where ν ranges over $N(X)$.

We can think of an extremal Beltrami differential $\mu \in B(X)$ as representing an infinitesimal quasiconformal mapping $f^{\epsilon\mu} : X \to X_{\epsilon\mu}$. The Teichmüller distance satisfies

$$d(X, X_{\epsilon\mu}) = \frac{1}{2}\log\frac{1+\epsilon\|\mu\|_T}{1-\epsilon\|\mu\|_T} \approx \epsilon\|\mu\|_T + o(\epsilon).$$

Using the definition of Finsler metric, one can show that $\|\cdot\|_T$ is the infinitesimal Finsler norm of the Teichmüller distance [10,22]. In other words, the Finsler metric dual to the L^1 norm on $Q^1(X)$ induces the Teichmüller metric on $\mathcal{T}(S)$.

In the following, we shall identify the tangent space of $\mathcal{T}(S)$ at X as $B(X)/N(X)$, endowed with either the Teichmüller norm or the quotient norm.

2.3. CAT(k) *space*

This section reviews some basic definitions and results in comparison geometry; for further background see [4].

Let (M, d) be a complete metric space. A *geodesic* in M is an isometric image of an interval of the real line. The isometric image of the positive real line is called a *geodesic ray* and the isometric image of a closed connected interval of the real line is called a *geodesic segment*. M is called a *geodesic metric space* if every two points in M are joined by a (not necessarily unique) geodesic. M is *straight* if any two points can be connected by a unique geodesic which extends uniquely to an isometric image of the real line.

Remark 2.1. It is well known that the Teichmüller space $\mathcal{T}(S)$ with Teichmüller metric is straight.

Given a real number k, the metric space $M^n_k, n \geq 2$ is defined by:

(i) If $k = 0$, then M^n_0 is the Euclidean space $\mathbb{E}^n$.
(ii) If $k > 0$, then M^n_k is obtained from the sphere $\mathbb{S}^n$ by multiplying the distance function by the constant $1/\sqrt{k}$.
(iii) If $k < 0$, then M^n_k is obtained from hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by the constant $1/\sqrt{-k}$.

In other words, M^n_k is the connected, simple connected Riemannian space with constant curvature k.

A *geodesic triangle* in (M, d) consists of three point $p, q, r \in M$, its *vertices*, and a choice of three geodesic segment $[p, q]$, $[q, r]$, $[r, q]$ joining them, its *sides*. Such a geodesic triangle will be denoted by $\Delta([p, q], [q, r], [r, q])$ or $\Delta(p, q, r)$. If a point $x \in M$ lies in the union of $[p, q]$, $[q, r]$ and $[r, q]$, then we denote $x \in \Delta(p, q, r)$.

A geodesic triangle $\overline{\Delta} = \overline{\Delta}(\bar{p}, \bar{q}, \bar{r})$ in M_k^2 is a *comparison triangle* for $\Delta = \Delta(p, q, r)$ if $d(\bar{p}, \bar{q}) = d(p, q)$, $d(\bar{q}, \bar{r}) = d(q, r)$ and $d(\bar{p}, \bar{r}) = d(p, r)$. In the case of $k > 0$, we require the perimeter $d(p, q) + d(q, r) + d(r, p)$ of Δ to be less than $2\pi/\sqrt{k}$. Then a comparison triangle $\overline{\Delta} \subset M_k^2$ always exists and it is unique up to isometry. A point $\bar{x} \in [\bar{q}, \bar{r}]$ is called *a comparison point* for $x \in [q, r]$ if $d(q, x) = d(\bar{q}, \bar{x})$. Comparison points on $[\bar{p}, \bar{q}]$ and $[\bar{p}, \bar{r}]$ are defined in the same way. The interior angle of $\overline{\Delta}(\bar{p}, \bar{q}, \bar{r})$ at $\bar{p}$ is called the *comparison angle* between q and r at p and is denoted by $\overline{\angle}_p(q, r)$.

A geodesic metric space (M, d) is called a CAT(k) *space* if all the geodesic triangles Δ satisfy

$$d(x, y) \leq d(\bar{x}, \bar{y}),$$

for all $x, y \in \Delta$ with comparison points $\bar{x}, \bar{y} \in \overline{\Delta}$.

A metric space (M, d) is said to be of *curvature* $\leq k$ if it is locally a CAT(k) space. If (M, d) is of curvature ≤ 0, then we say that (M, d) is of *non-positively curved.*

Next we introduce the concept of *Alexandrov angle* between geodesics in a geodesic metric space.

Let (M, d) be a geodesic metric space. Let $c : [0, a] \to M$ and $c' : [0, a'] \to M$, $c(0) = c'(0)$, be two geodesic segments. For $t \in (0, a)$ and $t' \in (0, a')$, we consider the comparison triangle $\overline{\Delta}(c(0), c(t), c'(t'))$, modeled on $\mathbb{E}^n$, and the comparison angle $\overline{\angle}_{c(0)}(c(t), c'(t'))$. We recall that

$$\cos \overline{\angle}_{c(0)}(c(t), c'(t')) = \frac{d(c(0), c(t))^2 + d(c(0), c'(t'))^2 - d(c(t), c'(t'))}{2d(c(0), c(t))d(c(0), c'(t'))}.$$

The *Alexandrov (upper) angle* between the geodesic segments c and c' is the number $\angle^A(c, c') \in [0, \pi]$ defined by

$$\angle^A(c, c') = \varlimsup_{t, t' \to 0} \overline{\angle}_{c(0)} \left(c(t), c'(t')\right).$$

(See [4].)

Definition 2.2 *(Angle).* With the above notation, if the limit

$$\angle(c, c') = \lim_{t, t' \to 0} \overline{\angle}_{c(0)}(c(t), c'(t'))$$

exists, then we say that the angle exists, and the limit is called the *angle between c and c'.*

We remark that the above notion of angles, if existed, would satisfy the following axiom of *generalized angle* (we denote by c, c', c'' a triple of geodesics issuing from a point):

(i) $A(c, c') \in [0, \pi]$;
(ii) $A(c, c') = A(c', c)$;
(iii) $A(c, c'') \leq A(c, c') + A(c', c'')$;
(iv) if c is the restriction of c' to an initial segment of its domain, then $A(c, c') = 0$;
(v) if $c : [-a, a] \to M$ is a geodesic and $c_-, c_+ : [0, a] \to M$ are defined by $c_-(t) = c(-t)$ and $c_+(t) = c(t)$, then $A(c_-, c_+) = \pi$.

The following result implies that in a (locally) CAT(k) space angles between geodesics always exist.

Proposition 2.3 *([4], II.3.1). Let M be a* CAT(K) *space and let $c : [0, a] \to X$ and $c' : [0, d] \to X$ be two geodesic segments issuing from the same point $c(0) = c'(0)$. Then the K-comparison angle $\overline{\angle}_{c(0)}(c(t), c'(t'))$ is a non-decreasing function of both $t, t' \geq 0$, and the Alexandrov angle $\angle^A(c, c')$ is equal to*

$$\lim_{t,t' \to 0} \overline{\angle}_{c(0)}(c(t), c'(t')) = \lim_{t \to 0} \overline{\angle}_{c(0)}(c(t), c'(t)).$$

3. Teichmüller distance and angles

Denote by d the distance function of the Teichmüller metric. Let $X \in \mathcal{T}(S)$ and $0 < s < 1$. For any Beltrami differential μ on X with norm $\|\mu\|_\infty = 1$, we consider the quasiconformal deformations of X with Beltrami differential $s\mu$ and denote the image of X by $X_{s\mu} = f^{s\mu}(X)$.

The first part of this section aims to establish the following estimation for the Teichmüller distance:

Theorem 3.1. *Let $X \in \mathcal{T}(S)$ and let μ, ν be a pair of extremal Beltrami differentials on X with $\|\mu\|_\infty = \|\nu\|_\infty = 1$. For small $s, t \in \mathbb{R}_+$, the Teichmüller distance between $X_{s\mu}$ and $X_{t\nu}$ satisfies*

$$d(X_{s\mu}, X_{t\nu}) = \sup_{\varphi} \mathrm{Re} \iint_X (s\mu - t\nu)\varphi + O((s + t)^2),$$

where the supremum is taken over the L^1 unit sphere of the space of holomorphic quadratic differentials on X.

The following formula of Hu-Shen [12] is a consequence of Theorem 3.1.

Corollary 3.2. *With the above notation, we have*

$$\begin{aligned} d(X_{t\mu}, X_{t\nu}) &= t \sup_{\varphi} \mathrm{Re} \iint_X (\mu - \nu)\varphi + o(t) \\ &= t\|\mu - \nu\|_T + o(t). \end{aligned}$$

Remark 3.3. The above formula in [12] was used to study the Teichmüller geodesic triangle through a notion of "symmetric angle" between Teichmüller geodesic rays. See also [14,7].

Let $X \in \mathcal{T}(S)$. For any $\mu \in B(X)$ and small $t \in \mathbb{R}_+$, we have (see [22, Page 370], [10, Page 133])

$$d(X, f^{t\mu}(X)) = t \sup_{\varphi \in Q^1(X)} \mathrm{Re} \iint_X \mu\varphi dxdy + O(t^2).$$

Earle [6] showed that d is a C^1-function on $d^{-1}((0,\infty))$ and obtained a formula for the first-order derivative of the Teichmüller distance.

Theorem 3.4 *(Earle). If $Y = f^\nu(X)$, with $\nu = k\frac{|\varphi|}{\varphi}$, $0 < k < 1$, $\varphi \in Q^1(X)$. Then*

$$\lim_{t \to 0} \frac{1}{t} \big(d(Y, f^{t\mu}(X)) - d(Y, X) \big) = -\mathrm{Re} \iint_X \mu\varphi dxdy.$$

The first variational formula was also proved by Gardiner [9] as an application of the Reich-Strebel inequality. More recently, Rees [20,21] proved that d is C^2 on $d^{-1}((0,\infty))$ but not $C^{2+\epsilon}$ for any $\epsilon > 0$.

Our proof of Theorem 3.1 follows the argument in [9,10]. As was pointed out by Gardiner, most of the known results about the infinitesimal theory of the Teichmüller metric can be derived from the Reich-Strebel inequality, without using deep theorems from real analysis.

3.1. Proof of Theorem 3.1

Let $\mu, \nu \in B_1(X)$ and let $s, t \in \mathbb{R}_+$ be small. Denote $f = f^{t\nu} \circ (f^{s\mu})^{-1}$. The Beltrami differential ζ of f satisfies

$$\zeta \circ f(z) \frac{\overline{\frac{\partial}{\partial z} f(z)}}{\frac{\partial}{\partial z} f(z)} = \frac{s\mu(z) - t\nu(z)}{1 - st\overline{\nu(z)}\mu(z)}$$

in local coordinate z on X. We set

$$\eta(z) = \frac{s\mu(z) - t\nu(z)}{1 - st\overline{\nu(z)}\mu(z)}.$$

Then we have $||\eta||_\infty = ||\zeta||_\infty$.

To estimate the Teichmüller distance $d(X_{s\mu}, X_{t\nu})$, we need the following variant of the Reich-Strebel inequality. See [10, §6.4].

Theorem 3.5 *(The fundamental inequalities). Let $X \in \mathcal{T}(S)$ and let f be the quasi-conformal map with Beltrami differential η on X. Let K_0 be the maximal dilatation of the extremal map in the class $[f]$. Then*

$$\frac{1}{K_0} \leq \iint_X \frac{|1 - \eta \frac{\varphi}{|\varphi|}|^2}{1 - |\eta|^2} |\varphi| dxdy, \tag{1}$$

for all $\varphi \in Q^1(X)$, and

$$K_0 \leq \sup_{\varphi \in Q^1(X)} \iint_X \frac{|1 + \eta \frac{\varphi}{|\varphi|}|^2}{1 - |\eta|^2} |\varphi| dxdy. \tag{2}$$

The fundamental inequalities give upper and lower estimates on the dilatation of an extremal representative of a given Teichmüller class of maps between Riemann surfaces.

In the following, $O((s+t)^2)$ denotes a function $g(s,t)$ satisfying

$$|g(s,t)| \leq C(s+t)^2$$

for some constants $C > 0$, $\delta > 0$ and for any $0 < s,\ t < \delta$.

Lemma 3.6. *For two extremal Beltrami differentials $\mu, \nu \in B_1(X)$, let $K = K(s,t)$ be the maximal dilatation of the extremal quasi-conformal map in the class of $f : X_{s\mu} \to X_{t\nu}$ for $s, t \in \mathbb{R}_{>0}$ sufficiently small, and let $k = \frac{K-1}{K+1}$. Then we have*

$$k = \sup_{\varphi \in Q^1(X)} Re \iint_X (s\mu - t\nu)\varphi dxdy + O((s+t)^2). \tag{3}$$

Proof. It is not hard to see that K is equal to the maximal dilatation of the extremal quasi-conformal map in the class of $f^\eta : X \to X_\eta$, that is, the Teichmüller distance $d(X_{s\mu}, X_{t\nu}) = d(X, X_\eta)$.

Recall that

$$\eta(z) = \frac{s\mu(z) - t\nu(z)}{1 - st\overline{\nu(z)}\mu(z)}.$$

Since $|\eta| = |\frac{s\mu - t\nu}{1 - st\overline{\nu}\mu}|$, we have $\eta = (s\mu - t\nu)(1 + O(st))$ and $O(\|\eta\|_\infty^2) = O((s+t)^2)$. There exists a unique $\varphi_\eta \in Q^1(X)$ realizing the supremum

$$\sup_{\varphi \in Q^1(X)} \langle \eta, \varphi \rangle = \sup_{\varphi \in Q^1(X)} \mathrm{Re} \iint_X \eta\varphi dxdy.$$

By definition,

$$\|\eta\|_T = \sup_{\varphi\in Q^1(X)} \langle \eta, \varphi\rangle.$$

Denote $\tilde{k} = \|\eta\|_T$ and $\tilde{\eta} = \eta - \tilde{k}\frac{|\varphi_\eta|}{\varphi_\eta}$. Then $\tilde{\eta}$ is infinitesimally trivial.

Then it follows from (2) that

$$\begin{aligned}
K &\le \sup_{\varphi\in Q^1(X)} \iint\limits_X \frac{|1+\eta\frac{\varphi}{|\varphi|}|^2}{1-|\eta|^2}|\varphi|dxdy \\
&\le \sup_{\varphi\in Q^1(X)} \iint\limits_X \frac{1+|\eta|^2}{1-|\eta|^2}|\varphi|dxdy + 2\sup_{\varphi\in Q^1(X)} \iint\limits_X \frac{\mathrm{Re}(\eta\varphi)}{1-|\eta|^2}dxdy \\
&\le \sup_{\varphi\in Q^1(X)} \iint\limits_X \frac{1+\|\eta\|_\infty^2}{1-\|\eta\|_\infty^2}|\varphi|dxdy + 2\sup_{\varphi\in Q^1(X)} \iint\limits_X \frac{\mathrm{Re}(\eta\varphi)}{1-\|\eta\|_\infty^2}dxdy \\
&= \frac{1+\|\eta\|_\infty^2}{1-\|\eta\|_\infty^2} + \frac{2}{1-||\eta||_\infty^2}\iint\limits_X \mathrm{Re}(\eta\varphi_\eta)dxdy \\
&= 1 + \frac{2\|\eta\|_\infty^2}{1-\|\eta\|_\infty^2} + \frac{2\langle\eta,\varphi_\eta\rangle}{1-\|\eta\|_\infty^2} \\
&= 1 + 2\|\eta\|_T + O(\|\eta\|_\infty^2).
\end{aligned}$$

On the other hand, by (1), we have

$$\begin{aligned}
\frac{1}{K} &\le \iint\limits_X \frac{|1-\eta\frac{\varphi_\eta}{|\varphi_\eta|}|^2}{1-|\eta|^2}|\varphi_\eta|dxdy \\
&\le \frac{1+\|\eta\|_\infty^2}{1-\|\eta\|_\infty^2} - 2\mathrm{Re}\iint\limits_X \frac{\eta\varphi_\eta}{1-|\eta|^2}dxdy \\
&= \frac{1+\|\eta\|_\infty^2}{1-\|\eta\|_\infty^2} - 2\mathrm{Re}\iint\limits_X \frac{(\tilde{k}\frac{|\varphi_\eta|}{\varphi_\eta}+\tilde{\eta})\varphi_\eta}{1-|\eta|^2}dxdy \\
&\le 1 + \frac{2\|\eta\|_\infty^2}{1-\|\eta\|_\infty^2} - 2\tilde{k} - 2\mathrm{Re}\iint\limits_X \frac{\tilde{\eta}\varphi_\eta}{1-|\eta|^2}dxdy \\
&\le 1 - 2\|\eta\|_T + O(\|\eta\|_\infty^2).
\end{aligned}$$

The last inequality follows since

$$\iint\limits_X \tilde{\eta}\varphi_\eta dxdy = 0$$

and

$$\frac{1}{1-|\eta|^2} = 1 + O(\|\eta\|_\infty^2).$$

By definition, $k = \frac{K-1}{K+1}$. With the above inequalities, we have

$$k \geq \frac{\|\eta\|_T - O(\|\eta\|_\infty^2)}{1 - (\|\eta\|_T - O(||\eta||_\infty^2))} = \|\eta\|_T + O(\|\eta\|_\infty^2)$$

and

$$k \leq \frac{\|\eta\|_T + O(\|\eta\|_\infty^2)}{1 + (\|\eta\|_T + O(||\eta||_\infty^2))} = \|\eta\|_T + O(\|\eta\|_\infty^2).$$

The estimation (3) follows. □

Proof of Theorem 3.1. Since $d(X_{s\mu}, X_{t\nu}) = \frac{1}{2}\log K$ and $K = \frac{1+k}{1-k}$, Theorem 3.1 follows from Lemma 3.6 immediately. □

3.2. Remark on Theorem 3.1

Theorem 3.1 is similar to a known result in Finsler geometry. In the work of Deng and Hou [5], they proved an analogue of the famous Myers-Steenord Theorem for Finsler space. Their proof depends on the following key result on distance function (see [5, Theorem 1.2]).

Theorem 3.7 *(Deng-Hou). Let $(\mathcal{M}, F, d)$ be a Finsler manifold and $x \in \mathcal{M}$. Suppose that the exponential map is a C^1 diffeomorphism from a small tangent ball $B_x(r)$ onto a neighborhood of x, then*

$$\frac{F(x, A-B)}{d(\exp_x A, \exp_x B)} \to 1$$

as $(A, B) \to (0, 0)$, where $A, B \in B_x(r)$.

The above result on the distance function is more general. However, it is often assumed in the literature that the Finsler structure is of least C^2 and the Hessian of F^2 is positive-definite. For instance, in the book of Bao-Chern-Shen (see [3, §5.3]), the exponential is defined using the equation of geodesic, where the Christoffel symbols appear.

In the setting of the Teichmüller metric, the exponential map is always defined using the Teichmüller extremal maps. To be more precise, let $X \in \mathcal{T}(S)$ and $Q^1(X)$ be the unit sphere of $Q(X)$ (equipped with the L^1-norm). We denote the unit ball of $Q(X)$ by $Q_{<1}(X)$. The nonzero set $Q_{<1}(X) \setminus \{0\}$ can be identified with $(0, 1) \times Q^1(X)$. The

Teichmüller map associated each $(k, \varphi), 0 < k < 1$, $\varphi \in Q^1(X)$ with a quasiconformal deformation of X:

$$\begin{aligned}\Phi : (0,1) \times Q^1(X) &\to \mathcal{T}(S) \\ k \cdot \varphi &\mapsto X_{k\cdot\varphi}\end{aligned}$$

such that the quasiconformal map from X to X_φ has Beltrami differential $k\frac{|\varphi|}{\varphi}$. The Teichmüller distance between X and $X_{k\cdot\varphi}$ is equal to

$$\frac{1}{2}\log\frac{1+k}{1-k}.$$

Then Φ defines a homeomorphism between $Q_{<1}(X)$ and $\mathcal{T}(S)$. And for each $\varphi \neq 0$ the ray

$$t \in [0,1) \mapsto \Phi(\frac{e^{2t}-1}{e^{2t}+1} \cdot \varphi)$$

is a Teichmüller geodesic ray.

To apply the theorem of Deng-Hou, we need to check that the exponential map is C^1 for the Teichmüller space with the Teichmüller metric. As we have mentioned before, the regularity of the Teichmüller distance is studied by Earle, Gardiner and Rees. Rees proved that the Teichmüller distance is C^2 on $d^{-1}((0,\infty))$. This result implies that (see [20, Theorem 2]).

Theorem 3.8 *(Rees). The "exponential map" Φ is a C^1 homeomorphism between $Q_{<1}(X) \setminus \{0\}$ and $\mathcal{T}(S) \setminus \{X\}$.*

Due to the knowledge of the authors, there is no published results to show that the exponential map is C^1 for the Teichmüller metric. As a result, we prefer a careful induction of Theorem 3.1 from the useful Reich-Strebel Inequalities. We regard the formula in Theorem 3.1 (and its proof) as a subsequence to earlier works done by Royden, Earle and Gardiner.

3.3. Nonexistence of angle

For any three distinct points X, Y and Z in $\mathcal{T}(S)$, we define the *comparison angle* between Y and Z at X by

$$\angle_X^{comp}(Y,Z) = \arccos\frac{d(X,Y)^2 + d(X,Z)^2 - d(Y,Z)^2}{2d(X,Y)d(X,Z)}.$$

For $X \in \mathcal{T}(S)$ and extremal Beltrami differentials $\mu, \nu \in B_1(X)$, we consider the Teichmüller geodesic rays $X_\mu(s) = f^{s\mu}(X)$ and $X_\nu(t) = f^{s\nu}(X)$. Applying Definition 2.2, we define the *angle between μ and ν at X* as

$$\angle(\mu,\nu)=\lim_{s,t\to 0}\angle_X^{comp}(X_\mu(s),X_\nu(t))$$

if the limit exists.

Remark 3.9. In [14,12,7], the authors studied the angle defined by

$$\lim_{t\to 0}\angle_X^{comp}(X_\mu(t),X_\nu(t)).$$

The existence is confirmed by [12].

Lemma 3.10. *Let $X\in\mathcal{T}(S)$ and let $\mu,\nu\in B_1(X)$ be two extremal Beltrami differentials. Let K be the maximal dilatation of the extremal map in the class of $f:X_{s\mu}\to X_{t\nu}$ and $k=\frac{K-1}{K+1}$. Then $\angle(\mu,\nu)$ exists if and only if*

$$\lim_{s,t\to 0}\frac{s^2+t^2-k^2}{2st}$$

exists. Moreover, when the limit exists,

$$\cos\angle(\mu,\nu)=\lim_{s,t\to 0}\frac{s^2+t^2-k^2}{2st}. \tag{4}$$

Proof. By the definition of comparison angle, we have

$$\cos\angle_X^{comp}(X_{s\mu},X_{t\nu})=\frac{d^2(X,X_{s\mu})+d^2(X,X_{t\nu})-d^2(X_{s\mu},X_{t\nu})}{2d(X,X_{s\mu})d(X,X_{t\nu})}.$$

Since μ is extremal and $||\mu||_\infty=1$, the Teichmüller distance between X and $X_{s\mu}$ is given by

$$\begin{aligned}d(X,X_{s\mu})&=\frac{1}{2}\log K(f_{s\mu})=\frac{1}{2}\log\frac{1+\|s\mu\|_\infty}{1-\|s\mu\|_\infty}=\frac{1}{2}\log\frac{1+s}{1-s}\\&=\sum_{n=0}^{\infty}(2n+1)^{-1}s^{2n+1}.\end{aligned}$$

We can represent $d^2(X,X_{s\mu})$ by a series

$$a_1s^2+a_2s^4+...+a_ns^{2n}+...$$

with $a_1=1$. Similarly, we have

$$d^2(X, X_{t\nu}) = \sum_{n=1}^{\infty} a_n t^{2n}$$

and

$$d^2(X_{s\mu}, X_{t\nu}) = \sum_{n=1}^{\infty} a_n k^{2n}.$$

Note that $k \to 0$ uniformly as s and t tend to 0. We have,

$$\lim_{s,t\to 0} \sum_{m=0}^{n-1} (s^2+t^2)^{n-1-m} k^m = 0$$

uniformly on n. It follows that

$$\begin{aligned}
&\lim_{s,t\to 0} \sum_{n=1}^{\infty} a_n (\sum_{m=0}^{n-1} (s^2+t^2)^{n-1-m} k^m) \\
&= \lim_{s,t\to 0} (a_1 + \sum_{n=2}^{\infty} a_n (\sum_{m=0}^{n-1} (s^2+t^2)^{n-1-m} k^m)) \\
&= a_1 = 1.
\end{aligned}$$

Then

$$\begin{aligned}
&\lim_{s,t\to 0} \frac{d^2(X, X_{s\mu}) + d^2(X, X_{t\nu}) - d^2(X_{s\mu}, X_{t\nu})}{2d(X, X_{s\mu})d(X, X_{t\nu})} \\
&= \lim_{s,t\to 0} \frac{\sum_{n=1}^{\infty} a_n (s^{2n} + t^{2n} - k^{2n})}{2st(1+O(s+t))} \\
&= \lim_{s,t\to 0} \frac{\sum_{n=1}^{\infty} a_n ((s^2+t^2)^n - k^{2n} + O(s^2t^2))}{2st} \\
&= \lim_{s,t\to 0} \frac{\sum_{n=1}^{\infty} a_n (s^2+t^2-k^2)(\sum_{m=0}^{n-1} (s^2+t^2)^{n-1-m} k^m)}{2st} + O(st) \\
&= \lim_{s,t\to 0} \frac{s^2+t^2-k^2}{2st} \cdot \left(\sum_{n=1}^{\infty} a_n (\sum_{m=0}^{n-1} (s^2+t^2)^{n-1-m} k^m) \right) \\
&= \lim_{s,t\to 0} \frac{s^2+t^2-k^2}{2st}.
\end{aligned}$$

The lemma follows. □

Proof of Theorem 1.2. Let $X \in \mathcal{T}(S)$ and $\mu, \nu \in B_1(X)$ be extremal Beltrami differentials. For any $\tilde{s}, \tilde{t} > 0$, $\epsilon > 0$, let $s = \epsilon\tilde{s}$, $t = \epsilon\tilde{t}$. Lemma 3.6 implies that

$$\begin{aligned} k &= \|t\mu - s\nu\|_T + O((t+s)^2) \\ &= \epsilon\|\tilde{t}\mu - \tilde{s}\nu\|_T + O(\epsilon^2). \end{aligned}$$

Now suppose that the angle $\angle(\mu, \nu)$ exists for any μ and ν, then

$$\begin{aligned} \cos\angle(\mu,\nu) &= \lim_{t,s\to 0} \cos\angle_X^{comp}(X_{t\mu}, X_{s\nu}) \\ &= \lim_{t,s\to 0} \frac{t^2+s^2-k^2}{2ts} \\ &= \lim_{\epsilon\to 0} \frac{\epsilon^2\tilde{t}^2+\epsilon^2\tilde{s}^2-(\epsilon\|\tilde{t}\mu-\tilde{s}\nu\|_T+O(\epsilon^2))^2}{2\epsilon^2\tilde{t}\tilde{s}} \\ &= \lim_{\epsilon\to 0} \frac{\|\tilde{t}\mu\|_T^2+\|\tilde{s}\nu\|_T^2-\|\tilde{t}\mu-\tilde{s}\nu\|_T^2}{2\|\tilde{t}\mu\|_T\|\tilde{s}\nu\|_T^2} + O(\epsilon)\times\frac{\|\tilde{t}\mu-\tilde{s}\nu\|_T}{2\tilde{t}\tilde{s}} \\ &= \frac{\|\tilde{t}\mu\|_T^2+\|\tilde{s}\nu\|_T^2-\|\tilde{t}\mu-\tilde{s}\nu\|_T^2}{2\|\tilde{t}\mu\|_T\|\tilde{s}\nu\|_T}. \end{aligned}$$

By identifying the tangent space $T_X\mathcal{T}(S)$ with $B(X)/N(X)$, it follows that the normed vector space $(T_X\mathcal{T}(S), \|\cdot\|_T)$ satisfies the law of cosine, that is,

$$\angle(u,v) = \arccos\frac{\tilde{t}^2\|u\|_T^2+\tilde{s}^2\|v\|_T^2-\|\tilde{t}u-\tilde{s}v\|_T^2}{2\tilde{t}\tilde{s}\|u\|_T\|v\|_T}$$

for any $u, v \in T_X\mathcal{T}(S)$ and $\tilde{t}, \tilde{s} > 0$.

Note that:

(i) If a normed vector space $(V, \|\cdot\|)$ satisfies the law of cosine, then the norm satisfies the equality

$$\|w+v\|^2 + \|w-v\|^2 = 2(\|v\|^2 + \|w\|^2) \tag{5}$$

for any $v, w \in (V, \|\cdot\|)$. See [4], I.4.5.

(ii) The equality (5) holds for the normed vector space $(V, \|\cdot\|)$ if and only if the norm $\|\cdot\|$ arises from a scalar product (see [4], I.4.4 for the proof).

As a result, the existence of angle $\angle(\mu, \nu)$ for any μ, ν contradicts with the fact that the Teichmüller metric is not a Riemannian metric. We are done. □

Theorem 1.1 is a direct corollary of Proposition 2.3 and Theorem 1.2. In fact, we have shown that

Corollary 3.11. *The Teichmüller space $\mathcal{T}(S)$ with the Teichmüller metric is not a locally* CAT(k) *space, i.e., a metric space of curvature $\leq k$, for any $k \in \mathbb{R}$.*

4. Characterization on the existence of angle

4.1. A necessary condition for the existence of angle

Let $0 < \lambda < \infty$ be a constant.

Lemma 4.1. *By setting $s = \lambda t$, the limit*

$$\angle^{\lambda}(\mu, \nu) = \lim_{t \to 0} \angle_X^{comp}(f^{\lambda t \mu}(X), f^{t\nu}(X))$$

exists and satisfies

$$\cos \angle^{\lambda}(\mu, \nu) = \frac{1}{2\lambda}\left(\lambda^2 + 1 - \sup_{\varphi \in Q^1(X)} \langle \lambda\mu - \nu, \varphi \rangle^2\right). \tag{6}$$

Proof. Let $k(t) = \frac{K(t)-1}{K(t)+1}$, where $K(t)$ be the maximal dilatation of the extremal quasiconformal map in the class of $X_{\lambda t \mu} \to X_{t\nu}$. Then by Lemma 3.10 we have

$$\begin{aligned}
&\lim_{t \to 0} \cos \angle_X^{comp}(f^{\lambda t \mu}(X), f^{t\nu}(X)) \\
&= \lim_{t \to 0} \frac{1}{2\lambda t^2}\left(\lambda^2 t^2 + t^2 - k(t)^2\right) \\
&= \lim_{t \to 0} \frac{1}{2\lambda}\left(\lambda^2 + 1 - (\sup_{\varphi \in Q^1(X)} \langle \lambda\mu - \nu, \varphi \rangle)^2 + O(t)^2\right) \\
&= \frac{1}{2\lambda}\left(\lambda^2 + 1 - (\sup_{\varphi \in Q^1(X)} \langle \lambda\mu - \nu, \varphi \rangle)^2\right). \quad \square
\end{aligned}$$

Remark 4.2. In a metric space, we say that two non-trivial geodesic rays issuing from a point are *equivalent* if their Alexandrov angle is zero. The Alexandrov angle $\angle^{Alex}$ induces a metric on the set of equivalence classes. The resulted metric space is called the *space of directions* at the point. The Euclidean cone over the space of direction is called the *tangent cone.*

For any $X \in \mathcal{T}(S)$ and two Teichmüller geodesic rays γ and γ' issuing from X, the corresponding Alexandrov angle

$$\angle^{Alex}(\gamma, \gamma') \geq \angle^{1}(\gamma, \gamma') > 0.$$

As a result, the tangent cone at X can be identified (in setwise) with $B(X)/N(X)$.

Let ϕ_λ be the quadratic differential in $Q^1(X)$ realizing the supremum

$$\sup_{\varphi \in Q^1(X)} \langle \lambda\mu - \nu, \varphi \rangle.$$

We can represent (6) as

$$\begin{aligned}
&\frac{1}{2\lambda}(\lambda^2 + 1 - \langle \lambda\mu - \nu, \phi_\lambda \rangle^2) \\
&= \langle \mu, \phi_\lambda \rangle \langle \nu, \phi_\lambda \rangle + \frac{1 - \langle \mu, \phi_\lambda \rangle^2}{2}\lambda + \frac{1 - \langle \nu, \phi_\lambda \rangle}{2} \frac{1 + \langle \nu, \phi_\lambda \rangle}{\lambda} \\
&= I + II + III.
\end{aligned}$$

By continuity, $\lim_{\lambda \to 0} \phi_\lambda = -\varphi_\nu$. This implies that

$$\lim_{\lambda \to 0} I = \langle \mu, \varphi_\nu \rangle.$$

It is obvious that

$$\lim_{\lambda \to 0} II = 0.$$

It follows from the definition of ϕ_λ that

$$\langle \lambda\mu - \nu, -\varphi_\nu \rangle \leq \langle \lambda\mu - \nu, \phi_\lambda \rangle.$$

Thus

$$1 + \langle \nu, \phi_\lambda \rangle \leq 1 + \langle \nu, -\varphi_\nu \rangle + \langle \lambda\mu, \phi_\lambda + \varphi_\nu \rangle = \lambda \langle \mu, \phi_\lambda + \varphi_\nu \rangle.$$

We have

$$III \leq \frac{1}{2} \langle \mu, \phi_\lambda + \varphi_\nu \rangle (1 - \langle \nu, \phi_\lambda \rangle).$$

Again, since $\lim_{\lambda \to 0} \phi_\lambda = -\varphi_\nu$, we have

$$\lim_{\lambda \to 0} III = 0.$$

By making $\lambda \to 0$, it follows from the above discussions that

$$\lim_{\lambda \to 0} \angle^\lambda(\mu, \nu) = \cos^{-1} \langle \mu, \varphi_\nu \rangle.$$

By interchanging μ and ν, or by letting $\lambda \to \infty$, we have shown that:

Proposition 4.3. *For $X \in \mathcal{T}(S)$, let μ and ν be two extremal Beltrami differentials with unit norm on X. Suppose that $\mu = \frac{|\varphi_\mu|}{\varphi_\mu}$ and $\nu = \frac{|\varphi_\nu|}{\varphi_\nu}$. If the angle $\angle(\mu, \nu)$ exists, then*

$$Re \iint\limits_X \mu\varphi_\nu = Re \iint\limits_X \nu\varphi_\mu. \tag{7}$$

Example 4.4. Let $B_1(X) = \{\mu \in B(X) \mid \|\mu\|_\infty = 1\}$. Given an extremal Beltrami differential $\mu \in B_1(X)$, the subset of $\mathcal{T}(S)$ defined by

$$\mathcal{D}(\mu) = \{f^{t\mu}(X) \mid t \in \mathbb{C}, |t| < 1\}$$

is called a *Teichmüller disk*. Endowed with the Teichnmüller distance, $\mathcal{D}(\mu)$ is isometric to the Poincaré disk.

If $\nu \in B_1(X)$ is extremal and $\mathcal{D}(\nu) = \mathcal{D}(\mu)$, then there exists some θ such that $\nu = e^{i\theta}\mu$. It is obvious that $\angle(\mu, \nu) = \theta$. By our definition,

$$\begin{aligned}\cos\angle^\lambda(\mu, \nu) &= \frac{1}{2\lambda}\left(\lambda^2 + 1 - \sup_{\varphi \in Q^1(X)} \langle(\lambda - e^{i\theta})\mu, \varphi\rangle^2\right) \\ &= \frac{1}{2\lambda}\left(\lambda^2 + 1 - |\lambda - \cos\theta - i\sin\theta|^2\right) \\ &= \cos\theta.\end{aligned}$$

Conjecture 4.5. *The angle between two extremal Beltrami differentials $\mu, \nu \in B_1(X)$ exists if and only if the Teichmüller disks $\mathcal{D}(\mu)$ and $\mathcal{D}(\nu)$ coincide.*

A confirmation of the above conjecture would imply a recent result of Antonakoudis [1] that every totally-geodesic isometry from the Poincaré disk to $\mathcal{T}(S)$, endowed with the Teichmüller metric, is a Teichmüller disk.

4.2. An equation related to the angle

We set

$$g(\lambda) = \|\lambda\mu - \nu\|_T^2.$$

Note that $g(\lambda)$ is C^1 [22]. By Lemma 4.1,

$$\cos\angle^\lambda(\mu, \nu) = \frac{1}{2\lambda}(\lambda^2 + 1 - g(\lambda)).$$

If $\angle(\mu, \nu)$ exists, we denote $\theta = \angle(\mu, \nu)$. In this case, $g(\lambda)$ satisfies the following equation:

$$\lambda^2 + 1 - g(\lambda) = 2\lambda\cos\theta. \tag{8}$$

Example 4.4 shows that

$$h(\lambda) = \|\lambda\mu - e^{i\theta}\mu\|_T^2$$

is a solution of the above equation.

Consider the derivative of (8), we have

$$2\lambda - g'(\lambda) = 2\cos\theta.$$

Thus $g'(\lambda) = h'(\lambda)$ and $g(\lambda) = h(\lambda) + C$. By (8) again, $C = 0$. As a result, $h(\lambda)$ is the unique C^1-solution, that is,

$$\|\lambda\mu - \nu\|_T^2 = \|\lambda\mu - e^{i\theta}\mu\|_T^2. \tag{9}$$

As a result, the Teichmüller norm $\|\lambda\mu - \nu\|_T$ (with μ, ν fixed and λ varied) is induced by a Riemannian inner product. This gives some evidence for Conjecture 4.5. In fact, Antonakoudis [2] announced that there is no complex linear isometric embedding from $(\mathbb{C}^2, \|\cdot\|_2)$ to $(Q(X), \|\cdot\|_1)$. We wish to have a complexification of the equation (9) and show that the existence of angle may lead to a complex linear isometry between $(\mathbb{C}^2, \|\cdot\|_2)$ and subspace of $(Q(X), \|\cdot\|_1)$ with complex dimension 2, when μ and ν span a complex space of dimension 2.

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