



Distinguished regular supercuspidal representations

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Abstract

Based on recent work of Kaletha, we apply Hakim–Murnaghan’s result to study distinguished regular supercuspidal representations of tamely ramified reductive p -adic groups. Assuming p is sufficiently large, we obtain a necessary and sufficient condition for regular supercuspidal representations to be distinguished. We also investigate the relation between the distinction problem and the Langlands functoriality, and confirm a conjecture of Lapid for regular depth-zero or epipelagic supercuspidal representations.

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1 Introduction

Overview

Supercuspidal representations are central in the representation theory of reductive groups over non-archimedean local fields, and the latter theory is crucial to the theory of automorphic representations. On the other hand, the properties of distinguished representations have become a main theme in the study of automorphic representations, especially after Jacquet and his collaborators' work on various automorphic periods. The basic question is to determine when the representations are distinguished, which we call the distinction problem for short. For tame supercuspidal representations, Hakim and Murnaghan [14] developed a general theory for this problem. Regular supercuspidal representations were recently introduced by Kaletha [22]. Our purpose is to apply Hakim–Murnaghan's work to determine distinguished regular supercuspidal representations, and study the relation between the distinction problem and the Langlands functoriality. Below we will give a brief summary of prior works related to this paper.

Our first goal is to give a natural criterion to detect the distinction of the regular supercuspidal representations, and our method relies heavily on the ways to construct supercuspidal representations. For a tamely ramified reductive p -adic group G , Yu [35], inspired by Adler's prior work [2], obtained a remarkable way to construct supercuspidal representations using generic cuspidal data. These representations are called tame supercuspidal representations. For $G = \mathrm{GL}_n$, Howe's classical result [15] shows that these representations can be parameterized by much simpler data. For general reductive groups, people are trying to find out a more explicit parametrization, e.g. see [28]. In his recent work [22], Kaletha considered a large subclass of tame supercuspidal representations which he called regular supercuspidal representations. He showed that these representations can be parameterized by simpler data (S, μ) , called tame regular elliptic pairs, than generic cuspidal data. This can be viewed as a generalization of Howe's parametrization mentioned above.

For tame supercuspidal representations, Hakim and Murnaghan [14] gave a criterion to detect the distinction with respect to symmetric spaces, and even obtained a multiplicity formula in terms of the generic cuspidal data. Using this theory together with Howe's construction, Hakim and his collaborators successfully obtained much simpler criteria in terms of Howe's data for several typical involutions of GL_n , e.g. see [8,11] and earlier work [12,13]. To apply Hakim–Murnaghan's result to concrete examples, it needs a further delicate analysis.

On the other hand, distinguished representations have a conjectural deep relation with the Langlands functoriality, which is the so-called relative Langlands program nowadays. We refer to Sakellaridis and Venkatesh's work [34] for a systematic exposition. To understand this relation, we have to assume the local Langlands correspondence. For regular supercuspidal representations, Kaletha [22] showed how to organize them into the L -packets attached to the regular supercuspidal L -parameters in the framework of rigid inner twists, which generalizes previous works

[4,5,18,19,31,32]. Our second goal is to apply our criterion for distinguished regular supercuspidal representations to the relative Langlands program.

Main results

Now we give a more detailed introduction to our results. Let F be a non-archimedean local field of residual characteristic p . We suppose that p is sufficiently large, and refer to Sect. 1 for the precise assumptions on p .

Let G be a tamely ramified connected reductive group over F , θ an involution of G defined over F , and $H = G^\theta$ the closed subgroup of fixed points of θ . For an irreducible admissible representation π of $G(F)$, we say that it is H -distinguished if the space $\text{Hom}_{H(F)}(\pi, \mathbf{1})$ is non-zero where $\mathbf{1}$ stands for the trivial representation of $H(F)$.

The first part of this article is concerned with the properties of distinguished regular supercuspidal representations in terms of the inducing data tame regular elliptic pairs. We refer to [22, §3] or Sect. 2.2.2 for the basic definitions and facts on regular supercuspidal representations.

To state our results clearly, let us first consider the depth-zero case, which is one of the cornerstones of the whole theory. In this case, all regular depth-zero supercuspidal representations of $G(F)$ are constructed from the data (S, μ) , where S is a maximally unramified elliptic maximal torus of G and μ a regular depth-zero character of $S(F)$. The construction is based on the Deligne–Lusztig representation $\kappa_{(S, \mu)}$ of the parahoric subgroup $G(F)_{x,0}$ of $G(F)$ determined by S . After extending $\kappa_{(S, \mu)}$ to a representation $\tilde{\kappa}_{(S, \mu)}$ of $S(F)G(F)_{x,0}$, we obtain the regular depth-zero supercuspidal representation $\pi_{(S, \mu)} = \text{ind}_{S(F)G(F)_{x,0}}^{G(F)} \tilde{\kappa}_{(S, \mu)}$. The isomorphism class of $\pi_{(S, \mu)}$ depends only on the $G(F)$ -conjugate class of (S, μ) .

Theorem 1.1 *The regular depth-zero supercuspidal representation $\pi_{(\dot{S}, \dot{\mu})}$ is H -distinguished if and only if $(\dot{S}, \dot{\mu})$ is $G(F)$ -conjugate to a pair (S, μ) such that S is θ -stable and*

$$\mu|_{S^\theta(F)} = \varepsilon_S.$$

Here, for a θ -stable maximally unramified elliptic maximal torus S , the character ε_S is a quadratic character of $S^\theta(F)$ (see Definition 3.4), whose appearance arises from Lusztig's solution [25] of the distinction problem for symmetric spaces over finite fields. Moreover the character ε_S satisfies the property that $\varepsilon_S|_{S^{\theta, \circ}(F)} = 1$ where $S^{\theta, \circ}$ is the identity component of S^θ . Due to this property, Theorem 1.1 implies the following relation between the contragredient representation π^\vee and the θ -twisted representation $\pi \circ \theta$ of π .

Corollary 1.2 (Corollary 3.17) *Suppose that π is an H -distinguished regular depth-zero supercuspidal representation of $G(F)$. Then we have $\pi^\vee \simeq \pi \circ \theta$.*

For general regular supercuspidal representations of $G(F)$ of positive depth, they are constructed from the data tame regular elliptic pairs (S, μ) , where S is a tame elliptic maximal torus of G and μ a character of $S(F)$ satisfying certain conditions (cf. [22, Definition 3.7.5] for the precise definition). If $G = \text{GL}_n$, these data coincide

with Howe's notion of admissible characters in [15]. There is a process, called *Howe factorization* of (S, μ) , which is a generalization of the Howe factorization lemma for GL_n ([15, Lemma 11 and Corollary]), to produce a cuspidal generic G -datum

$$\Psi = (\vec{G} = (G^0, \dots, G^d), \pi_{(S, \mu_0)}, \vec{\phi} = (\phi_0, \dots, \phi_d)), \quad (1)$$

such that $\pi_{(S, \mu_0)}$ is a regular depth-zero supercuspidal representation of $G^0(F)$. We refer to [22, §3.6 and §3.7] or Sect. 2.2.4 for more details. By Yu's construction, this G -datum Ψ gives rise to a regular supercuspidal representation $\pi_{(S, \mu)}$ whose isomorphism class depends only on the $G(F)$ -conjugate class of (S, μ) . The following is our main theorem on the distinction problem, which is a generalization of Theorem 1.1.

Theorem 1.3 (Theorem 3.16) *The regular supercuspidal representation $\pi_{(\dot{S}, \dot{\mu})}$ is H -distinguished if and only if $(\dot{S}, \dot{\mu})$ is $G(F)$ -conjugate to a tame regular elliptic pair (S, μ) such that S is θ -stable and*

$$\mu|_{S^\theta(F)} = \varepsilon_S \cdot \eta_S.$$

Here ε_S is the quadratic character of $S^\theta(F)$ introduced before, but with respect to (G^0, S) , and η_S is also a quadratic character of $S^\theta(F)$, whose appearance arises from Hakim–Murnaghan's work when the representation is of positive depth. The reason that we do not have a consequence of Theorem 1.3 as Corollary 1.2 is due to the character η_S . At this moment, we could not show $\eta_S|_{S(F)^{1+\theta}} = 1$, while this relation holds for all the examples in the literature as far as we know. In particular, η_S is the trivial character for epipelagic supercuspidal representations. Therefore the analog of Corollary 1.2 holds for epipelagic supercuspidal representations.

Let us outline the proof of Theorem 1.3. Roughly speaking, modulo $G(F)$ -conjugation, we can apply Hakim–Murnaghan's work to reduce the distinction problem of $\pi_{(S, \mu)}$ to that of $\tilde{\kappa}_{(S, \mu_0)}$ which is a representation of $S(F)G^0(F)_{x,0}$. To ensure that the distinction problem makes sense for $\tilde{\kappa}_{(S, \mu_0)}$, we have to show that $S(F)G^0(F)_{x,0}$ is θ -stable. This point is guaranteed by the distinction problem over finite fields (Lemma 3.21) and liftings of θ -stable tori over finite fields to those over p -adic fields (Lemma 3.2). The solution of the distinction problem of $\tilde{\kappa}_{(S, \mu_0)}$ relies on Proposition 3.23. The proof of Proposition 3.23 is a little complicated, due to the subtleness of the construction of $\tilde{\kappa}_{(S, \mu_0)}$.

Remark 1.4 After a preliminary version of this article was completed, we notice that Hakim's most recent work [9] and [10] provide a new approach to the construction of tame supercuspidal representations and its application to the distinction problem. One of the main features of [9] is that it eliminates Howe's factorizations in Yu's construction. This new construction improves the main results of [14], see [10, Theorem 2.0.1]. It would be interesting to see whether our results can be simplified or generalized when combined with these developments.

The second part of this article is concerned with the properties of distinguished regular supercuspidal representations in terms of the Langlands parameters. The philosophy of the relative Langlands program [34] is that if π is an H -distinguished representation then the L -parameter φ of π should have more symmetries. Roughly

speaking, φ should factor through the L -group of the symmetric space. In other words, π should be a Langlands functorial lift from a representation of some group other than $G(F)$. In the context of symmetric spaces, Lapid proposed a conjecture which is easier to state. We learnt of this conjecture from [7, Conjecture 1.1]. We believe that Lapid’s conjecture is closely related to the relative Langlands program, since it reflects certain information on the symmetry of the parameter φ . However it is beyond our scope to discuss this relationship. For Galois symmetric spaces, Prasad [30] formulated a more precise conjecture in terms of the refined L -parameters, see *loc. cit.* for more details.

Conjecture 1.5 (Lapid) Let G be a connected reductive group over a p -adic field F , θ an involution of G defined over F , and $H = G^\theta$. Let π be an admissible irreducible representation of $G(F)$ and $\Pi_\varphi(G)$ the conjectural L -packet containing π . Suppose that π is H -distinguished. Then we have

$$\{\tau \circ \theta \mid \tau \in \Pi_\varphi(G)\} = \{\tau^\vee \mid \tau \in \Pi_\varphi(G)\}.$$

In other words, $\Pi_\varphi(G)$ is invariant under $\tau \mapsto \tau^\vee \circ \theta$.

Now we return to the context of regular supercuspidal representations and retain the assumptions as before. We further assume that F has characteristic zero and G is quasi-split. Kaletha [22, §5] defined the notion *regular supercuspidal L -parameters* φ for G . For each rigid inner twist (G', ξ, z) of G , he also constructed L -packets $\Pi_\varphi(G')$ that consists of certain regular supercuspidal representations of $G'(F)$. In this paper, we consider not merely the distinction problem for G , but also for all other rigid inner twists (G', ξ, z) of G such that the fixed involution θ of G can be “transferred” to an involution θ' of G' . For this purpose, we introduce the notion *rigid inner twists* of (G, H, θ) in Sect. 4.2.2, which are denoted by (G', H', θ') where $H' = (G')^{\theta'}$. For each rigid inner twist (G', H', θ') , we can think about H' -distinction for the representations in $\Pi_\varphi(G')$. Motivated by Conjecture 1.5, we would like to know what the sets

$$\Pi_\varphi^\theta(G') := \{\pi \circ \theta' \mid \pi \in \Pi_\varphi(G')\} \quad \text{and} \quad \Pi_\varphi^\vee(G') := \{\pi^\vee \mid \pi \in \Pi_\varphi(G')\}$$

are, especially in terms of the L -parameters. The answer is not surprising and has been long expected. Let ${}^L C$ be the Chevalley involution of the L -group ${}^L G$ and ${}^L \theta$ the involution of ${}^L G$ dual to θ . Then ${}^L \theta \circ \varphi$ and ${}^L C \circ \varphi$ are also regular supercuspidal parameters. In Propositions 4.14 and 4.17, we show that

$$\Pi_\varphi^\theta(G') = \Pi_{L\theta \circ \varphi}(G') \quad \text{and} \quad \Pi_\varphi^\vee(G') = \Pi_{{}^L C \circ \varphi}(G').$$

Therefore, if π^\vee is isomorphic to $\pi \circ \theta'$ for some representation π in $\Pi_\varphi(G')$, the parameters ${}^L C \circ \varphi$ and ${}^L \theta \circ \varphi$ are \widehat{G} -conjugate, which implies $\Pi_\varphi^\theta(G') = \Pi_\varphi^\vee(G')$. In particular, if π is an H' -distinguished regular depth-zero or epipelagic supercuspidal representation then Conjecture 1.5 holds (see Corollary 4.18).

Remark 1.6 In a sequel [37] to this paper, we show that η_S is trivial if θ is an unramified Galois involution (cf. *loc. cit.* Proposition 4.1). Therefore Conjecture 1.5 holds for

distinguished regular supercuspidal representations with respect to unramified Galois involutions (cf. *loc. cit.* Corollary 4.2).

Organization of this article

The assumptions on the residual characteristic p , and necessary notation and convention are given in the rest of this section. We recollect some background materials in Sect. 2, including Yu's construction of tame supercuspidal representations, Hakim–Murnaghan's result on distinguished tame supercuspidal representations, and Kaletha's work on regular supercuspidal representations. Some details of these contents that we need will appear in latter sections or be referred to the references. Our main results on the distinction problem are stated in Sect. 3.1.4. Before that, we introduce the two characters ε_S and η_S , and analyze their properties in Sects. 3.1.1 and 3.1.2. The notion $(\theta, \varepsilon\eta)$ -symmetric pairs is introduced in Sect. 3.1.3 where its basic properties are also discussed. The proofs of Theorems 1.1 and 1.3 are given in Sect. 3.2. Kaletha's construction of regular supercuspidal L -packets Π_φ is reviewed in Sect. 4.1. Then we study the twisted L -packets Π_φ^θ and the contragredient L -packets Π_φ^\vee in Sects. 4.2 and 4.3 respectively.

Assumptions

Throughout this article, F is a non-archimedean local field of residual characteristic p . Moreover, we will require that F has characteristic zero in Sect. 4. The reason that we restrict to characteristic zero is that this assumption is needed in Kaletha's construction of L -packets. To apply the theories mentioned in the introduction, we have to make certain restrictions on p in different stages. We require that p satisfies all of the following conditions:

- (1) p is odd,
- (2) $p \nmid |\pi_1(G_{\text{der}})|$,
- (3) p is not a *bad prime* for G ,
- (4) $p \nmid |\pi_0(G)|$.

The first assumption is needed in Hakim–Murnaghan's work. The second one is used for the definition of regular supercuspidal representations and also to ensure the existence of Howe factorizations of tame regular elliptic pairs, see Remark 2.3. The third assumption is required for the proof of Lemma 3.9 and for the construction of regular supercuspidal L -packets. The last one is also for the construction of regular supercuspidal L -packets. We refer to [22, §2.1] for more detailed explanations and discussions on the roles that these assumptions play in his theory of regular supercuspidal representations, and also for a brief summarization of bad primes determined by the type of G .

Notation and convention

Let F be a non-archimedean local field as before, O_F the ring of integers of F , and k_F the residue field of F . We fix a separable closure F^s of F and denote by Γ the

Galois group $\text{Gal}(F^s/F)$. We write W_F for the Weil group of F , I_F for the inertia subgroup of W_F , and P_F for the tame inertia subgroup of I_F . Let F^u be the maximal unramified extension of F in F^s with residue field \bar{k}_F .

For a connected reductive group G defined over F , we denote by $Z(G)$ its center, by $A(G)$ its split central torus, by G_{der} its derived subgroup, by G_{ad} the adjoint quotient of G_{der} , and by \mathfrak{g} the Lie algebra of G . For an element $g \in G$ we will write $\text{Ad}(g)$ for the conjugation action of g on G , i.e., $\text{Ad}(g)(x) = gxg^{-1}$ for $x \in G$, and also for the adjoint action of g on \mathfrak{g} . When we mention a subgroup of G , we always assume that it is a closed algebraic subgroup defined over F . For a subgroup M of G , we use M° to denote its identity connected component. For any subset U of G , we use $C_G(U)$ to denote the identity component of its centralizer in G .

For an involution θ of G , we always mean that it is a non-trivial automorphism of order two and defined over F . We denote by G^θ the θ -fixed subgroup of G and by $G^{\theta,\circ}$ its identity component. Then both G^θ and $G^{\theta,\circ}$ are reductive subgroups of G . The group $G(F)$ has a natural action on the set of involutions, which is given by

$$g \cdot \theta := \text{Ad}(g) \circ \theta \circ \text{Ad}(g^{-1}).$$

Let M be a subgroup of G and ϕ a character of $M(F)$. For $g \in G(F)$ we denote

$${}^g M := g^{-1} M g \quad \text{and} \quad {}^g \phi := \phi \circ \text{Ad}(g),$$

where ${}^g \phi$ is a character of ${}^g M(F)$. We will use the following fact frequently. If M is $g \cdot \theta$ -stable then ${}^g M$ is θ -stable and $({}^g M)^\theta = {}^g (M^{g \cdot \theta})$. For a θ -stable subgroup U of $G(F)$, let

$$U^{1+\theta} := \{u\theta(u) \mid u \in U\} \subseteq U$$

denote the subgroup of norms with respect to θ . If (π, V_π) is a representation of $G(F)$ where V_π is the underlying space of π , we use $\pi \circ \theta$ to denote the representation of $G(F)$ with underlying space V_π and action given by $(\pi \circ \theta)(g)v = \pi(\theta(g))v$ for $v \in V_\pi$.

We will use similar notation as above when we discuss objects over finite fields.

For a maximal torus S of G , we denote by $N(S, G)$ the normalizer of S in G , by $\Omega(S, G) = N(S, G)/S$ the absolute Weyl group, and by $R(S, G)$ the corresponding set of roots. The Galois group Γ has a natural action on $R(S, G)$. For any $\alpha \in R(S, G)$, we denote by Γ_α (resp. $\Gamma_{\pm\alpha}$) the stabilizer of α (resp. $\{\alpha, -\alpha\}$) in Γ , and by F_α (resp. $F_{\pm\alpha}$) the corresponding fixed subfield of F^s . We call α symmetric if the degree of the extension $F_\alpha/F_{\pm\alpha}$ is 2, and call asymmetric otherwise. We call α ramified or unramified if the extension $F_\alpha/F_{\pm\alpha}$ is ramified or unramified respectively.

We denote by $\mathcal{B}^{\text{red}}(G, F)$ the reduced Bruhat–Tits building of $G(F)$, and by $\mathcal{A}^{\text{red}}(S, F)$ the reduced apartment of S in $\mathcal{B}^{\text{red}}(G, F)$ where S is a maximal torus of G that is maximally split. For $x \in \mathcal{B}^{\text{red}}(G, F)$, we write $G(F)_x$ for the stabilizer of x in $G(F)$, $G(F)_{x,0}$ for the parahoric subgroup of $G(F)$ attached to x , $G(F)_{x,0+}$ for its pro-unipotent radical, and G_x for the corresponding connected reductive group over k_F . More generally, we denote by $G(F)_{x,r}$ the Moy–Prasad filtration subgroups

for any $r \in \mathbb{R}_{\geq 0}$ and by $\mathfrak{g}(F)_{x,r}$ the filtration lattices of $\mathfrak{g}(F)$ for any $r \in \mathbb{R}$ (see [26]). Moreover, we write $G(F)_{x,r+} = \bigcup_{s>r} G(F)_{x,s}$, $\tilde{G}(F)_{x,r;s} = G(F)_{x,r}/G(F)_{x,s}$ and $\mathfrak{g}(F)_{x,r;s} = \mathfrak{g}(F)_{x,r}/\mathfrak{g}(F)_{x,s}$ for $s > r$. We use $\tilde{\mathbb{R}}$ to denote the set $\mathbb{R} \cup \{r + |r \in \mathbb{R}\} \cup \{\infty\}$.

Given a torus T defined over F , let \mathfrak{T} be the connected Neron model of T over O_F . We denote by $T(F)_0$ the subgroup $\mathfrak{T}(O_F)$ of $T(F)$. We write T^u for the maximal unramified subtorus of T . We can also define the Moy–Prasad filtration subgroups $T(F)_r$ for any $r \geq 0$. In particular, when $T = \text{Res}_{E/F} \mathbb{G}_m$, we have $E_0^\times = O_E^\times$ and $E_r^\times = 1 + \mathfrak{p}_E^{[er]}$ for $r > 0$, where \mathfrak{p}_E is the maximal ideal of O_E and e is the ramification index of the finite extension E/F .

2 Preliminaries

2.1 Yu’s construction

In this subsection we briefly review Yu’s construction of tame supercuspidal representations [35]. Let G be a tamely ramified connected reductive group over F .

2.1.1 Cuspidal G -data

Recall that a *cuspidal G -datum* is a 4-tuple $\Psi = (\vec{G}, x, \rho, \vec{\phi})$ that satisfies the following conditions:

- (1) \vec{G} is a tamely ramified twisted Levi sequence $\vec{G} = (G^0, \dots, G^d)$ in G such that $Z(G^0)/Z(G)$ is anisotropic.
- (2) x is a point in $\mathcal{A}^{\text{red}}(S, F)$, where S is a tame maximal torus of G^0 .
- (3) ρ is an irreducible representation of $K^0 := G^0(F)_x$ such that $\rho|_{G^0(F)_{x,0+}}$ is **1**-isotypic and the compactly induced representation $\pi_{-1} = \text{ind}_{K^0}^{G^0(F)}(\rho)$ is irreducible.
- (4) $\vec{\phi} = (\phi_0, \dots, \phi_d)$ is a sequence of quasicharacters, where ϕ_i is a quasicharacter of $G^i(F)$. We require that: if $d = 0$ then ϕ_0 is of depth $r_0 \geq 0$; if $d > 0$ and ϕ_d is non-trivial then ϕ_i is of depth r_i for $i = 0, \dots, d$ and $0 < r_0 < r_1 < \dots < r_{d-1} < r_d$; if $d > 0$ and ϕ_d is trivial then ϕ_i is of depth r_i for $i = 0, \dots, d - 1$ and $0 < r_0 < r_1 < \dots < r_{d-1}$. We will call $\vec{r} = (r_0, \dots, r_d)$ the depth of $\vec{\phi}$ for short, and denote $\phi := \prod_{i=0}^d \phi_i|_{G^0(F)}$.

Note that the condition on ρ implies that π_{-1} is supercuspidal and of depth zero. Conversely every irreducible depth-zero supercuspidal representation of $G^0(F)$ arises in this way. We call a triple $\Psi = (\vec{G}, \pi_{-1}, \vec{\phi})$ a *reduced cuspidal G -datum* if \vec{G} and $\vec{\phi}$ satisfy the condition 1 and 4 above respectively and π_{-1} is an irreducible depth-zero supercuspidal representation of $G^0(F)$. There is no essential difference between cuspidal G -datum and reduced cuspidal G -datum.

We say that a (reduced) cuspidal G -datum is *generic* if ϕ_i is G^{i+1} -generic for $i \neq d$. We refer to [14, Definition 3.9] for the notion of genericity. Throughout this article, we will only deal with generic (reduced) cuspidal G -data, and will call them *G -data* for short.

2.1.2 The representation $\pi(\Psi)$

Let Ψ be a cuspidal G -datum. Let $K^0 = G^0(F)_x$ and $K_+^0 = G^0(F)_{x,0+}$. For $0 \leq i \leq d - 1$, set $s_i = \frac{r_i}{2}$, and put

$$K^{i+1} = K^0 G^1(F)_{x,s_0} \cdots G^{i+1}(F)_{x,s_i}, \quad K_+^{i+1} = K_+^0 G^1(F)_{x,s_0+} \cdots G^{i+1}(F)_{x,s_i+}.$$

When Ψ is generic, Yu obtained an irreducible supercuspidal representation $\pi(\Psi)$ of $G(F)$, called *tame supercuspidal representation*, by a very technical process. The basic idea is first to construct a representation κ of K^d from ρ and generic quasicharacters ϕ_i , and then set $\pi(\Psi) = \text{ind}_{K^d}^{G(F)} \kappa$. We refer to [35] for more details. In summary we have a map from the set of G -data to that of tame supercuspidal representations.

2.1.3 G -equivalence

To study the dependence of $\pi(\Psi)$ on Ψ , Hakim and Murnaghan introduced three operations, which are called *refactorization*, *elementary transformation* and *G -conjugation*, on generic (reduced) cuspidal G -data. We refer to [14, Definition 4.19], [14, Definitions 5.2 and 6.2] and [14, page 110] for the definition of these three operations respectively. Note that these operations do not change the genericity. Two G -data are called *G -equivalent* if they can be obtained from each other by a finite sequence of these three operations. One of the main result of [14] is the following.

Theorem 2.1 *Let Ψ and $\dot{\Psi}$ be two generic (reduced) cuspidal G -data. Then $\pi(\Psi)$ and $\pi(\dot{\Psi})$ are isomorphic if and only if Ψ and $\dot{\Psi}$ are G -equivalent.*

Remark 2.2 Theorem 2.1 was proved in *loc. cit.* Theorem 6.6 under a hypothesis called $C(\tilde{G})$. We refer to *loc. cit.* page 47 for the precise statement of $C(\tilde{G})$. Recently this hypothesis was removed by Kaletha [22, Corollary 3.5.5].

2.2 Kaletha’s work

Kaletha’s recent work [22] provides a more elegant parametrization for most of the tame supercuspidal representations. These representations are called regular supercuspidal representations, which are the main objects of our paper. As before, G is a tamely ramified connected reductive group over F .

2.2.1 Maximally unramified elliptic tori and regular depth-zero characters

A maximal torus S of G is called *maximally unramified* if $S \times F^u$ is a minimal Levi subgroup of the quasi-split group $G \times F^u$. See [22, Fact 3.4.1] for other equivalent definitions. Now let S be a maximally unramified elliptic maximal torus. Recall that we denote by S^u the maximal unramified subtorus of S . The unique Frobenius-fixed point x in $\mathcal{A}^{\text{red}}(S, F^u)$ is a vertex of $\mathcal{B}^{\text{red}}(G, F)$ [22, Lemma 3.4.3]. We have ([22, Lemmas 3.1.6 and 3.4.6])

$$S^u(F)_0/S^u(F)_{0+} \xrightarrow{\cong} S(F)_0/S(F)_{0+}, \quad S(F)_0 = S(F) \cap G(F)_{x,0}.$$

The special fiber of the connected Neron model of S^u embeds in G_x as an elliptic maximal torus S^u , and the images of $S(F)_0$ and $S^u(F)_0$ in $G_x(k_F)$ are equal to $S^u(k_F)$. Conversely, for a vertex x of $\mathcal{B}^{\text{red}}(G, F)$, every elliptic maximal torus of G_x arises in this way. See *loc. cit.* Lemma 3.4.4.

For a vertex x of $\mathcal{B}^{\text{red}}(G, F)$ and an elliptic maximal torus S^u of G_x that corresponds to a maximally unramified elliptic maximal torus S of G , a character λ of $S^u(k_F)$ is called *regular* if its stabilizer in $N(S, G)(F)/S(F)$ is trivial. A depth-zero character μ of $S(F)$ is called *regular* if $\mu|_{S(F)_0}$ induces a regular character $\bar{\mu}$ of $S^u(k_F)$; see *loc. cit.* Definition 3.4.16. Note that if λ is a regular character of $S^u(k_F)$ then it is in general position which means that its stabilizer in $\Omega(S^u, G_x)(k_F)$ is trivial (cf. *loc. cit.* Fact 3.4.18).

2.2.2 Regular supercuspidal representations

Let $\Psi = (\vec{G}, \pi_{-1}, \vec{\phi})$ be a reduced generic cuspidal G -datum. Recall that π_{-1} is an irreducible depth-zero supercuspidal representation of $G^0(F)$. By [27, Proposition 6.8], there exists a vertex $x \in \mathcal{B}^{\text{red}}(G^0, F)$ such that $\pi_{-1}|_{G^0(F)_{x,0}}$ contains the inflation to $G^0(F)_{x,0}$ of an irreducible cuspidal representation κ of $G^0(F)_{x,0;0+} \simeq G_x^0(k_F)$.

We call Ψ *regular* if κ is a Deligne–Lusztig cuspidal representation $\pm R_{T,\lambda}$ attached to an elliptic maximal torus T of G_x^0 and a regular character λ of $T(k_F)$ (cf. [22, Definition 3.7.9]). Note that if Ψ is regular, then any G -datum in its G -equivalent class is also regular. We call an irreducible supercuspidal representation π of $G(F)$ *regular* if it is of the form $\pi(\Psi)$ for some regular generic reduced cuspidal G -datum Ψ . According to Theorem 2.1 the regularity of π is well defined.

Remark 2.3 The above definition of regular supercuspidal representations requires the assumption that $p \nmid |\pi_1(G_{\text{der}})|$. More generally, if p divides $|\pi_1(G_{\text{der}})|$, an irreducible supercuspidal representation π of $G(F)$ is called regular if its inflation to $\tilde{G}(F)$ is regular in the above sense, where $\tilde{G} \rightarrow G$ is a z -extension. See [22, §3.7.4] for more details. In this situation, there may exist regular supercuspidal representations which can not be constructed from Yu’s construction. One reason that we need the assumption $p \nmid |\pi_1(G_{\text{der}})|$ is that we want to apply Hakim–Murnaghan’s result that is valid for tame supercuspidal representations. Another reason is that we need the existence of Howe factorizations of tame regular elliptic pairs, see Sect. 2.2.4.

2.2.3 Parametrization: depth-zero case

Suppose that π is a regular depth-zero supercuspidal representation of $G(F)$. By [22, Proposition 3.4.27] there exists a maximally unramified elliptic maximal torus S of G and a regular depth-zero character μ of $S(F)$ such that π is of the form $\pi_{(S,\mu)}$.

First let us recall the definition of $\pi_{(S,\mu)}$. Let x be the vertex determined by S , and $\pm R_{S^u, \bar{\mu}}$ the Deligne–Lusztig cuspidal representation of $G_x(k_F)$ associated with S^u and $\bar{\mu}$. We denote by $\kappa_{(S,\mu)}$ the inflation of $\pm R_{S^u, \bar{\mu}}$ to $G(F)_{x,0}$. In [22, §3.4.4], Kaletha constructed a representation $\tilde{\kappa}_{(S,\mu)}$ of $G_S := S(F)G(F)_{x,0}$, which is an extension of $\kappa_{(S,\mu)}$. The technical issue is that in general $Z(F)S(F)_0$ is not equal to $S(F)$, which

makes the construction of $\tilde{\kappa}_{(S,\mu)}$ more subtle. According to *loc. cit.* Lemma 3.4.20, the representation

$$\rho_{(S,\mu)} := \text{ind}_{G_S}^{G(F)_x} \tilde{\kappa}_{(S,\mu)}$$

is irreducible and thus the representation

$$\pi_{(S,\mu)} := \text{ind}_{G(F)_x}^{G(F)} \rho_{(S,\mu)}$$

is a regular depth-zero supercuspidal representation.

Moreover, [22, Proposition 3.4.27] states that the isomorphism classes of regular depth-zero supercuspidal representations are parameterized by the conjugacy classes of the pairs (S, μ) . In summary, we have:

Lemma 2.4 *Each regular depth-zero supercuspidal representation is of the form $\pi_{(S,\mu)}$ for some pair (S, μ) as above. Two regular depth-zero supercuspidal representations $\pi_{(S_1,\mu_1)}$ and $\pi_{(S_2,\mu_2)}$ are isomorphic if and only if the pairs (S_1, μ_1) and (S_2, μ_2) are $G(F)$ -conjugate.*

2.2.4 Parametrization: general case

To obtain an analogous parametrization as Lemma 2.4 for general regular supercuspidal representations, Kaletha introduced the notion *tame regular elliptic pairs* (S, μ) , where S is a tame elliptic maximal torus of G and μ a character of $S(F)$ satisfying the conditions in [22, Definition 3.7.5].

Suppose that $\Psi = (\vec{G}, \pi_{(S,\mu_o)}, \vec{\phi})$ is a regular reduced generic cuspidal G -datum, where S is a maximally unramified elliptic maximal torus of G^0 and μ_o a regular depth-zero character of $S(F)$ with respect to G^0 . Then (S, μ) is a tame regular elliptic pair [22, Proposition 3.7.8], where

$$\mu = \mu_o \prod_{i=0}^d \phi_i|_{S(F)}.$$

Conversely, given a tame regular elliptic pair (S, μ) , a Howe factorization [22, §3.6] of (S, μ) provides a regular generic cuspidal G -datum Ψ and thus a regular supercuspidal representation

$$\pi_{(S,\mu)} := \pi(\Psi)$$

of $G(F)$. For later use, let us review the definition of Howe factorization. Let E be the splitting field of S . For each $r > 0$, the Levi subsystem

$$R_r = \{\alpha \in R(S, G) \mid \mu(N_{E/F}(\alpha^\vee(E_r^\times))) = 1\}$$

of $R(S, G)$ gives a filtration $r \mapsto R_r$ of $R(S, G)$. The Levi subsystems R_{r_i} 's associated to the breaks $r_{d-1} > r_{d-2} > \dots > r_0 > 0$ in the filtration of $R(S, G)$ give rise to a twisted Levi sequence

$$\vec{G} = (G^0, \dots, G^d = G), \tag{2}$$

where, for $0 \leq i \leq d - 1$, G^i is the twisted Levi subgroup of G such that S is a maximal torus and $R(S, G^i) = R_{r_i}$. Set $r_d = \text{depth}(\mu)$ and

$$\vec{r} = (r_0, \dots, r_d). \tag{3}$$

A *Howe factorization* of (S, μ) is a sequence of characters:

$$\mu_\circ : S(F) \rightarrow \mathbb{C}^\times, \quad \vec{\phi} = \left(\phi_i : G^i(F) \rightarrow \mathbb{C}^\times, 0 \leq i \leq d \right)$$

satisfying the conditions: μ_\circ is regular (with respect to G^0) and of depth zero, $\vec{\phi}$ is of depth \vec{r} such that $(\vec{G}, \pi_{(S, \mu_\circ)}, \vec{\phi})$ is a *normalized* (cf. [22, Definition 3.7.1]) regular reduced cuspidal generic G -datum and $\mu = \mu_\circ \prod_{i=0}^d \phi_i|_{S(F)}$. For convenience, we also call the above G -datum $(\vec{G}, \pi_{(S, \mu_\circ)}, \vec{\phi})$ a *Howe factorization* of (S, μ) . Under the assumption that $p \nmid |\pi_1(G_{\text{der}})|$, Howe factorizations always exist and differ by refactorizations; see *loc. cit.* Proposition 3.6.7 and Lemma 3.6.6. We remark that a Howe factorization is defined for any pair (S, μ) where S is a tame maximal torus of G and μ an arbitrary character of $S(F)$ in *loc. cit.* Definition 3.6.2. We refer to [22, §3.6] for more details.

In summary, [22, Proposition 3.7.8 and Corollary 3.7.10] establish a parametrization of regular supercuspidal representations in terms of tame regular elliptic pairs.

Lemma 2.5 *Each regular supercuspidal representation is of the form $\pi_{(S, \mu)}$ for some tame regular elliptic pair (S, μ) . Two regular supercuspidal representations $\pi_{(S_1, \mu_1)}$ and $\pi_{(S_2, \mu_2)}$ are isomorphic if and only if the pairs (S_1, μ_1) and (S_2, μ_2) are $G(F)$ -conjugate.*

2.3 Hakim–Murnaghan’s work

Let θ be an involution of G and $H = G^\theta$. Let $\Psi = (\vec{G}, x, \rho, \vec{\phi})$ be a generic cuspidal G -datum and $\pi = \pi(\Psi)$ the irreducible supercuspidal representation of $G(F)$ attached to Ψ (see §2.1.1). Hakim–Murnaghan’s result [14, Theorem 5.26] provides an explicit formula for $\dim \text{Hom}_{H(F)}(\pi, \mathbf{1})$. Later Hakim and Lansky [11] corrected some mistakes in [14] and improved the theory.

Definition 2.6 We say that Ψ is θ -symmetric if

- $\theta(\vec{G}) = \vec{G}$, i.e., $\theta(G^i) = G^i$ for any $0 \leq i \leq d$,
- $\vec{\phi} \circ \theta = \vec{\phi}^{-1}$, i.e., $\phi_i \circ \theta = \phi_i^{-1}$ for any $0 \leq i \leq d$,
- $\theta(x) = x$.

As in [11], we denote by $[\Psi]$ the set of refactorizations of Ψ and by $[\theta]$ the K^0 -orbit of θ . Recall that we denote $\phi = \prod_{i=0}^d \phi_i|_{G^0(F)}$. Set $\rho_{\text{nm}} := \rho \otimes (\phi|_{K^0})$, which is an invariant of $[\Psi]$. Note that $\phi|_{K^0_+}$ is also an invariant of $[\Psi]$.

Definition 2.7 We write $[\theta] \sim [\Psi]$ if

$$\theta(K^0) = K^0 \quad \text{and} \quad \phi|_{K^0_+} = 1. \tag{4}$$

We remark that $[\theta] \sim [\Psi]$ is well defined since the condition (4) depends only on $[\theta]$ and $[\Psi]$. The relation between Definitions 2.6 and 2.7 is as follows [11, Proposition 3.9].

Lemma 2.8 *We have $[\theta] \sim [\Psi]$ if and only if there exists $\check{\Psi} \in [\Psi]$ such that $\check{\Psi}$ is θ -symmetric.*

In particular $[\theta] \sim [\Psi]$ implies that $\theta(\vec{G}) = \vec{G}$ and $\theta(x) = x$. For our purpose on the distinction problem, we only need the following partial result derived from [14, Theorem 5.26] and [11, Theorem 3.10].

Theorem 2.9 *For a G -datum $\check{\Psi}$, the representation $\pi(\check{\Psi})$ is H -distinguished if and only if $\check{\Psi}$ is G -equivalent to a G -datum Ψ such that*

- (1) $[\theta] \sim [\Psi]$,
- (2) $\text{Hom}_{K^{0,\theta}}(\rho_{\text{nm}}, \eta_\theta) \neq 0$.

Here η_θ is a quadratic character of $K^{0,\theta}$, which is introduced in [14, §5.6] and denoted by η'_θ therein. We caution the reader that in §5.6 of *loc. cit.* there is another character denoted by η_θ . We apologize for the inconsistency of this notation. Now let us record the definition of η_θ . For each $0 \leq i \leq d - 1$, consider the subgroups

$$J^{i+1} = (G^i, G^{i+1})(F)_{x,(r_i,s_i)} \quad \text{and} \quad J_+^{i+1} = (G^i, G^{i+1})(F)_{x,(r_i,s_{i+})}$$

of $G^{i+1}(F)_{x,s_i}$ and $G^{i+1}(F)_{x,s_{i+}}$ respectively, whose definitions are given at the end of [35, §3] (also see [14, page 53]). The quotient group

$$W_i := J^{i+1}/J_+^{i+1}$$

is equipped with a structure of symplectic \mathbb{F}_p -vector space [35, Lemma 11.1]. Since $[\theta] \sim [\Psi]$, both J^{i+1} and J_+^{i+1} are θ -stable for each i . Thus θ induces a linear transformation on W_i , which is still denoted by θ . Set

$$W_i^\theta = \{w \in W_i \mid \theta(w) = w\}.$$

Then W_i^θ is stable by $K^{0,\theta}$ under the conjugate action. Let χ_i^θ be the quadratic character of $K^{0,\theta}$ defined by

$$\chi_i^\theta(k) = \det \left(\text{Ad}(k)|_{W_i^\theta} \right)^{\frac{p-1}{2}}. \tag{5}$$

Then the character η_θ is defined to be

$$\eta_\theta = \prod_{i=0}^{d-1} \chi_i^\theta, \tag{6}$$

and trivial if $d = 0$. Note that η_θ depends only on $[\Psi]$.

3 Distinction

3.1 Main results

Our main theorem is Theorem 3.16, whose statement is given in Sect. 3.1.4 and whose proof is delayed to Sect. 3.2. We first introduce two characters ε_S and η_S , which are involved in Theorem 3.16, in Sects. 3.1.1 and 3.1.2 respectively. A direct but important consequence (Corollary 3.17) of the main theorem is also stated in Sect. 3.1.4. Some examples are discussed in Sect. 3.1.5.

As before, we always assume that G is a tamely ramified connected reductive group over F and θ an involution of G .

3.1.1 The character ε_S

First we review the character ε_T introduced in [25, §2] for the distinction problem over finite fields. Let G be a connected reductive group over k_F and θ an involution of G defined over k_F . Suppose that T is a θ -stable maximal k_F -torus of G . Recall that we denote by $T^{\theta, \circ}$ the identity component of T^θ . As a map on $T^\theta(k_F)$, ε_T is given by

$$\varepsilon_T(t) = \sigma(C_G(T^{\theta, \circ})) \cdot \sigma(C_G(T^{\theta, \circ}) \cap C_G(t)),$$

where $\sigma(M)$ is set to be $(-1)^{\text{rank}_{k_F}(M)}$ for any connected reductive group M over k_F . By [25, Proposition 2.3 (b) and (c)] the map ε_T is actually a character, and satisfies

$$\varepsilon_T|_{T^{\theta, \circ}(k_F)} = 1. \tag{7}$$

Let T_{ad} be the image of T in G_{ad} . For $t \in T_{\text{ad}}$, we denote by $C_G(t)$ the identity component of the centralizer of t in G . The involution θ induces an involution, still denoted by θ , on G_{ad} . Let $(T_{\text{ad}})^{\theta, \circ}$ be the identity component of $(T_{\text{ad}})^\theta$, and $(T^{\theta, \circ})_{\text{ad}}$ the image of $T^{\theta, \circ}$ in G_{ad} .

For any $t \in (T_{\text{ad}})^\theta(k_F)$, $C_G(t)$ is also defined over k_F . Therefore we can define a map, still denoted by ε_T , on $(T_{\text{ad}})^\theta(k_F)$, which is given by the same formula

$$\varepsilon_T(t) = \sigma(C_G(T^{\theta, \circ})) \cdot \sigma(C_G(T^{\theta, \circ}) \cap C_G(t)).$$

Lemma 3.1 *The map ε_T is a character of $(T_{\text{ad}})^\theta(k_F)$. Moreover we have*

$$\varepsilon_T|_{(T_{\text{ad}})^{\theta, \circ}(k_F)} = 1.$$

Proof Recall that, according to [33, §2.2], for an arbitrary torus S over k_F equipped with an involution θ , we have $S^{\theta, \circ} = \{s\theta(s) | s \in S\}$ over an algebraic closure \bar{k}_F of k_F . Since $(T_{\text{ad}})^{\theta, \circ}(\bar{k}_F) = \{t\theta(t) | t \in T_{\text{ad}}(\bar{k}_F)\}$ and $T^{\theta, \circ}(\bar{k}_F) = \{t\theta(t) | t \in T(\bar{k}_F)\}$, the natural embedding $(T^{\theta, \circ})_{\text{ad}} \rightarrow (T_{\text{ad}})^{\theta, \circ}$ is also surjective and thus an isomorphism. For $t \in (T_{\text{ad}})^{\theta, \circ}(k_F) = (T^{\theta, \circ})_{\text{ad}}(k_F)$, take any lift $i \in T^{\theta, \circ}(\bar{k}_F)$. Then $C_G(t) = C_G(i) \supset C_G(T^{\theta, \circ})$, and thus $\varepsilon_T(t) = 1$. The rest of the proof is same as that of [25, Proposition 2.3]. □

Now we come back to the p -adic fields case. We first consider the depth-zero case. Let S be a maximally unramified elliptic maximal torus of G and S^u the maximal unramified subtorus of S . Let x be the vertex of $\mathcal{B}^{\text{red}}(G, F)$ attached to S . Suppose that $\theta(x) = x$. Thus both $G(F)_{x,0}$ and $G(F)_{x,0+}$ are θ -stable. Therefore θ induces an involution on G_x , which is still denoted by θ . We assume that S^u is θ -stable, where S^u is the elliptic maximal torus of G_x corresponding to S^u .

Lemma 3.2 *There exists $y \in G(F)_{x,0+}$ such that ${}^y S$ is θ -stable.*

Proof When S is unramified, the assertion follows from [11, Lemma A.2] and the results in [3, §2]. In general, using the same proof as that of [11, Lemma A.2], we can show that there exists a θ -stable maximally unramified elliptic maximal torus S_1 of G such that $x \in \mathcal{A}^{\text{red}}(S_1, F^u)$ and S^u also corresponds to S_1^u . According to [22, Lemma 3.4.5], S_1 and S are $G(F)_{x,0+}$ -conjugate. \square

Recall that we denote $G_S = S(F)G(F)_{x,0}$. We use the same notation $S(k_F)$ as [22, §3.4.4] to denote

$$S(k_F) := S(F)/S(F)_{0+},$$

which is a subgroup of

$$G_S(k_F) := G_S/G(F)_{x,0+}.$$

Corollary 3.3 *Both G_S and $S(k_F)$ are θ -stable.*

Proof According to Lemma 3.2, there exists a θ -stable torus S_1 which is $G(F)_{x,0+}$ -conjugate to S . Since $S(F)$ normalizes $G(F)_{x,0}$, we see G_S is also equal to $S_1(F)G(F)_{x,0}$ which is θ -stable. Moreover $S(k_F)$ coincides with the image of $S_1(F)$ in $G_S(k_F)$, which is also θ -stable. \square

According to the third paragraph of [22, §3.4.4], there is a natural homomorphism

$$\iota : S(k_F) \rightarrow S_{\text{ad}}^u(k_F)$$

where S_{ad}^u is the image of S^u in $[G_x]_{\text{ad}}$, and ι is given by the composition

$$S(F) \rightarrow S_{\text{ad}}(F) = S_{\text{ad}}(F)_0 \rightarrow S_{\text{ad}}(F)_{0:0+} = [S_{\text{ad}}]^u(k_F) \rightarrow S_{\text{ad}}^u(k_F)$$

where S_{ad} is the image of S in G_{ad} , $[S_{\text{ad}}]^u$ the elliptic maximal torus of $[G_{\text{ad}}]_x$ corresponds to $(S_{\text{ad}})^u$, and $[S_{\text{ad}}]^u \rightarrow S_{\text{ad}}^u$ given by the natural map $[G_{\text{ad}}]_x \rightarrow [G_x]_{\text{ad}}$. Therefore the image of $S(k_F)^\theta$ under ι lies in $(S_{\text{ad}}^u)^\theta(k_F)$. The character ε_S of $S(k_F)^\theta$ is defined to be

$$\varepsilon_S = \varepsilon_{S^u} \circ \iota. \tag{8}$$

Definition 3.4 Suppose that S is θ -stable. The character ε_S is defined to be the composition of the natural map $S^\theta(F) \rightarrow S(k_F)^\theta$ and ε_S .

Remark 3.5 For general depth case, suppose that (S, μ) is a tame regular elliptic pair of G such that S is θ -stable and $\mu|_{S(F)_{0+}^\theta} = 1$. Let G^0 be the 0th twisted Levi subgroup in the twisted Levi sequence \vec{G} of G determined by (S, μ) . Then G^0 is also θ -stable (see Lemma 3.7 below). Recall that S is a maximally unramified elliptic maximal torus of G^0 . We define the character ε_S of $S^\theta(F)$ as in Definition 3.4, but with respect to G^0 .

Lemma 3.6 Suppose that S is θ -stable and $\mu|_{S(F)_{0+}^\theta} = 1$. Then we have

$$\varepsilon_S|_{S^{\theta,\circ}(F)} = 1.$$

Proof It is obvious that the image of $S^{\theta,\circ}(F)$ in $(S_{\text{ad}}^u)^\theta(k_F)$ is actually in $(S_{\text{ad}}^u)^{\theta,\circ}(k_F)$. Hence the assertion follows from Lemma 3.1 immediately. \square

3.1.2 The character η_S

Lemma 3.7 Suppose that (S, μ) is a tame regular elliptic pair of G . Let \vec{G} be the twisted Levi sequence determined by (S, μ) , and x the vertex of $\mathcal{B}^{\text{red}}(G^0, F)$ attached to S . If S is θ -stable and $\mu|_{S(F)_{0+}^\theta} = 1$, then $\theta(\vec{G}) = \vec{G}$ and $\theta(x) = x$.

Proof Since S is θ -stable, θ acts on $R(S, G)$ by $\theta(\alpha) := \alpha \circ \theta$ and acts on $R(S, G)^\vee$ by $\theta(\alpha^\vee) = \theta \circ \alpha^\vee$. It is clear that $\theta(\alpha^\vee) = \theta(\alpha)^\vee$. If $\alpha \in R_r$, that is $\mu(N_{E/F}(\alpha^\vee(E_r^\times))) = 1$, we have

$$\begin{aligned} \mu(N_{E/F}(\theta(\alpha)^\vee(E_r^\times))) &= \mu(N_{E/F}(\theta \circ \alpha^\vee(E_r^\times))) \\ &= \mu(\theta \circ N_{E/F}(\alpha^\vee(E_r^\times))) \\ &= \mu(N_{E/F}(\alpha^\vee(E_r^\times)))^{-1} \\ &= 1. \end{aligned}$$

Hence R_r is θ -stable. Therefore each twisted Levi subgroup G^i is θ -stable. It is obvious that $\theta(x) = x$. \square

Now we assume that (S, μ) is a tame regular elliptic pair of G such that S is θ -stable and $\mu|_{S(F)_{0+}^\theta} = 1$. Let \vec{G} and \vec{r} be the twisted Levi sequence and the sequence of depths determined by (S, μ) ; see Eqs. (2) and (3) in Sect. 2.2.4. Let x be the vertex of $\mathcal{B}^{\text{red}}(G^0, F)$ attached to S . According to Lemma 3.7, it makes sense to let η_θ be the character of $K^{0,\theta}$ defined by the formula (6), where $K^0 = G^0(F)_x$. Note that $S(F) \subseteq K^0 = G^0(F)_x$.

Definition 3.8 Under the above conditions, the character η_S of $S^\theta(F)$ is defined to be

$$\eta_S = \eta_\theta|_{S^\theta(F)}.$$

Lemma 3.9 *Let (S, μ) be a tame regular elliptic pair such that S is θ -stable and $\mu|_{S(F)_{0+}^\theta} = 1$. Let S^u be the maximal unramified subtorus of S . Then we have*

$$\eta_S|_{(S^u)^{\theta, \circ}(F)} = 1.$$

Proof Let \vec{G} , \vec{r} and x be the above-mentioned objects determined by (S, μ) . Recall that the character η_θ is defined to be $\prod_{i=0}^{d-1} \chi_i^\theta$; see Eq. (6). We will show that

$$\chi_i^\theta|_{(S^u)^{\theta, \circ}(F)} = 1, \quad \forall 0 \leq i \leq d - 1.$$

To simplify the notation, we denote $r = r_i, s = \frac{r_i}{2}, J = J^{i+1}, J_+ = J_+^{i+1}, W = W_i, T = (S^u)^{\theta, \circ}, G = G^{i+1}, G' = G^i, H = G^{\theta, \circ}$ and $H' = (G')^{\theta, \circ}$. Let $\mathfrak{g} = \text{Lie}(G), \mathfrak{g}' = \text{Lie}(G'), \mathfrak{h} = \text{Lie}(H)$ and $\mathfrak{h}' = \text{Lie}(H')$. According to Sect. 1, our assumptions on p satisfy [2, Hypothesis 2.1.1]. Therefore there exists a $G(F)$ -invariant non-degenerate symmetric bilinear F -valued form B on $\mathfrak{g}(F)$. Denote by $\mathfrak{g}'(F)^\perp$ the orthogonal complement of $\mathfrak{g}'(F)$ in $\mathfrak{g}(F)$ and by $\mathfrak{g}'(F)_{x,t}^\perp$ the intersection $\mathfrak{g}'(F)^\perp \cap \mathfrak{g}(F)_{x,t}$ for any $t \in \tilde{\mathbb{R}}$. By [2, Proposition 1.9.3], we have

$$\mathfrak{g}(F)_{x,t} = \mathfrak{g}'(F)_{x,t} \oplus \mathfrak{g}'(F)_{x,t}^\perp.$$

Put

$$\tilde{\mathfrak{J}} = \mathfrak{g}'(F)_{x,r} \oplus \mathfrak{g}'(F)_{x,s}^\perp, \quad \tilde{\mathfrak{J}}_+ = \mathfrak{g}'(F)_{x,r} \oplus \mathfrak{g}'(F)_{x,s+}^\perp.$$

There is a natural $G'(F)_x$ -equivariant isomorphism from J/J_+ to $\tilde{\mathfrak{J}}/\tilde{\mathfrak{J}}_+$. Therefore, for $k \in G'(F)_x^\theta$, we have

$$\begin{aligned} \chi^\theta(k) &= \det(\text{Ad}(k)|_W^\theta) \\ &= \det(\text{Ad}(k)|_{(J/J_+)^\theta}) \\ &= \det(\text{Ad}(k)|_{(\tilde{\mathfrak{J}}/\tilde{\mathfrak{J}}_+)^\theta}). \end{aligned}$$

Note that

$$\begin{aligned} \tilde{\mathfrak{J}}/\tilde{\mathfrak{J}}_+ &= \mathfrak{g}'(F)_{x,s}^\perp / \mathfrak{g}'(F)_{x,s+}^\perp \\ &= \left(\mathfrak{g}'(F)_{x,s} \oplus \mathfrak{g}'(F)_{x,s}^\perp \right) / \left(\mathfrak{g}'(F)_{x,s} \oplus \mathfrak{g}'(F)_{x,s+}^\perp \right) \\ &= \left(\mathfrak{g}(F)_{x,s} / \mathfrak{g}(F)_{x,s+} \right) / \left(\left(\mathfrak{g}'(F)_{x,s} \oplus \mathfrak{g}'(F)_{x,s+}^\perp \right) / \mathfrak{g}(F)_{x,s+} \right) \\ &= \left(\mathfrak{g}(F)_{x,s} / \mathfrak{g}(F)_{x,s+} \right) / \left(\mathfrak{g}'(F)_{x,s} / \mathfrak{g}'(F)_{x,s+} \right) \\ &= \mathfrak{g}(F)_{x,s:s+} / \mathfrak{g}'(F)_{x,s:s+}. \end{aligned}$$

Due to [14, Lemma 2.11, Proposition 2.12], we can identify

$$\left(\mathfrak{g}(F)_{x,s:s+} / \mathfrak{g}'(F)_{x,s:s+} \right)^\theta = \mathfrak{g}(F)_{x,s:s+}^\theta / \mathfrak{g}'(F)_{x,s:s+}^\theta.$$

By [29, Lemma 2.8], for any $t \in \tilde{\mathbb{R}}$, we have

$$\mathfrak{g}(F)_{x,t}^\theta = \mathfrak{g}(F)_{x,t} \cap \mathfrak{h}(F) = \mathfrak{h}(F)_{x,t},$$

and

$$\mathfrak{g}'(F)_{x,t}^\theta = \mathfrak{g}'(F)_{x,t} \cap \mathfrak{h}(F) = \mathfrak{h}'(F)_{x,t}.$$

Hence

$$(\mathfrak{J}/\mathfrak{J}_+)^\theta = \mathfrak{h}(F)_{x,s:s+}/\mathfrak{h}'(F)_{x,s:s+}.$$

Note that $T \subseteq H' \subseteq H$. Since T is an unramified elliptic torus, according to [16, Lemma 7.1.1] we have $T(F) = A(T)(F) \cdot T(F)_0$. Recall that $A(\cdot)$ denotes the central split torus of the corresponding reductive group. Since S is an elliptic maximal torus in G' and G , we have $A(S) = A(G') = A(G)$. On the other hand, we have $A(T) \subseteq A(S)^\theta$, $A(G')^\theta \subseteq Z(H')$ and $A(G)^\theta \subseteq Z(H)$. Hence $A(T) \subseteq Z(H) \subseteq Z(H')$. Thus, to prove the lemma, it suffices to show that

$$\det(\text{Ad}(t)|_{\mathfrak{h}(F)_{x,s:s+}}) = \det(\text{Ad}(t)|_{\mathfrak{h}'(F)_{x,s:s+}}) = 1, \forall t \in T(F)_0. \tag{9}$$

Let \mathbb{T} be the special fiber of the connected Neron model of T , which is a subtorus of H'_x and H_x . Then (9) is equivalent to

$$\det(\text{Ad}(t)|_{\mathfrak{h}(F)_{x,s:s+}}) = \det(\text{Ad}(t)|_{\mathfrak{h}'(F)_{x,s:s+}}) = 1, \forall t \in \mathbb{T}(k_F). \tag{10}$$

Denote $V_{x,s} = \mathfrak{h}(F)_{x,s:s+}$, which is viewed as a k_F -affine space. The adjoint action of H_x on $V_{x,s}$ is an algebraic representation. Hence $\det(\text{Ad}(\cdot)|_{V_{x,s}})$ is an algebraic character of H_x . Since $H_x = Z(H_x)^\circ \cdot (H_x)_{\text{der}}$ and the restriction of this algebraic character to $Z(H_x)^\circ$ and $(H_x)_{\text{der}}$ is trivial, $\det(\text{Ad}(\cdot)|_{V_{x,s}})$ itself is the trivial character. By the same reason, $\det(\text{Ad}(\cdot)|_{\mathfrak{h}'(F)_{x,s:s+}})$ is also trivial. We conclude that (10) holds. \square

Remark 3.10 As pointed out by a referee, our proof of Lemma 3.9 is nearly identical to that of [8, Lemma 7.10], except that the case $G = \text{GL}_n$ is treated therein. We also remark that, according to the above proof, Lemma 3.9 holds under the Hypothesis 2.1.1 of [2] on p .

3.1.3 $(\theta, \varepsilon\eta)$ -symmetric pair

Definition 3.11 Let (S, μ) be a tame regular elliptic pair of G . We say that (S, μ) is $(\theta, \varepsilon\eta)$ -symmetric if:

- S is θ -stable,
- $\mu|_{S^\theta(F)} = \varepsilon_S \cdot \eta_S$.

Lemma 3.12 *Let (S, μ) be a $(\theta, \varepsilon\eta)$ -symmetric tame regular elliptic pair. Then there exists a Howe factorization $(\bar{G}, \pi_{(S, \mu_\circ)}, \bar{\phi})$ of (S, μ) such that*

$$\phi_i|_{(S^u)^{\theta, \circ}(F)} = 1 \tag{11}$$

and

$$\phi_i|_{G^0(F)_{x, 0+}^\theta} = 1 \tag{12}$$

for each $0 \leq i \leq d$.

Proof Denote $T = (S^u)^{\theta, \circ}$. According to Lemmas 3.6 and 3.9, we have $\mu|_{T(F)} = 1$. From the definition of ε_S and η_S , it is easy to see that $\mu|_{S(F)_{0+}^\theta} = 1$. To prove the assertion of this lemma, it suffices to plug the conditions (11) and (12) into the proof of [22, Proposition 3.6.7], which establishes the existence of Howe factorizations by a recursive construction, and check that the same recursion goes through in our situation. We just point out that the only necessary modification that we need is a stronger statement of [22, Lemma 3.6.9]: we further require that the character $\phi : G(F) \rightarrow \mathbb{C}^\times$ satisfies $\phi|_{T(F)} = 1$ and $\phi|_{G^0(F)_{x, 0+}^\theta} = 1$ besides $\phi|_{S(F)_r} = \mu|_{S(F)_r}$. Here we use the same notation as that in the proof of [22, Lemma 3.6.9]. Now let us prove this statement. Let M_1 and M_2 be the images of $T(F)$ and $G^0(F)_{x, 0+}^\theta$ in $D(F) = (G/G_{\text{der}})(F)$ respectively. Let M_3 be the subgroup of $D(F)$ generated by M_1 and $D(F)_r = S(F)_r/S_{\text{der}}(F)_r$, and M_4 the subgroup of $D(F)$ generated by M_2 and M_3 . Then μ descends to a non-trivial finite order character of M_3 which is trivial on M_1 and $D(F)_{r+}$. Since $G^0(F)_{x, 0+}^\theta \cap S(F) = S(F)_{0+}^\theta$, this character of M_3 can be extended uniquely to a character ϕ' of M_4 which is trivial on M_2 . Then ϕ' can be extended to a character ϕ of $D(F)$, whose pull-back to $G(F)$ satisfies our requirement. \square

Corollary 3.13 *Let (S, μ) be a $(\theta, \varepsilon\eta)$ -symmetric tame regular elliptic pair. Then any Howe factorization $(\bar{G}, \pi_{(S, \mu_\circ)}, \bar{\phi})$ of (S, μ) satisfies*

$$\phi|_{G^0(F)_{x, 0+}^\theta} = 1$$

where $\phi = \prod_{i=0}^d \phi_i$ is viewed as a character of $G^0(F)$.

Proof It is a direct consequence of Lemma 3.12 and the fact that $\phi|_{G^0(F)_{x, 0+}^\theta}$ are the same for all the Howe factorizations since they differ by refactorizations. \square

Remark 3.14 For the depth-zero case, i.e., when S is a maximally unramified elliptic maximal torus of G and μ a regular depth-zero character of $S(F)$, we abbreviate the notion $(\theta, \varepsilon\eta)$ -symmetric to be (θ, ε) -symmetric since η_S is trivial in this situation. In this case, if (S, μ) is (θ, ε) -symmetric, according to Lemma 3.6, (S, μ) is θ -symmetric, i.e., we have $\mu^{-1} = \mu \circ \theta$. This is because $S(F)^{1+\theta} \subseteq S^{\theta, \circ}(F)$.

Remark 3.15 For positive depth case, according to Lemma 3.6, the condition $\mu|_{S^\theta(F)} = \varepsilon_S \cdot \eta_S$ implies that

$$\mu|_{S^{1+\theta}(F)} = \eta_S|_{S^{1+\theta}(F)}.$$

However, we could not show $\eta_S|_{S^{1+\theta}(F)} = 1$, i.e., η_S is θ -symmetric. To the best of the author's knowledge, η_S is θ -symmetric for all the examples studied by Hakim and his collaborators. We speculate that it also holds in general.

3.1.4 Statement of the main theorem

Now we can state our main theorem on the distinction problem.

Theorem 3.16 *Let $\pi_{(S,\mu)}$ be a regular supercuspidal representation of $G(F)$. Then $\pi_{(S,\mu)}$ is H -distinguished if and only if (S, μ) is $G(F)$ -conjugate to a $(\theta, \varepsilon\eta)$ -symmetric tame regular elliptic pair.*

Corollary 3.17 *Let π be a regular depth-zero supercuspidal representation of $G(F)$. If π is H -distinguished then we have $\pi^\vee \simeq \pi \circ \theta$.*

Proof For a regular supercuspidal representation $\pi_{(S,\mu)}$, it follows from [14, Theorem 4.25, Corollary 4.26] that

$$\pi_{(S,\mu)}^\vee \simeq \pi_{(S,\mu^{-1})} \quad \text{and} \quad \pi_{(S,\mu)} \circ \theta \simeq \pi_{(\theta(S), \mu \circ \theta)}. \quad (13)$$

Now let π be an H -distinguished regular depth-zero supercuspidal representation. According to Theorem 3.16 and Remark 3.14, we can choose a (θ, ε) -symmetric maximally unramified regular elliptic pair (S, μ) such that $\pi \simeq \pi_{(S,\mu)}$. By Remark 3.14, the condition that (S, μ) is θ -symmetric implies the corollary. \square

3.1.5 Some examples

Galois involution Let H be a connected reductive group over F , E a quadratic field extension of F , and $G = R_{E/F}H$ the Weil restriction of H with respect to E/F . The non-trivial automorphism of $\text{Gal}(E/F)$ gives rise to an involution θ of G , which is called a *Galois involution*. If E/F is unramified, we call θ an *unramified Galois involution*.

Corollary 3.18 *Let θ be a Galois involution of G and $\pi_{(\dot{S}, \dot{\mu})}$ a regular supercuspidal representation of $G(F)$. Then $\pi_{(\dot{S}, \dot{\mu})}$ is H -distinguished if and only if $(\dot{S}, \dot{\mu})$ is $G(F)$ -conjugate to a pair (S, μ) such that S is θ -stable and $\mu|_{S^\theta(F)} = \eta_S$.*

Proof According to Theorem 3.16, it suffices to show that ε_S is trivial if S is θ -stable. Since S is θ -stable, by Galois descent, we have $S = R_{E/F}T$ where $T = S^\theta = S^{\theta \circ \theta}$ is a torus of H . Therefore, by Lemma 3.6, ε_S is trivial. \square

Remark 3.19 For Galois involutions, Prasad [30] stated a precise conjecture to give sufficient and necessary conditions for representations to be distinguished in terms of the Langlands parameters. Our prior work [36] verified a necessary condition of this conjecture for unramified Galois involutions, when the representations are regular depth-zero supercuspidal representations of unramified groups. The above corollary is a generalization of [36, Proposition 3.2]. Moreover, in a sequel [37] to this paper, we prove that η_S is indeed trivial for unramified Galois involutions.

Epipelagic supercuspidal representations The notion of epipelagic supercuspidal representations was first introduced by Reeder and Yu [32]. Kaletha [19] later studied the properties of epipelagic L -packets, including the endoscopic character identities. This kind of representations is a special case of a more general class of supercuspidal representations, called toral supercuspidal representations which were first considered by Adler [2]. We refer to [32, §2.5] for the definition of epipelagic supercuspidal representations, and to [22, §6] for a brief discussion on toral supercuspidal representations. In terms of Yu’s data, epipelagic supercuspidal representations are constructed from generic cuspidal *epipelagic* G -data

$$((G^0 = S, G^1 = G), x, \rho = 1, (\phi_0 = \mu, \phi_1 = 1)),$$

where $x \in \mathcal{B}^{\text{red}}(G, F)$ is a rational point of order e (cf. [32, §3.3]) and (S, μ) a tame regular elliptic pair satisfying [19, Conditions 3.3]. The resulting representations $\pi_{(S, \mu)}$ are called *epipelagic*. An important property of epipelagic G -data is that

$$G(F)_{x, \frac{1}{2e}+} = G(F)_{x, \frac{1}{2e}} = G(F)_{x, \frac{1}{e}}.$$

Therefore J^1/J^1_+ is automatically trivial, and thus $\eta_\theta = 1$ for any epipelagic G -datum such that $[\theta] \sim [\Psi]$. Then the following corollary is a direct consequence of Theorem 2.9 or [14, Proposition 5.31].

Corollary 3.20 *Let $\pi_{(\dot{S}, \dot{\mu})}$ be an epipelagic supercuspidal representation. Then $\pi_{(\dot{S}, \dot{\mu})}$ is H -distinguished if and only if $(\dot{S}, \dot{\mu})$ is $G(F)$ -conjugate to a pair (S, μ) such that S is θ -stable and $\mu|_{S^\theta(F)} = 1$. In particular, if $\pi_{(S, \mu)}$ is H -distinguished then we have $\pi_{(S, \mu)}^\vee \simeq \pi_{(S, \mu)} \circ \theta$.*

3.2 Proof of the main theorem

3.2.1 Distinction over field fields

Let G be a connected reductive group over k_F and T a maximal k_F -torus of G . Let λ be a character of $T(k_F)$ in general position and $\kappa_{(T, \lambda)} = \pm R_{T, \lambda}$ the Deligne–Lusztig representation of $G(k_F)$. Let θ be an involution of G defined over k_F , $H = G^\theta$, and η a character of $H(k_F)$. Denote

$$m = \dim \text{Hom}_{H(k_F)} (\kappa_{(T, \lambda)}, \eta).$$

We call (T, λ) a $(\theta, \varepsilon\eta)$ -symmetric pair if T is θ -stable and

$$\lambda|_{T^\theta(k_F)} = \varepsilon_T \cdot \eta|_{T^\theta(k_F)}.$$

Recall that the definition of ε_T is given at the beginning of Sect. 3.1.1. The following lemma is a partial summary of prior works [25], [11, §3.2] and [8, §8.2].

Lemma 3.21 *If the multiplicity m is non-zero, then (T, λ) is $G(k_F)$ -conjugate to a $(\theta, \varepsilon\eta)$ -symmetric pair. If we further assume that $\eta|_{T_1^{\theta, \circ}(k_F)} = 1$ for any θ -stable torus T_1 that is $G(k_F)$ -conjugate to T , the converse also holds.*

Remark 3.22 When λ is an arbitrary character and $\eta = 1$, Lusztig [25, Theorem 3.3] established an explicit formula for m . Hakim and Lansky [11, Theorem 3.11] generalized Lusztig's formula to arbitrary η . When λ is in general position, the multiplicity formula for m becomes much more simple, as discussed in [25, §10] and [8, §8.2]. The above lemma can be deduced directly from the multiplicity formula for m .

3.2.2 Distinction of $\tilde{\kappa}_{(S, \mu)}$

Let G be a tamely ramified connected reductive group over F and θ an involution of G . Let S be a maximally unramified elliptic maximal torus of G and S^u the maximal unramified subtorus of S . Let x be the vertex of $\mathcal{B}^{\text{red}}(G, F)$ attached to S . Let μ be a regular depth-zero character of $S(F)$ and $\tilde{\kappa}_{(S, \mu)}$ the representation of $G_S = S(F)G(F)_{x,0}$ introduced in Sect. 2.2.3. Suppose that $\theta(x) = x$ and G_S is θ -stable. Then $G(F)_{x,0}$ and $G(F)_{x,0+}$ are both θ -stable. Let η be a character of G_S^θ which is trivial on $Z^\theta(F)$ and $G(F)_{x,0+}^\theta$. One of the key steps to prove Theorem 3.16 is to determine when the multiplicity

$$m := \dim \text{Hom}_{G_S^\theta}(\tilde{\kappa}_{(S, \mu)}, \eta)$$

is non-zero. In this subsection, we say that (S, μ) is $(\theta, \varepsilon\eta)$ -symmetric if S is θ -stable and

$$\mu|_{S^\theta(F)} = \varepsilon_S \cdot \eta|_{S^\theta(F)}.$$

Proposition 3.23 *If the multiplicity m is non-zero, then (S, μ) is $G(F)_{x,0}$ -conjugate to a $(\theta, \varepsilon\eta)$ -symmetric pair. If we further assume that $\eta|_{(S_1^u)^{\theta, \circ}(F)_0} = 1$ for any θ -stable torus S_1 that is $G(F)_{x,0}$ -conjugate to S , the converse also holds.*

Proof For simplicity we will denote $\tilde{\kappa}_{(S, \mu)}$ by $\tilde{\kappa}$ when there is no confusion. First note that $Z(F)$ acts on the representation space V of $\tilde{\kappa}$ by the restriction of μ to $Z(F)$. Hence a necessary condition for the non-vanishing of m is

$$\mu|_{Z^\theta(F)} = 1. \tag{14}$$

From now on we assume (14). Denote

$$M = G_S^\theta / (Z(F)^\theta G(F)_{x,0+}^\theta).$$

Since $G(F)_{x,0+}$ acts trivially on V , we have

$$m = \dim \text{Hom}_M(\tilde{\kappa}, \eta).$$

We claim that M is a finite group. Note that

$$\begin{aligned} G_S^\theta \cap (Z(F)G(F)_{x,0+}) &= (Z(F)G(F)_{x,0+})^\theta \\ &= Z(F)^\theta G(F)_{x,0+}^\theta, \end{aligned}$$

where the last equality is due to [14, Lemma 2.11, Proposition 2.12]. Therefore M is a subgroup of $G_S / (Z(F)G(F)_{x,0+})$ and the latter group is obviously a finite group since $G_S / Z(F)$ is compact.

Since the group M is finite and the space V is finite dimensional, we have

$$m = \dim V^{(M,\eta)} = \frac{1}{|M|} \sum_{\gamma \in M} \Theta(\gamma)\eta^{-1}(\gamma),$$

where $V^{(M,\eta)}$ is the isotypical subspace of V on which M acts by η , and Θ is the character of the representation $\tilde{\kappa}$.

Now we review the character formula of Θ [22, Propositions 3.4.23 and 3.4.24]. The notation below is the same as that in Sect. 3.1.1. We view $\tilde{\kappa}$ as a representation of $G_S(k_F) = G_S / G(F)_{x,0+}$. For $\gamma = rg \in G_S(k_F)$ with $r \in S(k_F) = S(F) / S(F)_{0,+}$ and $g \in G_x(k_F) = G(F)_{x,0} / G(F)_{x,0+}$, there exists a Jordan decomposition $\gamma = \gamma_s \gamma_u$ given as follows; see the paragraph below *loc. cit.* Proposition 3.4.23 and the first two paragraphs of the proof of *loc. cit.* Propositions 3.4.23 and 3.4.24. Let \bar{r} be the image of r in $S_{\text{ad}}^u(k_F)$ and \dot{r} any lift of \bar{r} in $S^u(k_F)$. Let $\dot{r}g = su$ be the Jordan decomposition of $\dot{r}g$ in $G_x(k_F)$. In fact we have $\dot{r}^{-1}s \in G_x(k_F)$ and $u \in G_x(k_F)_{\text{unip}}$ where $G_x(k_F)_{\text{unip}}$ denotes the set of unipotent elements of $G_x(k_F)$. Set $\gamma_s = r\dot{r}^{-1}s \in G_S(k_F)$ and $\gamma_u = u \in G_x(k_F)$. This decomposition is independent of the choice of \dot{r} and thus is unique. Moreover $r\dot{r}^{-1}$ commutes with any element of $G_x(k_F)$ and $C_{G_x}(\gamma_s) = C_{G_x}(s)$ is defined over k_F . Then the character formula is

$$\Theta(\gamma) = \sigma(G_x)\sigma(S^u) \frac{1}{|C_{G_x}(\gamma_s)(k_F)|} \sum_{\substack{y \in G_x(k_F) \\ y^{-1}\gamma_s y \in S(k_F)}} \mu(y^{-1}\gamma_s y) Q_{yS^u_{y^{-1},1}}^{C_{G_x}(\gamma_s)}(\gamma_u), \tag{15}$$

where $Q_{yS^u_{y^{-1},1}}^{C_{G_x}(\gamma_s)}(\gamma_u)$ is the Green function.

We remark that the character formulas in *loc. cit.* Propositions 3.4.23 and 3.4.24 are valid for elements in G_S , and the formula (15) is valid for elements in $G_S(k_F)$. The proof of (15) can be read off from that of *loc. cit.* Propositions 3.4.23 and 3.4.24.

From now on, for convenience, we denote $G = G_x$. Passage from $G_S(k_F)$ to M , for $\gamma \in M$ we have Jordan decomposition $\gamma = \gamma_s \gamma_u$ with $\gamma_s \in G_S(k_F)/Z(F)^\theta$ and $\gamma_u \in G(k_F)_{\text{unip}}$. Since $\theta(\gamma) = \gamma$, by the uniqueness of Jordan decomposition, it has to be $\gamma_s \in M$ and $\gamma_u \in G(k_F)_{\text{unip}}^\theta$. Put

$$\bar{S}(k_F) = S(F)/Z(F)S(F)_{0+}.$$

We denote by M_{ss} the semisimple part of M . Set

$$\chi = \eta^{-1}.$$

The following computation of m is a modification of that in the proof of [11, Theorem 3.11] which is based on the proof of the main result of [25, Theorem 3.3]. First, by the Jordan decomposition, we have

$$\begin{aligned} m &= \frac{1}{|M|} \sum_{\gamma_s \gamma_u \in M} \Theta(\gamma_s \gamma_u) \chi(\gamma_s \gamma_u) \\ &= \frac{1}{|M|} \sum_{\gamma_s \in M_{\text{ss}}} \chi(\gamma_s) \sum_{\gamma_u \in C_G(\gamma_s)(k_F) \cap G(k_F)_{\text{unip}}^\theta} \Theta(\gamma_s \gamma_u) \\ &= \frac{\sigma(G)\sigma(S^u)}{|M|} \sum_{\gamma_s \in M_{\text{ss}}} \chi(\gamma_s) \sum_{\gamma_u \in C_G(\gamma_s)(k_F) \cap G(k_F)_{\text{unip}}^\theta} \frac{1}{|C_G(\gamma_s)(k_F)|} \\ &\quad \cdot \sum_{\substack{y \in G(k_F) \\ y^{-1}\gamma_s y \in \bar{S}(k_F)}} \mu(y^{-1}\gamma_s y) Q_{yS^u y^{-1}, 1}^{C_G(\gamma_s)}(\gamma_u) \\ &= \frac{\sigma(G)\sigma(S^u)}{|M|} \sum_{\gamma_s \in M_{\text{ss}}} \sum_{\substack{y \in G(k_F) \\ y^{-1}\gamma_s y \in \bar{S}(k_F)}} \frac{\mu(y^{-1}\gamma_s y) \chi(\gamma_s)}{|C_G(\gamma_s)(k_F)|} \\ &\quad \cdot \sum_{\gamma_u \in C_G(\gamma_s)(k_F) \cap G(k_F)_{\text{unip}}^\theta} Q_{yS^u y^{-1}, 1}^{C_G(\gamma_s)}(\gamma_u). \end{aligned}$$

By [25, Theorem 3.4], we have

$$\begin{aligned} &\sum_{\gamma_u \in C_G(\gamma_s)(k_F) \cap G(k_F)_{\text{unip}}^\theta} Q_{yS^u y^{-1}, 1}^{C_G(\gamma_s)}(\gamma_u) \\ &= \frac{\sigma(S^u)}{|S^u(k_F)|} \sum_{\substack{g \in C_G(\gamma_s)(k_F) \\ (y^{-1}g \cdot \theta)(S^u) = S^u}} \sigma\left(C_{C_G(\gamma_s)}\left((y^{-1}gS^u)^{\theta, \circ}\right)\right). \end{aligned}$$

Therefore

$$m = \frac{\sigma(\mathbf{G})}{|\mathbf{M}| \cdot |\mathbf{S}^u(k_F)|} \sum_{\gamma \in \mathbf{M}_{\text{ss}}} \sum_{\substack{y \in G(k_F) \\ y^{-1}\gamma y \in \bar{\mathbf{S}}(k_F)}} \frac{\mu(y^{-1}\gamma y)\chi(\gamma)}{|C_G(\gamma)(k_F)|} \cdot \sum_{\substack{g \in C_G(\gamma)(k_F) \\ (y^{-1}g \cdot \theta)(\mathbf{S}^u) = \mathbf{S}^u}} \sigma\left(C_{C_G(\gamma)}\left((y^{-1}g\mathbf{S}^u)^{\theta, \circ}\right)\right)$$

Changing variables $y^{-1}\gamma y \mapsto \gamma_1$ and $y^{-1}g \mapsto y_1$, we obtain

$$m = \frac{\sigma(\mathbf{G})}{|\mathbf{M}| \cdot |\mathbf{S}^u(k_F)|} \sum_{\substack{(\gamma_1, y_1) \in \bar{\mathbf{S}}(k_F) \times G(k_F) \\ y_1^{-1}\gamma_1 y_1 \in \mathbf{M} \\ (y_1 \cdot \theta)(\mathbf{S}^u) = \mathbf{S}^u}} \mu(\gamma_1)\chi(y_1^{-1}\gamma_1 y_1)\sigma\left(C_{C_G(y_1^{-1}\gamma_1 y_1)}\left((y_1\mathbf{S}^u)^{\theta, \circ}\right)\right).$$

For each $y_1 \in G(k_F)$ in the above summation, we choose an arbitrary lift $\dot{y}_1 \in G(F)_{x,0}$ of y_1 . The maximal torus of \mathbf{G} which corresponds to $\dot{y}_1\mathbf{S}$ is ${}^{\dot{y}_1}\mathbf{S}^u$. Hence by Lemma 3.2 there exists a θ -stable torus S_1 which is $G(F)_{x,0+}$ -conjugate to ${}^{\dot{y}_1}\mathbf{S}$ and thus $G(F)_{x,0}$ -conjugate to \mathbf{S} . We have $S_1(k_F) = y_1^{-1}\mathbf{S}(k_F)y_1$ which is θ -stable. Denote by $\varepsilon_{y_1\mathbf{S}}$ the character ε_{S_1} defined by (8). Note that $\varepsilon_{y_1\mathbf{S}}$ is well defined since it is independent of the choices of \dot{y}_1 and S_1 . According to the definition of the character $\varepsilon_{y_1\mathbf{S}}$, we have

$$\begin{aligned} \sigma\left(C_{C_G(y_1^{-1}\gamma_1 y_1)}\left((y_1\mathbf{S}^u)^{\theta, \circ}\right)\right) &= \sigma\left(C_G\left((y_1\mathbf{S}^u)^{\theta, \circ}\right) \cap C_G\left(y_1^{-1}\gamma_1 y_1\right)\right) \\ &= \varepsilon_{y_1\mathbf{S}}(y_1^{-1}\gamma_1 y_1)\sigma\left(C_G\left((y_1\mathbf{S}^u)^{\theta, \circ}\right)\right). \end{aligned}$$

Thus,

$$m = \frac{\sigma(\mathbf{G})}{|\mathbf{M}| \cdot |\mathbf{S}^u(k_F)|} \sum_{\substack{(\gamma_1, y_1) \in \bar{\mathbf{S}}(k_F) \times G(k_F) \\ y_1^{-1}\gamma_1 y_1 \in \mathbf{M} \\ (y_1 \cdot \theta)(\mathbf{S}^u) = \mathbf{S}^u}} \mu(\gamma_1)\chi(y_1^{-1}\gamma_1 y_1)\varepsilon_{y_1\mathbf{S}}(y_1^{-1}\gamma_1 y_1)\sigma\left(C_G\left((y_1\mathbf{S}^u)^{\theta, \circ}\right)\right).$$

Changing variables $y_1^{-1}\gamma_1 y_1 \mapsto \gamma_2$, we get

$$m = \frac{\sigma(\mathbf{G})}{|\mathbf{M}| \cdot |\mathbf{S}^u(k_F)|} \sum_{\substack{y_1 \in G(k_F) \\ (y_1 \cdot \theta)(\mathbf{S}^u) = \mathbf{S}^u}} \sigma\left(C_G\left((y_1\mathbf{S}^u)^{\theta, \circ}\right)\right) \cdot \sum_{\gamma_2 \in \left(y_1^{-1}\bar{\mathbf{S}}(k_F)y_1\right) \cap \mathbf{M}} ({}^{y_1}\mu)(\gamma_2)\chi(\gamma_2)\varepsilon_{y_1\mathbf{S}}(\gamma_2).$$

The term

$$m_{y_1} := \sum_{\gamma_2 \in (y_1^{-1} \bar{S}(k_F) y_1) \cap M} ({}^{y_1} \mu)(\gamma_2) \chi(\gamma_2) \varepsilon_{y_1 S}(\gamma_2)$$

is a positive integer precisely when

$${}^{y_1} \mu|_{({}^{y_1} S)^\theta(k_F)} = \varepsilon_{y_1 S} \cdot \eta|_{({}^{y_1} S)^\theta(k_F)}, \tag{16}$$

which is equivalent to

$$\mu_1|_{S_1^\theta(F)} = \varepsilon_{S_1} \cdot \eta|_{S_1^\theta(F)}, \tag{17}$$

where $\mu_1 = {}^g \mu$ and $g \in G(F)_{x,0}$ is such that $S_1 = {}^g S$. Otherwise m_{y_1} is zero. At this moment, we have proved the first assertion of the proposition.

To prove the second assertion, first note that the relation (16) implies that

$$\mu_1|_{(S_1^u)^\theta(k_F)} = \varepsilon_{S_1^u} \cdot \eta|_{(S_1^u)^\theta(k_F)}. \tag{18}$$

Therefore, according to (7) and the condition of the second assertion, (18) implies that

$$\mu_1|_{(S_1^{u,\circ})^\theta(k_F)} = 1. \tag{19}$$

By [25, Lemmas 10.4 and 10.5] and its slight generalization [8, Lemma 8.1], the condition (19) implies that

$$\sigma(C_G((S_1^u)^\theta, \circ)) = \sigma(G).$$

Therefore the multiplicity m is equal to

$$\frac{1}{|M| \cdot |S^u(k_F)|} \sum_{\substack{y \in G(k_F) \\ (y \cdot \theta)(S^u) = S^u}} m_y,$$

which implies the second assertion of the proposition directly. □

3.2.3 Proof of Theorem 3.16

Now let $\pi = \pi_{(S, \mu)}$ be a regular supercuspidal representation of $G(F)$ and $\Psi = (\bar{G}, \pi_{(S, \mu_\circ)}, \vec{\phi})$ a Howe factorization of (S, μ) . Before proving Theorem 3.16, let us remind the reader of the following notation that will be frequently used:

- $\phi = \prod_{i=0}^d \phi_i$, a character of $G^0(F)$,
- $x \in \mathcal{B}^{\text{red}}(G^0, F)$ is the vertex determined by S ,
- $K^0 = G^0(F)_x$ and $G_S^0 = S(F)G^0(F)_{x,0}$.

- $\kappa := \kappa_{(S, \mu_\circ)}$ and $\tilde{\kappa} := \tilde{\kappa}_{(S, \mu_\circ)}$ are representations of $G^0(F)_{x,0}$ and G_S^0 respectively, which are constructed from the depth-zero tame regular elliptic pair (S, μ_\circ) for G^0 , as described in Sect. 2.2.3, except that G is replaced by G^0 ,
- $\rho := \rho_{(S, \mu_\circ)} = \text{ind}_{G_S^0}^{K^0} \tilde{\kappa}_{(S, \mu_\circ)}$,
- we will abuse the notation to also denote by Ψ the generic cuspidal G -datum $(\vec{G}, x, \rho, \vec{\phi})$,
- $\rho_{\text{nm}} := \rho \otimes (\phi|_{K^0})$,
- $\tilde{\eta}_\theta := \eta_\theta \cdot (\phi^{-1}|_{K^{0,\theta}})$ if $[\theta] \sim [\Psi]$.

Sufficient condition Let us first prove the sufficient condition of Theorem 3.16. According to Lemma 2.5 we can and do assume that the tame regular elliptic pair (S, μ) is $(\theta, \varepsilon\eta)$ -symmetric, which implies that $[\theta] \sim [\Psi]$ by Corollary 3.13. By Lemma 3.12 we can further assume that $\phi|_{(S^u)^{\theta,\circ}(F)} = 1$. To show that π is H -distinguished, according to Theorem 2.9, it suffices to show

$$\text{Hom}_{K^{0,\theta}}(\rho_{\text{nm}}, \eta_\theta) \neq 0,$$

which is equivalent to

$$\text{Hom}_{K^{0,\theta}}(\rho, \tilde{\eta}_\theta) \neq 0.$$

Since $\rho = \text{ind}_{G_S^0}^{K^0} \tilde{\kappa}$, applying Mackey theory, we have

$$\text{Hom}_{K^{0,\theta}}(\rho, \tilde{\eta}_\theta) \cong \bigoplus_{g \in G_S^0 \backslash K^0 / K^{0,\theta}} \text{Hom}_{G_S^0 \cap g K^{0,\theta}}(\tilde{\kappa}, {}^g \tilde{\eta}_\theta). \tag{20}$$

Consider the case when $g = 1$. Note that $G_S^0 \cap K^{0,\theta} = G_S^{0,\theta}$. The condition that (S, μ) is $(\theta, \varepsilon\eta)$ -symmetric implies that (S, μ_\circ) is $(\theta, \varepsilon \cdot \tilde{\eta}_\theta)$ -symmetric in the sense of Sect. 3.2.2. By Proposition 3.23, Lemma 3.9 and the condition $\phi|_{(S^u)^{\theta,\circ}(F)} = 1$, we obtain that

$$\text{Hom}_{G_S^{0,\theta}}(\tilde{\kappa}, \tilde{\eta}_\theta) \neq 0.$$

Thus $\text{Hom}_{K^{0,\theta}}(\rho_{\text{nm}}, \eta_\theta)$ is non-zero.

Necessary condition Now let us turn to proving the necessary condition of Theorem 3.16. According to Theorem 2.9, Theorem 2.1 and Lemma 2.5, we can and do assume that the generic cuspidal G -datum Ψ associated to a Howe factorization of (S, μ) satisfies $[\theta] \sim [\Psi]$ and

$$\text{Hom}_{K^{0,\theta}}(\rho_{\text{nm}}, \eta_\theta) = \text{Hom}_{K^{0,\theta}}(\rho, \tilde{\eta}_\theta) \neq 0.$$

Due to the isomorphism (20), there exists $k \in K^0$ such that

$$\text{Hom}_{G_S^0 \cap k K^{0,\theta}}(\tilde{\kappa}, {}^k \tilde{\eta}_\theta) \neq 0. \tag{21}$$

Set $\theta' = k^{-1} \cdot \theta$. Then

$${}^k K^{0,\theta} = K^{0,\theta'} \quad \text{and} \quad {}^k \tilde{\eta}_\theta = \tilde{\eta}_{\theta'}.$$

Since $k \in K^0$, we have $[\theta'] = [\theta] \sim [\Psi]$, which implies that $G^0(F)_{x,0}$ is θ' -stable. Therefore,

$$G^0(F)_{x,0}^{\theta'} \subseteq G_S^0 \cap K^{0,\theta'}.$$

Recall that $\tilde{\kappa}$ is an extension of κ . Hence, by (21), we have

$$\text{Hom}_{G^0(F)_{x,0}^{\theta'}}(\kappa, \tilde{\eta}_{\theta'}) \neq 0. \tag{22}$$

By [14, Proposition 2.12], we see $G^0(F)_{x,0}^{\theta'}/G^0(F)_{x,0+}^{\theta'} = G_x^0(k_F)^{\theta'}$. Hence (22) is equivalent to

$$\text{Hom}_{G_x^0(k_F)^{\theta'}}(\kappa, \tilde{\eta}_{\theta'}) \neq 0.$$

Recall that $\kappa = \pm R_{S^u, \bar{\mu}_o}$. By Lemma 3.21 we know that there exists $\bar{y} \in G_x^0(k_F)$ such that $\bar{y}S^u$ is θ' -stable. Thus, by Lemma 3.2, there exists $y \in G^0(F)_{x,0}$ such that ${}^y S$ is θ' -stable. Note that G_S^0 is also equal to ${}^y S(F)G^0(F)_{x,0}$, which implies that G_S^0 is θ' -stable. We deduce from (21) that

$$\text{Hom}_{G_S^{0,\theta'}}(\tilde{\kappa}, \tilde{\eta}_{\theta'}) \neq 0.$$

According to Proposition 3.23, there exists $z \in G^0(F)_{x,0}$ such that ${}^z S$ is θ' -stable and

$$\begin{aligned} {}^z \mu_\circ|_{zS^{\theta'}(F)} &= \varepsilon_{zS} \cdot \tilde{\eta}_{\theta'}|_{zS^{\theta'}(F)} \\ &= \varepsilon_{zS} \cdot \eta_{\theta'} \cdot \phi^{-1}|_{zS^{\theta'}(F)}, \end{aligned}$$

where the character ε_{zS} is defined with respect to the involution θ' . Therefore we have

$${}^z \mu|_{zS^{\theta'}(F)} = \varepsilon_{zS} \cdot \eta_{\theta'}|_{zS^{\theta'}(F)}.$$

Recall that $\theta' = k' \cdot \theta$ with $k' = k^{-1} \in K^0$. Since ${}^z S$ is θ' -stable, we have that ${}^{zk'} S$ is θ -stable and

$${}^{zk'} \mu|_{zk'S^{\theta}(F)} = k' \varepsilon_{zS} \cdot k' \eta_{\theta'}|_{zk'S^{\theta}(F)}.$$

It is easy to see that

$$k' \eta_{\theta'} = \eta_\theta,$$

and

$${}^{k'}\varepsilon_{zS} = \varepsilon_{zk'S}$$

where the latter character is defined with respect to the involution θ . In summary we conclude that $({}^{zk'}S, {}^{zk'}\mu)$ is $(\theta, \varepsilon\eta)$ -symmetric.

4 Functoriality

In this section, we assume that F has characteristic zero. Let G be a connected tamely ramified quasi-split reductive group over F , and \widehat{G} the complex Langlands dual group of G . We fix Γ -invariant splittings $(T, B, \{X_\alpha\})$ of G and $(\widehat{T}, \widehat{B}, \{\widehat{X}_{\widehat{\alpha}}\})$ of \widehat{G} . Recall that a splitting of a reductive group G , also called a pinning, is a triple $(T, B, \{X_\alpha\})$ where T is maximal torus of G , $B \supset T$ is a Borel subgroup, and X_α is a non-zero vector in the root subspace \mathfrak{g}_α where α runs over the set of simple B -positive roots. Let ${}^L G = \widehat{G} \rtimes W_F$ be the Weil-form L -group.

4.1 Regular supercuspidal L -packets

In this subsection, we recall Kaletha’s construction [22, §5] of the compound L -packets Π_φ for regular supercuspidal L -parameters φ .

4.1.1 Regular supercuspidal L -parameters and L -packet data

The following Definitions 4.1 and 4.2 were introduced by Kaletha. See [22, Lemma 5.2.2, Definitions 5.2.3 and 5.2.4] and also the paragraph of *loc. cit.* below Definition 5.2.3.

Definition 4.1 We call a discrete L -parameter $\varphi : W_F \rightarrow {}^L G$ *regular supercuspidal* if it satisfies:

- (1) $\varphi(P_F)$ is contained in a torus of \widehat{G} .
- (2) $C := C_{\widehat{G}}(\varphi(I_F))$ is a torus.
- (3) If $n \in N(\widehat{T}, \widehat{M})$ projects onto a non-trivial element of $\Omega(\widehat{S}, \widehat{M})^\Gamma$, then n does not belong to the centralizer of $\varphi(I_F)$ in \widehat{G} . Here we set \widehat{M} to be $C_{\widehat{G}}(\varphi(P_F))$, \widehat{T} to be $C_{\widehat{M}}(C)$, and \widehat{S} to be the Γ -module with underlying abelian group \widehat{T} and the Γ -action given by $\text{Ad}(\varphi(-))$.

Definition 4.2 We call a 4-tuple $(S, \widehat{j}, \chi, \mu)$ a *regular supercuspidal L -packet datum* if it satisfies:

- (1) S is a torus over F of dimension equal to the absolute rank of G and splits over a tame extension of F ,
- (2) $\widehat{j} : \widehat{S} \rightarrow \widehat{G}$ is an embedding of complex reductive groups, whose \widehat{G} -conjugacy class is Γ -stable. Then \widehat{j} gives rise to a Γ -stable G -conjugacy class J of *admissible embeddings* $S \rightarrow G$. Choose a Γ -fixed element $j \in J$, which is defined over F ,

and identify S with its image $j(S)$ in G . We require that $S/Z(G)$ is anisotropic, which means that S is a tame elliptic maximal torus of G ,

- (3) μ is a character of $S(F)$ such that (S, μ) is a tame *extra* regular elliptic pair for G . The character μ determines a tamely ramified twisted Levi subgroup G^0 of G and a subgroup $\Omega(S, G^0)$ of $\Omega(S, G)$,
- (4) χ is $\Omega(S, G^0)(F)$ -invariant *minimally ramified* χ -data for $R(S, G)$.

We have to explain the terminology in Definition 4.2. The notion admissible embeddings is standard and is reviewed in [22, §5.1]. For a tame regular elliptic pair (S, μ) of G , it is called a tame extra regular elliptic pair if the stabilizer of $\mu|_{S(F)_0}$ in $\Omega(S, G^0)(F)$ is trivial. A set of χ -data is called minimally ramified if $\chi_\alpha = 1$ for asymmetric α , χ_α is unramified for unramified symmetric α , and χ_α is tamely ramified for ramified symmetric α .

Remark 4.3 We can define morphisms, which are indeed isomorphisms, between regular supercuspidal L -packet data. This enables us to view the set of regular supercuspidal L -packet data as a category. See *loc. cit.* Definition 5.2.5 for more details.

The relation between regular supercuspidal L -packet data and regular supercuspidal parameter is discussed in [22, Proposition 5.2.7]. First, given a regular supercuspidal L -packet datum $(S, \widehat{j}, \chi, \mu)$, let

$$\varphi_{S, \mu} : W_F \rightarrow {}^L S$$

be the Langlands parameter corresponding to the character μ of $S(F)$, and

$${}^L j_\chi : {}^L S \rightarrow {}^L G$$

the L -embedding extending \widehat{j} that is determined by the χ -data χ . Set $\varphi = {}^L j_\chi \circ \varphi_{S, \mu}$. It is shown that φ is a regular supercuspidal parameter. Conversely, given a regular supercuspidal parameter φ , Kaletha proved that there exists a regular supercuspidal L -packet datum $(S, \widehat{j}, \chi, \mu)$ such that $\varphi = {}^L j_\chi \circ \varphi_{S, \mu}$. The following is *loc. cit.* Proposition 5.2.7.

Proposition 4.4 *The above process provides an 1-1 correspondence between the isomorphism classes of regular supercuspidal L -packet data and the \widehat{G} -conjugacy classes of regular supercuspidal parameters.*

4.1.2 Rigid inner twists

Kaletha [20, §3.2] defined a set $Z^1(u \rightarrow W, Z \rightarrow G)$ of cocycles and a cohomology set $H^1(u \rightarrow W, Z \rightarrow G)$ for any finite central subgroup Z of G , where u is a multiplicative pro-algebraic group and W a fixed extension of Γ by u . We have natural maps

$$Z^1(u \rightarrow W, Z \rightarrow G) \rightarrow Z^1(\Gamma, G/Z) \rightarrow Z^1(\Gamma, G_{\text{ad}})$$

and

$$H^1(u \rightarrow W, Z \rightarrow G) \rightarrow H^1(\Gamma, G/Z) \rightarrow H^1(\Gamma, G_{\text{ad}}),$$

which are induced by the projection $G \rightarrow G/Z$. Recall that an inner twist $G \rightarrow G'$ of G gives rise to a cocycle $z \in Z^1(\Gamma, G_{\text{ad}})$, and the set of isomorphism classes of inner twists is parameterized by $H^1(\Gamma, G_{\text{ad}})$. We call the triple (G', ξ, z) a *rigid inner twist* of G if $\xi : G \rightarrow G'$ is an inner twist and z is a cocycle in $Z^1(u \rightarrow W, Z \rightarrow G)$ for some Z such that ξ corresponds to the image of z in $Z^1(\Gamma, G_{\text{ad}})$. The set of isomorphism classes of rigid inner twists given by $Z^1(u \rightarrow W, Z \rightarrow G)$ is parameterized by $H^1(u \rightarrow W, Z \rightarrow G)$.

4.1.3 Regular supercuspidal data and L -packets

Definition 4.5 We call a tuple $(S, \widehat{j}, \chi, \mu, (G', \xi, z), j)$ a *regular supercuspidal datum* if it satisfies:

- (1) $(S, \widehat{j}, \chi, \mu)$ is a regular supercuspidal L -packet datum,
- (2) (G', ξ, z) is a rigid inner twist of G ,
- (3) $j : S \rightarrow G'$ is an admissible embedding over F with respect to \widehat{j} .

Remark 4.6 The above definition is [22, Definition 5.3.2]. We can also make the set of regular supercuspidal data being a category. There is a natural forgetful functor from it onto the category of regular supercuspidal L -packet data. Given a regular supercuspidal L -packet datum $(S, \widehat{j}, \chi, \mu)$, the set of isomorphism classes of regular supercuspidal data mapping to it is a torsor under $H^1(u \rightarrow W, Z \rightarrow S)$. See *loc. cit.* Definition 5.3.3 and the paragraph below it.

According to [22, §5.3], for a regular supercuspidal datum $(S, \widehat{j}, \chi, \mu, (G', \xi, z), j)$, in order to define the associated L -packet, we have to first modify it to be a proper datum $(S, \widehat{j}, \chi^{\text{new}}, \mu^{\text{new}}, (G', \xi, z), j)$ which is isomorphic to $(S, \widehat{j}, \chi, \mu, (G', \xi, z), j)$. We refer to *loc. cit.*, especially pages 1076 and 1153, for the reason of this modification, and refer to *loc. cit.* Steps 1 and 2 in §5.3 for the definitions of χ^{new} and μ^{new} . The following two definitions are given at *loc. cit.* page 1154.

Definition 4.7 Given a regular supercuspidal datum $(S, \widehat{j}, \chi, \mu, (G', \xi, z), j)$, we set $\pi_{(S_j, \mu_j)}$ to be the regular supercuspidal representation of $G'(F)$ associated to the tame regular elliptic pair (S_j, μ_j) where

- S_j is the image of S in G' under j ,
- $\mu_j := (\mu^{\text{new}} \circ j^{-1}) \cdot \epsilon_{f, \text{ram}} \cdot \epsilon^{\text{ram}}$ is a character of $S_j(F)$. Here $\epsilon_{f, \text{ram}}$ and ϵ^{ram} are certain quadratic characters of $S_j(F)$, whose definitions are given in [22, Definition 4.7.3 and (4.3.3)] respectively.

To avoid confusion, we will also write $\epsilon_{f, \text{ram}, S_j}$ and $\epsilon_{S_j}^{\text{ram}}$ instead of $\epsilon_{f, \text{ram}}$ and ϵ^{ram} respectively to indicate that they are characters of $S_j(F)$.

Definition 4.8 Let φ be a regular supercuspidal L -parameter and $(S, \widehat{j}, \chi, \mu)$ a regular supercuspidal L -datum corresponding to φ . For each rigid inner twist (G', ξ, z) , we define the L -packet $\Pi_\varphi(G')$ to be

$$\Pi_\varphi(G') = \{\pi_j\}$$

where $(S, \widehat{j}, \chi, \mu, (G', \xi, z), j)$ runs over the set of isomorphism classes of regular supercuspidal data mapping to $(S, \widehat{j}, \chi, \mu)$ and $\pi_j := \pi_{(S_j, \mu_j)}$. We define the compound L -packet Π_φ to be the disjoint union

$$\Pi_\varphi = \bigsqcup \Pi_\varphi(G')$$

where (G', ξ, z) runs over the set of isomorphism classes of rigid inner twists of G .

4.2 Twisted regular supercuspidal L -packets

Now let θ be an involution of G and $H = G^\theta$. We denote by $\widehat{\theta}$ the involution of \widehat{G} dual to θ with respect to the fixed splittings. Note that $\widehat{\theta}$ commutes with the action of Γ on \widehat{G} and can be extended to an L -automorphism ${}^L\theta := \widehat{\theta} \times \text{id}_{W_F}$ of ${}^L G$. We refer to [24, §1.8] for the existence and the basic properties of $\widehat{\theta}$. We fix a regular supercuspidal L -parameter φ for G .

4.2.1 Twisted regular supercuspidal L -parameters and L -packet data

Suppose that S is a maximal torus of G , and $\chi = (\chi_\alpha)_{\alpha \in R(S, G)}$ is χ -data for $R(S, G)$. For $\alpha \in R(S, G)$, set $\theta(\alpha) = \alpha \circ \theta$, which is an algebraic character of $\theta(S)$. Then $\theta(\alpha)$ is in $R(\theta(S), G)$, whose root space is $\theta(\mathfrak{g}_\alpha)$. Hence we obtain a 1-1 correspondence

$$R(S, G) \longleftrightarrow R(\theta(S), G), \quad \alpha \leftrightarrow \theta(\alpha).$$

Since θ is defined over F , we have $\Gamma_\alpha = \Gamma_{\theta(\alpha)}$, and thus $F_\alpha = F_{\theta(\alpha)}$ and $F_{\pm\alpha} = F_{\pm\theta(\alpha)}$ for any $\alpha \in R(S, G)$. Therefore $\theta(\chi) := (\chi_{\theta(\alpha)})_{\alpha \in R(S, G)}$ is χ -data for $R(\theta(S), G)$, where $\chi_{\theta(\alpha)} : F_{\theta(\alpha)}^\times \rightarrow \mathbb{C}^\times$ is the character χ_α by identifying $F_{\theta(\alpha)} = F_\alpha$.

Lemma 4.9 *The L -parameter ${}^L\theta \circ \varphi$ is regular supercuspidal. Moreover, if $(S, \widehat{j}, \chi, \mu)$ is a regular supercuspidal L -packet datum corresponding to φ , then $(S, \widehat{\theta} \circ \widehat{j}, \theta(\chi), \mu)$ is a regular supercuspidal L -packet datum corresponding to ${}^L\theta \circ \varphi$.*

Proof The first assertion, that ${}^L\theta \circ \varphi$ is a regular supercuspidal L -parameter, can be easily verified by checking the definition.

For the second assertion, it is harmless to assume that the fixed splitting of \widehat{G} satisfies $\widehat{j}(\widehat{S}) = \widehat{T}$. Let $j : S \rightarrow G$ be an admissible embedding over F with respect to \widehat{j} . Then $\theta \circ j : S \rightarrow G$ is an admissible embedding over F with respect to $\widehat{\theta} \circ \widehat{j} : \widehat{S} \rightarrow \widehat{G}$. We view S as a tame regular elliptic maximal torus of G by the embedding j . Then

$(\theta(S), \mu \circ \theta)$ is a tame extra regular elliptic pair for G , and $\theta(\chi)$ is $\Omega(\theta(S), \theta(G^0))$ -invariant minimally ramified χ -data for $R(\theta(S), G)$. In summary, $(S, \widehat{\theta} \circ \widehat{j}, \theta(\chi), \mu)$ is a regular supercuspidal L -packet datum. It is routine to check that

$${}^L\theta \circ {}^Lj_\chi = {}^Lj_{\theta(\chi)}$$

where ${}^Lj_{\theta(\chi)}$ is the L -embedding ${}^LS \rightarrow {}^LG$ extending $\widehat{\theta} \circ \widehat{j}$ that is determined by the χ -data $\theta(\chi)$. Therefore

$${}^L\theta \circ \varphi = {}^Lj_{\theta(\chi)} \circ \varphi_{S,\mu},$$

and thus $(S, \widehat{\theta} \circ \widehat{j}, \theta(\chi), \mu)$ corresponds to ${}^L\theta \circ \varphi$. □

4.2.2 Rigid inner twists of symmetric spaces

Definition 4.10 (1) Let (G', ξ, z) be a rigid inner twist of G . We call (G', ξ, z) a *rigid inner twist of (G, H, θ)* if z lies in the image of $Z^1(u \rightarrow W, Z \rightarrow H)$ in $Z^1(u \rightarrow W, Z \rightarrow G)$.

(2) Let (G', ξ, z) be a rigid inner twist of (G, H, θ) . We define an involution θ' of G' by

$$\theta' = \xi \circ \theta \circ \xi^{-1}.$$

Lemma 4.11 *The involution θ' is defined over F .*

Proof Let \bar{z} be the image of z in $Z^1(\Gamma, G/Z)$, which is viewed as an element in $Z^1(\Gamma, H/Z)$ by the condition imposed on z . Then for any $\sigma \in \Gamma$ we have

$$\begin{aligned} \sigma \circ \theta' &= \sigma \circ \xi \circ \theta \circ \xi^{-1} \\ &= \xi \circ \text{Int}(\bar{z}_\sigma) \circ \sigma \circ \theta \circ \xi^{-1} \\ &= \xi \circ \text{Int}(\theta(\bar{z}_\sigma)) \circ \theta \circ \sigma \circ \xi^{-1} \\ &= \xi \circ \theta \circ \text{Int}(\bar{z}_\sigma) \circ \sigma \circ \xi^{-1} \\ &= \xi \circ \theta \circ \xi^{-1} \circ \sigma \\ &= \theta' \circ \sigma. \end{aligned}$$

Therefore θ' is defined over F . □

Remark 4.12 Note that, according to the definition of θ' , we have

$$\theta' \circ \xi = \xi \circ \theta.$$

Let $H' = (G')^{\theta'}$. Then we have

$$H'(\bar{F}) = G'(\bar{F})^{\theta'} = (\xi(G(\bar{F})))^{\theta'} = \xi(G(\bar{F})^\theta) = \xi(H(\bar{F})).$$

Thus the restriction of (ξ, z) onto H gives rise to a rigid inner twist $\xi_H : H \rightarrow H'$. If (G', ξ, z) is clear, we also call (G', H', θ') a rigid inner twist of (G, H, θ) .

4.2.3 Twisted regular supercuspidal L -packets

Definition 4.13 Let φ be a regular supercuspidal L -parameter and $(S, \widehat{j}, \chi, \mu)$ a regular supercuspidal L -datum corresponding to φ . For each rigid inner twist (G', H', θ') of (G, H, θ) , we define the *twisted L -packet* $\Pi_\varphi^\theta(G')$ to be

$$\Pi_\varphi^\theta(G') = \{\pi_j \circ \theta' | \pi_j \in \Pi_\varphi(G')\},$$

and define the *compound twisted L -packet* $\Pi_\varphi^{\theta, \circ}$ to be the disjoint union

$$\Pi_\varphi^{\theta, \circ} = \bigsqcup \Pi_\varphi^{\theta, \circ}(G')$$

where (G', H', θ') runs over the set of isomorphism classes of rigid inner twists of (G, H, θ) .

The way we define the twisted L -packet $\Pi_\varphi^{\theta, \circ}$ is on the level of representations, that is, we twist the representations in the L -packets by involutions. It is natural to ask whether the twisted L -packet $\Pi_\varphi^{\theta, \circ}$ is indeed a compound L -packet in some sense. The answer is yes. More precisely we have:

Proposition 4.14 *For each rigid inner twist (G', H', θ') we have*

$$\Pi_\varphi^\theta(G') = \Pi_{L_{\theta \circ \varphi}}(G').$$

Therefore we have $\Pi_\varphi^{\theta, \circ} \subseteq \Pi_{L_{\theta \circ \varphi}}$.

Proof Let $(S, \widehat{j}, \chi, \mu)$ be a regular supercuspidal L -datum corresponding to φ and $(S, \widehat{j}, \chi, \mu, (G', \xi, z), j)$ a regular supercuspidal datum such that (G', ξ, z) is a rigid inner twist of (G, H, θ) . According to Lemma 4.9, $(S, \widehat{\theta} \circ \widehat{j}, \theta(\chi), \mu)$ is a regular supercuspidal L -datum corresponding to ${}^L\theta \circ \varphi$. Choose a Γ -fixed admissible embedding $j_0 : S \rightarrow G$ with respect to \widehat{j} . Then $\theta \circ j_0 : S \rightarrow G$ is a Γ -fixed admissible embedding with respect to $\widehat{\theta} \circ \widehat{j}$. Since $j : S \rightarrow G'$ is admissible, there exists $g \in G$ such that $j = \xi \circ \text{Int}(g) \circ j_0$. We have

$$\begin{aligned} \theta' \circ j &= \theta' \circ \xi \circ \text{Int}(g) \circ j_0 \\ &= \xi \circ \theta \circ \text{Int}(g) \circ j_0 \\ &= \xi \circ \text{Int}(\theta(g)) \circ (\theta \circ j_0). \end{aligned}$$

Hence $\theta' \circ j : S \rightarrow G'$ is indeed an admissible embedding with respect to $\widehat{\theta} \circ \widehat{j}$. Therefore, for those rigid inner twists (G', ξ, z) of (G, H, θ) , the map

$$(S, \widehat{j}, \chi, \mu, (G', \xi, z), j) \mapsto (S, \widehat{\theta} \circ \widehat{j}, \theta(\chi), \mu, (G', \xi, z), \theta' \circ j)$$

establishes an 1-1 correspondence between regular supercuspidal data for φ with regular supercuspidal data for ${}^L\theta \circ \varphi$. We remark that if $(S, \widehat{j}, \chi^{\text{new}}, \mu^{\text{new}}, (G', \xi, z), j)$ is the modified datum of $(S, \widehat{j}, \chi, \mu, (G', \xi, z), j)$, then $(S, \widehat{\theta} \circ \widehat{j}, \theta(\chi^{\text{new}}), \mu^{\text{new}}, (G', \xi, z), \theta' \circ j)$ equals to the modified datum of $(S, \widehat{\theta} \circ \widehat{j}, \theta(\chi), \mu, (G', \xi, z), \theta' \circ j)$. To prove the proposition, it remains to show that

$$\pi_{(S_j, \mu_j)} \circ \theta' \simeq \pi_{(S_{\theta' \circ j}, \mu_{\theta' \circ j})}.$$

First we have

$$\pi_{(S_j, \mu_j)} \circ \theta' \simeq \pi_{(\theta'(S_j), \mu_j \circ \theta')}.$$

As a character of $\theta'(S_j)(F)$,

$$\mu_j \circ \theta' = (\mu^{\text{new}} \circ j^{-1} \circ \theta') \cdot (\epsilon_{f, \text{ram}, S_j} \circ \theta') \cdot (\epsilon_{S_j}^{\text{ram}} \circ \theta').$$

On the other hand, we have

$$\mu_{\theta' \circ j} = (\mu^{\text{new}} \circ j^{-1} \circ \theta') \cdot \epsilon_{f, \text{ram}, \theta'(S_j)} \cdot \epsilon_{\theta'(S_j)}^{\text{ram}}.$$

According to the definition of $\epsilon_{f, \text{ram}}$ and ϵ^{ram} , and the correspondence $R(S, G') \leftrightarrow R(\theta'(S), G')$ established before, it is straightforward to check that

$$\epsilon_{f, \text{ram}, S_j} \circ \theta' = \epsilon_{f, \text{ram}, \theta'(S_j)} \quad \text{and} \quad \epsilon_{S_j}^{\text{ram}} \circ \theta' = \epsilon_{\theta'(S_j)}^{\text{ram}},$$

which completes the proof. □

4.3 Contragredient regular supercuspidal L -packets

For a general L -parameter φ for G , it is conjectured by Adams and Vogan [1] on the level of packets, and by Prasad [30] and Kaletha [17] on the level of representations, that the contragredient of the L -packet Π_φ itself should be an L -packet and there should exist an explicit relation between the refined L -parameters of π and π^\vee for $\pi \in \Pi_\varphi$. Kaletha [17, §5] showed that this conjecture holds when φ is *tame regular semisimple elliptic* or *epipelagic*. In this subsection, we give a proof of this conjecture for regular supercuspidal parameters, following the arguments of [17, §5] closely. From now on, we fix a regular supercuspidal L -parameter φ for G .

Definition 4.15 For each rigid inner twist (G', ξ, z) of G , the *contragredient L -packet* $\Pi_\varphi^\vee(G')$ is defined to be

$$\Pi_\varphi^\vee(G') = \left\{ \pi_j^\vee \mid \pi_j \in \Pi_\varphi(G') \right\},$$

and the *compound contragredient L -packet* Π_φ^\vee is defined to be the disjoint union

$$\Pi_\varphi^\vee = \bigsqcup \Pi_\varphi^\vee(G')$$

where (G', ξ, z) runs over the set of isomorphism classes of rigid inner twists of G .

We fix a Γ -invariant splitting $(\widehat{T}, \widehat{B}, \{X_{\widehat{\alpha}}\})$ for \widehat{G} . The Chevalley involution \widehat{C} of \widehat{G} is uniquely determined by the following conditions:

- $\widehat{C}(\widehat{T}) = \widehat{T}$ and $\widehat{C}|_{\widehat{T}} = -1$ where -1 denotes the inverse map,
- $\widehat{C}(\widehat{B}) = \widehat{B}^{\text{op}}$ where \widehat{B}^{op} is the opposite Borel of \widehat{B} ,
- $C(X_{\widehat{\alpha}}) = X_{-\widehat{\alpha}}$ for $\widehat{\alpha} \in R(\widehat{T}, \widehat{G})$.

Note that \widehat{C} commutes with the action of Γ on \widehat{G} . Thus we can extend \widehat{C} to an L -automorphism ${}^L C = \widehat{C} \times \text{id}_{W_F}$ of ${}^L G$.

Let $(S, \widehat{j}, \chi, \mu)$ be a regular supercuspidal L -packet datum corresponding to φ . We assume that $\widehat{j}(\widehat{S}) = \widehat{T}$. Note that ${}^L C \circ \varphi$ is also regular supercuspidal. For the χ -data $\chi = (\chi_{\alpha})$ we denote by χ^{-1} the χ -data (χ_{α}^{-1}) .

Lemma 4.16 *The 4-tuple $(S, \widehat{j}, \chi^{-1}, \mu^{-1})$ is a regular supercuspidal L -packet datum and corresponds to ${}^L C \circ \varphi$.*

Proof It is straightforward to check that $(S, \widehat{j}, \chi^{-1}, \mu^{-1})$ is a regular supercuspidal L -packet datum. On the other hand, by [17, Lemma 4.1] the following diagram is commutative:

$$\begin{array}{ccc}
 {}^L S & \xrightarrow{-1} & {}^L S \\
 {}^L j_{\chi} \downarrow & & \downarrow {}^L j_{\chi^{-1}} \\
 {}^L G & \xrightarrow{{}^L C} & {}^L G
 \end{array}$$

Therefore, since $\varphi = {}^L j_{\chi} \circ \varphi_{S, \mu}$, we have

$$\begin{aligned}
 {}^L C \circ \varphi &= {}^L j_{\chi^{-1}} \circ (-1) \circ \varphi_{S, \mu} \\
 &= {}^L j_{\chi^{-1}} \circ \varphi_{S, \mu^{-1}},
 \end{aligned}$$

where $\varphi_{S, \mu^{-1}} : W_F \rightarrow {}^L S$ is the L -parameter attached to the character μ^{-1} of $S(F)$. This implies that $(S, \widehat{j}, \chi^{-1}, \mu^{-1})$ corresponds to ${}^L C \circ \varphi$. □

Proposition 4.17 *We have $\Pi_{\varphi}^{\vee} = \Pi_{{}^L C \circ \varphi}$.*

Proof Let $(S, \widehat{j}, \chi, \mu, (G', \xi, z), j)$ be a regular supercuspidal datum of φ . As before, we remark that if $(S, \widehat{j}, \chi^{\text{new}}, \mu^{\text{new}}, (G', \xi, z), j)$ is the modified datum of $(S, \widehat{j}, \chi, \mu, (G', \xi, z), \widehat{j})$, then $(S, \widehat{j}, (\chi^{\text{new}})^{-1}, (\mu^{\text{new}})^{-1}, (G', \xi, z), j)$ equals to the modified datum of $(S, \widehat{j}, \chi^{-1}, \mu^{-1}, (G', \xi, z), j)$. According to [14, §3], we have

$$\pi_{(S_j, \mu_j)}^{\vee} \simeq \pi_{(S_j, (\mu_j)^{-1})}.$$

Recall that $\mu_j = (\mu^{\text{new}} \circ j^{-1}) \cdot \epsilon_{f, \text{ram}} \cdot \epsilon^{\text{ram}}$, and both $\epsilon_{f, \text{ram}}$ and ϵ^{ram} are quadratic characters of $S_j(F)$. Hence we have

$$(\mu_j)^{-1} = ((\mu^{\text{new}})^{-1} \circ j^{-1}) \cdot \epsilon_{f, \text{ram}} \cdot \epsilon^{\text{ram}} = (\mu^{-1})_j.$$

Therefore, the representation $\pi_{(S_j, (\mu^{-1})_j)}$, which is attached to the regular supercuspidal datum $(S, \widehat{j}, \chi^{-1}, \mu^{-1}, (G', \xi, z), j)$ for ${}^L C \circ \varphi$, is isomorphic to $\pi_{(S_j, \mu_j)}^\vee$. \square

4.4 Consequences

Let θ be an involution of G . Let φ be a regular depth-zero or an epipelagic supercuspidal L -parameter for G , which is in particular a regular supercuspidal parameter. Regular depth-zero supercuspidal L -parameters were first introduced in [4, page 825], which are called tame regular semisimple elliptic L -parameters therein. Epipelagic supercuspidal L -parameters were first considered in [32, §7] and then discussed in [19, §5.1]. Recall the fact that regular depth-zero supercuspidal L -parameters correspond to regular depth-zero supercuspidal representations, and epipelagic supercuspidal L -parameters correspond to epipelagic supercuspidal representations. The following corollary is a direct consequence of Corollaries 3.17 and 3.20, Propositions 4.14 and 4.17.

Corollary 4.18 *Let (G', H', θ') be a rigid inner twist of (G, H, θ) , φ a regular depth-zero or an epipelagic supercuspidal L -parameter, and $\pi \in \Pi_\varphi(G')$. Suppose that π is H' -distinguished. Then the L -parameters ${}^L\theta \circ \varphi$ and ${}^L C \circ \varphi$ are \widehat{G} -conjugate, and thus $\Pi_{L\theta \circ \varphi} = \Pi_{L C \circ \varphi}$.*

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References

1. Adams, J., Vogan, D.: Contragredient representations and characterizing the local Langlands correspondence. *Am. J. Math.* **138**(3), 657–682 (2016)
2. Adler, J.D.: Refined anisotropic K -types and supercuspidal representations. *Pac. J. Math.* **185**(1), 1–32 (1998)
3. DeBacker, S.: Parameterizing conjugacy classes of maximal unramified tori via Bruhat–Tits theory. *Mich. Math. J.* **54**(1), 157–178 (2006)
4. DeBacker, S., Reeder, M.: Depth-zero supercuspidal L -packets and their stability. *Ann. Math.* (2) **169**(3), 795–901 (2009)
5. DeBacker, S., Reeder, M.: On some generic very cuspidal representations. *Compos. Math.* **146**(4), 1029–1055 (2010)
6. DeBacker, S., Spice, L.: Stability of character sums for positive-depth, supercuspidal representations. *J. Reine Angew. Math.* **742**, 47–78 (2018)
7. Glazer, I.: Representations of reductive groups distinguished by symmetric subgroups. *Math. Z.* **289**(1–2), 471–489 (2018)
8. Hakim, J.: Tame supercuspidal representations of GL_n distinguished by orthogonal involutions. *Represent. Theory* **17**, 120–175 (2013)
9. Hakim, J.: Constructing tame supercuspidal representations. *Represent. Theory* **22**, 45–86 (2018)
10. Hakim, J.: Distinguished cuspidal representations over p -adic and finite fields (**preprint**). [arXiv:1703.08861v1](https://arxiv.org/abs/1703.08861v1)

11. Hakim, J., Lansky, J.: Distinguished tame supercuspidal representations and odd orthogonal periods. *Represent. Theory* **16**, 276–316 (2012)
12. Hakim, J., Murnaghan, F.: Tame supercuspidal representations of $GL(n)$ distinguished by a unitary group. *Compos. Math.* **133**(2), 199–244 (2002)
13. Hakim, J., Murnaghan, F.: Two types of distinguished supercuspidal representations. *Int. Math. Res. Not. IMRN* **35**, 1857–1889 (2002)
14. Hakim, J., Murnaghan, F.: Distinguished tame supercuspidal representations. *Int. Math. Res. Pap. IMPR* 2008, no. 2, Art. ID rpn005
15. Howe, R.: Tamely ramified supercuspidal representations of GL_n . *Pac. J. Math.* **73**(2), 437–460 (1977)
16. Kaletha, T.: Endoscopic character identities for depth-zero supercuspidal L-packets. *Duke Math. J.* **158**(2), 161–224 (2011)
17. Kaletha, T.: Genericity and contragredience in the local Langlands correspondence. *Algebra Number Theory* **7**(10), 2447–2474 (2013)
18. Kaletha, T.: Supercuspidal L -packets via isocrystals. *Am. J. Math.* **136**(1), 203–239 (2014)
19. Kaletha, T.: Epipelagic L -packets and rectifying characters. *Invent. Math.* **202**(1), 1–89 (2015)
20. Kaletha, T.: Rigid inner forms of real and p -adic groups. *Ann. Math. (2)* **184**(2), 559–632 (2016)
21. Kaletha, T.: The local Langlands conjectures for non-quasi-split groups. *Families of Automorphic Forms and the Trace Formula. Simons Symposia*, pp. 217–257. Springer, Berlin (2016)
22. Kaletha, T.: Regular supercuspidal representations. *J. Am. Math. Soc.* **32**(4), 1071–1170 (2019)
23. Kim, J.-L.: Supercuspidal representations: an exhaustion theorem. *J. Am. Math. Soc.* **20**(2), 273–320 (2007)
24. Kottwitz, R.: Stable trace formula: cuspidal tempered terms. *Duke Math. J.* **51**(3), 611–650 (1984)
25. Lusztig, G.: Symmetric spaces over a finite field. *The Grothendieck Festschrift*, vol. III, pp. 57–81, *Progr. Math.*, vol. 88. Birkhäuser, Boston (1990)
26. Moy, A., Prasad, G.: Unrefined minimal K -types for p -adic groups. *Invent. Math.* **116**(1–3), 393–408 (1994)
27. Moy, A., Prasad, G.: Jacquet functors and unrefined minimal K -types. *Comment. Math. Helv.* **71**(1), 98–121 (1996)
28. Murnaghan, F.: Parametrization of tame supercuspidal representations. On certain L -functions, *Clay Math. Proc.*, vol 13. Amer. Math. Soc., Providence, pp. 439–469 (2011)
29. Portilla, R.: Finite order automorphisms and a parametrization of nilpotent orbits in p -adic Lie algebras. *Am. J. Math.* **136**(3), 551–598 (2014)
30. Prasad, D.: A ‘relative’ local Langlands correspondence (**preprint**). [arXiv:1512.04347v1](https://arxiv.org/abs/1512.04347v1)
31. Reeder, M.: Supercuspidal L -packets of positive depth and twisted Coxeter elements. *J. Reine Angew. Math.* **620**, 1–33 (2008)
32. Reeder, M., Yu, J.-K.: Epipelagic representations and invariant theory. *J. Am. Math. Soc.* **27**(2), 437–477 (2014)
33. Richardson, R.W.: Orbits, invariants, and representations associated to involutions of reductive groups. *Invent. Math.* **66**(2), 287–312 (1982)
34. Sakellaridis, Y., Venkatesh, A.: Periods and harmonic analysis on spherical varieties. *Astérisque* 396, viii+360 pp (2017)
35. Yu, J.-K.: Construction of tame supercuspidal representations. *J. Am. Math. Soc.* **14**(3), 579–622 (2001)
36. Zhang, C.: Distinction of regular depth-zero supercuspidal L -packets. *Int. Math. Res. Not. IMRN* **15**, 4579–4601 (2018)
37. Zhang, C.: Theta lifts and distinction for regular supercuspidal representations. *Math. Z.* (**published online**). <https://doi.org/10.1007/s00209-019-02391-w>