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Bessel F-isocrystals for reductive groups

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Abstract We construct the Frobenius structure on a rigid connection $\mathrm{Be}_{\check{G}}$ on $\mathbb{G}_m$ for a split reductive group $\check{G}$ introduced by Frenkel–Gross. These data form a $\check{G}$-valued overconvergent F-isocrystal $\mathrm{Be}^{\dagger}_{\check{G}}$ on $\mathbb{G}_{m,\mathbb{F}_p}$, which is the p-adic companion of the Kloosterman $\check{G}$-local system $\mathrm{Kl}_{\check{G}}$ constructed by Heinloth–Ngô–Yun. By studying the structure of the underlying differential equation, we calculate the monodromy group of $\mathrm{Be}^{\dagger}_{\check{G}}$ when $\check{G}$ is almost simple (which recovers the calculation of monodromy group of $\mathrm{Kl}_{\check{G}}$ due to Katz and Heinloth–Ngô–Yun), and prove a conjecture of Heinloth–Ngô–Yun on the functoriality between different Kloosterman $\check{G}$-local systems. We show that the Frobenius Newton polygons of $\mathrm{Kl}_{\check{G}}$ are generically ordinary for every $\check{G}$ and are everywhere ordinary on $|\mathbb{G}_{m,\mathbb{F}_p}|$ when $\check{G}$ is classical or G_2.

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1 Introduction

1.1 Bessel equations and Kloosterman sums

1.1.1. The classical Bessel differential equation (of rank n) with a parameter λ

$$\left(x\frac{d}{dx}\right)^n (f) - \lambda^n x \cdot f = 0 \tag{1.1.1.1}$$

has a unique solution which is holomorphic over the complex plane:

$$\oint_{(S^1)^{n-1}} \exp \lambda\left(z_1 + \cdots + z_{n-1} + \frac{x}{z_1 \cdots z_{n-1}}\right) \frac{dz_1 \cdots dz_{n-1}}{(2\pi i)^{n-1} z_1 \cdots z_{n-1}} \tag{1.1.1.2}$$
$$= \sum_{r \geq 0} \frac{1}{(r!)^n} (\lambda^n x)^r.$$

One may reinterpret this fact using the language of algebraic $\mathscr{D}$-modules as follows. Let K be a field of characteristic zero. The Bessel equation (1.1.1.1) can be converted to a connection Be_n on the rank n trivial bundle on the multiplicative group $\mathbb{G}_{m,K}$

$$\mathrm{Be}_n: \quad \nabla = d + \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \lambda^n x & 0 & 0 & \dots & 0 \end{pmatrix} \frac{dx}{x}, \tag{1.1.1.3}$$

which we call the *Bessel connection (of rank n)*. On the other hand, we consider the following diagram

$$\mathbb{G}_m \xleftarrow{\text{mult}} \mathbb{G}_m^n \xrightarrow{\text{add}} \mathbb{A}^1, \tag{1.1.1.4}$$

where add (resp. mult) denotes the morphism of taking sum (resp. product) of n coordinates of $\mathbb{G}_m^n$, and define the *Kloosterman $\mathscr{D}$-module* on $\mathbb{G}_{m,K}$ as

$$\mathrm{Kl}_n^{\mathrm{dR}} := \mathrm{R}^{n-1}\,\mathrm{mult}_!(\mathrm{add}^*(\mathsf{E}_\lambda)), \tag{1.1.1.5}$$

where $\mathsf{E}_\lambda = (\mathscr{O}_{\mathbb{A}^1_K}, \nabla = d - \lambda dx)$ is the *exponential $\mathscr{D}$-module* on $\mathbb{A}^1_K$. With these notations, the fact that (1.1.1.2) is a solution of (1.1.1.1) reflects an isomorphism of algebraic $\mathscr{D}$-modules on $\mathbb{G}_{m,K}$

$$\mathrm{Be}_n \simeq \mathrm{Kl}_n^{\mathrm{dR}}.$$

Its differential Galois group was calculated by Katz [49].

1.1.2. There is a parallel theory in positive characteristic. Let p a prime number. For every finite extension $\mathbb{F}_q/\mathbb{F}_p$ and $a \in \mathbb{F}_q^\times$, the *Kloosterman sum* $\mathrm{Kl}(n; a)$ in n-variables is defined by[1]

[1] The sum (1.1.2.1) is slightly different from the standard definition by a factor $(-\frac{1}{\sqrt{q}})^{n-1}$.

$$\mathrm{Kl}(n;a) = (\frac{-1}{\sqrt{q}})^{n-1} \times \sum_{z_i \in \mathbb{F}_q^\times} \exp\left(\frac{2\pi i}{p} \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}\big(z_1 + \cdots + z_{n-1} + \frac{a}{z_1 \ldots z_{n-1}}\big)\right). \tag{1.1.2.1}$$

It admits a sheaf-theoretic interpretation due to Deligne [33]. Namely, the analog of the exponential $\mathscr{D}$-module in positive characteristic is the Artin–Schreier sheaf AS_ψ on $\mathbb{A}^1_{\mathbb{F}_p}$ associated to a non-trivial character $\psi : \mathbb{F}_p \to \mathbb{Q}_\ell(\mu_p)^\times$. In [33], Deligne defined the *Kloosterman sheaf* Kl_n on $\mathbb{G}_{m,\mathbb{F}_p}$ as below:

$$\mathrm{Kl}_n = \mathrm{R}^{n-1}\,\mathrm{mult}_!(\mathrm{add}^*(\mathrm{AS}_\psi))\left(\frac{n-1}{2}\right), \tag{1.1.2.2}$$

and showed that Kl_n is a local system of rank n and of weight 0 and that the Frobenius trace of Kl_n is equal to the Kloosterman sum $\mathrm{Kl}(n;-)$ via an embedding $\iota : \mathbb{Q}_\ell(\mu_p) \to \mathbb{C}$ such that $\iota\psi(x) = \exp(2\pi i x/p)$ for $x \in \mathbb{F}_p$. In particular, this implies the Weil bound of the Kloosterman sum $|\mathrm{Kl}(n;a)| \leq n$.

In [50, § 11], Katz calculated the (global) geometric and arithmetic monodromy group of Kl_n as follow:

$$G_{\mathrm{geo}}(\mathrm{Kl}_n) = G_{\mathrm{arith}}(\mathrm{Kl}_n) = \begin{cases} \mathrm{Sp}_n & n \textit{ even}, \\ \mathrm{SL}_n & pn \textit{ odd}, \\ \mathrm{SO}_n & p=2, n \textit{ odd}, n \neq 7, \\ G_2 & p=2, n=7. \end{cases} \tag{1.1.2.3}$$

Surprisingly, the exceptional group G_2 appears as the monodromy group.

1.1.3. In 70's [40], Dwork and Sperber showed that there exists a Frobenius structure on the Bessel connection (1.1.1.3) whose Frobenius traces give the Kloosterman sum. Here a *Frobenius structure* is a horizontal isomorphism between the Bessel connection and its pullback by the "Frobenius endomorphism" $F : \mathbb{G}_{m,K} \to \mathbb{G}_{m,K}$ over K defined by $x \mapsto x^p$. Although the Bessel connection is algebraic, such a horizontal isomorphism is not algebraic but of p-adic analytic nature.

To explain their result, we recall *the ring of p-adic analytic functions on $\mathbb{P}^1$ overconvergent along* $\{\infty\}$ [19]. We set $K = \mathbb{Q}_p(\mu_p)$, equipped with a p-adic valuation |-| normalised by $|p| = p^{-1}$, and denote by $A^\dagger$ the ring of p-adic analytic functions with a radius of convergence > 1:

$$A^\dagger = \left\{ \sum_{n=0}^{+\infty} a_n x^n \mid a_n \in K, \exists\, \rho > 1, \lim_{n\to+\infty} |a_n|\rho^n = 0 \right\}. \tag{1.1.3.1}$$

We take an algebraic closure $\overline{K}$ of K and fix an isomorphism $\iota : \overline{K} \to \mathbb{C}$. There exists a unique element π of K satisfying $\pi^{p-1} = -p$, which corresponds to the character $\exp 2\pi i(\frac{-}{p}) : \mathbb{F}_p \to \mathbb{C}^\times$ (cf. 2.1.1(i)).

Theorem 1.1.4 (Dwork, Sperber [40,71,72]) *Let n be an integer prime to p and set $\lambda = -\pi$ as above. There exists a unique $\varphi(x) \in \mathrm{GL}_n(A^\dagger)$ satisfying the following properties.*

(i) *The matrix φ satisfies the differential equation:*

$$x\frac{d\varphi}{dx}\varphi^{-1} + \varphi \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \lambda^n x & 0 & 0 & \dots & 0 \end{pmatrix} \varphi^{-1} = p \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \lambda^n x^p & 0 & 0 & \dots & 0 \end{pmatrix}.$$

That is, φ defines a horizontal isomorphism $F^*(\mathrm{Be}_n) \xrightarrow{\sim} \mathrm{Be}_n$.

(ii) *For $a \in \mathbb{F}_q^\times$, we have $\iota \operatorname{Tr} \varphi_a = \mathrm{Kl}(n; a)$, where $\varphi_a = \prod_{i=0}^{\deg(a)-1} \varphi(\widetilde{a}^{p^i}) \in \mathrm{GL}_n(\overline{K})$ and $\widetilde{a} \in \overline{K}$ denotes the Teichmüller lifting of a.*

(iii) *If $\{\alpha_1, \cdots, \alpha_n\}$ denote the eigenvalues of φ_a, then we have $|\alpha_i| = p^{\frac{n+1-2i}{2}\deg(a)}$ after reordering α_i.*

The data (Be_n, φ) form an overconvergent F-isocrystal on $\mathbb{G}_{m,\mathbb{F}_p}$ (relative to K) [19], which we call the *Bessel F-isocrystal (of rank n)* and denote by $\mathrm{Be}_n^\dagger$. By (ii), $\mathrm{Be}_n^\dagger$ is the *p-adic companion* of the Kloosterman sheaf Kl_n in the sense of [3,34].

1.2 Generalization for reductive groups

1.2.1. Recently, there are two generalizations of above results, from different perspectives of the (geometric) Langlands program. The first one is the generalization of Bessel equations by Frenkel and Gross [42]. For each (split) reductive group $\check{G}$ over a field K of characteristic zero, Frenkel–Gross wrote down an explicit $\check{G}$-connection $\mathrm{Be}_{\check{G}}$ on $\mathbb{G}_m$, which specializes to Be_n when $\check{G} = \mathrm{GL}_n$. We will call $\mathrm{Be}_{\check{G}}$ the *Bessel connection of $\check{G}$* in this paper. Another one, due to Heinloth, Ngô and Yun [47], is the generalization of the Kloosterman sums. Namely, the authors explicitly constructed, for each (split) reductive group G over the rational function field $\mathbb{F}_p(t)$, a Hecke eigenform of G, and defined $\mathrm{Kl}_{\check{G}}$ as its Langlands parameter, which is an ℓ-adic $\check{G}$-local system on $\mathbb{G}_m$ that specializes to Kl_n if $\check{G} = \mathrm{GL}_n$. The authors called $\mathrm{Kl}_{\check{G}}$ the *Kloosterman sheaf of $\check{G}$*.

The original goals of our work are to prove a functoriality conjecture of Heinloth–Ngô–Yun relating Kloosterman sheaves for different groups [47, conjecture 7.3] and to study the arithmetic properties of exponential sums associated to $\mathrm{Kl}_{\check{G}}$. Although this conjecture is about ℓ-adic sheaves, it seems difficult to access it purely in the ℓ-adic framework. Our approach is based on our investigation of the p-adic aspects of the above story which unifies the previous two generalizations. Our main results can be summarized as follows:

(i) We construct the Frobenius structure on $\mathrm{Be}_{\check{G}}$ and obtain the Bessel F-isocrystal $\mathrm{Be}^{\dagger}_{\check{G}}$ of $\check{G}$, which is the p-adic companion of $\mathrm{Kl}_{\check{G}}$ in appropriate sense;
(ii) We calculate the monodromy group of $\mathrm{Be}^{\dagger}_{\check{G}}$ and then deduce a complete result on the monodromy group of $\mathrm{Kl}_{\check{G}}$; (Our approach is entirely different and more conceptual compared to that of Katz (1.1.2.3) and of Heinloth–Ngô–Yun [47].)
(iii) We prove the conjecture of Heinloth–Ngô–Yun on the functoriality of Kloosterman sheaves, and therefore obtain identities between different exponential sums associated to $\mathrm{Kl}_{\check{G}}$;
(iv) We show that the Frobenius Newton polygons of $\mathrm{Be}^{\dagger}_{\check{G}}$ (and therefore $\mathrm{Kl}_{\check{G}}$) are generically ordinary and when $\check{G}$ is classical or G_2 they are everywhere ordinary on $|\mathbb{G}_{m,\mathbb{F}_p}|$.

We discuss these results in more details in the sequel.

1.2.2. Let $\check{G}$ be a split almost simple group over a field K of characteristic zero. Fix a Borel subgroup $\check{B} \subset \check{G}$, and a principal nilpotent element N in $\check{\mathfrak{b}} = \mathrm{Lie}(\check{B})$. Let E denote a basis vector of the lowest root space in $\check{\mathfrak{g}} = \mathrm{Lie}(\check{G})$. In [42], Frenkel and Gross considered a connection on the trivial $\check{G}$-bundle over $\mathbb{G}_m$:

$$\mathrm{Be}_{\check{G}} = d + N\frac{dx}{x} + \lambda^h E\, dx,$$

where x is a coordinate of $\mathbb{G}_m$, $\lambda \in K$ is a parameter and h is the Coxeter number of $\check{G}$. This connection is rigid and has a regular singularity at 0 and an irregular singularity at ∞. We regard it as a tensor functor from the category of representations of $\check{G}$ to the category of connections on the trivial bundles over $\mathbb{G}_m$.

1.2.3. Let G be a split almost simple group over $\mathbb{F}_p(t)$ whose dual group is $\check{G}$. In [47], Heinloth–Ngô–Yun wrote down a cuspidal Hecke eigenform f on G, and defined the Kloosterman sheaf $\mathrm{Kl}_{\check{G}}$ for $\check{G}$ as the Langlands parameter of f. We recall their construction here under the assumption that G is simply-connected. If we fix opposite Iwahori subgroups $I(0)^{\mathrm{op}} \subset G(\mathscr{O}_0)$ and $I(0) \subset G(\mathscr{O}_\infty)$ at $0, \infty$, and a non-degenerate character $\phi : I(1)/I(2) \to \mathbb{Q}(\mu_p)^\times$, where $I(i)$

denotes the ith step in the Moy–Prasad filtration of $I(0)$, then f is the unique (up to scalar) non-zero function on $G(\mathbb{F}_p(t))\backslash G(\mathbb{A})$ that is,

- invariant under $G(\mathscr{O}_x)$ for every place $x \neq 0, \infty$;
- invariant under $I(0)^{\mathrm{op}}$ at 0;
- $(I(1), \phi)$-equivariant at ∞.

Then Heinloth–Ngô–Yun defined $\mathrm{Kl}_{\check{G}} : \mathbf{Rep}(\check{G}) \to \mathrm{LocSysm}(\mathbb{G}_{m,\mathbb{F}_p})$ as a tensor functor from the category of representations of $\check{G}$ (over $\overline{\mathbb{Q}}_\ell$) to the category of ℓ-adic local systems on $\mathbb{G}_{m,\mathbb{F}_p}$, such that for every $V \in \mathbf{Rep}(\check{G})$ and every $a \in |\mathbb{G}_{m,\mathbb{F}_p}|$,

$$T_{V,a}(f) = \mathrm{Tr}(\mathrm{Frob}_a, (\mathrm{Kl}_{\check{G},V})_{\overline{a}})f,$$

where $T_{V,a}$ is the Hecke operator associated to (V, a). The actual construction of $\mathrm{Kl}_{\check{G}}$ uses the geometric Langlands correspondence (see 4.1.9).

Our first main result is the existence of a Frobenius structure on Bessel connections for reductive groups.

Theorem 1.2.4 (4.4.4, 5.3.2) *Let $K = \mathbb{Q}_p(\mu_p)$, $\overline{K}$ an algebraic closure of K and set $\lambda = -\pi$ as in 1.1.4.*

(i) *There exists a unique $\varphi(x) \in \check{G}(A^\dagger)$ satisfying the differential equation*

$$x\frac{d\varphi}{dx}\varphi^{-1} + \mathrm{Ad}_\varphi(N + \lambda^h xE) = p(N + \lambda^h x^p E)$$

and such that via a (fixed) isomorphism $\overline{K} \simeq \overline{\mathbb{Q}}_\ell$, for every $a \in \mathbb{F}_q^\times$ and $V \in \mathbf{Rep}(\check{G})$

$$\mathrm{Tr}(\varphi_a, V) = \mathrm{Tr}(\mathrm{Frob}_a, (\mathrm{Kl}_{\check{G},V})_{\overline{a}}),$$

where $\varphi_a = \prod_{i=0}^{\deg(a)-1} \varphi(\widetilde{a}^{p^i}) \in \check{G}(\overline{K})$ and $\widetilde{a} \in \overline{K}$ denotes the Teichmüller lifting of a.

In particular, the analytification of the Bessel connection $\mathrm{Be}_{\check{G}}$ on $\mathbb{G}_{m,K}^{\mathrm{an}}$ is overconvergent and underlies a tensor functor from $\mathbf{Rep}(\check{G})$ to the category of overconvergent F-isocrystals on $\mathbb{G}_{m,\mathbb{F}_p}$:

$$\mathrm{Be}_{\check{G}}^\dagger : \mathbf{Rep}(\check{G}) \to F\text{-}\mathrm{Isoc}^\dagger(\mathbb{G}_{m,\mathbb{F}_p}/K),$$

which can be regarded as the p-adic companion of $\mathrm{Kl}_{\check{G}}$.

(ii) *Let $\rho \in \mathbb{X}_\bullet(\check{T})$ be the half sum of positive coroots. If $\check{G}$ is of classical type or G_2, for every $a \in |\mathbb{G}_{m,\mathbb{F}_p}|$, the set of p-adic order of eigenvalues of $\varphi_a \in \check{G}(\overline{K})$ (also known as the Frobenius slopes at a) is same as that of*

$\rho(p^{\deg(a)}) \in \check{G}(\overline{K})$. If $\check{G}$ is of other exceptional type, the same assertion holds generically on $|\mathbb{G}_{m,\mathbb{F}_p}|$.

Remark 1.2.5 (i) For a $\check{G}$-valued overconvergent F-isocrystal on a smooth variety X over $\mathbb{F}_p$, we say its Newton polygon is ordinary at a if the Frobenius slopes at a are given by ρ (in the above sense). We expect that the Newton polygons of $\mathrm{Be}^{\dagger}_{\check{G}}$ are always ordinary at each closed point of $\mathbb{G}_{m,\mathbb{F}_p}$.

(ii) V. Lafforgue [57] showed that ρ is the upper bound for the p-adic valuations of Hecke eigenvalues of Hecke eigenforms (cf. 5.3.1 for a precise statement). Drinfeld and Kedlaya [38] proved an analogous result for the Frobenius slopes of an indecomposable convergent F-isocrystal on a smooth scheme.

1.2.6. Global monodromy groups. In [42, Cor. 9,10], Frenkel and Gross calculate the differential Galois group G_{gal} of $\mathrm{Be}_{\check{G}}$ over $\overline{K}$, which we list in the following table (up to central isogeny):

$\check{G}$	G_{gal}
A_{2n}	A_{2n}
A_{2n-1}, C_n	C_n
$B_n, D_{n+1} (n \geq 4)$	B_n
E_7	E_7
E_8	E_8
E_6, F_4	F_4
B_3, D_4, G_2	$G_2.$

(1.2.6.1)

If G_{geo} denotes the geometric monodromy group of $\mathrm{Be}^{\dagger}_{\check{G}}$ over $\overline{K}$, there exists a canonical homomorphism

$$G_{\mathrm{geo}} \to G_{\mathrm{gal}}.$$

Theorem 1.2.7 (4.5.2) (i) *If either $\check{G}$ is not of type A_{2n}, or $p > 2$, the above morphism is an isomorphism.*

(ii) *If $p = 2$ and $\check{G} = \mathrm{SL}_{2n+1}$, then $G_{\mathrm{geo}} \simeq \mathrm{SO}_{2n+1}$ if $n \neq 3$ and $G_{\mathrm{geo}} \simeq G_2$ if $n = 3$.*

(iii) *The arithmetic monodromy group G_{arith} of $\mathrm{Be}^{\dagger}_{\check{G}}$ is isomorphic to G_{geo}.*

In fact, the second part of the theorem follows from the first part and Theorem 1.2.8(ii) below. By companion, this theorem allows us to recover Katz's result on the monodromy group of Kl_n (1.1.2.3) and Heinloth–Ngô–Yun's result on the monodromy group of $\mathrm{Kl}_{\check{G}}$ [47] (and remove the restriction of the characteristic of k in *loc. cit.*) in a different way. For instance, the G_2-symmetry on $\mathrm{Be}^{\dagger}_7$ when $p = 2$ (1.1.2.3) appears naturally in our approach, compared with Katz' original approach via point counting. In addition, we avoid difficult geometry related to quasi-minuscule and adjoint Schubert varieties, as analyzed in [47].

We also have partial results about the local monodromy of $\mathrm{Be}^{\dagger}_{\check{G}}$ (and $\mathrm{Kl}_{\check{G}}$) at ∞ (see Corollary 4.5.9 and Remark 4.5.10 for details).

As an application of our p-adic theory, we prove a conjecture of Heinloth–Ngô–Yun on certain functoriality between Kloosterman sheaves for different groups [47, conjecture 7.3].

Theorem 1.2.8 (5.1.4, 5.2.10(ii)) (i) *For $\check{G}' \subset \check{G}$ appearing in the same line in the left column of the above diagram, $\mathrm{Kl}_{\check{G}}$ is isomorphic to the push-out of $\mathrm{Kl}_{\check{G}'}$ along $\check{G}' \to \check{G}$.*

(ii) *If $p = 2$, $\mathrm{Kl}_{\mathrm{SL}_{2n+1}}$ is the push-out of $\mathrm{Kl}_{\mathrm{SO}_{2n+1}}$ along $\mathrm{SO}_{2n+1} \to \mathrm{SL}_{2n+1}$.*

1.2.9. The above theorem allows us to identify various exponential sums associated to Kloosterman sheaves defined by different groups. Here are some examples (see Corollary 5.2.11):

(i) When $\check{G} = \mathrm{SO}_3 \simeq \mathrm{PGL}_2$, we have the following identity for $a \in \mathbb{F}_q^{\times}$:

$$\Big(\sum_{x\in\mathbb{F}_q^{\times}} \psi(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x+\frac{a}{x}))\Big)^2 - q \tag{1.2.9.1}$$
$$= \begin{cases} \displaystyle\sum_{x_1,x_2\in\mathbb{F}_q^{\times}} \psi\Big(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x_1+x_2+\frac{a}{x_1x_2})\Big), & p=2, \\ \dfrac{1}{G(\psi^{-1},\rho)} \\ \quad\times \sum_{x_1x_2x_3=4ay,x_i\in\mathbb{F}_q^{\times}} \psi\Big(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x_1+x_2+x_3-y)\Big)\rho(y), & p>2 \end{cases}$$

where $\psi(-) = \exp\frac{2\pi i}{p}(-)$, ρ denotes the quadratic character of $\mathbb{F}_q^{\times}$ and $G(\psi^{-1}, \rho)$ the associated Gauss sum. The identity is due to Carlitz [24] when $p = 2$ and Katz [52, § 3] when $p > 2$. Our method is completely different from these works.

One can obtain other identities between different exponential sums whose sheaf-theoretic incarnations were considered [52].

(ii) For $n \geq 2$, via the inclusion $\mathrm{SO}_{2n+1} \to \mathrm{SO}_{2n+2}$, for $a \in \mathbb{F}_q^{\times}$, we have

$$\sum_{uv=a,u,v\in F_q^\times}\left(\left(\sum\nolimits_{x\in\mathbb{F}_q^\times}\psi(\operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x+\tfrac{u}{x}))\right)^2-q\right)$$
$$\times\left(\sum_{x_i\in\mathbb{F}_q^\times}\psi(\operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x_1+\cdots+x_{2n-3}+\tfrac{v}{x_1\cdots x_{2n-3}}))\right)$$
$$=\tfrac{-1}{\sqrt{q}}\left(\sum_{x_i\in\mathbb{F}_q^\times}\psi\left(\operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x_1+\cdots+x_{2n}+a\tfrac{x_1+x_2}{x_1x_2\cdots x_{2n}})\right)-q^{n-1}\right).$$

1.3 Strategy of the proof and the organization of the article

1.3.1. We outline the strategy of proofs. Theorem 1.2.4(i) follows by combining following three ingredients:

(i) We first mimic Heinloth–Ngô–Yun's construction to produce a $\check{G}$-valued overconvergent F-isocrystal $\mathrm{Kl}_{\check{G}}^{\mathrm{rig}}$ on $\mathbb{G}_{m,\mathbb{F}_p}$ and a $\check{G}$-bundle with connection $\mathrm{Kl}_{\check{G}}^{\mathrm{dR}}$ on $\mathbb{G}_{m,K}$ (Sect. 4.1). A key step is to develop the geometric Satake equivalence for arithmetic $\mathscr{D}$-modules, which we will discuss latter (1.3.5). Certain proofs are parallel to the ℓ-adic setting. We omit most of them and repeat some only for the notation purposes.

(ii) Then we show that the overconvergent isocrystal $\mathrm{Kl}_{\check{G}}^{\mathrm{rig}}$ is isomorphic to the analytification of the $\check{G}$-connection $\mathrm{Kl}_{\check{G}}^{\mathrm{dR}}$ (Sect. 4.2) by comparing certain relative de Rham cohomologies and relative rigid cohomologies.

(iii) We strengthen a result of the second author [80] to identify $\mathrm{Kl}_{\check{G}}^{\mathrm{dR}}$ with $\mathrm{Be}_{\check{G}}$ (Sect. 4.3).

1.3.2. The local monodromy of $\mathrm{Be}_{\check{G}}^{\dagger}$ at 0 is principal unipotent, which implies that its geometric monodromy G_{geo} contains a principal SL_2. This puts strong restrictions on the possible Dynkin diagrams of G_{geo} (cf. 4.5.5 for a possible list). A result of Baldassarri [14] (cf. [10] 3.2), which implies that the p-adic slope of $\mathrm{Be}_{\check{G}}^{\dagger}$ at ∞ is less or equal to the formal slope of $\mathrm{Be}_{\check{G}}$ at ∞, allows us to exclude the case $G_{\mathrm{geo}}=\mathrm{PGL}_2$ (or SL_2) in most cases. Together with certain symmetry on $\mathrm{Be}_{\check{G}}^{\dagger}$, this implies Theorem 1.2.7(i). We shall emphasize that being able to directly bound the p-adic slope of $\mathrm{Be}_{\check{G}}^{\dagger}$ at ∞ is one of the main advantages of our p-adic method over the ℓ-adic methods used in [47].

1.3.3. The analogous functoriality (Theorem 1.2.8(i)) for Bessel connections $\mathrm{Be}_{\check{G}}$ follows from their definition. Then we deduce the corresponding functoriality between $\mathrm{Be}_{\check{G}}^{\dagger}$'s by Theorems 1.2.4(i) and 1.2.7(i). For Theorem 1.2.8(ii) (and therefore Theorem 1.2.7(ii)), we construct an isomorphism between the maximal slope quotients of $\mathrm{Be}_{2n+1}^{\dagger}$ and $\mathrm{Be}_{\mathrm{SO}_{2n+1},\mathrm{Std}}^{\dagger}$ using a refinement of Dwork's congruences [39] in the 2-adic case. Then we conclude that $\mathrm{Be}^{\dagger}_{2n+1} \simeq \mathrm{Be}^{\dagger}_{\mathrm{SO}_{2n+1},\mathrm{Std}}$ by a recent theorem of Tsuzuki [74] (cf. appendix A). Since $\mathrm{Be}^{\dagger}_{\check{G}}$ is the p-adic companion of $\mathrm{Kl}_{\check{G}}$, Theorem 1.2.8 follows.

1.3.4. By functoriality, we reduce Theorem 1.2.4(ii) to the corresponding assertion for (Frobenius) Newton polygon of $\mathrm{Be}^{\dagger}_{\mathrm{SL}_n,\mathrm{Std}}$ and of $\mathrm{Be}^{\dagger}_{\mathrm{SO}_{2n+1},\mathrm{Std}}$, which are isomorphic to $\mathrm{Be}^{\dagger}_{n}$ and a hypergeometric overconvergent F-isocrystal [61] respectively. Then the assertion follows from the results of Dwork, Sperber and Wan [40,72,76].

1.3.5. As mentioned above, in order to carry through the first step of 1.3.1, we need to establish a version of the geometric Satake equivalence for arithmetic $\mathscr{D}$-modules. This is based on the recent development of the six functors formalism, weight theory and nearby/vanishing cycle functors for arithmetic $\mathscr{D}$-modules developed by Berthelot, Caro, Abe and etc [4–6,25,26].

To state our result, we introduce some notations. Let k be a finite field with $q = p^s$ elements and K a finite extension of $\mathbb{Q}_q$. Suppose that there exists an automorphism $\sigma : K \to K$, which extends the lifting of the t-th Frobenius automorphism of k to $\mathbb{Q}_q$ for some integer t. Let G be a split reductive group over k, $\check{G}$ its Langlands dual group over K, Gr_G the affine Grassmannian of G and L^+G the positive loop group of G.

For a k-scheme X, let $\mathrm{Hol}(X/K)$ be the category of holonomic arithmetic $\mathscr{D}$-modules on X and $\mathrm{Hol}(X/K_F)$ the category of objects of $\mathrm{Hol}(X/K)$ with a Frobenius structure. They are the analogues of the category of ℓ-adic sheaves on $X_{\overline{k}}$ and the category of Weil sheaves on X respectively. We denote by $\mathrm{Hol}_{L^+G}(\mathrm{Gr}_G/K)$ (resp. $\mathrm{Hol}_{L^+G}(\mathrm{Gr}_G/K_F)$) the category of L^+G-equivariant objects in $\mathrm{Hol}(\mathrm{Gr}_G/K)$ (resp. $\mathrm{Hol}(\mathrm{Gr}_G/K_F)$).

The geometric Satake equivalence (for geometric coefficients) states that the category $\mathrm{Hol}_{L^+G}(\mathrm{Gr}_G/K)$ is a neutral Tannakian category over K whose Tannakian group is $\check{G}$ (3.4.1). The Tannakian structure and the Frobenius structure on $\mathrm{Hol}_{L^+G}(\mathrm{Gr}_G/K_F)$ allows us to define a homomorphism $\iota : \mathbb{Z} \to \mathrm{Aut}(\check{G}(K))$ (3.4.3) and hence a semi-direct product $\check{G}(K) \rtimes \mathbb{Z}$.

Theorem 1.3.6 (i) (Geometric coefficients 3.4.1) *There exists a natural equivalence of monoidal categories between* $\mathrm{Hol}_{L^+G}(\mathrm{Gr}_G/K)$ *and* $\mathbf{Rep}(\check{G})$.

(ii) (Arithmetic coefficients 3.4.7) *There exists an equivalence of monoidal categories between* $\mathrm{Hol}_{L^+G}(\mathrm{Gr}_G/K_F)$ *and the category* $\mathbf{Rep}^{\circ}_{K,\sigma}(\check{G}(K) \rtimes \mathbb{Z})$ *of certain σ-semi-linear representations of $\check{G}(K) \rtimes \mathbb{Z}$ (cf.* 3.4.4*).*

Note that the formulation of the arithmetic coefficients geometric Satake here is different from the corresponding arithmetic version the ℓ-adic case [69,79].

Although the strategy of the proof of this theorem is same as the ℓ-adic case, we need to establish some foundational results in the setting of arithmetic

$\mathscr{D}$-modules. We introduce a notion of *universal local acyclicity* (ULA) for arithmetic $\mathscr{D}$-modules and discuss its relation with the nearby/vanishing cycle functors introduced by Abe-Caro and Abe [4,5] in Sect. 2.2.

Recall that there are motivic versions of geometric Satake [68,81]. The above theorem can be regarded as their p-adic realization. (But as far as we know, there is no general construction of the realization functor as we need so the above theorem is not a formal consequence of *loc. cit.*) On the other hand, there is a recent work of R. Cass [27] on the geometric Satake equivalence for perverse $\mathbb{F}_p$-sheaves. It would be interesting to see whether there is a version of geometric Satake for some $\mathbb{Z}_p$-coefficient sheaf theory, which after inverting p and mod p specializes to our version and Cass' version respectively.

We hope our article will lead further investigation of the p-adic aspect of the geometric Langlands program.

1.3.7. We briefly go over the organization of this article. Section 2 contains a review of and some complements on the theory of arithmetic $\mathscr{D}$-modules and overconvergent (F-)isocrystals. In Sect. 3, we establish the geometric Satake for arithmetic $\mathscr{D}$-modules (Theorem 1.3.6). Sections 4.1–4.4 are devoted to the proof of Theorem 1.2.4(i) (1.3.1). We calculate the monodromy group of $\mathrm{Be}^{\dagger}_{\check{G}}$ in Sect. 4.5 (Theorem 1.2.7 and 1.3.2). In Sect. 5.1, we prove the functoriality of Bessel F-isocrystals and of Kloosterman sheaves (Theorem 1.2.8(i) and 1.3.3). In Sect. 5.2, we identify the Bessel F-isocrystals for classical groups with certain hypergeometric differential equations studied by Katz and Miyatani [51,61] and then deduce identities in 1.2.9. In the last Sect. 5.3, we study the Frobenius Newton polygon of $\mathrm{Be}^{\dagger}_{\check{G}}$ and prove Theorem 1.2.4(ii). Appendix A is devoted to a proof of Theorem 1.2.8(ii) from the perspective of p-adic differential equations.

1.3.8. Notation. In this article, we fix a prime number p. Let s be a positive integer and set $q = p^s$. Let k be a perfect field of characteristic p, $\overline{k}$ an algebraic closure of k and R a complete discrete valuation ring with residue field k. We set $K = \mathrm{Frac}(R)$. We fix an algebraic closure $\overline{K}$ of K. We assume moreover that the s-th Frobenius endomorphism $k \xrightarrow{\sim} k$, $x \mapsto x^q$ lifts to an automorphism $\sigma : R \xrightarrow{\sim} R$.

By a *k-scheme* (resp. *R-scheme*), we mean a separated scheme of finite type over k (resp. over R).

We use the notation of arithmetic $\mathscr{D}$-modules [20,21]. For a smooth formal R-scheme $\mathfrak{X}$ and a divisor Z of the special fiber of $\mathfrak{X}$, let $\mathscr{O}_{\mathfrak{X},\mathbb{Q}}({}^{\dagger}Z)$ (resp. $\mathscr{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}({}^{\dagger}Z)$) denote the sheaf of rings of functions (resp. differential operators) on $\mathfrak{X}$ with singularities overconvergent along Z [20, 4.2.4]. We omit $({}^{\dagger}Z)$ if Z is empty. If we set $U = \mathfrak{X}_k - Z$, we denote $\mathscr{O}_{\mathfrak{X},\mathbb{Q}}({}^{\dagger}Z)$ (resp. $\mathscr{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}({}^{\dagger}Z)$) by $\mathcal{O}_U$ (resp. $\mathscr{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}(Z)$ (or $\mathscr{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}(\infty)$)) for short.

2 Review and complements on arithmetic $\mathscr{D}$-modules

2.1 Overconvergent (F-)isocrystals and arithmetic $\mathscr{D}$-modules

2.1.1. Let X be a k-scheme. We denote by $\mathrm{Isoc}^{\dagger}(X/K)$ (resp. $F\text{-}\mathrm{Isoc}^{\dagger}(X/K)$) the category of overconvergent isocrystals (resp. F-isocrystals) on X (relative to K) and refer to [19] for their definition. We denote by $\mathrm{Isoc}^{\dagger\dagger}(X/K)$ the thick full subcategory of $\mathrm{Isoc}^{\dagger}(X/K)$ generated by those that can be endowed with an s'-th Frobenius structure for some integer s' divisible by s.

A typical example used in this paper is the *Dwork F-isocrystal* [18]. Let $k = \mathbb{F}_p$ (i.e. $s = 1$), $K = \mathbb{Q}_p(\mu_p)$, $R = \mathscr{O}_K$ and $\sigma = \mathrm{id}$. We choose $\pi \in K$ such that $\pi^{p-1} = -p$ and take a frame $(\mathbb{P}^1_k, \widehat{\mathbb{P}^1_R})$ of $X = \mathbb{A}^1_k$ [6, definition 1.1.1]. If x denotes a coordinate of $\mathbb{A}^1$, the connection $d + \pi dx$ on $\mathcal{O}_{\mathbb{A}^1_k}$ forms an object of $\mathrm{Isoc}^{\dagger}(\mathbb{A}^1_k/K)$ and is called *Dwork isocrystal*, denoted by $\mathscr{A}_\pi$. Its Frobenius structure $\varphi : F^*_{\mathbb{A}^1_k}(\mathscr{A}_\pi) \to \mathscr{A}_\pi$ is the multiplication by $\theta_\pi(x) = \exp(\pi(x - x^p))$, which is a section of $\mathcal{O}_{\mathbb{A}^1_k}$.

There exists a unique nontrivial additive character $\psi : \mathbb{F}_p \to K^\times$ satisfying $\psi(1) = 1 + \pi \mod \pi^2$. For each $x \in \mathbb{F}_p$, we denote by $\widetilde{x}$ the Teichmüller lifting of x in $\mathbb{Q}_p$. Then $\theta_\pi(\widetilde{x}) = \psi(x)$ [18, 1.4]. So the Frobenius trace function of $\mathscr{A}_\pi$ is equal to $\psi \circ \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(-)$. We also denote $\mathscr{A}_\pi$ by $\mathscr{A}_\psi$, as it plays a similar role of Artin–Schreier sheaf associated to ψ in the ℓ-adic theory.

2.1.2. Let us recall basic notions of p-adic coefficients used in [3]. Let L be an extension of K in $\overline{K}$ and $\mathfrak{T} = \{k, R, K, L\}$ the associated geometric base tuple [3, 1.4.10, 2.4.14].

We will also work in the arithmetic setting (p-adic coefficients with Frobenius structure). For this purpose, we need to assume that there exists an automorphism $L \to L$ extending $\sigma : K \to K$ that we still denote by σ, and that there exists a sequence of finite extensions M_n of K in L satisfying $\sigma(M_n) \subset M_n$ and $\cup_n M_n = L$. Then we obtain an arithmetic base tuple $\mathfrak{T}_F = \{k, R, K, L, s, \sigma\}$ [3, 1.4.10, 2.4.14]. We set $L_0 = L^{\sigma=1}$.

Let X be a k-scheme. There exists an L-linear (resp. L_0-linear) triangulated category $\mathrm{D}(X/L)$ (resp. $\mathrm{D}(X/L_F)$) relative to the geometric base tuple $\mathfrak{T}$ (resp. arithmetic base tuple $\mathfrak{T}_F$). This category is denoted by $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(X/\mathfrak{T})$ or $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(X/L)$ (resp. $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(X/\mathfrak{T}_F)$ or $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(X/L_F)$) in [3, 1.1.1, 2.1.16]. There exists a *holonomic t-structure* on $\mathrm{D}(X/L_\blacktriangle)$, whose heart is denoted by $\mathrm{Hol}(X/L_\blacktriangle)$, called *category of holonomic modules*. We denote by $\mathcal{H}^*$ the cohomological functor for holonomic t-structure.

The six functor formalism for $\mathrm{D}(X/L)$ (resp. $\mathrm{D}(X/L_F)$) has been established recently. We refer to [5,6] and [3, 2.3] for details and to [3, 1.1.3] for a summary. Here we only collect some notations.

(i) Let $f : X \to Y$ be a morphism of k-schemes. For $\blacktriangle \in \{\emptyset, F\}$, there exist triangulated functors

$$f_+, f_! : \mathrm{D}(X/L_\blacktriangle) \to \mathrm{D}(Y/L_\blacktriangle), \qquad f^+, f^! : \mathrm{D}(Y/L_\blacktriangle) \to \mathrm{D}(X/L_\blacktriangle),$$

such that (f^+, f_+), $(f_!, f^!)$ are adjoint pairs.

(ii) The category $\mathrm{D}(X/L_\blacktriangle)$ is a closed symmetric monoidal category, namely it is equipped with a tensor product functor $\otimes$ and the unit object $L_X = \pi^+(L)$, where $\pi : X \to \mathrm{Spec}(k)$ is the structure morphism and L is the constant module in degree 0. The *internal Hom* functor $\mathscr{H}om_X$ is a right adjoint of $\otimes$.

(iii) There exists a duality functor $\mathbb{D}_X = \mathscr{H}om_X(-, p^! L)$ from $\mathrm{D}(X/L_\blacktriangle)$ to its opposite category [3, 1.1.4]. We set $(-)\widetilde{\otimes}(-) = \mathbb{D}_X(\mathbb{D}_X(-) \otimes \mathbb{D}_X(-))$.

2.1.3. For any object $\mathscr{M}$ of $\mathrm{D}(X/L)$ and the structural morphism $f : X \to \mathrm{Spec}(k)$, we set

$$\mathrm{H}^*(X, \mathscr{M}) = \mathcal{H}^* f_+(\mathscr{M}), \qquad \mathrm{H}^*_{\mathrm{c}}(X, \mathscr{M}) = \mathcal{H}^* f_!(\mathscr{M}),$$

and call them *cohomology groups, compact support cohomology groups of* $\mathscr{M}$, respectively. Note that they are finite dimensional L-vector spaces. If $\mathscr{M}$ is an object of $\mathrm{D}(X/L_F)$, then above cohomology groups are equipped with a Frobenius structure.

Suppose that there exists a finite filtration of closed subschemes $\{X_i\}_{i \in \mathbb{Z}}$ of X, with closed immersions $X_{i+1} \hookrightarrow X_i$ such that $X_i = X$ for i small enough and $X_i = \emptyset$ for i big enough. We deduce from the distinguished triangle [3, 1.1.3(10), 2.2.9] a spectral sequence (cf. [33] *2.5)

$$\mathrm{E}_1^{ij} = \mathrm{H}_{\mathrm{c}}^{i+j}(X_i - X_{i+1}, \mathscr{M}) \Rightarrow \mathrm{H}_{\mathrm{c}}^{i+j}(X, \mathscr{M}). \qquad (2.1.3.1)$$

2.1.4. Let X be a smooth k-scheme of dimension $d : \pi_0(X) \to \mathbb{N}$. There exists a full subcategory $\mathrm{Sm}(X/L_\blacktriangle)$ of $\mathrm{Hol}(X/L_\blacktriangle)[-d] \subset \mathrm{D}(X/L)$ consisting of *smooth objects* [3, 1.1.3(12) and 2.4.15]. In general, we say a complex $\mathscr{M} \in \mathrm{D}(X/L_\blacktriangle)$ is *smooth* if $\mathcal{H}^i(\mathscr{M})[-d]$ belongs to $\mathrm{Sm}(X/L_\blacktriangle)$ for every i.

When $L = K$, there exists an equivalence $\widetilde{\mathrm{Sp}}_*$ between $\mathrm{Sm}(X/K)$ (resp. $\mathrm{Sm}(X/K_F)$) and $\mathrm{Isoc}^{\dagger\dagger}(X/K)$ (resp. F-$\mathrm{Isoc}^{\dagger}(X/K)$). In the following, we identify these two categories by $\widetilde{\mathrm{Sp}}_*$ and we use alternatively these two notations. Suppose X admits a smooth compactification $\overline{X}$ such that $\overline{X}$ possesses a smooth lifting over R and that $\overline{X} - X$ is a divisor. If $\mathrm{H}^*_{\mathrm{rig}}(X, -)$ denotes the rigid cohomology [19], we have canonical isomorphisms for any object $\mathscr{M}$ of $\mathrm{Isoc}^{\dagger\dagger}(X/K)$ (resp. F-$\mathrm{Isoc}^{\dagger}(X/K)$) [1, 5.9]:

$$\mathrm{H}^*_{\mathrm{rig}}(X, \mathscr{M}) \simeq \mathrm{H}^*(X, \widetilde{\mathrm{Sp}}_*(\mathscr{M})), \qquad \mathrm{H}^*_{\mathrm{rig,c}}(X, \mathscr{M}) \simeq \mathrm{H}^*_{\mathrm{c}}(X, \widetilde{\mathrm{Sp}}_*(\mathscr{M})), \qquad (2.1.4.1)$$

as objects of Vec_K (resp. $F\text{-}\mathrm{Vec}_K$). In particular, we have $\mathrm{H}^0(\mathbb{A}^n, L) \simeq L$, $\mathrm{H}^i(\mathbb{A}^n, L) = 0$ for $i \neq 0$ and $\mathrm{H}^{2n}_{\mathrm{c}}(\mathbb{A}^n, L) \simeq L$, $\mathrm{H}^i_{\mathrm{c}}(\mathbb{A}^n, L) = 0$ for $i \neq 2n$.

2.1.5. Let X be a k-scheme. There exists a *constructible t-structure* (*c-t-structure* in short) on $\mathrm{D}(X/L)$ (cf. [3] 1.3, 2.2.23). When $X = \mathrm{Spec}(k)$, the constructible t-structure coincides with the holonomic one (2.1.3). If X a smooth k-scheme, any object of $\mathrm{Sm}(X/L)$ is constructible.

The heart of c-t-structure is denoted by $\mathrm{Con}(X)$, called the *category of constructible modules*. The cohomology functor of c-t-structure is denoted by ${}^{\mathrm{c}}\mathcal{H}^*$.

Let $f : X \to Y$ be a morphism between k-schemes. The functor f^+ is c-t-exact and f_+ is left c-t-exact. If i is a closed immersion, then i_+ is c-t-exact. If j is an open immersion, then $j_!$ is c-t-exact [3, 1.3.4].

Using constructible t-structure, we show an analogue of [16, 4.2.5] for arithmetic $\mathscr{D}$-modules.

Proposition 2.1.6 *Let $f : X \to Y$ be a smooth morphism of k-schemes of relative dimension d with geometrically connected fibers. Then the functor $f^+[d] : \mathrm{Hol}(Y/L_\blacktriangle) \to \mathrm{Hol}(X/L_\blacktriangle)$ (for $\blacktriangle \in \{\emptyset, F\}$) is fully faithful.*

Lemma 2.1.7 *Let $\mathscr{M}$ be an object of $\mathrm{D}^{\leq 0}(X/L)$ and $\mathscr{N}$ an object of $\mathrm{D}^{\geq 0}(X/L)$. Then $\mathscr{H}om_X(\mathscr{M}, \mathscr{N})$ belongs to ${}^{\mathrm{c}}\mathrm{D}^{\geq 0}(X/L)$ (2.1.5).*

Proof We prove by induction on the dimension of X. The assertion is clear if $\dim X = 0$. To prove the assertion, we can reduce to the case where $\mathscr{M}, \mathscr{N} \in \mathrm{Hol}(X/L)$. Then there exists a dense smooth open subscheme $j : U \to X$ such that $\mathscr{M}|_U, \mathscr{N}|_U$ are smooth. Let $i : Z \to X$ be the complement of U and consider the triangle

$$i_+ i^! \mathscr{H}om_X(\mathscr{M}, \mathscr{N}) \to \mathscr{H}om_X(\mathscr{M}, \mathscr{N}) \to j_+ j^+ \mathscr{H}om_X(\mathscr{M}, \mathscr{N}) \to .$$

Since $i^! \mathscr{H}om_X(\mathscr{M}, \mathscr{N}) \simeq \mathscr{H}om_X(i^+\mathscr{M}, i^!\mathscr{N})$ [3, 1.1.5], the first term belongs to ${}^{\mathrm{c}}\mathrm{D}^{\geq 0}(X/L)$ by induction hypotheses. Note that $\mathscr{H}om_U(\mathscr{M}|_U, \mathscr{N}|_U) \simeq \mathbb{D}_U(\mathscr{M}|_U \otimes \mathbb{D}_U(\mathscr{N}|_U))$ is a smooth module and of constructible degree 0. Then $j_+ j^+ \mathscr{H}om_X(\mathscr{M}, \mathscr{N})$ belongs to ${}^{\mathrm{c}}\mathrm{D}^{\geq 0}(X/L)$ and the assertion follows. □

2.1.8. *Proof of Proposition* 2.1.6. Since Frobenius pullback induces an equivalence of categories, it suffices to show the assertion for $\mathrm{Hol}(-/L)$. Let $\mathscr{M}, \mathscr{N}$ be two objects of $\mathrm{Hol}(Y/L)$. Since f is smooth, we deduce from $f^! \mathscr{H}om_Y(\mathscr{M}, \mathscr{N}) \simeq \mathscr{H}om_X(f^+\mathscr{M}, f^!\mathscr{N})$ [3, 1.1.5] an isomorphism

$$f^+ \mathscr{H}om_Y(\mathscr{M}, \mathscr{N}) \xrightarrow{\sim} \mathscr{H}om_X(f^+\mathscr{M}, f^+\mathscr{N}).$$

By applying ${}^{\mathrm{c}}\mathcal{H}^0 f_+ {}^{\mathrm{c}}\mathcal{H}^0(-)$ to the above isomorphism and Lemma 2.1.7, we have

$$\begin{aligned} &{}^{\mathrm{c}}\mathcal{H}^0 f_+ f^+\big({}^{\mathrm{c}}\mathcal{H}^0(\mathscr{H}om_Y(\mathscr{M},\mathscr{N}))\big) \\ &\quad \xrightarrow{\sim} {}^{\mathrm{c}}\mathcal{H}^0 f_+ {}^{\mathrm{c}}\mathcal{H}^0(\mathscr{H}om_X(f^+\mathscr{M}[d], f^+\mathscr{N}[d])). \end{aligned} \tag{2.1.8.1}$$

We claim that for any constructible module $\mathscr{F}$ on Y, there is a canonical isomorphism

$$\mathscr{F} \xrightarrow{\sim} {}^{\mathrm{c}}\mathcal{H}^0 f_+ f^+ \mathscr{F}. \tag{2.1.8.2}$$

Then, by Lemma 2.1.7, the proposition follows by applying $\mathrm{H}^0(Y,-)$ to the composition of (2.1.8.1) and (2.1.8.2).

By smooth base change, to prove (2.1.8.2), we can reduce to the case where Y is a point. After extending the scalar L and the base field k (cf. [3] 1.4.11), we may assume moreover that $Y=\mathrm{Spec}(k)$. In this case, the isomorphism (2.1.8.2) follows from the geometrical connectedness of X. □

2.1.9. Let $u: Y \to X$ be a locally closed immersion. We refer to [6, § 1.4] for the *intermediate extension functor* $u_{!+}: \mathrm{Hol}(Y/L_\blacktriangle) \to \mathrm{Hol}(X/L_\blacktriangle)$. Recall [6, 1.4.7] that if $\mathscr{M}$ is irreducible, then $u_{!+}(\mathscr{M})$ is the unique irreducible sub-object of $\mathcal{H}^0(u_+\mathscr{M})$ (resp. irreducible quotient of $\mathcal{H}^0(u_!\mathscr{M})$) in $\mathrm{Hol}(X/L_\blacktriangle)$.

Lemma 2.1.10 *Let $j: U \to X$ be an open subscheme of X and $i: Z \to X$ its complement.*

(i) *Given a holonomic module $\mathscr{M}$ on U, $j_{!+}(\mathscr{M})$ is the unique extension $\mathscr{F}$ of $\mathscr{M}$ to* $\mathrm{Hol}(X/L_\blacktriangle)$ *such that $i^+\mathscr{F} \in \mathrm{D}^{\leq -1}(Z/L_\blacktriangle)$ and that $i^!\mathscr{F} \in \mathrm{D}^{\geq 1}(Z/L_\blacktriangle)$.*

(ii) *If X is smooth and $\mathscr{F}$ is a smooth holonomic module on X, then $j_{!+}(\mathscr{F}|_U) \simeq \mathscr{F}$.*

Proof (i) Since $j_!, i^+$ are right exact [6, 1.3.2], $\mathcal{H}^0 i^+(\mathcal{H}^0(j_!(\mathscr{M}))) = 0$. By applying i^+ to $0 \to \mathrm{Ker}(\theta^0_{j,\mathscr{M}}) \to \mathcal{H}^0(j_!(\mathscr{M})) \to j_{!+}(\mathscr{M}) \to 0$, we obtain $i^+(j_{!+}(\mathscr{M})) \in \mathrm{D}^{\leq -1}(Z/L)$. We prove $i^!\mathscr{F} \in \mathrm{D}^{\geq 1}(Z/L)$ in a dual way.

Conversely, given such an extension $\mathscr{F}$, we can prove that the adjunction morphism $\mathcal{H}^0 j_!(\mathscr{M}) \to \mathscr{F}$ (resp. $\mathscr{F} \to \mathcal{H}^0 j_+(\mathscr{M})$) is surjective (resp. injective) by the Berthelot–Kashiwara theorem [3, 1.1.3(10), 2.2.9]. Assertion (i) follows.

(ii) The intermediate extension is stable under composition [6, 1.4.5]. Then we can reduce to the case where Z is smooth over k. In this case, assertion (ii) follows from (i) and [3, 2.4.15]. □

2.2 Universal local acyclicity and nearby/vanishing cycles

In the following, we write simply $\mathrm{D}(X)$ (resp. $\mathrm{Hol}(X)$) for $\mathrm{D}(X/L_{\blacktriangle})$ (resp. $\mathrm{Hol}(X/L_{\blacktriangle})$).

2.2.1. Following Braverman–Gaitsgory [23, 5.1], we propose a notion of *(universal) local acyclicity* for arithmetic $\mathscr{D}$-modules with respect to a morphism to a smooth target.

For a smooth k-scheme X, we denote by $d_X : \pi_0(X) \to \mathbb{N}$ the dimension of X. Let $g : X_1 \to X_2$ be a morphism of k-schemes and $\mathscr{F}, \mathscr{F}'$ two objects of $\mathrm{D}(X_2)$. We consider the composition

$$g_!(g^+(\mathscr{F}) \otimes g^!(\mathscr{F}')) \simeq \mathscr{F} \otimes g_!(g^!(\mathscr{F}')) \to \mathscr{F} \otimes \mathscr{F}'$$

and its adjunction: $g^+(\mathscr{F}) \otimes g^!(\mathscr{F}') \to g^!(\mathscr{F} \otimes \mathscr{F}')$.

Now let S be a smooth k-scheme and $f : X \to S$ a morphism of k-schemes. We set $X_1 = X$, $X_2 = X \times S$, $\mathscr{F}' = L_{X_2}$ and take g to be the graph of f. By Poincaré duality, we have $L_{X_1}(-d_S)[-2d_S] \xrightarrow{\sim} g^!(L_{X_2})$. Then, we obtain a canonical morphism

$$g^+(\mathscr{F}) \to g^!(\mathscr{F})(d_S)[2d_S].$$

By taking $\mathscr{F}$ to be $\mathscr{M} \boxtimes \mathscr{N}$ [6, 1.1.8, 1.3.3], we obtain a canonical morphism

$$\mathscr{M} \otimes f^+(\mathscr{N}) \to (\mathscr{M} \widetilde{\otimes} f^!(\mathscr{N}))(d_S)[2d_S]. \tag{2.2.1.1}$$

Definition 2.2.2 Let S be a smooth k-scheme and $f : X \to S$ a morphism of k-schemes. We say that an object $\mathscr{M}$ of $\mathrm{D}(X)$ is *locally acyclic (LA) with respect to* f, if the morphism (2.2.1.1) is an isomorphism for any object $\mathscr{N}$ of $\mathrm{D}(S)$. We say that $\mathscr{M}$ is *universally locally acyclic (ULA) with respect to* f, if for any morphism of smooth k-schemes $S' \to S$, the $+$-inverse image of $\mathscr{M}$ to $X \times_S S'$ is locally acyclic with respect to $X \times_S S' \to S'$.

Proposition 2.2.3 *Keep the notation of Definition 2.2.2 and let $\mathscr{M}$ be an object of* $\mathrm{D}(X)$.

(i) *Any object $\mathscr{M}$ of* $\mathrm{D}(X)$ *is ULA with respect to the structure morphism* $X \to \mathrm{Spec}(k)$.

(ii) *Let $g : Y \to X$ be a smooth (resp. smooth surjective) morphism. Then $g^+(\mathscr{M})$ on Y is LA with respect to $f \circ g$ if (resp. if and only if) $\mathscr{M}$ is LA with respect to f.*

(iii) *If $g : S \to S'$ is a smooth morphism of smooth k-schemes and $\mathscr{M}$ is LA with respect to a morphism $f : X \to S$, then $\mathscr{M}$ is LA with respect to $g \circ f$.*

(iv) *Let $h : Y \to S$ be a morphism of finite type and $g : X \to Y$ a proper S-morphism (resp. a closed immersion). Then $g_+(\mathscr{M})$ is LA with respect to h if (resp. if and only if) $\mathscr{M}$ is LA with respect to f.*

(v) *If $\mathscr{M}$ is LA with respect to f, then so is its dual $\mathbb{D}_X(\mathscr{M})$.*

Proof (i) Let S be a smooth k-scheme and $\mathscr{N}$ an object of $\mathrm{D}(S)$. We show that the canonical morphism

$$(\mathrm{id}_X \times \Delta)^+(\mathscr{M} \boxtimes p_2^+(\mathscr{N})) \to (\mathrm{id}_X \times \Delta)^!(\mathscr{M} \boxtimes p_2^+(\mathscr{N}))(d_S)[2d_S]$$

is an isomorphism, where $\Delta : S \to S \times S$ is the diagonal map and $p_2 : S \times S \to S$ is the projection in the second component. We can reduce to show that the canonical morphism

$$\mathscr{N} \to \Delta^!(p_2^+(\mathscr{N}))(2d_S)[2d_S]$$

is an isomorphism. After taking dual functor, the assertion follows from [3, 1.5.14].

The rest of the proposition follow from Poincaré duality, Berthelot–Kashiwara theorem and smooth descent [3, 2.1.13]. We left the proof to readers. □

2.2.4. In a recent work [4], Abe formulated the nearby and vanishing cycle functors for holonomic arithmetic $\mathscr{D}$-modules, based on the unipotent nearby and vanishing cycle functors introduced by himself and Caro [5].

Let $f : X \to \mathbb{A}^1_k$ be a morphism of k-schemes. We denote by $j : U = f^{-1}(\mathbb{G}_m) \to X$ the open immersion and by $i : X_0 = X - U \to X$ its complement. We first review the unipotent nearby cycle functor

$$\Psi_f^{\mathrm{un}} : \mathrm{Hol}(U) \to \mathrm{Hol}(X_0).$$

We set $\mathcal{O}_{\mathbb{G}_m} = \mathscr{O}_{\widehat{\mathbb{P}}^1_R,\mathbb{Q}}({}^\dagger\{0,\infty\})$ (see 1.3.8). For $n \geq 1$, we define a free $\mathcal{O}_{\mathbb{G}_m}$-module Log^n of rank n

$$\mathrm{Log}^n = \oplus_{i=0}^{n-1} \mathcal{O}_{\mathbb{G}_m} \cdot \log^{[i]},$$

generated by the symbols $\log^{[i]}$. There exists a unique $\mathscr{D}^\dagger_{\widehat{\mathbb{P}}^1_R,\mathbb{Q}}(\{0,\infty\})$-module structure on Log^n defined for $i \geq 0$ and $g \in \mathcal{O}_{\mathbb{G}_m}$ by

$$\nabla_{\partial_t}(g \cdot \log^{[i]}) = \partial_t(g) \cdot \log^{[i]} + \frac{g}{t} \cdot \log^{[i-1]},$$

where t is the local coordinate of $\mathbb{G}_m$ and $\log^{[j]} = 0$ for $j < 0$. There exists a canonical Frobenius structure on Log^n sending $\log^{[i]}$ to $q^i \log^{[i]}$. This defines an overconvergent F-isocrystal on $\mathbb{G}_m$ and then a smooth object of

$\mathrm{Hol}(\mathbb{G}_m/K_F)$. We still denote by Log^n the extension of scalars $\iota_{L/K}(\mathrm{Log}^n)$ in $\mathrm{Hol}(\mathbb{G}_m)$.

We set $\mathrm{Log}^n_f = f^+\mathrm{Log}^n \in \mathrm{Hol}(U)$ and define for $\mathscr{F} \in \mathrm{Hol}(U)$:[2]

$$\Psi^{\mathrm{un}}_f(\mathscr{F}) = \varinjlim_{n\geq 1} \mathrm{Ker}(j_!(\mathscr{F}\otimes \mathrm{Log}^n_f) \to j_+(\mathscr{F}\otimes \mathrm{Log}^n_f)). \tag{2.2.4.1}$$

This limit is representable in $\mathrm{Hol}(X_0)$ by [5, lemma 2.4].

The vanishing cycle functor Φ^{un}_f is defined in a similar way. The functors Ψ^{un}_f, Φ^{un}_f are exact [5, 2.7] and extend to triangulated categories. There exists a distinguished triangle $i^+[-1] \to \Psi^{\mathrm{un}}_f \to \Phi^{\mathrm{un}}_f \xrightarrow{+1}$.

To define nearby and vanishing cycles functors over a strict henselian trait, we consider $\mathbf{Pro}(k)$ the full subcategory of Noetherian schemes over k which can be representable by a projective limit of a projective system of k-schemes whose transition morphisms are affine and étale. The category $\mathbf{Pro}(k)$ is closed under henselization (resp. strict henselization) [4, 1.3]). Given an object X of $\mathbf{Pro}(k)$, the triangulated category $\mathrm{D}(X)$ of arithmetic $\mathscr{D}$-modules on X is well-defined and one can extend the definition of cohomological functors to $\mathrm{D}(X)$ (cf. [4] 1.4).

Let (S, s, η) be a strict henselian trait in $\mathbf{Pro}(k)$ and $f : X \to S$ a morphism of finite type. In this setting, Abe defined the (unipotent) nearby and vanishing cycles functors for f (cf. [4] 1.7, 1.8, 2.2)

$$\Psi_f, \Psi^{\mathrm{un}}_f, \Phi_f, \Phi^{\mathrm{un}}_f : \mathrm{Hol}(X) \to \mathrm{Hol}(X_s). \tag{2.2.4.2}$$

Proposition 2.2.5 *Keep the notation of Definition 2.2.2 and let D be a smooth effective divisor in S, $i : Z = f^{-1}(D) \to X$ the closed immersion and $j : U \to X$ its complement. Let $\mathscr{M}$ be an object of* $\mathrm{D}(X)$ *such that it is LA with respect to f and that $\mathscr{M}|_U$ is holonomic.*

(i) *There exists canonical isomorphisms:*

$$\mathscr{M} \simeq j_{!+}(\mathscr{M}|_U), \qquad i^+\mathscr{M}[-1] \xrightarrow{\sim} i^!\mathscr{M}(1)[1]. \tag{2.2.5.1}$$

In particular, $\mathscr{M}$ and $i^+\mathscr{M}[-1]$ are holonomic.

(ii) *The holonomic module $i^+\mathscr{M}[-1]$ is LA with respect to $f\circ i$ and $f|_Z : Z \to D$.*

Proof (i) By étale descent for holonomic modules [3, 2.1.13], we may assume that there is a smooth morphism $g : S \to \mathbb{A}^1$ such that $D = g^{-1}(0)$. By Proposition 2.2.3(iii), $\mathscr{M}$ is LA with respect to $g\circ f : X \to \mathbb{A}^1$. Then we

[2] We adopt the definition of [4], which is different from that of [5] by a Tate twist.

can reduce to the case $f : X \to \mathbb{A}^1$ and $Z = f^{-1}(0)$. We will show that $\Phi_f^{\mathrm{un}}(\mathscr{M}) = 0$, i.e. the canonical morphism

$$i^+\mathscr{M}[-1] \to \Psi_f^{\mathrm{un}}(\mathscr{M}) \tag{2.2.5.2}$$

is an isomorphism. We denote by $\overline{j} : \mathbb{G}_m \to \mathbb{A}^1$ the canonical morphism and abusively by f the restriction $f|_U : U \to \mathbb{G}_m$. By the projection formula, we have

$$j_!(\mathscr{M}|_U \otimes f^+ \mathrm{Log}^n) \xrightarrow{\sim} \mathscr{M} \otimes j_! f^+ \mathrm{Log}^n \simeq \mathscr{M} \otimes f^+ \overline{j}_! \mathrm{Log}^n .$$

On the other hand, by the projection formula and the LA property of $\mathscr{M}$, we have

$$\begin{aligned} j_+(\mathscr{M}|_U \otimes (f^+ \mathrm{Log}^n)) &\xrightarrow{\sim} j_+(\mathscr{M}|_U \widetilde{\otimes}(f^! \mathrm{Log}^n))(d_X)[2d_X] \\ &\xrightarrow{\sim} \mathscr{M}\widetilde{\otimes}(j_+ f^! \mathrm{Log}^n)(d_X)[2d_X] \\ &\simeq \mathscr{M} \otimes (f^+ \overline{j}_+ \mathrm{Log}^n). \end{aligned}$$

Via the above isomorphisms, the canonical morphism $j_!(\mathscr{M}|_U \otimes (f^+ \mathrm{Log}^n)) \to j_+(\mathscr{M}|_U \otimes (f^+ \mathrm{Log}^n))$ coincides with the canonical morphism

$$\mathscr{M} \otimes (f^+(\overline{j}_! \mathrm{Log}^n \to \overline{j}_+ \mathrm{Log}^n)).$$

To prove that (2.2.5.2) is an isomorphism, we can reduce to the case where f is the identity map of $\mathbb{A}^1$ and $\mathscr{M}$ is the constant module $L_{\mathbb{A}^1}[1]$ on $\mathbb{A}^1$. If we denote by N_n the action induced by $t\partial_t$ on the fiber $(\mathrm{Log}^n)_0$ of Log^n at 0 (called *residue morphism* in [6] 3.2.11), then $\mathrm{Ker}(j_!(\mathrm{Log}_n) \to j_+(\mathrm{Log}_n))$ is isomorphic to $\mathrm{Ker}(N_n)$ (cf. [5] proof of lemma 2.4). The connection of Log^n has a maximal unipotent monodromy at 0. Then $\mathrm{Ker}(N_n)$ is one-dimensional and transition map $\mathrm{Ker}(N_n) \to \mathrm{Ker}(N_{n+1})$ is an isomorphism. Hence (2.2.5.2) is an isomorphism in this case.

In particular $i^+\mathscr{M}[-1]$ is holonomic. By Proposition 2.2.3(v) and the commutation between nearby cycle and dual functors [5, 2.5], the second isomorphism of (2.2.5.1) follows from (2.2.5.2):

$$\Psi_f^{\mathrm{un}}(\mathscr{M}) \simeq \mathbb{D}_{X_0} \Psi_f^{\mathrm{un}}(\mathbb{D}_X(\mathscr{M}))(1) \xrightarrow{\sim} i^! \mathscr{M}(1)[1].$$

Then we deduce $\mathscr{M} \simeq j_{!+}(\mathscr{M}|_U)$ by 2.1.10. This finishes the proof of assertion (i).

Assertion (ii) follows from the six functor formalism. We left the proof to readers. □

Corollary 2.2.6 *If an object $\mathscr{M}$ of* $\mathrm{D}(X)$ *is ULA with respect to f, then, for any strict henselian trait T of* $\mathbf{Pro}(k)$ *and any morphism $g : T \to S$, we have $\Phi^{\mathrm{un}}_{f_T}(\mathscr{M}|_{X_T}) = 0$ and $\Phi_{f_T}(\mathscr{M}|_{X_T}) = 0$, where $f_T : X_T \to T$ is the base change of f by g.*

Proof By definition [4, 1.9], it suffices to show that the unipotent vanishing cycle $\Phi^{\mathrm{un}}_{f_T}(\mathscr{M}|_{X_T})$ vanishes.

There exists a smooth k-scheme S', a smooth effective divisor D of S' with generic point η_D and a morphism $h : S' \to S$ such that the strict henselization of S' at η_D is isomorphic to T and that g is induced by h. We denote by $f_{S'} : X_{S'} \to S'$ the base change of f by h. After shrinking S', we may assume that there exists a smooth morphism $\pi : S' \to \mathbb{A}^1_k$ with $D = \pi^{-1}(0)$.

By definition (cf. [4] 1.7-1.8), we reduce to show that $\Phi^{\mathrm{un}}_{\pi\circ f_{S'}}(\mathscr{M}|_{X_{S'}}) = 0$. But this follows from Proposition 2.2.3(iii) and the proof of (2.2.5.2). Then the assertion follows. □

2.2.7. In 4.1, we will use the notion of holonomic modules over an algebraic stack and apply cohomological functors of a *schematic* morphism of algebraic stacks, that we briefly explain in the following.

Let $\mathfrak{X}$ be an algebraic stack of finite type over k. Let $\mathrm{Hol}(\mathfrak{X})$ be the category of holonomic modules on $\mathfrak{X}$ [3, 2.1.16] and $\mathrm{D}(\mathfrak{X})$ its derived version (corresponds to the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$ in *loc. cit*). The dual functor $\mathbb{D}_{\mathfrak{X}}$ is defined in [3, 2.2]. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a schematic morphism, $Y_\bullet \to \mathfrak{Y}$ a simplicial algebraic space presentation. By pullback, we obtain a simplicial presentation $X_\bullet \to \mathfrak{X}$ and a Cartesian morphisms $f_\bullet : X_\bullet \to Y_\bullet$. Then the constructions of [3, 2.1.10 and 2.2.14] allow us to define cohomological functors:

$$f_+ : \mathrm{D}(\mathfrak{X}) \simeq \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(X_\bullet) \rightleftarrows \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(Y_\bullet) \simeq \mathrm{D}(\mathfrak{Y}) : f^!.$$

Given an object $\mathscr{M}$ of $\mathrm{D}(\mathfrak{X})$ and a morphism $g : \mathfrak{X} \to S$ to a smooth k-scheme S, we say $\mathscr{M}$ is ULA with respect to g if its $+$-pullback to a presentation $U \to \mathfrak{X}$ is ULA with respect to $U \to S$.

Suppose S is moreover a curve. Let s be a closed point of S and $S_{(s)}$ the strict henselian at s. Since nearby/vanishing cycle functors commute with smooth pullbacks, we can extend the definition of nearby/vanishing cycle functors for $g \times_S S_{(s)}$.

2.3 Complements on the local monodromy of an overconvergent F-isocrystal on a curve

2.3.1. We denote by $\mathcal{R}_K$ the Robba ring over K and by $\mathrm{MC}(\mathcal{R}_K/K)$ (resp. $\mathrm{MC}(\mathcal{R}/\overline{K})$) the category of ∇-modules finitely presented over $\mathcal{R}_K$ (resp. over $\mathcal{R} = \mathcal{R} \otimes_K \overline{K}$).

The full subcategory $\mathrm{MC}^{\mathrm{uni}}(\mathcal{R}/\overline{K})$ of $\mathrm{MC}(\mathcal{R}/\overline{K})$, consisting of unipotent objects, is a Tannakian category over $\overline{K}$ and its Tannakian group is isomorphic to $\mathbb{G}_a$ [60, 4.1]. There is an equivalence between the category $\mathrm{Vec}^{\mathrm{nil}}_{\overline{K}}$ of finite dimensional $\overline{K}$-vector space with a nilpotent endomorphism and $\mathrm{MC}^{\mathrm{uni}}(\mathcal{R}/\overline{K})$, given by the functor $(V_0, N) \mapsto (V_0 \otimes_{\overline{K}} \mathcal{R}, \nabla_N)$, where the connection ∇_N is defined by $\nabla_N(v \otimes 1) = Nv \otimes dx/x$.

We denote by $K\{x\}$ the K-algebra of analytic functions on the open unit disc $|x| < 1$, i.e.

$$K\{x\} = \left\{ \sum_{n\geq 0} a_n x^n \in K[\![x]\!];\ |a_n|\rho^n \to 0\ (n \to \infty)\ \forall \rho \in [0,1) \right\}. \tag{2.3.1.1}$$

Let $\Omega^1_{K\{x\}}(\log)$ be the free $K\{x\}$-module of rank 1 with basis dx/x and consider the following canonical derivation $d : K\{x\} \to \Omega^1_{K\{x\}}(\log),\ f \mapsto xf'(x)dx/x$. An unipotent object (M, ∇) of $\mathrm{MC}(\mathcal{R}_K/K)$ extends to a *log ∇-module* $(M^{\log}, \nabla^{\log})$ over $K\{x\}$. Then $(M^{\log}|_{x=0}, N = \mathrm{Res}\,\nabla^{\log})$ is the object of $\mathrm{Vec}^{\mathrm{nil}}_K$ associated to (M, ∇). There exists a canonical isomorphism between $\mathrm{Coker}(N)$ and the solution space $\mathrm{Sol}(M)$:

$$\mathrm{Coker}(N) \xrightarrow{\sim} \mathrm{Sol}(M) = \mathrm{Hom}_{K\{x\}}((M, \nabla), (K\{x\}, d))^{\nabla=0}. \tag{2.3.1.2}$$

If the connection ∇ is defined by a differential operator D, then $\mathrm{Sol}(M)$ is the solution space of D.

Let I (resp. P) the inertia (resp. wild inertia) subgroup of $\mathrm{Gal}(k(\!(t)\!)^{\mathrm{sep}}/k(\!(t)\!))$. The full subcategory $\mathrm{MCF}(\mathcal{R}/\overline{K})$ of $\mathrm{MC}(\mathcal{R}/\overline{K})$, consisting of objects admitting a Frobenius structure, is a Tannakian category over $\overline{K}$ and its Tannakian group is isomorphic to $I \times \mathbb{G}_a$ [11, 3.4, 7.1.1]. The $\mathbb{G}_a$-action is the same as a nilpotent monodromy operator commuting with I-action. By a theorem of Matsuda–Tsuzuki [60,73] (cf. [11] 7.1.2), the irregularity of an object M of $\mathrm{MCF}(\mathcal{R}/\overline{K})$, defined by p-adic slopes [29], is equal to the Swan conductor of the representation of I on a fiber of M.

2.3.2. Let X be a smooth curve over k, $i : \{x\} \to X$ a closed k-point and $j : U \to X$ its complement. There exists a canonical functor defined by restriction at x:

$$|_x : \mathrm{Isoc}^{\dagger\dagger}(U/K) \to \mathrm{MCF}(\mathcal{R}_{\overline{K}}/\overline{K}). \tag{2.3.2.1}$$

We refer to [70] and [53, § 6] for the definition of log convergent (F-)isocrystals on X with a log pole at x. Let $\mathscr{E}$ be an object of $\mathrm{Isoc}^{\dagger\dagger}(U/K)$ (resp. F-$\mathrm{Isoc}^{\dagger}(U/K)$). A log-extenbility criterion of Kedlaya [53, 6.3.2] says that if $\mathscr{E}|_x$ is unipotent, then $\mathscr{E}$ extends to a *log convergent isocrystal (resp. F-isocrystal)* $\mathscr{E}^{\log}$ on X with a log pole at x.

The fiber $\mathscr{E}_x^{\log}$ of $\mathscr{E}^{\log}$ at x is a K-vector space equipped with a nilpotent operator. If $\mathscr{E}$ moreover has a Frobenius structure, then $\mathscr{E}_x^{\log}$ is a (φ, N)-*module*, that is a K-vector space V equipped with a nilpotent operator $N : V \to V$ and a σ-semilinear automorphism $\varphi : V \to V$ such that $\varphi^{-1} N \varphi = qN$.

Proposition 2.3.3 *Keep the above assumption. Let $\phi : S \to X$ be the strict henselization at x, $\mathscr{E}$ a smooth holonomic module on U and V the $I_x \times \mathbb{G}_a$-representation associated to $\mathscr{E}|_x$. Then there exists a canonical isomorphism of inclusion of $\overline{K}$-vector spaces (resp. $\overline{K}$-vector spaces with Frobenius structure):*

$$(i^+(j_{!+}(\mathscr{E}))[-1] \hookrightarrow \Psi_{\mathrm{id}}(\phi^+(j_{!+}(\mathscr{E})))) \xrightarrow{\sim} (V^{I_x \times \mathbb{G}_a} \hookrightarrow V^{I_x}).$$

Proof We first prove the case where $\mathscr{E}|_x$ is unipotent. We may assume there exists a morphism $f : X \to \mathbb{A}^1$ étale outside $\{x\} = f^{-1}(0)$. As $\mathscr{E}|_x$ is unipotent, $\Psi_{\mathrm{id}}(\phi^+(j_{!+}(\mathscr{E})))$ is calculated by $\Psi_f^{\mathrm{un}}(\mathscr{E})$.

By [6, 3.4.19, cf. [5] 2.4(1)], we have a Frobenius equivariant isomorphism of vector spaces:

$$\begin{aligned}&\mathrm{Ker}(j_!(\mathscr{E} \otimes \mathrm{Log}_f^n) \to j_+(\mathscr{E} \otimes \mathrm{Log}_f^n))\\ &\quad \simeq \mathrm{Ker}(N^n : (\mathscr{E}^{\log} \otimes \mathrm{Log}_f^n)_x \to (\mathscr{E}^{\log} \otimes \mathrm{Log}_f^n)_x),\end{aligned}$$

where Log_f^n defined in 2.2.4 and $N^n = N_{\mathscr{E}_x^{\log}} \otimes \mathrm{id} + \mathrm{id} \otimes N_{\mathrm{Log}_{f,x}^n}$ is the tensor product of two nilpotent operators. In this case, the isomorphism $\Psi_f^{\mathrm{un}}(\mathscr{E}) \xrightarrow{\sim} \mathscr{E}_x^{\log}$ follows from $\Psi_f^{\mathrm{un}}(\mathscr{E}) \simeq \varinjlim_{n \geq 1} \mathrm{Ker}(N^n)$ and [59, lemma 2.10]. The assertion follows from isomorphisms

$$i_x^+(j_{!+}(\mathscr{E}))[-1] \xrightarrow{\sim} \mathrm{Ker}(j_!(\mathscr{E}) \to j_+(\mathscr{E})) \simeq (\mathscr{E}_x^{\log})^N.$$

In general, by Kedlaya's semistable reduction theorem [55], after shrinking X, we may choose $\pi : X' \to X$ a proper map of smooth curves, finite étale over U such that $\pi^+(\mathscr{E})$ is unipotent at $x' = \pi^{-1}(x)$. Let $j' : U' \to X'$ (resp. $\phi' : S' \to X'$) be the base change of j (resp. ϕ) by π. Then $\mathrm{Aut}(U'/U)$ is a quotient of I_x and ϕ' is the strict henselization at x'. By base change, we obtain I_x-equivariant isomorphisms:

$$\begin{aligned}i_x^+(j_{!+}(\pi_+\pi^+(\mathscr{E}))) &\xrightarrow{\sim} i_{x'}^+(j'_{!+}(\pi^+(\mathscr{E}))),\\ \Psi_{\mathrm{id}}(\phi^+ j_{!+}(\pi_+\pi^+(\mathscr{E}))) &\xrightarrow{\sim} \Psi_{\mathrm{id}}(\phi'^+ j'_{!+}(\pi^+(\mathscr{E}))).\end{aligned}$$

By taking I_x-invariants, we conclude the proposition from the unipotent case. □

Corollary 2.3.4 *Let $\mathscr{M}$ be a holonomic module on X which is smooth outside x. If $\phi : S \to X$ denotes the strict henselization at x and $\Phi_{\mathrm{id}}(\phi^+(\mathscr{M})) = 0$, then $\mathscr{M}$ is smooth.*

Proof For simplicity, we may assume $(X, x) = (\mathbb{A}^1, 0)$. Recall [4, 1.9] that

$$\Phi_{\mathrm{id}}(\phi^+(\mathscr{M})) = \varinjlim_{S' \in \mathrm{Hen}(S)} h_+ \Phi^{\mathrm{un}}_{\mathrm{id}}(h^+\phi^+(\mathscr{M})),$$

where $h : S' \to S$ is taking over the category of henselian traits over S, which are generically étale. The transition morphism in this inductive limit is injective [4, 1.9] and then each term is zero.

We choose a proper map $\pi : X' \to X$ such that $\mathscr{N} := \pi^+(\mathscr{M})$ is unipotent around $x' = \pi^{-1}(x)$ as in 2.3.3. Then, $\Phi^{\mathrm{un}}_{\pi}(\mathscr{N}) = \Phi^{\mathrm{un}}_{\mathrm{id}}(\phi'^+(\mathscr{N}))$ (c.f. [4] 1.7, 1.8) vanish. Hence $i_x^+(\mathscr{M})[-1] = i_{x'}^+(\mathscr{N})[-1] \xrightarrow{\sim} \Psi^{\mathrm{un}}_{\pi}(\mathscr{N})$ are holonomic and have the same rank as $\mathscr{N}$ by Proposition 2.3.3. By a dual argument, the rank of $i_x^!(\mathscr{M})$ and $\mathscr{M}$ are the same. Then we conclude the assertion by [8] lemma 4.1.4. □

Remark 2.3.5 The above proofs follow a similar line of [3] lemma 2.4.11 and are limited to the curve case. In [4] theorem 3.8, Abe proved an analogous result of Corollary 2.3.4 for constructible modules on a k-variety.

2.4 (Co)specialization morphism for de Rham and rigid cohomologies

In this subsection, we review the specialization and cospecialization morphisms between the de Rham and rigid cohomology following [15, § 1] and show the compatibility of these two morphisms in Proposition 2.4.5. We also study the specialization morphism in a relative setting. The results of this subsection will be used in Sect. 4.2.

2.4.1. In this subsection, X denotes a smooth R-scheme of pure relative dimension d and X_k (resp. X_K) its special (resp. generic) fiber. We use the corresponding calligraphic letter $\mathcal{X}$ to denote the rigid analytic space X_K^{an} associated to X_K and the corresponding gothic letter $\mathfrak{X}$ to denote the p-adic completion of X. We denote by $\mathfrak{X}^{\mathrm{rig}}$ the rigid generic fiber of $\mathfrak{X}$ and by $\varepsilon : \mathcal{X} \to X_K$ the canonical morphism of topoi.

Let (M, ∇) be a coherent $\mathscr{O}_{X_K}$-module endowed with an integrable connection (relative to K). We denote by $(M^{\mathrm{an}}, \nabla^{\mathrm{an}})$ its pullback to $\mathcal{X}$ along ε. Then the canonical morphism $\varepsilon^{-1}(M \otimes_{\mathscr{O}_{X_K}} \Omega^\bullet_{X_K}) \to M^{\mathrm{an}} \otimes_{\mathscr{O}_{\mathcal{X}}} \Omega^\bullet_{\mathcal{X}}$ induces a morphism from algebraic de Rham cohomology to analytic de Rham cohomology

$$\begin{aligned}\mathbb{R}\Gamma_{\mathrm{dR}}(X_K, (M, \nabla)) &= \mathbb{R}\Gamma(X_K, M \otimes_{\mathscr{O}_{X_K}} \Omega^{\bullet}_{X_K}) \\ &\to \mathbb{R}\Gamma(\mathcal{X}, M^{\mathrm{an}} \otimes_{\mathscr{O}_{\mathcal{X}}} \Omega^{\bullet}_{\mathcal{X}}) = \mathbb{R}\Gamma_{\mathrm{an}}(\mathcal{X}, (M^{\mathrm{an}}, \nabla^{\mathrm{an}})).\end{aligned} \tag{2.4.1.1}$$

2.4.2. We assume that there exists a smooth proper R-scheme $\overline{X}$ and an open immersion $j : X \to \overline{X}$. Let $\overline{\mathfrak{X}}$ be the p-adic completion of $\overline{X}$. Then the two rigid spaces $\overline{\mathfrak{X}}^{\mathrm{rig}}$ and $\overline{\mathcal{X}} = \overline{X}_K^{\mathrm{an}}$ are isomorphic, and $\mathfrak{X}^{\mathrm{rig}}$ is the tube $]X_k[_{\overline{\mathfrak{X}}}$ of X_k in $\overline{\mathcal{X}}$.

In particular, $\mathcal{X}$ is a strict neighborhood of $\mathfrak{X}^{\mathrm{rig}}$ in $\overline{\mathfrak{X}}^{\mathrm{rig}}$. We denote by $\mathrm{Conn}(X_K)$ (resp. $\mathrm{Conn}(\mathcal{X})$) the category of coherent $\mathscr{O}_{X_K}$-modules with an integrable connection. For any strict neighborhood V of $]X[_{\overline{\mathfrak{X}}}$ in $\overline{\mathfrak{X}}^{\mathrm{rig}}$, we refer to [19, 2.1.1] for the definition of functor $j^{\dagger}$ from the category $\mathbf{Ab}(V)$ of abelian sheaves on V to itself. We associate to M^{an} a $j^{\dagger}\mathscr{O}_{\overline{\mathfrak{X}}^{\mathrm{rig}}}$-module $M^{\dagger} = j^{\dagger}(M^{\mathrm{an}})$, endowed with the corresponding connection. In this setting, we have the following diagram:

$$\begin{tikzcd}
\mathrm{Conn}(X_K) \arrow[r, "(-)^{\mathrm{an}}"'] \arrow[rr, bend left, "(-)^{\dagger}"] & \mathrm{Conn}(\mathcal{X}) \arrow[r, "j^{\dagger}"'] & \mathrm{Conn}(j^{\dagger}\mathscr{O}_{\overline{\mathfrak{X}}^{\mathrm{rig}}}) \arrow[r, "|_{\mathfrak{X}^{\mathrm{rig}}}"] & \mathrm{Conn}(\mathscr{O}_{\mathfrak{X}^{\mathrm{rig}}}) \\
F\text{-}\mathrm{Isoc}^{\dagger}(X_k/K) \arrow[r] & \mathrm{Isoc}^{\dagger\dagger}(X_k/K) \arrow[r, hook] & \mathrm{Isoc}^{\dagger}(X_k/K) \arrow[r, "|_{\mathfrak{X}^{\mathrm{rig}}}"] \arrow[u, hook] & \mathrm{Isoc}(X_k/K) \arrow[u, hook]
\end{tikzcd} \tag{2.4.2.1}$$

where $\mathrm{Isoc}(X_k/K)$ denotes the category of convergent isocrystals on X_k/K and the vertical arrows are fully faithful [19, 2.2.5, 2.2.7]. When $\overline{X}_k \setminus X_k$ is a divisor, the functor $|_{\mathfrak{X}^{\mathrm{rig}}}$ is exact and faithful [20, 4.3.10].

2.4.3. In the following, we assume moreover that the connection on $M^{\dagger}$ is *overconvergent* (i.e. it is isomorphic to an object of $\mathrm{Isoc}^{\dagger}(X_k/K)$). The rigid cohomology $\mathbb{R}\Gamma_{\mathrm{rig}}(X_k/K, M^{\dagger})$ can be calculated by

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X_k/K, M^{\dagger}) \xrightarrow{\sim} \mathbb{R}\Gamma(\mathcal{X}, M^{\dagger} \otimes_{\mathscr{O}_{\mathcal{X}}} \Omega^{\bullet}_{\mathcal{X}}).$$

The adjoint morphism $\mathrm{id} \to j^{\dagger}$ induces a canonical morphism on $\mathcal{X}$

$$M^{\mathrm{an}} \otimes_{\mathscr{O}_{\mathcal{X}}} \Omega^{\bullet}_{\mathcal{X}} \to M^{\dagger} \otimes_{\mathscr{O}_{\mathcal{X}}} \Omega^{\bullet}_{\mathcal{X}}. \tag{2.4.3.1}$$

By composing with (2.4.1.1), we deduce a canonical morphism, denoted by ρ_M and called *specialization morphism* for de Rham and rigid cohomologies:

$$\rho_M : \mathbb{R}\Gamma_{\mathrm{dR}}(X_K, (M, \nabla)) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_k/K, M^\dagger). \tag{2.4.3.2}$$

Let $\mathbb{R}\underline{\Gamma}_{]X_k[}$ be the (derived) functor of local sections supported in the tube $]X_k[_{\mathfrak{X}}$ on $\mathcal{X}$ (or on $\overline{\mathcal{X}}$) [19, 2.1.6]. The rigid cohomology with compact supports and coefficients in $M^\dagger$ is defined as:

$$\mathbb{R}\Gamma_{\mathrm{rig,c}}(X_k/K, M^\dagger) := \mathbb{R}\Gamma(\mathcal{X}, \mathbb{R}\underline{\Gamma}_{]X_k[}(M^{\mathrm{an}} \otimes \Omega^\bullet_{\mathcal{X}})).$$

The canonical morphism

$$\mathbb{R}\underline{\Gamma}_{]X_k[}(M^{\mathrm{an}} \otimes \Omega^\bullet_{\mathcal{X}}) \to M^{\mathrm{an}} \otimes \Omega^\bullet_{\mathcal{X}} \tag{2.4.3.3}$$

and (2.4.3.1) induce a morphism

$$\iota_{\mathrm{rig}} : \mathbb{R}\Gamma_{\mathrm{rig,c}}(X_k/K, M^\dagger) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_k/K, M^\dagger). \tag{2.4.3.4}$$

Via (2.1.4.1), the canonical morphism $\mathrm{H}^*_{\mathrm{c}}(X, \widetilde{\mathrm{Sp}}_*(M)) \to \mathrm{H}^*(X, \widetilde{\mathrm{Sp}}_*(M))$ is compatible with ι_{rig}.

2.4.4. We recall the definition of de Rham cohomology with compact supports and coefficients in (M, ∇) and the cospecialization morphism, following [15, 1.8] and [12, Appendix D.2].

Let I be the ideal sheaf of the reduced closed subscheme $\overline{X}_K - X_K$ in $\overline{X}_K$. Take a coherent $\mathscr{O}_{\overline{X}_K}$-module $\overline{M}$ extending M. The connection ∇ extends to a connection on the pro-$\mathscr{O}_{\overline{X}_K}$-module $(I^n\overline{M})_n$ [12, D.2.12]. This allows us to define the de Rham pro-complex $I^\bullet\overline{M} \otimes_{\mathscr{O}_{\overline{X}_K}} \Omega^\bullet_{\overline{X}_K} := (I^n\overline{M})_n \otimes \Omega^\bullet_{\overline{X}_K}$. The algebraic de Rham cohomology with compact supports and coefficients in (M, ∇) is defined as [12, D.2.16]

$$\begin{aligned}\mathbb{R}\Gamma_{\mathrm{dR,c}}(X_K, (M, \nabla)) &:= \mathbb{R}\Gamma(\overline{X}_K, \mathbb{R}\varprojlim I^\bullet\overline{M} \otimes \Omega^\bullet_{\overline{X}_K}) \\ &\simeq \mathbb{R}\varprojlim \mathbb{R}\Gamma(\overline{X}_K, I^\bullet\overline{M} \otimes \Omega^\bullet_{\overline{X}_K}).\end{aligned}$$

If j_K denotes the open immersion $X_K \to \overline{X}_K$, we have a canonical isomorphism $j_K^*(\mathbb{R}\varprojlim(I^\bullet\overline{M} \otimes \Omega^\bullet_{\overline{X}_K})) \xrightarrow{\sim} M \otimes \Omega^\bullet_{X_K}$. Then its adjoint $\mathbb{R}\varprojlim(I^\bullet\overline{M} \otimes \Omega^\bullet_{\overline{X}_K}) \to \mathbb{R}j_{K*}(M \otimes \Omega^\bullet_{X_K})$ induces a canonical morphism:

$$\iota_{\mathrm{dR}} : \mathbb{R}\Gamma_{\mathrm{dR,c}}(X_K, (M, \nabla)) \to \mathbb{R}\Gamma_{\mathrm{dR}}(X_K, (M, \nabla)).$$

By the rigid GAGA, there are canonical isomorphisms

$$\begin{aligned}\mathbb{R}\varprojlim \mathbb{R}\Gamma(\overline{X}_K, I^\bullet\overline{M}\otimes\Omega^\bullet_{\overline{X}_K}) &\xrightarrow{\sim} \mathbb{R}\varprojlim \mathbb{R}\Gamma(\overline{\mathcal{X}}, I^\bullet\overline{M}^{\mathrm{an}}\otimes\Omega^\bullet_{\overline{\mathcal{X}}}) \qquad (2.4.4.1)\\ &\xrightarrow{\sim} \mathbb{R}\Gamma(\overline{\mathcal{X}}, \mathbb{R}\varprojlim I^\bullet\overline{M}^{\mathrm{an}}\otimes\Omega^\bullet_{\overline{\mathcal{X}}}).\end{aligned}$$

We denote the right hand side by $\mathbb{R}\Gamma_{\mathrm{an,c}}(\mathcal{X},(M^{\mathrm{an}},\nabla^{\mathrm{an}}))$. Let j^{an} be the inclusion $\mathcal{X}\to\overline{\mathcal{X}}$. Similarly, there exists a canonical morphism

$$\mathbb{R}\varprojlim(I^\bullet\overline{M}^{\mathrm{an}}\otimes\Omega^\bullet_{\overline{\mathcal{X}}}) \to \mathbb{R}j^{\mathrm{an}}_*(M^{\mathrm{an}}\otimes\Omega^\bullet_{\mathcal{X}}), \qquad (2.4.4.2)$$

which induces a morphism on analytic de Rham cohomologies

$$\iota_{\mathrm{an}}:\mathbb{R}\Gamma_{\mathrm{an,c}}(\mathcal{X},(M^{\mathrm{an}},\nabla^{\mathrm{an}}))\to\mathbb{R}\Gamma_{\mathrm{an}}(\mathcal{X},(M^{\mathrm{an}},\nabla^{\mathrm{an}})).$$

Since $(\mathcal{X},](\overline{X}-X)_k[_{\mathfrak{X}})$ is an admissible covering of $\overline{\mathcal{X}}$, the canonical morphisms

$$\begin{aligned}&\mathbb{R}\underline{\Gamma}_{]X_k[}(\mathbb{R}j^{\mathrm{an}}_*(E))\to\mathbb{R}j^{\mathrm{an}}_*(\mathbb{R}\underline{\Gamma}_{]X_k[}(E)), \qquad (2.4.4.3)\\ &\mathbb{R}\underline{\Gamma}_{]X_k[}(E)\to\mathbb{R}\underline{\Gamma}_{]X_k[}\mathbb{R}j^{\mathrm{an}}_*(j^{\mathrm{an}\,*}(E))\end{aligned}$$

are isomorphic for any complex of abelian sheaves E on $\mathcal{X}$ (resp. $\overline{\mathcal{X}}$). Then (2.4.4.2) induces an isomorphism

$$\mathbb{R}\underline{\Gamma}_{]X_k[}(\mathbb{R}\varprojlim(I^\bullet\overline{M}^{\mathrm{an}}\otimes\Omega^\bullet_{\overline{\mathcal{X}}}))\xrightarrow{\sim}\mathbb{R}\underline{\Gamma}_{]X_k[}(\mathbb{R}j^{\mathrm{an}}_*(M^{\mathrm{an}}\otimes\Omega^\bullet_{\mathcal{X}})). \qquad (2.4.4.4)$$

The *cospecialization morphism*, denoted by $\rho_{\mathrm{c},M}$, is defined as the composition

$$\begin{aligned}\rho_{\mathrm{c},M}:\mathbb{R}\Gamma_{\mathrm{rig,c}}(X_k/K,M^\dagger) &\overset{(2.4.4.3)}{\simeq} \mathbb{R}\Gamma(\overline{\mathcal{X}},\mathbb{R}\underline{\Gamma}_{]X_k[}\mathbb{R}j^{\mathrm{an}}_*(M^{\mathrm{an}}\otimes\Omega^\bullet_{\mathcal{X}}))\\ &\overset{(2.4.4.4)}{\simeq} \mathbb{R}\Gamma(\overline{\mathcal{X}},\mathbb{R}\underline{\Gamma}_{]X_k[}(\mathbb{R}\varprojlim(I^\bullet\overline{M}^{\mathrm{an}}\otimes\Omega^\bullet_{\overline{\mathcal{X}}})))\\ &\to \mathbb{R}\Gamma(\overline{\mathcal{X}},\mathbb{R}\varprojlim(I^\bullet\overline{M}^{\mathrm{an}}\otimes\Omega^\bullet_{\overline{\mathcal{X}}}))\\ &(= \mathbb{R}\Gamma_{\mathrm{an,c}}(\mathcal{X},(M^{\mathrm{an}},\nabla^{\mathrm{an}})))\\ &\simeq \mathbb{R}\Gamma_{\mathrm{dR,c}}(X_K,(M,\nabla)).\end{aligned}$$

Proposition 2.4.5 *With the above notation and assumption, the following diagram is commutative:*

$$\begin{tikzcd} \mathbb{R}\Gamma_{\mathrm{rig,c}}(X_k/K, M^{\dagger}) \arrow[r, "\iota_{\mathrm{rig}}"] \arrow[d, "\rho_{\mathrm{c},M}"'] & \mathbb{R}\Gamma_{\mathrm{rig}}(X_k/K, M^{\dagger}) \\ \mathbb{R}\Gamma_{\mathrm{dR,c}}(X_K, (M, \nabla)) \arrow[r, "\iota_{\mathrm{dR}}"] & \mathbb{R}\Gamma_{\mathrm{dR}}(X_K, (M, \nabla)). \arrow[u, "\rho_M"'] \end{tikzcd}$$

Proof The algebraic de Rham cohomology with compact supports is isomorphic to the analytic one (2.4.4.1). It suffices to show the following diagram is commutative

$$\begin{tikzcd} \mathbb{R}\Gamma_{\mathrm{rig,c}}(X_k/K, M^{\dagger}) \arrow[r, "\iota_{\mathrm{rig}}"] \arrow[d, "\rho_{\mathrm{c},M}"'] & \mathbb{R}\Gamma_{\mathrm{rig}}(X_k/K, M^{\dagger}) \\ \mathbb{R}\Gamma_{\mathrm{an,c}}(\mathcal{X}, (M^{\mathrm{an}}, \nabla^{\mathrm{an}})) \arrow[r, "\iota_{\mathrm{an}}"] & \mathbb{R}\Gamma_{\mathrm{an}}(\mathcal{X}, (M^{\mathrm{an}}, \nabla^{\mathrm{an}})). \arrow[u, "(2.4.3.1)"'] \end{tikzcd} \tag{2.4.5.1}$$

The morphism $\mathbb{R}\Gamma_{\mathrm{rig,c}}(X_k/K, M^{\dagger}) \to \mathbb{R}\Gamma_{\mathrm{an}}(\mathcal{X}, (M^{\mathrm{an}}, \nabla^{\mathrm{an}}))$ is induced by the composition on $\overline{\mathcal{X}}$:

$$\mathbb{R}\underline{\Gamma}_{]X_k[}(\mathbb{R}\varprojlim(I^{\bullet}\overline{M}^{\mathrm{an}} \otimes \Omega^{\bullet}_{\overline{\mathcal{X}}})) \to \mathbb{R}\varprojlim(I^{\bullet}\overline{M}^{\mathrm{an}} \otimes \Omega^{\bullet}_{\overline{\mathcal{X}}}) \xrightarrow{(2.4.4.2)} \mathbb{R}j^{\mathrm{an}}_{*}(M^{\mathrm{an}} \otimes \Omega^{\bullet}_{\mathcal{X}}).$$

The restriction of the above morphism to $\mathcal{X}$ coincides with the canonical morphism (2.4.3.3), which induces ι_{rig} (2.4.3.4). Then the commutativity of (2.4.5.1) follows. □

2.4.6. In the end, we present a generalization of the specialization morphism (2.4.3.2) in a relative situation using the direct image of arithmetic $\mathscr{D}$-modules.

Let $f : X = \mathrm{Spec}(B) \to S = \mathrm{Spec}(A)$ be a smooth morphism of affine smooth R-schemes of relative dimension d and let (M, ∇) be a coherent $\mathscr{O}_{X_K}$-module endowed with an integrable connection relative to K. Consider M as a $\mathscr{D}_{X_K}$-module. The direct image $f^{\mathrm{dR}}_{+}(M)$ of $\mathscr{D}$-modules is calculated by the relative de Rham complex $M \otimes \Omega^{\bullet}_{X/S}$. Since f is affine, the above complex is calculated by

$$\Gamma(S, f^{\mathrm{dR}}_{+}(M)) \simeq \mathrm{DR}_{B/A}(M, \nabla) = M \to M \otimes_B \Omega^1_{B/A} \to \cdots,$$

where we denote abusively by M the global section $\Gamma(X_K, M)$.

2.4.7. We assume moreover that f admits a *good compactification*, i.e. f can be extended to a smooth morphism $\overline{f} : \overline{X} \to \overline{S}$ of smooth *projective* R-schemes

$\overline{X}$, $\overline{S}$ such that $\overline{X}_k - X_k$, $\overline{S}_k - S_k$ are ample divisors. We keep the notation of 2.4.2 and assume that $M^\dagger = j^\dagger(M^{\mathrm{an}})$ is overconvergent as in 2.4.3. We denote abusively the $\mathscr{D}^\dagger_{\overline{\mathfrak{X}},\mathbb{Q}}(\infty)$-module $\mathrm{Sp}_*(M^\dagger)$ (2.1.4) by $M^\dagger$. The direct image of $M^\dagger$ along $f_k : X_k \to S_k$ is calculated by a relative de Rham complex:

$$f_{k,+}(M^\dagger) \xrightarrow{\sim} \mathbb{R}\overline{f}_{k,*}(\mathrm{Sp}_*(M^\dagger \otimes_{\mathscr{O}_{\mathcal{X}}} \Omega^\bullet_{\mathcal{X}/\mathcal{S}})).$$

The above complex is a complex of overholonomic (and hence coherent) $\mathscr{D}^\dagger_{\overline{\mathfrak{S}},\mathbb{Q}}(\infty)$-modules.

We set $A^\dagger = \Gamma(\overline{\mathfrak{S}}, \mathcal{O}_S)$, $B^\dagger = \Gamma(\overline{\mathfrak{X}}, \mathcal{O}_X)$ and $D^\dagger_{\overline{\mathfrak{S}}}(\infty) = \Gamma(\overline{\mathfrak{S}}, \mathscr{D}^\dagger_{\overline{\mathfrak{S}},\mathbb{Q}}(\infty))$ (1.3.8). By $\mathscr{D}^\dagger$-affinity [48, 5.3.3], the above complex is equivalent to a complex of coherent $D^\dagger_{\overline{\mathfrak{S}}}(\infty)$-modules:

$$\begin{aligned}\mathbb{R}\Gamma(\overline{\mathfrak{S}}, f_{k,+}(M^\dagger)) &\simeq \mathbb{R}\Gamma(\overline{\mathfrak{X}}, \mathrm{Sp}_*(M^\dagger \otimes_{\mathscr{O}_{\mathcal{X}}} \Omega^\bullet_{\mathcal{X}/\mathcal{S}}))\\ &\simeq (M \otimes_{B_K} B^\dagger) \otimes_B \Omega^\bullet_{B/A}.\end{aligned}$$

We denote the complex in the second line by $\mathrm{DR}^\dagger_{B/A}(M^\dagger)$, which is an $A^\dagger$-linear complex. If we set $D_{S_K} = \Gamma(S_K, \mathscr{D}_{S_K})$, there exists a canonical D_{S_K}-linear morphism, called the *(relative) specialisation morphism*

$$\mathrm{DR}_{B/A}(M, \nabla) \to \mathrm{DR}^\dagger_{B/A}(M^\dagger). \tag{2.4.7.1}$$

2.5 Equivariant holonomic $\mathscr{D}$-modules

In this subsection, we study the notion of *equivariant holonomic $\mathscr{D}$-modules* over a k-scheme (or an ind-scheme).

2.5.1. Let $X \to S$ be a morphism of k-schemes, H a smooth affine group scheme over S and $\mathrm{act} : H \times_S X \to X$ an action of H on X. We denote by $\mathrm{pr}_2 : H \times_S X \to X$ the projection. A H-equivariant holonomic module on X is a pair consisting of a holonomic module $\mathscr{M}$ on X and an isomorphism $\theta : \mathrm{act}^+(\mathscr{M}) \xrightarrow{\sim} \mathrm{pr}_2^+(\mathscr{M})$ in $\mathrm{D}(H \times_S X)$, satisfying:

(i) $e^+(\theta) = \mathrm{id}$, where $e : X \to H \times_S X$ is induced by the unit section of H;
(ii) a cocycle condition on $H \times_S H \times_S X$.

Morphisms are defined in a natural way. We denote by $\mathrm{Hol}_H(X)$ the category of H-equivariant holonomic modules on X, which is an abelian subcategory of $\mathrm{Hol}(X)$.

Suppose that H has geometrically connected fibers over S and that $[X/H]$ is represented by a separated scheme of finite type $\overline{X}$ over S. By smooth descent of holonomic modules [3, 2.1.13], the pullback functor along the canonical morphism $q : X \to \overline{X}$ induces an equivalence of categories:

$$q^+[d_H] : \mathrm{Hol}(\overline{X}) \xrightarrow{\sim} \mathrm{Hol}_H(X). \tag{2.5.1.1}$$

Using Proposition 2.1.6 and repeating the argument of [79, A.1.2], we deduce that the canonical functor $\mathrm{Hol}_H(X) \to \mathrm{Hol}(X)$ is fully faithful.

2.5.2. Let Y be a separated S-scheme of finite type and $\varpi : E \to Y$ an H-torsor over S with trivial action of H on Y. We denote by $Y \widetilde{\times}_S X$ the quotient of $E \times_S X$ by the diagonal action of H.

Let $\mathscr{M}$ be a holonomic module on Y and $\mathscr{N}$ an H-equivariant holonomic module on X. Assume that $\mathscr{M} \boxtimes_S \mathscr{N}$ is a holonomic module on $Y \times_S X$ (Note that it is true if the base $S = \mathrm{Spec}(k)$). Then $(\varpi^+\mathscr{M}[\dim H]) \boxtimes_S \mathscr{N}$ is holonomic on $E \times_S X$ and is H-equivariant by construction. By (2.5.1.1), it descends to a holonomic module on $Y \widetilde{\times}_S X$, denoted by $\mathscr{M} \widetilde{\boxtimes}_S \mathscr{N}$ and called the *twisted external product* of $\mathscr{M}$ and $\mathscr{N}$.

2.5.3. Let $\mathcal{X} \simeq \varinjlim_{i\in I} X_i$ be an ind-scheme over k [79, definition 0.3.4]. For a transition morphism $\varphi : X_i \to X_j$, the functor $\varphi_+ : \mathrm{D}(X_i) \to \mathrm{D}(X_j)$ is exact and fully faithful. We define a triangulated category $\mathrm{D}(\mathcal{X})$ as the 2-inductive limit

$$\mathrm{D}(\mathcal{X}) = \varinjlim_{i\in I} \mathrm{D}(X_i).$$

The definition is independent of the choice of a ind-presentation of $\mathcal{X}$. Since φ_+ is exact, $\mathrm{D}(\mathcal{X})$ is also equipped with a t-structure, whose heart is denoted by $\mathrm{Hol}(\mathcal{X})$. Note that $\mathrm{Hol}(\mathcal{X})$ coincides with the full abelian subcategory $\varinjlim_{i\in I} \mathrm{Hol}(X_i)$ of $\mathrm{D}(\mathcal{X})$.

Given a morphism $f = (f_i)_{i\in I} : \mathcal{X} = \varinjlim X_i \to S$ to a k-scheme S, the cohomology functors $f_{i,!}$'s and $f_{i,+}$'s allow us to define $f_!, f_+ : \mathrm{D}(\mathcal{X}) \to \mathrm{D}(S)$. If S is smooth, in view of Proposition 2.2.3(iv), we can define the notion of *LA (resp. ULA) with respect to* f for objects of $\mathrm{D}(\mathcal{X})$.

2.5.4. Let $\mathcal{X} = \varinjlim_{i\in I} X_i$ be an ind-scheme and $f : \mathcal{X} \to S$ a morphism to a k-scheme. Let $(H_j)_{j\in J}$ be a projective system of smooth affine S-group schemes with geometrically connected fibers, whose transition morphisms are quotient. We set $H = \varprojlim_{j\in J} H_j$ and assume that there exists an action of H on $f : \mathcal{X} \to S$ such that it stabilizes each subfunctor $f|_{X_i}$ and that the H-action factors through a quotient H_{j_i} on $X_i \to S$ for each $i \in I$. Then we define the *category* $\mathrm{Hol}_H(\mathcal{X})$ *of* H*-equivariant holonomic modules on* $\mathcal{X}$ as in [79, A.1.4]. Let $\mathcal{Y} = \varinjlim_{i\in I} Y_i$ an ind-scheme over S and $\varpi : E \to \mathcal{Y}$ an H-torsor. Let $\mathscr{M}$ be an object of $\mathrm{Hol}(\mathcal{Y})$ supported in Y_l and $\mathscr{N}$ an object of $\mathrm{Hol}_H(\mathcal{X})$ supported in X_i. We can define an ind-scheme $\mathcal{Y}\widetilde{\times}_S\mathcal{X}$ and the *twisted product* $\mathscr{M}\widetilde{\boxtimes}_S\mathscr{N}$ in $\mathrm{Hol}(\mathcal{Y}\widetilde{\times}_S\mathcal{X})$ as in [79, A.1.4].

2.6 Hyperbolic localization for arithmetic $\mathscr{D}$-modules

2.6.1. Let X be a quasi-projective k-scheme such that $X \otimes_k \overline{k}$ is connected and normal. We suppose that there exists an action $\mu : \mathbb{G}_m \times X \to X$ of the torus $\mathbb{G}_m$ over k. Following [37], we denote by X^0 the closed subscheme of fixed points of X [37, 1.3] by X^+ (resp. X^-) the attractor (resp. repeller) of X [37, 1.4, 1.8]. We have a commutative diagram

$$\begin{tikzcd}
& X^+ \arrow[dl, bend right, "\pi"'] \arrow[dr, "g"] & \\
X^0 \arrow[ur, "f"'] \arrow[dr, "f'"] & & X \\
& X^- \arrow[ul, bend left, "\pi'"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

where f, f' are closed immersions and are sections of π, π', respectively, the restriction of g (resp. g') to each connected component of X^+ (resp. X^-) is a locally closed immersion [37, 1.6.8].

We define *hyperbolic localization* functors $(-)^{!+}, (-)^{+!} : \mathrm{D}(X) \to \mathrm{D}(X^0)$, for $\mathscr{F} \in \mathrm{D}(X)$ by:

$$\mathscr{F}^{!+} = f^!(g^+(\mathscr{F})), \qquad \mathscr{F}^{+!} = f'^+(g'^!(\mathscr{F})).$$

We say an object $\mathscr{F}$ of $\mathrm{D}(X)$ is *weakly equivariant* if there exists an isomorphism $\mu^+(\mathscr{F}) \simeq \mathscr{L}[-1] \boxtimes \mathscr{F}$ for some smooth module $\mathscr{L}$ on $\mathbb{G}_m$.

Theorem 2.6.2 (Braden [22]) (i) *There exists a canonical morphism* $\iota_{\mathscr{F}} : \mathscr{F}^{+!} \to \mathscr{F}^{!+}$, *which is an isomorphism if* $\mathscr{F}$ *is weakly equivariant.*

(ii) *The canonical morphisms* $\pi_! \to f^!$, $\pi'_+ \to f'^+$ *induce morphisms*

$$\pi_! g^+\mathscr{F} \to \mathscr{F}^{!+}, \qquad \pi'_+ g'^!\mathscr{F} \to \mathscr{F}^{+!},$$

which are isomorphisms if $\mathscr{F}$ *is weakly equivariant.*

Braden's original proof only relies on the six functor formalism of ℓ-adic sheaves. We can apply the same argument and obtain the above theorem in the arithmetic $\mathscr{D}$-modules setting.

3 Geometric Satake equivalence for arithmetic $\mathscr{D}$-modules

In this section, we establish the geometric Satake equivalence for arithmetic $\mathscr{D}$-modules.

We assume that k is a finite field with $q = p^s$ elements and keep the notation in § 2. We work with holonomic modules (resp. complexes) over the geometric base tuple $\mathfrak{T} = \{k, R, K, L\}$ and we omit $/L$ from the notations $\mathrm{Hol}(-/L)$, $\mathrm{D}(-/L)$ for simplicity. We take an arithmetic base tuple $\mathfrak{T}_F = \{k, R, K, L, s, \mathrm{id}_L\}$ and an isomorphism $\overline{K} \simeq \mathbb{C}$ to apply the theory of weights and the decomposition theorem [6].

Let G be a split reductive group over k and T the abstract Cartan of G. We denote by $\mathbb{X}^\bullet = \mathbb{X}^\bullet(T)$ the weight lattice and by $\mathbb{X}_\bullet = \mathbb{X}_\bullet(T)$ the coweight lattice. Let $\Phi \subset \mathbb{X}^\bullet$ (resp. $\Phi^\vee \subset \mathbb{X}_\bullet$) the set of roots (resp. coroots). Let $\Phi^+ \subset \Phi$ be the set of positive roots and $\mathbb{X}_\bullet(T)^+ \subset \mathbb{X}_\bullet(T)$ the semi-group of dominant coweights, determined by a choice of B. (But they are independent of the choice of B.) Given $\lambda, \mu \in \mathbb{X}_\bullet(T)$, we define $\lambda \leq \mu$ if $\mu - \lambda$ is a non-negative integral linear combinations of simple coroots and $\lambda < \mu$ if $\lambda \leq \mu$ and $\lambda \neq \mu$. This defines a partial order on $\mathbb{X}_\bullet(T)$ (and on $\mathbb{X}_\bullet(T)^+$). We denote by $\rho \in \mathbb{X}^\bullet(T) \otimes \mathbb{Q}$ the half sum of all positive roots.

3.1 The Satake category

3.1.1. Recall that the *loop group* LG (resp. *positive loop group* L^+G) is the fpqc sheaf on the category of k-algebras associated to the functor $R \mapsto G(R(\!(t)\!))$ (resp. $R \mapsto G(R[\![t]\!])$). Then L^+G is a subsheaf of LG and the *affine Grassmannian* Gr_G is the fpqc-quotient $\mathrm{Gr}_G = LG/L^+G$, which is represented by an ind-projective ind-scheme over k. We write simply Gr instead of Gr_G, if there is no confusion.

For any dominant coweight $\mu \in \mathbb{X}_\bullet(T)^+$, we denote by Gr_μ the corresponding (L^+G)-orbit in Gr, which is smooth quasi-projective over k of dimension $2\rho(\mu)$ [79, 2.1.5]. Let $\mathrm{Gr}_{\leq\mu}$ be the reduced closure of Gr_μ in Gr, which is equal to $\cup_{\lambda\leq\mu} \mathrm{Gr}_\lambda$. Let $j_\mu : \mathrm{Gr}_\mu \to \mathrm{Gr}_{\leq\mu}$ be the open inclusion. We have an ind-presentation $\mathrm{Gr}_{\mathrm{red}} \simeq \varinjlim_{\mu\in\mathbb{X}_\bullet(T)^+} \mathrm{Gr}_{\leq\mu}$. Since we will work with holonomic modules, we can replace Gr by its reduced ind-subscheme [3, 1.1.3 lemma], and omit the subscript $_{\mathrm{red}}$ to simplify the notation.

For $i \geq 0$, let G_i be the i-th jet group defined by the functor $R \mapsto G(R[t]/t^{i+1})$. Then G_i is representable by a smooth geometrically connected affine group scheme over k and we have $L^+G \simeq \varprojlim_i G_i$. If we consider the left action of L^+G on Gr, then the action on $\mathrm{Gr}_{\leq\mu}$ factors through G_i for some i. We can define the category of (L^+G)-equivariant holonomic modules on

Gr (2.5.4), denoted as Sat_G and called *Satake category*. It is a full subcategory of Hol(Gr) (2.5.1).

Proposition 3.1.2 *The category* Sat_G *is semisimple with simple objects* $\mathrm{IC}_\mu := j_{\mu,!+}(L_{\mathrm{Gr}_\mu}[2\rho(\mu)])$ (2.1.9).

Lemma 3.1.3 *For* $\mu \in \mathbb{X}_\bullet(T)^+$*, the category* $\mathrm{Sm}(\mathrm{Gr}_\mu)$ (2.1.4) *is semisimple with simple object* L_{Gr_μ}.

Proof The (L^+G)-orbit Gr_μ is geometrically connected and satisfies $\pi_1^{\text{ét}}(\mathrm{Gr}_\mu \otimes_k \overline{k}) \simeq \{1\}$ (cf. [67] proof of proposition 4.1). Every irreducible object $\mathscr{M}$ of $\mathrm{Sm}(\mathrm{Gr}_\mu)$ has a Frobenius structure with finite determinant [2, 6.1]. By the companion theorem for overconvergent F-isocrystals over a smooth k-scheme [7, 4.2] and Čebotarev density [3, A.4], we deduce that $\mathscr{M} \simeq L_{\mathrm{Gr}_\mu}$ in $\mathrm{Sm}(\mathrm{Gr}_\mu)$. Alternatively, one can show a weaker statement that L_{Gr_μ} is the only L^+G-equivariant irreducible object of $\mathrm{Sm}(\mathrm{Gr}_\mu)$ using a similar argument of the proof of (2.5.1.1).

To show the semisimplicity, it suffices to show that $\mathrm{H}^1(\mathrm{Gr}_\mu, L) = 0$. There exists a morphism $\pi : \mathrm{Gr}_\mu \to G/P_\mu$ realizing Gr_μ as an affine bundle over the partial flag variety G/P_μ, where P_μ is the parabolic subgroup containing B and associated with $\{\alpha \in \Phi, (\alpha, \mu) = 0\}$. In view of the cohomology of affine spaces (2.1.4), the cohomology $\mathrm{H}^i(\mathrm{Gr}_\mu, L)$ is isomorphic to $\mathrm{H}^i(G/P_\mu, L)$. Since the partial flag variety admits a stratification of affine spaces, we deduce that $\mathrm{H}^i(G/P_\mu, L) = 0$ if i is odd by (2.1.3.1). Then the assertion follows. □

We prove the following parity result by the same argument of [44] A.7 (cf. [13] §4.2 for a detailed exposition) in the ℓ-adic case using the decomposition theorem [6, 4.3.1, 4.3.6], spectral sequence (2.1.3.1) and the parity of the compact support p-adic cohomology of affine spaces (2.1.4).

Lemma 3.1.4 *The constructible module* ${}^{\mathrm{c}}\mathcal{H}^i(\mathrm{IC}_\mu)$ *vanishes unless* $i \equiv \dim(\mathrm{Gr}_\mu) \pmod 2$.

3.1.5. *Proof of Proposition* 3.1.2. We follow the same line as in the ℓ-adic case (cf. [44] prop. 1). By 2.1.9(i), holonomic modules IC_μ are irreducible objects of Sat_G. Let $\mathscr{E}$ be an irreducible object of Sat_G. There exists an (L^+G)-orbit Gr_μ such that $\mathscr{E}|_{\mathrm{Gr}_\mu}$ is a non-zero smooth object. By 2.1.6 and 3.1.3, we deduce that $\mathscr{E}$ is isomorphic to IC_μ.

To prove the semisimplicity, it suffices to show that for $\lambda, \mu \in \mathbb{X}_\bullet(T)^+$, we have

$$\mathrm{Ext}^1_{\mathrm{Hol(Gr)}}(\mathrm{IC}_\lambda, \mathrm{IC}_\mu) = \mathrm{Hom}_{\mathrm{D(Gr)}}(\mathrm{IC}_\lambda, \mathrm{IC}_\mu[1]) = 0. \tag{3.1.5.1}$$

(i) In the case $\lambda = \mu$, (3.1.5.1) follows from $\mathrm{Ext}^1_{\mathrm{Hol}(\mathrm{Gr}_\mu)}(L_{\mathrm{Gr}_\mu}, L_{\mathrm{Gr}_\mu}) = \mathrm{H}^1(\mathrm{Gr}_\mu, L) = 0$ (Lemma 3.1.3).

(ii) Then we consider the case either $\lambda < \mu$ or $\mu < \lambda$. Since the dual functor $\mathbb{D}$ induces an anti-equivalence, we may assume that $\mu < \lambda$. We denote by $i : \mathrm{Gr}_{\leq\mu} \to \mathrm{Gr}_{\leq\lambda}$ the closed immersion and we have

$$\mathrm{Hom}_{\mathrm{D(Gr)}}(\mathrm{IC}_\lambda, i_+ \mathrm{IC}_\mu[1]) \simeq \mathrm{Hom}_{\mathrm{D}(\mathrm{Gr}_{\leq\mu})}(i^+ \mathrm{IC}_\lambda, \mathrm{IC}_\mu[1]).$$

Note that $i^+ \mathrm{IC}_\lambda$ has cohomological degrees ≤ -1 (2.1.10(i)). Each (L^+G)-equivariant holonomic module $\mathcal{H}^j(i^+ \mathrm{IC}_\lambda|_{\mathrm{Gr}_\mu})$ is smooth and hence is constant (3.1.3). If there exists a non-zero morphism $g : i^+ \mathrm{IC}_\lambda \to \mathrm{IC}_\mu[1]$, then it would induce a non-zero morphism $h : \mathcal{H}^{-1}(i^+ \mathrm{IC}_\lambda|_{\mathrm{Gr}_\mu}) \to L_{\mathrm{Gr}_\mu}[2\rho(\mu)]$. Given a closed point x of Gr_μ, the restriction of the fiber functor $i_x^+[-\dim \mathrm{Gr}_\mu]$ to smooth objects is exact [3, 2.4.15]. Then the fiber $i_x^+(\mathcal{H}^{-1}(i^+ \mathrm{IC}_\lambda|_{\mathrm{Gr}_\mu}))$ is isomorphic to $\mathcal{H}^{-1+\dim \mathrm{Gr}_\mu}(i_x^+ \mathrm{IC}_\lambda)$. If $\mathcal{H}^{-1}(i^+ \mathrm{IC}_\lambda|_{\mathrm{Gr}_\mu})$ is non-zero, then it contradicts to 3.1.4 as i_x^+ is c-t-exact. The equality (3.1.5.1) in this case follows.

(iii) In the case $\lambda \nleq \mu$ and $\mu \nleq \lambda$, we prove (3.1.5.1) by base change in the same way as in [13, 4.3]. □

3.1.6. We refer to [79, 1.2.12, 1.2.13] for the definition of the twisted product $\mathrm{Gr} \widetilde{\times} \mathrm{Gr}$ and of the convolution morphism $m : \mathrm{Gr} \widetilde{\times} \mathrm{Gr} \to \mathrm{Gr}$. The morphism m is ind-proper and (L^+G)-equivariant with respect to the left (L^+G)-actions.

Given two objects $\mathcal{A}_1, \mathcal{A}_2$ of Sat_G, we denote by $\mathcal{A}_1 \widetilde{\boxtimes} \mathcal{A}_2$ their external twisted product on $\mathrm{Gr} \widetilde{\times} \mathrm{Gr}$ (see 2.5.2 and 2.5.4), and define the *convolution product* by

$$\mathcal{A}_1 \star \mathcal{A}_2 = m_+(\mathcal{A}_1 \widetilde{\boxtimes} \mathcal{A}_2). \tag{3.1.6.1}$$

We will show that $\mathcal{A}_1 \star \mathcal{A}_2$ is an object of Sat_G and that $\star$ defines a symmetric monoidal structure on Sat_G. To do it, we will interpret the convolution product as the specialization of a fusion product on Beilinson–Drinfeld Grassmannians in the next subsection.

3.2 Fusion product

3.2.1. Let X be the affine line $\mathbb{A}_k^1$, n an integer ≥ 1 and X^n the n-folded self product of X over k. We denote by $q^n : \mathrm{Gr}_{G,X^n} \to X^n$ the Beilinson–Drinfeld Grassmannian associated to G over X^n [17], cf. [79, § 3]. If there is no confusion, we will write simply Gr_{X^n} instead of Gr_{G,X^n}.

We refer to [79, 3.1] the definition of global loop groups $(L^+G)_{X^n}$ and $(LG)_{X^n}$ over X^n. There exists a canonical isomorphism of fpqc-sheaves $(LG)_{X^n}/(L^+G)_{X^n} \xrightarrow{\sim} \mathrm{Gr}_{G,X^n}$. We consider the left action of $(L^+G)_{X^n}$ on Gr_{G,X^n} over X^n and denote by $\mathrm{Hol}_{(L^+G)_{X^n}}(\mathrm{Gr}_{X^n})$ the category of $(L^+G)_{X^n}$-equivariant holonomic modules on Gr_{X^n} (2.5.4).

There exists an isomorphism $\mathrm{Gr}_X \simeq \mathrm{Gr} \times X$. Given a holonomic module $\mathcal{A}$ on Gr, the holonomic module $\mathcal{A}_X = \mathcal{A} \boxtimes L_X[1]$ is ULA with respect to $q : \mathrm{Gr}_X \to X$ (2.5.3). If $\mathcal{A}$ is moreover (L^+G)-equivariant, then $\mathcal{A}_X$ is $(L^+G)_X$-equivariant. By Proposition 2.1.6, we obtain a fully faithful functor

$$\iota : \mathrm{D}(\mathrm{Gr}) \to \mathrm{D}(\mathrm{Gr}_X), \quad \mathcal{A} \mapsto \mathcal{A}_X. \tag{3.2.1.1}$$

We denote the essential image of Sat_G via ι by Sat_X, which is a full subcategory of $\mathrm{Hol}_{(L^+G)_X}(\mathrm{Gr}_X)$.

To define the fusion product on Sat_X, we will use the factorization structure of Beilinson–Drinfeld Grassmannians. Let U be complement of $\Delta : X \to X^2$. Then there exists a canonical isomorphism, called the *factorization isomorphism* [79, 3.2.1(iii)]

$$c : \mathrm{Gr}_{X^2} \times_{X^2} U \xrightarrow{\sim} (\mathrm{Gr}_X \times \mathrm{Gr}_X) \times_{X^2} U. \tag{3.2.1.2}$$

The involution $\sigma : X^2 \to X^2,\ (x, y) \mapsto (y, x)$, induces an involution $\Delta(\sigma) : \mathrm{Gr}_{X^2} \to \mathrm{Gr}_{X^2}$.

3.2.2. The morphism m (3.1.6) also admits a globalization. We refer to [79, 3.1.21] for the definition of *convolution Grassmannian* $\mathrm{Gr}_X \widetilde{\times} \mathrm{Gr}_X$ and convolution morphism $m : \mathrm{Gr}_X \widetilde{\times} \mathrm{Gr}_X \to \mathrm{Gr}_{X^2}$ over X^2.

Using a $(L^+G)_X$-torsor $\mathbb{E} \to \mathrm{Gr}_X \times X$ [79, 3.1.22], one can identify $\mathrm{Gr}_X \widetilde{\times} \mathrm{Gr}_X$ with the twisted product $(\mathrm{Gr}_X \times X) \widetilde{\times}_X \mathrm{Gr}_X$ (2.5.4). In summary, we have the following diagram over X^2

$$\mathrm{Gr}_X \times \mathrm{Gr}_X = (\mathrm{Gr}_X \times X) \times_X \mathrm{Gr}_X \leftarrow \mathbb{E} \times_X \mathrm{Gr}_X \to \mathrm{Gr}_X \widetilde{\times} \mathrm{Gr}_X \xrightarrow{m} \mathrm{Gr}_{X^2}. \tag{3.2.2.1}$$

Let $\mathcal{A}_1, \mathcal{A}_2$ be two objects of Sat_X. Note that $(\mathcal{A}_1 \boxtimes L_X) \boxtimes_X \mathcal{A}_2 \simeq \mathcal{A}_1 \boxtimes \mathcal{A}_2$ is holonomic. We denote by $\mathcal{A}_1 \widetilde{\boxtimes} \mathcal{A}_2$ the twisted product of $\mathcal{A}_1 \boxtimes L_X$ and $\mathcal{A}_2$ on $\mathrm{Gr}_X \widetilde{\times} \mathrm{Gr}_X$ (2.5.4).

Proposition 3.2.3 (i) *There exists a canonical isomorphism of holonomic modules on* Gr_{X^2}*:*

$$m_+(\mathcal{A}_1 \widetilde{\boxtimes} \mathcal{A}_2) \simeq j_{!+}(\mathcal{A}_1 \boxtimes \mathcal{A}_2|_U). \tag{3.2.3.1}$$

The left hand side, denoted by $\mathcal{A}_1 \circledast \mathcal{A}_2$, is ULA with respect to $q^2 : \mathrm{Gr}_{X^2} \to X^2$.

(ii) *There exists a canonical isomorphism of holonomic modules on* Gr_X*:*

$$\Delta^+[-1](\mathcal{A}_1 \circledast \mathcal{A}_2) \xrightarrow{\sim} \Delta^![1](\mathcal{A}_1 \circledast \mathcal{A}_2).$$

We denote one of the above module by $\mathcal{A}_1 \circledast \mathcal{A}_2$ *and call it fusion product of* $\mathcal{A}_1, \mathcal{A}_2$. *This holonomic module is ULA with respect to* $q : \mathrm{Gr}_X \to X$.

Proof (i) The holonomic module $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ on $\mathrm{Gr}_X \times \mathrm{Gr}_X$ is the inverse image of a holonomic module on $\mathrm{Gr} \times \mathrm{Gr}$ and hence is ULA with respect to the projection $\mathrm{Gr}_X \times \mathrm{Gr}_X \to X^2$. Recall that $\mathcal{A}_1 \widetilde{\boxtimes} \mathcal{A}_2$ is constructed by descent along a quotient by a smooth group scheme over X (2.5.4, 3.2.2.1). Hence it is ULA with respect to the projection to X^2 by Proposition 2.2.3(iii). Since m is ind-proper, then $m_+(\mathcal{A}_1 \widetilde{\boxtimes} \mathcal{A}_2)$ is ULA with respect to $q^2 : \mathrm{Gr}_{X^2} \to X^2$. Since $m|_U$ is an isomorphism [79, 3.1.23], under (3.2.1.2) we have

$$\mathcal{A}_1 \widetilde{\boxtimes} \mathcal{A}_2|_U = \mathcal{A}_1 \boxtimes \mathcal{A}_2|_U,$$

which is holonomic. Then we deduce the isomorphism (3.2.3.1) from Proposition 2.2.5(i).

Assertion (ii) follows from Proposition 2.2.5. □

By repeating the argument of [79, lemma 5.4.6, remark 5.4.7], we obtain the following corollary.

Corollary 3.2.4 *Let* $\mathcal{A}_1, \mathcal{A}_2$ *be two objects of* Sat_G.

(i) *There exists a canonical isomorphism on* Gr_X (3.1.6.1)

$$(\mathcal{A}_1 \star \mathcal{A}_2)_X \simeq \mathcal{A}_{1,X} \circledast \mathcal{A}_{2,X}.$$

(ii) *The convolution product* $\mathcal{A}_1 \star \mathcal{A}_2$ *is still holonomic and belongs to* Sat_G. *The category* Sat_G *equipped with the bifunctor* $\star$ *and the unit object* IC_0 *forms a monoidal category.*

3.3 Hypercohomology functor and semi-infinite orbits

Proposition 3.3.1 *The hypercohomology functor* H^*, *defined by*

$$\mathrm{H}^* : \mathrm{Sat}_G \to \mathrm{Vec}_L, \qquad \mathcal{A} \mapsto \bigoplus_{n \in \mathbb{Z}} \mathrm{H}^n(\mathrm{Gr}, \mathcal{A}), \tag{3.3.1.1}$$

is exact and monoidal.

Proof Since Sat_G is semisimple (3.1.2), H^* is exact. Let $\mathcal{A}$ be an object of Sat_G and $\pi : \mathrm{Gr} \to \mathrm{Spec}(k)$ the structure morphism. By the Künneth formula [3, 1.1.7], there exists a canonical isomorphism

$$q_+(\mathcal{A}_X)[-1] \simeq \pi_+(\mathcal{A}) \boxtimes L_X. \tag{3.3.1.2}$$

The monoidal property of H^* follows from Corollary 3.2.4(i), (3.3.1.2) and the following lemma □

Lemma 3.3.2 *Given two objects* $\mathcal{A}_1, \mathcal{A}_2$ *of* Sat_X*, there exists a canonical isomorphism*

$$q_+(\mathcal{A}_1 \circledast \mathcal{A}_2)[-1] \simeq (q_+(\mathcal{A}_1)[-1]) \otimes (q_+(\mathcal{A}_2)[-1]).$$

Proof It suffices to construct a canonical isomorphism

$$q_+^2(\mathcal{A}_1 \boxast \mathcal{A}_2) \simeq q_+(\mathcal{A}_1) \boxtimes q_+(\mathcal{A}_2). \tag{3.3.2.1}$$

By (3.2.3.1) and the Künneth formula [3, 1.1.7], such an isomorphism exists on $U = X^2 - \Delta(X)$.

Let $\tau : X^2 \to X$ be the morphism sending (x, y) to $x - y$. Both sides of (3.3.2.1) are ULA with respect to τ by Propositions 2.2.3 and 3.2.3. By Proposition 2.2.5, we deduce a canonical isomorphism on X

$$\Delta^!\big(q_+^2(\mathcal{A}_1 \boxast \mathcal{A}_2)\big) \simeq \Delta^!\big(q_+(\mathcal{A}_1) \boxtimes q_+(\mathcal{A}_2)\big).$$

Then the isomorphism (3.3.2.1) follows from the distinguished triangle $\Delta_+\Delta^! \to \mathrm{id} \to j_+j^+ \to$. □

Remark 3.3.3 For objects $\mathcal{A}_1, \mathcal{A}_2$ of Sat_G, we have

$$\mathrm{H}^*(\mathcal{A}_1 \star \mathcal{A}_2) \simeq \mathrm{H}^*(\mathcal{A}_1) \otimes \mathrm{H}^*(\mathcal{A}_2).$$

The above proof also applies to arithmetic $\mathscr{D}$-modules with Frobenius structures. If $\mathcal{A}_1, \mathcal{A}_2$ are equipped with Frobenius structures, the above isomorphism is compatible with Frobenius structures.

3.3.4. In the following, we study the p-adic cohomology of objects of Sat_G on semi-infinite orbits of Gr_G following Mirković and Vilonen [62]. Let B^{op} be the opposite Borel subgroup. The inclusion $B, B^{\mathrm{op}} \to G$ and projections $B, B^{\mathrm{op}} \to T$ induce morphisms

$$\mathrm{Gr}_T \xleftarrow{\pi} \mathrm{Gr}_B \xrightarrow{i} \mathrm{Gr}_G, \qquad \mathrm{Gr}_T \xleftarrow{\pi'} \mathrm{Gr}_{B^{\mathrm{op}}} \xrightarrow{i'} \mathrm{Gr}_G\,.$$

Via i, each connected component of Gr_B is locally closed in Gr_G.

The affine Grassmannian Gr_T is discrete, whose k-points are given by $L_\lambda = t^\lambda T(k[\![t]\!])/T(k[\![t]\!]) \in \mathrm{Gr}_T(k),\ \lambda \in \mathbb{X}_\bullet(T)$. For $\lambda \in \mathbb{X}_\bullet(T)$, we denote by S_λ (resp. T_λ) the ind-subscheme $i(\pi^{-1}(L_\lambda))$ (resp. $i'(\pi'^{-1}(L_\lambda))$ of Gr_G. The union $S_{\leq\lambda} = \cup_{\lambda'\leq\lambda} S_{\lambda'}$ is closed in Gr_G and S_λ is open and dense in $S_{\leq\lambda}$. We set cohomology functors $\mathrm{H}^*_{\mathrm{c}}(S_\lambda, -)$ and $\mathrm{H}^*_{T_\lambda}(\mathrm{Gr}_G, -)$ to be

$$\mathrm{H}^*_{\mathrm{c}}(S_\lambda, -) = \mathrm{H}^*((\pi_! i^+(-))_\lambda), \quad \mathrm{H}^*_{T_\lambda}(\mathrm{Gr}_G, -) = \mathrm{H}^*(\pi'_+ i'^!(-))_\lambda).$$

Proposition 3.3.5 (i) *For any object $\mathcal{A}$ of Sat_G, there exists a functorial isomorphism*

$$\mathrm{H}^i_{\mathrm{c}}(S_\lambda, \mathcal{A}) \simeq \mathrm{H}^i_{T_\lambda}(\mathrm{Gr}_G, \mathcal{A}).$$

Both sides vanish if $i \neq 2\rho(\lambda)$.

(ii) *For $\mu \in \mathbb{X}_\bullet(T)^+$, the dimension of $\mathrm{H}^{2\rho(\lambda)}_{\mathrm{c}}(S_\lambda, \mathrm{IC}_\mu)$ is equal to the number of geometrically irreducible components of $S_\lambda \cap \mathrm{Gr}_{G,\mu}$. If we set $\mathrm{IC}_\mu = j_{\mu,!+}(L_{\mathrm{Gr}_\mu}[2\rho(\mu)])$ as an object of $\mathrm{D}(\mathrm{Gr}, L_F)$, then the Frobenius acts on $\mathrm{H}^{2\rho(\lambda)}_{\mathrm{c}}(S_\lambda, \mathrm{IC}_\mu)$ by multiplication by $q^{\rho(\lambda+\mu)}$.*

(iii) *For any integer i, there exists a functorial isomorphism*

$$\mathrm{H}^i(\mathrm{Gr}_G, \mathcal{A}) \simeq \bigoplus_{\lambda\in\mathbb{X}_\bullet(T)} \mathrm{H}^i_{\mathrm{c}}(S_\lambda, \mathcal{A}).$$

(iv) *The hypercohomology functor H^* (3.3.1.1) is faithful.*

If we consider the action of $\mathbb{G}_m$ on Gr_G induced by $2\check{\rho}$, Gr_B (resp. $\mathrm{Gr}_{B^{\mathrm{op}}}$, Gr_T) is the attractor (resp. repeller, resp. closed subscheme of fixed points) of Gr_G (cf. 2.6.1). When the intersection $S_\lambda \cap \mathrm{Gr}_{G,\mu}$ is non-empty, it has pure dimension $\rho(\lambda+\mu)$. Then the proposition can be proved in the same way as in [62, 3.5, 3.6] by Braden's theorem (2.6.2). The faithfulness of H^* follows from the calculation of cohomologies (ii,iii).

Proposition 3.3.6 *Given two objects $\mathcal{A}_1, \mathcal{A}_2$ of Sat_G, there exists a canonical isomorphism*

$$\mathrm{H}^{2\rho(\lambda)}_{\mathrm{c}}(S_\lambda, \mathcal{A}_1 \star \mathcal{A}_2) \simeq \bigoplus_{\lambda_1+\lambda_2=\lambda} \mathrm{H}^{2\rho(\lambda_1)}_{\mathrm{c}}(S_{\lambda_1}, \mathcal{A}_1) \otimes \mathrm{H}^{2\rho(\lambda_2)}_{\mathrm{c}}(S_{\lambda_2}, \mathcal{A}_2). \tag{3.3.6.1}$$

Proof We consider the action of $\mathbb{G}_m$ on Gr_{G,X^n} induced by $2\check{\rho}$, which is compatible with the action of $\mathbb{G}_m$ on Gr_G on each fiber of $x \in |X^n|$. We denote the connected components of Gr^+_{G,X^n} (resp. Gr^-_{G,X^n}), parametrized by $\lambda \in \mathbb{X}_\bullet(T)$, by $S_\lambda(X^n)$ (resp. $T_\lambda(X^n)$) (cf. [62] 6.4). The fiber of $S_\lambda(X^2)$ (resp. $T_\lambda(X^2)$) at $x = (x,x) \in \Delta(X) \subset X^2$ is isomorphic to S_λ (resp. T_λ) and its fiber at $x=(x,y)\in X^2-\Delta(X)$ is isomorphic to $\prod_{\lambda_1+\lambda_2=\lambda} S_{\lambda_1}\times S_{\lambda_2}$ (resp. $\prod_{\lambda_1+\lambda_2=\lambda} T_{\lambda_1}\times T_{\lambda_2}$). Consider the following diagram of ind-schemes:

$$\begin{tikzcd} S_\lambda(X^2) \arrow[r, "j"] \arrow[rr, bend right, "i_\lambda"'] & S_{\leq\lambda}(X^2) \arrow[r, "\overline{i_\lambda}"] & \mathrm{Gr}_{G,X^2} \arrow[r, "q^2"] & X^2 \end{tikzcd} \tag{3.3.6.2}$$

Let $\mathcal{A}_1,\mathcal{A}_2$ be two objects of Sat_G, $\mathcal{A}_{1,X},\mathcal{A}_{2,X}$ their extensions to $\mathrm{Gr}_{G,X}$ (3.2.1.1) and $\mathcal{B}=\mathcal{A}_{1,X}\boxtimes\mathcal{A}_{2,X}$. For $i\in\mathbb{Z}$, we define the constructible module $\mathcal{L}^i_\lambda(\mathcal{A}_1,\mathcal{A}_2)$ on X^2 to be

$$\mathcal{L}^i_\lambda(\mathcal{A}_1,\mathcal{A}_2)={}^{\mathrm{c}}\mathcal{H}^i(q^2_+(i_{\lambda,!}(i^+_\lambda\mathcal{B})))\simeq{}^{\mathrm{c}}\mathcal{H}^i(q^2_+(i'_{\lambda,+}(i'^!_\lambda\mathcal{B}))),$$

where the second isomorphism follows from Braden's theorem (2.6.2). By 3.3.5, $\mathcal{L}^i_\lambda(\mathcal{A}_1,\mathcal{A}_2)$ vanishes unless $i=2\rho(\lambda)$ and the stalk of $\mathcal{L}^{2\rho(\lambda)}_\lambda(\mathcal{A}_1,\mathcal{A}_2)$ at a k-point (x_1,x_2) of X^2 is isomorphic to

$$\begin{aligned}&\mathcal{L}^{2\rho(\lambda)}_\lambda(\mathcal{A}_1,\mathcal{A}_2)_{(x_1,x_2)}\\ &\simeq\begin{cases}\mathrm{H}^{2\rho(\lambda)}_{\mathrm{c}}(S_\lambda,\mathcal{A}_1\star\mathcal{A}_2) & \text{if } x_1=x_2,\\ \bigoplus_{\lambda_1+\lambda_2=\lambda}\mathrm{H}^{2\rho(\lambda_1)}_{\mathrm{c}}(S_{\lambda_1},\mathcal{A}_1)\otimes\mathrm{H}^{2\rho(\lambda_2)}_{\mathrm{c}}(S_{\lambda_1},\mathcal{A}_2) & \text{if } x_1\neq x_2.\end{cases}\end{aligned} \tag{3.3.6.3}$$

The adjunction morphisms $\mathrm{id}\to\overline{i}_{\lambda,+}\overline{i}^+_\lambda$ and $j_!j^+\to\mathrm{id}$ (3.3.6.2) induce canonical morphisms

$${}^{\mathrm{c}}\mathcal{H}^{2\rho(\lambda)}(q^2_+(\mathcal{A}_{1,X}\boxtimes\mathcal{A}_{2,X}))\twoheadrightarrow{}^{\mathrm{c}}\mathcal{H}^{2\rho(\lambda)}((q^2\circ\overline{i}_\lambda)_+\overline{i}^+_\lambda\mathcal{B})\xleftarrow{\sim}\mathcal{L}^{2\rho(\lambda)}_\lambda(\mathcal{A}_1,\mathcal{A}_2), \tag{3.3.6.4}$$

where the first arrow is an epimorphism and the second arrow is an isomorphism in view of the calculation of their fibers (3.3.5). By applying a dual argument to $T_\lambda(X^2)$, we obtain a section $\mathcal{L}^{2\rho(\lambda)}_\lambda(\mathcal{A}_1,\mathcal{A}_2)\to{}^{\mathrm{c}}\mathcal{H}^{2\rho(\lambda)}(q^2_+(\mathcal{A}_{1,X}\boxtimes\mathcal{A}_{2,X}))$ of (3.3.6.4). In view of Proposition 3.3.5, we deduce a decomposition

$${}^{\mathrm{c}}\mathcal{H}^i(q^2_+(\mathcal{A}_{1,X}\boxtimes\mathcal{A}_{2,X}))\simeq\bigoplus_{2\rho(\lambda)=i}\mathcal{L}^i_\lambda(\mathcal{A}_1,\mathcal{A}_2).$$

The left side is a constant module with value $\mathrm{H}^i(\mathrm{Gr},\mathcal{A}_1\star\mathcal{A}_2)$ by (3.3.1.2, 3.3.2.1). Then each summand $\mathcal{L}^i_\lambda(\mathcal{A}_1,\mathcal{A}_2)$ is also constant and fibers of $\mathcal{L}^i_\lambda(\mathcal{A}_1,\mathcal{A}_2)$ (3.3.6.3) are isomorphic. The proposition follows. $\square$

3.4 Tannakian structure and the Langlands dual group

Theorem 3.4.1 (i) *The monoidal category* $(\mathrm{Sat}_G, \mathrm{IC}_0, *)$*, equipped with the constraints* c *defined below and the functor* H^* (3.3.1.1)*, forms a neutral Tannakian category over* L.

(ii) *The Tannakian group* $\widetilde{G} = \mathrm{Aut}^{\otimes}\mathrm{H}^*$ *of the Tannakian category* Sat_G *is a connected reductive group scheme over* L*. If* T *is a maximal torus of* G*, then* $\widetilde{T}$ *is a maximal torus of* $\widetilde{G}$.

(iii) *The reductive group* $\widetilde{G}$ *is the Langlands dual group of* G *over* L*. More precisely, the root datum of* $\widetilde{G}$ *with respect to* $\widetilde{T}$ *is dual to that of* (G, T).

We prove Theorem 3.4.1 in the same way as in [79, 5.2.9, § 5.3] using Propositions 3.3.5 and 3.3.6. We briefly review the construction of the constrains c in the following.

The permutation $\sigma : \{1, 2\} \to \{1, 2\}$ induces an involution $\Delta(\sigma) : \mathrm{Gr}_{X^2} \to \mathrm{Gr}_{X^2}$ over the involution $\sigma : X^2 \to X^2$, $(x, y) \mapsto (y, x)$ (3.2.1). Let $\mathcal{A}_1, \mathcal{A}_2$ be two objects of Sat_G. We deduce from the factorization isomorphism (3.2.1.2) and (3.2.3.1) a canonical isomorphism $\Delta(\sigma)^+(\mathcal{A}_{1,X} \boxtimes \mathcal{A}_{2,X}) \xrightarrow{\sim} \mathcal{A}_{2,X} \boxtimes \mathcal{A}_{1,X}$. Taking its fiber at (x, x), we obtain a canonical isomorphism $c'_{\mathcal{A}_1,\mathcal{A}_2} : \mathcal{A}_1 * \mathcal{A}_2 \simeq \mathcal{A}_2 * \mathcal{A}_1$.

We modify $c'_{\mathcal{A}_1,\mathcal{A}_2}$ by a sign as follows (see [62] after Remark 6.2). The morphism $p : \mathbb{X}_\bullet(T) \to \mathbb{Z}/2\mathbb{Z}$, $\mu \mapsto (-1)^{2\rho(\mu)}$ defines a $\mathbb{Z}/2\mathbb{Z}$-grading on simple objects of Sat_G. Given two simple objects $\mathcal{A}_1, \mathcal{A}_2$ of Sat_G, we define a new constraint $c_{\mathcal{A}_1,\mathcal{A}_2} = (-1)^{p(\mathcal{A}_1)p(\mathcal{A}_2)} c'_{\mathcal{A}_1,\mathcal{A}_2}$.

Since Sat_G is semisimple, the definition of $c_{\mathcal{A}_1,\mathcal{A}_2}$ extends to any pair $(\mathcal{A}_1, \mathcal{A}_2)$ of objects of Sat_G. Proposition 3.3.5 and the same argument of [79, proposition 5.2.6] allow us to deduce the following commutative diagram

$$\begin{tikzcd} \mathrm{H}^*(\mathcal{A}_1 * \mathcal{A}_2) \arrow[r, "c_{\mathcal{A}_1,\mathcal{A}_2}"] \arrow[d, "\wr"] & \mathrm{H}^*(\mathcal{A}_2 * \mathcal{A}_1) \arrow[d, "\wr"] \\ \mathrm{H}^*(\mathcal{A}_1) \otimes \mathrm{H}^*(\mathcal{A}_2) \arrow[r, "c_{\mathrm{Vec}}"] & \mathrm{H}^*(\mathcal{A}_2) \otimes \mathrm{H}^*(\mathcal{A}_1), \end{tikzcd}$$

where the isomorphism c_{Vec} is the usual commutativity constraint on vector spaces, i.e $c_{\mathrm{Vec}}(v \otimes w) = w \otimes v$.

3.4.2. For our applications of the geometric Satake equivalence for arithmetic $\mathscr{D}$-modules, it is important to consider the Frobenius structure on the Satake category. In the following, we study the full Langlands dual group constructed by the Satake category equipped with Frobenius structures.

We suppose that the geometric base tuple $\{k, R, K, L\}$ underlies to an arithmetic base tuple $\{k, R, K, L, t, \sigma\}$ (2.1.2), where t is an integer (which may be different from the degree s of k over $\mathbb{F}_p$).

The t-th Frobenius pullback functor $F^*_{\mathrm{Gr}} = F^+_{\mathrm{Gr}/k} \circ \sigma^* : \mathrm{Hol}(\mathrm{Gr}/L) \xrightarrow{\sim} \mathrm{Hol}(\mathrm{Gr}/L)$ [3, 1.1.3 remark] induces a σ-semi-linear equivalence of tensor categories $F^*_{\mathrm{Gr}} : \mathrm{Sat}_G \xrightarrow{\sim} \mathrm{Sat}_G$. We denote by $F\text{-}\mathrm{Sat}_G$ the category of pairs (X, φ) consisting of an object X of Sat_G and a Frobenius structure $\varphi : F^*_{\mathrm{Gr}}X \xrightarrow{\sim} X$. Morphisms are morphisms of Sat_G compatible with φ (cf. [3] 1.4.6). We will show that $F\text{-}\mathrm{Sat}_G$ is a Tannakian category.

3.4.3. We first study some general constructions in the Tannakian formalism following [69].

For $n \in \mathbb{Z}$, we denote abusively by σ^n the equivalence of categories $(-) \otimes_{L,\sigma^n} L : \mathrm{Vec}_L \xrightarrow{\sim} \mathrm{Vec}_L$.

Let $(\mathcal{C}, \omega)$ be a neutralized Tannakian over L. We suppose that, for each $n \in \mathbb{Z}$, there exists a σ^n-semi-linear equivalence of tensor categories

$$\tau_n : \mathcal{C} \to \mathcal{C}$$

and an isomorphism of tensor functors $\alpha_n : \omega \circ \tau_n \xrightarrow{\sim} \sigma^n \circ \omega$. For any pair $n, m \in \mathbb{Z}$, we suppose moreover that there exists an isomorphism of tensor functors $\varepsilon : \tau_m \circ \tau_n \simeq \tau_{m+n}$ such that

$$(\mathrm{id} \circ \alpha_n) \circ (\alpha_m \circ \mathrm{id}) = \alpha_{m+n} \circ \omega(\varepsilon) : \omega \circ \tau_m \circ \tau_n \simeq \sigma^{m+n} \circ \omega.$$

Since ω is faithful, such an isomorphism ε is unique.

Let H be the Tannakian group of $(\mathcal{C}, \omega)$. The above structure defines a homomorphism

$$\iota : \mathbb{Z} \to \mathrm{Aut}(H(L)), \tag{3.4.3.1}$$

by letting $\iota(n)$ send $h : \omega \to \omega$ to $\omega \xrightarrow{\alpha_n^{-1}} \sigma^{-n} \circ \omega \circ \tau_n \xrightarrow{h \circ \mathrm{id}} \sigma^{-n} \circ \omega \circ \tau_n \xrightarrow{\alpha_n} \omega$.

We define the category $\mathcal{C}^{\mathbb{Z}}$ of $\mathbb{Z}$-equivariant objects in $\mathcal{C}$ as follows. An object $(X, \{c_n\}_{n \in \mathbb{Z}})$ consists of an object X of $\mathcal{C}$ and isomorphisms $c_n : \tau_n(X) \xrightarrow{\sim} X$ satisfying cocycle conditions $c_{n+m} = c_n \circ \tau_n(c_m)$. A morphism between $(X, \{c_n\}_{n \in \mathbb{Z}})$ and $(X', \{c'_n\}_{n \in \mathbb{Z}})$ is a morphism of $\mathcal{C}$ compatible with c_n, c'_n.

3.4.4. Let Γ be an abstract group and $\varphi : \Gamma \to \mathbb{Z}$ a homomorphism. We say an action of Γ on an L-vector space V is *σ-semi-linear (with respect to φ)* if it is additive and satisfies $\gamma(av) = \sigma^{\varphi(\gamma)}(a)\gamma(v)$ for $\gamma \in \Gamma$, $a \in L$ and $v \in V$. We denote by $\mathbf{Rep}_{L,\sigma}(\Gamma)$ the category of σ-semi-linear representations of Γ on finite dimensional L-vector spaces.

We denote by $H(L) \rtimes \mathbb{Z}$ the semi-direct product of $H(L)$ and $\mathbb{Z}$ via ι (3.4.3.1). The short exact sequence $1 \to H(L) \to H(L) \rtimes \mathbb{Z} \to \mathbb{Z} \to 1$ allows us to define the category $\mathbf{Rep}_{L,\sigma}(H(L) \rtimes \mathbb{Z})$.

Proposition 3.4.5 *Keep the assumption and notation as above.*

(i) *The category $\mathcal{C}^{\mathbb{Z}}$ is a Tannakian category over $L_0 = L^{\sigma=1}$ neutralized by ω over L* [36, § 3].

(ii) *Suppose that the Tannakian group H of $(\mathcal{C}, \omega)$ is a split reductive group over L. Then ω induces an equivalence of tensor categories*

$$\mathcal{C}^{\mathbb{Z}} \xrightarrow{\sim} \mathbf{Rep}^{\circ}_{L,\sigma}(H(L) \rtimes \mathbb{Z}),$$

where $\mathbf{Rep}^{\circ}_{L,\sigma}(H(L) \rtimes \mathbb{Z})$ is the full subcategory of $\mathbf{Rep}_{L,\sigma}(H(L) \rtimes \mathbb{Z})$ (3.4.4) consisting of representations whose restriction to $H(L)$ is algebraic.

Proof (i) We define a monoidal structure on $\mathcal{C}^{\mathbb{Z}}$ by letting

$$(X, \{c_n\}) \otimes (X', \{c'_n\}) = (X'', \{c''_n\}),$$

where $X'' = X \otimes X'$ and c''_n is the composition $\tau_n(X'') \simeq \tau_n(X) \otimes \tau_n(X') \xrightarrow{c_n \otimes c'_n} X \otimes X'$. This defines a structure of symmetric monoidal category on $\mathcal{C}^{\mathbb{Z}}$.

We apply [35, 2.5] to show that $(\mathcal{C}^{\mathbb{Z}}, \otimes)$ is rigid. Given an object $(X, \{c_n\})$ of $\mathcal{C}^{\mathbb{Z}}$, we denote by $X^\vee$ be the dual of X in $\mathcal{C}$ and then we have $\tau_n(X^\vee) \simeq \tau_n(X)^\vee$. For each n, we have an isomorphism

$$c_n^\vee : X^\vee \xrightarrow{\sim} (\tau_n(X))^\vee \simeq \tau_n(X^\vee).$$

In view of [36, 1.6.5], the evaluation and coevaluation morphisms of X and of $\tau_n(X)$ are compatible via τ_n. So $(X^\vee, \{(c_n^\vee)^{-1}\})$ is the dual of $(X, \{c_n\})$ in $\mathcal{C}^{\mathbb{Z}}$, as the evaluation and the coevaluation morphisms of $(X, \{c_n\})$ in $\mathcal{C}^{\mathbb{Z}}$ satisfying the axiom of [35, 2.1.2]. Hence $\mathcal{C}^{\mathbb{Z}}$ is a rigid abelian tensor category.

Since τ_n is σ^n-semi-linear, we have $\mathrm{End}(\mathrm{id}_{\mathcal{C}^{\mathbb{Z}}}) \simeq L_0$. The forgetful tensor functor $\mathcal{C}^{\mathbb{Z}} \to \mathcal{C}$ is exact and faithful. Hence the fiber functor ω of $\mathcal{C}$ defines a fiber functor $\omega : \mathcal{C}^{\mathbb{Z}} \to \mathrm{Vec}_L$ [36, 3.1]. Then the assertion follows from [35, 1.10–1.13], see also [36, footnote 12].

(ii) It suffices to construct an equivalence of tensor categories

$$\Psi : \mathbf{Rep}_L(H)^{\mathbb{Z}} \xrightarrow{\sim} \mathbf{Rep}^{\circ}_{L,\sigma}(H(L) \rtimes \mathbb{Z}). \tag{3.4.5.1}$$

Let $((V, \rho), \{c_n\})$ be an object of $\mathbf{Rep}_L(H)^{\mathbb{Z}}$. Then we define a representation $(V, \widetilde{\rho})$ of $\mathbf{Rep}^{\circ}_{L,\sigma}(H(L) \rtimes \mathbb{Z})$, for any element $(h, n) \in H(L) \rtimes \mathbb{Z}$, by letting $\widetilde{\rho}(h, n)$ to be the composition

$$\sigma^n(\omega(V, \rho)) \xrightarrow{\alpha_n^{-1}} \omega(\tau_n(V, \rho)) \xrightarrow{h \circ \mathrm{id}} \omega(\tau_n(V, \rho)) \xrightarrow{c_n} \omega(V, \rho). \tag{3.4.5.2}$$

Using the cocycle condition, one checks that (3.4.5.2) defines a representation. Then we obtain the functor Ψ (3.4.5.1). The natural functor $\mathbf{Rep}_L(H) \to \mathbf{Rep}_L(H(L))$, $\rho \mapsto \rho(L)$, is fully faithful. In view of (3.4.5.2), we deduce that Ψ is fully faithful. We leave the verification of the essential surjectivity to readers. $\square$

3.4.6. The Frobenius pullback functor $F^*_{\mathrm{Gr}} = F^+_{\mathrm{Gr}/k} \circ \sigma^* : \mathrm{Sat}_G \xrightarrow{\sim} \mathrm{Sat}_G$ satisfies $\mathrm{H}^* \circ F^*_{\mathrm{Gr}} \simeq \sigma \circ \mathrm{H}^*$. We take for every integer n the tensor equivalence τ_n on Sat_G to be $|n|$-th composition of F^*_{Gr} (or a quasi-inverse of F^*_{Gr} if $n < 0$) (3.4.3). These functors satisfy the assumption of 3.4.3. With the notation of 3.4.3, $F\text{-}\mathrm{Sat}_G$ is equivalent to the category $\mathrm{Sat}_G^{\mathbb{Z}}$. In this case, we obtain the following result by 3.4.5.

Theorem 3.4.7 (i) *The category $F\text{-}\mathrm{Sat}_G$ is a Tannakian category over L_0, neutralized by the fiber functor H^* over L. If $t = s$ and $\sigma = \mathrm{id}_L$, then $F\text{-}\mathrm{Sat}_G$ is a neutral Tannakian category.*

(ii) *There exists a canonical equivalence of tensor categories*

$$F\text{-}\mathrm{Sat}_G \xrightarrow{\sim} \mathbf{Rep}^{\circ}_{L,\sigma}(\check{G}(L) \rtimes \mathbb{Z}),$$

compatible with fiber functors.

3.4.8. We work with the arithmetic tuple $\mathfrak{T}_F = \{k, R, K, L, s, \mathrm{id}_L\}$ and we suppose there exists a square-root $p^{1/2}$ of p in L. This allows to define half Tate twist functor $(\frac{n}{2})$ for $n \in \mathbb{Z}$ by sending each object $\mathscr{M} \in \mathrm{D}(X/L_F)$, equipped with the Frobenius structure Φ, to $(\mathscr{M}, p^{-sn/2} \cdot \Phi)$.

For $\mu \in \mathbb{X}_\bullet(T)$, we denote by $\mathrm{IC}^{\mathrm{Weil}}_\mu = j_{\mu,!+}(L_{\mathrm{Gr}_\mu})[2\rho(\mu)](\rho(\mu))$ the holonomic module in $F\text{-}\mathrm{Sat}_G$ with weight 0, and by $\mathcal{S}$ the full subcategory of $F\text{-}\mathrm{Sat}_G$ consisting of direct sums of $\mathrm{IC}^{\mathrm{Weil}}_\mu$'s.

The category $\mathcal{S}$ is closed under the convolution on $F\text{-}\mathrm{Sat}_G$, i.e. $\mathrm{IC}^{\mathrm{Weil}}_\lambda \star \mathrm{IC}^{\mathrm{Weil}}_\mu$ is isomorphic to a direct sum of $\mathrm{IC}^{\mathrm{Weil}}_\nu$. Indeed, by Proposition 3.3.5(ii), the Frobenius acts on the total cohomology $\mathrm{H}^*(\mathrm{IC}^{\mathrm{Weil}}_\mu)$ by a diagonalizable automorphism with eigenvalues $q^{n/2}$, $n \in \mathbb{Z}$. Since H^* is compatible with Frobenius structure (3.3.3), so is the Frobenius action on $\mathrm{H}^*(\mathrm{IC}^{\mathrm{Weil}}_\lambda \star \mathrm{IC}^{\mathrm{Weil}}_\mu)$. We have a decomposition $\mathrm{IC}_\lambda \star \mathrm{IC}_\mu \simeq \bigoplus \mathrm{IC}_\nu$. Then the claim follows from the fact that the the action of Frobenius on cohomology determines the isomorphism class of an object of $F\text{-}\mathrm{Sat}_G$ whose underlying holonomic module is isomorphic to a direct sum of IC_ν's.

The canonical functor $F\text{-}\mathrm{Sat}_G \to \mathrm{Sat}_G$ induces an equivalence of tensor categories $\mathcal{S} \xrightarrow{\sim} \mathrm{Sat}_G$. In particular, we obtain equivalences of tensor categories

$$\mathrm{Sat} : \mathbf{Rep}_L(\check{G}) \simeq \mathrm{Sat}_G \simeq \mathcal{S}. \tag{3.4.8.1}$$

3.4.9. We briefly review the action of outer automorphism group of G on Sat_G (resp. on $\mathcal{S}$).

Let $(\mathcal{C}, \omega)$ be a Tannakian category over L and H the associated Tannakian group. We denote by $\mathrm{Aut}^{\otimes}(\mathcal{C}, \omega)$ the set of isomorphism classes of pairs (τ, α) of a tensor equivalence $\tau : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ and an isomorphism of functors $\alpha : \omega \xrightarrow{\sim} \omega \circ \tau$. This set has a natural group structure. A similar construction as in 3.4.3 defines a canonical morphism $\mathrm{Aut}^{\otimes}(\mathcal{C}, \omega) \to \mathrm{Aut}(H)$, which is an isomorphism [47, lemma B.1]. We apply this to the Satake category $\mathcal{S}$ (or Sat_G) equipped with the fiber functor H^*. The action of $\mathrm{Aut}(G)$ on Gr_G induces an action on $(\mathcal{S}, \mathrm{H}^*)$, and therefore an action of $\mathrm{Aut}(G)$ on $\check{G}$, i.e. a homomorphism $\iota : \mathrm{Aut}(G) \to \mathrm{Aut}(\check{G})$.

Lemma 3.4.10 *There is a natural pinning $(\check{B}, \check{T}, N)$ of $\check{G}$ such that that map ι factors as* $\mathrm{Aut}(G) \twoheadrightarrow \mathrm{Out}(G) \xrightarrow{\sim} \mathrm{Aut}^{\dagger}(\check{G}, \check{B}, \check{T}, N) \subset \mathrm{Aut}(\check{G})$.

The lemma can be shown in the same way as in [47, lemma B.2] or [69, lemma A.6]). In particular, for $\sigma \in \mathrm{Aut}(G)$ and $V \in \mathbf{Rep}(\check{G})$, we have $\sigma^* \mathrm{Sat}(V) \simeq \mathrm{Sat}(\iota(\sigma)V)$.

4 Bessel F-isocrystals for reductive groups

In this section, we construct Bessel F-isocrystals for reductive groups and calculate their monodromy groups. We use notations from 1.3.8, with k being a finite field of $q = p^s$ elements. We assume moreover that there exists an element $\pi \in K$ satisfying $\pi^{p-1} = -p$ and a square root of p in K. We fix an arithmetic base tuple $\{k = \mathbb{F}_q, R, K, L, s, \mathrm{id}_L\}$ (2.1.2) and an isomorphism $\overline{K} \simeq \mathbb{C}$ (in order to talk about weight).

We fix $\{0, \infty\} \subset \mathbb{P}^1$ (over some base that we will specify in each subsection), and set $X = \mathbb{P}^1 - \{0, \infty\}$. Although $X \simeq \mathbb{G}_m$, it is more convenient to regard it as a curve with a simply transitive $\mathbb{G}_m$-action.

Throughout this section, let G be a split reductive group (over some base). We fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Let $U \subset B$ be the unipotent radical of B, and $U^{\mathrm{op}} \subset B^{\mathrm{op}}$ the opposite Borel and its unipotent radical. Let $T_{\mathrm{ad}} \subset B_{\mathrm{ad}} \subset G_{\mathrm{ad}}$ denote the quotients of $T \subset B \subset G$ by the center $Z(G)$ of G. We denote by $(\check{G}, \check{B}, \check{T})$ the Langlands dual group of G over L, constructed by the geometric Satake equivalence (3.4).

4.1 Kloosterman F-isocrystals for reductive groups

In this subsection, we follow the method of Heinloth–Ngô–Yun [47] to produce overconvergent F-isocrystals on X via the geometric Langlands correspondence.

We work with schemes over k. We will consider with both geometric coefficients and arithmetic coefficients, but for simplicity, we omit $L_\blacktriangle$ from the notation $\mathrm{Hol}(-/L_\blacktriangle)$, $\mathrm{D}(-/L_\blacktriangle)$ and L from $\mathbf{Rep}_L(-)$.

4.1.1. Let $\mathcal{G} = G \times \mathbb{P}^1$. For a coordinate x on $\mathbb{P}^1$, so $y = x^{-1}$ is a local coordinate around ∞, we denote by

$$\begin{aligned} I(0) &= \{g \in G(k[\![y]\!]) \mid g(0) \in B\} \quad \text{the Iwahori subgroup,}\\ I(1) &= \{g \in G(k[\![y]\!]) \mid g(0) \in U\} \quad \text{the unipotent radical of } I(0),\\ Z(G)(1) &= \{g \in Z(G)(k[\![y]\!]) \mid g(0) \equiv 1 \mod y\},\\ I(2) &= Z(G)(1)[I(1), I(1)],\\ I(i)^{\mathrm{op}} &\subset G(k[\![x]\!]) \text{ the analogous groups obtained by opposite Borel subgroup.} \end{aligned}$$

If G is semisimple, $I(2) = [I(1), I(1)]$. On the other hand, if G is a torus, then $I(2) = I(1)$. (So our definition of $I(2)$ is slightly different from [47] 1.2 when G is not semisimple, but for $G = \mathrm{GL}_n$ coincides with the one in [47] 3.1.) These groups are independent of the choice of x.

By abuse of notations, we use the same notations for the corresponding (ind)-group schemes over k. Then

$$I(1)/I(2) \simeq \bigoplus_{\alpha \text{ affine simple}} U_\alpha,$$

where $U_\alpha(k) \subset G(k[\![s]\!])$ is the root subgroup corresponding to α. We also write

$$\Omega = N_{G(k(\!(x)\!))}(I(0)^{\mathrm{op}})/I(0)^{\mathrm{op}},$$

which is regarded as a discrete group over k.

We denote by $\mathcal{G}(m,n)$ the group scheme over $\mathbb{P}^1$ such that [47, 1.2]

$$\begin{aligned} &\mathcal{G}(m,n)|_X = G \times X,\\ &\mathcal{G}(m,n)(\mathscr{O}_0) = I(m)^{\mathrm{op}} \subset G(\mathscr{O}_0), \mathcal{G}(m,n)(\mathscr{O}_\infty) = I(n) \subset G(\mathscr{O}_\infty). \end{aligned}$$

We denote by $\mathrm{Bun}_{\mathcal{G}(m,n)}$ the moduli stack of $\mathcal{G}(m,n)$-bundles on $\mathbb{P}^1$. Let $\mathrm{Bun}^0_{\mathcal{G}(m,n)}$ denote its connected component containing the trivial $\mathcal{G}(m,n)$-bundle $\star : \mathrm{Spec}(k) \to \mathrm{Bun}_{\mathcal{G}(m,n)}$. For each $\gamma \in \Omega$, there is a canonical isomorphism $\mathrm{Hk}_\gamma : \mathrm{Bun}_{\mathcal{G}(0,n)} \simeq \mathrm{Bun}_{\mathcal{G}(0,n)}$ given by the Hecke modification of $\mathcal{G}(0,n)$-bundles at $0 \in \mathbb{P}^1$ corresponding to γ [47, Corollary 1.2]. This induces a canonical bijection between Ω and the set of connected components of $\mathrm{Bun}_{\mathcal{G}(0,n)}$ (and therefore all $\mathrm{Bun}_{\mathcal{G}(m,n)}$). Let $\mathrm{Bun}^\gamma_{\mathcal{G}(m,n)}$ denote the connected component corresponding to γ under the bijection. For $\gamma \in \Omega$, let $i_\gamma = \mathrm{Hk}_\gamma(\star) : \mathrm{Spec}(k) \to \mathrm{Bun}^\gamma_{\mathcal{G}(0,n)}$.

There is also the action of $I(1)/I(2)$ on $\mathrm{Bun}_{\mathcal{G}(0,2)}$ by modifying $\mathcal{G}(0,2)$-bundles at ∞. Let

$$j : \Omega \times I(1)/I(2) \to \mathrm{Bun}_{\mathcal{G}(0,2)}, \tag{4.1.1.1}$$

be the open immersion of the big cell, defined by applying the action of $I(1)/I(2) \times \Omega$ to the trivial $\mathcal{G}(0,2)$-bundle [47, Corollary 1.3]. Let $j_\gamma : I(1)/I(2) \to \mathrm{Bun}^{\gamma}_{\mathcal{G}(0,2)}$ denote its restriction to the component corresponding to γ.

4.1.2. The stack of Hecke modifications $\mathrm{Hecke}^X_{\mathcal{G}(m,n)}$ of $\mathcal{G}(m,n)$-torsors (over X) classifies quadruples $(\mathscr{E}_1, \mathscr{E}_2, x, \beta)$, where $\mathscr{E}_i \in \mathrm{Bun}_{\mathcal{G}(m,n)}$, $x \in X$ and $\beta : \mathscr{E}_1|_{X-x} \xrightarrow{\sim} \mathscr{E}_2|_{X-x}$. There exist natural morphisms

$$\mathrm{Bun}_{\mathcal{G}(m,n)} \xleftarrow{\mathrm{pr}_1} \mathrm{Hecke}^X_{\mathcal{G}(m,n)} \xrightarrow{\mathrm{pr}_2} \mathrm{Bun}_{\mathcal{G}(m,n)} \times X, \tag{4.1.2.1}$$

where pr_1 (resp. pr_2) sends $(\mathscr{E}_1, \mathscr{E}_2, x, \beta)$ to $\mathscr{E}_1$ (resp. $(\mathscr{E}_2, x)$).

Following [47], we denote by GR the Beilinson–Drinfeld Grassmannian of $\mathcal{G}(m,n)$ with modifications on X. Note that $\mathrm{GR} \simeq \mathrm{Gr}_{G,X} \simeq \mathrm{Gr}_G \times X$ and therefore is independent of (m,n). There exists a smooth atlas $\varpi : U \to \mathrm{Bun}_{\mathcal{G}(m,n)}$ such that [47, remark 4.1]

$$\begin{aligned} U \times_{\mathrm{Bun}_{\mathcal{G}(m,n)},\mathrm{pr}_1} \mathrm{Hecke}^X_{\mathcal{G}(m,n)} &\simeq U \times \mathrm{GR}, \\ (U \times X) \times_{(\mathrm{Bun}_{\mathcal{G}(m,n)} \times X),\mathrm{pr}_2} \mathrm{Hecke}^X_{\mathcal{G}(m,n)} &\simeq U \times \mathrm{GR}\,. \end{aligned} \tag{4.1.2.2}$$

For $V \in \mathbf{Rep}(\check{G})$, we associate a holonomic module $\mathrm{Sat}(V)$ on Gr_G by the geometric Satake equivalence (3.4.8.1). We denote abusively by IC_V the holonomic module on $\mathrm{Hecke}^X_{\mathcal{G}(m,n)}$ defined by smooth descent of $K_{U\times X} \boxtimes \mathrm{Sat}(V)$ on $U \times X \times \mathrm{Gr}_G$ (supported in a subscheme $U \times X \times \mathrm{Gr}_{G,V}$). Then IC_V is supported in a substack $\mathrm{Hecke}^X_{\mathcal{G}(m,n),V}$ of $\mathrm{Hecke}^X_{\mathcal{G}(m,n)}$.

The geometric Hecke operators is defined as a functor

$$\begin{aligned} &\mathrm{Hk} : \mathbf{Rep}(\check{G}) \times \mathrm{D}(\mathrm{Bun}_{\mathcal{G}(m,n)}) \to \mathrm{D}(\mathrm{Bun}_{\mathcal{G}(m,n)} \times X), \\ &(V, \mathscr{M}) \mapsto \mathrm{Hk}_V(\mathscr{M}) := \mathrm{pr}_{2,!}\big(\mathrm{pr}^+_{1,V}(\mathscr{M}) \otimes \mathrm{IC}_V\big). \end{aligned}$$

Here $\mathrm{pr}_{1,V} : \mathrm{Hecke}^X_{\mathcal{G}(m,n),V} \to \mathrm{Bun}_{\mathcal{G}(m,n)}$ and $\mathrm{pr}_2\,|_{\mathrm{Hecke}^X_{\mathcal{G}(m,n),V}} : \mathrm{Hecke}^X_{\mathcal{G}(m,n),V} \to \mathrm{Bun}_{\mathcal{G}(m,n)} \times X$ are schematic (4.1.2.2), which allows us to apply cohomological functors of $\mathrm{pr}_{1,V}$, pr_2 (2.2.7).

We call a tensor functor

$$E : \mathbf{Rep}(\check{G}) \to \mathrm{Sm}(X/L) \quad (\text{resp.}\ \mathrm{Sm}(X/L_F)), \quad V \mapsto E_V$$

$\check{G}$-valued overconvergent isocrystal (resp. F-isocrystal) E on X. A Hecke eigen-module with eigenvalue E is a holonomic module $\mathscr{M}$ on $\mathrm{Bun}_{\mathcal{G}(m,n)}$ together with isomorphisms $\mathrm{Hk}_V(\mathscr{M}) \xrightarrow{\sim} \mathscr{M} \boxtimes E_V$, $V \in \mathbf{Rep}(\check{G})$, which are compatible with tensor structure on $\mathbf{Rep}(\check{G})$ and composition of Hecke operator. We refer to [17, 5.4.2] for the precise definition and detailed discussions.

4.1.3. We take a non-trivial additive character $\psi : \mathbb{F}_p \to K^\times$ and denote by $\pi \in K$ the associated element satisfying $\pi^{p-1} = -p$ (2.1.1). Let $\mathscr{A}_\psi$ be the Dwork F-isocrystal on $\mathbb{A}^1$ (2.1.1).

We fix a generic linear function ϕ of $I(1)/I(2)$, that is, a homomorphism $\phi : I(1)/I(2) \to \mathbb{A}^1$ of algebraic group over k whose restriction to each U_α is an isomorphism

$$\phi_\alpha := \phi|_{U_\alpha} : U_\alpha \simeq \mathbb{A}^1. \tag{4.1.3.1}$$

Let $\mathscr{A}_{\psi\phi} = \phi^+(\mathscr{A}_\psi)$. (Note that our notation is slightly abusive as this sheaf depends only on the character $\psi \circ \mathrm{tr}_{k/\mathbb{F}_p} \circ \phi$ of $I(1)/I(2)$ as a p-group). We denote by $\mathrm{Hol}(\mathrm{Bun}_{\mathcal{G}(0,2)})^{I(1)/I(2),\mathscr{A}_{\psi\phi}}$ the category of holonomic modules on $\mathrm{Bun}_{\mathcal{G}(0,2)}$ which are $(I(1)/I(2), \mathscr{A}_{\psi\phi})$-equivariant.

By repeating the argument of [47, 2.3], we obtain a parallel result for holonomic modules.

Lemma 4.1.4 [47, 2.3] (i) *The canonical morphism $j_{\gamma,!}(\mathscr{A}_{\psi\phi}) \xrightarrow{\sim} j_{\gamma,+}(\mathscr{A}_{\psi\phi})$ is an isomorphism.*

(ii) *The functor*

$$\mathrm{Hol}(X) \to \mathrm{Hol}(\mathrm{Bun}^{\gamma}_{\mathcal{G}(0,2)} \times X)^{I(1)/I(2),\mathscr{A}_{\psi\phi}}, \quad \mathscr{M} \mapsto j_{\gamma,!}(\mathscr{A}_{\psi\phi}) \boxtimes \mathscr{M}$$

is an equivalence of categories, with a quasi-inverse given by

$$\mathscr{N} \mapsto (i_\gamma \times \mathrm{id}_X)^+(\mathscr{N}) \simeq (i_\gamma \times \mathrm{id}_X)^!(\mathscr{N}).$$

We denote by $A_{\psi\phi}$ the object of $\mathrm{Hol}(\mathrm{Bun}_{\mathcal{G}(0,2)})^{I(1),\mathscr{A}_{\psi\phi}}$ defined by $(j_{\gamma,!}(\mathscr{A}_{\psi\phi})[\dim \mathrm{Bun}_{\mathcal{G}(0,2)}])_{\gamma\in\Omega}$.

Theorem 4.1.5 (i) *For $(m,n) = (0,2)$, the holonomic module $A_{\psi\phi}$ (4.1.4) is a Hecke eigen-module with Hecke eigenvalue a $\check{G}$-valued overconvergent F-isocrystal*

$$\mathrm{Kl}^{\mathrm{rig}}_{\check{G}}(\psi\phi) : \mathbf{Rep}(\check{G}) \to \mathrm{Sm}(X/L_F). \tag{4.1.5.1}$$

(ii) *For every representation V of $\check{G}$, $\mathrm{Kl}^{\mathrm{rig}}_{\check{G},V}(\psi\phi)$ is pure of weight zero.*

If ψ (resp. ψ and ϕ) is clear from the context, we simply write $\mathrm{Kl}^{\mathrm{rig}}_{\check{G}}(\psi\phi)$ by $\mathrm{Kl}^{\mathrm{rig}}_{\check{G}}(\phi)$ (resp. $\mathrm{Kl}^{\mathrm{rig}}_{\check{G}}$). In the remainder of this section, we skecth the proof of the above theorem by repeating the strategy in the ℓ-adic case, following [47]. Using [6, 1.3.13] and the cleanness of $\mathscr{A}_{\psi\phi}$ (4.1.4), one can show the holonomicity.

Lemma 4.1.6 [47, 4.1] *For every $V\in\mathbf{Rep}(\check{G})$, the complex* $\mathrm{Hk}_V(A_{\psi\phi})[1]$ *is holonomic.*

Proof of 4.1.5 (i) The action of $I(1)/I(2)$ on $\mathrm{Bun}_{\mathcal{G}(0,2)}$ extends to an action on the diagram (4.1.2.1). For each $\gamma\in\Omega$, $\mathrm{Hk}_V(A_{\psi\phi})|_{\mathrm{Bun}^{\gamma}_{\mathcal{G}(0,2)}\times X}$ is $(I(1)/I(2),\mathscr{A}_{\psi\phi})$-equivariant. By 4.1.4, for each $\gamma\in\Omega$, we have

$$\mathrm{Hk}_V(A_{\psi\phi})|_{\mathrm{Bun}^{\gamma}_{\mathcal{G}(0,2)}\times X}\simeq A^{\gamma}_{\psi\phi}\boxtimes E^{\gamma}_V,$$

where $E^{\gamma}_V[1]$ is a holonomic module on X. By the same argument as in [47, 4.2], we show that E^{γ}_V is canonically isomorphic to E^0_V. So we will drop the index γ in the following.

Since IC_V is ULA with respect to the projection $\mathrm{GR}\to X$ (3.2.1), we have $\Phi(\mathrm{IC}_V)=0$ (2.2.6). Since taking vanishing cycle functor commutes with smooth pull-back and proper push-forward [4, 2.6], we deduce that

$$\begin{aligned}A_{\psi\phi}\boxtimes\Phi(E_V)&\simeq\Phi(A_{\psi\phi}\boxtimes E_V)\simeq\mathrm{pr}_{2,!}(\Phi(\mathrm{pr}^{+}_{1,V}(A_{\psi\phi})\otimes\mathrm{IC}_V))\\&\simeq\mathrm{pr}_{2,!}(\mathrm{pr}^{+}_{1,V}(A_{\psi\phi})\otimes\Phi(\mathrm{IC}_V))=0.\end{aligned}$$

By Corollary 2.3.4, E_V is smooth. Then assertion (i) follows.

(ii) In the following, we present a concrete way to calculate the Hecke eigenvalue.

We denote by $\star\in\mathrm{Bun}_{\mathcal{G}(0,2)}$ the base point corresponding to the trivial bundle $\mathcal{G}(0,2)$. The base change of convolution diagram (4.1.2.1) to $\star\times X$ can be written as

$$\mathrm{Bun}_{\mathcal{G}(0,2)}\xleftarrow{p_1}\mathrm{GR}\xrightarrow{p_2}X. \tag{4.1.6.1}$$

We denote by $\mathrm{GR}_V\subset\mathrm{GR}\simeq\mathrm{Gr}\times X$ the support of $\mathrm{Sat}(V)\boxtimes L_X$, by GR° the inverse image of the big cell $j(I(1)/I(2)\times\Omega)$ by p_1, and by $\mathrm{GR}^{\circ}_V=$

$\mathrm{GR}_V \cap \mathrm{GR}^\circ$. Consider the following diagram:

$$\begin{tikzcd} & & \mathrm{GR}_V^\circ \arrow[dl, "p_{1,V}^\circ"'] \arrow[r, hook, "j'"'] \arrow[rr, bend left, "p_2^\circ"] & \mathrm{GR}_V \arrow[dl, "p_{1,V}"'] \arrow[dr, "p_2"] & \\ \mathbb{A}^1 & I(1)/I(2) \times \Omega \arrow[l, "\phi"'] \arrow[r, hook, "j"] & \mathrm{Bun}_{\mathcal{G}(0,2)} & & X. \end{tikzcd} \tag{4.1.6.2}$$

The argument of [47, § 4.1-4.2] shows that the following canonical morphism is an isomorphism

$$p_{2,!}^\circ(p_{1,V}^{\circ,+}(\mathscr{A}_{\psi\phi}) \otimes \mathrm{IC}_V\,|_{\mathrm{GR}^\circ}) \xrightarrow{\sim} p_{2,+}^\circ(p_{1,V}^{\circ,+}(\mathscr{A}_{\psi\phi}) \otimes \mathrm{IC}_V\,|_{\mathrm{GR}^\circ}). \tag{4.1.6.3}$$

The overconvergent F-isocrystal E_V can be calculated by one of the above pushforward and is therefore pure of weight zero. In particular, Theorem 4.1.5(ii) follows. □

4.1.7. There is the following "trivial" functoriality between Kloosterman F-isocrystals. We fix ψ. Let $G' \to G$ be a homomorphism of reductive groups inducing the same adjoint quotient $G'_{\mathrm{ad}} \xrightarrow{\sim} G_{\mathrm{ad}}$. Then it induces an isomorphism $I'(1)/I'(2) \simeq I(1)/I(2)$, and therefore we can abusively use the notation ϕ to denote the "same" linear functions on these spaces under the identification. On the other hand, it induces a homomorphism of dual groups $\check{G} \to \check{G}'$ and therefore a tensor functor $\mathrm{Res} : \mathbf{Rep}(\check{G}') \to \mathbf{Rep}(\check{G})$ by restrictions. Then $\mathrm{Kl}_{\check{G}'}^{\mathrm{rig}}$ is the push-out of $\mathrm{Kl}_{\check{G}}^{\mathrm{rig}}$ along $\check{G} \to \check{G}'$. Concretely, this means that there is a canonical isomorphism of tensor functors (we omit both ψ and ϕ from the notations)

$$\mathrm{Kl}_{\check{G}'}^{\mathrm{rig}} \simeq \mathrm{Kl}_{\check{G}}^{\mathrm{rig}} \circ \mathrm{Res} : \mathbf{Rep}(\check{G}') \to \mathrm{Sm}(X/L_F)$$

This allows use to reduce certain questions of $\mathrm{Kl}_{\check{G}}^{\mathrm{rig}}$ to the case when $\check{G}$ is simply-connected. We also obtain the following exceptional isomorphisms (due to coincidences of Dynkin diagrams in low rank cases)

$$\mathrm{Kl}_{\mathrm{SL}_2,\mathrm{Sym}^2}^{\mathrm{rig}} \simeq \mathrm{Kl}_{\mathrm{SO}_3,\mathrm{Std}}^{\mathrm{rig}}, \qquad \mathrm{Kl}_{\mathrm{Sp}_4,\ker(\wedge^2\to\mathbf{1})}^{\mathrm{rig}} \simeq \mathrm{Kl}_{\mathrm{SO}_5,\mathrm{Std}}^{\mathrm{rig}} \tag{4.1.7.1}$$

$$\mathrm{Kl}_{\mathrm{SO}_4,\mathrm{Std}}^{\mathrm{rig}} \simeq \mathrm{Kl}_{\mathrm{SL}_2\times\mathrm{SL}_2,\mathrm{Std}\boxtimes\mathrm{Std}}^{\mathrm{rig}}, \qquad \mathrm{Kl}_{\mathrm{SO}_6,\mathrm{Std}}^{\mathrm{rig}} \simeq \mathrm{Kl}_{\mathrm{SL}_4,\wedge^2}^{\mathrm{rig}}, \tag{4.1.7.2}$$

where **1** denotes the trivial representation, Std the standard representation, $\mathrm{Sym}^\bullet$ and $\wedge^\bullet$ the symmetric powers and wedge powers of the standard representation.

4.1.8. There is a natural action of $\mathbb{G}_m$ on $X \subset \mathbb{P}^1$. On the other hand, the group of automorphisms $\mathrm{Aut}(G, B, T)$ acts on $\mathcal{G}(m,n)$. It follows that $\mathbb{G}_m \times \mathrm{Aut}(G, B, T)$ acts on (4.1.2.1), and therefore on (4.1.6.1). It also acts on $I(1)/I(2) \times \Omega$ as group automorphisms such that the open embedding (4.1.1.1) is $\mathbb{G}_m \times \mathrm{Aut}(G, B, T)$-equivariant. Recall that the natural action of $\mathrm{Aut}(G)$ on the Satake category induces a homomorphism $\iota : \mathrm{Aut}(G) \to \mathrm{Aut}(\check{G}, \check{B}, \check{T}, N)$ (3.4.10). Given $\delta = (a, \sigma) \in (\mathbb{G}_m \times \mathrm{Aut}(G, B, T))(k)$ and $V \in \mathbf{Rep}(\check{G})$, then there is a canonical isomorphism

$$\mathrm{Kl}^{\mathrm{rig}}_{\check{G},V}(\psi(\phi \circ \delta)) \simeq a^+ \,\mathrm{Kl}^{\mathrm{rig}}_{\check{G},\iota(\sigma^{-1})V}(\psi\phi), \tag{4.1.8.1}$$

given by the composition

$$\begin{aligned} p_{2,!}(p_{1,V}^+(j_!(\phi\circ\delta)^+\mathscr{A}_\psi) \otimes \mathrm{IC}_V) &\simeq p_{2,!}(\delta^+ p_{1,V}^+(j_!\phi^+\mathscr{A}_\psi) \otimes \mathrm{IC}_V) \\ &\simeq a^+ p_{2,!}(p_{1,V}^+(j_!\phi^+\mathscr{A}_\psi) \otimes (\delta^{-1})^+ \mathrm{IC}_V) \\ &\simeq a^+ p_{2,!}(p_{1,V}^+(j_!\phi^+\mathscr{A}_\psi) \otimes (\mathrm{IC}_{\iota(\sigma^{-1})V})). \end{aligned}$$

In particular, given $t \in T_{\mathrm{ad}}(k) \subset \mathrm{Aut}(G, B, T)$, the element $\delta = (1, t)$ induces an isomorphism

$$\mathrm{Kl}^{\mathrm{rig}}_{\check{G}}(\psi(\phi \circ \delta)) \simeq \mathrm{Kl}^{\mathrm{rig}}_{\check{G}}(\psi\phi). \tag{4.1.8.2}$$

That is, $\mathrm{Kl}^{\mathrm{rig}}_{\check{G}}(\psi\phi)$ depends only on the T_{ad}-orbit of ϕ. On the other hand, let a be an element of $\mathbb{G}_m(k)$, ψ_a the additive character defined by $\psi_a(-) = \psi(a-)$, $t_a \in T_{\mathrm{ad}}$ the unique element such that $\alpha(t_a) = a$ for every simple root α of G and h the Coxeter number of G. By applying $\delta = (a^h, t_a)$ in (4.1.8.1), we deduce that

$$\mathrm{Kl}^{\mathrm{rig}}_{\check{G}}(\psi_a\phi) \simeq \mathrm{Kl}^{\mathrm{rig}}_{\check{G}}(\psi(\phi \circ \delta)) \simeq (a^h)^+ \,\mathrm{Kl}^{\mathrm{rig}}_{\check{G}}(\psi\phi). \tag{4.1.8.3}$$

In addition, given a generic linear function ϕ of $I(1)/I(2)$, the collection $\{\phi_\alpha\}$ from (4.1.3.1) for those α being simple roots of G, provide a pinning of (G, B, T), and therefore induces a splitting $\mathrm{Out}(G) \to \mathrm{Aut}(G, B, T)$. If G is almost simple, not of type A_{2n}, then every element $\sigma \in \mathrm{Out}(G)$ fixes the remaining ϕ_α. If G is of type A_{2n}, the unique non-trivial element $\sigma_0 \in \mathrm{Out}(G)$ send the remaining ϕ_α to $-\phi_\alpha$. Therefore, if either $\check{G}$ is almost simple not of type A_{2n}, or if $p = 2$, then for every $\sigma \in \mathrm{Out}(G)$, we have $\phi \circ (1, \sigma) = \phi$

and a canonical isomorphism

$$\mathrm{Kl}^{\mathrm{rig}}_{\check{G},V}(\psi\phi) \simeq \mathrm{Kl}^{\mathrm{rig}}_{\check{G},\iota(\sigma^{-1})V}(\psi\phi),$$

compatible with the tensor structures. On the other hand, if G is almost simple of A_{2n} and if $p > 2$, then the element $\delta = (-1, \sigma_0)$ induces a canonical isomorphism (4.1.8.1)

$$\mathrm{Kl}^{\mathrm{rig}}_{\check{G},V}(\psi\phi) \simeq (-1)^{+}\,\mathrm{Kl}^{\mathrm{rig}}_{\check{G},V^\vee}(\psi\phi), \tag{4.1.8.4}$$

where $V^\vee$ denotes the dual representation of V, compatible with the tensor structures.

4.1.9. Let ℓ be a prime different from p. We take an isomorphism $\iota : \overline{K} \simeq \overline{\mathbb{Q}}_\ell$. Using the ℓ-adic Artin–Schreier sheaf AS_ψ on $\mathbb{A}^1_k$ associated to ψ, Heinloth, Ngô and Yun construct a ℓ-adic $\check{G}$ local system

$$\mathrm{Kl}^{\text{ét},\ell}_{\check{G}}(\psi\phi) : \mathbf{Rep}(\check{G}) \to \mathrm{LocSysm}(X). \tag{4.1.9.1}$$

By the trace formula [41], [6, 4.3.9] and Gabber–Fujiwara's ℓ-independence [6, 4.3.11], the Frobenius traces of $\mathrm{Kl}^{\text{ét},\ell}_{\check{G},V}(\psi\phi)$ and of $\mathrm{Kl}^{\mathrm{rig}}_{\check{G},V}(\psi\phi)$ at each closed point of X_k coincide via ι.

4.1.10. There is a variant of Heinloth–Ngô–Yun's construction using algebraic $\mathscr{D}$-modules instead of ℓ-adic sheaves to produce a $\check{G}$-connection on X_K in zero characteristic [47, 2.6], as all the geometric objects used in the construction has analogues over K. Namely, we replace the Artin–Schreier sheaf AS_ψ on $\mathbb{A}^1_k$ by the *exponential $\mathscr{D}$-module* $\mathsf{E}_\lambda = K\langle x, \partial_x\rangle/(\partial_x - \lambda)$ with parameter $\lambda \in K$ on $\mathbb{A}^1_K$. Then we have a tensor functor

$$\mathrm{Kl}^{\mathrm{dR}}_{\check{G}}(\lambda\phi) : \mathbf{Rep}(\check{G}) \to \mathrm{Conn}(X_K).$$

Here we identify homomorphisms $\phi : I(1)/I(2) \to \mathbb{A}^1$ of algebraic group over K with $\mathrm{Hom}_K(\mathrm{Lie}\, I(1)/I(2), K)$ via differentiation, so $\lambda\phi$ is regarded as a linear function on $\mathrm{Lie}(I(1)/I(2))$.

4.2 Comparison between $\mathrm{Kl}^{\mathrm{dR}}_{\check{G}}$ and $\mathrm{Kl}^{\mathrm{rig}}_{\check{G}}$

In this subsection, we work with schemes over R and we keep the notation of 4.1. We say a linear function $\phi : I(1)/I(2) \to \mathbb{A}^1$ over R is *generic*, if it is generic modulo the maximal ideal of R. We take such a function ϕ and we denote abusively its base change to k (resp. K) by ϕ. The following theorem is our main result of this subsection.

Theorem 4.2.1 *We set $L = K$. For every representation V of $\check{G}$, there exists a canonical isomorphism of $\mathcal{O}_X$-modules (1.3.8) with connection* (2.4.2)

$$\iota_V : (\mathrm{Kl}^{\mathrm{dR}}_{\check{G},V}(-\pi\phi))^{\dagger} \xrightarrow{\sim} \mathrm{Kl}^{\mathrm{rig}}_{\check{G},V}(\phi), \tag{4.2.1.1}$$

compatible with tensor structures.

Remark 4.2.2 There is a variant of the construction of $\mathrm{Kl}^{\mathrm{dR}}_{\check{G}}(\lambda\phi)$ (resp. $\mathrm{Kl}^{\mathrm{rig}}_{\check{G}}(\psi\phi)$) with multiplicative characters: Kummer $\mathscr{D}$-modules (resp. Kummer isocrystals), which slightly generalizes $A_{\psi\phi}$ (cf. [47] remark 2.5). In this setting, one can also compare de Rham and p-adic local systems as above by the same argument, if the corresponding multiplicative characters match.

4.2.3. We first consider the case where V is associated to a minuscule coweight λ. In this case, Gr_λ is isomorphic to a partial flag variety and is smooth and projective, and IC_V is isomorphic to $K_{\mathrm{Gr}_\lambda}[\dim \mathrm{Gr}_\lambda]$ supported on $\mathrm{GR}_V \simeq \mathrm{Gr}_\lambda \times X$. We show the above theorem by comparing the relative twisted de Rham cohomologies and the relative twisted rigid cohomologies along the morphism

$$p_2^\circ : \mathrm{GR}_V^\circ \to X$$

in (4.1.6.2). To do so, we first show that the associated de Rham and rigid cohomologies at each fiber of X are isomorphic.

We regard (4.1.6.2) as a diagram of schemes over R. We denote $M := p_{1,V}^{\circ,+} \circ \phi^+(\mathsf{E}_{-\pi})[\dim \mathrm{Gr}_\lambda]$, which is a line bundle with connection on $\mathrm{GR}^\circ_{V,K}$. The bundle with connection $M^\dagger$ on $(\mathrm{GR}^\circ_{V,K})^{\mathrm{an}}$ (cf. 2.4.2) is overconvergent and underlies to the arithmetic $\mathscr{D}$-module $p_{1,V}^{\circ,+} \circ \phi^+(\mathscr{A}_\psi)[\dim \mathrm{Gr}_\lambda]$ on $\mathrm{GR}^\circ_{V,k}$, denoted by $\mathscr{M}$.

Lemma 4.2.4 *Let s be a point of $X(R)$. If M_s (resp. $\mathscr{M}_s$) denotes the $+$-pullback of M (resp. $\mathscr{M}$) along the fiber at s, then the specialisation morphism* (2.4.3.2)

$$\mathrm{H}^*_{\mathrm{dR}}((\mathrm{GR}^\circ_{V,s})_K, M_s) \to \mathrm{H}^*_{\mathrm{rig}}((\mathrm{GR}^\circ_{V,s})_k, \mathscr{M}_s) \tag{4.2.4.1}$$

is an isomorphism. Moreover, these cohomology groups vanish except for the middle degree 0.

Proof We set $Y = \mathrm{GR}^\circ_{V,s}$ and we write M (resp. $\mathscr{M}$) instead of M_s (resp. $\mathscr{M}_s$). Since Y admits a smooth compactification Gr_λ whose boundary is a divisor, we can calculate above cohomology groups by direct image of corresponding algebraic (resp. arithmetic) $\mathscr{D}$-modules (2.1.4). Note that $\mathrm{Kl}^{\mathrm{dR}}_{\check{G},V}$

(resp. $\mathrm{Kl}^{\mathrm{rig}}_{\check{G},V}$) is a bundle with connection (resp. overconvergent F-isocrystal) of rank $\dim V$. By the base change, cohomology groups in (4.2.4.1) vanish except for the middle degree and have dimension $\dim V$ in the middle degree. By (4.1.6.3), the canonical morphism $\iota_{\mathrm{rig}} : \mathrm{H}^*_{\mathrm{rig,c}}(Y_k, \mathscr{M}) \to \mathrm{H}^*_{\mathrm{rig}}(Y_k, \mathscr{M})$ is an isomorphism. In view of Proposition 2.4.5, we deduce that the specialisation morphism (4.2.4.1) is surjective. Then the assertion follows. □

4.2.5. *Proof of Theorem* 4.2.1 *in the minuscule case*. Now we use the relative specialization morphism (2.4.7.1) to compare $(\mathrm{Kl}^{\mathrm{dR}}_{\check{G},V})^\dagger$ and $\mathrm{Kl}^{\mathrm{rig}}_{\check{G},V}$. Let $\mathrm{Gr}_{\mathbb{P}^1} \to \mathbb{P}^1$ be the Beilinson–Drinfeld Grassmannian of G over $\mathbb{P}^1$ and $\varpi : \mathrm{Gr}_{\lambda,\mathbb{P}^1} \to \mathbb{P}^1$ the closed subscheme associated to λ. Note that ϖ is a locally trivial fibration over $\mathbb{P}^1$ with smooth projective fibers Gr_λ and defines a good compactification of p_2° in the sense of 2.4.7.

We take again the notation of 2.4.7 for the smooth R-morphism p_2°. We set $A = \Gamma(X, \mathscr{O}_X)$, $A_K = A[\frac{1}{p}]$, $A^0 = \widehat{A}[\frac{1}{p}]$ the ring of analytic functions on $\widehat{X}^{\mathrm{rig}}$ and $A^\dagger = \Gamma(\mathbb{P}^1_k, \mathcal{O}_X)$. We have inclusions $A_K \subset A^\dagger \subset A^0$. If D_{X_K} denotes the ring of algebraic differential operators on X_K, there exists a canonical D_{X_K}-linear specialization morphism (2.4.7.1)

$$\Gamma(X_K, \mathrm{Kl}^{\mathrm{dR}}_{\check{G},V}) \to \Gamma(X_k, \mathrm{Kl}^{\mathrm{rig}}_{\check{G},V}),$$

where the left (resp. right) hand side is coherent over A_K (resp. $A^\dagger$). The above morphism induces a horizontal $A^\dagger$-linear morphism

$$\iota_V : \Gamma(X_K, \mathrm{Kl}^{\mathrm{dR}}_{\check{G},V}) \otimes_{A_K} A^\dagger \to \Gamma(X_k, \mathrm{Kl}^{\mathrm{rig}}_{\check{G},V}),$$

which gives rise to the morphism (4.2.1.1). Recall that the homomorphism $A^\dagger \to A^0$ is faithfully flat [20, 4.3.10)]. To prove ι_V is an isomorphism, it suffices to show that the induced horizontal A^0-linear morphism:

$$\iota_V \otimes_{A^\dagger} A^0 : \Gamma(X_K, \mathrm{Kl}^{\mathrm{dR}}_{\check{G},V}) \otimes_{A_K} A^0 \to \Gamma(X_k, \mathrm{Kl}^{\mathrm{rig}}_{\check{G},V}) \otimes_{A^\dagger} A^0 \quad (4.2.5.1)$$

is an isomorphism. Let $\widehat{A} \to R$ be a continuous homomorphism and $s : A \to R$ the associated R-point of $\mathbb{G}_m$. By base change and [3, 2.4.15], the fiber $\iota \otimes_{A^\dagger} K$ coincides with the morphism (4.2.4.1) associated to the point $s \in X(R)$ and is an isomorphism (4.2.4). Since both sides of (4.2.5.1) are coherent A^0-modules, the morphism $\iota_V \otimes_{A^\dagger} A^0$ is an isomorphism and the assertion follows. □

4.2.6. Next, we consider the case where V is associated to the quasi-minuscule coweight λ. In this case, $\mathrm{Gr}_{\leq\lambda}$ contains a smooth open subscheme Gr_λ whose complement is isomorphic to $\mathrm{Spec}(R)$, and admits a desingularisation $\widetilde{\mathrm{Gr}}_{\leq\lambda}$

(cf. [63] § 7). We take an isomorphism $\mathrm{GR}_V \simeq X \times \mathrm{Gr}_{\leq\lambda}$ and set $\mathrm{GR}_V^{\circ\circ} = \mathrm{GR}_V^{\circ} \cap (X \times \mathrm{Gr}_\lambda)$ to be the smooth locus of GR_V° (4.1.6.2). We denote by $j : \mathrm{GR}_V^{\circ\circ} \to \mathrm{GR}_V^{\circ}$ the open immersion and by

$$\tau = p_2^{\circ} \circ j : \mathrm{GR}_V^{\circ\circ} \to X \tag{4.2.6.1}$$

the canonical morphism, which admits a good compactification $\widetilde{\mathrm{Gr}}_{\leq\lambda} \times \mathbb{P}^1 \to \mathbb{P}^1$ in the sense of 2.4.7. Indeed, $\mathrm{GR}_V^{\circ} \hookrightarrow X \times \mathrm{Gr}_{\leq\lambda}$ is defined by the nonvanishing of sections of some line bundles [47, remark 4.2] and so is $\mathrm{GR}_V^{\circ\circ} \hookrightarrow X \times \mathrm{Gr}_\lambda$. Since $\widetilde{\mathrm{Gr}}_{\leq\lambda} \to \mathrm{Gr}_{\leq\lambda}$ is the blowup outside Gr_λ, then $\mathrm{GR}_V^{\circ\circ} \hookrightarrow X \times \widetilde{\mathrm{Gr}}_{\leq\lambda}$ is the comoplement of some ample divisors. Hence, $\mathrm{GR}_V^{\circ\circ}$ is affine and so is τ.

Let M be the line bundle with connection $p_1^{+}(\mathsf{E}_{-\pi})[\dim \mathrm{Gr}_\lambda]|_{\mathrm{GR}_{V,K}^{\circ\circ}}$ and $\mathscr{M}$ the smooth arithmetic $\mathscr{D}$-module $p_1^{+}(\mathscr{A}_\psi)[\dim \mathrm{Gr}_\lambda]|_{\mathrm{GR}_{V,k}^{\circ\circ}}$. The holonomic module IC_V is constant on $\mathrm{GR}_V^{\circ\circ}$. Then we deduce that

$$j_{!+}(M) \simeq p_1^{+}(\mathsf{E}_{-\pi}) \otimes \mathrm{IC}_V|_{\mathrm{GR}_{V,K}^{\circ}}, \quad j_{!+}(\mathscr{M}) \simeq p_1^{+}(\mathscr{A}_\psi) \otimes \mathrm{IC}_V|_{\mathrm{GR}_{V,k}^{\circ}}.$$

Note that $j_{!+}(M)[1]$, $j_{!+}(\mathscr{M})[1]$ are holonomic.

Lemma 4.2.7 (i) *The complex $\tau_{k,+}(\mathscr{M})[1]$ (resp. $\tau_{K,+}(M)[1]$) is holonomic.*

(ii) *Let s be a point of $X(k)$. We choose a lifting in $X(R)$ and still denote it by s. If we denote by M_s (resp. $\mathscr{M}_s$) the $+$-pullback of M (resp. $\mathscr{M}$) along the fiber at s, then the specialisation morphism* (2.4.3.2)

$$\mathrm{H}^*_{\mathrm{dR}}((\mathrm{GR}_{V,s}^{\circ\circ})_K, M_s) \to \mathrm{H}^*_{\mathrm{rig}}((\mathrm{GR}_{V,s}^{\circ\circ})_k, \mathscr{M}_s)$$

induces an isomorphism

$$\mathrm{H}^0_{\mathrm{dR}}((\mathrm{GR}_{V,s}^{\circ})_K, j_{!+}(M_s)) \xrightarrow{\sim} \mathrm{H}^0_{\mathrm{rig}}((\mathrm{GR}_{V,s}^{\circ})_k, j_{!+}(\mathscr{M}_s)). \tag{4.2.7.1}$$

Proof (i) Let $i : Z \to \mathrm{GR}_V^{\circ}$ be the complement of $\mathrm{GR}_V^{\circ\circ}$ in GR_V°, which is isomorphic to X. Consider the distinguished triangle on $\mathrm{GR}_{\leq\lambda,k}^{\circ}$

$$j_{!+}(\mathscr{M})[1] \to j_{+}(\mathscr{M})[1] \to C \to .$$

By 2.1.10, $C \simeq i^{!}(j_{!+}(\mathscr{M}))[2]$ has degree ≥ 0 and is supported on Z. Applying $p_{2,+}^{\circ}$ to the above triangle, we obtain

$$p_{2,+}^{\circ}(j_{!+}(\mathscr{M}))[1] \to \tau_{+}(\mathscr{M})[1] \to p_{2,+}^{\circ}(C) \to,$$

where the first term is holonomic (cf. 4.1.6), and the second term has cohomological degrees ≤ 0 because τ is affine and the last term has cohomological degrees ≥ 0 since $p_2^\circ|_Z$ is the identity. Then we deduce that each term in the above triangle is holonomic.

(ii) We set $Y = \mathrm{GR}^\circ_{V,s}$, $U = \mathrm{GR}^{\circ\circ}_{V,s}$ and we write simply M (resp. $\mathscr{M}$) instead of M_s (resp. $\mathscr{M}_s$). By applying the argument (resp. a dual argument) of (i), we deduce that the canonical morphism of cohomology groups $\mathrm{H}^0_{\mathrm{rig}}(Y_k, j_{!+}(\mathscr{M})) \to \mathrm{H}^0_{\mathrm{rig}}(U_k, \mathscr{M})$ is injective. (resp. $\mathrm{H}^0_{\mathrm{rig,c}}(U_k, \mathscr{M}) \to \mathrm{H}^0_{\mathrm{rig,c}}(Y_k, j_{!+}(\mathscr{M}))$ is surjective). In summary, we have a sequence whose composition is the canonical morphism ι_{rig}:

$$\mathrm{H}^0_{\mathrm{rig,c}}(U_k, \mathscr{M}) \twoheadrightarrow \mathrm{H}^0_{\mathrm{rig,c}}(Y_k, j_{!+}(\mathscr{M})) \xrightarrow{\sim} \mathrm{H}^0_{\mathrm{rig}}(Y_k, j_{!+}(\mathscr{M})) \hookrightarrow \mathrm{H}^0_{\mathrm{rig}}(U_k, \mathscr{M}), \tag{4.2.7.2}$$

where the middle isomorphism is due to the cleanness (4.1.6.3).

We construct an analogue sequence of (4.2.7.2) for de Rham cohomology of M on U_K. These two sequences fit into a commutative diagram (2.4.5)

$$\begin{tikzcd}
\mathrm{H}^0_{\mathrm{dR,c}}(U_K, M) \arrow[r, two heads] & \mathrm{H}^0_{\mathrm{dR,c}}(Y_K, j_{!+}(M)) \arrow[r, "\sim"] & \mathrm{H}^0_{\mathrm{dR}}(Y_K, j_{!+}(M)) \arrow[r, hook] & \mathrm{H}^0_{\mathrm{dR}}(U_K, M) \arrow[d, "\rho_M"] \\
\mathrm{H}^0_{\mathrm{rig,c}}(U_k, \mathscr{M}) \arrow[u, "\rho_{M,\mathrm{c}}"] \arrow[r, two heads] & \mathrm{H}^0_{\mathrm{rig,c}}(Y_k, j_{!+}(\mathscr{M})) \arrow[r, "\sim"] & \mathrm{H}^0_{\mathrm{rig}}(Y_k, j_{!+}(\mathscr{M})) \arrow[r, hook] & \mathrm{H}^0_{\mathrm{rig}}(U_k, \mathscr{M})
\end{tikzcd} \tag{4.2.7.3}$$

Let E be the image of $\mathrm{H}^0_{\mathrm{rig,c}}(U_k, \mathscr{M}) \to \mathrm{H}^0_{\mathrm{dR}}(Y_K, j_{!+}(M))$. Then the specialisation morphism ρ_M sends E surjectively to the subspace $\mathrm{H}^0_{\mathrm{rig}}(Y_k, j_{!+}(\mathscr{M}))$. Since $\dim E \leq \dim \mathrm{H}^0_{\mathrm{dR}}(Y_K, j_{!+}(M)) = \dim \mathrm{H}^0_{\mathrm{rig}}(Y_k, j_{!+}(\mathscr{M}))$, we deduce that $E = \mathrm{H}^0_{\mathrm{dR}}(Y_K, j_{!+}(M))$ and that ρ_M induces an isomorphism (4.2.7.1). □

4.2.8. *Proof of Theorem* 4.2.1 *in the quasi-minuscule case*. By 4.2.7(i), we have a diagram of D_{X_K}-modules

$$\begin{tikzcd}
\Gamma(X_K, \mathrm{Kl}^{\mathrm{dR}}_{\check{G},V}) \arrow[r] & \Gamma(X_K, \tau_{K,+}(M)) \arrow[d] \\
\Gamma(X_k, \mathrm{Kl}^{\mathrm{rig}}_{\check{G},V}) \arrow[r] & \Gamma(X_k, \tau_{k,+}(\mathscr{M}))
\end{tikzcd}$$

where the vertical arrow is the relative specialization morphism (2.4.7.1). Let U be an open dense subscheme of X_k such that $\tau_{k,+}(\mathscr{M})|_U$ is smooth, $\mathfrak{U}$ the corresponding formal open subscheme of $\widehat{X}$ and $Z = \mathbb{P}^1_k \setminus U$.

By 4.2.7, (4.2.7.3) and the same argument of 4.2.5, the above diagram induces an injective morphism of $\mathcal{O}_U$-modules with connection $(\mathrm{Kl}^{\mathrm{dR}}_{\check{G},V})^\dagger \otimes_{\mathcal{O}_X}$

$\mathcal{O}_U \to \tau_+(\mathscr{M}) \otimes_{\mathcal{O}_X} \mathcal{O}_U$ and then induces an isomorphism of $\mathcal{O}_U$-modules with connection:

$$(\mathrm{Kl}^{\mathrm{dR}}_{\check{G},V})^\dagger \otimes_{\mathcal{O}_X} \mathcal{O}_U \xrightarrow{\sim} \mathrm{Kl}^{\mathrm{rig}}_{\check{G},V} \otimes_{\mathcal{O}_X} \mathcal{O}_U. \tag{4.2.8.1}$$

In particular, the left hand side is overconvergent along Z. Since the convergency of an $\mathscr{O}_{\widehat{X}^{\mathrm{rig}}}$-module with connection can be checked by restricting to a dense open subscheme of X_k [64, 2.16], the $\mathscr{O}_{\widehat{X}^{\mathrm{rig}}}$-module with connection $(\mathrm{Kl}^{\mathrm{dR}}_{\check{G},V})^\dagger|_{\widehat{X}^{\mathrm{rig}}}$ is convergent. Then we deduce that the $\mathcal{O}_X$-module with connection $(\mathrm{Kl}^{\mathrm{dR}}_{\check{G},V})^\dagger$ is overconvergent along $\{0,\infty\}$. The restriction functor $\mathrm{Isoc}^\dagger(X_k/K) \to \mathrm{Isoc}^\dagger(U/K)$ is fully faithful (cf. [53] 6.3.2). Then the isomorphism (4.2.8.1) gives rise to an isomorphism (4.2.1.1) and the assertion follows. □

4.2.9. In the end, we show the general case of Theorem 4.2.1. Let $V_1,\dots,V_n$ be minuscule and quasi-minuscule representations of $\check{G}$. Then we have a decomposition of representations

$$V_1 \otimes V_2 \otimes \cdots \otimes V_n \simeq \bigoplus_{W\in\mathbf{Rep}(\check{G})} m_W W,$$

where m_W denotes the multiplicity of W. Each representation W of $\mathbf{Rep}(\check{G})$ appears as a summand of the above decomposition for some minuscule and quasi-minuscule representations $V_1,\dots,V_n$.

Then we obtain the associated decomposition of bundles with connection on X_K and of overconvergent F-isocrystals on X_K respectively:

$$\bigotimes_{i=1}^n \mathrm{Kl}^{\mathrm{dR}}_{\check{G},V_i} \simeq \bigoplus_{W\in\mathbf{Rep}(\check{G})} m_W\, \mathrm{Kl}^{\mathrm{dR}}_{\check{G},W}, \qquad \bigotimes_{i=1}^n \mathrm{Kl}^{\mathrm{rig}}_{\check{G},V_i} \simeq \bigoplus_{W\in\mathbf{Rep}(\check{G})} m_W\, \mathrm{Kl}^{\mathrm{rig}}_{\check{G},W}. \tag{4.2.9.1}$$

Theorem 4.2.1 in the minuscule and quasi-minuscule cases provides an isomorphism of overconvergent isocrystals

$$(\otimes_{i=1}^n \mathrm{Kl}^{\mathrm{dR}}_{\check{G},V_i})^\dagger \xrightarrow{\sim} \otimes_{i=1}^n \mathrm{Kl}^{\mathrm{rig}}_{\check{G},V_i}. \tag{4.2.9.2}$$

By [19, 2.2.7(iii)], the connection on left hand side, restricted on each component $(\mathrm{Kl}^{\mathrm{dR}}_{\check{G},W})^\dagger$, is overconvergent. We denote abusively the associated overconvergent isocrystal on X_k by $(\mathrm{Kl}^{\mathrm{dR}}_{\check{G},W})^\dagger$.

The isomorphism (4.2.9.2) induces a commutative diagram

$$\begin{tikzcd} & \mathrm{End}_{\mathbf{Rep}(\check{G})}(\bigotimes_{i=1}^{n} V_i) \arrow[dl, "\mathrm{Kl}^{\mathrm{dR}}_{\check{G}}"'] \arrow[dr, "\mathrm{Kl}^{\mathrm{rig}}_{\check{G}}"] & \\ \mathrm{End}_{\mathrm{Conn}(X_K)}(\bigotimes_{i=1}^{n} \mathrm{Kl}^{\mathrm{dR}}_{\check{G},V_i}) \arrow[rr] & & \mathrm{End}_{\mathrm{Sm}(X_k/K)}(\bigotimes_{i=1}^{n} \mathrm{Kl}^{\mathrm{rig}}_{\check{G},V_i}) \end{tikzcd} \tag{4.2.9.3}$$

Indeed, choose a k-point s of X_k and a lift $\widetilde{s}$ to $X(K)$. The isomorphism (4.2.9.2) induces an isomorphism between fibers $(\mathrm{Kl}^{\mathrm{dR}}_{\check{G},V_i})_{\widetilde{s}}$ and $(\mathrm{Kl}^{\mathrm{rig}}_{\check{G},V_i})_s$. The composition of the functor $\mathrm{Kl}^{\mathrm{dR}}_{\check{G}}$ (resp. $\mathrm{Kl}^{\mathrm{rig}}_{\check{G}}$) with the fiber functor at $\widetilde{s}$ (resp. s) is the forgetful functor $\mathbf{Rep}(\check{G}) \to \mathrm{Vec}_K$. Since fiber functors are faithful, we deduce the commutativity of (4.2.9.3) by considering their fibers.

If e denotes the idempotent of $\mathrm{End}_{\mathbf{Rep}(\check{G})}(\otimes_{i=1}^{n} V_i)$ corresponding to a summand W, then its image via left (resp. right) vertical arrow is the idempotent corresponding to $\mathrm{Kl}^{\mathrm{dR}}_{\check{G},W}$ (resp. $\mathrm{Kl}^{\mathrm{rig}}_{\check{G},W}$) (4.2.9.1. By (4.2.9.2) and (4.2.9.3), we deduce a canonical isomorphism of overconvergent isocrystals on X_k

$$\iota_W : (\mathrm{Kl}^{\mathrm{dR}}_{\check{G},W})^{\dagger} \xrightarrow{\sim} \mathrm{Kl}^{\mathrm{rig}}_{\check{G},W}.$$

One verifies that the above isomorphism is independent of the choice of idempotent e and then of the choice of minuscule representations $\{V_i\}_{i=1}^{n}$. Isomorphisms ι_W are compatible with tensor structures due to (4.2.9.2). Now Theorem 4.2.1 follows. □

4.3 Comparison between $\mathrm{Kl}^{\mathrm{dR}}_{\check{G}}$ and $\mathrm{Be}_{\check{G}}$

In this subsection, we recall the Bessel connection $\mathrm{Be}_{\check{G}}(\check{\xi})$ of $\check{G}$ on X constructed by Frenkel and Gross [42] and identify it with $\mathrm{Kl}^{\mathrm{dR}}_{\check{G}}(\phi)$ (4.1.10).

We work with schemes over K. Let $(\check{\mathfrak{g}}, \check{\mathfrak{b}}, \check{\mathfrak{t}})$ denote the Lie algebras of $(\check{G}, \check{B}, \check{T})$ over K.

4.3.1. Let A_K denote the ring of algebraic functions of X. There exists a grading on the affine Lie algebra $\check{\mathfrak{g}}_{\mathrm{aff}} := \check{\mathfrak{g}} \otimes A_K$, which on $\check{\mathfrak{g}}$-part is given by $\mathrm{Ad}\,\rho(\mathbb{G}_m)$, and on A_K-part is given by the $\check{h}$-multiple of the grading induced by the natural action of $\mathbb{G}_m$ on X. Here as before $\rho \in \mathbb{X}^{\bullet}(T) \otimes \mathbb{Q}$ is the half sum of positive roots of G (and therefore is a cocharacter of $\check{G}_{\mathrm{ad}}$), and $\check{h}$ is the Coxeter number of $\check{G}$.

Let $\check{\mathfrak{g}}_{\mathrm{aff}}(1) \subset \check{\mathfrak{g}}_{\mathrm{aff}}$ be the subspace of degree 1. Then

$$\check{\mathfrak{g}}_{\mathrm{aff}}(1) = \bigoplus_{\check{\alpha} \text{ affine simple}} \check{\mathfrak{g}}_{\mathrm{aff},\check{\alpha}},$$

where $\check{\mathfrak{g}}_{\mathrm{aff},\check{\alpha}}$ is the root subspace corresponding to the affine simple root $\check{\alpha}$ of $\check{\mathfrak{g}}_{\mathrm{aff}}$. Let $\check{\xi} \in \check{\mathfrak{g}}_{\mathrm{aff}}(1)$ be a *generic* element, by which we mean each of its $\check{\alpha}$-component $\check{\xi}_{\check{\alpha}} \neq 0$. In [42], Frenkel and Gross defined a $\check{\mathfrak{g}}$-valued connection on the trivial $\check{G}$-bundle on X by the following formula:

$$\mathrm{Be}_{\check{G}}(\check{\xi}) = d + \check{\xi}\frac{dx}{x}. \tag{4.3.1.1}$$

Here x is a coordinate of $X \cup \{0\} \simeq \mathbb{A}^1$. Note that $\frac{dx}{x}$ itself is independent of the choice of the coordinate x, and is a generator of the module of log differentials on $X \cup \{0\}$ with logarithmic pole at 0.

We may write $N = \sum_{\check{\alpha}} \check{\xi}_{\check{\alpha}}$, where the sum is taken over simple roots of $\check{\mathfrak{g}}$ (instead of $\check{\mathfrak{g}}_{\mathrm{aff}}$). This is a principal nilpotent element of $\check{\mathfrak{g}}$. The remaining affine root subspaces are of the form $x\check{\mathfrak{g}}_{-\check{\theta}_i}$, where x is a coordinate as above and $\check{\theta}_i$ is the highest root of the simple factor $\check{\mathfrak{g}}_i$ of $\check{\mathfrak{g}}$. So we may write the sum of the remaining affine root vectors as xE for some $E \in \sum \check{\mathfrak{g}}_{-\check{\theta}_i}$. Then the connection can be written as

$$\mathrm{Be}_{\check{G}}(\check{\xi}) = d + (N + xE)\frac{dx}{x}, \tag{4.3.1.2}$$

which is the form as used in [42]. This connection is regular singular with a principal unipotent monodromy at 0 and has an irregular singularity at ∞, with maximal formal slope $1/\check{h}$ [42, §5].

We regard $\mathrm{Be}_{\check{G}}(\check{\xi})$ as a tensor functor from the category $\mathbf{Rep}(\check{G})$ of representations of $\check{G}$ to the category $\mathrm{Conn}(X)$ of bundles with connection on X.

4.3.2. We will identify $\mathrm{Kl}^{\mathrm{dR}}_{\check{G}}(\lambda\phi)$ and $\mathrm{Be}_{\check{G}}(\check{\xi})$ as $\check{G}$-bundles with integrable connections on X. For this purpose, we need to discuss how these connections depend on parameters. We identify the dual space $\mathfrak{g}^*_{\mathrm{aff}}$ of $\mathfrak{g}_{\mathrm{aff}} := \mathfrak{g} \otimes A_K$ with $\mathfrak{g}^* \otimes \omega_X$ via the canonical residue pairing

$$(\mathfrak{g} \otimes A_K) \otimes (\mathfrak{g}^* \otimes \omega_X) \to K, \quad (\xi \otimes f, \check{\xi} \otimes g) = (\xi, \check{\xi})\mathrm{Res}_{x=\infty} fg\frac{dx}{x}.$$

Recall that $\lambda\phi$ is a linear function $\mathrm{Lie}(I(1)/I(2)) \to K$. We identify $\mathrm{Hom}_K(\mathrm{Lie}\, I(1)/I(2), K)$ with

$$\mathfrak{g}^*_{\mathrm{aff}}(1) = \bigoplus_{\alpha \text{ affine simple}} \mathfrak{g}^*_\alpha.$$

where $\mathfrak{g}^*_\alpha \subset \mathfrak{g}^*_{\mathrm{aff}}$ is the dual of the root subspace corresponding to α.

By (4.1.8.2) (applied to the $\mathscr{D}$-module setting), $\mathrm{Kl}^{\mathrm{dR}}_{\check{G}}(\lambda\phi)$ depends only on the T_{ad}-orbit of the functional $\lambda\phi$. In addition, T_{ad}-orbits of generic linear functions on $\mathrm{Lie}(I(1)/I(2))$ are parameterized by the GIT quotient $\mathfrak{g}^*_{\mathrm{aff}}(1)/\!\!/T_{\mathrm{ad}}$.

On the other hand, the group $\mathbb{G}_m \times \mathrm{Aut}(\check{G}, \check{B}, \check{T})$ acts on $\check{\mathfrak{g}}_{\mathrm{aff}}$ preserving the grading. For $\check{\delta} = (a, \check{\sigma})$, a gauge transform implies that the analogue of (4.1.8.1) holds, namely

$$\mathrm{Be}_{\check{G},V}(\check{\delta}(\check{\xi})) \simeq a^+ \mathrm{Be}_{\check{G},\check{\sigma}V}(\check{\xi}). \tag{4.3.2.1}$$

It follows that the analogue of (4.1.8.2) and of (4.1.8.3) also hold for Bessel connections. In particular, $\mathrm{Be}_{\check{G}}(\check{\xi})$ only depends on the $\check{T}_{\mathrm{ad}}$-orbit of $\check{\xi}$. Again, $\check{T}_{\mathrm{ad}}$-orbits of generic $\check{\xi}$ are parameterized by the GIT quotient $\check{\mathfrak{g}}_{\mathrm{aff}}(1)/\!\!/\check{T}_{\mathrm{ad}}$.

Here is the main theorem of this subsection. When $\check{G}$ is of adjoint type, a weaker version of this theorem was the main result of [80].

Theorem 4.3.3 *There exists a canonical isomorphism of affine schemes*

$$\mathfrak{g}^*_{\mathrm{aff}}(1)/\!\!/T \xrightarrow{\sim} \check{\mathfrak{g}}_{\mathrm{aff}}(1)/\!\!/\check{T}, \tag{4.3.3.1}$$

such that if the T_{ad}-orbit through $\lambda\phi$ and the $\check{T}_{\mathrm{ad}}$-orbit through $\check{\xi}$ match under this isomorphism, then

$$\mathrm{Kl}^{\mathrm{dR}}_{\check{G}}(\lambda\phi) \simeq \mathrm{Be}_{\check{G}}(\check{\xi})$$

as $\check{G}$-bundles with connection on X.

4.3.4. We first explain the isomorphism (4.3.3.1). Let ω_X denote the canonical bundle on X and by abuse of notation, we sometimes also use it to denote the space of its global sections. Via the open embedding $j_\gamma : I(1)/I(2) \hookrightarrow \mathrm{Bun}^\gamma_{\mathcal{G}(0,2)}$, we identify $I(1)/I(2) \times \mathfrak{g}^*_{\mathrm{aff}}(1)$ with $T^*\,\mathrm{Bun}^\gamma_{\mathcal{G}(0,2)}\,|_{j_\gamma(I(1)/I(2))}$. The Hitchin map (e.g. see [17] Sect. 2, and [80])

$$h^{cl} : T^*\,\mathrm{Bun}^\gamma_{\mathcal{G}(0,2)} \to \mathrm{Hitch}(X) := \Gamma(X, \mathfrak{c}^* \times^{\mathbb{G}_m} \omega_X)$$

induces a closed embedding $h^{cl} : \mathfrak{g}^*_{\mathrm{aff}}(1)/\!\!/T \hookrightarrow \mathrm{Hitch}(X)$, where $\mathfrak{c}^* := \mathfrak{g}^*/\!\!/G$ is the GIT quotient of $\mathfrak{g}^*$ by the adjoint action of G, equipped with a $\mathbb{G}_m$-action induced by the natural $\mathbb{G}_m$-action on $\mathfrak{g}^*$. (For an explicit description of the image of the map when $\mathfrak{g}$ is simple, see the discussions before [80] lemma 18).

On the other hand, there exists a canonical morphism

$$\check{\mathfrak{g}}_{\mathrm{aff}}(1)\frac{dx}{x} \subset \check{\mathfrak{g}} \otimes \omega_X \to \Gamma(X, \check{\mathfrak{c}} \times^{\mathbb{G}_m} \omega_X)$$

where $\check{\mathfrak{c}} := \check{\mathfrak{g}} /\!\!/ \check{G}$, which also induces a closed embedding $\check{\mathfrak{g}}_{\mathrm{aff}}(1)/\!\!/\check{T} \to \Gamma(X, \check{\mathfrak{c}} \times^{\mathbb{G}_m} \omega_X)$. The identification $(\mathrm{Lie}\, T)^* = \mathrm{Lie}\, \check{T}$ induces a canonical isomorphism $\mathfrak{c}^* \xrightarrow{\sim} \check{\mathfrak{c}}$. One checks easily that there is a unique isomorphism $\mathfrak{g}^*_{\mathrm{aff}}(1)/\!\!/T \xrightarrow{\sim} \check{\mathfrak{g}}_{\mathrm{aff}}(1)/\!\!/\check{T}$ that fits into the following commutative diagram

$$\begin{tikzcd}
\mathfrak{g}^*_{\mathrm{aff}}(1)/\!\!/T \arrow[r, "\sim"] \arrow[d, hook] & \check{\mathfrak{g}}_{\mathrm{aff}}(1)/\!\!/\check{T} \arrow[d, hook] \\
\Gamma(X, \mathfrak{c}^* \times^{\mathbb{G}_m} \omega_X) \arrow[r, "\sim"] & \Gamma(X, \check{\mathfrak{c}} \times^{\mathbb{G}_m} \omega_X)
\end{tikzcd}$$

where the bottom isomorphism is induced by $\mathfrak{c}^* \xrightarrow{\sim} \check{\mathfrak{c}}$.

In the case G and $\check{G}$ are almost simple, unveiling the definition, we see that $\lambda\phi$ and $\check{\xi}$ match to each other if the following holds: Let r be the rank of G and $\check{G}$. Recall that the ring of invariant polynomials on $\mathfrak{g}^*$ (resp. $\check{\mathfrak{g}}$) has a generator P_r (resp. $\check{P}_r$), homogeneous of degree $h = \check{h}$. We choose them to match each other as functions on $\mathfrak{c}^* \simeq \check{\mathfrak{c}}$. Then $\lambda\phi$ matches $\check{\xi}$ if and only if

$$\lambda^h P_r(\phi) = P_r(\lambda\phi) = \check{P}_r(\check{\xi}). \tag{4.3.4.1}$$

This condition is independent of the choice of P_r and $\check{P}_r$ (as soon as they match to each other).

For concrete computations, it is convenient to fix a coordinate $x \in \mathbb{A}^1 \subset \mathbb{P}^1$, and a pinning $N = \sum_{\check{\alpha} \in \check{\Delta}} \check{\xi}_{\check{\alpha}}$ of $(\check{G}, \check{B}, \check{T})$. Then we may rewrite (4.3.3.1) as an isomorphism

$$\mathfrak{g}^*_{\mathrm{aff}}(1)/\!\!/T \simeq \check{\mathfrak{g}}_{\mathrm{aff}}(1)/\!\!/\check{T} \simeq N + x \sum_i \check{\mathfrak{g}}_{-\check{\theta}_i} \simeq x \sum_i \check{\mathfrak{g}}_{-\check{\theta}_i}. \tag{4.3.4.2}$$

4.3.5. We prove Theorem 4.3.3 by quantizing (4.3.3.1) and applying the Galois-to-automorphic direction of geometric Langlands correspondence. By descent, it suffices to prove the theorem after base change from K to $\overline{K}$. So we assume that all the geometric objects below are defined over $\overline{K}$, and omit the subscript. Let $\check{G}_{\mathrm{ad}}$ denote the adjoint group of $\check{G}$.

We denote by $\mathrm{Op}_{\check{\mathfrak{g}}}(X)$ the ind-affine scheme of $\check{G}_{\mathrm{ad}}$-opers on X [17, 3.1.11]. We consider the subscheme of $\mathrm{Op}_{\check{\mathfrak{g}}} := \mathrm{Op}_{\check{\mathfrak{g}}}(\mathbb{P}^1)_{(0,\varpi(0)),(\infty,1/\check{h})} \subset \mathrm{Op}_{\check{\mathfrak{g}}}(X)$, which is the moduli of $\check{G}_{\mathrm{ad}}$-opers on X which are

- regular singular with principal unipotent monodromy at 0;
- possibly irregular of maximal formal slope $\leq 1/\check{h}$ at ∞.

See the discussions before [80, lemma 20] (where slightly different notations were used). In this case, the action of $\Gamma(X, \check{\mathfrak{c}} \times^{\mathbb{G}_m} \omega_X)$ on $\mathrm{Op}_{\check{\mathfrak{g}}}(X)$ induces a free and transitive action of $x \sum_i \check{\mathfrak{g}}_{-\check{\theta}_i} \simeq \check{\mathfrak{g}}_{\mathrm{aff}}(1) /\!\!/ \check{T}$ (4.3.4.2) on $\mathrm{Op}_{\check{\mathfrak{g}}}$. In particular, $\mathrm{FunOp}_{\check{\mathfrak{g}}}$ has a natural filtration whose associated graded is $(\mathrm{Fun}\check{\mathfrak{g}}_{\mathrm{aff}}(1))^{\check{T}}$.

On the other hand, the space $\mathrm{Op}_{\check{\mathfrak{g}}}$ has a distinguished point, corresponding to the $\check{G}_{\mathrm{ad}}$-oper that is tame at both 0 and ∞. Therefore, we obtain a canonical isomorphism $x \sum_i \check{\mathfrak{g}}_{-\check{\theta}_i} \in \check{\mathfrak{g}}_{\mathrm{aff}}(1) /\!\!/ \check{T} \simeq \mathrm{Op}_{\check{\mathfrak{g}}}(X)$. Explicitly, this isomorphism sends $xE \in x \sum_i \check{\mathfrak{g}}_{-\check{\theta}_i}$ to the connection $d + (N + xE)\frac{dx}{x}$ on the trivial $\check{G}$-bundle which has a natural oper form. Now the quantization of (4.3.3.1) gives a canonical isomorphism of filtered algebras [80, lemma 21]

$$U(\mathrm{Lie}\, I(1)/I(2))^T \simeq \mathrm{FunOp}_{\check{\mathfrak{g}}},$$

whose associated graded gives back to (4.3.3.1). Here $U(V)$ is the universal enveloping algebra of $V = \mathrm{Lie}\, I(1)/I(2)$, equipped with the usual filtration. As V is abelian, it is also canonically isomorphic to $(\mathrm{Fun}V^*)^T$. Putting all the above isomorphisms together, we obtain the following commutative diagram

$$\begin{tikzcd} (\mathrm{Fun}\mathfrak{g}^*_{\mathrm{aff}}(1))^T \arrow[r, "\sim"] \arrow[d, "\sim"'] & (\mathrm{Fun}\check{\mathfrak{g}}_{\mathrm{aff}}(1))^{\check{T}} \arrow[d, "\sim"] \\ U(\mathrm{Lie}\, I(1)/I(2))^T \arrow[r, "\sim"] & \mathrm{FunOp}_{\check{\mathfrak{g}}} \end{tikzcd}$$

Together with the main result of [80], we obtain the proof of Theorem 4.3.3 in the case when $\check{G} = \check{G}_{\mathrm{ad}}$.

4.3.6. Next, we explain how to extend it to allow G to be a general semisimple group.

One approach is to generalize the work of [17] to allow certain level structures, as what [80] did for simply-connected groups. In this approach, one must deal with the subtle question of the construction of "square root" of the canonical bundle on the moduli of $\mathcal{G}$-bundles. In our special case, we have another short and direct approach, using the isomorphism $\mathrm{Kl}^{\mathrm{dR}}_{\check{G}_{\mathrm{ad}}}(\lambda\phi) \simeq \mathrm{Be}_{\check{G}_{\mathrm{ad}}}(\check{\xi})$ just established.

First, we claim that up to isomorphism, there exists a unique de Rham $\check{G}$-local system on X, which induces $\mathrm{Be}_{\check{G}_{\mathrm{ad}}}(\check{\xi})$, and has unipotent monodromy at 0. Indeed, any two such de Rham $\check{G}$-local systems differ by a de Rham $\check{Z}$-local system on $X \cup \{0\} \simeq \mathbb{A}^1$ (i.e. one is obtained from the other by twisting a de Rham $\check{Z}$-local system). As $\check{Z}$ is a finite group, the wild part of the differential Galois group at ∞ of this local system must be trivial, and therefore this local system itself is trivial.

Now since both $\mathrm{Kl}_{\check{G}}^{\mathrm{dR}}(\lambda\phi)$ and $\mathrm{Be}_{\check{G}}(\check{\xi})$ have the property as in the claim (to see that $\mathrm{Kl}_{\check{G}}^{\mathrm{dR}}(\lambda\phi)$ has unipotent monodromy at 0, one uses the same argument as [47] theorem 1 (2)), they must be isomorphic. □

4.4 Bessel F-isocrystals for reductive groups

In this subsection, we construct Bessel F-isocrystals for reductive groups, by putting the above ingredients together. We keep the notation of 4.2.

4.4.1. We take a non-trivial additive character $\psi : \mathbb{F}_p \to K^\times$ and a generic linear function $\phi : I(1)/I(2) \to \mathbb{A}^1$ over R (4.2). We set $\lambda = -\pi \in K$ corresponding to ψ (as in 2.1.1). Let $\check{\xi} \in \check{\mathfrak{g}}_{\mathrm{aff}}(1)$ match $-\pi\phi$ under the isomorphism (4.3.3.1).

We write $\mathrm{Be}_{\check{G}}(\check{\xi})$ more explicitly as follows. Choose a coordinate x of $X \cup \{0\}$ over R, and a pinning $N = \sum_{\check{\alpha} \in \check{\Delta}} \check{\xi}_{\check{\alpha}}$ of $(\check{G}, \check{B}, \check{T})$. By (4.3.4.2), there is a unique element $E = E_\phi \in \sum_i \check{\mathfrak{g}}_{-\check{\theta}_i}$ such that

$$\mathrm{Kl}_{\check{G}}^{\mathrm{dR}}(1 \cdot \phi) \simeq d + (N + xE)\frac{dx}{x}, \tag{4.4.1.1}$$

By (4.3.4.1), we deduce that

$$\mathrm{Kl}_{\check{G}}^{\mathrm{dR}}(-\pi\phi) \simeq d + (N + (-\pi)^h xE)\frac{dx}{x} = \mathrm{Be}_{\check{G}}(\check{\xi}).$$

Now we can define the object appearing in the title of the paper. Let $\mathrm{Be}^\dagger_{\check{G}}(\check{\xi})$ denote the composition of $\mathrm{Be}_{\check{G}}(\check{\xi}) : \mathbf{Rep}(\check{G}) \to \mathrm{Conn}(X_K)$ with the $(-)^\dagger$-functor from (2.4.2.1). By Theorem 4.2.1, a choice of above isomorphism endows $\mathrm{Be}^\dagger_{\check{G}}(\check{\xi})$ with a Frobenius structure, i.e. a lifting of $\mathrm{Be}^\dagger_{\check{G}}(\check{\xi})$ as a functor $\mathbf{Rep}(\check{G}) \to F\text{-}\mathrm{Isoc}^\dagger(X_k/K)$, or alternatively, an isomorphism of tensor functors

$$\varphi : F^*_{X_k} \circ \mathrm{Be}^\dagger_{\check{G}}(\check{\xi}) \xrightarrow{\sim} \mathrm{Be}^\dagger_{\check{G}}(\check{\xi}) : \mathbf{Rep}(\check{G}) \to \mathrm{Isoc}^\dagger(X_k/K),$$

where $F^*_{X_k}: \mathrm{Isoc}^\dagger(X_k/K) \to \mathrm{Isoc}^\dagger(X_k/K)$ denotes the s-th Frobenius pull-back functor. From the calculation of the differential Galois group of $\mathrm{Be}_{\check{G}}$ in [42] coro. 9, coro. 10 (see (1.2.6.1)) that the automorphism group of $\mathrm{Be}_{\check{G}}$ is $Z_G(K)$. Therefore, the Frobenius structure on $\mathrm{Be}^\dagger_{\check{G}}(\check{\xi})$ is independent of the choice of the isomorphism $\mathrm{Be}_{\check{G}}(\check{\xi}) \simeq \mathrm{Kl}^{\mathrm{dR}}_{\check{G}}(\lambda\phi)$. We use $(\mathrm{Be}^\dagger_{\check{G}}(\check{\xi}), \varphi)$ (or simply $\mathrm{Be}^\dagger_{\check{G}}(\check{\xi})$ if there is no confusion) to denote the $\check{G}$-valued overconvergent F-isocrystal

$$(\mathrm{Be}^\dagger_{\check{G}}(\check{\xi}), \varphi) : \mathbf{Rep}(\check{G}) \to F\text{-}\mathrm{Isoc}^\dagger(X_k/K), \tag{4.4.1.2}$$

which we call the *Bessel F-isocrystal of $\check{G}$*.

4.4.2. For each representation $\rho : \check{G} \to \mathrm{GL}(V)$, the restriction of $\mathrm{Be}^\dagger_{\check{G},V}(\check{\xi})$ at 0 defines an object $\mathrm{Be}^\dagger_{\check{G},V}(\check{\xi})|_0$ of $\mathrm{MC}(\mathcal{R}_K/K)$ (2.3.1) equipped with a Frobenius structure and is therefore is solvable at 1 [54, 12.6.1]. By (4.3.1.1), the p-adic exponents of $\mathrm{Be}^\dagger_{\check{G},V}(\check{\xi})|_0$ are 0. Then it is equivalent to the connection $d + d\rho(N)$ over the Robba ring by [54, 13.7.1]. Hence, $\mathrm{Be}^\dagger_{\check{G},V}(\check{\xi})|_0$ satisfies the Robba condition (i.e. it has zero p-adic slope [29]) and is unipotent.

We denote by $F\text{-}\mathrm{Isoc}^{\mathrm{log,uni}}\big((\mathbb{A}^1_k, 0)/K\big)$ the category of log convergent F-isocrystals on $\mathbb{A}^1_k$ with a log pole at 0 relative to K and nilpotent residue, and are overconvergent along ∞ (2.3.2). By [53, 6.3.2], this category is equivalent to the full subcategory of $F\text{-}\mathrm{Isoc}^\dagger(X_k/K)$ consisting of objects which are unipotent at 0. Then the $\check{G}$-valued overconvergent F-isocrystal $(\mathrm{Be}^\dagger_{\check{G}}(\check{\xi}), \varphi)$ (4.4.1.2) factors through:

$$(\mathrm{Be}^\dagger_{\check{G}}(\check{\xi}), \varphi) : \mathbf{Rep}(\check{G}) \to F\text{-}\mathrm{Isoc}^{\mathrm{log,uni}}\big((\mathbb{A}^1_k, 0)/K\big).$$

4.4.3. Here is a more concrete description of the Frobenius structure on $\mathrm{Be}^\dagger_{\check{G}}(\check{\xi})$. Note that its underlying bundles of $\mathrm{Be}^\dagger_{\check{G},V}(\check{\xi})$ are free $\mathcal{O}_{\mathbb{A}^1_k}$-modules (1.3.8). If we set $A^\dagger = \Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{A}^1_k})$ (1.1.3.1), by the Tannakian formalism, the Frobenius structure on $\mathrm{Be}^\dagger_{\check{G}}(\check{\xi})$ is equivalent to an element $\varphi \in \check{G}(A^\dagger)$ satisfying

$$x\frac{d\varphi}{dx}\varphi^{-1} + \mathrm{Ad}_\varphi(N + (-\pi)^h xE) = q(N + (-\pi)^h x^q E). \tag{4.4.3.1}$$

Given a point $a \in |\mathbb{A}^1_k|$ and $\widetilde{a} : A^\dagger \to \overline{K}$ its Teichmüller lifting, we denote by $\varphi_a = \prod_{i=0}^{\deg(a)-1} \varphi(\widetilde{a}^{q^i})$. When $a \neq 0$, the Frobenius trace of $(\mathrm{Be}^\dagger_{\check{G}}(\check{\xi}), \varphi)$ at a

can be calculated by the trace of φ_a. Now we rephrase the above discussions as follows, which is the first main result of our article.

Theorem 4.4.4 *There is a unique element $\varphi \in \check{G}(A^\dagger)$ satisfying the differential equation (4.4.3.1) such that via a (fixed) isomorphism $\overline{K} \simeq \overline{\mathbb{Q}}_\ell$, for every $a \in |X|$ and $V \in \mathbf{Rep}(\check{G})$*

$$\mathrm{Tr}(\varphi_a, V) = \mathrm{Tr}(\mathrm{Frob}_a, \mathrm{Kl}^{\text{ét},\ell}_{\check{G},V,\bar{a}}(\psi\phi)).$$

When $a = 0$, we can describe φ_0 more precisely.

Proposition 4.4.5 *Let 2ρ be the sum of positive coroots in $\mathbb{X}_\bullet(\check{T})$. Then $\varphi_0 = 2\rho(\sqrt{q})$ in the semisimple conjugacy classes $\mathrm{Conj}^{\mathrm{ss}}(\check{G}(\overline{K}))$ of $\check{G}(\overline{K})$.*

Proof The Frobenius endomorphism φ_0 at 0 satisfies $\varphi_0^{-1} N \varphi_0 = qN$ (4.3.1). Since N is principal nilpotent and $\mathrm{Ad}_{\rho(q)} N = q^{-1}N$, we deduce that $\varphi_0 = \varepsilon\rho(q)$ in $\mathrm{Conj}^{\mathrm{ss}}(\check{G}(\overline{K}))$ for some element ε in the center $Z_{\check{G}}(\overline{K})$.

To show $\varepsilon = \mathrm{id}$, it suffices to investigate Frobenius eigenvalues of $\Psi(\mathrm{Be}^\dagger_{\check{G},V})$ (2.3.3) for $V \in \mathbf{Rep}(\check{G})$, which is same as those of $\Psi(\mathrm{Kl}^{\text{ét},\ell}_{\check{G},V})$ by 4.1.10 and Gabber–Fujiwara's ℓ-independence [3, 4.3.11]. By a result of Görtz and Haines [45], the i-th graded piece of the weight filtration of $\Psi(\mathrm{Kl}^{\text{ét},\ell}_{\check{G},V})$ has the same dimension as the dimension of $\mathrm{H}^{2i}(\mathrm{Gr}_G, \mathrm{IC}_V)$ and is equipped with a Frobenius action by $\times q^i$ (cf. [47] 4.3). Then we deduce that $\varepsilon = \mathrm{id}$. □

4.5 Monodromy groups

4.5.1. In this subsection, we keep the notation of 4.4 and we take L to be $\overline{K}$. We drop $\phi\psi$ from the notation.

We denote by $\langle \mathrm{Be}^\dagger_{\check{G}} \rangle$ (resp. $\langle \mathrm{Be}^\dagger_{\check{G}}, \varphi \rangle$, resp. $\langle \mathrm{Be}_{\check{G}} \rangle$) the full subcategory of $\mathrm{Sm}(X_k/\overline{K})$ (resp. $\mathrm{Sm}(X_k/\overline{K}_F)$, resp. $\mathrm{Conn}(X_{\overline{K}})$) whose objects are all the sub-quotients of objects $\mathrm{Be}^\dagger_{\check{G},V}$ (resp. $(\mathrm{Be}^\dagger_{\check{G},V}, \varphi)$, resp. $\mathrm{Be}_{\check{G},V}$) for $V \in \mathbf{Rep}(\check{G})$. Then $\langle \mathrm{Be}^\dagger_{\check{G}} \rangle$ (resp. $\langle \mathrm{Be}^\dagger_{\check{G}}, \varphi \rangle$, resp. $\langle \mathrm{Be}_{\check{G}} \rangle$) forms a Tannakian category over $\overline{K}$ and we denote by G_{geo} (resp. G_{arith}, resp. G_{gal}) the associated Tannakian group (with respect to a fiber functor ω, but is independent of the choice of the fiber functor up to isomorphism [35]). The tensor functors on the left side of the following diagrams induce closed immersions of algebraic groups on the right side

$$\begin{tikzcd}[column sep=small] & \langle \mathrm{Be}^{\dagger}_{\check{G}}, \varphi\rangle \arrow[dl] & \\ \langle \mathrm{Be}^{\dagger}_{\check{G}}\rangle & & \mathbf{Rep}(\check{G}) \arrow[ul] \arrow[dl] \\ & \langle \mathrm{Be}_{\check{G}}\rangle \arrow[ul] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & G_{\mathrm{arith}} \arrow[dr] & \\ G_{\mathrm{geo}} \arrow[ur] \arrow[dr] & & \check{G} \\ & G_{\mathrm{gal}} \arrow[ur] & \end{tikzcd}$$

In [42, Cor. 9, 10], Frenkel and Gross showed that the differential Galois group G_{gal} of the $\check{G}$-connection $\mathrm{Be}_{\check{G}}:\mathbf{Rep}(\check{G})\to \mathrm{Conn}(X_{\overline{K}})$ is a connected closed subgroup of $\check{G}$ and explicitly calculated it when $\check{G}$ is almost simple (cf. (1.2.6.1)). The main theorem of this subsection is as follows.

Theorem 4.5.2 *Let G be a split almost simple group over R and $\check{G}$ its Langlands dual group over $\overline{K}$. We denote by Σ the outer automorphism group of $\check{G}$ and by $\mathrm{Out}(\check{\mathfrak{g}})$ the outer automorphism group of $\check{\mathfrak{g}}$.*

(i) *If $\check{G}$ is not of type A_{2n} or $\mathrm{char}(k)>2$, then $G_{\mathrm{geo}}\to G_{\mathrm{gal}}$ is an isomorphism. In particular,*

- $G_{\mathrm{geo}}\xrightarrow{\sim}\check{G}^{\Sigma,\circ}$, *if $\check{G}$ is not type A_{2n} $(n\geq 2)$ or B_3 or D_{2n} $(n\geq 2)$ with $\Sigma\neq \mathrm{Out}(\check{\mathfrak{g}})$.*
- $G_{\mathrm{geo}}=\check{G}$, *if $\check{G}$ is of type A_{2n},*
- $G_{\mathrm{geo}}\xrightarrow{\sim}G_2$, *if $\check{G}$ is of type B_3 or of type D_4.*
- $G_{\mathrm{geo}}\xrightarrow{\sim}\mathrm{Spin}_{4n-1}$ *if $\check{G}$ is of type D_{2n} with $\Sigma\simeq\{1\}$ $(n\geq 3)$.*

(ii) *If $\check{G}=\mathrm{SL}_{2n+1}$ and $\mathrm{char}(k)=2$, then $G_{\mathrm{geo}}(\mathrm{Be}^{\dagger}_{\mathrm{SL}_{2n+1}})=G_{\mathrm{geo}}(\mathrm{Be}^{\dagger}_{\mathrm{SO}_{2n+1}})$. In particular,*

- $G_{\mathrm{geo}}\xrightarrow{\sim}\mathrm{SO}_{2n+1}$, *if $n\neq 3$,*
- $G_{\mathrm{geo}}\xrightarrow{\sim}G_2$, *if $n=3$.*

In particular, $G_{\mathrm{geo}}\neq G_{\mathrm{gal}}$ in this case.

(iii) *The map $G_{\mathrm{geo}}\to G_{\mathrm{arith}}$ is always an isomorphism.*

4.5.3. We first study the local monodromy at 0 and ∞.

In view of 4.4.2, the restriction functor at 0 (2.3.2.1) induces

$$\mathbf{Rep}(\check{G})\to\langle \mathrm{Be}^{\dagger}_{\check{G}}\rangle\xrightarrow{|_0}\mathrm{MC}^{\mathrm{uni}}(\mathcal{R}/\overline{K})\xrightarrow{\sim}\mathrm{Vec}^{\mathrm{nil}}_{\overline{K}},$$

sending $\rho : \check{G} \to \mathrm{GL}(V)$ to $(V, d\rho(N)) \in \mathrm{Vec}^{\mathrm{nil}}_{\overline{K}}$. Then, it induces closed immersions of Tannakian groups

$$\mathbb{G}_a \to G_{\mathrm{geo}} \to \check{G}, \tag{4.5.3.1}$$

whose composition sends $1 \in \overline{K} \simeq \mathrm{Lie}(\mathbb{G}_a)$ to $N \in \check{\mathfrak{g}}$.

Lemma 4.5.4 *The restriction functor* $|_\infty : \langle \mathrm{Be}^{\dagger}_{\check{G}} \rangle \to \mathrm{MCF}(\mathcal{R}/\overline{K})$ (2.3.2.1) *at* $\infty \in \mathbb{P}^1_k$ *induces a homomorphism* $I_\infty \times \mathbb{G}_a \to G_{\mathrm{geo}}$ *which is non-trivial on* P_∞.

Proof If the image P_∞ in $\check{G}_{\mathrm{geo}}$ were trivial, by the Grothendieck–Ogg–Shafarevich formula, $\mathrm{Kl}^{\text{ét},\ell}_{\check{G}}$ would also be tame at $0, \infty$. Then the associated ℓ-adic representation $\pi_1(X_{\overline{k}}) \to \check{G}$ would factor through the tame quotient $\pi_1^{\mathrm{tame}}(X_{\overline{k}})$, which is isomorphic to I_∞^{tame} as $X \simeq \mathbb{G}_m$. Since $\mathrm{Kl}^{\text{ét},\ell}_{\check{G},V}$ is pure of weight zero for every $V \in \mathbf{Rep}(\check{G})$, the geometric monodromy group of $\mathrm{Kl}_{\check{G}}$ would be semisimple and then finite. This contradicts to fact that $\mathrm{Kl}^{\text{ét},\ell}_{\check{G}}$ has a principal unipotent monodromy at 0 [47, Thm. 1]. □

4.5.5. Since every overconvergent F-isocrystal $\mathrm{Be}^{\dagger}_{\check{G},V}$ is pure of weight 0 and is therefore geometrically semi-simple [6, 4.3.1], the neutral component G°_{geo} is semi-simple [31]. Therefore, (4.5.3.1) implies that it contains a principal unipotent element and hence its projection to the adjoint group $\check{G}_{\mathrm{ad}}$ of $\check{G}$ contains a principal PGL_2. Then it is almost simple and its Lie algebra appears in one of the following chains:

$$\mathfrak{sl}_2 \longrightarrow \mathfrak{sp}_{2n} \longrightarrow \mathfrak{sl}_{2n}$$

$$\begin{tikzcd} & & \mathfrak{sl}_{2n+1} \\ \mathfrak{sl}_2 \arrow[r] & \mathfrak{so}_{2n+1} \arrow[ur] \arrow[dr] & \\ & & \mathfrak{so}_{2n+2} \end{tikzcd}$$

$$\begin{tikzcd} & & & \mathfrak{sl}_{7} \\ \mathfrak{sl}_2 \arrow[r] & \mathfrak{g}_2 \arrow[r] & \mathfrak{so}_{7} \arrow[ur] \arrow[dr] & \\ & & & \mathfrak{so}_{8} \end{tikzcd}$$

$$\mathfrak{sl}_2 \longrightarrow \mathfrak{f}_4 \longrightarrow \mathfrak{e}_6$$

$$\mathfrak{sl}_2 \longrightarrow \mathfrak{e}_7$$

$$\mathfrak{sl}_2 \longrightarrow \mathfrak{e}_8$$

Lemma 4.5.6 *If $\check{G}$ is not of type A_1, and not of type A_2 when $p = 2$, the image $G_{\text{geo}} \to \check{G}_{\text{ad}}$ cannot be contained in a principal* PGL_2 *of $\check{G}_{\text{ad}}$.*

Proof The image of the wild inertia group P_∞ (resp. I_∞) in PGL_2 is a finite p-group (resp. a solvable group). In view of the all possible finite groups contained in PGL_2, there are two possibilities:

(a) the image of P_∞ is contained in $\mathbb{G}_m \subset \mathrm{PGL}_2$;

(b) $p = 2$ and the image of I_∞ (resp. P_∞) is isomorphic to the alternating group A_4 (resp. the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$).

To prove the lemma, we follow a similar argument of [47, 6.8], but with the quasi-minuscule representation replaced by the adjoint representation Ad. In any case, by a result of Baldassarri [14] (cf. [10] 3.2), the maximal p-adic slope of $\mathrm{Be}^{\dagger}_{\check{G},\mathrm{Ad}}$ is less or equal to the maximal formal slope $1/\check{h}$ of $\mathrm{Be}_{\check{G},\mathrm{Ad}}$ (4.3.1). Let r be the rank of $\check{G}$ and h the Coxeter number of $\check{G}$. Then we deduce that

$$\mathrm{Irr}_\infty(\mathrm{Be}^{\dagger}_{\check{G},\mathrm{Ad}}) \leq \frac{\text{rank Ad}}{\check{h}} = \frac{\check{h}+1}{\check{h}} r < r+1, \tag{4.5.6.1}$$

and hence $\mathrm{Irr}_\infty(\mathrm{Be}^\dagger_{\check{G},\mathrm{Ad}}) \leq r$.

On the other hand, we have a decomposition $\mathrm{Ad} \simeq \oplus_{i=1}^r S^{2\ell_i}$ as representations of principal PGL_2, where $\{\ell_1+1, \ldots, \ell_r+1\}$ is the set of exponents of $\check{\mathfrak{g}}$.

Case (a). Since $\mathrm{Irr}_\infty(\mathrm{Be}^\dagger_{\check{G}}) \neq 0$, the image of P_∞ in PGL_2 contains μ_p and the image of I_∞ is contained in $N(\mathbb{G}_m)$. By a similar argument of [47, 6.8], we deduce $\mathrm{Irr}_\infty(S^{2\ell}) \geq \ell - \lfloor \ell/p \rfloor \geq 1$. Under our assumption, $\max_i\{\ell_i, p\} > 2$, so there is least one i such that $\ell_i - \lfloor \ell_i/p \rfloor > 1$. Then $\mathrm{Irr}_\infty(\mathrm{Be}^\dagger_{\check{G},\mathrm{Ad}}) > r$. Contradiction!

Case (b). Recall that there are four irreducible representations of A_4: id, two non-trivial one dimensional representation V_1', V_1'', the standard representation V_3. Via the inclusion $A_4 \to \mathrm{PGL}_2$, we have

$$\begin{aligned}
&S^2 \simeq V_3, \quad S^4 \simeq V_1' \oplus V_1'' \oplus V_3, \quad S^6 \simeq \mathrm{id} \oplus V_3^{\oplus 2},\\
&S^8 \simeq \mathrm{id} \oplus V_1' \oplus V_1'' \oplus V_3^{\oplus 2},\\
&S^{10} \simeq V_1' \oplus V_1'' \oplus V_3^{\oplus 3}, \quad S^{12} \simeq \mathrm{id}^{\oplus 2} \oplus V_1' \oplus V_1'' \oplus V_3^{\oplus 3},\\
&S^{14} \simeq \mathrm{id} \oplus V_1' \oplus V_1'' \oplus V_3^{\oplus 4}.
\end{aligned}$$

In particular, we have $\mathrm{Irr}_\infty(S^{2\ell}) \geq 2$ for $\ell = 3, 4, 5, 6, 7$. In general, I_∞ acts non-trivially on S^{2n} and we have $\mathrm{Irr}_\infty(S^{2\ell}) \geq 1$. Then we deduce that $\mathrm{Irr}_\infty(\mathrm{Ad}) \geq r(G) + 1$. Contradiction! □

4.5.7. *Proof of Theorem* 4.5.2. By the "trivial" functoriality (4.1.7), it is enough to prove the theorem when $\check{G}$ is simply-connected, so that $\check{G}^\Sigma$ is connected.

(a) The case where $\check{G}$ is not of type A_{2n}. In view of lemma 4.5.6, and the calculation of G_{gal} (1.2.6.1), we deduce that $G^\circ_{\mathrm{geo}} \to G_{\mathrm{geo}} \to G_{\mathrm{gal}}$ are isomorphisms. Using 4.1.8, we see $G_{\mathrm{arith}} \subset \check{G}^\Sigma$. This implies that $G_{\mathrm{arith}} = G_{\mathrm{geo}}$ unless $\check{G}$ is of type B_3. In this last case, if $\check{G} = \mathrm{Spin}_7$, and $G_{\mathrm{arith}} \subset G_2 \times Z(\check{G})$. Taking into account of the Frobenius at 0 (4.4.5), we see that $G_{\mathrm{arith}} = G_{\mathrm{geo}}$.

(b) The case where $\check{G}$ is of type A_{2n} and $p > 2$. It suffices to exclude that G_{geo} is contained in SO_{2n+1}. Suppose it is true by contrast. Let σ_0 be the generator of Σ and $\check{\delta} = (-1, \sigma_0)$ in $\mathbb{G}_m \times \mathrm{Aut}(G, B, T)$. Then we deduce isomorphisms of overconvergent isocrystals on X_k

$$\mathrm{Be}^\dagger_{\mathrm{SL}_{2n+1},\mathrm{Std}}(\check{\xi}) \simeq (-1)^+ \, \mathrm{Be}^\dagger_{\mathrm{SL}_{2n+1},\mathrm{Std}^\vee}(\check{\xi}) \simeq (-1)^+ \, \mathrm{Be}^\dagger_{\mathrm{SL}_{2n+1},\mathrm{Std}}(\check{\xi}),$$

where the first isomorphism follows from (4.1.8.4), and the second one is due to $\mathrm{Std}^\vee \simeq \mathrm{Std}$ as representations of SO_{2n+1}. Since char $k > 2$, this isomorphism provides a "descent datum" so that $\mathrm{Be}^\dagger_{\mathrm{SL}_{2n+1},\mathrm{Std}}(\check{\xi})$ descends to $\mathbb{G}_m/\mu_2$. It follows that its Swan conductor at ∞ is at least two, if non-zero. On the other hand, using Lemma 4.5.4 and the result of Baldassarri [14] (cf. [10] 3.2) again, the Swan conductor of $\mathrm{Be}^{\dagger}_{\mathrm{SL}_{2n+1},\mathrm{Std}}(\check{\xi})$ at ∞ is 1, contradiction!

(c) The case where $\check{G}$ is of type A_{2n} and $p=2$. In appendix (A1), we will identify $\mathrm{Be}^{\dagger}_{\mathrm{SO}_{2n+1},\mathrm{Std}}$ with $\mathrm{Be}^{\dagger}_{\mathrm{SL}_{2n+1},\mathrm{Std}}$. Then we reduce to the case (a). □

We end this section by some corollaries of our calculation of the monodromy groups.

Corollary 4.5.8 *Assume that $\check{G}$ is almost simple. The monodromy groups G^{ℓ}_{geo}, $G^{\ell}_{\mathrm{arith}}$ of the $\mathrm{Kl}^{\text{ét},\ell}_{\check{G}}(\psi\phi)$ over $\overline{\mathbb{Q}}_{\ell}$ (4.1.9.1) are calculated as in Theorem 4.5.2.*

Note that this gives a different proof of the main result of [47] theorem 3.

Proof The monodromy group $G^{\ell}_{\mathrm{arith}}$ (resp. G_{arith}) can be calculated by that of $\mathrm{Kl}^{\text{ét},\ell}_{\check{G},V}$ (resp. $\mathrm{Be}^{\dagger}_{\check{G},V}$) for a faithful representation V of $\check{G}$. The semisimplification of $\mathrm{Kl}^{\text{ét},\ell}_{\check{G},V}$ and $\mathrm{Be}^{\dagger}_{\check{G},V}$ are semi-simple and have same Frobenius traces. Then by [32, 4.1.1, 4.3.2], there exists a surjective morphism $G^{\ell}_{\mathrm{arith}} \twoheadrightarrow G_{\mathrm{arith}}$. Since they are both closed subgroups of $\check{G}$, they must be isomorphic to each other and the assertion follows. □

Corollary 4.5.9 *Assume that $\check{G}$ is almost simple. Let* Ad *be the adjoint representation of $\check{G}$.*

(i) *We have* $\mathrm{H}^i(\mathbb{P}^1, j_{!+}(\mathrm{Be}^{\dagger}_{\check{G},\mathrm{Ad}}))=0$ *for all i.*

(ii) *We have* $\mathrm{Irr}_{\infty}(\mathrm{Be}^{\dagger}_{\check{G},\mathrm{Ad}})=r(\check{G})$*, the rank of $\check{G}$. In addition,* $\mathrm{Ad}^{I_{\infty}}=0$*, and the nilpotent monodromy operator* $N_{\infty}=0$ (4.5.3). *Therefore, the local Galois representation $I_{\infty}\to\check{G}$ is a simple wild parameter in the sense of Gross–Reeder [46, § 6].*

Proof The corresponding assertions for the algebraic connection $\mathrm{Be}_{\check{G},\mathrm{Ad}}$ are proved in [42, §14]. Set $\mathscr{E}=\mathrm{Be}^{\dagger}_{\check{G},\mathrm{Ad}}$, which is self dual. We have $\mathrm{H}^0(X,\mathscr{E})=\mathrm{Ad}^{G_{\mathrm{geo}}}=0$ and $\mathrm{H}^2(X,\mathscr{E})=0$ by $\mathscr{D}^{\dagger}$-affinity. We obtain $\mathrm{H}^i_{\mathrm{c}}(X,\mathscr{E})=0$ for $i=0,2$ by the Poincaré duality. By the Grothendieck–Ogg–Shafarevich formula and (4.5.6.1), we have

$$\mathrm{H}^1_{\mathrm{c}}(X,\mathscr{E})=\mathrm{Irr}_{\infty}(\mathscr{E})\leq r(\check{G}).$$

Let $j : X \to \mathbb{P}^1$ be the inclusion. The distinguished triangle $j_!(\mathscr{E}) \to j_{!+}(\mathscr{E}) \to i_0^+ j_{!+}(\mathscr{E}) \oplus i_\infty^+ j_{!+}(\mathscr{E}) \to$ induces a long exact sequence:

$$\begin{aligned} &0 \to \mathrm{H}^0(\mathbb{P}^1, j_{!+}(\mathscr{E})) \to i_0^+ j_{!+}(\mathscr{E}) \oplus i_\infty^+ j_{!+}(\mathscr{E}) \xrightarrow{d} \mathrm{H}^1_c(X, \mathscr{E}) \to \\ &\mathrm{H}^1(\mathbb{P}^1, j_{!+}(\mathscr{E})) \to 0 \to \mathrm{H}^2_c(X, \mathscr{E}) = 0 \to \mathrm{H}^2(\mathbb{P}^1, j_{!+}(\mathscr{E})) \to 0. \end{aligned} \tag{4.5.9.1}$$

By the Poincaré duality, we conclude that $\mathrm{H}^i(\mathbb{P}^1, j_{!+}(\mathscr{E})) = 0$ for $i = 0, 2$.

For $x \in \{0, \infty\}$, the restriction of $\mathscr{E}$ at x gives rise to an action of the inertia group I_x on Ad and a commuting nilpotent monodromy operator $\mathcal{N}_x : \mathrm{Ad} \to \mathrm{Ad}$ (2.3.1). By Proposition 2.3.3, we have:

$$i_x^+(j_{!+}(\mathscr{E})) \simeq \mathrm{Ad}^{I_x, \mathcal{N}_x} := \mathrm{Ker}(\mathcal{N}_x : \mathrm{Ad}^{I_x} \to \mathrm{Ad}^{I_x}).$$

The Bessel isocrystal is unipotent at 0 with $\mathcal{N}_0 = [-, N]$ (4.4.2). We have $\mathrm{Ad}^{I_0, \mathcal{N}_0} = \mathrm{Ad}^N$, which has dimension $r(\check{G})$. Then the morphism d in (4.5.9.1) is both injective and surjective. We deduce that

$$\mathrm{Ad}^{I_\infty, \mathcal{N}_\infty} = 0, \qquad \mathrm{H}^1(\mathbb{P}^1, j_{!+}(\mathscr{E})) = 0.$$

Since $\mathcal{N}_\infty$ is still a nilpotent operator on Ad^{I_∞}, we conclude assertions (i) and (ii). □

Remark 4.5.10 (i) By Corollary 4.5.8 and the same arguments, we recover [47] prop. 5.3 on the analogous statements for $\mathrm{Kl}_{\check{G}}$ (and remove the restriction of the characteristic of k in *loc. cit.*).

(ii) It follows from [46] prop. 5.6 that when p does not divide the order $\sharp W$ of Weyl group, the only non-zero break of $\mathrm{Be}^\dagger_{\check{G},\mathrm{Ad}}$ (and $\mathrm{Kl}_{\check{G}}$) at ∞ is $1/\check{h}$. Indeed, the local Galois representation $I_\infty \to \check{G}$ is described explicitly in [46] prop. 5.6 and § 6.2.

(iii) It is expected that the description in (ii) of the local monodromy of $\mathrm{Be}^\dagger_{\check{G}}$ (and $\mathrm{Kl}_{\check{G}}$) at ∞ should hold when $(p, \check{h}) = 1$. When $\check{G} = \mathrm{GL}_n$, this is indeed the case. For Kl_n, this was proved by Fu and Wan [43, Theorem 1.1]. For $\mathrm{Be}^\dagger_n$, this can be shown by studying the solutions of Bessel differential equation (1.1.1.1) at ∞. We omit details and refer to [65, 6.7] for a treatment in the case when $n = 2$.

(iv) Using Theorem 4.5.2 (ii), which will be proved in the Appendix A1, we see that when $p = 2$ and n is an odd integer, the associated local Galois representation of $\mathrm{Be}^\dagger_{\mathrm{SO}_n}$ at ∞ coincides with the simple wild parameter constructed by Gross–Reeder [46] § 6.3. In particular, the image of the inertia group I_∞ in the case $\check{G} = \mathrm{SO}_3$ is isomorphic to A_4. Together with $\mathrm{Be}^\dagger_{\mathrm{SO}_3,\mathrm{Std}} \simeq \mathrm{Be}^\dagger_{\mathrm{SL}_2,\mathrm{Sym}^2}$

(4.1.7.1), this allows us to recover André's result on the local monodromy group of $\mathrm{Be}_2^\dagger$ at ∞ in the case $p = 2$ [10, § 7, 8].

5 Applications

In this section, we give some applications of our study of Bessel F-isocrystals for reductive groups.

5.1 Functoriality of Bessel F-isocrystals

We may ask all possible Frobenius structure on $\mathrm{Be}^\dagger_{\check{G}}(\check{\xi})$ (not necessarily the one from 4.4.1), i.e. all possible isomorphisms of tensor functors $\varphi : F_X^* \circ \mathrm{Be}^\dagger_{\check{G}} \xrightarrow{\sim} \mathrm{Be}^\dagger_{\check{G}}$.

Lemma 5.1.1 *The Frobenius structure on* $\mathrm{Be}^\dagger_{\check{G}}(\check{\xi})$ *is unique up to an element in the center* $Z_{\check{G}}(\overline{K})$ *of* $\check{G}$.

Proof Given two Frobenius structures $\varphi_1, \varphi_2, u := \varphi_2 \circ \varphi_1^{-1}$ is an isomorphism of tensor functors $\mathrm{Be}^\dagger_{\check{G}}(\check{\xi}) \xrightarrow{\sim} \mathrm{Be}^\dagger_{\check{G}}(\check{\xi})$. If ω denotes a fiber functor of $\langle \mathrm{Be}^\dagger_{\check{G}}(\check{\xi}) \rangle$, then $\omega \circ u$ is an element in $\check{G}(\overline{K})$ commuting with $G_{\mathrm{geo}}(\overline{K})$ by the Tannakian formalism. Then the assertion follows from $Z_{\check{G}}(G_{\mathrm{geo}}) = Z_{\check{G}}$. □

5.1.2. Let G, G' be two split, almost simple groups over R whose Langlands dual groups $\check{G}' \subset \check{G}$ over $\overline{K}$ appear in the same line in the left column of the (1.2.6.1). Up to conjugation, we can assume that the inclusion $\check{G}' \subset \check{G}$ preserves the pinning. Then it induces a natural inclusion $\check{\mathfrak{g}}'_{\mathrm{aff}}(1) \subset \check{\mathfrak{g}}_{\mathrm{aff}}(1)$. Let ϕ' be a generic linear function of G' over R (4.4.1) and $\check{\xi}$ the generic element in $\check{\mathfrak{g}}'_{\mathrm{aff}}(1)$ corresponding to $-\pi\phi'$ (4.3.3.1). Note that $\check{\xi}$ is also a generic element in $\check{\mathfrak{g}}_{\mathrm{aff}}(1)$.

Proposition 5.1.3 (i) *There exists a generic linear function* ϕ *of* G *over* R *such that* $-\pi\phi$ *matches* $\check{\xi} \in \check{\mathfrak{g}}_{\mathrm{aff}}(1)$ *under the isomorphism* (4.3.3.1).

(ii) *Let* $(\mathrm{Be}^\dagger_{\check{G}'}(\check{\xi}), \varphi')$ *(resp.* $(\mathrm{Be}^\dagger_{\check{G}}(\check{\xi}), \varphi)$*) be the Bessel* F*-isocrystal of* $\check{G}'$ *(resp.* $\check{G}$*) constructed by* ϕ' *(resp.* ϕ*) in* 4.4.1. *Then* $(\mathrm{Be}^\dagger_{\check{G}}(\check{\xi}), \varphi)$ *is the push-out of* $(\mathrm{Be}^\dagger_{\check{G}'}(\check{\xi}), \varphi')$.

Proof (i) Let ϕ be the generic linear function of G over K such that $-\pi\phi$ corresponds to $\check{\xi}$ under the isomorphism (4.3.3.1). We will show that ϕ is naturally integral.

By construction, $\mathrm{Be}_{\check{G}}(\check{\xi})$ is the push-out of $\mathrm{Be}_{\check{G}'}(\check{\xi})$. In particular, for $V \in \mathbf{Rep}(\check{G})$, the connection $(\mathrm{Be}_{\check{G},V}(\check{\xi}))^{\dagger}$ has a Frobenius structure and is overconvergent. Let χ be a generic linear function of G over R and $\check{\eta} \in \check{\mathfrak{g}}_{\mathrm{aff}}(1)$ the corresponding generic element. Then there exists an element $c \in K^{\times}$ such that we can rewrite two Bessel connections for the adjoint representation of $\check{G}$ as follow (4.3.1.2):

$$\mathrm{Be}_{\check{G},\mathrm{Ad}}(\check{\eta}) = d + (N + xE)\frac{dx}{x}, \quad \mathrm{Be}_{\check{G},\mathrm{Ad}}(\check{\xi}) = d + (N + cxE)\frac{dx}{x}.$$

Via (4.3.3.1), it suffices to show that $c \in R^{\times}$.

Both the above two connections admit Frobenius structures and decompose in the categories $\mathrm{Conn}(X_{\overline{K}})$, $\mathrm{Sm}(X_k/\overline{K})$ and $\mathrm{Sm}(X_k/\overline{K}_F)$ in the same way Theorem 4.5.2. Let V be a non-trivial irreducible component of Ad in $\mathbf{Rep}(\check{G}')$ and $V(\check{\eta})$, $V(\check{\xi})$ the corresponding overconvergent F-isocrystal. Since $V(\check{\eta})|_0$ is unipotent, if $\{e_i\}$ denotes a basis of V, there exists a solution

$$u : e_i \mapsto f_i(x) \in \mathrm{Sol}(V(\check{\eta})|_0) \quad (2.3.1.2)$$

whose convergence domain is the open unit disc of radius 1. Then $u_c : e_i \mapsto f_i(cx)$ belongs to $\mathrm{Sol}(V(\check{\xi})|_0)$ and has the same convergent radius. If c is not a p-adic unit, then $V(\check{\eta})$ (or $V(\check{\xi})$) admits the trivial overconvergent isocrystal on X_k as a quotient, which contracts to their irreducibility. The assertion follows.

(ii) By (i), the $\check{G}$-valued overconvergent isocrystal $\mathrm{Be}^{\dagger}_{\check{G}}(\check{\xi})$ is the push-out of $\mathrm{Be}^{\dagger}_{\check{G}'}(\check{\xi})$. It remains to identify two Frobenius structures on $\check{G}$-valued overconvergent isocrystals $\mathrm{Be}^{\dagger}_{\check{G}}(\check{\xi}) \simeq \mathrm{Be}^{\dagger}_{\check{G}'}(\check{\xi}) \times^{\check{G}'} \check{G}$, which are different by an element ε in the center $Z_{\check{G}}(\overline{K})$ by (5.1.1). Taking account of the extension of Frobenius structures to 0 (4.4.5), we deduce that $\varepsilon = \mathrm{id}$ and the assertion follows. □

Now we can prove the following conjecture of Heinloth–Ngô–Yun [47, conjecture 7.3].

Theorem 5.1.4 *We keep the notation of 5.1.2 and fix a non-trivial additive character ψ. Assume that $\check{G}' \subset \check{G}$ over $\overline{\mathbb{Q}}_\ell$ appear in the same line in the left column of the (1.2.6.1). For every generic linear function ϕ' of G' over k, there is a generic linear function ϕ of G over k such that $\mathrm{Kl}^{\mathrm{\acute{e}t},\ell}_{\check{G}}(\psi\phi)$ is isomorphic to the push-out of $\mathrm{Kl}^{\mathrm{\acute{e}t},\ell}_{\check{G}'}(\psi\phi')$ along $\check{G}' \subset \check{G}$ as ℓ-adic $\check{G}$-local systems on X_k.*

Proof By the "trivial" functoriality (4.1.7), we may assume that $\check{G}$ is simply connected. We lift ϕ to be a generic linear function of G' over R and take ϕ'

as in 5.1.3. We need to show that $\mathrm{Kl}_{\check{G}}^{\text{ét},\ell}(\psi\phi) \simeq \mathrm{Kl}_{\check{G}'}^{\text{ét},\ell}(\psi\phi') \times^{\check{G}'} \check{G}$ as $\check{G}$-local systems. It follows from Theorem 4.4.4 and Proposition 5.1.3 that for every representation $V \in \mathbf{Rep}(\check{G})$, regarded as a representation of $\check{G}'$, and every $a \in |X_k|$, we have

$$\mathrm{Tr}(\mathrm{Frob}_a | \mathrm{Kl}_{\check{G},V,\bar{a}}^{\text{ét},\ell}) = \mathrm{Tr}(\mathrm{Frob}_a | \mathrm{Kl}_{\check{G}',V,\bar{a}}^{\text{ét},\ell}).$$

Note that if Σ is the group of pinned automorphisms of $\check{G}$, the closed embedding $\check{G}^{\Sigma} \to \check{G}$ induces a surjective homomorphism of K-rings $K(\mathbf{Rep}(\check{G})) \otimes \overline{\mathbb{Q}}_\ell \to K(\mathbf{Rep}(\check{G}^{\Sigma})) \otimes \overline{\mathbb{Q}}_\ell$. Then the homomorphism $K(\mathbf{Rep}(\check{G})) \otimes \overline{\mathbb{Q}}_\ell \to K(\mathbf{Rep}(G_{\mathrm{geo}})) \otimes \overline{\mathbb{Q}}_\ell$ is also surjective. It follows that if we replace V by any representation W of G_{geo} ($\subset \check{G}' \subset \check{G}$), the above equality holds. This implies that Frobenius conjugacy classes of $\mathrm{Kl}_{\check{G}}^{\text{ét},\ell}$ and of $\mathrm{Kl}_{\check{G}'}^{\text{ét},\ell}$ have the same image in $G_{\mathrm{geo}} /\!\!/ G_{\mathrm{geo}}$. Therefore, for a faithful representation W of G_{geo}, two representations $\mathrm{Kl}_{\check{G}}^{\text{ét},\ell}, \mathrm{Kl}_{\check{G}'}^{\text{ét},\ell} : \pi_1(X_k, \overline{x}) \to G_{\mathrm{geo}}(\overline{\mathbb{Q}}_\ell)$ are conjugated in $\mathrm{GL}(W)$ by an element g, which induces an automorphism of G_{geo}. It fixes every Frobenius conjugacy class and therefore fixes $G_{\mathrm{geo}} /\!\!/ G_{\mathrm{geo}}$. Then g must be inner. That is these two representations are conjugate in G_{geo} and the assertion follows. $\square$

5.2 Bessel F-isocrystals for classical groups

5.2.1. The Kloosterman sheaf and the Bessel F-isocrystal for ($G = \mathrm{GL}_n$, $\check{G} = \mathrm{GL}_n$) have been extensively studied. As usual, let I_n denote the identity matrix and E_{ij} denote the $n \times n$-matrix with the (i, j)-entry 1 and all other entries 0. We choose the standard Borel B of the upper triangular matrices and the standard torus T of the diagonal matrices. We choose a coordinate x of $\mathbb{A}^1$. Then there is a canonical isomorphism

$$\mathbb{G}_a^n \simeq I(1)/I(2), \quad (a_1, \dots, a_n) \mapsto I_n + \sum_{i=1}^{n-1} a_i E_{i,i+1} + a_n x^{-1} E_{n,1}.$$

We choose $\phi : \mathbb{G}_a^n \to \mathbb{G}_a$ to be the addition map. Under the isomorphism (4.3.3.1) and (4.3.4.1), ϕ corresponds to $\check{\xi} = N + Edx$ (4.4.1.1) with

$$N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{5.2.1.1}$$

On the other hand, by 4.1.7 and [47, §3], we have $\mathrm{Kl}^{\text{ét}}_{\mathrm{SL}_n,\mathrm{Std}} \simeq \mathrm{Kl}^{\text{ét}}_{\mathrm{GL}_n,\mathrm{Std}} \simeq \mathrm{Kl}^{\text{ét}}_n$. Therefore, the Kloosterman connection is isomorphic to the classical Bessel connection (1.1.1, 1.1.4)

$$\mathrm{Kl}^{\mathrm{dR}}_{\mathrm{SL}_n,\mathrm{Std}}(\lambda\phi) \simeq \mathrm{Be}_n, \qquad \mathrm{Kl}^{\mathrm{rig}}_{\mathrm{SL}_n,\mathrm{Std}}(\phi) \simeq \mathrm{Be}^{\dagger}_n.$$

Recall that the connection Be_n corresponds to the Bessel differential equation (1.1.1.1).

5.2.2. Consider

$$G = \mathrm{SO}_{2n+1}, \quad \check{G} = \mathrm{Sp}_{2n} = \{A \in \mathrm{SL}_{2n} \mid AJA^T = J\},$$

where J is the anti-diagonal matrix with $J_{ij} = (-1)^i \delta_{i,2n+1-j}$. Then matrices (N, E) as in (5.2.1.1) are in $\check{\mathfrak{g}}$ and $\mathrm{Be}_{\check{G}}(\check{\xi})$ is given by the same formula as GL_{2n} case. Then we deduce an isomorphism of overconvergent F-isocrystals $\mathrm{Be}^{\dagger}_{\mathrm{Sp}_{2n},\mathrm{Std}}(\check{\xi}) \simeq \mathrm{Be}^{\dagger}_{2n}$ by (5.1.3).

5.2.3. Consider

$$G = \mathrm{SO}_{2n} \;\; \check{G} = \mathrm{SO}_{2n} = \{A \in \mathrm{SL}_{2n}, AJA^T = J\},$$

where J is the anti-diagonal matrix with $J_{ij} = (-1)^{\max\{i,j\}}\delta_{i,2n+1-j}$. There exists a canonical isomorphism

$$\begin{aligned} &\mathbb{G}_a^{n+1} \simeq I(1)/I(2),\\ &(a_1, \ldots, a_{n+1}) \mapsto I_{2n} + \sum_{i=1}^{n-1}(E_{i,i+1} + E_{2n-i,2n-i+1})\\ &\quad +(E_{n-1,n+1} + E_{n,n+2}) + x^{-1}(E_{1,2n-1} + E_{2,2n}). \end{aligned}$$

Then we take $\phi : \mathbb{G}_a^{n+1} \to \mathbb{G}_a$ to be the addition map. When $n \geq 3$, under the isomorphism (4.3.3.1) and (4.3.4.1), $\lambda\phi$ corresponds to $\check{\xi} = N + \lambda^{2n-2}Ex$ (4.4.1.1) with

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots\dots\dots & 0 \\ \vdots & \ddots & \ddots & & \dots & \vdots \\ \vdots & \ddots & 0 & 1 & 1 & 0 \dots 0 \\ & & \dots & 0 & 0 & 1 & \vdots \\ & & & & 0 & 1 & \vdots \\ 0 & & & & & 0 & \ddots & \vdots \\ 0 & 0 & & & & & \ddots & 1 \\ 0 & 0 & 0 & \dots\dots & & & & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots\dots\dots & 0 \\ \vdots & \ddots & \ddots & & \dots & \vdots \\ \vdots & \ddots & 0 & 0 & 0 & 0 \dots 0 \\ & & \dots & 0 & 0 & 0 & \vdots \\ & & & & 0 & 0 & \vdots \\ 0 & & & & & 0 & \ddots & \vdots \\ 1 & 0 & & & & & \ddots & 0 \\ 0 & 1 & 0 & \dots\dots & & & & 0 \end{pmatrix}. \tag{5.2.3.1}$$

The corresponding Bessel connection is written as

$$\mathrm{Be}_{\mathrm{SO}_{2n},\mathrm{Std}}(\check{\xi}) = d + (N + \lambda^{2n-2} E x)\frac{dx}{x}.$$

If $e_1, \dots, e_{2n}$ denote a basis for the above connection matrix, the restriction of the above connection to the subbundle generated by $e_n - e_{n+1}$ is trivial. The other horizontal subbundle, generated by $e_n + e_{n+1}$ and other basis vectors, is isomorphic to the Bessel connection $\mathrm{Be}_{\mathrm{SO}_{2n-1},\mathrm{Std}}(\check{\xi})$ discussed below (5.2.6.4).

5.2.4. In [58], T. Lam and N. Templier identified the diagram (4.1.6.2) with the Laudau–Ginzburg model for quadrics [66] and used it to calculate the associated Kloosterman $\mathscr{D}$-modules. We briefly recall this construction following [66, § 3]. Let $Q_{2n-2} = G/P$ be the $(2n-2)$-dimensional quadric and let $(p_0 : \dots : p_{n-1} : p'_{n-1} : p_n : \dots : p_{2n-2})$ be the Plücker coordinates of Q_{2n-2} satisfying

$$p_{n-1}p'_{n-1} - p_{n-2}p_n + \dots + (-1)^{n-1}p_0p_{2n-2} = 0. \tag{5.2.4.1}$$

Consider the open subscheme

$$Q^\circ_{2n-2} = Q_{2n-2} - D,$$

with the complement $D = D_0 + D_1 + \dots + D_{n-1} + D'_{n-1}$, where D_i is defined by

$$\begin{aligned} &D_0 := \{p_0 = 0\}, \\ &D_\ell := \left\{\sum_{k=0}^{\ell}(-1)^k p_{\ell-k}p_{2n-2-\ell+k} = 0\right\} \text{ for } 1 \le \ell \le n-3 \\ &D_{n-2} := \{p_{2n-2} = 0\}, \quad D_{n-1} := \{p_{n-1} = 0\}, \quad D'_{n-1} := \left\{p'_{n-1} = 0\right\} \end{aligned}$$

The divisor D is anti-canonical in Q_{2n-2}. For simplicity, we set

$$\delta_\ell = \sum_{k=0}^{\ell}(-1)^k p_{\ell-k}p_{2n-2-\ell+k}, \quad \text{for } 0 \le \ell \le n-3.$$

If x denotes a coordinate of $\mathbb{G}_m$, we define a regular function $W : Q^\circ_{2n-2} \times \mathbb{G}_m \to \mathbb{A}^1$ to be

$$W(p_i : p'_{n-1}; x) = \frac{p_1}{p_0} + \sum_{\ell=1}^{n-3}\frac{p_{\ell+1}p_{2n-2-\ell}}{\delta_\ell} + \frac{p_n}{p_{n-1}} + \frac{p_n}{p'_{n-1}} + x\frac{p_1}{p_{2n-2}}. \tag{5.2.4.2}$$

The Kloosterman overconvergent F-isocrystal and connection are calculated by

$$\begin{aligned}\mathrm{Kl}^{\mathrm{rig}}_{\mathrm{SO}_{2n},\mathrm{Std}}(\phi) &\simeq \mathrm{pr}_{2,!}(W^*(\mathscr{A}_\psi))[2(n-1)](n-1),\\ \mathrm{Kl}^{\mathrm{dR}}_{\mathrm{SO}_{2n},\mathrm{Std}}(\lambda\phi) &\simeq \mathrm{pr}_{2,!}(W^*(\mathsf{E}_\lambda))[2(n-1)].\end{aligned}$$

We deduce that the Frobenius trace $\mathrm{Kl}_{\mathrm{SO}_{2n},\mathrm{Std}}$ of $\mathrm{Kl}^{\mathrm{rig}}_{\mathrm{SO}_{2n},\mathrm{Std}}(\phi)$ is defined for $a \in \mathbb{F}_q^\times$ by

$$\begin{aligned}\mathrm{Kl}_{\mathrm{SO}_{2n},\mathrm{Std}}(a) = \frac{1}{q^{n-1}} \sum_{(p_i,p'_{n-1})\in Q^\circ_{2n-2}(\mathbb{F}_q)} \psi\Bigg(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}\Bigg(\frac{p_1}{p_0} + \sum_{\ell=1}^{n-3}\frac{p_{\ell+1}p_{2n-2-\ell}}{\delta_\ell}\\ + \frac{p_n}{p_{n-1}} + \frac{p_n}{p'_{n-1}} + a\frac{p_1}{p_{2n-2}}\Bigg)\Bigg).\end{aligned} \tag{5.2.4.3}$$

Proposition 5.2.5 (i) *When $n = 2$, we have* $\mathrm{Kl}_{\mathrm{SO}_4,\mathrm{Std}}(a) = \mathrm{Kl}(2; a)^2$.

(ii) *When $n \ge 3$, we can simplify above sum as*

$$\begin{aligned}&\mathrm{Kl}_{\mathrm{SO}_{2n},\mathrm{Std}}(a)\\ &= \frac{1}{q^{n-1}}\Bigg(\sum_{x_i\in\mathbb{F}_q^\times}\psi\Bigg(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x_1+\cdots+x_{2n-2}+a\frac{x_1+x_2}{x_1\ldots x_{2n-2}})\Bigg)+(q-1)q^{n-2}\Bigg).\end{aligned} \tag{5.2.5.1}$$

Proof Assertion (i) is easy to prove and is left to readers. It also follows from (4.1.7.2).

(ii) The equality follows from subdividing the sum (5.2.4.3) into the following parts:

(a) Case $p_n, p_{n+1}, \dots, p_{2n-3} \neq 0$: we replace p_i, p'_{n-1} by $x_i, y_i \in \mathbb{F}_q^\times$ as follows:

$$p_k = \begin{cases} 1 & \text{if } k = 0, \\ x_1 \dots x_{k-1}(x_k + y_k) & \text{if } 1 \leq k \leq n-2, \\ x_1 \dots x_{n-2} x_{n-1} & \text{if } k = n-1, \\ x_1 \dots x_{n-2} x_{n-1} y_{n-1} & \text{if } k = n, \\ x_1 \dots x_{n-2} x_{n-1} y_{n-1} y_{n-2} \dots y_{2n-1-k}, & \text{otherwise}, \end{cases}$$
$$p'_{n-1} = x_1 \dots x_{n-2} y_{n-1}.$$

Then the sum (5.2.4.3) becomes the toric exponential sum in (5.2.5.1).

(b) Case $p_n = 0$ and $p_{2n-2-\ell} \neq 0$ for some $\ell \in \{1, \dots, n-3\}$: we assume ℓ is maximal. By dividing p'_{n-1}, we consider the affine coordinates $p_0, \dots, p_{2n-2}$ and we replace p_{n-1} by the equation (5.2.4.1). Since $p_n, \dots, p_{2n-2-\ell-1} = 0$, $p_{\ell+1}$ can be taken in $\mathbb{F}_q$ regardless of the condition $\delta_\ell \neq 0$. Then we have $\sum_{p_{\ell+1} \in \mathbb{F}_q} \psi(\frac{p_{\ell+1} p_{2n-2-\ell}}{\delta_\ell}) = 0$ and that the sum (5.2.4.3) equals to zero in this case.

(c) Case $p_n = p_{n+1} = \dots = p_{2n-3} = 0$: it is easy to show that the sum (5.2.4.3) equals to $\frac{q-1}{q}$, which is the constant part of (5.2.5.1). $\square$

5.2.6. Consider

$$G = \mathrm{Sp}_{2n}, \quad \check{G} = \mathrm{SO}_{2n+1} = \{A \in \mathrm{SL}_{2n+1} \mid AJA^T = J\},$$

where J is the anti-diagonal matrix with $J_{ij} = (-1)^i \delta_{i,2n+2-j}$. There exists a canonical isomorphism

$$\mathbb{G}_a^{n+1} \simeq I(1)/I(2),$$
$$(a_1, \dots, a_{n+1}) \mapsto I_{2n} + \sum_{i=1}^{n-1} (E_{i,i+1} + E_{2n-i,2n-i+1}) + E_{n-1,n} + x^{-1} E_{2n,1}.$$

Then we take $\phi : \mathbb{G}_a^{n+1} \to \mathbb{G}_a$ to be the addition map. Under isomorphisms (4.3.3.1) and (4.3.4.1), $\lambda\phi$ corresponds to $\check{\xi} = N + \lambda^{2n} E x$ (4.4.1.1) with N as in (5.2.1.1), which belongs to $\check{\mathfrak{g}}$, and

$$E = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & & \\ 2 & 0 & \dots & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \end{pmatrix} \in \check{\mathfrak{g}}. \qquad (5.2.6.1)$$

Then we can write the Bessel connection as

$$\mathrm{Kl}^{\mathrm{dR}}_{\mathrm{SO}_{2n+1},\mathrm{Std}}(\lambda\phi) \simeq \mathrm{Be}_{\mathrm{SO}_{2n+1},\mathrm{Std}}(\check{\xi}) = d + (N + \lambda^{2n} E x)\frac{dx}{x}.$$

After taking a gauge transformation by the matrix

$$\begin{pmatrix} 1 & 0 & \dots\dots & 0 \\ 0 & 1 & \dots\dots & 0 \\ \dots & \dots\dots & \dots\dots & \dots \\ 0 & \dots\dots & \dots\dots & \dots \\ 2\lambda^{2n}x & 0 & \dots\ 0 & 1 \end{pmatrix}.$$

we obtain the scalar differential equation associated to $\mathrm{Be}_{\mathrm{SO}_{2n+1},\mathrm{Std}}(\check{\xi})$:

$$(x\frac{d}{dx})^{2n+1} - \lambda^{2n}x(4x\frac{d}{dx} + 2) = 0. \qquad (5.2.6.2)$$

When $n \geq 2$, we can rewrite $\check{\xi}$ as

$$\check{\xi} = \begin{pmatrix} 0 & 1 & 0 & \dots & & 0 & 0 \\ 0 & \ddots & \ddots & & & & 0 \\ \vdots & & 0 & \sqrt{2} & 0 & & \vdots \\ \vdots & & 0 & 0 & \sqrt{2} & & \vdots \\ \vdots & & 0 & 0 & 0 & \ddots & 0 \\ 0 & \dots & & & & & 1 \\ 0 & 0 & \dots & & & & 0 \end{pmatrix} + \lambda^{2n} \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ 0 & \dots & & & 0 \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} x, \qquad (5.2.6.3)$$

where $\sqrt{2}$ is a square root of 2 in $\overline{K}$ and appears in positions $(n, n+1)$ and $(n+1, n+2)$. Via the natural inclusion $\mathfrak{so}_{2n+1} \to \mathfrak{so}_{2n+2}$ the above element $\check{\xi} \in (\mathfrak{so}_{2n+1})_{\mathrm{aff}}(1)$ corresponds to $\check{\xi} \in (\mathfrak{so}_{2n+2})_{\mathrm{aff}}(1)$ defined in (5.2.3.1). The standard $(2n+2)$-dimensional representation of $\mathfrak{so}_{2n+2}$ decomposes as a direct sum of the trivial representation and the standard $(2n+1)$-dimensional representation of $\mathfrak{so}_{2n+1}$ as representations of $\mathfrak{so}_{2n+1}$. Then we obtain decompositions of Bessel connections and Bessel F-isocrystals by Proposition 5.1.3

$$\begin{aligned}\mathrm{Be}_{\mathrm{SO}_{2n+2},\mathrm{Std}}(\check{\xi}) &\simeq \mathrm{Be}_{\mathrm{SO}_{2n+1},\mathrm{Std}}(\check{\xi}) \oplus (\mathscr{O}_{\mathbb{G}_{m,K}}, d),\\ \mathrm{Be}^{\dagger}_{\mathrm{SO}_{2n+2},\mathrm{Std}}(\check{\xi}) &\simeq \mathrm{Be}^{\dagger}_{\mathrm{SO}_{2n+1},\mathrm{Std}}(\check{\xi}) \oplus (\mathcal{O}_{\mathbb{G}_m}, d).\end{aligned} \tag{5.2.6.4}$$

In the remaining of this subsection, we omit $\check{\xi}$ from the notation.

Remark 5.2.7 The fact that matrix E in (5.2.6.1) takes value 2 in its non-zero entries is delicate. On the one hand, it comes from the calculation of invariant polynomials. On the other hand, it ensures the existence of a Frobenius structure on the differential equation (5.2.6.2) with parameter $\lambda = -\pi$. For instance, for every prime number p, the convergence domain of the unique solution of (5.2.6.2) ($\lambda = -\pi$) at 0 :

$$F(x) = \sum_{r\geq 0} \frac{(2r-1)!!}{(r!)^{2n+1}} (2\pi^{2n})^r x^r,$$

is the open unit disc of radius 1. In particular, $F(x)$ belongs to $K\{x\}$ (2.3.1.1) and it justifies (2.3.1.2).

5.2.8. The equation (5.2.6.2) is closely related to the hypergeometric differential equations and hypergeometric F-isocrystals studied by Katz [51] and Miyatani [61]. We briefly recall them in the following.

Let $n > m$ be two non-negative integers, $\pi \in K$ associated to ψ (2.1.1) and $\underline{\beta} = (\beta_1, \ldots, \beta_m)$ a sequence of elements of $\frac{1}{q-1}\mathbb{Z} - \mathbb{Z}$. We denote by $\mathrm{Hyp}_{\pi}(n, \underline{\beta})$ the p-adic hypergeometric differential operator on $\mathbb{G}_m$

$$\mathrm{Hyp}_{\pi}(n, \underline{\beta}) = (x\frac{d}{dx})^n - (-1)^{n+mp}\pi^{n-m} x \prod_{i=1}^{m} (x\frac{d}{dx} - \beta_j). \tag{5.2.8.1}$$

We denote by $\mathscr{H}yp_{\pi}(n, \underline{\beta})$ the $\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}}^1,\mathbb{Q}}(\{0,\infty\})$-module

$$\mathscr{H}yp_{\pi}(n, \underline{\beta}) = \mathscr{D}^{\dagger}_{\widehat{\mathbb{P}}^1,\mathbb{Q}}(\{0,\infty\}) / (\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}}^1,\mathbb{Q}}(\{0,\infty\})\mathrm{Hyp}_{\pi}(n, \underline{\beta})).$$

In [61], Miyatani showed that $\mathscr{H}yp_{\pi}(n, \underline{\beta})$ underlies to a pure overconvergent F-isocrystal on $\mathbb{G}_{m,k}$ of rank n and weight $n+m-1$. Moreover, the the overconvergent isocrystal $\mathscr{H}yp_{\pi}(n, \underline{\beta})$ is irreducible and admits a unique (up to a scalar) Frobenius structure.

5.2.9. Normalised Hypergeometric sum. The overconvergent F-isocrystal $\mathscr{H}yp_{\pi}(n, \underline{\beta})$ has a maximal unipotent monodromy at zero. If N_0 denotes this monodromy action, then the space $(\mathscr{H}yp_\pi(n,\underline{\beta})|_0)^{N_0}$ of N_0-invariants is one-dimensional on which Frob_k acts as $\alpha = (-1)^m \prod_{j=1}^m G(\psi^{-1},\rho_j^{-1})$ (cf. [52] 2.6.1 for a proof in the ℓ-adic case), where $G(\psi^{-1},\rho_j^{-1})$ denotes the Gauss sum associated to ψ^{-1} and ρ_j^{-1}, and ρ_i is defined for $\xi \in k^\times$ and $\widetilde{\xi}$ the Teichmüller lifting of ξ, by $\rho_i(\xi) = \widetilde{\xi}^{(q-1)\beta_i}$.

Any lifting F_0 in the decomposition group D_0 at 0 of the Frobenius automorphism has eigenvalues set $\{\alpha, q\alpha, \dots, q^{2n}\alpha\}$ (cf. [50] 7.0.7). After twisting a geometrically constant rank one overconvergent F-isocrystal, we denote by $\widetilde{\mathscr{H}yp}_\psi(n,\underline{\rho})$ the normalised hypergeometric F-isocrystal whose the Frobenius eigenvalues at 0 is $\{q^{-(n-1)/2},\dots,q^{(n-1)/2}\}$. Then by [61, proposition 4.1.6], its Frobenius trace function, called the *normalised hypergeometric sum* $\widetilde{\mathrm{H}}_\psi(n,\underline{\rho})$ is defined for $a \in \mathbb{F}_q^\times$ by

$$\begin{aligned}
&\widetilde{\mathrm{H}}_\psi(n,\underline{\rho})(a)\\
&\quad= \frac{1}{(-\sqrt{q})^{n-1}\prod_{j=1}^m G(\psi^{-1},\rho_j^{-1})}\\
&\qquad\times\left(\sum \psi\left(\mathrm{Tr}_{k'/\mathbb{F}_p}\Big(\sum_{i=1}^n x_i - \sum_{j=1}^m y_j\Big)\right)\cdot \prod_{j=1}^m \rho_j^{-1}(\mathrm{Nm}_{k'/k}(y_j))\right),
\end{aligned}$$

where the sum take over $(x_1,\dots,x_n,y_1,\dots,y_m) \in (k'^\times)^{m+n}$ satisfying $\prod_{i=1}^n x_i = a\prod_{j=1}^m y_j$.

When $m=0$, we have $\widetilde{\mathscr{H}yp}_\psi(n,\emptyset) = \mathrm{Be}_n^\dagger$ (1.1.4).

Proposition 5.2.10 (i) *When $p>2$, there exists an isomorphism of overconvergent F-isocrystals* (5.2.9)

$$\mathrm{Be}^\dagger_{\mathrm{SO}_{2n+1},\mathrm{Std}} \simeq [x \mapsto 4x]^+ \widetilde{\mathscr{H}yp}_\psi(2n+1,\rho), \tag{5.2.10.1}$$

where ρ denotes the quadratic character of $k^\times$.

(ii) *When $p=2$, there exists an isomorphism of overconvergent F-isocrystals* $\mathrm{Be}^\dagger_{\mathrm{SO}_{2n+1},\mathrm{Std}} \simeq \mathrm{Be}^\dagger_{2n+1}$. *In particular, the* SL_{2n+1}*-overconvergent F-isocrystals* $\mathrm{Be}^\dagger_{\mathrm{SL}_{2n+1}}$ *is the push-out of* $\mathrm{Be}^\dagger_{\mathrm{SO}_{2n+1}}$ *along* $\mathrm{SO}_{2n+1} \to \mathrm{SL}_{2n+1}$.

Proof (i) If we rescale x by $x \mapsto \frac{1}{4}x$, the differential equation (5.2.6.2) turns to the hypergeometric differential equation $\mathrm{Hyp}_\psi(2n+1;\rho)$ associated to ρ (5.2.8.1). Frobenius structures on two sides of (5.2.10.1) are of weight zero and have Frobenius eigenvalues $\{q^{-n},\dots,q^{-1},0,q,\dots,q^n\}$ at 0 (4.4.5, 5.2.9). Then these two Frobenius structures coincide by 5.2.8 and the isomorphism (5.2.10.1) follows.

(ii) We will prove the assertion in Appendix A. ◻

Corollary 5.2.11 (i) *The Frobenius trace function* $\mathrm{Kl}_{\mathrm{SO}_{2n+1},\mathrm{Std}}$ *of* $\mathrm{Be}^{\dagger}_{\mathrm{SO}_{2n+1},\mathrm{Std}}$ *is equal to*

$$\begin{aligned}\mathrm{Kl}_{\mathrm{SO}_{2n+1},\mathrm{Std}}(a) &= \sum_{x,y\in k^{\times},xy=a} \mathrm{Kl}_{\mathrm{SO}_3,\mathrm{Std}}(x)\,\mathrm{Kl}(2n-2;y)\\ &= \begin{cases}\mathrm{Kl}(2n+1;a), & p=2,\\ \widetilde{\mathrm{H}}_{\psi}(2n+1;\rho)(4a), & p>2.\end{cases}\end{aligned}$$

(ii) *We have an identity of exponential sums (5.2.5.1)*

$$\mathrm{Kl}_{\mathrm{SO}_{2n+2},\mathrm{Std}}(a)-1=\mathrm{Kl}_{\mathrm{SO}_{2n+1},\mathrm{Std}}(a).$$

Proof (i) Let $\star$ denote the *(multiplicative) convolution* of arithmetic $\mathscr{D}$-modules [61, 2.1.1]. By the convolution interpretation of hypergeometric overconvergent F-isocrystals [61, Main theorem (ii) and 3.3.3], we deduce an isomorphism of overconvergent F-isocrystals

$$\mathrm{Be}^{\dagger}_{\mathrm{SO}_{2n+1},\mathrm{Std}} \simeq \mathrm{Be}^{\dagger}_{\mathrm{SO}_3,\mathrm{Std}} \star \mathrm{Be}^{\dagger}_{2n-2}.$$

Then the first equality follows. The second one follows from 5.2.10(i–ii).

(ii) It follows from Proposition 5.2.5 and (5.2.6.4). ◻

In particular, by (4.1.7.1) and Corollary 5.2.11(i), we obtain (1.2.9.1). Using the trivial functoriality 4.1.7 and the exceptional isomorphism for groups of low ranks (4.1.7.1, 4.1.7.2), one can similarly obtain other identities between exponential sums, whose sheaf-theoretic incarnations were obtained by Katz [52].

5.3 Frobenius slopes of Bessel F-isocrystals

5.3.1 We first recall the definition of the *Newton polygon* of a conjugacy class in $\check{G}(\overline{K})$. Let $\mathbb{X}_{\bullet}(\check{T})^{+}$ be the set of dominant coweights of $\check{G}$ and $\mathbb{X}_{\bullet}(\check{T})^{+}_{\mathbb{R}}$ the positive Weyl chamber, equipped with the following partial order $\leq$: $\mu\leq\lambda$ if $\lambda-\mu$ can be written as a linear combination of positive coroots of $\check{G}$ with coefficients in $\mathbb{R}_{+}$. We identify $(\mathbb{X}_{\bullet}(\check{T})\otimes_{\mathbb{Z}}\mathbb{R})/W$ and $\mathbb{X}_{\bullet}(\check{T})^{+}_{\mathbb{R}}$. Recall that ρ denotes the half sum of positive roots of G

$$\rho=\frac{1}{2}\sum_{\alpha\in\Phi^{+}}\alpha\in\mathbb{X}^{\bullet}(T)=\mathbb{X}_{\bullet}(\check{T}).$$

Let $v:\overline{K}\to\mathbb{Q}\cup\{\infty\}$ be the p-adic order, normalised by $v(q)=1$. It induces a homomorphism of groups $v:\check{T}(\overline{K})\to\mathbb{X}_{\bullet}(\check{T})\otimes_{\mathbb{Z}}\mathbb{R}$. By identifying

$\check{T}(\overline{K})/W$ and the set of semisimple conjugacy classes $\mathrm{Conj}^{\mathrm{ss}}(\check{G}(\overline{K}))$ in $\check{G}(\overline{K})$, we deduce a homomorphism:

$$\mathrm{NP}:\mathrm{Conj}^{\mathrm{ss}}(\check{G}(\overline{K}))=\check{T}(\overline{K})/W\to(\mathbb{X}_{\bullet}(\check{T})\otimes_{\mathbb{Z}}\mathbb{R})/W=\mathbb{X}_{\bullet}(\check{T})^{+}_{\mathbb{R}}.$$

When $\check{G}=\mathrm{GL}_n$, NP is equivalent to the classical p-adic Newton polygon (cf. [57] § 1).

Theorem 5.3.2 *Let $x\in|\mathbb{A}^1_k|$ be a closed point and $\varphi_x\in\check{G}(\overline{K})$ the Frobenius automorphism of $(\mathrm{Be}^{\dagger}_{\check{G}},\varphi)$ at x (4.4.3). Let v be the p-adic order normalised by $v(q^{\deg(x)})=1$ and NP defined as above.*

(i) *Except for finitely many closed points of $|\mathbb{A}^1_k|$, we have $\mathrm{NP}(\varphi_x)=\rho$.*

(ii) *Suppose that $\check{G}$ is of type A_n, B_n, C_n, D_n or G_2, then we have $\mathrm{NP}(\varphi_x)=\rho$ for every $x\in|\mathbb{A}^1_k|$.*

Proof (i) In [57, 2.1], V. Lafforgue shows that the Newton polygon of the Hecke eigenvalue of a cuspidal function is $\leq\rho$. In particular, we deduce that $\mathrm{NP}(\varphi_x)\leq\rho$ for all $x\in|\mathbb{G}_{m,k}|$. By 4.4.5, we have $\mathrm{NP}(\varphi_0)=\mathrm{NP}(\rho(q))=\rho$. That is the Newton polygon achieves the upper bound ρ at 0. We take a finite set of tensor generators $\{V_1,\ldots,V_n\}$ of $\mathbf{Rep}(\check{G})$. Then the assertion follows by applying Grothendieck–Katz' theorem (cf. [30] 1.6) to log convergent F-isocrystals $\mathrm{Be}^{\dagger}_{\check{G},V_i}$.

(ii) (a) The case where $\check{G}$ is of type A_n, C_n. By functoriality (5.1.3), we reduce to study the Frobenius slope of Bessel F-isocrystal $\mathrm{Be}^{\dagger}_n$ of rank n (1.1.4). In this case, the assertion follows from the work of Dwork, Sperber and Wan [40,72,76].

(b) The case where $\check{G}$ is of type B_n, D_n, G_2. By functoriality (5.1.3), we reduce to show that the Frobenius slope set of $\mathrm{Be}^{\dagger}_{\mathrm{SO}_{2n+1},\mathrm{Std}}$ at each closed point is equal to $\{-n,-n+1,\ldots,n\}$. If $p=2$, it follows from 5.2.10(ii) and the case (a). If $p>2$, in view of 5.2.9 and 5.2.10(i), it follows from the following lemma. □

Lemma 5.3.3 *The Frobenius slope set of $\mathscr{H}yp_{\psi}(2n+1;\rho)$ (5.2.8) at each closed point is equal to*

$$\left\{\frac{1}{2},\frac{3}{2},\ldots,2n+\frac{1}{2}\right\}.$$

Proof We deduce this fact from Wan's results on Frobenius slope of certain toric exponential sums [76,77].

For $a \in \mathbb{F}_q^\times$ and a divisor d of $p-1$, consider the following Laurent polynomial in $\mathbb{F}_q[x_1^\pm, \dots, x_{2n+1}^\pm]$

$$f_d(x_1, \dots, x_{2n+1}) = x_1 + \cdots + x_{2n} - x_{2n+1}^d + \frac{a x_{2n+1}^d}{x_1 x_2 \dots x_{2n}}.$$

For $m \geq 1$, we denote by $S_m(f_d)$ the exponential sum associated a Laurent polynomial:

$$S_m(f_d) = \sum_{x_i \in \mathbb{F}_{q^m}^\times} \psi\Big(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_p} f_d(x_1, \dots, x_{2n+1})\Big).$$

Then we have an identity

$$\begin{aligned} S_m(f_2) = S_m(f_1) + \sum_{\substack{x_1 \dots x_{2n+1} = ay \\ x_i \in \mathbb{F}_{q^m}^\times}} \psi\Big(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_p}(x_1 + \cdots + x_{2n+1} - y)\Big) \\ \cdot \rho^{-1}\big(\mathrm{Nm}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(y)\big), \end{aligned} \tag{5.3.3.1}$$

where the last term is the Frobenius trace of $\mathscr{H}yp_\psi(2n+1; \rho)$.

The L-function associated to these exponential sums is a rational function:

$$\mathrm{L}(f_d, T) = \exp\Big(\sum_{m \geq 1} S_m(f_d) \frac{T^m}{m}\Big).$$

We denote by $\Delta(f_d)$ the convex closure in $\mathbb{R}^{2n+1}$ generated by the origin and lattices defined by exponents appeared in f_d: $\{(0, \dots, 0), (1, \dots, 0), \dots, (0, \dots, 1, 0), (0, \dots, 0, d), (-1, \dots, -1, d)\}$. The polyhedron $\Delta(f_d)$ is $(2n+1)$-dimensional and has volume $\frac{d}{2n!}$. The Laurent polynomials f_d is non-degenerate (cf. [77] Def. 1.1). After Adolphson–Sperber [9], the L-function $\mathrm{L}(f_d, T)$ is a polynomial of degree $d(2n+1)$.

We denote by $\mathrm{NP}(f_d)$ the (Frobenius) Newton polygon associated to L-functions $\mathrm{L}(f_d, T)$ (cf. [77] 1.1) and by $\mathrm{HP}(f_d)$ the Hodge polygon defined in term of the polyhedron $\Delta(f_d)$ (cf. [77] 1.2). The (multi-)set of slopes of $\mathrm{HP}(f_d)$ is

$$\Big\{0, \frac{1}{d}, \frac{2}{d}, \dots, 2n + \frac{d-1}{d}\Big\}. \tag{5.3.3.2}$$

The Newton polygon lies above the Hodge polygon [9]. A Laurent polynomial is called *ordinary* if these two polygons coincide. Let δ be a co-dimension 1 face of Δ which does not contain the origin and f_d^δ the restriction of f_d to δ which is also non-degenerate. The Laurent polynomial f_d^δ is diagonal in the sense of [77, § 2]. If $V_1, \dots, V_{2n+1}$ denote the vertex of δ written as column vectors, the set $S(\delta)$ of solutions of

$$(V_1, \dots, V_{2n+1}) \begin{pmatrix} r_1 \\ \vdots \\ r_{2n+1} \end{pmatrix} \equiv 0 \pmod 1, \quad r_i \text{ rational}, 0 \leq r_i < 1,$$

forms an abelian group of order d (cf. [77] 2.1). Since d is a divisor of $p-1$, we deduce that for each δ, f_d^δ is ordinary by [77, Cor. 2.6]. By Wan's criterion for the ordinariness [76] (cf. [77] Thm. 3.1), f_d is ordinary.

In view of (5.3.3.1) and the slope sets of $\mathrm{HP}(f_1), \mathrm{HP}(f_2)$ (5.3.3.2), the assertion follows. □

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Appendix A. A 2-adic proof of Carlitz's identity and its generalization

As mentioned in introduction, Carlitz [24] proved the following identity between Kloosterman sums:

$$\mathrm{Kl}(3; a) = \mathrm{Kl}(2; a)^2 - 1, \qquad \forall\, a \in \mathbb{F}_{2^s}^\times.$$

In this appendix, we reprove and generalize this identity by establishing an isomorphism between two Bessel F-isocrystals $\mathrm{Be}_{2n+1}^\dagger$ and $\mathrm{Be}_{\mathrm{SO}_{2n+1},\mathrm{Std}}^\dagger$. The following is a restatement of Proposition 5.2.10(ii).

Proposition A1 *There exists an isomorphism between following two overconvergent F-isocrystals on $\mathbb{G}_{m,\mathbb{F}_2}$:*

$$\mathrm{Be}_{2n+1}^\dagger : \left(x\frac{d}{dx}\right)^{2n+1} + 2^{2n+1}x = 0, \tag{A11}$$
$$\mathrm{Be}_{\mathrm{SO}_{2n+1},\mathrm{Std}}^\dagger : \left(x\frac{d}{dx}\right)^{2n+1} - 2^{2n+1}x\left(2x\frac{d}{dx} + 1\right) = 0.$$

Our strategy is first to show that their maximal slope quotient convergent F-isocrystals are isomorphic. Then we conclude the proposition by a dual version of a minimal slope conjecture (proposed by Kedlaya [56] and recently proved by Tsuzuki [74]) that we briefly recall in the following.

Let X be a smooth k-scheme and $\mathscr{M}^\dagger$ an overconvergent F-isocrystal on X/K. We denote the associated convergent F-isocrystal on X/K by $\mathscr{M}$. When the (Frobenius) Newton polygons of $\mathscr{M}$ are constant on X, $\mathscr{M}$ admits a *(dual) slope filtration*, that is a decreasing filtration

$$\mathscr{M}=\mathscr{M}^0 \supsetneq \mathscr{M}^1 \supsetneq \mathscr{M}^2 \supsetneq \cdots \supsetneq \mathscr{M}^{r-1} \supsetneq \mathscr{M}^r=0 \tag{A12}$$

of convergent F-isocrystals on X/K such that

- $\mathscr{M}^i/\mathscr{M}^{i+1}$ is isoclinic of slope s^i and
- $s^0>s^1>\cdots>s^{r-1}$.

Theorem A2 (Tsuzuki, [74] theorem 1.3) *Let X be a smooth connected curve over k. Let $\mathscr{M}^\dagger$, $\mathscr{N}^\dagger$ be two irreducible overconvergent F-isocrystals such that the corresponding convergent F-isocrystals $\mathscr{M}$, $\mathscr{N}$ admit the slope filtrations $\{\mathscr{M}^i\}$, $\{\mathscr{N}^i\}$ respectively. Suppose there exists an isomorphism $h:\mathscr{N}/\mathscr{N}^1 \xrightarrow{\sim} \mathscr{M}/\mathscr{M}^1$ of convergent F-isocrystals between the maximal slope quotients. Then there exists a unique isomorphism $g^\dagger:\mathscr{N}^\dagger \xrightarrow{\sim} \mathscr{M}^\dagger$ of overconvergent F-isocrystals, which is compatible with h as morphisms of convergent F-isocrystals.*

A3 Following Dwork's strategy [39, § 1-3], we study the maximal slope quotients of $\mathrm{Be}_{2n+1}^\dagger$ and of $\mathrm{Be}_{\mathrm{SO}_{2n+1},\mathrm{Std}}^\dagger$ in terms of their unique solutions at 0.

In the following, we assume $k=\mathbb{F}_p$. We first recall Dwork's congruences and show a refinement of his result in the 2-adic case. Consider for every $i\geq 0$, a map $B^{(i)}(-):\mathbb{Z}_{\geq 0}\to K^\times$ and the following congruence relation for $0\leq a<p$ and $n,m,s\in\mathbb{Z}_{\geq 0}$:

(a) $B^{(i)}(0)$ is a p-adic unit for all $i\geq 0$,

(b) $\dfrac{B^{(i)}(a+np)}{B^{(i+1)}(n)}\in R$ for all $i\geq 0$,

(c) $\dfrac{B^{(i)}(a+np+mp^{s+1})}{B^{(i+1)}(n+mp^s)}\equiv\dfrac{B^{(i)}(a+np)}{B^{(i+1)}(n)} \mod p^{s+1}$ for all $i\geq 0$.

(c') When $p=2$, $\dfrac{B^{(i)}(a+n2+m2^{s+1})}{B^{(i+1)}(n+m2^s)}\equiv u(i,s,m)\dfrac{B^{(i)}(a+n2)}{B^{(i+1)}(n)}$ mod 2^{s+1} for all $i\geq 0$, where $u(i,s,m)=1$ if $s\neq 1$ and $u(i,1,m)=1$ or -1 depending on i and m.

If conditions (a–c) (or (a,b,c')) are satisfied, then $B^{(i)}(n) \in R$ for all $i, n \geq 0$. We set

$$F^{(i)}(x) = \sum_{j=0}^{\infty} B^{(i)}(j)x^j \in K[\![x]\!],$$

$$F^{(i)}_{m,s}(x) = \sum_{j=mp^s}^{(m+1)p^s-1} B^{(i)}(j)x^j \in K[x], \quad s \geq 0.$$

We write $F^{(i)}_{0,s}$ by $F^{(i)}_s$ for simplicity.

Theorem A4 (i) [39, theorem 2] *If conditions (a–c) are satisfied, then*

$$F^{(0)}(x)F^{(1)}_{m,s}(x^p) \equiv F^{(0)}_{m,s+1}(x)F^{(1)}(x^p) \qquad \text{mod } p^{s+1}B^{(s+1)}(m)[\![x]\!]. \tag{A41}$$

(i') If conditions (a,b,c') are satisfied (in particular $p = 2$), then

$$F^{(0)}(x)F^{(1)}_{m,s}(x^2) \equiv F^{(0)}_{m,s+1}(x)F^{(1)}(x^2) \qquad \text{mod } 2^s B^{(s+1)}(m)[\![x]\!]. \tag{A42}$$

(ii) [39, theorem 3] *Under the assumption of (i) or (i') and suppose moreover that*

(d) $B^{(i)}(0) = 1$ *for* $i \geq 0$.
(e) $B^{(i+r)} = B^{(i)}$ *for all* $i \geq 0$ *and some fixed* $r \geq 1$.

Let U be the open subscheme of $\mathbb{A}^1_k$ defined by $F^{(i)}_1(x) \neq 0$, for $i = 0, 1, \ldots, r-1$. Then the limit

$$f(x) = \lim_{s\to\infty} F^{(0)}_{s+1}(x)/F^{(1)}_s(x^p) \tag{A43}$$

defines a global function on the formal open subscheme $\mathfrak{U}$ of $\widehat{\mathbb{A}}^1_R$ associated to U, which takes p-adic unit value at each rigid point of $\mathfrak{U}^{\mathrm{rig}}$.

We prove assertion (i') in the end (A11). We briefly explain Dwork's result (ii) in the language of formal schemes. The assumption implies that $F^{(i)}_s \neq 0$ on U (cf. [39] 3.4). For $s \geq 1$, the congruences (A41) or (A42) imply that

$$F^{(0)}_{s+1}(x)/F^{(1)}_s(x^p) = F^{(0)}_s(x)/F^{(1)}_{s-1}(x^p) \quad \in \Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{U}}/p^{s-1}\mathscr{O}_{\mathfrak{U}}).$$

This allows us to use (A43) to define a global function f of $\mathscr{O}_{\mathfrak{U}}$.

A5 Let $F(x) = \sum_{j\geq 0} B(j)x^j$ be a formal power series in $R[\![x]\!]$. We say F *satisfies Dwork's congruences* if by setting $B^{(i)}(j) = B(j)$ for every $i \geq 0$, conditions of Theorem A4(ii) are satisfied.

We take such a function F and then we obtain a function $f \in \Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{U}})$ coinciding with $F(x)/F(x^p)$ in $K\{x\}$ (2.3.1.1) (i.e. the open unit disc). Moreover, by [39, lemma 3.4(ii)], there exists a function $\eta \in \Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{U}})$ coinciding with $F'(x)/F(x)$ in $K\{x\}$ defined by

$$\eta(x) \equiv F'_{s+1}(x)/F_{s+1}(x) \mod p^s.$$

The functions $f(x)$ and $\eta(x)$ satisfy a differential equation:

$$\frac{f'(x)}{f(x)} + px^{p-1}\eta(x^p) = \eta(x).$$

Note that $f(0) = F(0)/F(0) = 1$. Then we deduce that the following corollary.

Corollary A6 *The connection $d-\eta$ on the trivial bundle $\mathscr{O}_{\mathfrak{U}^{\mathrm{rig}}}$ and the function f form a unit-root convergent F-isocrystal $\mathscr{E}_F$ on U/K, whose Frobenius eigenvalue at 0 is 1.*

A7. Let $\mathscr{M}^\dagger$ be an overconvergent F-isocrystal on $\mathbb{G}_{m,k}$ over K of rank r whose underlying bundle is trivial and the connection is defined by a differential equation:

$$P(\delta) = \delta^r + p_r\delta^{r-1} + \cdots + p_1 = 0,$$

where $\delta = x\frac{d}{dx}$, $p_i \in \Gamma(\widehat{\mathbb{A}}^1_R, \mathscr{O}_{\widehat{\mathbb{A}}^1_R})[\frac{1}{p}]$. We assume moreover that $\mathscr{M}^\dagger$ is *unipotent at* 0 *with a maximal unipotent local monodromy*. Then $\mathscr{M}^\dagger$ extends to a log convergent F-isocrystal $\mathscr{M}^{\log}$ on $(\mathbb{A}^1, 0)$ and its Frobenius slopes at 0 are

$$s^0 > s^1 = s^0 - 1 > \cdots > s^{r-1} = s^0 - (r-1).$$

Note that $\mathscr{M}^\dagger$ is indecomposable in $F\text{-}\mathrm{Isoc}^\dagger(\mathbb{G}_{m,k}/K)$ and so is $\mathscr{M}$ in $F\text{-}\mathrm{Isoc}(\mathbb{G}_{m,k}/K)$. Then by Drinfeld–Kedlaya's theorem on the generic Frobenius slopes [38], we deduce property (i):

(i) The generic Frobenius slopes (mult-)set is $\{s^0, \ldots, s^{r-1}\}$ with $s^i = s^0 - (i-1)$.

(ii) In view of (2.3.1.2), the differential equation $D = 0$ admits a unique solution at 0:

$$F(x) = \sum_{n\geq 0} A(n)x^n \quad \in K\{x\}, \quad \text{with } A(0) = 1.$$

Proposition A8 *Suppose the function $F(x)$ satisfies Dwork's congruences (A5) and let $\mathscr{E}_F$ be the associated unit-root convergent F-isocrystal on $U \subset \mathbb{A}^1_k$. Then*

(i) *There exists an epimorphism of log convergent isocrystals $\mathscr{M}^{\log} \to \mathscr{E}_F$ on $(U, 0)$.*

(ii) *As convergent isocrystals, $\mathscr{E}_F$ coincides with the maximal slope quotient $\mathscr{M}^{\log}/\mathscr{M}^{\log,1}$ of $\mathscr{M}^{\log}$* (A12).

Proof (i) We set $A = \Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{U}})[\frac{1}{p}]$. We claim that there exists a decomposition of differential operators:

$$P(\delta) = Q(\delta)(\delta - x\eta), \quad Q(\delta) = \delta^{r-1} + q_{r-1}\delta^{r-2} + \cdots + q_1, \quad q_i \in A. \tag{A81}$$

Indeed, by the Euclidean algorithm [54, 5.5.2], there exists $r \in A$ such that $P = Q(\delta - x\eta) + r$. By evaluating the above identity at F (in the ring $K\{x\}$ containing A), we obtain

$$P(\delta)(F) = 0 = Q(\delta)(\delta - x\eta)(F) + rF = rF.$$

Then we deduce $r = 0$ and (A81) follows.

Let $e_1, \ldots, e_r$ be a basis of $\mathscr{M}$ such that $\nabla_\delta(e_i) = e_{i+1}, 1 \leq i \leq r-1$ and $\nabla_\delta(e_r) = -(p_r e_r + \cdots + p_1 e_1)$. We consider a free $\mathscr{O}_{\mathfrak{U}^{\mathrm{rig}}}$-module with a log connection $\mathscr{N}$ with a basis $f_1, \ldots, f_{r-1}$ and the connection defined by $\nabla_\delta(f_i) = f_{i+1}$, $\nabla_\delta(f_{r-1}) = -(q_{r-1}f_{r-1} + \cdots + q_1 f_1)$. By (A81), the morphism $f_1 \mapsto e_2 - x\eta e_1$ induces a horizontal monomorphism $\mathscr{N} \to \mathscr{M}^{\log}$ whose cokernel is isomorphic to $\mathscr{E}_F$.

(ii) Note that $\mathrm{Pic}(\mathfrak{U}^{\mathrm{rig}}) \simeq \mathrm{Pic}(U)$ [75, 3.7.4] is trivial. Then the rank one convergent isocrystal $\mathscr{M}^{\log}/\mathscr{M}^{\log,1}$ can be represented as a connection $d - \lambda$ on the trivial bundle $\mathscr{O}_{\mathfrak{U}^{\mathrm{rig}}}$.

Since $\mathscr{M}^{\log}$ has a maximal unipotent at 0, the rank one quotient of the restriction $\mathscr{M}^{\log}|_0$ of $\mathscr{M}^{\log}$ at the open unit disc around 0 is unique (2.3.1). In particular, $d - \lambda$ kills the unique solution F of $P(\delta) = 0$. By analytic continuation, we have $\lambda = \eta$ and the assertion follows. □

Remark A9 The unique solution $F(x)$ belongs to the ring $K[\![x]\!]_0 = R[\![x]\!] \otimes_R K$ of bounded functions on open unit disc, which is a subring of $K\{x\}$. Assertion (ii) can be viewed as an example of Dwork–Chiarellotto–Tsuzuki conjecture on the comparison between the log-growth filtration (of solutions) and Frobenius slope filtration [28]. This conjecture was recently proved by Ohkubo [65].

Proof of Proposition A1 We set $k = \mathbb{F}_2$ and apply the above discussions to overconvergent F-isocrystals $\mathscr{M}^\dagger = \mathrm{Be}^\dagger_{2n+1}$ and $\mathscr{N}^\dagger = \mathrm{Be}^\dagger_{\mathrm{SO}_{2n+1},\mathrm{Std}}$ on

$\mathbb{G}_{m,\mathbb{F}_2}/K$ (A11). Their unique solutions at 0 are:

$$F(x)=\sum_{r\geq 0}\frac{(-2)^{(2n+1)r}}{(r!)^{2n+1}}x^r,\qquad G(x)=\sum_{r\geq 0}\frac{2^{(2n+1)r}(2r-1)!!}{(r!)^{2n+1}}x^r.$$

In the following lemma, we show that F and G satisfy Dwork's congruences and that the associated maximal slope quotients $\mathscr{E}_F$ and $\mathscr{E}_G$ (A8) are isomorphic. Then Proposition A1 follows from theorem A2 and the following lemma. □

Lemma A10 (i) *The functions $F(x)$ and $G(x)$ satisfy Dwork's congruences (A5) and define unit-root convergent F-isocrystals $\mathscr{E}_F$ and $\mathscr{E}_G$ on $\mathbb{A}^1_k$ respectively.*

(ii) *The function $F(x)/G(x)$ extends to a global function of $\widehat{\mathbb{A}}^1_R$ and induces an isomorphism $\mathscr{E}_G \xrightarrow{\sim} \mathscr{E}_F$.*

Proof (i) Conditions (a,b,d,e) are easy to verified. The coefficients of $F(x)$ (resp. $G(x)$) satisfy condition (c') (resp. (c)), that is

$$\begin{aligned}&\frac{(-2)^{(2n+1)(a+\ell 2+m2^{s+1})}/((a+\ell 2+m2^{s+1})!)^{2n+1}}{(-2)^{(2n+1)(\ell+m2^s)}/((\ell+m2^s)!)^{2n+1}}\\ &\quad\equiv u(s,m)\frac{(-2)^{(2n+1)(a+\ell 2)}/((a+\ell 2)!)^{2n+1}}{(-2)^{(2n+1)\ell}/(\ell!)^{2n+1}}\mod 2^{s+1},\end{aligned}$$

where $u(1,m)=(-1)^m$ and $u(s,m)=1$ if $s\neq 1$, and

$$\begin{aligned}&\frac{(2(a+\ell 2+m2^{s+1})-1)!!2^{(2n+1)(a+\ell 2+m2^{s+1})}/((a+\ell 2+m2^{s+1})!)^{2n+1}}{(2(\ell+m2^s)-1)!!2^{(2n+1)(\ell+m2^s)}/((\ell+m2^s)!)^{2n+1}}\\ &\equiv\frac{(2(a+\ell 2)-1)!!2^{(2n+1)(a+\ell 2)}/((a+\ell 2)!)^{2n+1}}{(2\ell-1)!!2^{(2n+1)\ell}/(\ell!)^{2n+1}}\mod 2^{s+1}.\end{aligned}$$

Since $F_1(x)\equiv G_1(x)\equiv 1 \mod 2$, the F-isocrystals $\mathscr{E}_F$, $\mathscr{E}_G$ are defined over $\mathbb{A}^1_k$.

(ii) We set $B^{(0)}(r)=\frac{(-2)^{(2n+1)r}}{(r!)^{2n+1}}$ and $B^{(1)}(r)=\frac{2^{(2n+1)r}(2r-1)!!}{(r!)^{2n+1}}$ and $B^{(i+2)}=B^{(i)}$. Then these sequences satisfy conditions (a,b,c',d,e). For condition (c'), the constants $u(i,1,m)$ are given by

$$u(0,1,m)=1,\quad u(1,1,m)=(-1)^m,\quad u(i+2,1,m)=u(i,1,m).$$

Since $F_1(x)\equiv G_1(x)\equiv 1 \mod 2$, $F(x)/G(x^2)$ extends to a global function of $\mathscr{O}_{\widehat{\mathbb{A}}^1_R}$ by Theorem A4 and so is $F(x)/G(x)$. Then the assertion follows. □

A11 *Proof of Theorem* A4(i'). We prove assertion (i') by modifying the argument of [39, theorem 2]. Note that condition (c') implies the following congruence relation:

$$\frac{B^{(i)}(a+n2+m2^{s+1})}{B^{(i+1)}(n+m2^s)} \equiv \frac{B^{(i)}(a+n2)}{B^{(i+1)}(n)} \mod 2^s. \tag{A111}$$

When $n<0$, we set $B^{(i)}(n)=0$. We set $A=B^{(0)}$, $B=B^{(1)}$ and for $a\in\{0,1\}$, $j,N\in\mathbb{Z}$, we set

$$U_a(j,N)=A(a+2(N-j))B(j)-B(N-j)A(a+2j),$$
$$H_a(m,s,N)=\sum_{j=m2^s}^{(m+1)2^s-1}U_a(j,N).$$

Then the assertion is equivalent to

$$H_a(m,s,N)\equiv 0 \mod 2^sB^{(s+1)}(m), \quad \text{for } s\geq 0, m\geq 0, N\geq 0. \tag{A112}$$

By condition (b), we have $A(a+2m)/B(m)\in R$ and hence

$$U_a(m,N)\equiv 0 \mod B(m).$$

Then equation (A112) for $s=0$ follows from the fact that $H_a(m,0,N)=U_a(m,N)$.

We now prove by induction on s. We write the induction hypothesis

$$\alpha_s: H_a(m,u,N)\equiv 0 \mod 2^uB^{(u+1)}(m), \quad \text{for } u\in[0,s), m,N\geq 0.$$

We may assume α_s for fixed $s\geq 1$. The main step is to show for $0\leq t\leq s$ that

$$\beta_{t,s}: v(s,t,m)H_a(m,s,N+m2^s)$$
$$\equiv \sum_{j=0}^{2^{s-t}-1} B^{(t+1)}(j+m2^{s-t})H_a(j,t,N)/B^{(t+1)}(j) \mod 2^sB^{(s+1)}(m),$$

where $v(s,t,m)=1$ or -1 depending on s,t,m.

We list some elementary facts (cf. [39, 2.5–2.7])

$$\sum_{m=0}^{T}H_a(m,s,N)=0 \quad \text{if } (T+1)2^s>N \tag{A113}$$

$$H_a(m,s,N)=H_a(2m,s-1,N)+H_a(1+2m,s-1,N)\quad \text{if } s\geq 1 \tag{A114}$$

$$B^{(t)}\left(i+m2^s\right)\equiv 0 \mod B^{(s+t)}(m)\quad \text{if } 0\leq i\leq 2^s-1,\ s,t\geq 0. \tag{A115}$$

We first prove $\beta_{0,s}$. We have

$$H_a\left(m,s,N+m2^s\right)=\sum_{j=0}^{2^s-1}U_a\left(j+m2^s,N+m2^s\right), \tag{A116}$$

$$\begin{aligned}&U_a\left(j+m2^s,N+m2^s\right)\\&\quad=A(a+2(N-j))B(j+m2^s)-B(N-j)A\left(a+2j+m2^{s+1}\right).\end{aligned}$$

By (A111), we have

$$A\left(a+2j+m2^{s+1}\right)=A(a+2j)B(j+m2^s)/B(j)+X_j2^sB(j+m2^s),$$

where $X_j\in R$. Then the right hand side of (A116) is

$$B\left(j+m2^s\right)\Big(U_a(j,N)/B(j)-2^sX_jB(N-j)\Big).$$

Since $U_a(j,N)=H_a(j,0,N)$, we obtain

$$\begin{aligned}H_a\left(m,s,N+m2^s\right)=&\sum_{j=0}^{2^s-1}B\left(j+m2^s\right)H_a(j,0,N)/B(j)\\&-2^s\sum_{j=0}^{2^s-1}X_jB\left(j+m2^s\right)B(N-j).\end{aligned}$$

Since $X_jB(N-j)\in R$, it follows from (A115) ($B=B^{(1)}$) that the second sum is congruent to zero modulo $2^sB^{(s+1)}(m)$. This proves $\beta_{0,s}$ with $v(s,0,m)=1$.

With s fixed, $s\geq 1$, t fixed, $0\leq t\leq s-1$, we show that $\beta_{t,s}$ together with α_s imply $\beta_{t+1,s}$. To do this we put $j=\mu+2i$ in the right side of $\beta_{t,s}$ and write it in the form

$$\sum_{\mu=0}^{1}\sum_{i=0}^{2^{s-t-1}}B^{(t+1)}\left(\mu+2i+m2^{s-i}\right)H_a(\mu+2i,t,N)/B^{(t+1)}(\mu+2i).$$

By condition (c'), we have,

$$\begin{aligned}&B^{(t+1)}\left(\mu+2i+m2^{s-t}\right)\\&\quad=u(t+1,s-t-1,m)\left(B^{(t+1)}(\mu+2i)B^{(t+2)}\left(i+m2^{s-t-1}\right)/B^{(t+2)}(i)\right)\\&\qquad+X_{i,\mu}2^{s-t}B^{(t+2)}\left(i+m2^{s-t-1}\right),\end{aligned}$$

where $X_{i,\mu}\in R$. Thus the general term in the above double sum is

$$u(t+1,s-t-1,m)\left(B^{(t+2)}(i+m2^{s-t-1})H_a(\mu+2i,t,N)/B^{(t+2)}(i)\right)+Y_{i,\mu},$$

where the error term:

$$Y_{i,\mu}=X_{i,\mu}2^{s-t}B^{(t+2)}\left(i+m2^{s-t-1}\right)H_a(\mu+2i,t,N)/B^{(t+1)}(\mu+2i).$$

For this error term, since $t<s$, we can apply α_s to conclude that

$$Y_{i,\mu}\equiv 0 \mod B^{(t+2)}(i+m2^{s-t-1})2^s.$$

Then we can use (A115) to conclude that

$$Y_{i,\mu}\equiv 0 \mod 2^s B^{(s+1)}(m).$$

After modulo $2^s B^{(s+1)}(m)$, the right side of $\beta_{t,s}$ is equal to

$$\begin{aligned}&u(t+1,s-t-1,m)\sum_{\mu=0}^{1}\sum_{i=0}^{2^{s-t-1}-1}B^{(t+2)}\left(i+m2^{s-t-1}\right)\\&\quad\times H_a(\mu+2i,t,N)/B^{(t+2)}(i).\end{aligned}$$

By reversing the order of summation and using (A114), the above sum is the same as

$$\begin{aligned}&u(t+1,s-t-1,m)\sum_{i=0}^{2^{s-t-1}-1}B^{(t+2)}\left(i+m2^{s-t-1}\right)\\&\quad\times H_a(i,t+1,N)/B^{(t+2)}(i),\end{aligned}$$

which proves $\beta_{t+1,s}$. In particular, we obtain $\beta_{s,s}$, which states

$$\begin{aligned}&v(s,s,m)H_a(m,s,N+m2^s)\\&\quad\equiv B^{(s+1)}(m)H_a(0,s,N)/B^{(s+1)}(0) \mod 2^s B^{(s+1)}(m).\end{aligned}\tag{A117}$$

We now consider the statement (with s fixed before)

$$\gamma_N : H_a(0, s, N) \equiv 0 \mod 2^s.$$

We know that γ_N is true for $N < 0$. Let N' (if it exists) be the minimal value of N for which $\gamma_{N'}$ fails. For $m \geq 1$, since $B^{(s+1)}(0)$ is a unit, we have by (A117)

$$\begin{aligned} H_a(m, s, N') &\equiv v(s, s, m) B^{(s+1)}(m) H_a(0, s, N' - m2^s)/B^{(s+1)}(0) \\ &\equiv 0 \mod 2^s. \end{aligned}$$

Applying this to (A113), we obtain that

$$H_a(0, s, N') \equiv 0 \mod 2^s.$$

Thus γ_N is valid for all N and Eq. (A117) implies α_{s+1}. This proves assertion (i′). □

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