

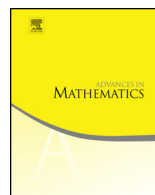


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Rigidity of the Bonnet-Myers inequality for graphs with respect to Ollivier Ricci curvature



D. Cushing^a, S. Kamtue^a, J. Koolen^b, S. Liu^b, F. Münch^c,
N. Peyerimhoff^{a,*}

^a *Department of Mathematical Sciences, Durham University, United Kingdom of Great Britain and Northern Ireland*

^b *School of Mathematical Sciences, University of Science and Technology of China, and Wu Wen-Tsun Key Laboratory of Mathematics of CAS, Hefei, China*

^c *Institute of Mathematics, Universität Potsdam, Germany*

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ABSTRACT

We introduce the notion of Bonnet-Myers and Lichnerowicz sharpness in the Ollivier Ricci curvature sense. Our main result is a classification of all self-centered Bonnet-Myers sharp graphs (hypercubes, cocktail party graphs, even-dimensional demi-cubes, Johnson graphs $J(2n, n)$, the Gosset graph and suitable Cartesian products). We also present a purely combinatorial reformulation of this result. We show that Bonnet-Myers sharpness implies Lichnerowicz sharpness and classify all distance-regular Lichnerowicz sharp graphs under the additional condition $\theta_1 = b_1 - 1$. We also relate Bonnet-Myers sharpness to an upper bound of Bakry-Émery ∞ -curvature, which motivates a general conjecture about Bakry-Émery ∞ -curvature.

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* Corresponding author.

E-mail address: norbert.peyerimhoff@durham.ac.uk (N. Peyerimhoff).

1. Introduction and statement of results

A fundamental question in geometry is in which way local properties determine the global structure of a space. A famous result of this kind is the Bonnet-Myers Theorem [17] for complete n -dimensional Riemannian manifolds M with $K = \inf \text{Ric}_M(v) > 0$, where the infimum is taken over all unit tangent vectors v of M . Under this condition, M is compact and its diameter satisfies

$$\text{diam}(M) \leq \pi \sqrt{\frac{n-1}{K}}. \quad (1.1)$$

Moreover, Cheng's Rigidity Theorem [4] states that this diameter estimate is sharp if and only if M is the n -dimensional round sphere. Note that inequality (1.1) can be reformulated as an upper bound on the infimum of the Ricci curvature in terms of the diameter, and this reformulation is the viewpoint we will assume in this paper.

In the discrete setting of graphs there are several analogs of Ricci curvature notions providing Bonnet-Myers type theorems (see, e.g., [9,10,13,15,19]). In view of Cheng's rigidity result, it is natural to ask for which graphs the Bonnet-Myers estimate is sharp. We call such graphs *Bonnet-Myers sharp* graphs. For example, in the case of Bakry-Émery ∞ -curvature, Bonnet-Myers sharp graphs have been fully characterised and are only the hypercubes (see [14]).

The motivation of this paper is to study Bonnet-Myers sharpness with respect to another curvature notion, namely, *Ollivier Ricci curvature*. (In fact, we will consider a modification of Ollivier's definition introduced in [15].) Henceforth, all graphs $G = (V, E)$ with vertex set V and edge set E will be simple (loopless without multiple edges) and edges can be identified with 2-element subsets of V . In this paper, we will only formulate and derive our results for regular graphs, that is, all vertices have the same valency, even though similar questions can be posed for non-regular graphs.

Ollivier Ricci curvature $\kappa(x, y)$ is a notion based on optimal transport and is defined on pairs of different vertices $x, y \in V$. The precise definition requires a longer introduction and is given in Subsection 2.2. Generally, $\kappa(x, y)$ is positive if the average distance between corresponding neighbours of x and y is smaller than $d(x, y)$.

Let us now state the discrete Bonnet-Myers Theorem for Ollivier Ricci curvature and introduce the associated notion of Bonnet-Myers sharpness for this curvature notion:

Theorem 1.1 (*Discrete Bonnet-Myers, see [19,15]*). *Let $G = (V, E)$ be a connected D -regular graph and $\inf_{x \sim y} \kappa(x, y) > 0$. Then G has finite diameter $L = \text{diam}(G) < \infty$ and*

$$\inf_{x \sim y} \kappa(x, y) \leq \frac{2}{L}. \quad (1.2)$$

*We say that such a graph G is **(D,L)-Bonnet-Myers sharp** (with respect to Ollivier Ricci curvature) if (1.2) holds with equality.*

To provide the readers with some understanding of the curvature condition $\kappa(x, y) = \frac{2}{L}$, we give the following sufficient combinatorial condition:

Proposition 1.2. *Let $G = (V, E)$ be a D -regular graph of diameter L and $e = \{x, y\} \in E$. Assume that e is contained in precisely $\frac{2D}{L} - 2$ triangles and there is a perfect matching between the neighbours of x and the neighbours of y which are not involved in these triangles. Then we have*

$$\kappa(x, y) = \frac{2}{L}.$$

This result follows directly from the later Proposition 2.7 by choosing $m = \frac{2D}{L} - 2$.

In the Riemannian Geometry setting, another fundamental result between local and global properties of a closed n -dimensional Riemannian manifold M is Lichnerowicz' Theorem [12, p. 135] which states that, under the condition $K = \inf \text{Ric}_M(v) > 0$, the smallest positive Laplace-Beltrami eigenvalue $\lambda_1(M)$ satisfies

$$\frac{n}{n - 1}K \leq \lambda_1.$$

The associated rigidity result is Obata's Theorem [18], which states that this eigenvalue estimate is sharp if and only if M is the n -dimensional round sphere.

There is a discrete analogue of Lichnerowicz' Theorem for Ollivier Ricci Curvature and the normalized Laplacian $\Delta_G = D^{-1}A_G - \text{Id}$ of an arbitrary graph $G = (V, E)$, where A_G denotes the adjacency matrix of G and D is here a diagonal matrix whose entries are the valencies d_x of the vertices $x \in V$. In the case of a D -regular graph this matrix is just given by $D \cdot \text{Id}$. Alternatively, Δ_G can be written as an operator acting on functions f defined on the vertices V via

$$\Delta_G f(x) = \frac{1}{d_x} \sum_{y \sim x} (f(y) - f(x)). \tag{1.3}$$

The discrete Lichnerowicz' Theorem gives naturally rise to the definition of Lichnerowicz sharpness:

Theorem 1.3 *(Discrete Lichnerowicz, see [15]). Let $G = (V, E)$ be a finite connected D -regular graph. Then we have for the smallest positive solution λ_1 of $\Delta_G f + \lambda f = 0$ with Δ_G given by (1.3),*

$$\inf_{x \sim y} \kappa(x, y) \leq \lambda_1. \tag{1.4}$$

We say that the graph G is **Lichnerowicz sharp** (with respect to Ollivier Ricci curvature) if (1.4) holds with equality.

Let us now state the main results of this paper. Some of our results require the additional condition of self-centeredness. Note that a graph $G = (V, E)$ is called *self-centered* if, for every vertex $x \in V$, there exists a vertex $\bar{x} \in V$ such that $d(x, \bar{x}) = \text{diam}(G)$.

- (a) *Cartesian products:* $G_1 \times G_2 \times \cdots \times G_k$ is Bonnet-Myers sharp if and only if all factors G_i are Bonnet-Myers sharp and satisfy

$$\frac{D_1}{L_1} = \frac{D_2}{L_2} = \cdots = \frac{D_k}{L_k}, \quad (1.5)$$

where D_i and L_i are the vertex degrees and the diameters of the graphs G_i , respectively (see Theorem 3.2).

- (b) Every Bonnet-Myers sharp graph is Lichnerowicz sharp (see Theorem 1.5).
 (c) *Classification of self-centered Bonnet-Myers sharp graphs:* Self-centered Bonnet-Myers sharp graphs are precisely the following ones: Hypercubes, cocktail party graphs, the Johnson graphs $J(2n, n)$, even-dimensional demi-cubes, the Gosset graph and Cartesian products of them satisfying (1.5) (see Theorem 1.6).
 (d) *Classification of distance-regular Lichnerowicz sharp graphs with $\theta_1 = b_1 - 1$:* These graphs are precisely the following ones: cocktail party graphs, Hamming graphs, all Johnson graphs, all demi-cubes, the Schläfli and the Gosset graph (see Theorem 6.4).
 (e) Self-centered (D, L) -Bonnet-Myers sharp graphs are Bakry-Émery ∞ -curvature sharp in all vertices with normalized ∞ -curvature value $\frac{1}{D} + \frac{1}{L}$ (see Theorem 1.8).

We provide more detailed information about these results in the next subsection. In particular, result (c) above is based on another result which can be reformulated in purely combinatorial terms. This combinatorial reformulation is derived in Subsection 1.2.

1.1. Our results on Bonnet-Myers sharp graphs

It is useful to know that the vertex degree D and the diameter L of a Bonnet-Myers sharp graph cannot be arbitrary:

Theorem 1.4. *Any (D, L) -Bonnet-Myers sharp graph satisfies $L \leq D$. Moreover L must divide $2D$.*

This theorem is proved in Section 5.2.

We have the following relation between the two sharpness properties (Bonnet-Myers and Lichnerowicz sharpness), proved in Section 6:

Theorem 1.5. *Every Bonnet-Myers sharp graph is Lichnerowicz sharp.*

Note, however, that the family of all Lichnerowicz sharp graphs is much larger than the family of all Bonnet-Myers sharp graphs (see Remark 3.3). In Section 6, we classify a special subclass of Lichnerowicz sharp graphs, namely, all distance regular Lichnerowicz sharp graphs with the additional spectral condition $\theta_1 = b_1 - 1$.

Our main theorem is the following classification result of self-centered Bonnet-Myers sharp graphs:

Theorem 1.6. *Self-centered Bonnet-Myers sharp graphs are precisely the following graphs:*

1. hypercubes Q^n , $n \geq 1$;
2. cocktail party graphs $CP(n)$, $n \geq 3$;
3. the Johnson graphs $J(2n, n)$, $n \geq 3$;
4. even-dimensional demi-cubes $Q_{(2)}^{2n}$, $n \geq 3$;
5. the Gosset graph;

and Cartesian products of 1.-5. satisfying the condition (1.5).

Let us explain the proof of Theorem 1.6: In Section 4 we show that all graphs in the list of Theorem 1.6 are self-centered Bonnet-Myers sharp. In Sections 7 and 8 we show that every self-centered Bonnet-Myers sharp graph is strongly spherical (this is the statement of Theorem 8.8; the notion of strongly spherical graphs is introduced in Definition 1.11). Then we use the classification result [11] (see Theorem 1.14 below), which states that strongly spherical graphs are precisely the Cartesian products in the list provided in Theorem 1.6, thus completing the proof. \square

Finally, we like to present connections between Bonnet-Myers sharpness with respect to Ollivier Ricci curvature and normalized Bakry-Émery ∞ -curvature. Generally, relations between different curvature notions of discrete spaces are a very interesting and challenging topic. Normalized Bakry-Émery ∞ -curvature is defined on the vertices of a graph $G = (V, E)$ and denoted by $\mathcal{K}_{G,x}^n(\infty)$, $x \in V$. For further details and the precise definition, we refer the readers to Subsection 10.1. We have the following results:

Theorem 1.7. *Every (D, L) -Bonnet-Myers sharp graph $G = (V, E)$ satisfies*

$$\inf_{x \in V} \mathcal{K}_{G,x}^n(\infty) \leq \frac{1}{D} + \frac{1}{L}. \tag{1.6}$$

This result follows immediately from Theorem 10.5 later in the paper.

As a consequence of Theorem 1.6 we also obtain

Theorem 1.8. *Let $G = (V, E)$ be a self-centered (D, L) -Bonnet-Myers sharp graph. Then G is Bakry-Émery ∞ -curvature sharp at all vertices $x \in V$ and we have*

$$\mathcal{K}_{G,x}^n(\infty) = \frac{1}{D} + \frac{1}{L}.$$

The proof of this theorem is given in Subsection 10.2.

These last two results provide a better understanding why Bonnet-Myers sharpness with respect to Bakry-Émery ∞ -curvature is much more restrictive: the only graphs with this property are the hypercubes (see [14]). This result classifies all D -regular graphs of diameter L satisfying the condition

$$\inf_{x \in V} \mathcal{K}_{G,x}^n(\infty) = \frac{2}{L},$$

which is much stronger than the equality condition of (1.6), that is

$$\inf_{x \in V} \mathcal{K}_{G,x}^n(\infty) = \frac{1}{D} + \frac{1}{L}, \tag{1.7}$$

since $L \leq D$, by Theorem 1.4. By considering the weaker condition (1.7) we encounter other graphs like the ones given in the list of Theorem 1.6. It is an open question whether these graphs and suitable Cartesian products of them are the only self-centered examples of D -regular graphs of diameter L satisfying (1.7).

Moreover, we are not aware of any D -regular graph G of diameter L which violates the condition (1.6). This led us to formulate the following conjecture:

Conjecture 1.9. Let $G = (V, E)$ be a finite connected D -regular graph of diameter L . Then we have

$$\inf_{x \in V} \mathcal{K}_{G,x}^n(\infty) \leq \frac{1}{D} + \frac{1}{L}.$$

Let us compare this conjecture with the combinatorial Bonnet-Myers Theorem for normalized Bakry-Émery ∞ -curvature proved in [13], namely,

$$\inf_{x \in V} \mathcal{K}_{G,x}^n(\infty) \leq \frac{2}{L}. \tag{1.8}$$

Our conjecture can be viewed as a strengthening of (1.8) in the case $L \leq D$.

1.2. A combinatorial result

For readers interested in combinatorial graph theory, this subsection provides a combinatorial reformulation of Theorem 8.8. The combinatorial version is given in Theorem 1.13 below and we think that it is of own interest.

We start with a combinatorial property closely related to Ollivier Ricci curvature (see Proposition 2.7 below for the connection).

Definition 1.10. Let $G = (V, E)$ be a regular graph. We say G satisfies $\Lambda(m)$ at an edge $e = \{x, y\} \in E$ if the following holds:

- (i) e is contained in at least m triangles and
- (ii) there is a perfect matching between the neighbours of x and the neighbours of y not involved in these triangles.

We say that G satisfies $\Lambda(m)$ if it satisfies $\Lambda(m)$ at each edge.

Next, we introduce the notions of antipodal and strongly spherical graphs, which are based on *intervals* $[x, y]$ which are the set of all vertices lying on geodesics from x to y .

Definition 1.11. A graph $G = (V, E)$ is called *antipodal* if, for every vertex $x \in V$, there exists another vertex $\bar{x} \in V$ satisfying $[x, \bar{x}] = V$.

A graph $G = (V, E)$ is called *strongly spherical* if G is antipodal and the induced subgraphs of all its intervals are antipodal. That is, the following two properties are necessary and sufficient conditions for strongly spherical graphs $G = (V, E)$:

- For every $x \in V$ there exists $\bar{x} \in V$ such that $V = [x, \bar{x}]$.
- For every pair $x, y \in V$, $x \neq y$, and every $z \in [x, y]$ there exists $\bar{z} \in [x, y]$ such that $[x, y] = [z, \bar{z}]$.

To reformulate Theorem 8.8 (self-centered Bonnet-Myers sharp graphs are strongly spherical) in purely combinatorial terms we need the following result which allows us to replace the curvature condition by a combinatorial condition.

Proposition 1.12. *Let G be a D -regular finite connected graph of diameter L . The following are equivalent:*

- G is self-centered Bonnet-Myers sharp.
- G is self-centered and satisfies $\Lambda\left(\frac{2D}{L} - 2\right)$.

Moreover, if any of these equivalent properties holds, then every edge of G lies in precisely $\frac{2D}{L} - 2$ triangles.

This proposition is a consequence of Corollary 5.10 and Proposition 2.7 later in the paper.

Using Proposition 1.12, Theorem 8.8 translates then into the following equivalent combinatorial result:

Theorem 1.13. *Let G be a D -regular finite connected graph of diameter L . Assume that G is self-centered and satisfies $\Lambda\left(\frac{2D}{L} - 2\right)$. Then G is strongly spherical.*

It is an interesting question whether the condition of self-centeredness in Theorem 1.13 can be removed. If this were possible, we could view this result as another example where *local* properties have a strong *global* implication.

Finally, we would like to present the following classification of strongly spherical graphs.

Theorem 1.14 (*Classification of strongly spherical graphs, see [11]*). *Strongly spherical graphs are precisely the Cartesian products $G_1 \times G_2 \times \cdots \times G_k$, where each factor G_i is either a hypercube, a cocktail party graph, a Johnson graph $J(2n, n)$, an even dimensional demi-cube, or the Gosset graph.*

1.3. Outline of the paper

For the reader's information, more details of certain arguments can be found in the slightly longer arXiv version [6]. The paper is organised as follows:

- Section 2: Basic graph theoretical notation and introduction into Ollivier Ricci curvature.
- Section 3: Cartesian products preserve Bonnet-Myers sharpness under particular conditions.
- Section 4: Presentation of all known examples of Bonnet-Myers sharp graphs.
- Section 5: Discussion of various consequences of a useful relation between the Laplacian and Ollivier Ricci curvature. For example, all vertices of a Bonnet-Myers sharp graph lie on geodesics between any pair of antipoles.
- Section 6: Proof that Bonnet-Myers sharp graphs are Lichnerowicz sharp and classification of all distance-regular Lichnerowicz sharp graphs under an additional spectral condition.
- Sections 7 and 8: Proof of our main result: Classification of all self-centered Bonnet-Myers sharp graphs.
- Section 9: Classification of all Bonnet-Myers sharp graphs with extremal diameters $L = 2$ and $L = D$.
- Section 10: Relations between Bonnet-Myers sharpness and an upper bound on the Bakry-Émery ∞ -curvature.

2. Basic definitions, concepts and notation

Throughout this paper, we restrict our graphs $G = (V, E)$ to be undirected, simple, unweighted, finite, and connected.

2.1. Graph theoretical notation

We write $x \sim y$ if there exists an edge between the vertices x and y . The degree of a vertex $x \in V$ is denoted by d_x . For a set of vertices $A \subseteq V$, the induced subgraph $\text{Ind}_G(A)$ is the subgraph of G whose vertex set is A and whose edge set consists of all edges in G that have both endpoints in A .

For any two vertices $x, y \in V$, the (combinatorial) distance $d(x, y)$ is the length (i.e. the number of edges) in a shortest path from x to y . Such paths of minimal length are also called *geodesics* from x to y . An *interval* $[x, y]$ is the set of all vertices lying on geodesics from x to y , that is

$$[x, y] = \{z \in V \mid d(x, z) + d(z, y) = d(x, y)\}.$$

The *diameter* of G is denoted by $\text{diam}(G) = \max_{x, y \in V} d(x, y)$. A vertex $x \in V$ is called a *pole* if there exists a vertex $y \in V$ such that $d(x, y) = \text{diam}(G)$, in which case y will be called an *antipole* of x (with respect to G). A graph G is called *self-centered* if every vertex is a pole.

For $k \in \mathbb{N}$ and $x \in V$ we define the k -sphere of x as $S_k(x) = \{z : d(z, x) = k\}$ and the k -ball of x as $B_k(x) = \{z : d(z, x) \leq k\}$. In particular, $S_1(x)$ is also denoted as N_x , the set of all neighbours of x . Denote also $N_{xy} = S_1(x) \cap S_1(y)$, the common neighbours of x and y . Especially when $d(x, y) = 2$, the induced subgraph $\text{Ind}_G(N_{xy})$ is called μ -graph of x and y . In case that μ -graphs are isomorphic to a graph H for all x, y with $d(x, y) = 2$, we call the graph H the μ -graphs of G .

For a vertex $x \in V$, let $\#_\Delta(x)$ denote the number of triangles containing x . Similarly, for an edge $e = \{x, y\} \in E$, let $\#_\Delta(x, y)$ denote the number of triangles containing e . We have the following relation:

$$\#_\Delta(x) = \frac{1}{2} \sum_{y: y \sim x} \#_\Delta(x, y). \tag{2.1}$$

For $x, y \in V$, we also define

$$\begin{aligned} d_x^-(y) &= |\{z \sim y : d(x, y) = d(x, z) + 1\}|, \\ d_x^0(y) &= |\{z \sim y : d(x, y) = d(x, z)\}|, \\ d_x^+(y) &= |\{z \sim y : d(x, y) = d(x, z) - 1\}|, \end{aligned}$$

which we call the *in degree*, *spherical degree*, *out degree* of y , respectively. The averages of these degrees are then defined as:

$$\begin{aligned} av_k^-(x) &= \frac{1}{|S_k(x)|} \sum_{y \in S_k(x)} d_x^-(y), \\ av_k^0(x) &= \frac{1}{|S_k(x)|} \sum_{y \in S_k(x)} d_x^0(y), \\ av_k^+(x) &= \frac{1}{|S_k(x)|} \sum_{y \in S_k(x)} d_x^+(y). \end{aligned} \tag{2.2}$$

By abuse of notation, we sometimes identify $S_k(x)$ and $B_k(x)$ with the induced subgraphs $\text{Ind}_G(S_k(x))$ and $\text{Ind}_G(B_k(x))$, respectively.

2.2. Ollivier Ricci curvature and Kantorovich duality

A fundamental concept in Optimal Transport Theory is the Wasserstein distance defined on probability measures.

Definition 2.1. Let $G = (V, E)$ be a graph. Let μ_1, μ_2 be two probability measures on V . The Wasserstein distance $W_1(\mu_1, \mu_2)$ between μ_1 and μ_2 is defined as

$$W_1(\mu_1, \mu_2) := \inf_{\pi \in \Pi(\mu_1, \mu_2)} \text{cost}(\pi), \tag{2.3}$$

where π runs over all transport plans in

$$\Pi(\mu_1, \mu_2) = \left\{ \pi : V \times V \rightarrow [0, 1] : \mu_1(x) = \sum_{y \in V} \pi(x, y), \mu_2(y) = \sum_{x \in V} \pi(x, y) \right\}.$$

and the cost of π is defined as

$$\text{cost}(\pi) := \sum_{x \in V} \sum_{y \in V} d(x, y) \pi(x, y).$$

The transportation plan π in the above definition moves a mass distribution given by μ_1 into a mass distribution given by μ_2 , and $W_1(\mu_1, \mu_2)$ is a measure for the minimal effort which is required for such a transition. If π attains the infimum in (2.3) we call it an *optimal transport plan* transporting μ_1 to μ_2 . We define the following probability measures μ_x^p for any $x \in V, p \in [0, 1]$:

$$\mu_x^p(z) = \begin{cases} p & \text{if } z = x, \\ \frac{1-p}{d_x} & \text{if } z \sim x, \\ 0 & \text{otherwise.} \end{cases}$$

In this paper we will pay particular attention to D -regular graphs and the idleness parameter $p = \frac{1}{D+1}$ and write μ_x for $\mu_x^{\frac{1}{D+1}}$, for simplicity.

Definition 2.2 ([19]). Let $p \in [0, 1]$. The p -Ollivier Ricci curvature between two different vertices $x, y \in V$ is

$$\kappa_p(x, y) = 1 - \frac{W_1(\mu_x^p, \mu_y^p)}{d(x, y)},$$

where p is called the *idleness*.

If G is D -regular then we define the curvature, κ , as

$$\kappa(x, y) = \frac{D+1}{D} \kappa_{\frac{1}{D+1}}(x, y). \tag{2.4}$$

Remark 2.3. The motivation for (2.4) is that $\kappa(x, y)$ agrees with the modified curvature $\kappa_{LLY}(x, y)$ introduced by Lin/Lu/Yau in [15] for neighbours $x \sim y$, due to [1, Theorem 1.1]. That is, we have for neighbours $x \sim y$ in D -regular graphs,

$$\kappa(x, y) = \frac{\kappa_{\frac{1}{D+1}}(x, y)}{1 - \frac{1}{D+1}} = \lim_{p \rightarrow 1} \frac{\kappa_p(x, y)}{1 - p} =: \kappa_{LLY}(x, y). \tag{2.5}$$

A straightforward consequence of the fact that Ollivier Ricci curvature is defined via a distance function (Wasserstein distance) is the following:

$$\inf_{x \sim y} \kappa(x, y) \leq \inf_{z \neq w} \kappa(z, w), \tag{2.6}$$

namely that it makes no difference to take the infimum of the curvature over all pairs of different vertices or only over neighbours. Moreover, the Discrete Bonnet-Myers Theorem 1.1 follows directly from (2.6) and the following stronger inequality for any pair of different vertices $z, w \in V$ (see [19]):

$$\kappa(z, w) \leq \frac{2}{d(z, w)}, \tag{2.7}$$

by choosing a pair of vertices $z, w \in V$ of maximal distance, that is, $d(z, w) = \text{diam}(G)$.

Another fundamental concept in Optimal Transport Theory is Kantorovich duality. First we recall the notion of 1-Lipschitz functions and then state the Kantorovich Duality Theorem.

Definition 2.4. Let $G = (V, E)$ be a graph and $\phi : V \rightarrow \mathbb{R}$. We say that ϕ is 1-Lipschitz if

$$|\phi(x) - \phi(y)| \leq d(x, y) \quad \text{for all } x, y \in V.$$

We denote the set of all 1-Lipschitz functions by $1\text{-Lip}(V)$.

For an arbitrary graph $G = (V, E)$, we denote by $\ell_\infty(V)$ the space of all bounded function $f : V \rightarrow \mathbb{R}$ and by $C_c(V)$ the subspace of all functions with finite support.

Theorem 2.5 (*Kantorovich duality [21]*). *Let $G = (V, E)$ be a graph. Let μ_1, μ_2 be two probability measures on V . Then*

$$W_1(\mu_1, \mu_2) = \sup_{\phi \in 1\text{-Lip}(V) \cap \ell_\infty(V)} \sum_{x \in V} \phi(x)(\mu_1(x) - \mu_2(x)).$$

If ϕ attains the supremum we call it an optimal Kantorovich potential transporting μ_1 to μ_2 .

The following fact from [16, Theorem 2.1] is at the heart of all the results presented in Section 5. Moreover, it provides an alternative definition of Ollivier Ricci curvature as a notion induced by a given Laplace operator. This alternative viewpoint allows a more general definition of Ollivier Ricci curvature for various kinds of Laplacians.

Theorem 2.6 (see [16]). *Let $G = (V, E)$ be a connected graph and $f : V \rightarrow \mathbb{R}$ be a function on the vertices. We define the gradient of f as*

$$\nabla_{xy}f = \frac{f(x) - f(y)}{d(x, y)} \quad \text{for all } x, y \in V, x \neq y.$$

Let Δ_G be the normalized Laplacian of G defined in (1.3). Then, for any pair $x, y \in V$ of different vertices, we have

$$\kappa_{LLY}(x, y) = \inf_{\substack{\phi \in 1\text{-Lip}(V) \cap C_c(V) \\ \nabla_{yx}\phi=1}} \nabla_{xy}(\Delta_G\phi), \tag{2.8}$$

where κ_{LLY} was defined in (2.5).

2.3. Transport plans based on triangles and a perfect matching

Recall the definition of the combinatorial property $\Lambda(m)$ given in Definition 1.10. The following proposition explains its curvature implication and is useful for our curvature calculations in Section 4. As mentioned before, it also implies Proposition 1.2 directly by choosing $m = \frac{2D}{L} - 2$. Besides presenting this proposition and its proof, we also introduce in this subsection the notion of a *transport plan based on triangles and a perfect matching*.

Proposition 2.7. *Let $G = (V, E)$ be a D -regular graph and $e = \{x, y\} \in E$ an edge. Then we have*

$$\kappa(x, y) \leq \frac{2 + \#\Delta(x, y)}{D} \tag{2.9}$$

Moreover, the following are equivalent:

- (a) $\kappa(x, y) = \frac{2 + \#\Delta(x, y)}{D}$;
- (b) there is a perfect matching between the vertex sets $N_x \setminus (N_{xy} \cup \{y\})$ and $N_y \setminus (N_{xy} \cup \{x\})$.

Proof. Assume that $G = (V, E)$ is D -regular and $e = \{x, y\} \in E$ is contained in $m = \#\Delta(x, y)$ triangles. Then we have

$$s = |N_x \setminus (N_{xy} \cup \{y\})| = |N_y \setminus (N_{xy} \cup \{x\})| = D - 1 - m.$$

Since all masses are equal to $\frac{1}{D+1}$, we are faced with a Monge Problem and there exists an optimal transport plan π transporting μ_x to μ_y with $\pi(z, w) \in \{0, \frac{1}{D+1}\}$ for all $z, w \in V$. (That is, π is induced by a bijective optimal transport map $T : B_1(x) \rightarrow B_1(y)$; for more details see Subsection 7.1.) By [1, Lemma 4.1], we can choose π to satisfy $\pi(z, z) = \frac{1}{D+1}$ for all $z \in N_{xy} \cup \{x, y\}$, that is there is no mass transport on these vertices, and enumerate the vertices in $N_x \setminus (N_{xy} \cup \{y\})$ by $\{x_i\}_{i=1}^s$ and in $N_y \setminus (N_{xy} \cup \{x\})$ by $\{y_j\}_{j=1}^s$ such that $\pi(x_i, y_j) = \frac{1}{D+1} \delta_{ij}$. Explicitly, we have

$$\pi(u, v) = \begin{cases} \frac{1}{D+1}, & \text{if } (u, v) = (x_i, y_i), \\ \frac{1}{D+1}, & \text{if } u = v \in N_{xy} \cup \{x, y\}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.10}$$

Then

$$W_1(\mu_x, \mu_y) = \sum_{u \in V} \sum_{v \in V} \pi(u, v) d(u, v) \geq \frac{s}{D+1} = \frac{D-1-m}{D+1}, \tag{2.11}$$

with equality iff $d(x_i, y_i) = 1$, that is, there exists a perfect matching between the vertex sets $N_x \setminus (N_{xy} \cup \{y\})$ and $N_y \setminus (N_{xy} \cup \{x\})$. Consequently, we have the following curvature estimate with the same matching property in case of equality:

$$\kappa(x, y) = \frac{D+1}{D} (1 - W_1(\mu_x, \mu_y)) \leq \frac{2+m}{D}. \quad \square \tag{2.12}$$

The transport plan π chosen in the proof is of a particular structure which will also be important later. Therefore we introduce the following definition:

Definition 2.8. Let $e = \{x, y\} \in E$ be an edge of a D -regular graph $G = (V, E)$ with the following property: There is a perfect matching between the sets $N_x \setminus (N_{xy} \cup \{y\})$ and $N_y \setminus (N_{xy} \cup \{x\})$ given by $x_i \sim y_i$, where x_i and y_i are defined as in the above proof. We say that the transport plan π defined by (2.10) is *based on triangles and a perfect matching*.

3. Cartesian products

In this section we show under what conditions Bonnet-Myers sharpness is preserved under taking Cartesian products. First we recall the following result of Lin, Lu and Yau from [15].

Theorem 3.1 ([15]). *Let $G = (V_G, E_G)$ be a d_G -regular graph and $H = (V_H, E_H)$ be a d_H -regular graph. Let $x_1, x_2 \in V_G$ with $x_1 \sim x_2$ and $y_1, y_2 \in V_H$ with $y_1 \sim y_2$. Then*

$$\kappa^{G \times H}((x_1, y_1), (x_2, y_1)) = \frac{d_G}{d_G + d_H} \kappa^G(x_1, x_2),$$

$$\kappa^{G \times H}((x_1, y_1), (x_1, y_2)) = \frac{d_H}{d_G + d_H} \kappa^H(y_1, y_2),$$

Theorem 3.2. Let $\{G_i = (V_i, E_i)\}_{i=1}^N$ be a family of regular graphs where G_i has valency D_i for each i . Let L_i be the diameter of G_i . Let $G = G_1 \times \dots \times G_N$. The following are equivalent:

- (i) G is Bonnet-Myers sharp.
- (ii) Each G_i is Bonnet-Myers sharp and $\frac{D_1}{L_1} = \dots = \frac{D_N}{L_N}$.

Proof. Since $G_1 \times \dots \times G_N = G_1 \times (G_2 \times \dots \times G_N)$ we may assume that $N = 2$ and use induction.

By Theorem 3.1,

$$\begin{aligned} \inf_{\substack{u, v \in V(G) \\ u \sim v}} \kappa^G(u, v) &= \min \left\{ \inf_{\substack{x_1, x_2 \in V_1 \\ x_1 \sim x_2 \\ y \in V_2}} \kappa^G((x_1, y), (x_2, y)), \inf_{\substack{y_1, y_2 \in V_2 \\ y_1 \sim y_2 \\ x \in V_1}} \kappa^G((x, y_1), (x, y_2)) \right\} \\ &= \min \left\{ \inf_{\substack{x_1, x_2 \in V_1 \\ x_1 \sim x_2}} \frac{D_1}{D_1 + D_2} \kappa^{G_1}(x_1, x_2), \inf_{\substack{y_1, y_2 \in V_2 \\ y_1 \sim y_2}} \frac{D_2}{D_1 + D_2} \kappa^{G_2}(y_1, y_2) \right\}. \end{aligned} \tag{3.1}$$

First we prove that (i) implies (ii). Since G is Bonnet-Myers sharp, we have, by Bonnet-Myers theorem applied on G_1 :

$$\frac{2}{L_1 + L_2} = \inf_{\substack{u, v \in V(G) \\ u \sim v}} \kappa^G(u, v) \leq \inf_{\substack{x_1, x_2 \in V_1 \\ x_1 \sim x_2}} \frac{D_1}{D_1 + D_2} \kappa^{G_1}(x_1, x_2) \leq \frac{D_1}{D_1 + D_2} \cdot \frac{2}{L_1},$$

which is equivalent to $\frac{D_2}{L_2} \leq \frac{D_1}{L_1}$.

On the other hand, Bonnet-Myers on G_2 gives

$$\frac{2}{L_1 + L_2} = \inf_{\substack{u, v \in V(G) \\ u \sim v}} \kappa^G(u, v) \leq \inf_{\substack{y_1, y_2 \in V_2 \\ y_1 \sim y_2}} \frac{D_2}{D_1 + D_2} \kappa^{G_2}(y_1, y_2) \leq \frac{D_2}{D_1 + D_2} \cdot \frac{2}{L_2},$$

which is equivalent to $\frac{D_1}{L_1} \leq \frac{D_2}{L_2}$.

Therefore, we can conclude that $\frac{D_1}{L_1} = \frac{D_2}{L_2}$, and all the inequalities above are sharp, that is G_1 and G_2 are Bonnet-Myers sharp as well.

To prove (ii) implies (i): we simply plug into (3.1)

$$\inf_{\substack{x_1, x_2 \in V_1 \\ x_1 \sim x_2}} \kappa^{G_1}(x_1, x_2) = \frac{2}{L_1} \quad \text{and} \quad \inf_{\substack{y_1, y_2 \in V_2 \\ y_1 \sim y_2}} \kappa^{G_2}(y_1, y_2) = \frac{2}{L_2}$$

and use the assumption that $\frac{D_1}{L_1} = \frac{D_2}{L_2}$. As a result, we obtain $\inf_{\substack{u,v \in V(G) \\ u \sim v}} \kappa^G(u, v) = \frac{2}{L_1 + L_2}$. \square

Remark 3.3. In contrast to the necessary and sufficient condition for Bonnet-Myers sharpness in Theorem 3.2, much less is required for the Cartesian product $G = G_1 \times G_2$ to be Lichnerowicz sharp. In fact, G is Lichnerowicz sharp already if G_1 is Lichnerowicz sharp and G_2 is an arbitrary graph with its curvature lower bound large enough, as explained in the following argument.

Let $\lambda_1^{G_1}, \lambda_1^{G_2}, \lambda_1^G$ be the smallest positive eigenvalues of the Laplacians on G_1, G_2, G . We have

$$\begin{aligned} \inf_{\substack{u,v \in V(G) \\ u \sim v}} \kappa^G(u, v) &= \min \left\{ \inf_{\substack{x_1, x_2 \in V_1 \\ x_1 \sim x_2}} \frac{D_1}{D_1 + D_2} \kappa^{G_1}(x_1, x_2), \inf_{\substack{y_1, y_2 \in V_2 \\ y_1 \sim y_2}} \frac{D_2}{D_1 + D_2} \kappa^{G_2}(y_1, y_2) \right\} \\ &\leq \min \left\{ \frac{D_1}{D_1 + D_2} \lambda_1^{G_1}, \frac{D_2}{D_1 + D_2} \lambda_1^{G_2} \right\} = \lambda_1^G. \end{aligned} \tag{3.2}$$

where the inequality comes from Lichnerowicz' Theorem on each graph G_i : $\inf \kappa^{G_i} \leq \lambda_1^{G_i}$. In order to obtain the equality in (3.2), a sufficient condition is $\inf \kappa^{G_1} = \lambda_1^{G_1}$ (i.e. G_1 is Lichnerowicz sharp) and $\frac{D_1}{D_2} \inf \kappa^{G_1} \leq \inf \kappa^{G_2}$.

4. Examples of Bonnet-Myers sharp graphs

Here we present various examples of Bonnet-Myers sharp graphs and study their properties. Interestingly, the μ -graphs in each of the following examples are cocktail party graphs and the 1-spheres are strongly regular. Note that a finite simple graph $G = (V, E)$ is called *strongly regular* with parameters (ν, k, λ, μ) if G is not a complete graph, V has cardinality ν , every vertex has degree k , each pair of adjacent vertices has precisely λ common neighbours, and each pair of non-adjacent vertices has precisely μ common neighbours. In contrast to the usual definition, we also consider a set of n isolated points to be a strongly regular graph with parameters $(\nu, k, \lambda, \mu) = (n, 0, *, 0)$ where $*$ can be any integer. We say that a strongly regular graph G with these parameters is $\text{srg}(\nu, k, \lambda, \mu)$.

Our examples are the following:

1. hypercubes Q^n with 2^n vertices indexed by $\{0, 1\}^n$ and edges between them if Hamming distance equals one;
2. cocktail party graphs $CP(n)$ obtained by removal of a perfect matching from the complete graph K_{2n} ;
3. the Johnson graphs $J(2n, n)$ with vertices corresponding to n -subsets of $\{1, 2, \dots, 2n\}$ and edges between them if they overlap in $n - 1$ elements;

Table 1
Examples of (D, L) -Bonnet-Myers sharp graphs.

| G | $ V $ | (D, L) | $\#_{\Delta}(x, y)$ | μ -graph | $S_1(x)$ |
|----------------|-----------------|-----------------|---------------------|--------------|------------------|
| Q^n | 2^n | (n, n) | 0 | $CP(1)$ | n points |
| $CP(n)$ | $2n$ | $(2n - 2, 2)$ | $2n - 4$ | $CP(n - 1)$ | $CP(n - 1)$ |
| $J(2n, n)$ | $\binom{2n}{n}$ | (n^2, n) | $2n - 2$ | $CP(2)$ | $K_n \times K_n$ |
| $Q_{(2)}^{2n}$ | 2^{2n-1} | $(2n^2 - n, n)$ | $4n - 4$ | $CP(3)$ | $J(2n, 2)$ |
| Gosset | 56 | $(27, 3)$ | 16 | $CP(5)$ | Schläfli |

- 4. even-dimensional demi-cubes $Q_{(2)}^{2n}$: one of the two isomorphic connected components of the vertex set $\{0, 1\}^{2n}$ and edges between them if Hamming distance equals two;
- 5. the Gosset graph with 56 vertices: the vertices are in one-one correspondence with the edges $\{i, j\}$ and $\{i, j\}'$ of two disjoint copies of K_8 , respectively, and $\{i, j\} \sim \{k, l\}$ if $|\{i, j\} \cap \{k, l\}| = 1$ and $\{i, j\} \sim \{k, l\}'$ if $\{i, j\} \cap \{k, l\} = \emptyset$.

It is straightforward to check that all these examples are self-centered and satisfy the conditions (a), (b) from Proposition 2.7 with a constant value $\#_{\Delta}(x, y)$ for all $x \sim y$. So we can apply this proposition and conclude that

$$\kappa(x, y) = \frac{2 + \#_{\Delta}(X, y)}{D}. \tag{4.1}$$

Specific information about each of these graphs is listed in Table 1.

Using (4.1), we check that $\kappa(x, y) = \frac{2}{L}$ in all examples above, confirming that they are all (D, L) -Bonnet-Myers sharp. Moreover, all these examples are irreducible (with the exception of the hypercubes $Q^n, n \geq 2$) and distance-regular with symmetric intersection array, that is,

$$c_j = b_{L-j} \quad \text{for } 1 \leq j \leq L.$$

A D -regular connected finite graph $G = (V, E)$ of diameter L is called *distance-regular* if there are integers b_j, c_j such that for any two vertices $x, y \in V$ with $d(x, y) = j$ we have

$$d_x^-(y) = c_j \quad \text{and} \quad d_x^+(y) = b_j. \tag{4.2}$$

The sequence

$$\{b_0, \dots, b_{L-1}; c_1 = 1, \dots, c_L\}$$

is called the *intersection array* of the distance regular graph G . Moreover, μ denotes the number of common neighbours of a pair of vertices at distance 2, i.e., $\mu = c_2$. Distance regular graphs can be also defined as those graphs $G = (V, E)$ with the property that, for any choice of integers $k, l, m \geq 0$, the cardinality of $B_k(x) \cap B_l(y)$ for $x, y \in V$ with $d(x, y) = m$ depends only on the integers k, l, m .

By direct inspection and the fact that the second largest adjacency eigenvalue of $\text{srg}(\nu, k, \lambda, \mu)$ is given by (see [3, Theorem 9.1.2])

$$\frac{1}{2} \left(\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)} \right),$$

we derive the following result, which will be useful in Section 10.

Proposition 4.1. *Let $G = (V, E)$ be one of the examples in Table 1. Then every μ -graph is isomorphic to the cocktail party graph $CP(m)$ with*

$$m = \frac{D - L}{L(L - 1)} + 1$$

and all 1-spheres $S_1(x)$ in G are strongly regular with parameters

$$\text{srg} \left(D, \frac{2D}{L} - 2, \frac{D - 1}{L - 1} - 3, \frac{2(D - L)}{L(L - 1)} \right).$$

Moreover, the adjacency matrix of $S_1(x)$ has second largest eigenvalue equals $\frac{(D-L)(L-2)}{L(L-1)}$.

5. General facts about Bonnet-Myers sharp graphs

5.1. A useful inequality and its applications

Henceforth, $\Delta = \Delta_G$ denotes the normalized Laplacian on a graph $G = (V, E)$ defined in (1.3). We start with the following lemma.

Lemma 5.1. *Let $G = (V, E)$ be a finite connected D -regular graph and $u, v \in V, p \in [0, 1]$. Then for any function $f : V \rightarrow \mathbb{R}$,*

$$\sum_{z \in V} f(z) (\mu_u^p(z) - \mu_v^p(z)) = f(u) - f(v) + (1 - p)\Delta f(u) - (1 - p)\Delta f(v).$$

In particular, if $f \in 1\text{-Lip}(V)$, then

$$W_1(\mu_u^p, \mu_v^p) \geq f(u) - f(v) + (1 - p)\Delta f(u) - (1 - p)\Delta f(v),$$

with equality iff f is an optimal Kantorovich potential transporting μ_u^p to μ_v^p .

Proof. The first statement is a simple calculation relating μ_u^p and μ_v^p to the Laplacian Δ :

$$\begin{aligned} \sum_{z \in V} f(z) (\mu_u^p(z) - \mu_v^p(z)) &= \left[pf(u) + \frac{1-p}{D} \sum_{z \sim u} f(z) \right] - \left[pf(v) + \frac{1-p}{D} \sum_{w \sim v} f(w) \right] \\ &= f(u) + (1 - p)\Delta f(u) - f(v) - (1 - p)\Delta f(v). \end{aligned}$$

In particular, the second statement follows immediately from Theorem 2.5 (Kantorovich Duality). \square

The following result from [16] will prove very useful in our investigations.

Theorem 5.2 ([16]). *Let $G = (V, E)$ be a finite connected D -regular graph with diameter L . Let $x, y \in V$ with $d(x, y) = L$, and z be a vertex lying on a geodesic from x to y and $f \in 1\text{-Lip}(V)$ satisfy $f(y) - f(x) = L$. Then*

$$\Delta f(z) \leq 1 - \kappa(x, z)d(x, z).$$

A very similar result was stated in [16, Theorem 4.1] for the specific function $f = d(x, \cdot)$. The proof is a straightforward consequence of the alternative definition (2.8) of Ollivier Ricci curvature. For the reader’s convenience, we provide a direct proof not making use of (2.8).

Proof. Observe that the negative function $-f$ lies in $1\text{-Lip}(V)$, too. Choosing $p = \frac{1}{D+1}$, Lemma 5.1 implies

$$W_1(\mu_x, \mu_z) \geq f(z) - f(x) - \frac{D}{D+1}\Delta f(x) + \frac{D}{D+1}\Delta f(z).$$

Thus

$$\Delta f(z) \leq \frac{D+1}{D}W_1(\mu_x, \mu_z) + \frac{D+1}{D}(f(x) - f(z)) + \Delta f(x).$$

Note that, since $f \in 1\text{-Lip}(V)$ and since $f(y) - f(x) = L = d(y, x)$, $f(z) - f(x) = d(x, z)$ and $\Delta f(x) \leq 1$. Therefore

$$\begin{aligned} \Delta f(z) &\leq \frac{D+1}{D}(W_1(\mu_x, \mu_z) - d(x, z)) + 1 \\ &= 1 - \kappa(x, z)d(x, z). \quad \square \end{aligned}$$

Lemma 5.3. *Let $G = (V, E)$ be a (D, L) -Bonnet-Myers sharp graph and let $x, y \in V$ with $d(x, y) = L$. Let z, w be two different vertices lying on a geodesic from x to y with $d(x, z) + d(z, w) + d(w, y) = L$. Then*

$$\kappa(z, w) = \frac{2}{L}.$$

Proof. We have to show that $\kappa_{\frac{1}{D+1}}(z, w) = \frac{D}{D+1}\frac{2}{L}$. The triangle inequality tells us that

$$\begin{aligned} &W_1(\mu_x, \mu_y) \\ &\leq W_1(\mu_x, \mu_z) + W_1(\mu_z, \mu_w) + W_1(\mu_w, \mu_y) \end{aligned}$$

$$\begin{aligned} &= d(x, z)(1 - \kappa_{\frac{1}{D+1}}(x, z)) + d(z, w)(1 - \kappa_{\frac{1}{D+1}}(z, w)) + d(w, y)(1 - \kappa_{\frac{1}{D+1}}(w, y)) \\ &= L - d(x, z)\kappa_{\frac{1}{D+1}}(x, z) - d(z, w)\kappa_{\frac{1}{D+1}}(z, w) - d(w, y)\kappa_{\frac{1}{D+1}}(w, y). \end{aligned}$$

Bonnet-Myers sharpness means that

$$\kappa_{\frac{1}{D+1}}(u, v) \geq \frac{D}{D+1} \frac{2}{L},$$

for all $u, v \in V, u \neq v$. Assume that $\kappa_{\frac{1}{D+1}}(z, w) > \frac{2D}{D+1} \frac{1}{L}$. Then

$$W_1(\mu_x, \mu_y) < L - \frac{D}{D+1} \frac{2}{L} (d(x, z) + d(z, w) + d(w, y)) = L - \frac{2D}{D+1},$$

and so

$$\kappa(x, y) > \frac{2}{L},$$

which is a contradiction to (2.7). Thus

$$\kappa(z, w) = \frac{2}{L}. \quad \square$$

Note that we can choose $z = x$ or $w = y$ in the statement of the lemma above. This, together with Theorem 5.2, leads to the following result:

Theorem 5.4. *Let $G = (V, E)$ be a (D, L) -Bonnet-Myers sharp graph with diameter L . Let $x, y \in V$ with $d(x, y) = L$, and z be a vertex lying on a geodesic from x to y , and $f \in 1$ -Lip satisfy $f(y) - f(x) = L$. Then*

$$\Delta f(z) = 1 - \frac{2d(x, z)}{L}. \tag{5.1}$$

Proof. From Lemma 5.3, $\kappa(x, z) = \kappa(y, z) = \frac{2}{L}$. We obtain, from Theorem 5.2

$$\Delta f(z) \leq 1 - \frac{2d(x, z)}{L},$$

and

$$\Delta(-f(z)) \leq 1 - \frac{2d(y, z)}{L} = 1 - \frac{2(L - d(x, z))}{L} = \frac{2d(x, z)}{L} - 1.$$

Thus

$$\Delta f(z) = 1 - \frac{2d(x, z)}{L},$$

as required. \square

Theorem 5.5. *Let $G = (V, E)$ be a (D, L) -Bonnet-Myers sharp graph and $x, y \in V$ with $d(x, y) = L$. Then any vertex $u \in V$ lies on a geodesic from x to y , that is, we have $[x, y] = V$.*

Proof. Let $f := d(x, \cdot)$, $g := L - d(\cdot, y)$. We must show that $f = g$. Note that

$$f(z) - g(z) = d(x, z) + d(z, y) - L \geq 0.$$

Thus $f \geq g$. Suppose $f \neq g$. Then there exists a vertex z closest to x with $f(z) > g(z)$ with $z \neq x$. Hence there exists a vertex $w \sim z$ with $d(x, w) < d(x, z)$. By our assumptions we have $f(w) = g(w)$. Therefore w is on a geodesic from x to y . Thus, by Theorem 5.4, we have $\Delta f(w) = \Delta g(w)$. However $f(z) > g(z)$ and so

$$\begin{aligned} D\Delta f(w) &= \sum_{u \sim w} (f(u) - f(w)) \\ &= f(z) - f(w) + \sum_{\substack{u \sim w \\ u \neq z}} (f(u) - f(w)) \\ &> g(z) - g(w) + \sum_{\substack{u \sim w \\ u \neq z}} (g(u) - g(w)) \\ &= D\Delta g(w) \end{aligned}$$

which is a contradiction. Thus $f = g$, completing the proof. \square

Corollary 5.6. *Let $G = (V, E)$ be Bonnet-Myers sharp. Then every vertex in V has at most one antipole.*

Proof. Assume $x \in V$ has two different antipoles y_1 and y_2 . Then y_1 lies on a geodesic from x to y_2 , by Theorem 5.5 and, since $y_1 \neq y_2$,

$$d(x, y_1) < d(x, y_2) = \text{diam}(G),$$

which contradicts to the assumption that y_1 is an antipole of x . \square

Theorem 5.7. *Let $G = (V, E)$ be (D, L) -Bonnet-Myers sharp graph and $x \in V$ be a pole. Then we have for every edge $\{x, y\} \in E$:*

- (a) *The edge $\{x, y\}$ lies in precisely $\frac{2D}{L} - 2$ triangles, or, in other words, the induced subgraph $S_1(x)$ is $(\frac{2D}{L} - 2)$ -regular;*
- (b) *There is a perfect matching between the sets $N_x \setminus (N_{xy} \cup \{y\})$ and $N_y \setminus (N_{xy} \cup \{x\})$;*
- (c) *There is an optimal transport plan π transporting μ_x to μ_y which is based on triangles and a perfect matching (see Definition 2.8) with the cost*

$$\text{cost}(\pi) = \frac{1}{D+1} \left(D + 1 - \frac{2D}{L} \right).$$

Proof. Let $x \in V$, and define $f(w) = d(x, w) - \frac{L}{2}$. By (5.1), we have $\Delta f + \frac{2}{L}f = 0$. Let $y \in S_1(x)$. We need to show that $d_x^0(y) = \frac{2D}{L} - 2$. Now since $f(z) - f(y) = 0$ if $z \sim y$, $z \in S_1(x)$,

$$\begin{aligned} 0 &= \Delta f(y) + \frac{2}{L}f(y) = \frac{1}{D} \left[(f(x) - f(y)) + \sum_{z \sim y, z \in S_2(x)} (f(z) - f(y)) \right] + \frac{2}{L}f(y) \\ &= \frac{1}{D}(d_x^+(y) - 1) + \frac{2}{L} \left(1 - \frac{L}{2} \right). \end{aligned}$$

Rearranging gives

$$d_x^+(y) = D + 1 - \frac{2D}{L}$$

and, therefore,

$$d_x^0(y) = D - d_x^+(y) - d_x^-(y) = D - (D + 1 - \frac{2D}{L}) - 1 = \frac{2D}{L} - 2.$$

We now show that there is an optimal transport plan transporting μ_x to μ_y which is based on triangles and a perfect matching of the neighbours of x and y . We will prove this indirectly. Suppose that there is no optimal transport plan based on triangles and a perfect matching. Thus, in any optimal transport plan, there must be some mass in $N_{xy} \setminus N_x$ that travels with the distance more than 1, so the total cost is

$$W_1(\mu_x, \mu_y) > \frac{1}{D+1} \left(D + 1 - \frac{2D}{L} \right),$$

and thus

$$\kappa(x, y) = \frac{D+1}{D} \kappa_{\frac{1}{D+1}}(x, y) < \frac{2}{L},$$

which contradicts to the fact that G is Bonnet-Myers sharp (see Lemma 5.3).

Therefore, there is an optimal transport plan π based on triangles and a perfect matching with the cost:

$$\text{cost}(\pi) = W_1(\mu_x, \mu_y) = \frac{1}{D+1} \left(D + 1 - \frac{2D}{L} \right). \quad \square$$

We now give relations between the in, out and spherical degrees of vertices inside Bonnet-Myers sharp graphs.

Theorem 5.8. *Let $G = (V, E)$ be a (D, L) -regular Bonnet-Myers sharp graph and $x \in V$ be a pole. Let $y \in S_k(x)$, where $k \in \mathbb{N}$. Then*

$$d_x^+(y) - d_x^-(y) = D \left(1 - \frac{2k}{L} \right), \tag{5.2}$$

$$2d_x^+(y) + d_x^0(y) = 2D \left(1 - \frac{k}{L} \right), \tag{5.3}$$

$$2d_x^-(y) + d_x^0(y) = \frac{2kD}{L}. \tag{5.4}$$

Proof. Define $f(w) = d(x, w) - \frac{L}{2}$. By (5.1), we have $\Delta f + \frac{2}{L}f = 0$. Thus

$$\begin{aligned} \Delta f(y) + \frac{2}{L}f(y) &= \frac{1}{D} \left[\sum_{z \sim y} (f(z) - f(y)) \right] + \frac{2}{L}f(y) \\ &= \frac{1}{D} [d_x^+(y) - d_x^-(y)] + \frac{2}{L} \left(k - \frac{L}{2} \right). \end{aligned}$$

Rearranging gives

$$d_x^+(y) - d_x^-(y) = D \left(1 - \frac{2k}{L} \right).$$

The rest of the formula are obtained by using $D = d_x^+(y) + d_x^0(y) + d_x^-(y) = D$ and algebraic manipulation. \square

We end this subsection with the proof of Theorem 1.4.

Theorem 1.4. *Any (D, L) -Bonnet-Myers sharp graph satisfies $L \leq D$. Moreover L must divide $2D$.*

Proof of Theorem 1.4. Let x be a pole of G . Choosing $k = 1$ in Theorem 5.8, we conclude from (5.4) that L must divide $2D$. Therefore, in the case $L > D$, we must have $L = 2D$. Choosing $k = L - 1$ in (5.3) would lead to

$$2d_x^+(y) + d_x^0(y) = 2D \left(1 - \frac{L-1}{L} \right) = 2D - (L-1) = 1,$$

which would imply $d_x^+(y) = 0$ and $d_x^0(y) = 1$, which cannot be since some $y \in S_{L-1}(x)$ must be a neighbour of the antipole of x . Therefore, we have ruled out $L = 2D$ and we conclude $L \leq D$. \square

5.2. Self-centered Bonnet-Myers sharp graphs

This subsection provides two immediate consequences of the results from the previous subsection under the extra assumption of self-centeredness, i.e. every vertex is a pole.

Theorem 5.9. *Let $G = (V, E)$ be a self-centered (D, L) -Bonnet-Myers sharp graph. Then we have for any pair z, w of different vertices*

$$\kappa(z, w) = \frac{2}{L},$$

that is, G has constant Ollivier-Ricci curvature $\frac{2}{L}$.

Proof. Let z' be the antipole of z in G . Then w lies on a geodesic from z to z' , by Theorem 5.5 and Lemma 5.3 yields

$$\kappa(z, w) = \frac{2}{L}. \quad \square$$

Corollary 5.10. *Let $G = (V, E)$ be a self-centered (D, L) -Bonnet-Myers sharp graph. Then we have for every edge $\{x, y\} \in E$:*

- (a) *The edge $\{x, y\}$ lies in precisely $\frac{2D}{L} - 2$ triangles, or, in other words, the induced subgraph $S_1(x)$ is $(\frac{2D}{L} - 2)$ -regular;*
- (b) *There is a perfect matching between the sets $N_x \setminus (N_{xy} \cup \{y\})$ and $N_y \setminus (N_{xy} \cup \{x\})$;*
- (c) *There is an optimal transport plan π transporting μ_x to μ_y which is based on triangles and a perfect matching with the cost*

$$\text{cost}(\pi) = \frac{1}{D+1} \left(D + 1 - \frac{2D}{L} \right).$$

Proof. It follows immediately from Theorem 5.7, since every vertex of a self-centered graph is a pole. \square

6. Curvatures and eigenvalues

The following result agrees with Theorem 1.5 and it provides additional information about the choice of a suitable eigenfunction.

Theorem 1.5. *Every (D, L) -Bonnet-Myers sharp graph $G = (V, E)$ is Lichnerowicz sharp with Laplace eigenfunction $f = d(x, \cdot) - \frac{L}{2}$, where x is a pole of G .*

Proof. Let $\text{diam}(G) = L$ and $x, y \in V$ satisfy $d(x, y) = L$. Let $f = d(x, \cdot) - \frac{L}{2}$. By Theorem 5.5 every vertex $z \in V$ lies on a geodesic from x to y . Thus, by Lemma 5.3, $\kappa(x, z) = \frac{2}{L}$. Then, by Theorem 5.4, we have

$$\Delta f + \frac{2}{L}f = 0.$$

Therefore $\lambda_1 \leq \frac{2}{L}$. By the Discrete Lichnerowicz Theorem 1.3 and Bonnet-Myers sharpness, we have

$$\lambda_1 \geq \inf_{\substack{u,v \in V \\ u \neq v}} \kappa(u, v) = \frac{2}{L}.$$

Thus $\lambda_1 = \frac{2}{L} = \inf_{u,v \in V, u \neq v} \kappa(u, v)$, completing the proof. \square

We also have the following general relation between eigenfunctions, curvature and Kantorovich potentials:

Theorem 6.1. *Let $G = (V, E)$ be a finite connected D -regular graph. Let $f : V \rightarrow \mathbb{R}$ be a Laplace eigenfunction, which is also an optimal Kantorovich potential transporting μ_u^p to μ_v^p for some $u, v \in V, u \neq v, p \in (\frac{1}{2}, 1)$. Then $\lambda = \kappa_{LLY}(u, v)$ with κ_{LLY} defined in (2.5).*

Proof. Since f is an optimal Kantorovich potential transporting μ_u^p to μ_v^p and $\Delta f + \lambda f = 0$, Lemma 5.1 yields

$$W_1(\mu_u^p, \mu_v^p) = \sum_{z \in V} f(z) (\mu_u^p(z) - \mu_v^p(z)) = (1 - (1 - p)\lambda)(f(u) - f(v)).$$

Moreover, since $p > \frac{1}{2}$, every optimal transport plan π from μ_u^p to μ_v^p must satisfy $\pi(u, v) > 0$, which then implies by complementary slackness that $f(u) - f(v) = d(u, v)$ (see, e.g., [1, Lemma 3.1], which states this fact for neighbours $x, y \in V$, but it is also true for arbitrary pairs of different vertices).

Substituting $f(u) - f(v) = d(u, v)$ in the above equation yields

$$\kappa_p(u, v) = 1 - \frac{W_1(\mu_u^p, \mu_v^p)}{d(u, v)} = (1 - p)\lambda,$$

which implies

$$\frac{\kappa_p(u, v)}{1 - p} = \lambda.$$

Since $p \in (\frac{1}{2}, 1)$, we conclude from [5, Corollary 3.4]

$$\kappa_{LLY}(u, v) = \lim_{p \rightarrow 1} \frac{\kappa_p(u, v)}{1 - p} = \frac{\kappa_p(u, v)}{1 - p},$$

finishing the proof. \square

The following result can be derived via similar arguments (but a different logic in its proof):

Theorem 6.2. *Let $G = (V, E)$ be Lichnerowicz sharp with a Laplace eigenfunction $f \in 1\text{-Lip}(V)$ associated to the eigenvalue $\lambda_1 = \inf_{x \sim y} \kappa(x, y)$. Then, for any pair of different vertices $u, v \in V$ with $f(u) - f(v) = d(u, v)$, f is an optimal Kantorovich potential transporting μ_u to μ_v .*

Proof. Assume $f \in 1\text{-Lip}(V)$, $\Delta f + \lambda_1 f = 0$ and $p = \frac{1}{D+1}$. Lemma 5.1 yields

$$W_1(\mu_u, \mu_v) \geq (1 - (1 - p)\lambda_1)(f(u) - f(v)),$$

with equality iff f is an optimal Kantorovich potential transporting μ_u to μ_v . This is equivalent to

$$\kappa_p(u, v) = 1 - \frac{W_1(\mu_u, \mu_v)}{d(u, v)} \leq (1 - p)\lambda_1 \frac{f(u) - f(v)}{d(u, v)} = (1 - p)\lambda_1,$$

using $f(u) - f(v) = d(u, v)$. This, in turn, is equivalent to

$$\kappa(u, v) = \frac{\kappa_p(u, v)}{1 - p} \leq \lambda_1. \tag{6.1}$$

Our assumption $\lambda_1 = \inf_{x \sim y} \kappa(x, y)$ then implies equality in (6.1) and, therefore, f is an optimal Kantorovich potential transporting μ_u to μ_v . \square

Finally, let us identify all Lichnerowicz sharp graphs within an interesting family of distance regular graphs. More precisely, there is the following classification of all distance-regular graphs with second largest adjacency eigenvalue $\theta_1 = b_1 - 1$ (see Theorem 6.3). Recall that b_1 is a parameter of the intersection array which was defined in (4.2).

Theorem 6.3 ([2]). *Let $G = (V, E)$ be a distance-regular graph with second largest eigenvalue $\theta_1 = b_1 - 1$. Then at least one of the following holds:*

- (i) G is a strongly regular graph with smallest adjacency eigenvalue -2 ;
- (ii) $\mu = 1$, i.e., G has numerical girth at least 5;
- (iii) $\mu = 2$, and G is a Hamming graph, a Doob graph, or a locally Petersen graph;
- (iv) $\mu = 4$, and G is a Johnson graph;
- (v) $\mu = 6$, and G is a demi-cube;
- (vi) $\mu = 10$, and G is the Gosset graph.

The Lichnerowicz sharp graphs within this class of distance-regular graphs are identified in the following theorem.

Table 2
Distance-regular graphs with second largest eigenvalue $\theta_1 = b_1 - 1$ and $\mu \neq 1$ from Theorem 6.3.

| G | $ V $ | D | $\theta_1 = b_1 - 1$ | λ_1 | $\inf_{x \sim y} \kappa(x, y)$ | $\dim E_{\lambda_1}$ |
|---------------|----------------|----------------|------------------------|---------------------|--------------------------------|----------------------|
| $CP(n)$ | $2n$ | $2n - 2$ | 0 | 1 | 1 | n |
| Shrikhande | 16 | 6 | 2 | $\frac{2}{3}$ | $\frac{1}{3}$ | 6 |
| Chang | 28 | 12 | 4 | $\frac{2}{3}$ | $\frac{1}{3}$ | 7 |
| Petersen | 10 | 3 | 1 | $\frac{2}{3}$ | 0 | 5 |
| Schläfli | 27 | 16 | 4 | $\frac{3}{4}$ | $\frac{3}{4}$ | 6 |
| $(K_n)^d$ | n^d | $(n - 1)d$ | $n(d - 1) - d$ | $\frac{n}{(n-1)d}$ | $\frac{n}{(n-1)d}$ | $(n - 1)d$ |
| Doob n,m | 4^{n+2m} | $3(n + 2m)$ | $3(n + 2m) - 4$ | $\frac{4}{3(n+2m)}$ | $\frac{2}{3(n+2m)}$ | $3(n + 2m)$ |
| (7, 2)-Kneser | 21 | 10 | 3 | $\frac{7}{10}$ | $\frac{1}{2}$ | 6 |
| Conway-Smith | 63 | 10 | 5 | $\frac{1}{2}$ | $-\frac{1}{10}$ | 12 |
| Hall | 65 | 10 | 5 | $\frac{1}{2}$ | $-\frac{1}{10}$ | 13 |
| $J(n, k)$ | $\binom{n}{k}$ | $k(n - k)$ | $k(n - k) - n$ | $\frac{n}{k(n-k)}$ | $\frac{n}{k(n-k)}$ | $n - 1$ |
| $Q_{(2)}^n$ | 2^{n-1} | $\binom{n}{2}$ | $\frac{(n-4)(n-1)}{2}$ | $\frac{4}{n}$ | $\frac{4}{n}$ | n |
| Gosset | 56 | 27 | 9 | $\frac{2}{3}$ | $\frac{2}{3}$ | 7 |

Theorem 6.4. *The Lichnerowicz sharp distance-regular graphs with second largest adjacency eigenvalue $\theta_1 = b_1 - 1$ are precisely the following ones:*

1. *the cocktail party graphs $CP(n)$ (also Bonnet-Myers sharp);*
2. *the Hamming graphs $H(n, d) = (K_n)^d$ (only Bonnet-Myers sharp if $n = 2$, that is $H(n, d) = Q^d$);*
3. *the Johnson graphs $J(n, k)$ (only Bonnet-Myers sharp if $n = 2k$);*
4. *the demi-cubes $Q_{(2)}^n$ (only Bonnet-Myers sharp if n is even);*
5. *the Schläfli graph (not Bonnet-Myers sharp);*
6. *the Gosset graph (also Bonnet-Myers sharp).*

Proof. Graphs satisfying $\mu = 1$ (case (ii) of Theorem 6.3) cannot be Lichnerowicz sharp due to Proposition 6.5 following this proof.

Combining the classification [3, Theorem 9.2.1] of strongly regular graphs with smallest adjacency eigenvalue -2 (case (i) of Theorem 6.3) with the cases (iii)-(vi) of Theorem 6.3 we are left with the task to check Lichnerowicz sharpness of all the graphs in Table 2. Since $\Delta_G = D^{-1}A_G - \text{Id}$ the eigenvalues θ_1 and λ_1 are related by

$$\lambda_1 = 1 - \frac{\theta_1}{D}.$$

The table contains also information about the multiplicity $\dim E_{\lambda_1}$ of the smallest positive eigenvalue of Δ_G . Note furthermore that Chang stands for any one of the three Chang graphs and that the Doob graphs are given by $\text{Doob}^{n,m} = K_4^n \times \text{Shk}^m$ with $n, m \geq 1$, where Shk denotes the Shrikhande graph, and the (7, 2)-Kneser, Conway-Smith graph and Hall graph are the three locally Petersen graphs. The curvatures $\inf_{x \sim y} \kappa(x, y)$ were determined with the help of the curvature calculator [7] at

<http://www.mas.ncl.ac.uk/graph-curvature/>

□

Proposition 6.5. *A distance-regular graph with second largest adjacency eigenvalue $\theta_1 = b_1 - 1$ and $\mu = 1$ cannot be Lichnerowicz sharp.*

Proof. Let $G = (V, E)$ be a distance-regular graph of vertex degree D and satisfying $\mu = 1$, and $x, z \in V$ with $d(x, z) = 2$. We denote the unique common neighbour of x and z by y . Then we have $0 \leq d_x^0(z) =: \alpha \leq D - 1$ and $b_1 = d_x^+(z) = D - 1 - \alpha$. The second largest adjacency eigenvalue is then $\theta_1 = b_1 - 1 = D - 2 - \alpha$ and, consequently, the smallest positive Laplace eigenvalue is

$$\lambda_1 = 1 - \frac{D - 2 - \alpha}{D} = \frac{2 + \alpha}{D} > 0.$$

Let us now estimate $\kappa(x, z)$. We have

$$W_1(\mu_x, \mu_z) \geq \frac{1}{D + 1} (2 + 2(D - 1 - \alpha) + \alpha) = \frac{2D - \alpha}{D + 1},$$

and, therefore,

$$\kappa_{\frac{1}{D+1}}(x, z) = 1 - \frac{W_1(\mu_x, \mu_z)}{2} \leq \frac{1 + \frac{\alpha}{2}}{D + 1}.$$

This implies that

$$\inf_{u \sim v} \kappa(u, v) \leq \kappa(x, z) = \frac{D + 1}{D} \kappa_{\frac{1}{D+1}}(x, z) \leq \frac{1 + \frac{\alpha}{2}}{D} = \frac{\lambda_1}{2} < \lambda_1.$$

This shows that G cannot be Lichnerowicz sharp. □

Remark 6.6. As pointed out by Paul Terwilliger, there is a surprising connection between our classification in Theorem 6.4 and the classification of all *complete connected hypermetric spaces* classified in [20]. These are precisely Cartesian products of cocktail party graphs, Johnson graphs, demi-cubes, the Schläfli graph and the Gosset graph which can also be realised as minimal saturated spaces (ms-spaces) of the irreducible root lattices A_n, D_n, E_6, E_7 . Note that the ranks of these root lattices coincide with the dimension $\dim E_{\lambda_1}$ of the Lichnerowicz eigenvalue λ_1 of the corresponding graphs given in Table 2.

7. Transport geodesics of self-centered Bonnet-Myers sharp graphs

This section together with the next one is dedicated to the proof that the examples in Section 4 and suitable Cartesian products of them are the only self-centered Bonnet-Myers sharp graphs. Of crucial importance in this proof are transport geodesic techniques. In view of this result, it is natural to ask the following:

Question. Are there any Bonnet-Myers sharp graphs which are not self-centered?

We assume that all Bonnet-Myers sharp graphs are self-centered, but this is currently still an open problem. Let us now start to introduce the relevant tools to achieve the above mentioned goal.

7.1. Concatenation of transport maps

Let $G = (V, E)$ be a simple, connected, D -regular graph and μ_0, μ_1 be probability measures on V . A transport plan $\pi \in \Pi(\mu_0, \mu_1)$ is induced by a *transport map* $T : \text{supp}(\mu_0) \rightarrow \text{supp}(\mu_1)$ if $\mu_1(T(x)) = \mu_0(x)$ for all $x \in V$ and

$$\pi(x, y) = \begin{cases} \mu_0(x), & \text{if } x \in \text{supp}(\mu_0) \text{ and } y = T(x), \\ 0, & \text{otherwise.} \end{cases}$$

We define the *cost* of a transport map T as the cost of its induced transport plan $\pi : V \times V \rightarrow [0, 1]$:

$$\text{cost}(T) := \text{cost}(\pi) = \sum_{x \in \text{supp}(\mu_0)} d(x, T(x))\mu_0(x).$$

T is called an *optimal transport map* from μ_0 to μ_1 if its induced plan $\pi \in \Pi(\mu_0, \mu_1)$ is an optimal transport plan. The existence of optimal transport maps for given probability measures μ_0, μ_1 is known as the *Monge Problem*.

Let T_1, T_2 be transport maps from μ_0 to μ_1 and from μ_1 to μ_2 , respectively. These transport maps can be concatenated to a transport map from μ_0 to μ_2 via

$$T_2 \circ T_1 : \text{supp}(\mu_0) \rightarrow \text{supp}(\mu_2).$$

The following fact about concatenation will be useful henceforth.

Proposition 7.1 (*transport geodesic*). *Let $G = (V, E)$ be a simple, connected D -regular graph and $\mu_0, \mu_1, \dots, \mu_k$ be probability measures on V . For $1 \leq j \leq k$, let T_j be a transport map from μ_{j-1} to μ_j , and T^j be the concatenated map*

$$T^j := T_j \circ T_{j-1} \circ \dots \circ T_1 : \text{supp}(\mu_0) \rightarrow \text{supp}(\mu_j).$$

Then we have

$$\text{cost}(T^k) \leq \sum_{j=1}^k \text{cost}(T_j). \tag{7.1}$$

Assume that we have equality in (7.1). Then, for each $z \in \text{supp}(\mu_0)$, the sequence of vertices

$$z, T^1(z), T^2(z), \dots, T^k(z)$$

lies on a geodesic from z to $T^k(z)$, that is,

$$d(z, T^k(z)) = d(z, T^1(z)) + d(T^1(z), T^2(z)) + \dots + d(T^{k-1}(z), T^k(z)). \tag{7.2}$$

Such sequence of vertices $z, T^1(z), T^2(z), \dots, T^k(z)$ is hence called a **transport geodesic**.

Proof. Setting $T^0 = \text{Id}_{\text{supp}(\mu_0)}$, we have

$$\begin{aligned} \text{cost}(T^k) &= \sum_{z \in \text{supp}(\mu_0)} d(z, T^k(z)) \mu_0(z) \\ &\stackrel{\Delta}{\leq} \sum_{z \in \text{supp}(\mu_0)} \left(\sum_{j=1}^k d(T^{j-1}(z), T^j(z)) \right) \mu_0(z) \\ &= \sum_{j=1}^k \left(\sum_{z \in \text{supp}(\mu_0)} d(T^{j-1}(z), T_j \circ T^{j-1}(z)) \mu_0(z) \right) \\ &= \sum_{j=1}^k \left(\sum_{x \in \text{supp}(\mu_{j-1})} d(x, T_j(x)) \mu_{j-1}(x) \right) = \sum_{j=1}^k \text{cost}(T_j), \end{aligned}$$

with equality if and only if (7.2) for all $z \in \text{supp}(\mu_0)$. \square

7.2. Transport geodesics of a self-centered Bonnet-Myers sharp graph

In this subsection and henceforth, we always assume that our (D, L) -Bonnet-Myers sharp graph $G = (V, E)$ has the extra condition of self-centeredness.

Let us start with a full-length (i.e. of length L) geodesic g , and denote the vertices along this geodesic by

$$g : \quad x_0 \sim x_1 \sim x_2 \sim x_3 \sim \dots \sim x_L.$$

For every $1 \leq j \leq L$, since $x_{j-1} \sim x_j$ consider an optimal transport map T_j from $\mu_{x_{j-1}}$ to μ_{x_j} based on triangles and a perfect matching (see Theorem 5.10(c)), that is:

$$T_j : B_1(x_{j-1}) \rightarrow B_1(x_j)$$

is a bijective function and satisfies

1. $x = T_j(x)$ if $x \in B_1(x_{j-1}) \cap B_1(x_j)$, and
2. $x \sim T_j(x)$ if $x \in B_1(x_{j-1}) \setminus B_1(x_j)$.

For simplicity, we will write 1. and 2. together as $x \simeq T_j(x)$, where the symbol \simeq means “adjacent or equal to”.

Moreover, for each $z \in B_1(x_0)$, we define $z(0) := z$ and for $1 \leq j \leq L$,

$$z(j) := T^j(z) := T_j \circ \dots \circ T_1(z) \in B_1(x_j).$$

Note that, in particular, we have $x_0(1) = x_0(0) = x_0$ by condition 1.

Proposition 7.2. *Let $G = (V, E)$ be a self-centered (D, L) -Bonnet-Myers sharp. Given a full-length geodesic g and maps T_j and T^j (for $1 \leq j \leq L$) defined as above. Then for every $z \in B_1(x_0)$, the sequence of vertices*

$$z(0) \simeq z(1) \simeq z(2) \simeq \dots \simeq z(L)$$

*is a transport geodesic. Since this transport geodesic follows closely the geodesic g , we call it a **transport geodesic along g** and denote it by g_z .*

Remark 7.3. The definition of a transport geodesic g_z depends on a full-length geodesic g and sets of transport maps $\{T_j\}_{j=1}^L$. Each T_j is a priori not uniquely defined, since the definition of T_j is based on triangles and a perfect matching, the latter of which is not necessarily unique. We will see later (cf. Remark 7.10) that in fact the maps $\{T_j\}_{j=1}^L$ are already uniquely determined by g in the case of self-centered Bonnet-Myers sharp graphs.

Proof. Note that T^L induces a transport plan from μ_{x_0} to μ_{x_L} , and together with Theorem 5.10(c), we have

$$W_1(\mu_{x_0}, \mu_{x_L}) \leq \text{cost}(T^L) \leq \sum_{j=1}^L \text{cost}(T_j) = L \cdot \frac{1}{D+1} \left(D+1 - \frac{2D}{L} \right) = L - \frac{2D}{D+1}.$$

On the other hand,

$$\frac{2}{L} \geq \kappa(x_0, x_L) = \frac{D+1}{D} \kappa_{\frac{1}{D+1}}(x_0, x_L) = \frac{D+1}{D} \left(1 - \frac{W_1(\mu_{x_0}, \mu_{x_L})}{L} \right)$$

implies that

$$L - \frac{2D}{D+1} \leq W_1(\mu_{x_0}, \mu_{x_L}).$$

Bringing these inequalities together, we conclude that

$$\text{cost}(T^L) = \sum_{j=1}^L \text{cost}(T_j) = L - \frac{2D}{D+1}. \tag{7.3}$$

Then by Proposition 7.1, for every $z \in B_1(x_0)$, the sequence of vertices $z, T^1(z), T^2(z), \dots, T^L(z)$ is a transport geodesic. This sequence is indeed the same as

$$z(0) \simeq z(1) \simeq z(2) \simeq \dots \simeq z(L)$$

since by definition $z(j) = T^j(z)$ and $z(j - 1) \simeq T_j(z(j - 1)) = z(j)$ for all j . \square

Proposition 7.4. *Given the same setup as in Proposition 7.2. Then for every $z \in B_1(x_0)$, the corresponding transport geodesic g_z , namely*

$$g_z : z(0) \simeq z(1) \simeq \dots \simeq z(L)$$

has the length

$$\ell(g_z) := d(z(0), z(L)) = \begin{cases} L - 1 & , \text{ if } z(0) = x_0 \text{ or } z(L) = x_L \\ L - 2 & , \text{ otherwise.} \end{cases}$$

As an immediate consequence, every geodesic g_z can be extended to the geodesic

$$\text{ext}(g_z) : x_0 \simeq z(0) \simeq z(1) \simeq \dots \simeq z(L) \simeq x_L.$$

Proof. Since $z(0) \in B_1(x_0)$ and $z(L) \in B_1(x_L)$, triangle inequality gives

$$\begin{aligned} L = d(x_0, x_L) &\leq d(x_0, z(0)) + d(z(0), z(L)) + d(z(L), x_L) \\ &= \mathbb{1}_{\{x_0 \neq z(0)\}} + \ell(g_z) + \mathbb{1}_{\{x_L \neq z(L)\}}. \end{aligned}$$

Note also that $z(0) = x_0$ and $z(L) = x_L$ cannot happen simultaneously. Otherwise, it means that $\ell(g_{x_0}) = L$ which would imply that the geodesic g_{x_0} contains all distinct vertices $x_0(0) \sim x_0(1) \sim \dots \sim x_0(L)$, contradicting to the repetition $x_0 = x_0(0) = x_0(1)$. Therefore,

$$\ell(g_z) \geq \begin{cases} L - 1 & , \text{ if } z(0) = x_0 \text{ or } z(L) = x_L \\ L - 2 & , \text{ otherwise.} \end{cases} \tag{7.4}$$

and

$$\sum_{z \in B_1(x_0)} \ell(g_z) \geq 2(L - 1) + (D - 1)(L - 2). \tag{7.5}$$

On the other hand, from (7.3), T^L has the cost of $L - \frac{2D}{D+1}$. It follows that

$$\frac{1}{D + 1} \sum_{z \in B_1(x_0)} \ell(g_z) = \sum_{z \in B_1(x_0)} d(z, T^L(z)) \frac{1}{D + 1}$$

$$\begin{aligned} &= \text{cost}(T^L) = L - \frac{2D}{D + 1} \\ &= \frac{1}{D + 1} \left(2(L - 1) + (D - 1)(L - 2) \right). \end{aligned}$$

which implies that the equality holds true in (7.5), and also in (7.4) as desired. \square

7.3. Antipoles of intervals in a self-centered Bonnet-Myers sharp graph

We still assume that our graph $G = (V, E)$ is a self-centered (D, L) -Bonnet-Myers sharp graph. Henceforth we will use the following notation related to intervals: Given an interval $[x, y] \subset V$ in G and a vertex $z \in [x, y]$, we call a vertex $\bar{z} \in [x, y]$ an *antipole of z w.r.t. $[x, y]$* if $d(x, y) = d(z, \bar{z})$. Note that antipoles were already introduced for graphs and this definition simply means that z and \bar{z} are antipoles of the induced subgraph of $[x, y]$. We now focus on identifying antipoles w.r.t. intervals via the method of transport geodesics.

Theorem 7.5. *Let $G = (V, E)$ be a self-centered (D, L) -Bonnet-Myers sharp, and given a full-length geodesic g :*

$$g : x_0 \sim x_1 \sim \dots \sim x_L.$$

Then for any $2 \leq k \leq L$, x_1 has a unique antipole w.r.t. the interval $[x_0, x_k]$, which we will then denote as $\text{ant}_{[x_0, x_k]}(x_1)$. In fact, we show that

$$\text{ant}_{[x_0, x_k]}(x_1) = x_0(k) = T^k(x_0) \in B_1(x_k)$$

for any fixed $\{T_j, T^j\}_{j=1}^L$ defined in Subsection 7.2.

Proof. First, fix a set of transport maps $\{T_j, T^j\}_{j=1}^L$ associated to g . Suppose that there exists $z \in [x_0, x_k]$ which is an antipole of x_1 w.r.t. $[x_0, x_k]$, that is $z \in [x_0, x_k]$ and $d(x_1, z) = d(x_0, x_k) = k$. Since $x_1 \sim x_0$ and $d(x_1, z) = k$, we have $d(x_0, z) \geq k - 1$. Since $z \in [x_0, x_k]$ and $z \neq x_k$, we must have $d(x_0, z) = k - 1$ and $d(z, x_k) = 1$. Since $z \in B_1(x_k)$, there is a unique $a \in B_1(x_0)$ such that $a(k) = z$, that is $a = (T^k)^{-1}(z)$ (because T^k is a bijective map). By Proposition 7.4,

$$\begin{aligned} x_0 &\simeq a(0) \simeq a(1) \simeq \dots \simeq a(k) \\ &\qquad\qquad\qquad\parallel \\ &\qquad\qquad\qquad z \end{aligned}$$

is part of the geodesic $\text{ext}(g_a)$, so it is also a geodesic. Therefore it satisfies

$$k - 1 = d(x_0, z) = d(x_0, a(0)) + d(a(0), a(1)) + d(a(1), z). \tag{7.6}$$

On the other hand, since $d(x_1, z) = k$ and $a(1) \in B_1(x_1)$, triangle inequality gives $d(a(1), z) \geq k - 1$. Equation (7.6) then implies $x_0 = a(0) = a(1)$, which means $a = x_0$. Thus $z = a(k) = x_0(k)$. So far we have shown that, for every $2 \leq k \leq L$, $x_0(k)$ is the only candidate for an antipole of x_1 w.r.t. $[x_0, x_k]$. It remains to show that $x_0(k)$ is in fact the antipole of x_1 w.r.t. $[x_0, x_k]$.

In particular, when $k = L$, the antipole of x_1 w.r.t. $[x_0, x_L] = V$ exists by the assumption that G is self-centered. Denote this antipole by \bar{x}_1 . By the previous argument, $x_0(L)$ must be \bar{x}_1 , $d(x_1, x_0(L)) = L$.

Consider the transport geodesic g_{x_0} :

$$g_{x_0} : \quad \begin{array}{ccc} x_0(0) = x_0(1) \simeq x_0(2) \simeq \dots \simeq x_0(L) & & \\ \parallel & & \parallel \\ x_0 & & \bar{x}_1 \end{array}$$

Since g_{x_0} has length $L - 1$ (by Proposition 7.4), all the “ \simeq ” in g_{x_0} must be strict “ \sim ”. Thus g_{x_0} can be written as

$$g_{x_0} : \quad \begin{array}{ccc} x_0(0) = x_0(1) \sim x_0(2) \sim \dots \sim x_0(L) & & \\ \parallel & & \parallel \\ x_0 & & \bar{x}_1 \end{array}$$

Moreover, since g_{x_0} has length $L - 1$ and $d(x_1, \bar{x}_1) = L$, the geodesic g_{x_0} can then be extended (by adding x_1 to the left) to another geodesic g' :

$$g' : \quad \begin{array}{ccc} x_1 \sim x_0(0) = x_0(1) \sim x_0(2) \sim \dots \sim x_0(L) & & \\ \parallel & & \parallel \\ x_0 & & \bar{x}_1 \end{array}$$

Consequently, we can read off from the geodesic g' that for every $k \in \{2, \dots, L\}$

1. $d(x_1, x_0(k)) = k$,
2. $x_0(k) \in [x_0, x_k]$, because $k = d(x_0, x_k) \leq d(x_0, x_0(k)) + d(x_0(k), x_k) \leq (k - 1) + 1$.

Therefore, $x_0(k)$ is the unique antipole of x_1 w.r.t. $[x_0, x_k]$ as desired. \square

Let us first discuss an immediate consequence of Theorem 7.5. Note that the theorem implies that there is a well-defined antipole map

$$\text{ant}_{[x,y]} : [x, y] \cap B_1(x) \rightarrow [x, y] \cap B_1(y).$$

Existence and uniqueness of antipoles for neighbours of x w.r.t. $[x, y]$ implies the following result:

Corollary 7.6. *Let $G = (V, E)$ be a self-centered Bonnet-Myers sharp graph, $x, y \in V$ be two different vertices. Then the antipole map*

$$\text{ant}_{[x,y]} : [x, y] \cap B_1(x) \rightarrow [x, y] \cap B_1(y)$$

is bijective and, consequently,

$$|[x, y] \cap B_1(x)| = |[x, y] \cap B_1(y)|.$$

Remark 7.7. Let $x, y \in V$ be two different vertices and $x' \in [x, y] \cap B_1(x)$ with its antipole $y' = \text{ant}_{[x,y]}(x')$. Observe that then $x, y \in [x', y']$ and $y = \text{ant}_{[x',y']}(x)$.

Another immediate consequence of Theorem 7.5 is the following corollary.

Corollary 7.8. Let $G = (V, E)$ be a self-centered Bonnet-Myers sharp graph. Then all μ -graphs of G are cocktail party graphs.

Proof. Let $x, y \in V$ with $d(x, y) = 2$ and $z \in N_{xy}$. Since G is self-centered Bonnet-Myers sharp, $x = x_0$ has an antipole $x_L \in V$, and we can find a geodesic g from x_0 to x_L passing through $z = x_1$ and $y = x_2$ by Theorem 5.5:

$$g : \quad x_0 \sim x_1 \sim x_2 \sim \dots \sim x_L.$$

$$\quad \quad \parallel \quad \parallel \quad \parallel$$

$$\quad \quad x \quad \quad z \quad \quad y$$

Applying Theorem 7.5 with $k = 2$, we conclude that there is a unique $z' \in N_{xy}$ satisfying $d(z, z') = 2$. This shows that the μ -graph of x and y is a cocktail party graph. \square

Remark 7.9. The fact that all μ -graphs of G are cocktail party graphs allows us to naturally introduce a *switching map*, defined as follows. Consider a pair $x, y \in V$ with $d(x, y) = 2$. Then the switching map $\sigma_{xy} : N_{xy} \rightarrow N_{xy}$ is defined by $\sigma_{xy}(z) := \text{ant}_{[x,y]}(z)$ and satisfies $\sigma_{xy}^2 = \text{Id}_{N_{xy}}$.

Remark 7.10. Recall from Remark 7.3 that for general Bonnet-Myers sharp graphs the perfect matchings defining the maps T_j are not necessarily unique. However, under the additional condition of self-centeredness, the fact that all μ -graphs of G are cocktail party graphs implies the uniqueness of these perfect matchings and the associated transport maps T_j . Therefore, the definition of a transport geodesic g_z depends only on the geodesic g .

In particular, the transport geodesic g_{x_0} containing all antipoles of x_1 w.r.t. increasing intervals $[x_0, x_k]$ (see Theorem 7.5) can be also understood as been generated via the following recursive process of switching maps, as illustrated in Fig. 1:

$$x_0(2) = \sigma_{x_0x_2}(x_1),$$

$$x_0(3) = \sigma_{x_0(2)x_3}(x_2),$$

$$\vdots$$

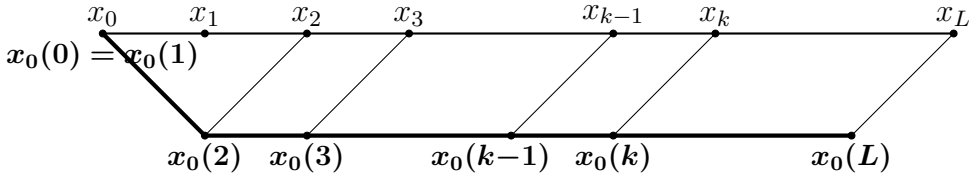


Fig. 1. Transport geodesic g_{x_0} (along g) shown in bold.

$$\begin{aligned}
 x_0(k) &= \sigma_{x_0(k-1)x_k}(x_{k-1}), \\
 &\vdots \\
 x_0(L) &= \sigma_{x_0(L-1)x_L}(x_{L-1}).
 \end{aligned}$$

8. Self-centered Bonnet-Myers sharp implies strongly spherical

The ultimate goal of this section is to prove that all self-centered Bonnet-Myers sharp graphs are strongly spherical (as stated in Theorem 8.8 below). Let us recall the definition of self-centeredness, antipodal, and strongly spherical (introduced in Definition 1.11 and in Subsection 2.1). For a finite connected graph $G = (V, E)$:

- G is self-centered if for every $x \in V$ there exists $\bar{x} \in V$ such that $d(x, \bar{x}) = \text{diam}(G)$.
- G is antipodal if for every $x \in V$ there exists $\bar{x} \in V$ such that $[x, \bar{x}] = V$. The vertex \bar{x} is then called an *antipode* of x .
- G is strongly spherical if G is antipodal, and the induced subgraph of every interval of G is antipodal.

Remark 8.1. It is important to notice the distinction between the notions “antipole” and “antipode”. Here are basic facts about antipodes:

- Antipodes are also antipoles: Let \bar{x} be an antipode of x in G , that is, $[x, \bar{x}] = V$. We choose arbitrary $y, z \in V$ such that $d(y, z) = \text{diam}(G)$. Then we have by $y, z \in [x, \bar{x}]$ and the triangle inequality

$$\begin{aligned}
 \text{diam}(G) \geq d(x, \bar{x}) &= \frac{1}{2}(d(x, y) + d(y, \bar{x})) + \frac{1}{2}(d(x, z) + d(z, \bar{x})) \\
 &= \frac{1}{2}(d(y, x) + d(x, z)) + \frac{1}{2}(d(y, \bar{x}) + d(\bar{x}, z)) \geq d(y, z) = \text{diam}(G).
 \end{aligned}$$

- Antipodes are necessarily unique: Assume \bar{x}_1 and \bar{x}_2 are antipodes of x . Then \bar{x}_2 lies on a geodesic from x to \bar{x}_1 . Since $d(x, \bar{x}_1) = d(x, \bar{x}_2)$, this implies $\bar{x}_1 = \bar{x}_2$.

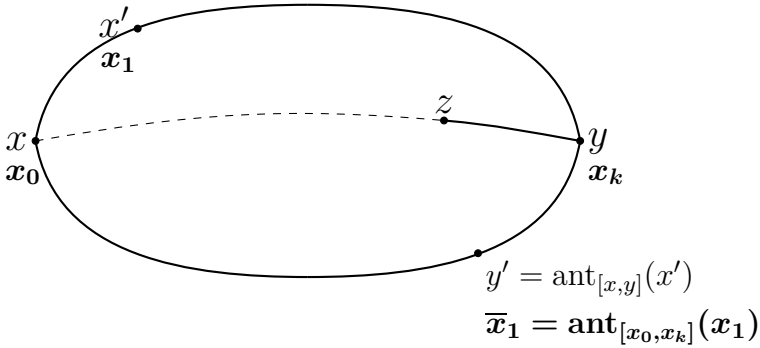


Fig. 2. The interval $[x, y]$ with the re-labelled vertices in bold, and $z \in [x_0, x_k] \cap B_1(x_k)$.

For the reader’s convenience, let us start with a brief overview of the proof that self-centered Bonnet-Myers sharp graphs are strongly spherical. Note first that every self-centered Bonnet-Myers sharp graph coincides with the interval of any pair of antipoles (by Theorem 5.5). Therefore, it suffices to prove that every interval $[x, y]$, $x, y \in V$, of a self-centered Bonnet-Myers sharp graph $G = (V, E)$ is antipodal. This proof is divided into the following four steps:

- Step 1:** Let $x' \in [x, y] \cap S_1(x)$ with antipole $y' = \text{ant}_{[x,y]}(x')$. We prove for every $z \in [x, y] \cap B_1(x)$ that $z \in [x', y']$ (see Theorem 8.2).
- Step 2:** Let $x' \in [x, y] \cap S_1(x)$ with antipole $y' = \text{ant}_{[x,y]}(x')$. We prove for every $z \in [x, y]$ that $z \in [x', y']$ (see Theorem 8.6).
- Step 3:** Let $x' \in [x, y] \cap S_1(x)$ with antipole $y' = \text{ant}_{[x,y]}(x')$. We prove that $[x, y] = [x', y']$ (see Corollary 8.7).
- Step 4:** Let $x' \in [x, y]$. We prove that there exists $y' \in [x, y]$ such that $[x, y] = [x', y']$ (see Theorem 8.8).

Let us now start to prove each of these steps in order. Recall that the existence of antipoles of vertices in $[x, y] \cap B_1(x)$ w.r.t. $[x, y]$ is guaranteed by Corollary 7.6.

Theorem 8.2. *Let $G = (V, E)$ be self-centered Bonnet-Myers sharp. Let $x, y \in V$ be two different vertices, and consider any $x' \in [x, y] \cap S_1(x)$ with its antipole $y' = \text{ant}_{[x,y]}(x')$. Then every $z \in [x, y] \cap B_1(y)$ satisfies $z \in [x', y']$.*

We start with the set-up and introduce particular sets A, A_1, A_2, Z, Z_1, Z_2 and a function F which will be important for the proof of the above theorem.

Let $k = d(x, y)$. We re-label the vertices as $x = x_0$ and $y = x_k$ and $x' = x_1$ and $y' = \bar{x}_1$, as illustrated in Fig. 2. Keep in mind that $\bar{x}_1 = \text{ant}_{[x_0, x_k]}(x_1)$ and $x_0 \sim x_1$ and $x_k \sim \bar{x}_1$.

We define the following sets

$$\begin{aligned} A &:= [x_0, x_k] \cap B_1(x_0), & Z &:= [x_0, x_k] \cap B_1(x_k), \\ A_1 &:= A \cap S_1(x_1) \setminus \{x_0\}, & Z_1 &:= Z \cap S_1(\bar{x}_1) \setminus \{x_k\}, \\ A_2 &:= A \cap S_2(x_1), & Z_2 &:= Z \cap S_2(\bar{x}_1). \end{aligned}$$

Note that the sets A and Z can be partitioned into

$$A = \{x_0, x_1\} \sqcup A_1 \sqcup A_2 \quad \text{and} \quad Z = \{x_k, \bar{x}_1\} \sqcup Z_1 \sqcup Z_2.$$

Now fix an arbitrary full-length geodesic g from x_0 to x_L (the antipole of x_0) which passes through x_1 and x_k (this can be done since $x_1 \in [x_0, x_k]$ and $x_k \in [x_0, x_L]$ by Theorem 5.5), namely

$$g : \quad x_0 \sim x_1 \sim x_2 \sim \dots \sim x_k \sim x_{k+1} \sim \dots \sim x_L.$$

Consider the transport map $T^k : B_1(x_0) \rightarrow B_1(x_k)$ introduced in Subsection 7.2. Recall that T^k is bijective. Then define a function $F : Z \rightarrow A$ to be $F(z) := (T^k)^{-1}(z)$ for all $z \in Z \subset B_1(x_k)$. Lemma 8.3 below guarantees that $F(Z) \subseteq A$, hence F is well-defined.

In order to conclude Theorem 8.2, we need to prove that $\forall z \in Z : z \in [x_1, \bar{x}_1]$, which is divided into Lemma 8.4 (dealing with the case $z \in Z_2 \sqcup \{x_k, \bar{x}_1\}$) and Lemma 8.5 (dealing with the case $z \in Z_1$).

Lemma 8.3. $F(Z) \subseteq A$ and $F : Z \rightarrow A$ is bijective.

Lemma 8.4. $F(Z_2 \sqcup \{x_k, \bar{x}_1\}) = A_2 \sqcup \{x_0, x_1\}$ and $\forall z \in Z_2 \sqcup \{x_k, \bar{x}_1\} : z \in [x_1, \bar{x}_1]$.

Lemma 8.5. $F(Z_1) = A_1$ and $\forall z \in Z_1 : z \in [x_1, \bar{x}_1]$.

Now we will prove the above three lemmas in order, and then conclude Theorem 8.2.

Proof of Lemma 8.3. First we show that $F(z) \in A$ for all $z \in Z$. Let $a := F(z) \in B_1(x_0)$, that is $a = a(0)$ and $z = a(k)$. By Proposition 7.4 we know that

$$x_0 \simeq a(0) \simeq a(1) \simeq \dots \simeq a(k)$$

is a geodesic (as a part of $\text{ext}(g_a)$). Moreover, since $a(k) = z \in [x_0, x_k]$, this geodesic can be extended to another geodesic γ , namely

$$\begin{aligned} \gamma : \quad x_0 &\simeq a(0) \simeq a(1) \simeq \dots \simeq a(k) \simeq x_k. & (8.1) \\ &\parallel & \parallel \\ &a & z \end{aligned}$$

Therefore, a must lie in the interval $[x_0, x_k]$, which means $a \in A$ and we have $F(Z) \subseteq A$.

Next, note that the function $F : Z \rightarrow A$, which is a restriction of $(T^k)^{-1}$, must be injective (because T^k is bijective). Note also that $|A| = |Z|$ because of Corollary 7.6. Therefore, F must be bijective. \square

Proof of Lemma 8.4. A main feature of the following proof is to show $A_2 \sqcup \{x_0, x_1\} \subseteq F(Z_2 \sqcup \{x_k, \bar{x}_1\})$. For that reason we start with an element $a \in A_2 \sqcup \{x_0, x_1\}$. Then there exists a unique $z \in Z$ with $F(z) = a$. Consequently, $z = a(k)$ and $z \in [x_0, x_k]$. Consider the following two cases.

Case $a = x_0$: From Theorem 7.5, we have $a = x_0 = (T^k)^{-1}(\bar{x}_1) = F(\bar{x}_1)$, so $z = \bar{x}_1$ and $z \in [x_1, \bar{x}_1]$.

Case $a \in A_2 \sqcup \{x_1\}$: As in the proof of Lemma 8.3, we have the following geodesic γ of length k (referred to the one in (8.1)):

$$\gamma : \quad x_0 \sim a(0) \simeq a(1) \simeq \dots \simeq a(k) \simeq x_k.$$

$$\qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel$$

$$\qquad \qquad \qquad a \qquad \qquad \qquad z$$

From this geodesic γ and an observation that

$$d(x_0, a(1)) = \begin{cases} 1, & \text{if } a = x_1 \text{ (and therefore also } a(1) = x_1), \\ 2, & \text{if } a \in A_2 \text{ (and therefore } a(1) \neq a(0)), \end{cases}$$

we conclude

$$d(a(1), x_k) = \begin{cases} k - 1, & \text{if } a = x_1, \\ k - 2, & \text{if } a \in A_2. \end{cases} \tag{8.2}$$

Now we extend the geodesic

$$a(1) \simeq \dots \simeq a(k) \simeq x_k$$

$$\qquad \qquad \qquad \parallel$$

$$\qquad \qquad \qquad z$$

to

$$x_1 \simeq a(1) \simeq \dots \simeq a(k) \simeq x_k \sim \bar{x}_1,$$

$$\qquad \qquad \qquad \parallel$$

$$\qquad \qquad \qquad z$$

which is, again a geodesic because of (8.2) (and recall that $d(x_1, \bar{x}_1) = k$). We can then read off from the above geodesic that $z = x_k$ or $z \in S_2(\bar{x}_1) \cap [x_0, x_k] = Z_2$ and $z \in [x_1, \bar{x}_1]$.

We conclude from both cases that $A_2 \sqcup \{x_0, x_1\} \subseteq F(Z_2 \sqcup \{x_k, \bar{x}_1\})$. Since F is bijective, it follows that $|A_2| \leq |Z_2|$. By switching the roles between x_0 and x_k and between

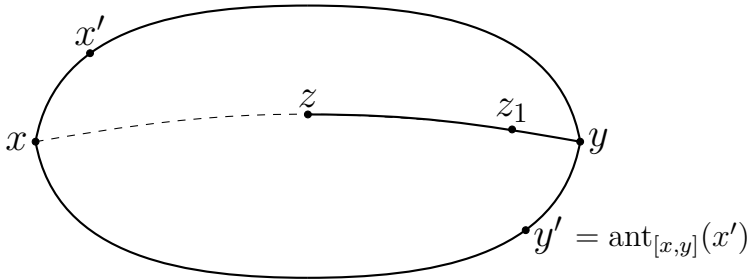


Fig. 3. The interval $[x, y]$ and $z \in [x, y]$ with $d(z, y) = j \geq 2$ and $z_1 \in [z, y] \cup S_1(y)$.

the antipoles x_1 and \bar{x}_1 w.r.t. $[x_0, x_k]$, we obtain the opposite inequality $|Z_2| \leq |A_2|$. Therefore, we have $|Z_2| = |A_2|$, and thus $A_2 \sqcup \{x_0, x_1\} = F(Z_2 \sqcup \{x_k, \bar{x}_1\})$, as desired.

Consequently, if we consider any $z \in Z_2 \sqcup \{x_k, \bar{x}_1\}$, then $a \in A_2 \sqcup \{x_0, x_1\}$ falls into one of the above cases, in which we have shown $z \in [x_1, \bar{x}_1]$. \square

Proof of Lemma 8.5. Since $F : Z \rightarrow A$ is bijective and $F(Z_2 \sqcup \{x_k, \bar{x}_1\}) = A_2 \sqcup \{x_0, x_1\}$, we conclude $F(Z_1) = A_1$.

Moreover, consider $z \in Z_1$. It follows that $z \sim \bar{x}_1$ and $d(x_1, z) \leq k - 1$, because $z \neq \bar{x}_1 = \text{ant}_{[x_0, x_k]}(x_1)$. Therefore

$$d(x_1, z) + d(z, \bar{x}_1) = d(x_1, z) + 1 \leq (k - 1) + 1 = k,$$

which means $z \in [x_1, \bar{x}_1]$. \square

Proof of Theorem 8.2. Recalling the original set-up and notation, we only need to show that $z \in [x_1, \bar{x}_1]$. This follows immediately from Lemma 8.4 and Lemma 8.5. \square

The next theorem generalized Theorem 8.2 by removing the restriction $z \in B_1(y)$.

Theorem 8.6. Let $G = (V, E)$ be self-centered Bonnet-Myers sharp. Let $x, y \in V$ be two different vertices, and consider any $x' \in [x, y] \cap S_1(x)$ with its antipole $y' = \text{ant}_{[x,y]}(x')$. Then every $z \in [x, y]$ satisfies $z \in [x', y']$.

Proof of Theorem 8.6. Let $d_1 = d(x, y)$ and $d_2 = d(z, y)$ (note that $0 \leq d_2 \leq d_1$). We will prove the statement of the theorem by induction on d_1 and d_2 .

Base step: For any value of d_1 , the cases $d_2 = 0, 1$ are both covered by Theorem 8.2.

Inductive step: Assume that the statement is true for $d_1 = k - 1$ and all d_2 , and assume that the statement is true for $d_1 = k$ and $d_2 = j - 1$ for some $2 \leq j \leq k - 1$. Now consider $d(x, y) = k$ and $z \in [x, y] \cap S_j(y)$. Choose an arbitrary $z_1 \in [z, y] \cap S_1(y)$. Hence x, z, z_1, y lies in a geodesic, see Fig. 3. In particular, $z \in [x, z_1]$.

Now consider the following three cases whether $d(z_1, y')$ is 0, 1, or 2.

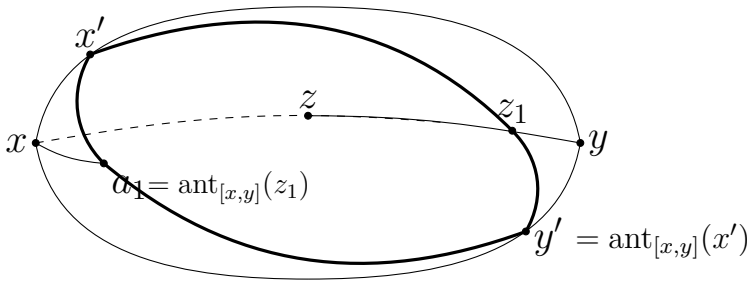


Fig. 4. Picture for Case $z_1 \sim y'$. The bold cycle represents the fact that a_1 and z_1 are antipoles w.r.t. not only $[x, y]$ but also $[x', y']$.

Case $z_1 = y'$: It follows immediately that $z \in [x, z_1] = [x, y'] \subseteq [x', y']$ where the last inclusion is due to $x \in [x', y']$.

Case $z_1 \sim y'$: Since $z_1 \in [x, y]$, by Theorem 7.5 there is a unique $a_1 = \text{ant}_{[x,y]}(z_1) \in [x, y]$. Since $a_1 \in [x, y] \cap B_1(x)$ and $z_1 \in [x, y] \cap B_1(y)$, by Theorem 8.2, $z_1, a_1 \in [x', y']$.

The fact that $a_1, z_1 \in [x', y']$ and that $d(a_1, z_1) = d(x, y) = d(x', y')$ altogether implies that a_1 must be the unique antipole $\text{ant}_{[x',y']}(z_1)$ by Corollary 7.6 since $z_1 \sim y'$. This is illustrated in Fig. 4. By Remark 7.7, it implies that $y' = \text{ant}_{[a_1,z_1]}(x')$.

Observe also that $z \in [x, z_1] \subset [a_1, z_1]$ with $d(z, z_1) = j - 1$.

We are now in a position to apply the induction hypothesis for the interval $[a_1, z_1]$ (instead of $[x, y]$) and $z \in [a_1, z_1]$ with $d(z, z_1) = j - 1$. Note that $d(a_1, z_1) = k$. Note also that $x' \in [a_1, z_1] \cap S_1(a_1)$ and $y' = \text{ant}_{[a_1,z_1]}(x')$. Then the induction hypothesis implies $z \in [x', y']$, finishing this case.

Case $d(z_1, y') = 2$: Since $z_1 \in [x, y] \cap B_1(y)$, by Theorem 8.2, we have $z_1 \in [x', y']$. The condition $d(z_1, y') = 2$ then implies that $d(x', z_1) = d(x', y') - 2 = k - 2$. It follows that

$$d(x, x') + d(x', z_1) + d(z_1, y) = 1 + (k - 2) + 1 = k = d(x, y)$$

which means that x' and z_1 lie on a geodesic from x to y . Let us denote this geodesic by g^* :

$$g^* : x \sim x' \sim \dots \sim z_1 \sim y.$$

In particular, $x' \in [x, z_1]$. Then $y'' := \text{ant}_{[x,z_1]}(x')$ exists by Corollary 7.6. The situation is illustrated in Fig. 5.

Next we apply the induction hypothesis for the interval $[x, z_1]$ (instead of $[x, y]$) and $z \in [x, z_1]$ with $d(z, z_1) = j - 1$. Note that $d(x, z_1) = k - 1$. Note also that $x' \in [x, z_1] \cap S_1(x)$ and $y'' = \text{ant}_{[x,z_1]}(x')$. Then the induction hypothesis implies $z \in [x', y'']$.

So far we have that $d(x', z) + d(z, y'') = d(x', y'') = d(x, z_1) = k - 1$. It remains to show that $d(y'', y') = 1$ which would imply

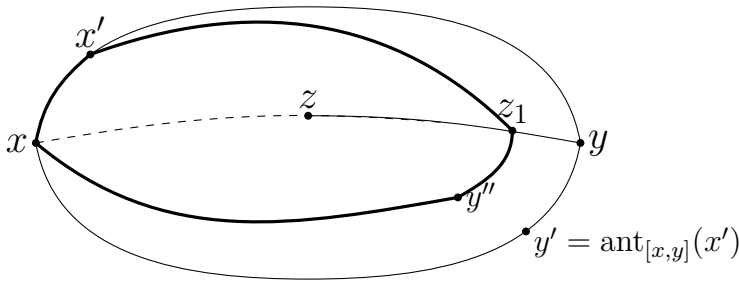


Fig. 5. Picture for Case $d(z_1, y') = 2$. The bold cycle represents $y'' = \text{ant}_{[x, z_1]}(x')$.

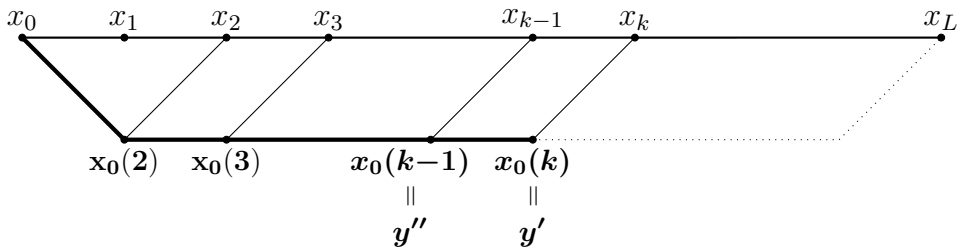


Fig. 6. Transport geodesic g_{x_0} and the antipoles of x_1 w.r.t. increasing intervals $[x_0, x_m]$.

$$k = d(x', y') \leq d(x', z) + d(z, y'') + d(y'', y') = (k - 1) + 1 = k,$$

that is $z \in [x', y']$, as desired.

To prove $d(y'', y') = 1$ we use transport geodesic techniques. Therefore, we relabel the vertices of the geodesic g^* and extend g^* to a full-length geodesic g in G starting from $x = x_0$ as follows:

$$g : \quad x \sim x' \sim \dots \sim z_1 \sim y \sim x_{k+1} \sim \dots \sim x_L,$$

$$\begin{array}{ccccccc} & \parallel & & \parallel & & \parallel & \parallel \\ & x_0 & & x_1 & & x_{k-1} & x_k \end{array}$$

and consider the transport geodesic along g starting at x_0 .

Theorem 7.5 guarantees that $x_0(m) = \text{ant}_{[x_0, x_m]}(x_1)$ for all $2 \leq m \leq L$. In particular, we have $y'' = \text{ant}_{[x_0, x_{k-1}]}(x_1) = x_0(k - 1)$ and $y' = \text{ant}_{[x_0, x_k]}(x_1) = x_0(k)$. Therefore, $y'' = x_0(k - 1)$ and $y' = x_0(k)$ must be adjacent vertices (as illustrated in Fig. 6), thus completing the proof. \square

An immediate but important consequence of the above theorem is the following corollary.

Corollary 8.7. *Let $G = (V, E)$ be self-centered Bonnet-Myers sharp. Let $x, y \in V$ be two different vertices, and consider any $x' \in [x, y] \cap S_1(x)$ with its antipole $y' = \text{ant}_{[x, y]}(x')$. Then $[x', y'] = [x, y]$.*

Proof of Corollary 8.7. Theorem 8.6 can be rephrased as $[x, y] \subseteq [x', y']$. Since $y = \text{ant}_{[x', y']}(x)$ by Remark 7.7, we can interchange the roles of x, y and x', y' to obtain the opposite inclusion $[x', y'] \subseteq [x, y]$. Therefore $[x', y'] = [x, y]$, as desired. \square

Now, we are ready to conclude the ultimate result of this section by using Corollary 8.7 inductively.

Theorem 8.8. *Let $G = (V, E)$ be self-centered Bonnet-Myers sharp. Then for two different vertices $x, y \in V$, the induced subgraph of the interval $[x, y]$ is antipodal. Therefore G is strongly spherical.*

Proof of Theorem 8.8. Let $x' \in [x, y]$. The existence of a vertex $y' \in [x, y]$ satisfying

$$[x, y] = [x', y']$$

is proved via induction on $d(x, x')$.

Base step: The case $d(x, x') = 0$ is trivial and $d(x, x') = 1$ is covered by Corollary 8.7.

Inductive step: We assume the statement of the Theorem is true for all $d(x, x') \leq m - 1$ with $2 \leq m \leq \text{diam}(G)$. Let $x' \in [x, y]$ with $d(x, x') = m$. We choose a vertex $x_1 \in [x, y] \cap S_1(x)$ such that x_1, x' lie on a geodesic from x to y . By the induction hypothesis, there exists $y_1 \in [x, y]$ such that

$$[x, y] = [x_1, y_1].$$

Since $d(x_1, x') = m - 1$, the induction hypothesis again implies the existence of $y' \in [x_1, y_1] = [x, y]$ such that

$$[x_1, y_1] = [x', y'] = [x, y].$$

This finishes the proof. \square

9. Classification of Bonnet-Myers sharp graphs with extremal diameters

In this section we show that there are no Bonnet-Myers sharp graphs of extremal diameter (that is $L = 2$ and $L = D$) which are not self-centered. In other words, the only Bonnet-Myers sharp graphs with diameter $L = 2$ are cocktail party graphs and the only Bonnet-Myers sharp graphs with diameter $L = D$ are hypercubes.

9.1. Characterisation of sharpness for $L = 2$

Theorem 9.1. *Let $G = (V, E)$ be a $(D, 2)$ -Bonnet Myers sharp graph. Then G is isomorphic to a cocktail party graph $CP(n)$ for $n \geq 2$.*

Proof. Let us first show that all Bonnet-Myers sharp graphs of diameter $L = 2$ are necessarily self-centered: if there were a $(D, 2)$ -Bonnet-Myers sharp graph which is not self-centered, it would have a vertex adjacent to all other vertices and, by D -regularity, would have to be the complete graph K_{D+1} , which does not have diameter 2. Now we employ the classification Theorem 1.6 for self-centered Bonnet-Myers sharp graphs and conclude that all Bonnet-Myers sharp graphs of diameter $L = 2$ are cocktail party graphs $CP(n)$, $n \geq 2$. \square

9.2. Characterisation of sharpness for $L = D$

We now show that the only Bonnet-Myers sharp graphs with diameter equal to their degree are the hypercubes.

Lemma 9.2. *Let $G = (V, E)$ be a D -regular graph. Suppose an edge $\{x, y\} \in E$ is contained in no triangle and satisfies $\kappa(x, y) \geq \frac{2}{D}$. Then, every pair of adjacent edges $w \sim x \sim y$ is contained in a 4-cycle.*

Proof. This follows immediately from Proposition 2.7. \square

We recall the definitions of the small sphere structure and the non-clustering property from [14] which have been the key concepts to prove the rigidity result under sharpness of Bonnet-Myers in the Bakry-Émery ∞ -curvature (see [14]).

Definition 9.3. Let $G = (V, E)$ be a D -regular graph and let $x \in V$.

(SSP) We say x satisfies the *small sphere property* (SSP) if

$$|S_2(x)| \leq \binom{D}{2}.$$

(NCP) We say x satisfies the *non-clustering property* (NCP) if, whenever $d_x^-(z) = 2$ holds for all $z \in S_2(x)$, one has that for all distinct $y_1, y_2 \in S_1(x)$ there is at most one $z \in S_2(x)$ satisfying $y_1 \sim z \sim y_2$.

Remark 9.4. For the arguments below, it is useful to understand structural properties of the hypercube Q^n . We view the vertices of Q^n as the elements of $\{0, 1\}^n$ which are connected if their Hamming distance is equal to 1, and assume without loss of generality that $x = (0, 0, \dots, 0)$. Then for $1 \leq k \leq n$,

$$S_k(x) = \{(a_i)_i \in \{0, 1\}^n \mid \sum_i a_i = k\},$$

which gives $\#S_k(x) = \binom{D}{k}$. In particular, Q^n satisfies (SSP).

Moreover, for distinct $y_1, y_2 \in S_k(x)$ we always have $d(y_1, y_2) \geq 2$. In the case $d(y_1, y_2) = 2$, the entries of y_1 and y_2 differ in precisely two places and, consequently, y_1, y_2 has precisely one common neighbour in $S_{k-1}(x)$ (if $k \geq 1$) and one common neighbour in $S_{k+1}(x)$ (if $k \leq n - 1$). Therefore, Q^n satisfies also (NCP).

Lemma 9.5. *Let $G = (V, E)$ be a D -regular graph. If $x \in V$ belongs to no triangle and if $\kappa(x, y) \geq 2/D$ for all $y \sim x$, then x satisfies (SSP) and (NCP).*

Proof. We first show $|S_2(x)| \leq \binom{D}{2}$. Let $z \in S_2(x)$ and let $y \in V$ s.t. $x \sim y \sim z$. Due to Lemma 9.2, z is connected to at least two vertices from $S_1(x)$. By double counting, we have

$$2|S_2(x)| \leq \sum_{z \in S_2(x)} d_x^-(z) = \sum_{y \in S_1(x)} d_x^+(y) \leq (D - 1)|S_1(x)| = (D - 1)D,$$

which implies $|S_2(x)| \leq \binom{D}{2}$. Therefore x satisfies (SSP).

Next we prove (NCP) at x : For all distinct $y_1, y_2 \in S_1(x)$ the pair of adjacent edges $y_1 \sim x \sim y_2$ is contained in a 4-cycle by Lemma 9.2, which means that there is $z \in S_2(x)$ with $y_1 \sim z \sim y_2$. Since $|S_2(x)| \leq \binom{D}{2}$, there is at most one such z for each such pair $y_1, y_2 \in S_1(x)$. \square

Now we state the main theorem of this section.

Theorem 9.6. *Let $G = (V, E)$ be (D, L) -Bonnet-Myers sharp with $L = D$. Then G is the hypercube Q^D .*

Proof. Let x be a pole. We write $B_N := B_N(x)$ and $S_N := S_N(x)$.

By Theorem 5.7(a), x is not contained in any triangles, therefore, $B_1(x)$ is isomorphic to the 1-ball in Q^D .

Now suppose, the N -ball B_N is isomorphic to the N -ball of the hypercube with $1 \leq N < D$. We want to show that the $(N + 1)$ -ball B_{N+1} is then isomorphic to the $(N + 1)$ -ball of the hypercube, which would prove the theorem by induction.

By (5.3) in Theorem 5.8 and the fact that $d_x^0(z) = 0$ for all $z \in S_N$ (because of the structure of the N -ball in the hypercube), we observe $d_x^+(z) = D - N$ for all $z \in S_N$. Let $M := \{\{v, w\} \subset S_N : d(v, w) = 2\}$. Since the N -ball B_N is isomorphic to the N -ball of the hypercube, we have

$$|M| \geq \binom{D}{N - 1} \binom{D - N + 1}{2}. \tag{9.1}$$

Again, due to the structure of the N -ball in the hypercube, any pair $\{v, w\} \in M$ cannot have any common neighbours in S_N and can have at most one common neighbour in S_{N-1} . Therefore, due to Lemma 9.2, since $\kappa \geq \frac{2}{D}$, there exists $p : M \rightarrow S_{N+1}$ satisfying $v \sim p(\{v, w\}) \sim w$ for all $\{v, w\} \in M$.

By (5.4) in Theorem 5.8, every $z \in S_{N+1}$ satisfies $d_x^-(z) \leq N + 1$. We classify the vertices in S_{N+1} by their backwards degree. Let a_s be the number of $z \in S_{N+1}$ with $d_x^-(z) = s$. Remark $a_s = 0$ for $s > N + 1$. Therefore, the set $E(S_{N+1}, S_N)$ of all edges joining S_N and S_{N+1} , satisfies

$$|E(S_{N+1}, S_N)| = \sum_{s \leq N+1} sa_s.$$

If $d_x^-(z) = s$ for some $z \in S_{N+1}$, then there are at most $\binom{s}{2}$ pairs $\{v, w\} \in M$ with $p(\{v, w\}) = z$. Thus,

$$|M| \leq \sum_{N+1 \geq s \geq 2} a_s \binom{s}{2} \leq \frac{N}{2} \sum_{s \leq N+1} sa_s = \frac{N}{2} |E(S_{N+1}, S_N)|. \tag{9.2}$$

Note that the second inequality in (9.2) is an equality iff $a_s = 0$ for all $s < N + 1$. Therefore, using (9.1) and (9.2),

$$|E(S_{N+1}, S_N)| \geq \frac{2}{N} |M| \geq \frac{2}{N} \binom{D}{N-1} \binom{D-N+1}{2} = \binom{D}{N} (D-N) = |E(S_{N+1}, S_N)|$$

where the last equality follows since $|S_N| = \binom{D}{N}$ and since every $z \in S_N$ satisfies $d_x^+(z) = D - N$. Therefore, we have sharpness everywhere which means $a_s = 0$ if $s \neq N + 1$, i.e., $d_x^-(z) = N + 1$ for all $z \in S_{N+1}$. This implies from (5.4) in Theorem 5.8 that

$$d_x^0(z) = 0 \quad \text{and} \quad d_x^+(z) = D - N - 1 \quad \text{for all } z \in S_{N+1} \tag{9.3}$$

and $|S_{N+1}| = \binom{D}{N+1}$.

Since B_N is isomorphic to the N -ball of the hypercube, any $y \in B_{N-1}$ is not contained in a triangle of G . Thus, we can apply Lemma 9.5 and conclude that (SSP) and (NCP) are satisfied for all $y \in B_{N-1}$. Using this fact and (9.3), we can apply [14, Theorem 6.2] (with $k = N + 1$) and conclude that B_{N+1} is isomorphic to the $(N + 1)$ -ball of the hypercube Q^D .

Note the following slight subtlety in this last argument: [14, Theorem 6.2] requires bipartiteness of G , which is a priori not known. Instead, we apply this theorem to a modification of G . This can be done by the following gluing process of the induced graph $B_{N+1}(x)$ with an $(L - N - 1)$ -ball of the hypercube Q^D : Let $B'_{L-N-1}(x')$ be an $(L - N - 1)$ -ball of a hypercube Q^D centered at x' and $S'_{L-N-1}(x')$ be the corresponding $(L - N - 1)$ -sphere. Since $|S_{N+1}(x)| = \binom{D}{N+1} = |S'_{L-N-1}(x')|$ and

$$d_x^+(z) = D - N - 1 = d_{x'}^-(z') \quad \text{and} \quad d_x^0(z) = 0 = d_{x'}^0(z')$$

for all $z \in S_{N+1}(x)$ and $z' \in S'_{L-N-1}(x')$, we can glue these two graphs via a bijective identification of the vertex sets of $S_{N+1}(x)$ and $S'_{L-N-1}(x')$. This guarantees that the new graph is bipartite and D -regular.

The proof is now finished by the induction principle. \square

10. Bonnet-Myers sharp graphs and Bakry-Émery curvature

10.1. Bakry-Émery curvature

Bakry-Émery curvature is a notion based on a fundamental identity in Riemannian Geometry, called *Bochner’s Formula*, involving the Laplace-Beltrami operator. This definition allows to introduce Bakry-Émery curvature also on other spaces with a well-defined Laplacian. The (normalized) Laplacian in our particular discrete setting of a graph G was given in (1.3). In this section, we will recall some fundamental properties which will be relevant for relating Bonnet-Myers sharpness in the sense of Ollivier Ricci curvature and Bakry-Émery curvature. More general details about Bakry-Émery curvature can be found in [8]. We start with Bakry-Émery’s Γ -calculus:

Definition 10.1 (*Γ and Γ_2 operators*). Let $G = (V, E)$ be a finite simple graph. For any two functions $f, g : V \rightarrow \mathbb{R}$, we define $\Gamma(f, g) : V \rightarrow \mathbb{R}$ and $\Gamma_2(f, g) : V \rightarrow \mathbb{R}$ by

$$\begin{aligned} 2\Gamma(f, g) &:= \Delta(fg) - f\Delta g - g\Delta f; \\ 2\Gamma_2(f, g) &:= \Delta\Gamma(f, g) - \Gamma(f, \Delta g) - \Gamma(\Delta f, g). \end{aligned}$$

We write $\Gamma(f) := \Gamma(f, f)$ and $\Gamma_2(f, f) := \Gamma_2(f)$, for short.

Definition 10.2 (*Bakry-Émery curvature*). Let $G = (V, E)$ be a finite simple graph. Let $\mathcal{K} \in \mathbb{R}$ and $\mathcal{N} \in (0, \infty) \cup \{\infty\}$. We say that a vertex $x \in V$ satisfies the *curvature-dimension inequality* $CD(\mathcal{K}, \mathcal{N})$ if, for any $f : V \rightarrow \mathbb{R}$, we have

$$\Gamma_2(f)(x) \geq \frac{1}{\mathcal{N}}(\Delta f(x))^2 + \mathcal{K}\Gamma(f)(x). \tag{10.1}$$

We call \mathcal{K} a lower Ricci curvature bound of G at x , and \mathcal{N} a dimension parameter. The graph $G = (V, E)$ satisfies $CD(\mathcal{K}, \mathcal{N})$ (globally), if all its vertices satisfy $CD(\mathcal{K}, \mathcal{N})$. Let $\mathcal{K}_{G,x}(\infty)$ be the largest real number such that the vertex x satisfies $CD(\mathcal{K}_{G,x}(\infty), \infty)$.

We now recall results from [8] which we will need for the rest of this section. Note that the Bakry-Émery curvature $\mathcal{K}_{G,x}$ in [8] is based on the non-normalized Laplacian which, in the case of D -regular graphs, can be easily translated into the normalized setting presented here. Henceforth, we will denote the Bakry-Émery curvature associated to the normalized Laplacian by $\mathcal{K}_{G,x}^n$ (for D -regular graphs, we have $\mathcal{K}_{G,x}^n = \frac{1}{D}\mathcal{K}_{G,x}$).

Let $G = (V, E)$ be a D -regular graph. Theorem 3.1 of [8] tells us that

$$\mathcal{K}_{G,x}^n(\infty) \leq \frac{2}{D} + \frac{\#\Delta(x)}{D^2} = \frac{3 + D - av_1^+(x)}{2D}, \tag{10.2}$$

for every $x \in V$, where $av_1^+(x)$ was defined in (2.2).

We say, as in [8], that a D -regular graph $G = (V, E)$ is ∞ -curvature sharp at $x \in V$ if (10.2) holds true with equality.

We now provide the following general result on Cartesian product, which will be useful in the next subsection.

Lemma 10.3. *Let $G_i = (V_i, E_i)$, $i = 1, 2$, be two connected, simple D_i -regular graphs with diameters L_i , respectively. Assume we have*

$$\mathcal{K}_{G_i, x_i}^n(\infty) \leq \frac{1}{D_i} + \frac{1}{L_i} \tag{10.3}$$

at $x_i \in V_i$, $i = 1, 2$. Then we have

$$\mathcal{K}_{G_1 \times G_2, (x_1, x_2)}^n(\infty) \leq \frac{1}{D_1 + D_2} + \frac{1}{L_1 + L_2}. \tag{10.4}$$

Moreover, if (10.3) holds with equality for $i = 1, 2$ and we have $\frac{D_1}{L_1} = \frac{D_2}{L_2}$, then (10.4) holds also with equality.

Proof. Let G_i be D_i -regular with diameter L_i , $i = 1, 2$ and $x_i \in V_i$ be the vertices satisfying

$$\mathcal{K}_{G_i, x_i}^n(\infty) \leq \frac{1}{D_i} + \frac{1}{L_i}.$$

Then we have, using [8, equation (7.26)],

$$\begin{aligned} \mathcal{K}_{G_1 \times G_2, (x_1, x_2)}^n(\infty) &= \frac{1}{D_1 + D_2} \min_{i=1,2} D_i \mathcal{K}_{G_i, x_i}^n(\infty) \\ &\leq \frac{1}{D_1 + D_2} \min_{i=1,2} \left(1 + \frac{D_i}{L_i} \right) \\ &\leq \frac{1}{D_1 + D_2} \left(1 + \frac{D_1 + D_2}{L_1 + L_2} \right) \\ &= \frac{1}{D_1 + D_2} + \frac{1}{L_1 + L_2}. \end{aligned}$$

It is easy to see that in the case of equality in (10.3) for $i = 1, 2$, the same calculation leads to equality in (10.4). \square

10.2. The Bakry-Émery curvature of Bonnet-Myers sharp graphs

As a straightforward consequence of Theorem 3.2 and Lemma 10.3, the following proposition show that the ∞ -curvature sharpness is preserved under taking Cartesian products of Bonnet-Myers sharp graphs (of the same ratios $\frac{D_i}{L_i}$).

Proposition 10.4. *Let $G_i = (V_i, E_i)$, $i = 1, 2$, be two (D_i, L_i) -Bonnet-Myers sharp graphs with $\mathcal{K}_{G_i, x_i}^n(\infty) = \frac{1}{D_i} + \frac{1}{L_i}$ at $x_i \in V_i$. Assume furthermore that $\frac{D_1}{L_1} = \frac{D_2}{L_2}$. Then the Cartesian product $G_1 \times G_2$ is also Bonnet-Myers sharp with*

$$\mathcal{K}_{G_1 \times G_2, (x_1, x_2)}^n(\infty) = \frac{1}{D_1 + D_2} + \frac{1}{L_1 + L_2}.$$

For Bonnet-Myers sharp graphs, we have the following ∞ -curvature estimate at poles.

Theorem 10.5. *Let $G = (V, E)$ be a (D, L) -Bonnet-Myers sharp graph. Then we have at every pole $x \in V$:*

$$\mathcal{K}_{G, x}^n(\infty) \leq \frac{1}{D} + \frac{1}{L}. \tag{10.5}$$

Moreover, equality in (10.5) is equivalent to the fact that G is Bakry-Émery ∞ -curvature sharp at x .

Proof. Let $x \in V$ be a pole of G . Using (5.2) in Theorem 5.8, we have for every $y \in S_1(x)$:

$$d_x^+(y) = 1 + D \left(1 - \frac{2}{L} \right).$$

This shows that x is S_1 -out regular with $av_1^+(x) = d_x^+(y) = 1 + D - \frac{2D}{L}$. We know from (10.2) that

$$\mathcal{K}_{G, x}^n(\infty) \leq \frac{3 + D - av_1^+(x)}{2D} = \frac{1}{2D} \left(2 + \frac{2D}{L} \right) = \frac{1}{D} + \frac{1}{L}. \tag{10.6}$$

Equality is equivalent to Bakry-Émery ∞ -curvature sharpness. \square

Since every graph has a pole, Theorem 10.5 immediately implies Theorem 1.7.

In case of self-centered Bonnet-Myers sharp graphs, Theorem 10.5 can be strengthened, where inequality (10.5) becomes equality at all vertices, resulting in Theorem 1.8:

Theorem 1.8. *Let G be a self-centered (D, L) -Bonnet-Myers sharp graph. Then G is Bakry-Émery ∞ -curvature sharp at all vertices $x \in V$ and*

$$\mathcal{K}_{G, x}^n(\infty) = \frac{1}{D} + \frac{1}{L}. \tag{10.7}$$

Before we enter the proof of this theorem, we first recall a method from [8] that allows us to check if a graph $G = (V, E)$ is ∞ -curvature sharp at a vertex x . Let $x \in V$ be an S_1 -out regular vertex, that is, $d_x^+(y)$ is constant for all $y \sim x$. Let $\{y_1, \dots, y_d\}$ be the vertices of $S_1(x)$. We now define two relevant (weighted) Laplacians $\Delta_{S_1(x)}$ and $\Delta_{S_1'(x)}$ on functions $f : S_1(x) \rightarrow \mathbb{R}$ as follows:

$$\Delta_{S_1(x)}f(y_i) = \sum_{y_j: y_j \sim y_i} (f(y_j) - f(y_i)),$$

that is, $\Delta_{S_1(x)}$ be the non-normalized Laplacian of the induced subgraph $S_1(x)$. Let $S'_1(x)$ be the graph with the same vertex set $\{y_1, \dots, y_d\}$ and an edge between y_i and y_j iff $|\{z \in S_2(x) \mid y_i \sim z \sim y_j\}| \geq 1$, where \sim describes adjacency in the original graph G . We introduce the following weights $w'_{y_i y_j}$ on the edges of $S'_1(x)$:

$$w'_{y_i y_j} = \sum_{z \in S_2(x)} \frac{w_{y_i z} w_{z y_j}}{d_x^-(z)}. \tag{10.8}$$

Where $w_{uv} = 1$ if $u \sim v$ and 0 otherwise.

The corresponding weighted Laplacian is then given by

$$\Delta_{S'_1(x)}f(y_i) = \sum_{j: j \neq i} w'_{y_i y_j} (f(y_j) - f(y_i)).$$

Let $S''_1(x) = S_1(x) \cup S'_1(x)$, i.e., the vertex set of $S''_1(x)$ is $\{y_1, \dots, y_d\}$ and the edge set is the union of the edge sets of $S_1(x)$ and $S'_1(x)$. Then the sum $\Delta_{S_1(x)} + \Delta_{S'_1(x)}$ can be understood as the weighted Laplacian $\Delta_{S''_1(x)}$ on $S''_1(x)$ with weights $w'' = w + w'$. Note that all our Laplacians Δ are defined on functions on the vertex set of $S_1(x)$. Let $\lambda_1(\Delta_{S''_1(x)})$ denote the smallest non-zero eigenvalue of $\Delta_{S''_1(x)}$.

Theorem 9.1 of [8] tells us that an S_1 -out regular vertex x in a D -regular graph G is ∞ -curvature sharp if and only if $\lambda_1(\Delta_{S''_1(x)}) \geq \frac{D}{2}$.

With this tool at hand, we can now start with the proof of Theorem 10.6:

Proof. Due to the result on Cartesian products (see Proposition 10.4), it suffices to prove this theorem only for the graphs in the list of Theorem 1.6. We therefore start with a graph $G = (V, E)$ in the list of Theorem 1.6 and prove (10.7) for every vertex. Without loss of generality, we can assume $L \geq 2$ since $L = 1$ implies $G = K_2$ which follows immediately from $\mathcal{K}_{K_2, x}^n(\infty) = 2$.

We know from Proposition 4.1 that all μ -graphs of G are cocktail party graphs $CP(m)$ with

$$m = \frac{D - L}{L(L - 1)} + 1,$$

and all 1-spheres of G are strongly regular with parameters

$$\text{srg} \left(D, \frac{2D}{L} - 2, \frac{D - 1}{L - 1} - 3, 2 \frac{D - L}{L(L - 1)} \right).$$

Note that $\lambda = \frac{D-1}{L-1} - 3$ can be understood as the number of triangles within any 1-sphere containing a fixed edge.

Let $x \in V$. Since every vertex of G is a pole, we know from (10.6) that

$$\mathcal{K}_{G,x}(\infty) \leq \frac{1}{D} + \frac{1}{L}$$

with equality iff x is ∞ -curvature sharp. We have already seen in the proof of Theorem 10.5 that x is S_1 -out regular. So it only remains to show $\lambda_1(\Delta_{S_1''(x)}) \geq \frac{D}{2}$.

Let A denote the adjacency matrix of $S_1(x)$. Then, by Theorem 5.7(a), $\Delta_{S_1(x)} = A - (\frac{2D}{L} - 2)\text{Id}$. Let $y \sim x$. Then, by (5.2) in Theorem 5.8, $d_x^+(y) = \frac{(L-2)D}{L} + 1$. Since every μ -subgraph of G is $CP(m)$, we have $d_x^-(z) = 2m = \frac{2}{L-1}(\frac{D}{L} + L - 2)$ for all $z \in S_2(x)$.

By counting arguments based on the above number λ we derive the weights of (10.8) as

$$\forall y, y' \in S_1(x), y \neq y' : \quad \omega'_{yy'} = \begin{cases} \frac{1}{2m}, & \text{if } y \sim y', \\ \left(\frac{2D}{L} - \frac{D-1}{L-1}\right) \frac{1}{2m}, & \text{if } y \not\sim y', \end{cases}$$

and consequently

$$\begin{aligned} \Delta_{S_1''(x)} &= \Delta_{S_1(x)} + \Delta_{S_1'(x)} \\ &= \frac{1}{2m} \left(\left(\frac{2D}{L} - \frac{D-1}{L-1} + 2m - 1 \right) \left(A - \left(\frac{2D}{L} - 2 \right) \text{Id} \right) - D \cdot \text{Id} + J \right), \end{aligned}$$

where J is the all-one matrix.

By Proposition 4.1, the second largest eigenvalue of A is $\frac{(D-L)(L-2)}{L(L-1)}$. Note that the eigenvector w of the second largest eigenvalue is orthogonal to $\mathbf{1}$ and, therefore, $Jw = 0$. Thus to complete the claim it remains to show that

$$\begin{aligned} \lambda_1(\Delta_{S_1''(x)}) &= \frac{-1}{2m} \left(\left(\frac{2D}{L} - \frac{D-1}{L-1} + 2m - 1 \right) \underbrace{\left(\frac{(D-L)(L-2)}{L(L-1)} - \left(\frac{2D}{L} - 2 \right) \right)}_{=-\frac{D-L}{L-1}} - D \right) \\ &\geq \frac{D}{2}. \end{aligned}$$

After multiplying by $2m$ and inserting $m = \frac{D-L}{L(L-1)} + 1$ into the expression, this simplifies to

$$\frac{1}{L(L-1)^2} (L(L-2) + D)(D-L) \geq 0,$$

which is obviously true since $L \leq D$ and $L \geq 2$. \square

10.3. A conjecture about Bakry-Émery curvature

In this subsection, let us revisit the following conjecture mentioned in the Introduction:

Conjecture 1.9. *Let $G = (V, E)$ be a connected, simple D -regular graph with diameter L . We then have*

$$\inf_{x \in V} \mathcal{K}_{G,x}^n(\infty) \leq \frac{1}{D} + \frac{1}{L}. \tag{10.9}$$

A simple argument provides the following general estimate. The challenge of the conjecture is thus to remove the final term in (10.10).

Theorem 10.6. *Let $G = (V, E)$ be a D -regular graph of diameter L . Then we have*

$$\inf_{x \in V} \mathcal{K}_{G,x}^n(\infty) \leq \frac{1}{D} + \frac{1}{L} + \frac{1}{2D^2} \max_{x \in V} \#_{\Delta}(x). \tag{10.10}$$

Proof. The proof is a combination of the inequalities (1.8) and (10.2). \square

Here is a list of examples providing supporting evidence for this conjecture:

1. All graphs with $D \leq L$: This is an immediate consequence of

$$\inf_{x \in V} \mathcal{K}_{G,x}^n(\infty) \leq \frac{2}{L}$$

proved in [13, Corollary 2.2].

2. All Bonnet-Myers sharp graphs: This follows immediately from Theorem 10.5.
3. All strongly regular graphs: Note that a strongly regular graph $G = (V, E)$ with parameters (ν, D, λ, μ) satisfies, as all vertices $x \in V$,

$$\#_{\Delta}(x) = \frac{D\lambda}{2} \leq \frac{D(D-2)}{2},$$

since $\lambda \leq D - 2$ (G cannot be the complete graph). Using (10.2), this implies

$$\mathcal{K}_{G,x}^n(\infty) \leq \frac{2}{D} + \frac{D-2}{2D} = \frac{1}{D} + \frac{1}{2}.$$

4. All complete graphs: Note that the complete graph $G = K_n$ has degree $D = n - 1$ and Bakry-Émery ∞ -curvature (see [8, Example 5.17])

$$\mathcal{K}_{G,x}^n(\infty) = \frac{D+3}{2D} \leq \frac{1}{D} + 1$$

in all vertices x .

5. All demi-cube graphs: The even-dimensional demi-cubes $Q_{(2)}^{2n}$ satisfies $\mathcal{K}_{G,x}^n(\infty) = \frac{1}{D} + \frac{1}{L}$ for all vertices x (due to Theorem 1.8 as it is self-centered Bonnet-Myers sharp). On the other hand, the odd-dimensional demi-cube $Q_{(2)}^{2n+1}$ has Bakry-Émery ∞ -curvature

$$K_{G,x}^n(\infty) \leq \frac{3 + D - av_1^+(x)}{2D} = \frac{1}{n} = \frac{1}{L} < \frac{1}{D} + \frac{1}{L},$$

where the upper bound $\frac{1}{D} + \frac{1}{L}$ will never be achieved.

6. All Johnson graphs: The Johnson graph $G = J(n, k)$ has the following Bakry-Émery ∞ -curvature (see [8, Example 9.7]) in all vertices x

$$\mathcal{K}_{G,x}^n(\infty) = \frac{n+2}{2k(n-k)} \leq \frac{1}{D} + \frac{1}{L},$$

with vertex degree $D = k(n-k)$ and diameter $L = \min\{k, n-k\}$.

7. All triangle-free graphs: Since $\#\Delta(x) = 0$ for all $x \in V$, (10.10) implies that

$$\inf_{x \in V} \mathcal{K}_{G,x}^n(\infty) \leq \frac{1}{D} + \frac{1}{L}.$$

8. Cartesian products: If (10.9) holds for the graphs $G_i = (V_i, E_i)$, $i = 1, 2$, then (10.9) holds also for the Cartesian product $G_1 \times G_2$ due to Lemma 10.3.

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