A DIOPHANTINE PROBLEM FROM MATHEMATICAL PHYSICS

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Abstract

In this paper, we study a Diophantine problem from mathematical physics and prove that for every positive integer k, there exists infinitely many sets of k $n-$tuples of positive integers with the same sum and the same sum of their cubes. Each set of k $n-$tuples is "primitive" in the sense that the greatest common divisor of all kn elements is 1. We reduce the corresponding Diophantine system to a family of elliptic curves and apply Nagell's algorithm, Nagell-Lutz theorem and the theorem of Poincaré and Hurwitz to deal with it. In the end, we raise two open questions about this Diophantine problem.

Key words: Diophantine system, Diophantine chains, $n-$tuples, primitive set, elliptic curves.

1 Introduction

In mathematical physics, a Racah operator is a linear operator acting on a particular abstract Hilbert space and gives rise to the Racah coefficients. A full discussion could be found in [1], you could also see the motivation and the importance of the study of the Racah coefficients in Quantum Theory. Considerable interest has been shown in the nontrivial zeros of the Racah coefficients, because these determine vector spaces belonging to the null space of a Racah operator and accordingly give structural information concerning the operator itself.

In 1985, Brudno and Louck in [4] found the relation between the all nontrivial zeros of weight 1 $6j$ Racah coefficients and the all non-negative integer solutions of the Diophantine system

$$\begin{cases} x_1 + x_2 + x_3 = y_1 + y_2 + y_3, \\ x_1^3 + x_2^3 + x_3^3 = y_1^3 + y_2^3 + y_3^3. \end{cases} \tag{1}$$

They mentioned that the special parametric solution given by Gerardin ([8], P. 713) in 1916 was very useful for their problem.

In 1986, Bremner in [2] got more solutions including Gerardin's. In the same year, complete solutions were given in terms of cubic polynomials in four variables by Bremner and Brudno in [3], as well as by Labarthe in [9], and the parameter solutions obtained by them are different in the form.

In 1991, a complete solution in terms of eight variables was given by Choudhry in [5]. In 2010, Choudhry in [6] gave a complete four-parameter solution in terms of quadratic polynomials. Of course, these two parameter solutions are different from the previous ones.

In this paper, we consider the positive integer solutions of the Diophantine chains

$$\begin{cases} \sum_{j=1}^{n} x_{1j} = \sum_{j=1}^{n} x_{2j} = \cdots = \sum_{j=1}^{n} x_{kj} = A, \\ \sum_{j=1}^{n} x_{1j}^3 = \sum_{j=1}^{n} x_{2j}^3 = \cdots = \sum_{j=1}^{n} x_{kj}^3 = B, \\ n \geq 2, \ k \geq 1, \end{cases} \tag{2}$$

where A, B are positive integers, which are determined by k $n-$tuples $(x_{i1}, x_{i2}, \cdots, x_{in}), i = 1, \cdots, k$. For $n = 2, k = 2$, it has been shown in [11] that (2) have no nontrivial integer solutions, so we consider $n \geq 3$. For $n = 3, k = 2$, (2) reduce to (1). For $n = 3, k \geq 3$, Choudhry in [6] gave a parameter solution in rational numbers of (2), but the solutions are not all positive, i.e., there are arbitrarily long Diophantine chains of the form (2) with $n = 3$.

The Diophantine chains (2) can be transformed into the following Diophantine system

$$\begin{cases} x_{i1} + \cdots + x_{in} = A, \\ x_{i1}^3 + \cdots + x_{in}^3 = B, \\ x_{ij} > 0, \ A > 0, \ B > 0, \\ i = 1, \cdots, k, \ j = 1, \cdots, n, \ n \geq 2, \ k \geq 1. \end{cases} \tag{3}$$

In 2011, Zhang and Cai in [13] studied a Diophantine system which is similar to (3), the method of this paper is inspired by their paper's, but we use the Nagell's algorithm to get a family of elliptic curves.

We mainly investigate the positive integer solutions of (2) or (3) for $n \geq 3, k \geq 1$, and prove the following theorem by using the theory of elliptic curves, including Nagell's algorithm, Nagell-Lutz theorem and the theorem of Poincaré and Hurwitz. The method used here is different from the methods used by Choudhry in [5], [6] and the result is stronger than Choudhry's.

Theorem 1. *For $n \geq 3, k \geq 1$, the Diophantine chains (2) have infinitely many coprime positive integer solutions. Equivalently, for every positive integer k, there exists infinitely many primitive sets of k $n-$tuples of positive integers with the same sum and the same sum of their cubes.*

A set S of $n-$tuples of positive integers is called primitive if the greatest common divisor of all elements of all $n-$tuples of S is 1.

In geometry, for each i, we can consider (3) as the intersection of a hyperplane and a hypersurface. To find the positive integer points on their intersection, we fix $n-3$ variables in the $n-$tuples, then the problem is transformed into finding positive integer points on a family of cubic curves, which is essentially a family of elliptic curves. Hence, we can use the theory of elliptic curves to deal with the new problem. The exact process will be given in sections 3 and 4.

2 The theory of elliptic curves

The Diophantine equation is one of the oldest branches of number theory, which deals with the solutions of polynomial equations or systems of equations in integers or rational numbers. One of the fascinations of the Diophantine equation is that the problems of it are usually easy to state, but sometimes very difficult to solve. When they can be solved, they always need extremely sophisticated mathematical theories and tools. The typical example is the Fermat's Last Theorem. The problem about Diophantine equation is called Diophantine problem.

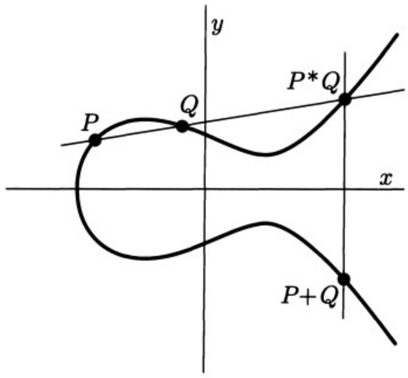

Figure 1: group law for $y^2 = x^3 + ax^2 + bx + c$

Elliptic curve is a very useful tool to study Diophantine equation, it is not only an important investigated object of number theory, but also a basic investigated object of algebraic geometry. We shall give the definition and some basic properties of elliptic curve. There are many definitions of it, the simplest form is defined by the Weierstrass equation in the field of rational numbers $\mathbb{Q}$:

$$E: \ y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \ (a_i \in \mathbb{Q}),$$

where the discriminant of this equation is not zero, which can reduce to $y^2 = x^3 + ax^2 + bx + c$ or $y^2 = x^3 + ax + b$.

A very beautiful property of elliptic curve is the group law, i.e., the all rational points on the elliptic curve form a group under an operator, then the theory of group can be applied on the elliptic curve. The group law is illustrated in figure 1, through any two rational points on the elliptic curve, say P and Q, we can get the third point, denoted by $P * Q$. The reflective point of $P * Q$ about the x axis is denoted by $P + Q$, where the symbol " + " is the operator of the group. When $Q = P$, we can get $P + P$.

Let $E(\mathbb{Q}) = \{(x, y)|\ y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6\} \bigcup \{O\}$, it is an abelian group under the operator " + ", where O denotes the point at infinity on the elliptic curve, which is the zero element of the group. Then we have the following famous theorem.

Mordell's Theorem. The group $E(\mathbb{Q})$ is finitely generated.

When we have the group law and the operator, the order of an element is defined, i.e., the order of a rational point. A rational point P on the elliptic curve is said to have order m, if

$$[m]P = \underbrace{P + \cdots + P}_{m} = O,$$

and $m'P \neq O,\ 1 \leq m' \leq m$. If such an m exists, then P has finite order; otherwise it has infinite order.

Next, we give the process of calculating $[2]P = P + P$, let $P = (x_0, y_0)$ be a rational point on the elliptic curve $E: \ y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \ (a_i \in \mathbb{Q})$, then the slope of the tangent line at P is

$$k = \frac{3x_0^2 + 2a_2x_0 + a_4 - a_1y_0}{2y_0 + a_1x_0 + a_3},$$

the tangent line is $y = k(x - x_0) + y_0$, substitute it into the equation of E, we get a cubic equation of x, from the relation between the roots and coefficients, we obtain the coordinates of the intersection point (x_1, y_1), then we have $[2]P = (x_1, -y_1 - a_1x_1 - a_3)$.

To prove our theorem, we need the following two profound theorems.

Nagell-Lutz Theorem ([10], P. 56). Let the equation of the elliptic curve be

$$y^2 = x^3 + ax^2 + bx + c \ (a, b, c \in \mathbb{Z}),$$

the discriminant of the cubic polynomial is $\Delta = -4a^3c + a^2b^2 + 18abc - 4b^3 - 27c^3$, let $P = (x, y)$ be a rational point of finite order, then x and y are integers; and either $y = 0$ or else $y|\Delta$.

From this theorem, we know that if x or y is not an integer, then $P = (x, y)$ is a rational point of infinite order, hence there are infinitely many rational points on the elliptic curve.

The theorem of Poincaré and Hurwitz ([12], P. 78). If the elliptic curve has infinitely many rational points, then it has infinitely many rational points in every neighborhood of any one of them.

3 Two propositions

In this section, we give two propositions, where the proposition 3 is the key step to prove our theorem. And the proofs of these two propositions are the applications of the theory of elliptic curves and need many calculations.

In fact, in order to prove the theorem, we only need the case for $n = 3$ of proposition 2. However, the proposition 2 and its proof are of interest for their own sake, so it's worth including them even though they provide more information than it is needed.

Proposition 2. *For $n \geq 3$, the Diophantine system*

$$\begin{cases} x_1 + \cdots + x_n = \dfrac{n(n+1)}{2}, \\ x_1^3 + \cdots + x_n^3 = \dfrac{n^2(n+1)^2}{4}, \end{cases} \tag{4}$$

has infinitely many rational solutions.

Proof. It's easy to see that $x_1 = 1, x_2 = 2, \cdots, x_n = n$ is a solution of (4). Taking $x_1 = 1, x_2 = 2, \cdots, x_{n-3} = n - 3$, we have

$$\begin{cases} x_{n-2} + x_{n-1} + x_n = 3(n-1), \\ x_{n-2}^3 + x_{n-1}^3 + x_n^3 = 3(n-1)(n^2 - 2n + 3). \end{cases}$$

Eliminating x_{n-2}, we get

$$\begin{aligned} &3x_{n-1}^2 x_n + 3x_{n-1}x_n^2 + 9(1-n)x_{n-1}^2 + 9(1-n)x_n^2 + 18(1-n)x_{n-1}x_n \\ &+ 27(n-1)^2(x_{n-1} + x_n) - 6(n-1)(2n-1)(2n-3) = 0, \end{aligned}$$

leading to

$$\begin{aligned} &3\frac{x_n}{x_{n-1}} + 3\left(\frac{x_n}{x_{n-1}}\right)^2 + 9(1-n)\frac{1}{x_{n-1}} + 9(1-n)\left(\frac{x_n}{x_{n-1}}\right)^2 \frac{1}{x_{n-1}} + 18(1-n)\frac{x_n}{x_{n-1}}\frac{1}{x_{n-1}} \\ &+ 27(n-1)^2\left(\frac{1}{x_{n-1}^2} + \frac{x_n}{x_{n-1}}\frac{1}{x_{n-1}^2}\right) - 6(n-1)(2n-1)(2n-3)\frac{1}{x_{n-1}^3} = 0. \end{aligned}$$

Put

$$u = \frac{x_n}{x_{n-1}}, \ v = \frac{1}{x_{n-1}},$$

we have

$$
\begin{aligned}
&-6(n-1)(2n-1)(2n-3)v^3+9(1-n)u^2v+27(n-1)^2uv^2+3u^2+18(1-n)uv\\
&+27(n-1)^2v^2+3u+9(1-n)v=0.
\end{aligned}
$$

Next, we use the Nagell's algorithm ([7], P. 115) to transform the above equation into the Weierstrass equation. Let both sides of the above equation be divided by v^3, and let $t=\frac{u}{v}$, we get

$$
\begin{aligned}
&\big(9(1-n)t^2+27(n-1)^2t-6(n-1)(2n-1)(2n-3)\big)v^2\\
&+(3t^2+18(1-n)t+27(n-1)^2)v+3t+9(1-n)=0.
\end{aligned}
$$

Because of the coefficient $9(1-n)t^2+27(n-1)^2t-6(n-1)(2n-1)(2n-3)$ is not zero for $n\geq 3$ and any $t\in\mathbb{Q}$, we can consider it as a quadratic equation of v, if it has rational solutions, the discriminant should be a perfect square, i.e.,

$$\Delta(t)=9(t-3n+3)(t^3+(3n-3)t^2-9(n-1)^2t+(n-1)(5n^2-10n-3))$$

is a square of some rational number.

Let

$$\rho=\tau^4\Delta(t)=9(t-3n+3)(t^3+(3n-3)t^2-9(n-1)^2t+(n-1)(5n^2-10n-3))\tau^4,$$

putting $t=3n-3+\frac{1}{\tau}$, we have

$$\rho=72(n-1)(2n-1)(2n-3)\tau^3+324(n-1)^2\tau^2+108(n-1)\tau+9.$$

Taking the transformation

$$(\tau,\rho)=\left(\frac{X}{c},\frac{Y^2}{c^2}\right),$$

where $c=72(n-1)(2n-1)(2n-3)$, we get a family of elliptic curves

$$
\begin{aligned}
E_n: Y^2=&X^3+324(n-1)^2X^2+7776(n-1)^2(2n-1)(2n-3)X\\
&+216^2(n-1)^2(2n-1)^2(2n-3)^2,
\end{aligned}
$$

where $n\geq 3$ is a positive integer.

The birational transformation of this process is

$$
\begin{cases}
x_{n-1}=\dfrac{-Y-216(n-1)(2n-1)(2n-3)}{6X},\\
x_n=\dfrac{3(n-1)(X+24(2n-1)(2n-3))}{X},
\end{cases}
\tag{5}
$$

the inverse transformation is

$$
\begin{cases}
X=\dfrac{72(n-1)(2n-1)(2n-3)}{x_n-3n+3},\\
Y=\dfrac{216(n-1)(2n-1)(2n-3)(3n-3-2x_{n-1}-x_n)}{x_n-3n+3}.
\end{cases}
\tag{6}
$$

The discriminant of E_n is $\Delta(n)=58773123072(n-1)^4(2n-1)^3(2n-3)^3$, where $n\geq 3$, it's easy to see that $\Delta(n)\neq 0$, i.e., E_n is nonsingular.

Noting that $x_1=1,x_2=2,\cdots,x_n=n$ is a solution of (4), let $x_{n-1}=n-1,x_n=n$ in (6), we get

$$X=-72(n-1)(2n-1),Y=216(n-1)(2n-1).$$

It means that the point $P = (-72(n-1)(2n-1), 216(n-1)(2n-1))$ lies on E_n. Using the group law on the elliptic curve, we obtain the points

$$[2]P = (144(n-1)(2n-1), -216(n-1)(2n-1)(18n-17)),$$
$$[3]P = (-4(6n-5)(6n-7), 8(108n^2 - 216n + 109)),$$
$$[4]P = (X_4, Y_4),$$

where

$$X_4 = \frac{288(n-1)(2n-1)(18n-19)}{(18n-17)^2},$$
$$Y_4 = \frac{216(n-1)(2n-1)(11664n^4 - 42768n^3 + 57456n^2 - 33084n + 6731)}{(18n-17)^3}.$$

To prove that there are infinitely many rational points on E_n, it is enough to find a rational point on E_n with $x-$coordinate not in $\mathbb{Z}$. We consider the $x-$coordinate of the point $[4]P$, when the numerator of the $x-$coordinate of it is divided by the denominator, the remainder equals

$$r = -704n + \frac{2080}{3},$$

for $n \geq 3$, r is not an integer, and the denominator $(18n-17)^2$ is an integer, then X_4 is not an integer. By the Nagell-Lutz theorem ([10], P. 56), $[4]P$ is a point of infinite order, hence E_n has infinitely many rational points for $n \geq 3$. From the birational transformation (5), we have

$$x_{n-2} = \frac{Y - 216(n-1)(2n-1)(2n-3)}{6X},$$

then the Diophantine system (4) has infinitely many rational solutions. □

Next, we state the proposition 3, and the proof is relatively simpler than the proposition 2, which is due to the theorem of Poincaré and Hurwitz, this is the key point in our paper.

Proposition 3. *For $n \geq 3$, the Diophantine system (4) has infinitely many positive rational solutions.*

Proof. Because of $x_1 = 1, x_2 = 2, \cdots, x_{n-3} = n-3$, to prove that there are infinitely many $x_j > 0, j = 1, \cdots, n$, we only need to prove $x_j > 0, j = n-2, n-1, n$. From (5) and x_{n-2}, we have the following equivalent condition

$$\begin{cases} x_{n-2} = \dfrac{Y - 216(n-1)(2n-1)(2n-3)}{6X} > 0, \\ x_{n-1} = \dfrac{-Y - 216(n-1)(2n-1)(2n-3)}{6X} > 0, \\ x_n = \dfrac{3(n-1)(X + 24(2n-1)(2n-3))}{X} > 0, \end{cases}$$
$$\Longleftrightarrow X < -24(2n-1)(2n-3), \ |Y| < 216(n-1)(2n-1)(2n-3). \quad (7)$$

In virtue of the theorem of Poincaré and Hurwitz ([12], P. 78), E_n has infinitely many rational points in every neighborhood of any one of them. Hence, if we find a rational point satisfies (7), we can prove that there are infinitely many rational points satisfy (7). It's easy to check that for $n \geq 3$, the points P and $[3]P$ satisfy (7). Therefore, there are infinitely many rational points on E_n satisfying (7), then we prove that (4) has infinitely many positive rational solutions. □

Example for $n = 3$, from the points

$$(X, Y) = (-432, 1296), (-572, 364), \left(\frac{-97511580}{190969}, \frac{-243727681320}{83453453}\right),$$

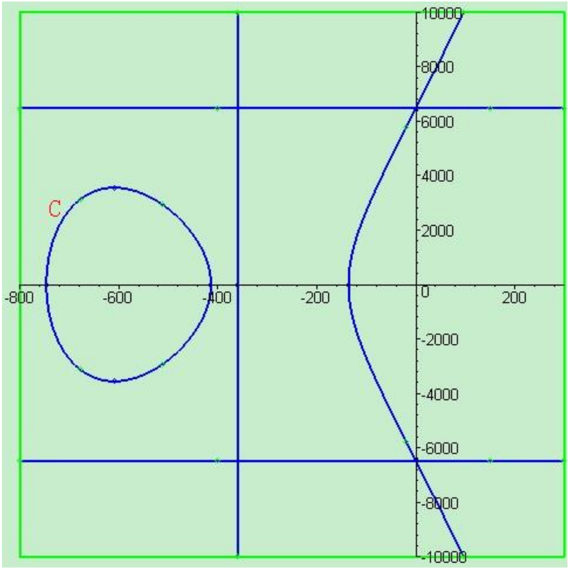

Figure 2: E_3, $x = -360$ and $y = \pm 6480$

we get

$$(x_1, x_2, x_3) = (1, 2, 3), \left(\frac{318}{143}, \frac{29}{333}, \frac{113}{39}\right), \left(\frac{319586}{180577}, \frac{674461}{219811}, \frac{182271}{156883}\right).$$

In figure 2, we display the elliptic curve E_3 and the three lines of (7), from it we find that the rational points, lie on the closed curve C, satisfy (7).

4 The proof of Theorem 1

Proof. Take any k positive rational solutions in (4), denote $(x_{i1}, \cdots, x_{in}), i = 1, \cdots, k$, where $x_{i1} = 1, x_{i2} = 2, \cdots, x_{i,n-3} = n-3$. Let d be the least common denominator of all the numbers x_{ij} $(j = 1, \cdots, n, i \leq k)$, we have

$$x_{ij} = \frac{a_{ij}}{d}, \ a_{ij} \in \mathbb{Z}^+, \ \big(gcd_{i,j}\ (a_{ij}), d\big) = 1,$$

where $a_{i1} = d, a_{i2} = 2d, \cdots, a_{i,n-3} = (n-3)d$.

Then

$$\sum_{i=1}^{n} a_{ij} = \frac{n(n+1)}{2} d, \ \sum_{i=1}^{n} a_{ij}^3 = \frac{n^2(n+1)^2}{4} d^3 \ (i \leq k),$$

hence

$$gcd_{i,j}\ (a_{ij}) = 1.$$

For two sets of solutions $\{(x_{i1}, \cdots, x_{in}), i \leq k\}$ and $\{(x'_{i1}, \cdots, x'_{in}), i \leq k\}$, if the sets of n-tuples of positive integers $\{(a_{i1}, \cdots, a_{in}), i \leq k\}$ and $\{(a'_{i1}, \cdots, a'_{in}), i \leq k\}$ coincide, then $d = d'$. Hence, the sets of solutions themselves coincide.

By proposition 3, there are infinitely many choices of k $n-$tuples from an infinite set, and $gcd_{i,j}\ (a_{ij}) = 1$, hence for every positive integer k, there exists infinitely many primitive sets of k $n-$tuples of positive integers with the same sum and the same sum of their cubes. □

Example for $n = 3$, from the positive rational triples

$$(x_1, x_2, x_3) = (1, 2, 3), \left(\frac{318}{143}, \frac{29}{333}, \frac{113}{39}\right), \left(\frac{319586}{180577}, \frac{674461}{219811}, \frac{182271}{156883}\right),$$

we have $d = 33853311921$, then the three triples of positive integers

$$(33853311921, 67706623842, 101559935763),$$
$$(75282190146, 29749880173, 98087801207),$$
$$(59913746178, 39331712277, 103874413071)$$

have the same sum 203119871526 and the same sum of their cubes 1396709184949924985734645154986596.

5 Two open questions

When we communicated with Professor Michael Zieve, he posed some questions, where the following two are interesting.

Question 4. *Whether there are infinitely many $n-$tuples of positive integers have no common element with the same sum and the same sum of their cubes for $n \geq 4$?*

In this paper, we do it for $n = 3$ by using (5), x_{n-2} and some calculations. But for $n = 4$, we get the rational quadruples which all have the form $(1, x, y, z)$, there is a common element 1 for all rational quadruples. It's natural to use a more restrictive definition of "primitive", i.e., the all $n-$tuples have no common element and the greatest common divisor of all elements is 1. Then question 4 is whether there are infinitely many "primitive" $n-$tuples of positive integers with the same sum and the same sum of their cubes for $n \geq 4$.

We conjecture that the answer to question 4 is yes, but we can't prove it for $n \geq 4$. There are some examples for $n = 4$, such as $(1, 2, 13, 24)$ and $(4, 5, 6, 25)$ have the same sum 40 and the same sum of their cubes 16030, $(1, 2, 17, 20)$ and $(3, 6, 8, 23)$ have the same sum 40 and the same sum of their cubes 12922, $(1, 2, 19, 24)$ and $(4, 6, 9, 27)$ have the same sum 46 and the same sum of their cubes 20692.

Question 5. *For which triples (i, j, k) of positive integers such that the Diophantine system*

$$\begin{cases} x + y + z = i + j + k, \\ x^3 + y^3 + z^3 = i^3 + j^3 + k^3, \end{cases} \tag{8}$$

has infinitely many rational solutions?

To this problem, we get an incomplete result but very interesting. Eliminating z of (8), we get

$$(i + j + k - y)x^2 - (i + j + k - y)^2 x + (i + j + k)y^2 - (i + j + k)^2 y + (i + j)(j + k)(k + j) = 0.$$

Noting that $(x, y, z) = (i, j, k)$ is a solution of (8), let $y = t(x - i) + j$ in the above equation, we have

$$\begin{aligned}(x - i)\big((t^2 + t)x^2 - ((2i + j + k)t^2 + 2(i + k)t + i + k)x \\ + (i^2 + ik + ij)t^2 + (i^2 - 2ik - j^2 + k^2)t + (i + k)k\big) = 0.\end{aligned}$$

Solving it, we get

$$x = i, \frac{(2i + j + k)t^2 + 2(i + k)t + i + k \pm \sqrt{\Delta}}{2(t^2 + t)},$$

where

$$\Delta = (j+k)^2t^4 + 4j(j+k)t^3 + 2(2i^2+ij+ik+jk+2j^2-k^2)t^2 + 4i(i+k)t + (i+k)^2.$$

If x is a rational number, we need Δ to be a perfect square. Following the usual procedure described by Dickson ([8], P. 639), we can find values of t that would make Δ a perfect square. One such value of t is given by

$$t = -\frac{i^2-k^2}{j^2-k^2},$$

and this leads to a rational solution of (8) as following

$$\begin{cases} x_1(i,j,k) = x = \dfrac{i^3+j^3+k^3-ijk-ij^2-ik^2}{(i-j)(i-k)}, \\ y_1(i,j,k) = y = -\dfrac{i^3+j^3+k^3-ijk-i^2j-jk^2}{(i-j)(j-k)}, \\ z_1(i,j,k) = z = \dfrac{i^3+j^3+k^3-ijk-i^2k-j^2k}{(i-k)(j-k)}. \end{cases} \tag{9}$$

By the symmetry of i, j, k in (9), we know that for $i \neq j \neq k$,

$$x_1(i,j,k) \neq y_1(i,j,k) \neq z_1(i,j,k).$$

From (9), we get an identity

$$\begin{cases} x_1(i,j,k) + y_1(i,j,k) + z_1(i,j,k) = i+j+k, \\ x_1(i,j,k)^3 + y_1(i,j,k)^3 + z_1(i,j,k)^3 = i^3+j^3+k^3, \end{cases}$$

where $i \neq j \neq k$ are arbitrary positive integers, replace i, j, k by $x_1(i,j,k), y_1(i,j,k)$ and $z_1(i,j,k)$, respectively, to get another identity

$$\begin{cases} x_2(i,j,k) + y_2(i,j,k) + z_2(i,j,k) = x_1(i,j,k) + y_1(i,j,k) + z_1(i,j,k), \\ x_2(i,j,k)^3 + y_2(i,j,k)^3 + z_2(i,j,k)^3 = x_1(i,j,k)^3 + y_1(i,j,k)^3 + z_1(i,j,k)^3. \end{cases}$$

In fact, we can repeat this process any times to get an arbitrarily long Diophantine chains of the type

$$\begin{cases} x_n(i,j,k) + y_n(i,j,k) + z_n(i,j,k) = \cdots = x_1(i,j,k) + y_1(i,j,k) + z_1(i,j,k) = i+j+k, \\ x_n(i,j,k)^3 + y_n(i,j,k)^3 + z_n(i,j,k)^3 = \cdots = x_1(i,j,k)^3 + y_1(i,j,k)^3 + z_1(i,j,k)^3 = i^3+j^3+k^3, \end{cases}$$

where $n = 1, 2, \cdots$.

However, we can't prove the chains don't have cycles after some steps. On the other hand, the rational solutions, we get in this form, are not all positive.

Example, let $(i,j,k) = (1,2,3)$, from (9) we have

$$(x_1, y_1, z_1) = \left(\frac{17}{2}, -10, \frac{15}{2}\right),$$

then

$$(x_2, y_2, z_2) = \left(\frac{-5237}{148}, \frac{7834}{1295}, \frac{4947}{140}\right).$$

In the future study, we shall deal with these two questions and more related problems from mathematical physics by using the methods of number theory.

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