Maximum of Summations of Numbers of Minesweeping Games

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摘 要

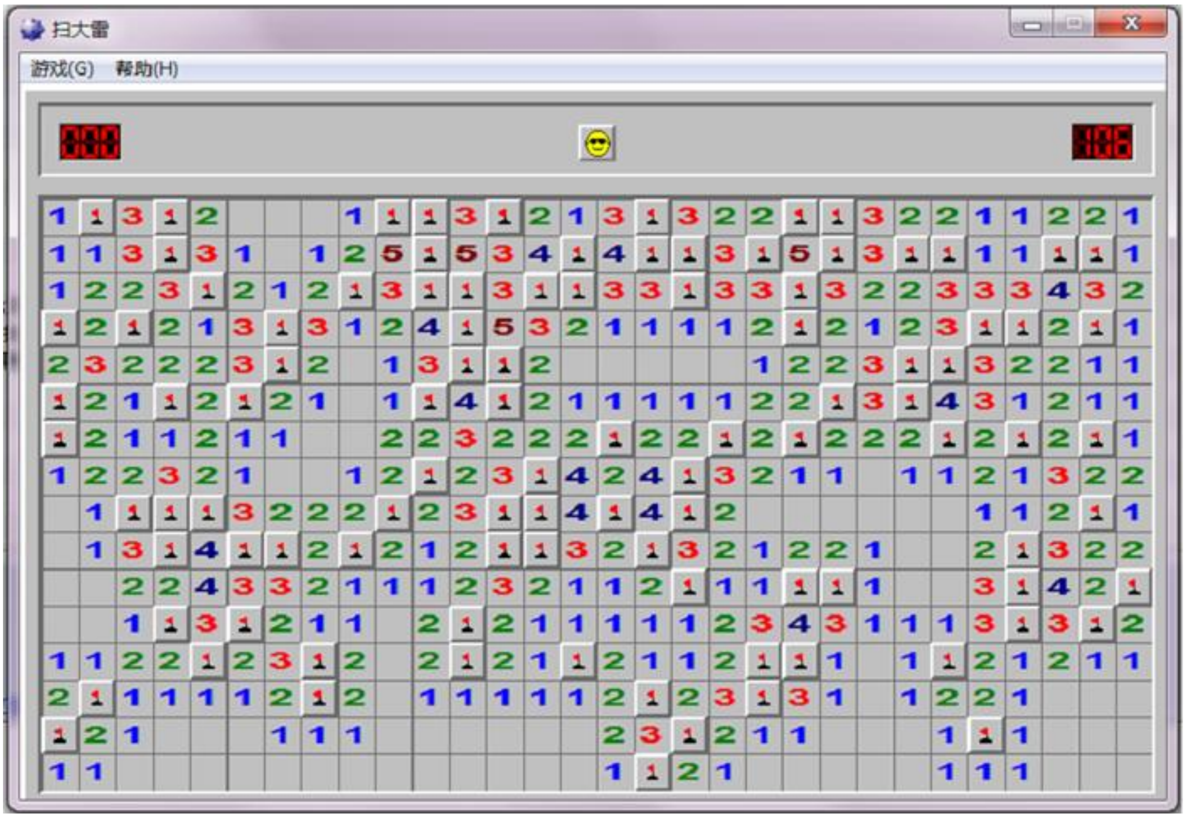
本文研究了 WINDOWS 扫雷游戏中数字和的最大值，这个数字和先通过组合学表示出来，然后利用放缩得到了最大值，并且通过放缩中的取等条件得到了取到最大值时雷的排布。

Abstract

This paper mainly studies about the summation of numbers appearing in the Windows “Minesweeping” game. The summation of the numbers is first written down through the methods of combinatorics. The maximum of this summation is figured out, and the situations in which the maximum reached is then found.

1. An Introduction of the Game Minesweeping

Minesweeping is a popular computer game. A rectangle is divided into a matrix of squares. Every square is assigned with a mine or an integer, the number of mines around it. The goal of the game is to click out all the squares not assigned mines. The playing of the game is as follows. Click a square which is known not assigned a mine to get the number of mines around this square. Then, using the gotten numbers, speculate a new square which is not a mine. Repeat the process until all the squares not filled with mines be clicked. If, on any step, a mine is clicked, the game fails.



If a square is not assigned a mine, the number of mines around it is called its number. For a square assigned a mine, we denote its number as 0. The summation of all the numbers of the squares is called the summation of the assignment.

2. The Main results

In this paper, we study the summation of the mine assignments. We obtain the upper bound of summations of assignments for ordinary minesweeping games and a kind of generalized games which we call cyclic minesweeping games. In addition, we give the conditions in which the summations reach the upper bounds.

3. Descriptions and Proofs

3.1 The upper bound of summation of numbers for ordinary case

For ordinary minesweeping game, the number for a square is determined by the cases of the adjacent squares assigned mines or not. *The summation of an assignment is equal to the number of pairs of adjacent squares being assigned different*. Suppose that the grid of the game is with *m* rows and *n* columns. Denote the summation of the numbers with $A_{m,n}$, and the number of pairs of adjacent squares assigned same (both be mines or neither be mine) as $B_{m,n}$. Then the total number of pairs of adjacent squares is

$$S = n(m-1)+m(n-1)+2(m-1)(n-1) = 4mn-3m-3m+2.$$

Thus we have $A_{m,n}+B_{m,n}=S=4mn-3m-3m+2.$

Denote $B_{m,n}-A_{m,n}=X_{m,n}$. Then we have $2A_{m,n}=S-X_{m,n}$. This means that finding the upper bound of $A_{m,n}$ is equivalent to find the lower bound of $X_{m,n}$.

To discuss the value of $X_{m,n}$, we need to differentiate the case of being assigned mine or not. Define

$$a_{i,j}=\begin{cases}1, & \text{if the square of ith row and jth column is assigned a mine,}\\ -1, & \text{otherwise.}\end{cases}$$

This implies that for each pair of adjacent squares of row i_1, column j_1 and row i_2, column j_2, $a_{i_1,j_1}a_{i_2 j_2}=1$ if and only if they are assigned same and $a_{i_1,j_1}a_{i_2 j_2}=-1$ if and only if they are assigned different. Thus we have

$$X_{m,n}=\sum_{j=1}^{n}\sum_{i=1}^{m-1}a_{i,j}a_{i+1,j}+\sum_{i=1}^{m}\sum_{j=1}^{n-1}a_{i,j}a_{i,j+1}+\sum_{j=1}^{n-1}\sum_{i=1}^{m-1}a_{i,j}a_{i+1,j+1}+\sum_{j=2}^{n}\sum_{i=1}^{m-1}a_{i,j}a_{i+1,j-1}.$$

$$X_{m,n}=\sum_{j=1}^{n}\sum_{i=1}^{m-1}a_{i,j}a_{i+1,j}+\sum_{i=1}^{m}\sum_{j=1}^{n-1}a_{i,j}a_{i,j+1}+\sum_{j=1}^{n-1}\sum_{i=1}^{m-1}a_{i,j}a_{i+1,j+1}+\sum_{j=2}^{n}\sum_{i=1}^{m-1}a_{i,j}a_{i+1,j-1}$$

First, we prove

Proposition 3.1 For a minesweeping game of m rows and n columns. If $m\geq n$, then $X_{m,n}\geq X_{n,n}-(m-n)(2n-1)$.

Proof We have

$$X_{m,n}=\sum_{j=1}^{n}\sum_{i=1}^{m-1}a_{i,j}a_{i+1,j}+\sum_{i=1}^{m}\sum_{j=1}^{n-1}a_{i,j}a_{i,j+1}+\sum_{j=1}^{n-1}\sum_{i=1}^{m-1}a_{i,j}a_{i+1,j+1}+\sum_{j=2}^{n}\sum_{i=1}^{m-1}a_{i,j}a_{i+1,j-1}$$

$$=\sum_{j=1}^{n}\sum_{i=1}^{m-2}a_{i,j}a_{i+1,j}+\sum_{j=1}^{n}a_{m-1,j}a_{m,j}+\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_{i,j}a_{i,j+1}+\sum_{j=1}^{n-1}a_{m,j}a_{m,j+1}+\sum_{j=1}^{n-1}\sum_{i=1}^{m-2}a_{i,j}a_{i+1,j+1}$$

$$+\sum_{j=1}^{n-1}a_{m-1,j}a_{m,j+1}+\sum_{j=2}^{n}\sum_{i=1}^{m-2}a_{i,j}a_{i+1,j-1}+\sum_{j=1}^{n-1}a_{m-1,j+1}a_{m,j}$$

$$=\sum_{j=1}^{n}\sum_{i=1}^{m-2}a_{i,j}a_{i+1,j}+\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_{i,j}a_{i,j+1}+\sum_{j=1}^{n-1}\sum_{i=1}^{m-2}a_{i,j}a_{i+1,j+1}+\sum_{j=2}^{n}\sum_{i=1}^{m-2}a_{i,j}a_{i+1,j-1}+\sum_{j=1}^{n-1}a_{m-1,j}a_{m,j}$$

$$+\sum_{j=1}^{n-1}a_{m,j}a_{m,j+1}+\sum_{j=1}^{n-1}a_{m-1,j}a_{m,j+1}+\sum_{j=1}^{n-1}a_{m-1,j+1}a_{m,j}+a_{m-1,n}a_{m,n}$$

$$=X_{m-1,n}+\sum_{j=1}^{n-1}(a_{m-1,j}a_{m,j}+a_{m,j}a_{m,j+1}+a_{m-1,j}a_{m,j+1}+a_{m-1,j+1}a_{m,j})+a_{m-1,n}a_{m,n}.$$

Rearranging the terms we obtain that

$$a_{m-1,j}a_{m,j}+a_{m,j}a_{m,j+1}+a_{m-1,j}a_{m,j+1}+a_{m-1,j+1}a_{m,j}$$
$$=\frac{1}{2}(a_{m-1,j}+a_{m,j}+a_{m,j+1})^2-\frac{1}{2}(a_{m-1,j}{}^2+a_{m,j}{}^2+a_{m,j+1}{}^2)+a_{m-1,j+1}a_{m,j}$$
$$=\frac{1}{2}(a_{m-1,j}+a_{m,j}+a_{m,j+1})^2+a_{m-1,j+1}a_{m,j}-\frac{3}{2}.$$

Remember that

$a_{m-1,j}+a_{m,j}+a_{m,j+1}=1$, 3, -1 or -3, and $a_{m-1,j+1}a_{m,j}=1$ or -1, we have that

$$\frac{1}{2}(a_{m-1,j}+a_{m,j}+a_{m,j+1})^2+a_{m-1,j+1}a_{m,j}-\frac{3}{2}\geq\frac{1}{2}-1-\frac{3}{2}=-2.$$

Thus

$$X_{m-1,n}+\sum_{j=1}^{n-1}(a_{m-1,j}a_{m,j}+a_{m,j}a_{m,j+1}+a_{m-1,j}a_{m,j+1}+a_{m-1,j+1}a_{m,j})+a_{m-1,n}a_{m,n}$$
$$\geq X_{m-1,n}-1-2\ (n-1)=X_{m-1,n}-2n+1.$$

Equivalently, we have $X_{m,n}\geq X_{m-1,n}-2n+1.$

Similarly, $X_{i,n}\geq X_{i-1,n}-2n+1$ holds for all $i\geq n+1$. Adding the inequalities we get

$$\sum_{i=n+1}^{m}X_{i,n}\geq\sum_{i=n+1}^{m}(X_{i-1,n}-2n+1),\ \text{or}\quad X_{m,n}\geq X_{n,n}-(m-n)(2n-1).$$

Proposition 3.2 $X_{n,n}\geq -2n^2+4n-2.$

Proof We have

$$X_{n,n}=\sum_{j=1}^{n}\sum_{i=1}^{n-1}a_{i,j}a_{i+1,j}+\sum_{i=1}^{n}\sum_{j=1}^{n-1}a_{i,j}a_{i,j+1}+\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}a_{i,j}a_{i+1,j+1}+\sum_{j=2}^{n}\sum_{i=1}^{n-1}a_{i,j}a_{i+1,j-1}$$
$$+\sum_{i=1}^{n-2}a_{i,n-1}a_{i+1,n}+\sum_{i=1}^{n-2}a_{i,n}a_{i+1,n-1}+\sum_{j=1}^{n-2}a_{n,j}a_{n-1,j+1}+a_{n-1,n-1}a_{n,n}+a_{n-1,n}a_{n,n-1}+a_{n-1,n-1}a_{n,n-1}$$
$$+a_{n-1,n-1}a_{n-1,n}+a_{n-1,n}a_{n,n}+a_{n,n-1}a_{n,n}$$
$$=X_{n-1,n-1}+\sum_{j=1}^{n-1}a_{n-1,j}a_{n,j}+\sum_{i=1}^{n-1}a_{i,n}a_{i+1,n}+\sum_{i=1}^{n-1}a_{i,n-1}a_{i,n}+\sum_{j=1}^{n-1}a_{n,j}a_{n,j+1}+\sum_{j=1}^{n-2}a_{n-1,j}a_{n,j+1}$$
$$+\sum_{i=1}^{n-2}a_{i,n-1}a_{i+1,n}+\sum_{i=1}^{n-2}a_{i,n}a_{i+1,n-1}+\sum_{j=1}^{n-2}a_{n,j}a_{n-1,j+1}+a_{n-1,n-1}a_{n,n}+a_{n-1,n}a_{n,n-1}$$
$$=X_{n-1,n-1}+\sum_{j=1}^{n-2}a_{n-1,j}a_{n,j}+\sum_{i=1}^{n-2}a_{i,n}a_{i+1,n}+\sum_{i=1}^{n-2}a_{i,n-1}a_{i,n}+\sum_{j=1}^{n-2}a_{n,j}a_{n,j+1}+\sum_{j=1}^{n-2}a_{n-1,j}a_{n,j+1}$$
$$+\sum_{i=1}^{n-2}a_{i,n-1}a_{i+1,n}+\sum_{i=1}^{n-2}a_{i,n}a_{i+1,n-1}+\sum_{j=1}^{n-2}a_{n,j}a_{n-1,j+1}+a_{n-1,n-1}a_{n,n}+a_{n-1,n}a_{n,n-1}+a_{n-1,n-1}a_{n,n-1}$$
$$+a_{n-1,n-1}a_{n-1,n}+a_{n-1,n}a_{n,n}+a_{n,n-1}a_{n,n}$$
$$=X_{n-1,n-1}+\sum_{j=1}^{n-2}(a_{n-1,j}a_{n,j}+a_{n,j}a_{n,j+1}+a_{n-1,j}a_{n,j+1}+a_{n,j}a_{n-1,j+1})$$

$$+\sum_{i=1}^{n-2}(a_{i,n}a_{i+1,n}+a_{i,n-1}a_{i,n}+a_{i,n-1}a_{i+1,n}+a_{i,n}a_{i+1,n-1})+a_{n-1,n-1}a_{n,n}+a_{n-1,n}a_{n,n-1}+a_{n-1,n-1}a_{n,n-1}$$

$$+a_{n-1,n-1}a_{n-1,n}+a_{n-1,n}a_{n,n}+a_{n,n-1}a_{n,n}.$$

Using the similar reasoning as in the proof of the above proposition, we know that

$$a_{n-1,j}a_{n,j}+a_{n,j}a_{n,j+1}+a_{n-1,j}a_{n,j+1}+a_{n,j}a_{n-1,j+1}\geq -2,$$

$$a_{i,n}a_{i+1,n}+a_{i,n-1}a_{i,n}+a_{i,n-1}a_{i+1,n}+a_{i,n}a_{i+1,n-1}\geq -2,$$

$$a_{n-1,n-1}a_{n,n}+a_{n-1,n}a_{n,n-1}+a_{n-1,n-1}a_{n,n-1}+a_{n-1,n-1}a_{n-1,n}+a_{n-1,n}a_{n,n}+a_{n,n-1}a_{n,n}\geq -2.$$

Thus $X_{n,n}\geq X_{n-1,n-1}-4n+6.$

Similarly, $X_{i,i}\geq X_{i-1,i-1}-4i+6$ for all $i\in N^{+}$.

Adding the inequalities, we obtain $X_{n,n}\geq X_{2,2}+6(n-2)-2n^2-2n+12.$

Counting directly, we get that

$$X_{2,2}=a_{1,1}a_{2,2}+a_{1,2}a_{2,1}+a_{1,1}a_{2,1}+a_{1,1}a_{1,2}+a_{1,2}a_{2,2}+a_{2,1}a_{2,2}\geq -2.$$

Substituting in the above inequality, we get what we wanted

$$X_{n,n}\geq -2n^2+4n-2.$$

Combining Proposition 3.1 and 3.2, we obtain one of our main theorem.

Theorem 3.3 Assume $m\geq n$. For an ordinary minesweeping game of *m* rows and *n* columns, the summation of numbers $A_{m,n}\leq(3n-2)(m-1)$.

Proof From Proposition 3.1 and 3.2, we know that

$$X_{m,n}\geq -2n^2+4n-2-(m-n)(2n-1).$$

Using this inequality and the equality $2A_{m,n}=S-X_{m,n}$, we get that

$$2A_{m,n}\leq 4mn-3m-3m+2+2n^2-4n+2+(m-n)(2n-1)=2(3n-2)(m-1).$$

Thus $A_{m,n} \le (3n-2)(m-1)$.

Remark It is obvious that the inequality holds when the game is of *n* columns and *m* rows.

Theorem 3.4 Assume $m \ge n$. For an ordinary minesweeping game of *m* rows and *n* columns, there exist assignments such that $A_{m,n} = (3n-2)(m-1)$.

Proof We need only to give an example with $A_{m,n} = (3n-2)(m-1)$. Assign a mine to a square if and only if it is in an odd row. Then the summation $A_{m,n}$ of the numbers is

$$A_{m,n} = \begin{cases} 6(n-2)(\frac{m}{2}-1) + 4\times 2\times(\frac{m}{2}-1) + 3(n-2) + 4, & \text{if } m \text{ is even,} \\ 6(n-2)\frac{m-1}{2} + 4\times 2\times \frac{m-1}{2}, & \text{if } m \text{ is odd.} \end{cases}$$

Thus $A_{m,n} = (3n-2)(m-1)$ in both cases.

Remark The equality holds when the roles of rows and columns exchange.

Remark If we use the notations $a_{i,j}$'s to express the assignments, the example in the proof of Proposition 3.4 is of the case $a_{i,j} = (-1)^i$.

3.2 The assignments for ordinary minesweeping games whose summations of numbers achieving the upper bound

In the following discussions, we still use the notation

$$a_{i,j} = \begin{cases} 1, & \text{if the square of ith row and jth column is assigned a mine,} \\ -1, & \text{otherwise.} \end{cases}$$

Moreover, two situations are said to be *equivalent* if one of them can be obtained from the other through the following operations:

(1) rotation;

(2) flip;

(3) multiplying all the entries $a_{i,j}$ with -1.

3.2.1 One of the entries m and n is equal to 2.

Suppose n=2. Under this assumption, the situations are not unique. For example, the maximum can be reached if we let $a_{i,1}a_{i,2}=-1$ $(\forall i\in\{1,2,\cdots,m\})$. Besides, there are also other situations in which the maximum can be reached for a given m. As an example, consider $m=3,n=2$. The equality holds if we let $a_{1,1}=a_{1,2}=a_{3,1}=-a_{2,1}=-a_{2,2}=-a_{3,2}=1$.

3.2.2 $m\geq n\geq 3$

In this case, the maximum can only be reached by letting $a_{i,j}=(-1)^i$. Next, we give an inductive proof on this statement. First, assume that $m=n$. For $n=3$, we can show the fact holds by enumeration. Suppose the fact holds for $n=k$. From the previous proof of inequality $X_{n,n}\geq X_{n-1,n-1}-4n+6$, we know that the equality holds if the equations

$$\begin{cases} a_{i,n}a_{i+1,n-1}=-1, i=1,2...n-1 \\ a_{n,j}a_{n-1,j+1}=-1, j=1,2...n-1 \\ a_{n-1,n-1}+a_{n-1,n}+a_{n,n-1}+a_{n,n}=0 \end{cases},$$

hold. As a result, the statement holds for $n=k+1$.

Next we inductively show that the statement holds for $m\geq n+1$. Assume $m=n+1$. From the previous proof of inequality $X_{m,n}\geq X_{m-1,n}-2n+1$, we know that the equality holds if the equations

$$\begin{cases} a_{m,j}a_{m-1,j+1}=-1, j=1,2...n-1 \\ a_{m-1,n}a_{m,n}=-1 \end{cases}$$

hold, which means $a_{i,j}=(-1)^i$.

Suppose the statement holds for $m=k$. From the previous proof of inequality $X_{m,n}\ge X_{m-1,n}-2n+1$, we know that the equality holds if the equations

$$\begin{cases} a_{m,j}a_{m-1,j+1}=-1, j=1,2...n-1 \\ a_{m-1,n}a_{m,n}=-1 \end{cases}$$

hold, which means $a_{i,j}=(-1)^i$.

Hence, the statement holds.

3.2.3 $n\ge m+1\ge 4$

Using the similar methods as above, we know the maximum can be reached if and only if $a_{i,j}=(-1)^j$ holds.

3.3 The upper bound of summation of numbers for cyclic case

Replacing the $m\times n$ rectangle of an ordinary minesweeping game by an $m\times n$ torus, we get a cyclic one. In this case the board is cyclic. Thus it does not have boundaries. In this case, it is easy to compute the total number of pairs of squares, $S=\frac{8mn}{2}=4mn.$

There are two subcases to be considered.

3.3.1 mn is even

Theorem 3.5 Suppose that mn is even. For a cyclic minesweeping game of m rows and n columns, the summation of number $A_{m,n}\le 3mn.$

Proof We still use the notations $A_{m,n},B_{m,n},X_{m,n}$ as in the above subsections. Then we have

$$X_{m,n}=\sum_{j=1}^{n}\sum_{i=1}^{m}(a_{i,j}a_{i+1,j}+a_{i,j}a_{i,j+1}+a_{i,j}a_{i+1,j+1}+a_{i,j+1}a_{i+1,j}).$$

Where $a_{m+1,j}=a_{1,j},a_{i,n+1}=a_{i,1}$. Using the inequality

$$a_{i,j}a_{i+1,j}+a_{i,j}a_{i,j+1}+a_{i,j}a_{i+1,j+1}+a_{i,j+1}a_{i+1,j}\geq -2,$$

we obtain $X_{m,n}\geq -2mn$. This means that $A_{m,n}\leq 3mn$.

Example If $2|m$, let $a_{i,j}=(-1)^i$. Then $A_{m,n}=3mn.$ Similarly, if $2|n$, let $a_{i,j}=(-1)^j$. Then $A_{m,n}=3mn.$

3.3.2 mn is odd

In this case, we still have $A_{m,n}\leq 3mn.$ But, till now, we have not found an assignment whose summation $A_{m,n}$ reaching the upper bound $3mn$. The maximum of $A_{m,n}$ we obtained is $3mn-4n+5$. We conjecture that this is the maximum of $A_{m,n}$ in this case. This assertion has not been proved now.

3.4 The assignments for cyclic minesweeping games whose summations of numbers achieving the upper bound

As in section 3.2, we still use the notations

$$a_{i,j}=\begin{cases}1, & \text{if the square of ith row and jth column is assigned a mine,}\\ -1, & \text{otherwise.}\end{cases}$$

Moreover, two situations are said to be *equivalent* if one of them can be obtained from the other through the following operations:

(1) rotation;

(2) flip;

(3) multiplying all the entries $a_{i,j}$ with -1.

Since the summations of numbers of equivalent situations are equal, we do not distinguish equivalent assignments.

We only discuss the cases with $2 \mid mn$ and $m \geq n$.

3.4.1 Both *m* and *n* are even and $m \neq n$

Theorem 3.6 In the case $2 \mid m, 2 \mid n$ and $m \neq n$, the summation of numbers of cyclic minesweeping game of size $m \times n$ achieves the upper bound $3mn$ if and only if $a_{i,j} = (-1)^i$ or $a_{i,j} = (-1)^j$.

Proof Computing directly, we get that $A_{m,n} = 3mn$ in both cases. To prove that these cases are the only cases such that $A_{m,n} = 3mn$, we first give another proof of the inequality $X_{m,n} \geq -2mn$.

As above, we know that

$$2X_{m,n} = 2\sum_{j=1}^{n}\sum_{i=1}^{m}(a_{i,j}a_{i+1,j} + a_{i,j}a_{i,j+1} + a_{i,j}a_{i+1,j+1} + a_{i,j+1}a_{i+1,j}).$$

Using the conventions $a_{m+1,j} = a_{1,j}, a_{i,n+1} = a_{i,1}$, we can get

$$2X_{m,n} = \sum_{j=1}^{n}\sum_{i=1}^{m}(2a_{i,j}a_{i+1,j} + 2a_{i,j}a_{i,j+1} + 2a_{i,j}a_{i+1,j+1} + 2a_{i,j+1}a_{i+1,j})$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m}(a_{i,j}a_{i+1,j} + a_{i,j+1}a_{i+1,j+1} + a_{i,j}a_{i,j+1} + a_{i+1,j}a_{i+1,j+1} + a_{i,j}a_{i+1,j+1} + a_{i,j+1}a_{i+1,j})$$

$$+ \sum_{j=1}^{n}\sum_{i=1}^{m}(a_{i,j}a_{i+1,j+1} + a_{i,j+1}a_{i+1,j}).$$

Thus $2X_{m,n} \geq -2mn - 2mn = -4mn.$

Here we use inequalities

$a_{i,j}a_{i+1,j} + a_{i,j+1}a_{i+1,j+1} + a_{i,j}a_{i,j+1} + a_{i+1,j}a_{i+1,j+1} + a_{i,j}a_{i+1,j+1} + a_{i,j+1}a_{i+1,j} \geq$ and

$$a_{i,j}a_{i+1,j+1} + a_{i,j+1}a_{i+1,j} \geq -2.$$

In the first inequality, the "=" holds only if, for every i and j,

$$a_{i,j} + a_{i+1,j} + a_{i,j+1} + a_{i+1,j+1} = 0. \qquad (*)$$

In the second inequality, the "=" holds only if

$$a_{1,1} = a_{1,2} = -a_{2,1} = -a_{2,2} = 1 \text{ or } a_{1,1} = a_{2,1} = -a_{1,2} = -a_{2,2} = 1.$$

Repeatedly use the equation (*), we get that

$$a_{i,j} = (-1)^i a_{i,j} = (-1)^j \text{ or } a_{i,j} = (-1)^j.$$

3.4.2 *m* is even and $m = n$

In this case, the summation of numbers of cyclic minesweeping game of size $m \times m$ achieves the upper bound $3m^2$ if and only if $a_{i,j} = (-1)^j$.

3.4.3 *m* is even and *n* is odd

In this case, the summation of numbers of cyclic minesweeping game of size $m \times n$ achieves the upper bound $3mn$ if and only if $a_{i,j} = (-1)^i$.

3.4.4 *m* is odd and *n* is even

In this case, the summation of numbers of cyclic minesweeping game of size $m \times n$ achieves the upper bound $3mn$ if and only if $a_{i,j} = (-1)^j$.

4. Future works

For cyclic minesweeping in an $m \times n$ rectangle (mn is odd), the problem that how the maximum can be achieved is unsolved in this paper and left as a future work.

5. Concluding Remarks

In this paper, we obtain that the maximum of the summations of numbers of ordinary minesweeping games of size $m \times n$ is $(3n-2)(m-1)$ when $m \geq n$. Moreover, for the case of cyclic games, the maximum of the summations is $3mn$ when mn is even. Recently, a new version of minesweeping game, i.e., three-dimensional minesweeping, has been developed. The methods in this paper can be generalized in the analysis of three-dimensional minesweeping and the similar results can be obtained. This is also the topic of our future work.

6. Reference

[1] http://baike.baidu.com/view/30088.htm