论两个函数方程解析解的渐近性质

On the Asymptotic Properties of the Analytical Solutions to Two Functional Equations

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摘要

本文研究了两个函数方程，阐明了它们解析解的渐近性质. 它们可以看成是某些较为特殊的函数方程的一般化. 本文将两个函数方程转化为算子方程，然后使用微分方程逼近算子方程，研究了函数方程解析解的存在性. 而解析解的渐近性质则通过复分析方法阐明. 本文的结果能够简化迭代序列的渐近估计，也可能在动力系统 (尤其是离散动力系统的嵌入流) 方面发挥一些作用. 另外，本文的结果也以某种不同的视角揭示了算子方程同动力系统之间的联系.

Abstract

This paper aims at studying two functional equations, and the asymptotic properties of their solutions are also expounded. These two equations can be viewed as the generalization of some specific ones. The existence of analytical solutions is discussed through transforming the functional equations into operator equations, which are then approximated by differential equations. Theorems on the asymptotic properties of the solutions are derived from methods of complex analysis. The results in this paper may simplify the asymptotic estimation for sequences given by recursion formulae, which may also be useful for researches on dynamical systems (especially on the embedding flows of discrete dynamical systems). Furthermore, they may also reveal the relations between operator equations and dynamical systems in a different way.

1 Introduction

In mathematical problems functional equations should be dealt with. As an example, a dynamical system defined by an iteration relation can be identified as a functional equation: if

$$z_{n+1} = G(z_n), z_0 = a,$$

then the solution to the functional equation $f(z+1) = G(f(z)), f(0) = a$ is the embedding flow of the dynamical system $\{z_n\}$. Generally speaking, it is very hard to solve most of those functional equations. However, sometimes, revealing the asymptotic properties of the solutions is more important than finding a precise expression. For instance, factorial, a function on the set of positive integers, is widely used in statistics, probability theories and combinatorics. However, calculating the factorial of large numbers can be very difficult. So Stirling's Formula comes into use, as stated in [1], [2] and [3]:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

This formula makes it much easier to estimate the value of factorial for large numbers. The analogue of factorial for complex numbers is Euler's Γ function. Euler's Γ function has an asymptotic expression of the similar form to that for positive integers. This formula turns out to be useful in mathematical physics and analytical number theories, as stated in [1], [2]. Euler's Γ function satisfies the Gamma Equation:

$$\Gamma(z+1) = z\Gamma(z), z \in \mathbb{C}$$

From the instance of factorial and Γ function the importance of asymptotic properties can be seen when dealing with functional equations.

The aim of this paper is to clarify the asymptotic properties of two functional equations, using some methods of complex analysis. These two functional equations can be viewed as some kind of generalization of the Gamma Equation. Numbers of embedding flows take the form of these two functional equations. Section 2 is the preparation part for the whole research. Some lemmas necessary for the study are proved, including the asymptotic estimation for some simple complex functions. The lemmas on the existence of the analytical solutions to these two functions reveal the connections between this problem and operator equations. In Subsection 3.1 a functional equation of much simplicity is solved, and thus its oscillating behaviours emerges, which turns to be important for further studies. In Subsection 3.2 the functional equation is approximated by a differential equation. Finally, an example of asymptotic estimation is given in Section 4. Some discussions about further studies are also made here, suggesting the relation between the study and dynamical systems. The results of the study can provide a valuable reference for related researches.

2 Symbols, Hypothesis and Lemmas

2.1 Symbols and Hypotheses

Some hypotheses necessary for the research are listed below.

Two functional equations are studied in this paper:

$$g(1+z) = A(z)(g(z))^k, g(\tau) = a \tag{1}$$

$$f(1+z) = A(z)(f(z))^k + B(f(z), z), f(\tau) = a \tag{2}$$

Here τ, a are both complex constants, $a \neq 0$. Equation (2) can be viewed as the embedding flow of the dynamical system given by the recursion formula $z_{n+1} = A(n)z_n^k + B(z_n, n)$. We assume that $k > 1$, and $A(z)$ satisfies the following conditions: $\frac{A'}{A}(z) = \frac{\mathrm{d}}{\mathrm{d}z} \log A(z)$ is a meromorphic function with finite many poles, with its Mittag-Leffler expansion taking the form

$$\frac{A'}{A}(z) = Q(z) + \sum_{r=1}^{p} g_r(z) \tag{3}$$

where $Q(z)$ is a polynomial with its degree lower than m, and

$$g_r(z) = \sum_{\alpha=1}^{p_r} \frac{c_\alpha^{(r)}}{(z-b_r)^\alpha} \tag{4}$$

are the principle parts of the Laurent expansion of the function $\frac{A'}{A}(z)$, over all the poles. Set the collection of the poles of the function $\frac{A'}{A}(z)$ to be $\{b_r | r = 1, 2, ...p\}$, and $\mathrm{Re}\tau > \max\{\mathrm{Re}b_r\}$. Also assume that the function $g(z)$, solving equation (1), has no zero or singularity in the region $x > \mathrm{Re}\tau$, and there exists a vertical strip $L : \rho < x < \chi$ wider than 1, ρ sufficiently large, in which the function $f(z)$ has no zero or singularity. The function $B(\zeta, z)$ is analytical for each variable, and for every fixed ζ there is no zero or singularity of the function $B(\zeta, z)$. When $\zeta \to 0$ the function B keeps bounded, and it too satisfies

$$\frac{B(\zeta, z)}{A(z)\zeta^k} = \frac{h(z)}{\zeta^\delta} + o\left\{\frac{h(z)}{\zeta^\delta}\right\}, \mathrm{Re}z > \tau, \zeta \to \infty \tag{5}$$

where δ is a positive constant, and $h(z)$ is bounded in the region $\mathrm{Re}z > \tau$.

2.2 Lemmas

Some lemmas are proved here.

Lemma 1 The series written below converges uniformly to a meromorphic function:

$$\sum_{l=1}^{\infty} \frac{A'}{A}(z+l)/k^{z+l+1} \tag{6}$$

Proof. Consider a compact subset of $\mathbb{C}$, which does not contain any poles of the function $\frac{A'}{A}(z+l)$. In this set it is obvious that

$$\left|\sum_{l=1}^{\infty} \frac{A'}{A}(z+l)/k^{z+l+1}\right| = k^{-x}\left|\sum_{l=1}^{\infty} \frac{A'}{A}(z+l)/k^{l+1}\right|,$$

where $z = x+iy$. According to the hypothesis about $A(z)$ in equation (3) , the following inequality should be true for N sufficiently large:

$$\left|\sum_{l=N}^{\infty} \frac{A'}{A}(z+l)/k^{l+1}\right| \leqslant C\sum_{l=N}^{\infty}\left|\frac{(z+l)^m}{k^{l+1}}\right| \leqslant C\sum_{l=N}^{\infty}\frac{l^{m+1}}{k^{l+1}}, \tag{7}$$

where C is a constant. Due to the absolute convergence of the series on the right side of inequality (7), the series (6) is uniformly convergent. In response to Weierstrass's Theorem, as stated in [2] and [3], the series (6) represents an analytical function in the set considered. The compact subset is in fact arbitrarily chosen, so it is apparent that the function has no singularities except the poles in the whole complex plane, and thus is meromorphic. ■

Lemma 2 The following asymptotic expressions hold for all $x = \mathrm{Re}z > 0$:

$$\sum_{l=1}^{\infty} \frac{1}{k^l}\log(z+l) = \frac{1}{k-1}\log z + O\left(\frac{1}{x}\right)$$

$$\sum_{l=1}^{\infty} \frac{1}{k^l}(z+l)^{-s} = O\left(\frac{1}{x^s}\right)$$

Here $s > 1$ is a positive number, $\log z = \log|z| + i\arg z, \arg z \in (-\pi,\pi]$, and $k^z = e^{z\log k}, z^a = e^{a\log z}$.

Proof. Use the following expression, as stated in [1], [2] and [3]:

$$\frac{1}{z^s} = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}e^{-zt}\mathrm{d}t, x = \mathrm{Re}z > 0.$$

It is obvious that the integral is uniformly convergent. Fix $s = 1$, and take integral along the tangential path connecting 0 and z. Then the following expression is obtained:

$$\log z = \int_0^{\infty} \frac{e^{-t} - e^{-zt}}{t}\mathrm{d}t, \mathrm{Re}z > 0.$$

Due to the uniform convergence of the integral, the following equality is true:

$$\sum_{l=1}^{\infty} \frac{1}{k^l} \log(z+l) - \frac{1}{k-1} \log z = \int_0^{\infty} \frac{e^{-zt}}{t} \left(\frac{1}{k-1} - \frac{1}{ke^t - 1} \right) \mathrm{d}t, \mathrm{Re} z > 0. \quad (8)$$

The uniform convergence of the series can be proved in the same way as in Lemma 1. When $s > 1$ the following expression holds:

$$\sum_{l=1}^{\infty} \frac{1}{k^l} (z+l)^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-zt}}{ke^t - 1} \mathrm{d}t, \mathrm{Re} z > 0. \quad (9)$$

For the integral in (8), the function $h(t) = \frac{1}{t} \left(\frac{1}{k-1} - \frac{1}{ke^t - 1} \right)$ has a limit $\frac{k}{(k-1)^2}$ at the point 0, which ensures the convergence of the integral. It is obvious that $|h(t)|$ is bounded, say, $|h(t)| \leqslant A, \forall t > 0$. So the following estimation holds for all $x = \mathrm{Re} z > 0$:

$$\left| \int_0^{\infty} \frac{e^{-zt}}{t} \left(\frac{1}{k-1} - \frac{1}{ke^t - 1} \right) \mathrm{d}t \right| \leqslant A \int_0^{\infty} e^{-xt} \mathrm{d}t = \frac{A}{x}.$$

As for the integral in (9) the following estimation holds:

$$\begin{aligned} \left| \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-zt}}{ke^t - 1} \mathrm{d}t \right| &\leqslant \frac{1}{\Gamma(s)} \int_0^{\infty} \left| \frac{t^{s-1} e^{-zt}}{ke^t - 1} \right| \mathrm{d}t \\ &\leqslant \frac{1}{\Gamma(s)} \frac{1}{k-1} \int_0^{\infty} t^{s-1} e^{-xt} \mathrm{d}t = \frac{1}{k-1} \frac{1}{x^s}. \end{aligned}$$

And the proof is completed. ■

Lemma 3 Define a linear operator

$$e^D := \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n}.$$

If $G(z)$ is holomorphic and $z+1$ is also in the definition domain of $G(z)$, then $e^D G(z) = G(1+z)$.

Proof. As G is analytical, it can be expressed by using Cauchy's formula, as stated in [2] and [3]:

$$\frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} G(z) = \frac{1}{2\pi i} \oint_L \frac{G(\zeta)}{(\zeta - z)^{n+1}} \mathrm{d}\zeta.$$

Take summation:

$$e^D G(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_L \frac{G(\zeta)}{(\zeta - z)^{n+1}} \mathrm{d}\zeta = \frac{1}{2\pi i} \oint_L \mathrm{d}\zeta \sum_{n=0}^{\infty} \frac{G(\zeta)}{(\zeta - z)^{n+1}}$$

$$= \frac{1}{2\pi i} \oint_L \frac{G(z)}{\zeta - (z+1)} \mathrm{d}\zeta = G(z+1).$$

The analyticity of G ensures the commutation of taking summation and integration. The Lemma is proved true. ■

In accordance to Lemma 3, equation (2) can be rewritten as (if taking infinite summation is legal)

$$e^D f(z) = A(z)(f(z))^k + B(f(z), z).$$

This is an operator equation. It is equivalent to

$$\sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} f(z) = A(z)(f(z))^k + B(f(z), z).$$

This suggests the possibility of approximating the solution by a sequence of differential equations, by cutting off the higher ordered terms:

$$\sum_{n=0}^{N} \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} f_N = A(z) f_N^k + B(f_N, z).$$

This ordinary differential equation is equivalent to the following equation, as stated in [4]:

$$\frac{\mathrm{d}}{\mathrm{d}z} \mathbf{w}_N = \mathbf{H}_N(z, \mathbf{w}_N), \tag{10}$$

where

$$\mathbf{w}_N = (w_1, w_2, ..., w_N), w_1(z) = f_N(z)$$

is a $N-$component (complex) vector, and

$$\mathbf{H}_N(z, \mathbf{w}_N) = \left(w_2, ..., w_N, A(z) w_N^k + B(w_N, z) - \sum_{n=0}^{N-1} \frac{1}{n!} w_n \right)$$

is a map: $\mathbb{C}^N \to \mathbb{C}^N$.

Lemma 4 Suppose that $U(r)$ is the open disk: $|z - \tau| < 1/2 + r$, where $r > 0$ is a real number. Define a norm $\| \cdot \|$, where

$$\|\mathbf{w}_N(z)\| = \sup_{z \in U(r)} |\mathbf{w}_N(z)|.$$

Also, assume that $\Pi_N \subset \mathbb{C}^N$ is a region, and $\mathbf{z}_N^0 \in \mathbb{C}^N$ a N-complex vector. For all $\mathbf{w}_N \in \Pi_N$, the following vector is in Π_N:

$$\mathbf{z}_N^0 + \int_\tau^z \mathbf{H}_N(\zeta, \mathbf{w}_N) \mathrm{d}\zeta;$$

and

$$|\mathbf{H}_N(z, \mathbf{w}_N) - \mathbf{H}_N(z, \omega_N)| \leqslant M_N |\mathbf{w}_N - \omega_N|$$

holds for all $\mathbf{w}_N, \omega_N \in \Pi_N$. Under these assumptions, if

A) $\overline{\lim}_{N\to\infty}\{M_N\} < 2$; and

B) the initial condition $\mathbf{z}_N^0$ can be so chosen that the family $\{f_N\}$ is uniformly bounded in $U(r)$ for N sufficiently large and some $r > 0$(r does not rely on N),

then there exists a solution to functional equation (2) in the region $B = \bigcup_n B_n$, where B_n is the open disk centered at $\tau + n$ with a radius $1/2 + r$.

Proof. For condition A), rewrite the differential equations (10):

$$\mathbf{w}_N = \mathbf{z}_N^0 + \int_\tau^z \mathbf{H}_N(\zeta, \mathbf{w}_N)\mathrm{d}\zeta.$$

In the form of operators, it becomes

$$\mathbf{w}_N = J_N \mathbf{w}_N.$$

As the inequality $|\mathbf{H}_N(z, \mathbf{w}_N) - \mathbf{H}_N(z, \omega_N)| \leqslant M_N|\mathbf{w}_N - \omega_N|$ is true, the following estimation for the operator J_N holds:

$$\|J_N\mathbf{w}_N - J_N\omega_N\| \leqslant |z - \tau| M_N \|\mathbf{w}_N - \omega_N\|.$$

Consider the open disk U_h:

$$U_h = \left\{ z : |z - \tau| < \frac{h}{M_N} = \frac{h}{2} + hr_N \right\},$$

where $h < 1$ is fixed. Due to the condition $\overline{\lim}_{N\to\infty}\{M_N\} < 2$, the inequality $r = \inf_{N>N_0}\{r_N\} > 0$ is true for N_0 sufficiently large. Also, the space of analytical functions on U_h can be proved complete under the norm $\|\cdot\|$. The map $J_N : \Pi_N \to \Pi_N$ turns to be a contracting map on a complete normed space for all $z \in U_h$. In accordance to the theories on differential equations, as stated in [4], there exists a unique solution to equations (10) in U_h, given any initial condition $\mathbf{z}_N^0 \in \mathbb{C}^N$. But now $h < 1$ is arbitrary. Choosing $r = \inf_{N>N_0}\{r_N\}$, the existence of analytical solutions to the differential equations for in $U(r)$ and $N > N_0$, or in other words, the existence of the family $\{f_N(z)\}$, turns to be obvious.

As for condition B), because of the existence and uniform boundness of the family $\{f_N(z)\}$ in the disk $U(r)$, one can always extract a uniformly convergent subsequence from the family, according to Montel's principle, as stated in [2] and [6]. The limit function $f(z)$ of this subsequence is analytical in $U(r)$ and, it solves equation (2), for the limit of the sequence of differential equations is the operator equation:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} f(z) = A(z)(f(z))^k + B(f(z), z),$$

which has been discussed before.

Now $f(z)$ is analytical in an open disk whose radius is strictly larger than $1/2$, therefore the analytical continuation $f(1+z) = A(z)(f(z))^k + B(f(z), z)$ in the region $B = \bigcup_n B_n$ is well defined. The solution in B does exist. Thus the proof is completed. ■

Lemma 3 and 4 suggest that the research on functional equation can be carried through the methods of operator equations, and can thus be transformed into the problem of differential equations.

Lemma 5 Suppose that $G(z)$ is analytical in a strip wider than 1, and $G(z \pm iu)/e^u$ tends to 0 when $u \to \infty$. Then there exists an analytical function $S(z)$ in the region $\mathrm{Re}z > \tau$ which satisfies the following functional equation:

$$S(z) - S(z-1) = G(z).$$

Proof. According to Abel-Plana's formula, as stated in [1], for all $\mathrm{Re}z > \tau$ the following expression holds:

$$\frac{1}{2}\{G(z) + G(z-1)\}$$

$$= \int_{z-1}^{z} G(\zeta)\mathrm{d}\zeta - i \int_0^{\infty} \frac{G(z+iu) - G(z-1+iu) - G(z-iu) + G(z-1-iu)}{e^{2\pi u} - 1} \mathrm{d}u,$$

where the integrals are taken along straight ways connecting the beginning and the ending point. It is apparent that the function $S(z)$ defined by the following expression is analytical, due to the uniform convergence of the improper integrals:

$$S(z) = \frac{1}{2}\{G(z) + G(\tau)\}$$

$$+ \int_{\tau}^{z} G(\zeta)\mathrm{d}\zeta - i \int_0^{\infty} \frac{G(z+iu) - G(\tau+iu) - G(z-iu) + G(\tau-iu)}{e^{2\pi u} - 1} \mathrm{d}u.$$

Replace z by $z-1$:

$$S(z) - S(z-1) = \frac{1}{2}\{G(z) - G(z-1)\}$$

$$+ \int_{z-1}^{z} G(\zeta)\mathrm{d}\zeta - i \int_0^{\infty} \frac{G(z+iu) - G(z-1+iu) - G(z-iu) + G(z-1-iu)}{e^{2\pi u} - 1} \mathrm{d}u.$$

A functional equation which is equivalent to $S(z)-S(z-1)=G(z)$ is obtained:

$$S(z)-S(z-1)=\frac{1}{2}\{G(z)-G(z-1)\}+\frac{1}{2}\{G(z)+G(z-1)\}.$$

Therefore the proof is completed. ■

3 Asymptotic Properties of the Analytical Solutions

3.1 The Equation Without the Remainder Term

Theorem 3.1 There is an analytical $g(z)$ that solves functional equation(including the initial condition)(1), the asymptotic expression of which takes the following form:

$$g(z+1)=\exp\left(C_1+Ck^z+Q_1(z)\right)P(z)\left\{1+O\left(\frac{1}{x}\right)\right\},$$

where $z=x+iy$ is in the region $\mathrm{Re}z>\tau$, $Q_1(z)$ is a polynomial, and

$$C_1=-\sum_{l=1}^{\infty}\frac{\log A(\tau)}{k^l}+Q_1(\tau),$$

$$C=\frac{\log a}{k^\tau}+\frac{1}{k^\tau}\sum_{l=0}^{\infty}\frac{\log A(l+\tau)}{k^l},$$

$$P(z)=\prod_r(z-b_r)^{\frac{c_1^{(r)}}{1-k}}.$$

The value of logarithmic function is taken a particular branch.

Proof. Take logarithm on both sides of the equation and differentiate it, then multiply both sides of the equation by$1/k^{z+1}$:

$$\frac{g'}{g}(1+z)/k^{1+z}=\frac{A'}{A}(z)/k^{1+z}+\frac{g'}{g}(z)/k^z. \tag{11}$$

Take summation of (11):

$$\frac{g'}{g}(1+n+z)/k^{1+n+z}=\sum_{l=1}^{n}\frac{A'}{A}(z+l)/k^{z+l+1}+\frac{g'}{g}(1+z)/k^{1+z}.$$

In response to Lemma 1, the series listed below is uniform convergent and thus represents a meromorphic function:

$$\sum_{l=1}^{\infty} \frac{A'}{A}(z+l)/k^{z+l+1}.$$

So the function $g(z)$ defined by the following expansion is a solution to functional equation (1):

$$\frac{g'}{g}(1+z) = C'k^z - \sum_{l=1}^{\infty} \frac{A'}{A}(z+l)/k^{l+1}, C' = const.$$

As stated in [3], this is the most simple solution to (1). Taking integral along the tangential path connecting 0 and z, we get

$$\log g(1+z) = \frac{1}{\log k} C'k^z - \sum_{l=1}^{\infty} \frac{\log A(z+l)}{k^{l+1}}, \tag{12}$$

where the value of logarithmic function is taken to balance the equation. Considering the initial condition, the constant C' in (11) should be determined by the following expression:

$$C' = \frac{\log a}{k^{\tau}} \log k + \frac{\log k}{k^{\tau}} \sum_{l=0}^{\infty} \frac{\log A(l+\tau)}{k^l} = C \log k.$$

For the meromorphic function $\sum_{l=1}^{\infty} \frac{A'}{A}(z+l)/k^{l+1}$, we divide it into two parts, the polynomial part $\sum_{l=1}^{\infty} Q(z+l)/k^{l+1}$ and the main part $\sum_{l=1}^{\infty} \sum_{r=1}^{p} g_r(z+l)/k^{l+1}$, in response to equation (3). For the polynomial part, the series written below is still a polynomial (the convergence of its coefficients is obvious):

$$\sum_{l=1}^{\infty} \frac{1}{k^l}(z+l)^m = \sum_{l=1}^{\infty} \frac{1}{k^l} \left\{ \sum_{j=1}^{m} C_j^m z^j l^{m-j} \right\}$$

(m is an integer). By taking summation and integral we obtain the polynomial $Q_1(z)$. As for the main part, we see that after integration this part becomes

$$\sum_{l=1}^{\infty} \left\{ \frac{1}{k^l} \sum_{r=1}^{p_r} \int_{\tau}^{z+l} g_r(z) \mathrm{d}z \right\}.$$

In response to Lemma 2,

$$\sum_{l=1}^{\infty} \left\{ \frac{1}{k^l} \sum_{r=1}^{p_r} \int_{\tau}^{z+l} g_r(z) \mathrm{d}z \right\} =$$

$$\sum_{l=1}^{\infty} \left\{ \frac{1}{k^l} \sum_{\alpha=2}^{p} \frac{c_{\alpha}^{(r)}}{(\alpha-1)(z-b_r)^{\alpha-1}} \right\} + \sum_{r} \frac{c_1^{(r)} \log(z-b_r)}{k-1} + C_1 + R_1(z),$$

where $C_1 = -\sum_{l=1}^{\infty} \frac{\log A(\tau)}{k^l} + Q_1(\tau)$ and $|R_1(z)| \leqslant \sum_r \frac{A}{|x-b_r|}$.

To sum up, we obtain the following expression:

$$\log g(1+z) = C_1 + Ck^z + Q_1(z) - \sum_r \frac{c_1^{(r)} \log(z-b_r)}{k-1} + O\left(\frac{1}{x}\right). \tag{13}$$

For sufficiently large Re$z = x$ the inequality $|R_1(z)| \leqslant 1$ holds. Taking exponent on both sides of (12), we complete the proof. ■

Corollary 3.2 If both ReC and ImC are not 0, then the function $g(z)$ has different asymptotic values for different $y = \mathrm{Im}z$ as $x = \mathrm{Re}z \to \infty$.

Proof. Suppose that $C = re^{i\psi}, \theta = y \log k$, then the real part of the function Ck^z is determined by the following expression:

$$\mathrm{Re}\{Ck^z\} = rk^x \cos(\theta + \psi). \tag{14}$$

This function is apparently periodic for the variable y. After taking exponent on both sides of (13), we see that the modulus of the function $g(z)$ is

$$|g(z)| = \exp\{rk^x \cos(\theta+\psi) + \mathrm{Re}C_1 + \mathrm{Re}Q_1(z)\} |P(z)| \left\{1 + O\left(\frac{1}{x}\right)\right\}.$$

For x sufficiently large, this function is oscillatory in the region Re$z > \tau$: when $\cos(\theta+\psi) > 0$, $|g(z)|$ diverges to ∞, and when $\cos(\theta+\psi) < 0$, $|g(z)|$ tends to 0. For $\cos(\theta+\psi) = 0$ the property of $|g(z)|$ depends on the polynomial $Q_1(z)$. ■

3.2 The Equation with the Remainder Term

The asymptotic properties of the functional equation (2) are studied here. But the theorem given in this section is not the eventual form, for it calls for much more effort.

Under the conditions of Lemma 3 and 4, we assume that the analytical solutions to (2) exists. We first derive some propositions necessary for the asymptotic formula.

Proposition I There exists a non-bounded sequence in the region $x > \mathrm{Re}\tau$ along which $f(z)$ tends to ∞.

Proof. Dividing (2) by (1), one can rewrite the functional equation to be in the following form:

$$\frac{f(1+z)}{g(1+z)} = \left(\frac{f(z)}{g(z)}\right)^k \left\{1 + \frac{B(f(z),z)}{A(z)(f(z))^k}\right\}. \tag{15}$$

So

$$\frac{|f(z)|^k}{|f(1+z)|} = \left|\frac{A(z)(f(z))^k}{A(z)(f(z))^k + B(f(z),z)}\right| \frac{|g(z)|^k}{|g(1+z)|}. \tag{16}$$

According to equality (5), the factor

$$\frac{|A(z)(f(z))^k|}{|A(z)(f(z))^k + B(f(z),z)|}$$

in (16) is bounded, in other words, not exceeding a constant $M > 0$. So the following inequality is true:

$$\frac{|f(z)|^k}{|f(1+z)|} \leqslant M \frac{1}{|A(z)|}.$$

Suppose that $g_0(z)$ is a solution to the following functional equation, with some particular initial condition:

$$g_0(1+z) = \frac{1}{M} A(z)(g_0(z))^k.$$

The inequality combined with this functional equation imply $|f(z)| \geqslant |g_0(z)|$. But in response to Theorem 3.1 and Corollary 3.2, there are strips in which $|g_0(z)|$ tends to ∞. So $|f(z)|$ tends to ∞ in these strips, too. ∎

Proposition II requires some added assumptions: a) the solution to equation (2) does exist on some right half plane; b) Abel-Plana's formula is available.

Proposition II Suppose that

$$\varphi(z) = \log \frac{f(z)}{g(z)}, F(z) = \log \left\{1 + \frac{B(f(z),z)}{A(z)(f(z))^k}\right\}.$$

Then the function $\varphi(z)$ can be expressed as the following, if z is in a vertical strip which contains no zero or singularity of $f(z)$:

$$\varphi(z) = \frac{1}{2}\left(-\frac{F(z)}{k} + k^{z-\tau} F(\tau)\right) + k^z I_1 - i\frac{I_2}{k} - \frac{k^z}{k^{1+\tau}} i I_3, \tag{17}$$

here

$$I_1 = \int_\tau^z \frac{F(\zeta)}{k^\zeta} \mathrm{d}\zeta,$$

$$I_2 = \int_0^\infty \frac{k^{-iu}F(z+iu) - k^{iu}F(z-iu)}{e^{2\pi u}-1}\mathrm{d}u,$$
$$I_3 = \int_0^\infty \frac{k^{iu}F(\tau-iu) - k^{-iu}F(\tau+iu)}{e^{2\pi u}-1}\mathrm{d}u,$$

where the definition of logarithm is taken to balance these equations.

Proof. First assume that $z \in L : \rho < x < \chi$. By taking logarithm on both sides of equation (15), we obtain the following:

$$\varphi(z+1) = k\varphi(z) + F(z). \tag{18}$$

The given initial condition is thus transformed into $\varphi(\tau) = 0$. By dividing both sides of (18) by the function k^{1+z}, the following equation is obtained:

$$\frac{\varphi(z+1)}{k^{1+z}} = \frac{\varphi(z)}{k^z} + \frac{F(z)}{k^{1+z}}.$$

Suppose that $S(z) = \varphi(1+z)/k^{1+z}$ and $G(z) = F(z)/k^{1+z}$. Using Lemma 5, one immediately gets (17). Due to Proposition I, for $z \in L$ and $u \in \mathbb{R}$ the function $|f(z \pm iu)|$ for the variable u grows no faster than $e^{A|u|^m}$ $(A = const.)$, and thus the improper integrals here are all uniformly convergent ones.

If $z \notin L$, then we can use analytical continuation for the function $\varphi(z)$ defined in (17). Through the continuation we see that the expression (17) still holds if z is in a vertical strip which contains no zero or singularity of $f(z)$. ■

We see that the function $\varphi(z)$ defined in (17) does solve the functional equation. It is obvious that two of the terms in (17) are constants:

$$F(\tau), \int_0^\infty \frac{k^{iu}F(\tau-iu) - k^{-iu}F(\tau+iu)}{e^{2\pi u}-1}\mathrm{d}u.$$

Now we will provide a theorem about the asymptotic properties of $f(z)$ here.

Theorem 3.3 There exist a $f(z)$ that solves (2), for which the asymptotic expression written below holds, if the integral I_2 defined in Proposition II could be proved $O(1)$:

$$f(1+z) = \exp\left(C_1 + Ck^z + Q_1(z)\right)P(z)\exp\{w(z) + O(1)\}\left\{1 + O\left(\frac{1}{x}\right)\right\},$$

here $w(z)$ is a particular solution to the following differential equation :

$$\frac{\mathrm{d}}{\mathrm{d}z}w(z) = w(z)\log k + \left(1 + \frac{\log k}{2k} - \frac{1}{2k}\frac{\mathrm{d}}{\mathrm{d}z}\right)\log\left\{1 + \frac{B(g(z)e^{w(z)}, z)}{A(z)g^k(z)e^{kw(z)}}\right\}.$$

Proof. We have

$$\varphi(z) = Ak^z + k^z \int_\tau^z \frac{F(\zeta)}{k^\zeta} \mathrm{d}\zeta - \frac{F(z)}{2k} - i\frac{I_2}{k}, \tag{19}$$

where the constants before k^z have already been emitted into one constant A. Divide equation (19) by k^z on both sides and differentiate the resulting expression. Take a small circle $\mathcal{L}$ surrounding the point z, and assume that $R(z) = \frac{i}{k^z} I_2$. Using Cauchy's theorem we obtain

$$R'(z) = \frac{1}{2\pi i} \oint_{\mathcal{L}} \frac{R(\zeta)}{(\zeta - z)^2} \mathrm{d}\zeta.$$

According to Cauchy's inequality for the derivatives of analytical functions, we see that

$$|R'(z)| \leqslant \frac{\max R(z)|_{z \in \mathcal{L}}}{r_{\mathcal{L}}} = O\left(\frac{1}{k^z}\right).$$

Thus

$$\varphi'(z) = \varphi(z) \log k + F(z) - \frac{1}{2k} F'(z) + \frac{1}{2k \log k} F(z) + O(1). \tag{20}$$

Equation (20) suggests that $\varphi(z)$ holds the same asymptotic property as the following ordinary differential equation does:

$$\frac{\mathrm{d}}{\mathrm{d}z} w(z) = w(z) \log k + \left(1 + \frac{\log k}{2k} - \frac{1}{2k} \frac{\mathrm{d}}{\mathrm{d}z}\right) \log \left\{1 + \frac{B(g(z)e^{w(z)}, z)}{A(z) g^k(z) e^{kw(z)}}\right\},$$

where $w(z)$ is a particular solution to the differential equation. The initial condition for the differential equation can always be chosen to satisfy $w(z) - \varphi(z) = O(1)$. So we can obtain such a result:

$$\varphi(z) = \log \frac{f(z)}{g(z)} = w(z) + O(1). \tag{21}$$

Taking exponent on both sides of equation (21) and using the result in Theorem 3.1, the asymptotic formula for $f(z)$ is obtained. Thus the proof of Theorem 3.3 is completed. ■

4 Further Discussion and Prospects

This section is to give some further discussion on the results of this paper.

4.1 An Example of Asymptotic Estimating

The results of Theorem 3.1, 3.2 and 3.3 are useful when asymptotic estimation is needed. Here is an example.

Consider a sequence $\{b_n\}$, which is given by recursion formula:

$$b_1 = 1, b_{n+1} = e^{\frac{1}{n+1}} n^2 b_n^2.$$

Rewrite it as a functional equation:

$$g(1) = 1, g(z+1) = e^{\frac{1}{z+1}} z^2 g(z).$$

Using Theorem 3.1, one can immediately obtain

$$b_{n+1} = g(n+1) = \frac{1}{n^2} \exp\{C2^n\} \left\{1 + O\left(\frac{1}{n}\right)\right\},$$

where

$$C = \frac{1}{2} \sum_{l=1}^{\infty} \left(\frac{\log l}{2^l} - \frac{1}{(l+2)2^l} \right).$$

The sequence $\{b_n\}$ grows extremely fast.

4.2 About Solving the Functional Equation

As stated in Theorem 3.3, the differential equation for $w(z)$ can be used as an approximation only if the error term, say, the integral I_2, is proved to be $O(1)$. But how can this error term be calculated precisely enough, before finding the solution to functional equation (2)? This needs more thoroughly studies on the iteration sequence, under the conception of dynamical systems. Take polynomial as a special case, say, assume functional equation (2) to be

$$f(1+z) = P(f(z)) = a_n(f(z))^n + a_{n-1}(f(z))^{n-1} + ... + a_1 f(z) + a_0. \quad (22)$$

As claimed in Corollary 3.2, the solution to this functional equation is oscillatory if the all the coefficients except a_n vanish. Then a question naturally emerges: what if we are trying to study the general form of functional equation (22)? Does the property of oscillation remain, or transform into some behaviours harder to figure out? Proposition I in 3.2 suggests that the oscillatory property is general and essential for these functional equations. Also, as stated in [5], the Julia set of a polynomial $J(P) \neq \mathbb{C}$. Intuitively speaking, this might be the analogue of the "oscillation" to the general form (22). If all the properties of the solution to (21) are figured out, then the dynamical systems generated by the iteration of $P(z)$ can be just identified as the values of one single function f on different sequences.

The conditions in Lemma 4 imply the existence of analytical solutions to (2)(in an unbounded region). The idea of transforming the functional equation into operator equations might be not unworthy of some more thorough researches. The analogue of the linear operator e^D is

$$E(D) = \sum_{n=0}^{\infty} A_n(z) \frac{\mathrm{d}^n}{\mathrm{d}z^n}.$$

Here $\{A_n(z)\}$ is a family of analytical functions. The discussions about the "ordinary differential equation of infinite order" can be generalized as

$$E(D)f(z) = f(\varphi(z)),$$

where $\varphi(z)$ is analytical. Suppose that $E(D)f(z) = f(\varphi(z))$. Use Cauchy's formula:

$$E(D)f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_L \frac{G(\zeta)A_n(z)}{(\zeta - z)^{n+1}} \mathrm{d}\zeta = \frac{1}{2\pi i} \oint_L \frac{f(\zeta)}{\zeta - \varphi(z)} \mathrm{d}\zeta.$$

The functions $A_n(z)$ can be so chosen to make the equality hold, or, in other words, just determined by comparing the series to the expansion of $(\zeta - \varphi(z))^{-1}$. Thus the functional equation containing the term $f \circ \varphi$ can be transformed into an operator equation, or an "differential equation of infinite order". Then it is possible that the operator equation can be approximated by a sequence of differential equations, which means to ignore the differentiation terms with high orders, as claimed in Section 2.2. The problem of convergence emerges here, and thus waiting for more profound and thorough studies.

Further more, the singularities of $f(z)$ may be worthy of more consideration. Generally speaking, the point ∞ is the limit of the singularities of $f(z)$. Actually, if $A(z)$ has singularities, then so does $f(z)$: if z_0 is a singularity of $A(z)$, then generally it is the singularity of $f(1+z)$. Then $z_0 - 1$ is the singularity of $f(z)$... Generally speaking, $f(z)$ has infinite many singularities, and ∞ is a limit point of them. As stated in Proposition I in 3.2, the point ∞ behaves just like an essential singularity, though it is not isolated. So there will be a question: is there any analogue of Julia's theorem on asymptotic values, as stated in [6], for this condition? Intuitively speaking, the solution to this problem may not be similar to that for meromorphic functions. For meromorphic functions the problem of asymptotic values is related to Picard's Theorem, as stated in [6]. If the asymptotic property of the generalized equation (2) is clarified, then one might be able to find out the error term I_2 more precisely. Also, this will be useful for the studied on chaotic dynamical systems.

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