Solve the Energy-saving Problem of Household Water Heaters with Calculus and Differential Equations

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Problem

With the development of science and technology, electric appliances perform much better. They serve us in a more and more convenient and efficient way. Take cookers for example. As long as you put rice into a cooker and add water to the predefined level, you can expect a delicious meal after work. Washers have analogous function. However, water heaters do not feature such convenience. Currently, most new houses have been well decorated before sale and of course the water heater has been installed somewhere in the wall. Hence owners can only see a switch and a control panel outside which presents the time and the temperature (we call it t-t panel for short), as shown in the picture. We all know that water can be heated up to $100°C$ while we have no idea about the volume of the water heater. Certainly, you can always heat the water to $100°C$, for instance, to ensure enough hot water for a single person, but the waste of energy is egregious at the same time!

If you turn off the switch, there will be no display on the t-t panel. If you turn on it, the t-t panel will show alternatively Beijing time and the water temperature, each for several seconds. When temperatures appear, the number on the right is the setted temperature, suggesting the temperature we predefine, and the number on the left side is the real temperature which equals to the current water temperature in the heater. See the picture below. $75°C$is the setted temperature, $70°C$is the real temperature and 17:43 indicates 5:43 pm, Beijing time.

Figure 1 t-t panel

Now turn on the switch, then the t-t panel shows numbers. Suppose the real temperature at present is $42°C$ and the time is 8:06. And we make the setted temperature be$56°C$. Ten minutes later, i.e. 8:16, the t-t panel suggests that the real temperature of water has risen to $52°C$. Appropriate to have a shower! We turn off the switch for security and turn on it again immediately after the bath. The setted temperature remains $56°C$ of course, but the real temperature has changed to $48°C$ and the time is 8:20. With all these conditions, we try to

（1）model the change of water temperature in the water heater during your bath time;

（2）interpret the relationship between the flow of water and the volume of the heater water;

（3）find the lowest real temperature that is appropriate for bath if a man takes a shower for about 10 minutes with the water to $40°C$ minimum.

（4）find the lowest real temperature that is appropriate for bath if the water heater is reliable enough and a man can take a shower with the switch on.

CONTENTS

Chapter 1

Analysis, assumptions and notations

1.1 Analysis

Hybrid electric water heaters work on the following principle:

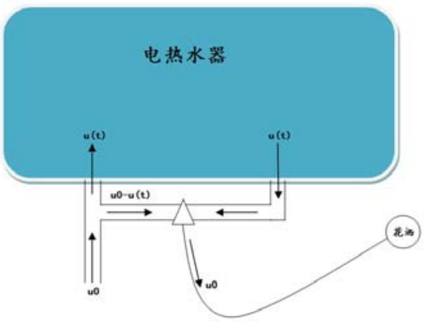

Figure 1.1 The Mode of Operation of Hybrid Electric Water Heaters

A quantity of cold water enters the water main, we denote it by u_0. A part of it flows through the intake into the water heater. Since the water temperature in the heater varies with time, the amount of this part is time-dependent as well, denoted by $u(t)$. It mixes with the hot water in the heater and then hot water is forced out. Next, the other part of the cold water mixes with the hot water at the water valve to reach the final ideal temperature. We denote this part by $u_0 - u(t)$.

To balance the pressure of the water tank, we further assume that the flow and its velocity at the intake is the same with that at the outlet. Then this problem has been reduced to problems of heat exchange in the water tank and at the water valve, which observe the law of conservation of energy.

1.2 Assumptions and Notations

One. Assumptions about the Model

1. The density of water is constant during the whole process;
2. The water temperature in the cold pipe is always $25°C$;
3. Pipes have the same diameter;
4. The flow at the valve varies continuously, keeping the temperature of water from the shower head at a comfortable point.（$40°C$in this problem）;
5. Heat exchange takes place mainly in the mix process. Ignore those happening between the tank wall and the air;
6. The quantity of inflow equals to that of the outflow;
7. Neglect any heat effect of the heating cord when the power is off;
8. Water flows continuously during the bath;
9. The quantity of the flow in pipes is a constant.

Two. Notations in the Model

- E .... The energy of water
- u_0 .... The quantity of the flow
- $T(t)$ .... Water temperature in the tank at time t
- T_0 .... Water temperature in the cold pipe
- V .... Volume of the water tank
- ρ .... The density of water
- $u(t)$ .... The quantity of flow at time t
- C .... The heat capacity
- P .... Power of the heating cord

Chapter 2

Preliminaries

2.1 Definitions of Derivatives and Definite Integrals

One. Definitions of Derivatives

Suppose that $f(x)$is defined on a neighborhood of x_0. If the limit of ratio between the increment of dependent variable

$$\triangle y = f(x_0 + \triangle x) - f(x_0)$$

and that of independent variable $\triangle x = x - x_0$exists, we say that $f(x)$isdifferentiableat x_0 and we call this limit the derivative of $f(x)$atx_0, denoted by

$$y'|_{x=x_0} = f'(x_0) = \lim_{\triangle x \to 0} \frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x}$$
$$= \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x}.$$

Two. Definitions of Definite Integrals

Let $f(x)$be a bounded function defined on $[a, b]$. There are $n + 1$ points in $[a, b]$, for instance $\{x_i\}_{i=0}^{n}$, forming a partition of $[a, b]$, i.e.

$$P : a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

. Denote the length of $[x_{i-1}, x_i]$by

$$\Delta x_i = x_i - x_{i-1}$$

and let $\lambda = \max\limits_{1\le i\le n}(\Delta x_i)$. If for any $\xi_i \in [x_{i-1}, x_i]$, the limit

$$\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

exists and has nothing to do with the partition P and the choice of ξ_i， then we say that $f(x)$is Riemann integrable on $[a, b]$. The sum

$$s_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

is called Riemann sum and its limit I is called the definit integral of $f(x)$ on $[a, b]$, denoted by

$$I = \int_a^b f(x)dx$$

. Here a and b are called the lower limit and the upper limit of the integral, respectively.

We set

$$I = \int_a^b f(x)dx = -\int_b^a f(x)dx$$

and

$$\int_a^a f(x)dx = 0$$

. This definition can also be expressed by "$\varepsilon - \delta$ language" as follows:

"We call a number I the definite of $f(x)$ on $[a, b]$ if for any $\varepsilon > 0$, there exists $\delta > O$, such that for any partition

$$P : a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

and any $\xi_i \in [x_{i-1}, x_i]$,

$$|\sum_{i=1}^{n} f(\xi_i)\Delta x_i - I| < \varepsilon$$

so long as $\lambda = \max\limits_{1\le i\le n}\Delta x_i < \delta$. And $f(x)$ is Riemann integrable on $[a, b]$ in this situation. With no confusion, we simply call Riemann integrable integrable.(We will learn in the future that there are other sorts of integral.) Remarkably, "definite integral" has nothing to do with the partition P and the choice of ξ_i.

2.2 Differential Equations and Correlative Modelings

One. Definition of Differential Equations

An identity consisting of an unknown function and its derivative is called a differential equation。

Especially, if there is only one independent variable in the function, this identity is an ordinary differential equation, differential equationfor short.

Two. Model with differential equations

The process of modeling with differential equations includes three steps:

First, interpret a practical issue into a mathematical problem which involves differential equations;

Secondly, solve the problem. Find its approximate solution or numeric solution, or study the property of solution with qualitative methods;

Last, go back to real life, uncover laws and make full use of these laws.

How to establish a differential equation?

1、Write down an identity according to known laws

There are many laws in mathematics, mechanics, physics, chemistry and so on.

Some of them, such as Newton's laws of motion, Newton's laws of cooling, laws of radiation and properties of tangent line, have been proved by experiments. We usually use them to find relationship between the rate of change of some variables and other changes, namely, write down a differential equation.

General steps are:

First, ascertain critical elements in the practical issue, such as independent variables, the unknown function, necessary parameters and constants and the coordinate system;

Secondly, recall relevant laws;

Thirdly, abstract a differential equation according to these laws and simplify it into a standard form.

2、Infinitesimal analysis

Establish an equation involving infinitely small quantities(those of independent variables and dependent variables).

Unlike the method of modeling according to laws, this method first deals with relationships between infinitesimals instead of listing out equations about change of rates directly.

When you use this method, please pay attention to critical elements in the practical issue and to laws about their infinitesimals, write down an identity and simplify it into a differential equation.

3、Approximate undeterminate simulative calibration

Many issues arising in biology or economics or other domains are very complicated. We can only settle these problems through approximate undeterminate simulative calibration.

To be specific, we build approximate models under different assumptions and then solve these models mathematically or study the properties of their solutions qualitatively, then we contrast our conclusion with real results and correct our models according to the difference.

Just be aware of three points. First, different assumptions yield different models. Secondly, the model that best explains real problems is one we should select and we need to improve it through correction. Thirdly, a model which is ideal under certain situation may be not so appropriate once situations change.

Chapter 3

Solutions

3.1 Solution to Question One

The heat exchange in the water tank complies with the law of conservation of energy. If the switch is off, then

$$\begin{aligned} & \text{energy of cold water flowing in} E_0 + \text{energy of hot water in the tank} E(t) \\ = \; & \text{energy of mixed water in the tank} E(t+\triangle t) + \text{energy of hot water flowing out} E \end{aligned} \tag{3.1}$$

By the formula of heat, energy E and temperature T satisfy:

$$E = C\rho VT$$

in which $C$is the heat capacity, ρ is the density of water, V is the volume of an object and T is its temperature.

If the flow at time t is $u(t)$, then the mass of cold water that enters the tank during $[t, t+\Delta t]$ is

$$m = \int_t^{t+\triangle t} \rho u(s) ds$$

To balance the pressure in the tank, the quantities of inflow and outflow are both m.

Suppose the temperature of water in the tank at time t is T(t) and cold water is constantly $T_0 = 25C$. For items in (3.1)we have:

$$\text{energy of cold water flowing in} E_0 = C\rho \int_t^{t+\Delta t} u(s) ds T_0,$$

$$\text{energy of hot water in the tank} E(t) = C\rho VT(t),$$

$$\text{energy of mixed water in the tank} E(t + \triangle t) = C\rho VT(t + \Delta t),$$

$$\text{energy of hot water flowing out} E = C\rho \int_t^{t+\Delta t} u(s)T(s)ds.$$

By the law of conservation of energy, we get

$$C\rho \int_t^{t+\Delta t} u(s)dsT_0 + C\rho VT(t) = C\rho VT(t+\Delta t) + C\rho \int_t^{t+\Delta t} u(s)T(s)ds$$

which equivalently is

$$V\left(T(t+\Delta t) - T(t)\right) = T_0 \int_t^{t+\Delta t} u(s)ds - T(t) \int_t^{t+\Delta t} u(s)ds$$

More specifically,

$$V\frac{T(t+\Delta t) - T(t)}{\Delta t} = T_0 \frac{\int_t^{t+\Delta t} u(s)ds}{\Delta t} - \frac{\int_t^{t+\Delta t} u(s)T(s)ds}{\Delta t} \tag{3.2}$$

Now let

$$U(t) = \int_{t_0}^{t} u(s)ds.$$

This is a variable upper limit function, whose derivative is

$$U'(t) = u(t)$$

Hence,

$$\lim_{\Delta t\to 0} \frac{\int_t^{t+\Delta t} u(s)ds}{\Delta t} = \frac{U(t+\Delta t) - U(t)}{\Delta t} = U'(t) = u(t)$$

Take the limit $\triangle t \to 0$ of both sides in (3.2), then we obtain a differential equation of $T(t)$

$$V\frac{dT(t)}{dt} = -u(t)T(t) + T_0 u(t) \tag{3.3}$$

Besides, the valve connects with three pipes: （1）The outlet. Denote by E_1 the energy of outflow during $[t, t+\Delta t]$;（2）The cold water pipe. Denote by E_2 the energy of cold water that passes during $[t, t+\Delta t]$;（3）The pipe leading to shower head. Denote by E_3 the energy of hot water that finally flow out (40°)during $[t, t+\Delta t]$

Therefore, the law of conservation of energy is expressed as (see picture 1)

$$E_1 + E_2 = E_3.$$

Now we expand E_1, E_2 and E_3. Since the water mixes in a very short time, the temperature of hot water that is forced out is $T(t)$. Then we have

$$E_1 = C\rho \int_t^{t+\Delta t} u(s)T(s)ds$$

And

$$E_2 = C\rho \left(u_0 \Delta t - \int_t^{t+\Delta t} u(s)ds \right) T_0$$

So

$$E_3 = C\rho u_0 \Delta t 40°C$$

and

$$C\rho \int_t^{t+\Delta t} u(s)T(s)ds + C\rho \left(u_0 \Delta t - \int_t^{t+\Delta t} u(s)ds \right) T_0 = C\rho u_0 \Delta t 40°C$$

Divide both side of the equation by Δt, it follows

$$\frac{\int_t^{t+\Delta t} u(s)T(s)ds}{\Delta t} + u_0 T_0 - T_0 \frac{\int_t^{t+\Delta t} u(s)ds}{\Delta t} = u_0 40°C$$

Take the limit $\Delta t \to 0$ then we get

$$u(t)T(t) = u_0 40° - u_0 T_0 + u(t)T_0 \tag{3.4}$$

Substitute (3.4) into (3.3)

$$\frac{dT(t)}{dt} = -\frac{u_0}{V}(40°C - T_0) \tag{3.5}$$

3.2 Solution to Question Two

By previous assumptions, $T_0 = 25°$. Thus, model (3.5) changes to

$$\frac{dT(t)}{dt} = -15\frac{u_0}{V} \tag{3.6}$$

then

$$T(t) - T(t_0) = -15\frac{u_0}{V}(t - t_0)$$

Meanwhile conditions suggest $T(t) = 48°C, T(t_0) = 52°C, t - t_0 = \frac{1}{15}$. By simple calculation we get

$$\frac{u_0}{V} = 4$$

3.3 Solution to Question Three

One. Analysis

We need to find the lowest real temperature that is appropriate for a man who takes a shower for about 10 minutes with a preference on water of $40°C$ and above. By results in question one， the water temperature decreases linearly over time. Thus to maximum the efficiency, we can set a temperature which happens to fall to $40°C$ in 10 minutes.

Two. Solution

According to analysis above, we have

$$\frac{dT(t)}{dt} = -60$$

We also have $T(t) = 40°C$， $t - t_0 = \frac{1}{6}$. Thus

$$40° - T(t_0) = -10$$

namely,

$$T(t_0) = 50°C$$

3.4 Solution to Question Four

One. Analysis

Apart from the consideration to question one, we must take into account the effect of the heating cord as well.

Two. Obtain the power of the water heater

The power of the heater(P)is the ratio of the work (W)to time($\triangle t$), and W equals to the energy increment of water($\triangle E$). The formula can be expressed as

$$P = \frac{W}{\triangle t} = \frac{\triangle E}{\triangle t}, \tag{3.7}$$

The energy and the temperature satisfy:

$$\triangle E = C\rho V \triangle T, \tag{3.8}$$

then together with(3.7)、(3.8) we have

$$P = \frac{CM\triangle T}{\triangle t}, \tag{3.9}$$

Given conditions, water temperature can rise by $40°C$ after the heating cord works for 10 minutes. Substitute(3.9)then it follows

$$P = \frac{10}{\frac{1}{6}} C\rho V = 60C\rho V。 \tag{3.10}$$

Three. Bathe with the switch on

By the law of conservation of energy

$$\begin{aligned} & \text{energy of cold water flowing in} E_0 + \text{energy of hot water in the tank} E(t) + \text{work of the heating cord} W \\ = \; & \text{energy of mixed water in the tank} E(t+\triangle t) + \text{energy of hot water flowing out} \end{aligned}$$

Expand each item

$$\text{energy of cold water flowing in} E_0 = C\int_t^{t+\Delta t} \rho u(s) ds T_0$$

$$\text{energy of hot water in the tank} E(t) = C\rho V T(t)$$

$$\text{work of the heating cord} W = 60 C\rho V \triangle t$$

$$\text{energy of mixed water in the tank} E(t+\triangle t) = C\rho V T(t+\Delta t)$$

$$\text{energy of hot water flowing out} = C\int_t^{t+\Delta t} \rho u(s) T(s) ds$$

Thus we have

$$\begin{aligned} & C\int_t^{t+\Delta t} \rho u(s) ds T_0 + C\rho V T(t) + 60C\rho V\triangle t \\ & = C\rho V T(t+\Delta t) + C\int_t^{t+\Delta t} \rho u(s) T(s) ds \end{aligned}$$

Divided both sides by Δt and take the limit $\Delta t \to 0$

$$V\frac{dT}{dt} = u(t)T_0 + 60V - u(t)T(t)$$

using(3.4)again we obtain

$$\frac{dT}{dt} = 60 - \frac{u_0}{V}(40° - T_0) = 0$$

which indicates we only need to make the setted temperature be $40°C$ with the switch on.

Chapter 4

Extended Topics

4.1 An Unimaginable Conclusion

Our model has indicated a linear relationship between water temperature and the time, which means whatever the initial value is, the temperature decrease by the same amount over the same time. That seems to contradict the reality. Now, we use two methods to prove this conclusion.

Prove One

Let $u_0$be a constant and Δt be the time spent. Suppose the water temperature is T_1 before bath and T_2 after bath, and is T_0 in the cold water pipe. By the law of conservation of energy,

$$\text{energy change in the tank} + \text{energy of cold water} = \text{energy of water of } 40^\circ$$

namely,

$$C\rho VT_1 - C\rho VT_2 + C\rho u_0 \Delta t T_0 = C\rho u_0 \Delta t \times 40^\circ$$

This is equivalent to

$$T_1 - T_2 = \frac{(40^\circ C - T_0)u_0}{V} \times \Delta t$$

which draws the conclusion.

Prove Two

Let's start from a weighted average point of view.

We still use the notations is Prove One. Suppose the volume of the tank is V. Now consider a very short time. If the volumes of inflow are V_1 and V_2 respectively, the corresponding weighted average temperatures are

$$\overline{T_1} = \frac{T_1(V - V_1) + T_0 V_1}{V},$$

and

$$\overline{T_2} = \frac{T_2(V - V_2) + T_0 V_2}{V},$$

The differentce between the highest temperature and the weighted average temperature is

$$T_i - \overline{T_i} = \frac{(T_i - T_0)V_i}{V}, \quad i = 1, 2$$

which concludes

$$\frac{T_1 - \overline{T_1}}{T_2 - \overline{T_2}} = \frac{(T_1 - T_0)V_1}{(T_2 - T_0)V_2}$$

The volume of used water must be the same over the same lengh of time, so we can denote it by V_0 Then

$$\overline{T_3} = \frac{T_0(V_0 - V_1) + T_1 V_1}{V_0},$$

and

$$\overline{T_3} = \frac{T_0(V_0 - V_2) + T_2 V_2}{V_0},$$

Hence,

$$(T_1 - T_0)V_1 = (T_2 - T_0)V_2$$

Finally,

$$\frac{T_1 - \overline{T_1}}{T_2 - \overline{T_2}} = \frac{(T_1 - T_0)V_1}{(T_2 - T_0)V_2} = 1$$

This indicates the water temperature decreases by the same scale over very short periods of time. Therefore, by adding these periods together, we reach the desired conclusion.

4.2 Suggestions for Water Heater Factories

Suggestion One: Improve the precision of the t-t panel;

Suggestion Two: Make valve adjust automatically and label a recommended temperature on the heater to realize the efficiency.

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