q-Symmetric Polynomials and nilHecke Algebras

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Abstract

Symmetric functions appear in many areas of mathematics and physics, including enumerative combinatorics and the representation theory of symmetric groups. A q-bialgebra of "q-symmetric functions" generalizing the symmetric functions was defined by Ellis and Khovanov as a quotient of the quantum noncommutative symmetric functions. In the $q = -1$ (or "odd") case of these q-symmetric functions, they and Lauda introduced odd divided difference operators and an odd nilHecke algebra, used in the categorification of quantum groups. Using diagrammatic techniques, we study relations for the q-symmetric functions when q is a root of unity other than 1 or -1. We then use q-analogues of divided difference operators to define a q-nilHecke algebra and describe its properties. In addition, Wang and Khongsap introduced an odd analogue of Dunkl operators, which have connections to Macdonald polynomials in the even case. We find a connection between the odd Dunkl operator and the odd nilHecke algebra, and show that a variant of the odd Dunkl operator can be used in constructing operators that generate the Lie algebra $\mathfrak{sl}_2$.

Contents

1 Introduction

The set of $n!$ permutations of $1, 2, 3, \ldots, n$ forms a group with multiplication being the composition of elements and the identity permutation being the multiplicative identity. This group is called the *symmetric group*, which we will denote S_n. *Symmetric polynomials* are polynomials in n independent, commutative variables $x_1, x_2, \ldots, x_n$ that are invariant under the action of any permutation acting on the indices. As n tends to infinity, one obtains the ring of symmetric functions, Λ, which is a graded $\mathbb{C}$-algebra. The bases of this ring, including the elementary, complete, monomial, and Schur symmetric polynomials, have been well-studied. Symmetric polynomials arise in various contexts in mathematics and physics. They are important in enumerative combinatorics, where one may study the important bases through the tools of Young tableaux and algebraic combinatorics as in [10]. They have a fundamental role in the representation theory of the symmetric group, which itself plays a role in quantum mechanics of identical particles. The space of symmetric polynomials in n variables, Λ_n, may be identified by a Hopf algebra structure with an adjoint bilinear form, which we will utilize in our work.

The theory of noncommutative symmetric functions, where in general $x_i x_j \neq x_j x_i$ for $1 \leq i \neq j \leq n$, has also been developed and analogues of objects in the commutative case have been found. Here, one requires the notion of a quasideterminant to generalize the concept of a determinant with noncommutative entries, in order to express transition matrices between bases of the noncommutative symmetric functions [5]. The quantum case, where $x_j x_i = q x_i x_j$ for $j > i$, has recently been introduced by Ellis and Khovanov. These polynomials are inherently connected, in the $q = -1$ case, to superalgebras.

Superalgebras, which arise from supersymmetry in physics, are the direct sum of two spaces K_0 and K_1, where $K_i K_j \subseteq K_{i+j}$ and indices are read modulo 2. In other words, a superalgebra may be considered as a $\mathbb{Z}_2$-graded algebra, with the same operations as an algebra (the unit and the multiplication). We can also induce a braiding τ by

$$\tau : V \otimes W \to W \otimes V$$
$$v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$$

for two vector spaces V and W, where $|f|$ is the degree of f. We can also define a multiplication on the tensor product two superalgebras A and B by:

$$(w \otimes x)(y \otimes z) = (-1)^{|x||y|} wy \otimes xz$$

for homogenous elements x and y.

In [3], Ellis and Khovanov introduced the "odd symmetric polynomials", which are polynomials in the n variables $x_1, \ldots, x_n$ such that $x_i x_j + x_j x_i = 0$ for $i \neq j$. These polynomials can be interepreted through a -1-bialgebra structure with a similar bilinear form as in the "even" case, where $x_i x_j = x_j x_i$ for all indices i, j. The odd symmetric polynomials have several important bases, and have similar combinatorial properties with their even counterparts. We will highlight one example through the elementary symmetric functions. In the even case,

$$e_k(x_1, \ldots, x_n) = \sum_{1 \le i_1 < \ldots < i_k \le n} x_{i_1} \cdots x_{i_n}$$

and $e_i e_j = e_j e_i$ for all indices i, j. These polynomials come up in Vieta's formulae, in the coefficients of x in the polynomial $(x - x_1) \ldots (x - x_n)$. In the odd case, these polynomials may be defined in the same way (but with anticommuting variables) such that the relations become:

$$e_i e_j = e_j e_i \text{ for } i + j \text{ even}$$
$$e_i e_j + (-1)^i e_j e_i = e_{j-1} e_{i+1} + (-1)^i e_{i+1} e_{j-1} \text{ for } i + j \text{ odd}$$

Thus, if the ground ring has characteristic 2, then all commutators vanish. Ellis and Khovanov also found various other properties of the odd symmetric functions, including an odd RSK correspondence which may be used to study odd analogues of Schur functions and their orthonormality properties with respect to the standard bilinear form.

Further work in this direction by Ellis, Khovanov, and Lauda has led to a categorication of the positive half of the quantum Lie algebra $\mathfrak{sl}_2$. Categorification is an idea, originally suggested by Louis Crane and Igor Frenkel, by which one could replace algebras and representations by graded additive categories and higher categories to obtain quantum 4-manifold invariants from quantum 3-manifold invariants. A classic example is that Khovanov homology, a bigraded abelian group, categorifies the one-variable Jones polynomial. In a recent work, Khovanov and Lauda, as well as Rouquier, introduced the so called KLR algebras which categorify the postive half of quantum Kac-Moody algebras (used to construct quantum 3-manifold invariants). These algebras generalize the nilHecke algebra introduced by Kostant and Kumar in the 80's, in the context of geometric representation theory.

The even nilHecke algebra NH_n is generated by n commuting variables $x_1, \ldots, x_n$ and n divided difference operators ∂_i, such that $\partial_i = (x_i - x_{i+1})^{-1}(1 - s_i)$, where s_i is the simple transposition in the symmetric group that swaps x_i and x_{i+1}. This algebra is Morita equivalent to the symmetric polynomials in n variables. Ellis, Khovanov, and Lauda categorified the positive half (much more recently, both halves) of $\mathfrak{sl}_2$ by introducing an odd analogue of the nilHecke algebra. Their odd analogue retains many of the properties that one sees in the even case, including a Leibniz rule for the odd divided difference operator, an interpretation of the odd nilHecke algebra as a matrix algebra over the odd symmetric polynomials, and the use of a diagrammatic calculus. The odd nilHecke algebra has since found many other applications in representation theory. It is related to affine Hecke-Clifford superalgebras, and has been used to construct odd analogues of the cohomology groups of Grassmannians and Springer varieties. In this work, we will introduce q-analogues of various results in this context. We study relations for the q-symmetric polynomials in the variables $x_1, \ldots, x_n$ such that $x_j x_i = q x_i x_j$ for $j > i$ and q is a root of unity in $\mathbb{C}$. We also develop a q-nilHecke algebra and discuss its properties. It would be interesting to study whether q-nilHecke algebras categorify an interesting Lie theoretic algebra, and whether they can be used to construct invariants of links or other geometric structures

There are still open questions relating to the even, odd, and general q cases, including an interpretation of power sums, Grothendieck polynomials, Macdonald polynomials, coinvariant algebras, and so on. Along these lines, Wang and Khongsap introduced an odd analogue of Dunkl operators, and developed an odd double affine Hecke algebra as well. In the even case, one may define a reduced root system and a Coxeter group generated by complex reflections over the hyperplane. The Dunkl operators then may be interepreted as differential-difference operators that generalize the concept of a partial derivative. The Dunkl operator η_i commutes ($\eta_i \eta_j = \eta_j \eta_i$), and satisfies various other interesting properties. These operators have a major role in mathematical physics and conformal field theory where they relate to the study of quantum many-body problems in the Calogero-Moser model. They are used to show the integrability of the non-periodic Calogero-Moser-Sutherland system, and are related to various other operators and symmetric polynomials, including the Cherednik operators and Jack polynomials. The Dunkl operators can also be used to define three operators, which play important roles in physics and harmonic analysis, that generate the Lie algebra $\mathfrak{sl}_2$. This result plays a crucial role in Fischer decomposition, which is of importance in representation theory and the Dirichlet problem. In this work, we find a connection between the odd nilHecke algebra introduced by Ellis, Khovanov, and Lauda, and the odd Dunkl operator introduced by Wang and Khongsap. We also find three operators, based on a variant of the odd Dunkl operator, that generate $\mathfrak{sl}_2$, as in the even case. These results should have an important role in better understanding the representation theory of odd symmetric functions.

2 Background

2.1 Dunkl Operators

In the even case, we work over the ring $\mathbb{C}\langle x_1, \ldots, x_n\rangle$, where $x_i x_j = x_j x_i$ for all $1 \le i, j \le n$. Dunkl [2] introduced the remarkable operator

$$\eta_i^{\text{even}} = \frac{\partial}{\partial x_i} + \alpha \sum_{k \neq i} \partial_{i,k}^{\text{even}} \tag{2.1}$$

where $\dfrac{\partial}{\partial x_i}$ is the partial derivative with respect to x_i, $\alpha \in \mathbb{C}$, and $\partial_{i,k}^{\text{even}}$ is the even divided difference operator:

$$\partial_{i,k}^{\text{even}} = (x_i - x_k)^{-1}(1 - s_{i,k}). \tag{2.2}$$

Since $x_i - x_k$ always divides $f - s_{i,k}(f)$ for $f \in \mathbb{C}\langle x_1, \ldots, x_n\rangle$, $\partial_{i,k}$ sends polynomials to polynomials. As usual, $s_{i,k}$ is the transposition in S_n that swaps x_i and x_k and satisfies the relations of the symmetric group.

These Dunkl operators have various properties, the most important of which is that they commute ($\eta_i \eta_j = \eta_j \eta_i$). They also satisfy the properties that [1]:

$$\eta_i(fg) = \frac{\partial}{\partial x_i}(f)g + f\eta_i(g) \tag{2.3}$$

$$\eta_i x_j + x_i \eta_j = \eta_j x_i + x_j \eta_i \tag{2.4}$$

In [9], Khongsap and Wang introduced an *odd* Dunkl operator which anti-commutes. These operators have similar commutation relations with x_i and x_j that the even Dunkl operators do. In Section 3, we will develop the connection between this operator and the odd nilHecke algebra introduced in [4].

Now introduce operators r^2, E (the Euler operator) and Δ_k (the Dunkl Laplacian):

$$r^2 = \sum_{i=1}^{n} x_i^2 \tag{2.5}$$

$$E = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + \frac{\mu}{2} \tag{2.6}$$

$$\Delta_k = \sum_{i=1}^{n} \eta_i^2 \tag{2.7}$$

where μ is the *Dunkl dimension*, which is defined by the relation $\eta_i |x|^2 = 2\mu$.

Let $[p, q] = pq - qp$ be the commutator. Heckman showed that r^2, E, and Δ_k satisfy the defining relations of the Lie algebra $\mathfrak{sl}_2$ [6]:

$$[E, r^2] = 2r^2 \tag{2.8}$$

$$[E, \Delta_k] = -2\Delta_k \tag{2.9}$$

$$[r^2, \Delta_k] = 4E \tag{2.10}$$

If one were to replace Δ_k with the classical Laplacian (replacing the Dunkl operator with the partial derivative), these three operators still satisfy the relations for $\mathfrak{sl}_2$. Combined with the fact that Dunkl operators commute, one can see that the Dunkl operators represent a meaningful generalization of the partial derivative.

In Section 4, we will focus on finding analagous results in the odd case.

2.2 Introduction to q-Bialgebras

We begin by an introduction to the terminology of bialgebras that will be used later. The notion of A being an $\mathbb{C}$-algebra entails that there are two maps: the unit and the multiplication.

$$\eta : \mathbb{C} \to A$$

$$m : A \otimes A \to A$$

The algebra A also has an identity map $1_A : A \to A$. We denote the degree of an element v by $|v|$. For homogenous elements v and w of A, define the braiding:

$$\tau_A : A \otimes A \to A \otimes A$$
$$v \otimes w \to q^{|v||w|} w \otimes v$$

One may define the multiplication map m_2 from $A^{\otimes 4} \to A^{\otimes 2}$ as:

$$m_2 = (m \otimes m)(1_A \otimes \tau_A \otimes 1_A)$$

A coalgebra is a structure with the following maps, the counit and comultiplication:

$$\epsilon : A \to \mathbb{C}$$
$$\Delta : A \to A \otimes A$$

A bialgebra B is equipped with all of the four maps $(\eta, m, \epsilon, \Delta)$, with the added compatibility that the comultiplication is a homomorphism of algebras. This condition implies that:

$$\Delta \circ m = m_2 \circ (\Delta \otimes \Delta)$$

and by the definition of m_2, we have the following bialgebra axiom:

$$\Delta \circ m = (m \otimes m) \circ (1_B \otimes \tau_B \otimes 1_B) \circ (\Delta \otimes \Delta)$$

3 Odd Dunkl operators and the Odd nilHecke algebra

3.1 Introduction to the Odd nilHecke Algebra

We work over $\mathbb{C}\langle x_1, \ldots, x_n \rangle / \langle x_j x_i + x_i x_j = 0 \text{ for } i \neq j \rangle$. We can define linear operators, called the odd divided difference operators, as below:

Definition 3.1. For $i = 1, \ldots .n-1$, the i-th *odd divided difference operator* ∂_i is the linear operator $\mathbb{C}\langle x_1, \ldots, x_n \rangle \to \mathbb{C}\langle x_1, \ldots, x_n \rangle$ defined by

$$\partial_i(x_j) = \begin{cases} 1 & j = i \\ 1 & j = i+1 \\ 0 & j \neq i, i+1 \end{cases} \tag{3.1}$$

$$\partial_i(fg) = \partial_i(f)g + (-1)^{|f|} s_i(f) \partial_i(g). \tag{3.2}$$

where $s_i(f)$ is the type A transposition:

$$s_i(x_j) = \begin{cases} x_{i+1} & j = i \\ x_i & j = i+1 \\ x_j & j \neq i, i+1 \end{cases} \tag{3.3}$$

It is shown in [4] that the odd divided difference operators can be used to construct an odd nilHecke algebra, generated by x_i and ∂_i for $1 \leq i \leq n$, subject to the following relations.

$$\partial_i^2 = 0$$
$$\partial_i \partial_j + \partial_j \partial_i = 0 \text{ for } |i-j| \geq 2$$
$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$$
$$x_i x_j + x_j x_i = 0 \text{ for } i \neq j$$
$$x_i \partial_i + \partial_i x_{i+1} = 1, \partial_i x_i + x_{i+1} \partial_i = 1$$
$$x_i \partial_j + \partial_j x_i = 0 \text{ for } |i-j| \geq 2$$

Due to [7], we have the following explicit definition of the odd divided difference operator:

$$\partial_i(f) = (x_{i+1}^2 - x_i^2)^{-1}[(x_{i+1} - x_i)f - (-1)^{|f|}s_i(f)(x_{i+1} - x_i)] \tag{3.4}$$

Although this formula a priori involves denominators, it does take skew polynomials to skew polynomials. We extend this definition to non-consecutive indices by allowing $i+1$ to equal any index $k \neq i$, for $1 \leq k \leq n$, and by replacing the simple transposition s_i with $s_{i,k}$, which swaps x_i and x_k.

$$\partial_{i,k}(f) = (x_k^2 - x_i^2)^{-1}[(x_k - x_i)f - (-1)^{|f|}s_{i,k}(f)(x_k - x_i)] \tag{3.5}$$

This extended odd divided difference operator satisfies the Leibniz rule $\partial_{i,k}(fg) = \partial_{i,k}(f)g + (-1)^{|f|}s_{i,k}\partial_{i,k}(g)$ [4].

3.2 Some operations on skew polynomials

First, we introduce a common operator in the study of Dunkl operators:

Definition 3.2. Let the -1-shift operator τ_i send $f(x_1, x_2, \ldots, x_i, \ldots, x_n)$ to $f(x_1, x_2, \ldots, -x_i, \ldots, x_n)$.

This operator satisfies the below properties, where f is a function in $\mathbb{C}\langle x_1, \ldots, x_n\rangle$:

$$s_{i,j}\tau_i = \tau_j s_{i,j} \tag{3.6}$$
$$s_{i,j}\tau_j = \tau_i s_{i,j} \tag{3.7}$$
$$s_{i,j}\tau_k = \tau_k s_{i,j} \text{ if } k \neq i, j \tag{3.8}$$
$$fx_i = (-1)^{|f|}x_i\tau_i(f) \tag{3.9}$$

Remark 3.3. Since skew polynomials are not super-commutative, we cannot say that $fg = (-1)^{|f||g|}gf$. But the operator τ_i allows us to track the discrepancy from super commutativity, since $x_i f = (-1)^{|f|}\tau_i(f)x_i$, which is why it becomes useful in this context.

We now introduce the operator $r_{i,k} = \partial_{i,k}s_{i,k}$ for $k \neq i$, which will serve as another odd divided difference operator that we will use to study odd Dunkl operators. For simplicity, let $r_i = r_{i,i+1}$. In the following lemma, we study the action of the transposition and -1-shift operator on $r_{i,k}$:

Lemma 3.4. For $i = 1$ to n, $s_{i,k}$ acts on $r_{i,k}$ as follows:

$$s_i r_{i,k} = r_{i+1,k}s_i \text{ if } k \neq i+1 \tag{3.10}$$
$$s_i r_i = r_i s_i \tag{3.11}$$
$$s_i r_{i+1} = r_{i,i+2}s_i \tag{3.12}$$
$$s_{i+1}r_i = r_{i,i+2}s_{i+1} \tag{3.13}$$
$$s_i r_j = r_j s_i \text{ for } |i-j| \geq 2 \tag{3.14}$$
$$\tau_i r_j = r_j \tau_i \text{ for } |i-j| \geq 2 \tag{3.15}$$

Proof. Note that

$$\begin{aligned} s_i r_{i,k}(f) &= s_i(x_k^2 - x_i^2)^{-1}[(x_k - x_i)f - (-1)^{|f|}s_{i,k}(f)(x_k - x_i)]s_{i,k} \\ &= (x_k^2 - x_{i+1}^2)^{-1}[(x_k - x_{i+1})f - (-1)^{|f|}s_{i+1,k}(f)(x_k - x_{i+1})]s_{i+1,k}s_i \\ &= r_{i+1,k}s_i(f) \end{aligned}$$

since $s_i s_{i,k} = s_{i+1,k}s_i$. The next four properties can be deduced similarly. The final property follows from $\tau_i s_j = s_j \tau_i$ and the fact that $\tau_i(x_j) = x_j$ for $i \neq j$. □

Recall the following relationship between $\partial_{i,j}$ and $s_{k,\ell}$ for $i \neq j$ and $k \neq \ell$, from Lemma 2.19 (1) of [4]:

$$\partial_{i,j}s_{k,\ell} = s_{k,\ell}\partial_{s_{k,\ell}(i,j)} \tag{3.16}$$

where $s_{k,\ell}(i,j)$ is the result of applying $s_{k,\ell}$ to the pair (i,j).

Remark 3.5. The result from Lemma 2.19 in [4] includes a sign, but this is because their $s_{i,j}$ is the *odd* transposition, $(-1)^{|f|}s_{i,j}^{\text{even}}$. Since the odd divided difference operator reduces the degree by 1, we obtain the correct expression above in terms of the even transposition $s_{k,\ell}$.

We now show that the properties of the $r_{i,k}$ are similar to those of the odd divided difference operator $\partial_{i,k}$:

Lemma 3.6. For $i = 1$ to n, we have

$$r_i^2 = 0 \tag{3.17}$$
$$r_i r_j + r_j r_i = 0 \tag{3.18}$$
$$r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1} \tag{3.19}$$
$$r_{i,k}(fg) = r_{i,k}(f)s_{i,k}(g) + (-1)^{|f|} f r_{i,k}(g) \tag{3.20}$$
$$r_i x_{i+1} + x_{i+1} r_i = s_i \tag{3.21}$$
$$r_i x_i + x_i r_i = s_i \tag{3.22}$$
$$r_j x_i + x_j r_i = 0 \tag{3.23}$$

Proof. Since $s_i r_i = r_i s_i$ and $r_i = \partial_i s_i$, it follows that $s_i \partial_i = \partial_i s_i$. Then, since $\partial_i^2 = 0$,

$$r_i^2 = \partial_i s_i \partial_i s_i = \partial_i \partial_i s_i s_i = 0$$

We also have from 3.14 that $s_i r_j = r_j s_i$ for $|i-j| \geq 2$, so $s_i \partial_j = \partial_j s_i$. Thus, r_i and r_j anti-commute since

$$r_i r_j = \partial_i s_i \partial_j s_j = \partial_i \partial_j s_i s_j = -\partial_j \partial_i s_i s_j = -\partial_j s_j \partial_i s_i = r_j r_i$$

because $\partial_i \partial_j + \partial_j \partial_i = 0$. The operators r_i also braid, which we show by inductively reducing to $i = 1$, and then using 3.16 and $s_i \partial_i = \partial_i s_i$ repeatedly:

$$r_1 r_2 r_1 = s_1 \partial_{12} \partial_2 s_1 \partial_1 = s_1 s_2 \partial_{1,3} s_1 \partial_{1,3} \partial_1 = s_1 s_2 s_1 \partial_{2,3} \partial_{1,3} \partial_{1,2}$$
$$r_2 r_1 r_2 = s_2 \partial_2 s_1 \partial_1 s_2 \partial_2 = s_2 s_1 \partial_{1,3} s_2 \partial_{1,3} \partial_2 = s_2 s_1 s_2 \partial_{1,2} \partial_{1,3} \partial_{2,3}$$

Therefore, $r_1 r_2 r_1 = r_2 r_1 r_2$ since the s_i braid, and since $\partial_{2,3}\partial_{1,3}\partial_{1,2} = \partial_{1,2}\partial_{1,3}\partial_{2,3}$ by symmetry. Next we find a Leibniz rule for $r_{i,k}$ using the Leibniz rule for $\partial_{i,k}$.

$$\partial_{i,k} s_{i,k}(f,g) = \partial_{i,k} s_{i,k}(f) s_{i,k}(g) + (-1)^{|f|} f \partial_{i,k} s_{i,k}(g) = r_{i,k}(f) s_{i,k}(g) + (-1)^{|f|} f r_{i,k}(g)$$

Note that $r_i(x_i) = r_i(x_{i+1}) = 1$ and $r_i(x_j) = 0$ for $j \neq i, i+1$. Equations 3.21 through 3.23 then follow from the Leibniz rule 3.20. □

We also desire an explicit definition of the $r_{i,k}$ analogous to that of the odd divided difference operator of [4]. To find such an expression, we utilize a preparatory lemma:

Lemma 3.7. For $i = 1$ to n, $s_{i,k} x_i \tau_i(f) - s_{i,k} x_k \tau_k(f) = (-1)^{|f|} s_{i,k}(f)(x_i - x_k)$ for $f \in \mathbb{C}\langle x_1, \ldots, x_n \rangle$.

Proof. It suffices to prove the result for a monomial $x^\lambda = x_1^{\lambda_1} \ldots x_i^{\lambda_i} \ldots x_k^{\lambda_k} \ldots x_n^{\lambda_n}$, where $i < k$. We will proceed by direct computation:

$$s_{i,k} x_i \tau_i(x^\lambda) = (-1)^{\lambda_i} x_k s_{i,k}(x_1^{\lambda_1} \ldots x_i^{\lambda_i} \ldots x_k^{\lambda_k} \ldots x_n^{\lambda_n}) = (-1)^{\lambda_1 + \ldots + \lambda_i} x_1^{\lambda_1} \ldots x_k^{\lambda_i + 1} \ldots x_i^{\lambda_k} \ldots x_n^{\lambda_n}$$
$$s_{i,k} x_k \tau_k(x^\lambda) = (-1)^{\lambda_k} x_i s_{i,k}(x_1^{\lambda_1} \ldots x_i^{\lambda_i} \ldots x_k^{\lambda_k} \ldots x_n^{\lambda_n}) = (-1)^{\lambda_1 + \ldots + \lambda_{k-1}} x_1^{\lambda_1} \ldots x_k^{\lambda_i} \ldots x_i^{\lambda_k + 1} \ldots x_n^{\lambda_n}$$
$$s_{i,k}(x_1^{\lambda_1} \ldots x_i^{\lambda_i} \ldots x_k^{\lambda_k} \ldots x_n^{\lambda_n}) x_i = (-1)^{\lambda_{k+1} + \ldots + \lambda_n} x_1^{\lambda_1} \ldots x_k^{\lambda_i} \ldots x_i^{\lambda_{k+1}} \ldots x_n^{\lambda_n}$$
$$s_{i,k}(x_1^{\lambda_1} \ldots x_i^{\lambda_i} \ldots x_k^{\lambda_k} \ldots x_n^{\lambda_n}) x_k = (-1)^{\lambda_{i+1} + \ldots + \lambda_n} x_1^{\lambda_1} \ldots x_k^{\lambda_{i+1}} \ldots x_i^{\lambda_k} \ldots x_n^{\lambda_n}$$

Since $|f| = \lambda_1 + \ldots + \lambda_n$, the desired result follows. □

We can now express $r_{i,k}$ explicitly as

Lemma 3.8. $r_{i,k} = (x_i^2 - x_k^2)^{-1}[(x_i - x_k)s_{i,k} - x_i \tau_i + x_k \tau_k]$

Proof. Follows from Lemma 3.7 and the formula of [7]. □

We will now connect the above results to the odd Dunkl operator introduced by Khongsap and Wang in [9].

Definition 3.9. Define an operator δ_i by $\delta_i = (2x_i)^{-1}(1-\tau_i)$

The above super-derivative can also be defined inductively, by imposing that $\delta_i(x_j) = 1$ if $i = j$ and 0 otherwise. We then extend the action to monomials as follows:

$$\delta_i(x_{a_1}x_{a_2}\dots x_{a_\ell}) = \sum_{k=1}^{\ell}(-1)^{k-1}x_{a_1}\dots\delta_i(x_{a_k})x_{a_{k+1}}\dots x_{a_\ell} \tag{3.24}$$

The operator δ_i is a priori from Laurent skew polynomials to Laurent skew polynomials, but it is easy to check that it preserves the subalgebra of skew polynomials. Khongsap and Wang found an odd analogue of the Dunkl operator:

$$\eta_i^{\text{odd}} = t\delta_i + u\sum_{k\neq i}(x_i^2 - x_k^2)^{-1}[(x_i - x_k)s_{i,k} - x_i\tau_i + x_k\tau_k] = t\delta_i + u\sum_{k\neq i} r_{i,k} \tag{3.25}$$

where $t, u \in \mathbb{C}$. Their operator anti-commutes; $\eta_i\eta_j + \eta_j\eta_i = 0$ for $i \neq j$.

By Lemma 3.8, this odd Dunkl operator may be expressed as:

$$\eta_i^{\text{odd}} = t\delta_i + u\sum_{k\neq i}\partial_{i,k}s_{i,k} \tag{3.26}$$

By analogy with the commutative case, discussed in section 1, the operator $r_{i,k}$ plays the same role in the odd theory that the even divided difference operator plays in the even theory.

4 Properties of Another Odd Dunkl Operator

In this section, we will show that a close variant of the odd Dunkl operator introduced by Khongsap and Wang can be used in the construction of three operators that satisfy the defining relations of the Lie algebra $\mathfrak{sl}_2$. First, we will consider a different operator p_i, which in some ways is more natural than δ_i:

Definition 4.1. The operator p_i acts on monomials as follows:

$$p_i(x_1^{\lambda_1}\dots x_i^{\lambda_i}\dots x_n^{\lambda_n}) = \lambda_i(-1)^{\lambda_1+\dots+\lambda_{i-1}}x_1^{\lambda_1}\dots x_i^{\lambda_i-1}\dots x_n^{\lambda_n} \tag{4.1}$$

Now consider a modified version of η_i:

Definition 4.2. Let

$$D_i = tp_i + u\sum_{k\neq i} r_{i,k} \tag{4.2}$$

Note that this operator substitutes p_i for δ_i in the odd Dunkl operator of Wang and Khongsap. Similar to the even case, define the Euler, r^2, and odd Dunkl Laplacian operators as below:

$$r^2 = (2t)^{-1}\sum_{i=1}^{n} x_i^2 \tag{4.3}$$

$$E = \sum_{i=1}^{n} x_ip_i + \frac{n}{2} + \frac{u}{t}\sum_{k\neq i} s_{i,k} \tag{4.4}$$

$$\Delta = -(2t)^{-1}\sum_{i=1}^{n} D_i^2 \tag{4.5}$$

As usual, let $[p,q] = pq - qp$ be the commutator. Note that in Heckman's paper, $t = 1$ [6]. We will consider t to be a fixed constant in $\mathbb{C}$.

Remark 4.3. The commutator in the setting of superalgebras is usually defined as $[a, b] = ab - (-1)^{|a||b|}ba$, where $|a|$ and $|b|$ are the degrees of a and b, respectively. However, since all of the operators we will be considering have even degree, there is no need to distinguish between commutators and super-commutators.

To demonstrate the relationship between these operators and $\mathfrak{sl}_2$ we will require a series of lemmas regarding the action of portions of the Euler operator E. We first investigate the action of the second term of the Euler operator:

Lemma 4.4. The operator $\sum_{i=1}^{n} x_i p_i$ acts by multiplication by $|f|$ on the space of homogenous functions f.

Proof. It suffices to show the result for a monomial $x^\lambda = x_1^{\lambda_1} \dots x_i^{\lambda_i} \dots x_n^{\lambda_n}$. Note that

$$x_i p_i(x^\lambda) = \lambda_i x_i (-1)^{\lambda_1 + \dots + \lambda_{i-1}} x_1^{\lambda_1} \dots x_i^{\lambda_i - 1} \dots x_n^{\lambda_n} = \lambda_i x^\lambda$$

By summing over all indices i, we obtain that

$$\sum_{i=1}^{n} x_i p_i(x^\lambda) = (\lambda_1 + \lambda_2 + \dots + \lambda_n) x^\lambda$$

which implies the desired result. □

The above lemma holds true in the even case as well, where p_i is replaced by the partial derivative with respect to x_i. We now prove some properties about the action of the third term of the Euler operator on r^2 and Δ:

Lemma 4.5. $\left[\sum_{k \neq i} s_{i,k}, \Delta\right] = 0$

Proof. We will first show that

$$s_{j,k} p_i = \begin{cases} p_i s_{j,k} & \text{if } i \neq j \neq k \\ p_j s_{j,k} & \text{if } i = k \\ p_k s_{j,k} & \text{if } i = j \end{cases} \tag{4.6}$$

Indeed, these relations can be verified by checking if they are true for $x_i^a x_j^b x_k^c$, $a, b, c \in \mathbb{Z}_+$, and then extending by linearity. We prove that $s_{j,k} p_i = p_j s_{j,k}$ if $i = k$, and the other two cases are similar. Without loss of generality, let $j < k$.

$$s_{j,k} p_k (x_j^a x_k^b) = b(-1)^a s_{j,k}(x_j^a x_k^{b-1}) = b(-1)^a x_k^a x_j^{b-1} = b(-1)^{ab} x_j^{b-1} x_k^a$$
$$p_j s_{j,k}(x_j^a x_k^b) = (-1)^{ab} p_j (x_j^b x_k^a) = b(-1)^{ab} x_j^{b-1} x_k^a$$

As a result of (4.6), we therefore have the action of the transposition on p_i. We will next need its action on $r_{\ell,m}$. By the work in the previous section, we find that

$$s_{j,k} r_{\ell,m} = r_{s_{j,k}(\ell,m)} s_{i,j} \tag{4.7}$$

where $s_{j,k}(\ell, m)$ is the result of applying the transposition $s_{j,k}$ to the pair (ℓ, m). Combining 4.6 and 4.7, we find that

$$s_{j,k} D_i = D_{s_{j,k}(i)} s_{j,k} \tag{4.8}$$

where $s_{j,k}(i)$ is the result of applying the transposition $s_{j,k}$ to i. By an easy induction, we now have that

$$s_{j,k} \Delta = \Delta s_{j,k}$$

Using the above equation multiple times proves the desired result. □

We have a similar result for the action of the third term of the Euler operator and r^2:

Lemma 4.6. $\left[\sum_{k \neq i} s_{i,k}, r^2\right] = 0$

Proof. Follows from the observation that

$$s_{j,k}x_i = \begin{cases} x_i s_{j,k} & \text{if } i \neq j \neq k \\ x_j s_{j,k} & \text{if } i = k \\ x_k s_{j,k} & \text{if } i = j \end{cases}$$

so that $s_{j,k}r^2 = r^2 s_{j,k}$ □

We are now ready to obtain two commutativity relations between the Euler, r^2, and odd Dunkl Laplacian operators:

Theorem 4.7. The odd Euler operator and r^2 satisfy the following commutation relations:

$$[E, r^2] = 2r^2 \tag{4.9}$$

$$[E, \Delta] = -2\Delta \tag{4.10}$$

Proof. Since r^2 has degree 2 and Δ has degree -2, the theorem follows from lemmas 4.4, 4.5, and 4.6. □

We also need to investigate what the third commutativity relation $[r^2, \Delta]$ turns out to be. We will prove one lemma before doing so.

Lemma 4.8. For $i = 1$ to n, the equation $x_i D_i + D_i x_i = 2t x_i p_i + t + u \sum_{k \neq i} s_{i,k}$ holds.

Proof. Recall that

$$D_i = tp_i + u \sum_{k \neq i} (x_i^2 - x_k^2)^{-1} [(x_i - x_k)s_{i,k} - x_i\tau_i + x_k\tau_k]$$

Therefore, since $p_i x_i = x_i p_i + 1$,

$$D_i x_i = t x_i p_i + t + u \sum_{k \neq i} (x_i^2 - x_k^2)^{-1} (x_i x_k - x_k^2) s_{i,k} + \sum_{k \neq i} (x_i^2 - x_k^2) [x_i^2 \tau_i - x_i x_k \tau_k]$$

$$x_i D_i = t x_i p_i + u \sum_{k \neq i} (x_i^2 - x_k^2)^{-1} (x_i^2 - x_i x_k) s_{i,k} + \sum_{k \neq i} (x_i^2 - x_k^2)^{-1} [-x_i^2 \tau_i + x_i x_k \tau_k]$$

Adding, we obtain the desired result. □

We now have the tools to find the third relation between r^2, E, and Δ:

Theorem 4.9. $[r^2, \Delta] = E$

Proof. We will first find $[r^2, D_i]$. The derivative p_i, much like the partial derivative in the even case, satisfies the properties:

$$p_i x_j = -x_j p_i \text{ for } i \neq j \tag{4.11}$$

$$p_i x_i = x_i p_i + 1 \tag{4.12}$$

Now, suppose that $i \neq j$. Then,

$$\begin{aligned} D_i x_j^2 &= tp_i x_j^2 + u x_j^2 \sum_{k \neq i \neq j} r_{i,k} + x_k^2 (x_i^2 - x_k^2)^{-1} [-x_i \tau_i + x_k \tau_k] + x_i^2 (x_i^2 - x_k^2)^{-1} [(x_i - x_k) s_{i,k}] \\ &= t x_j^2 p_i + u x_j^2 \sum_{k \neq i} r_{i,k} + (x_i - x_j) s_{i,j} \end{aligned}$$

Now, we will find $D_i x_i^2$:

$$\begin{aligned} D_i x_i^2 &= tp_i x_i^2 + \sum_{k \neq i} x_k^2 (x_i^2 - x_k^2)^{-1} (x_i - x_k) s_{i,k} + x_i^2 \sum_{k \neq i} (x_i^2 - x_k^2)^{-1} [-x_i \tau_i + x_k \tau_k] \\ & t x_i^2 p_i + 2t x_i + x_i^2 \sum_{k \neq i} r_{i,k} - \sum_{k \neq i} (x_i - x_k) s_{i,k} \end{aligned}$$

Therefore, $\left[\sum_{i=1}^{n} x_i^2, D_i\right] = -2tx_i$. This implies that

$$r^2 D_i - D_i r^2 = -x_i \tag{4.13}$$

Now,

$$\begin{aligned}
[r^2, \Delta] &= -(2t)^{-1} \sum_{i=1}^{n} [r^2, D_j^2] = -(2t)^{-1} \sum_{i=1}^{n} (r^2 D_j^2 - D_j^2 r^2) \\
&= -(2t)^{-1} \sum_{i=1}^{n} [(D_j r^2 D_j - x_j D_j) - (D_j r^2 D_j + D_j x_j)] \\
&= (2t)^{-1} \sum_{i=1}^{n} (x_i D_i + D_i x_i)
\end{aligned}$$

where we have used 4.13. Now, by Lemma 4.8,

$$[r^2, \Delta] = \sum_{i=1}^{n} x_i p_i + \frac{n}{2} + \frac{u}{t} \sum_{k \neq i} s_{i,k} = E$$

as desired. □

To summarize, we have found operators E,r^2, and Δ, similar to their even counterparts, which satisfy the defining relations of the Lie algebra $\mathfrak{sl}_2$:

$$\begin{aligned}
[E, r^2] &= 2r^2 \\
[E, \Delta] &= -2\Delta \\
[r^2, \Delta] &= E
\end{aligned}$$

Remark 4.10. If one uses the odd Dunkl operator η_i as found in [9] instead of the D_i introduced here, the r^2, E, and Δ operators then seem to generate an abelian Lie algebra rather than $\mathfrak{sl}_2$.

Remark 4.11. Although our results hold true for all t and u in $\mathbb{C}$, one typically sets $t = 1$ and $u = \alpha^{-1}$, since without loss of generality one of t and u may equal 1.

5 Relations in the q-Analogue of Symmetric Polynomials

5.1 Introduction to the q-Analogue of Noncommutative Symmetic Functions

Let $N\Lambda^q$ be a free, associative, $\mathbb{Z}$-graded $\mathbb{C}$-algebra with generators h_m for $m \geq 0$. We define $h_0 = 1$ and $h_m = 0$ for $m < 0$, and let $q \in \mathbb{C}$. The homogenous part of $N\Lambda^q$ of degree ℓ has a basis $\{h_\alpha\}_{\alpha \vDash k}$, where

$$h_\alpha = h_{\alpha_1} \cdots h_{\alpha_z} \text{ for a composition } \alpha = (\alpha_1, \ldots, \alpha_z) \text{ of } \ell.$$

Define a multiplication for homogenous x and y on $N\Lambda^{q \otimes 2}$ as follows, where deg(x) denotes the degree of x:

$$(w \otimes x)(y \otimes z) = q^{\deg(x)\deg(y)}(wy \otimes xz)$$

We can make $N\Lambda^q$ into a q-bialgebra by letting the comultiplication on generators be:

$$\Delta(h_n) = \sum_{k=0}^{n} h_k \otimes h_{n-k}$$

and by letting the counit be $\epsilon(x) = 0$ if x is homogenous and deg(x)> 0.

We can impose, through the braiding structure, that:

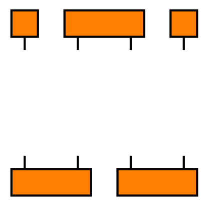
$$\Delta(h_a h_b) = \sum_{j=0}^{a}\sum_{k=0}^{b}(h_j \otimes h_{a-j})(h_k \otimes h_{b-k}) = \sum_{j=0}^{a}\sum_{k=0}^{b} q^{k(a-j)}(h_j h_k \otimes h_{a-j}h_{b-k})$$

For any partitions λ and μ of n, consider the set of double cosets of subroups S_λ and S_μ of S_n: $S_\lambda \backslash S_n / S_\mu$. For every C in this set, let w_C be the minimal length representative of C and let $\ell(w_C)$ be the length of this minimal length representative. We will now attribute a bilinear form to $N\Lambda^q$:

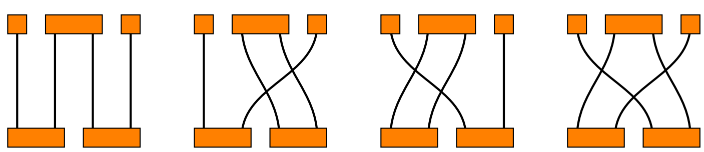
$$(h_\lambda, h_\mu) = \sum_{C \in S_\lambda \backslash S_n / S_\mu} q^{\ell(w_C)},$$

The bilinear form admits a diagrammatic description. Let h_n be an orange platform with n non-intersecting strands coming out of it. When computing (h_λ, h_μ), with $\ell(\lambda) = z$ and $\ell(\mu) = y$, draw z orange platforms at the top of the diagram, representing λ_1, λ_2,$\cdots$,λ_z. Draw y orange platforms at the bottom of the diagram, representative of μ_1, μ_2,$\cdots$, μ_y. We require that $|\lambda| = |\mu|$, so that the top platforms and bottom platforms have the same number of strands.

Consider the example (h_{121}, h_{22}). In the following diagram, snippets of the strands from each platform are shown.



Every strand must start at one platform at the top and end on another platform at the bottom. No strands that have originated from one platform may intersect. The strands themselves have no critical points with respect to the height function, no two strands ever intersect more than once, and there are no triple-intersections where three strands are concurrent. Diagrams are considered up to isotopy. Without any restrictions, there would be $n!$ such diagrams if $|\lambda| = n$, since there would be no limitations on the ordering of the strands. However, due to the above rules, there are only 4 possible diagrams in the computation of (h_{121}, h_{22}), shown below.



Define

$$(h_\lambda, h_\mu) = \sum_{\text{all diagrams D representing } (h_\lambda, h_\mu)} q^{\text{number of of crossings in } D}. \tag{5.1}$$

In the above example, $(h_{121}, h_{22}) = 1 + 2q^2 + q^3$.

We can extend the bilinear form to $N\Lambda^{q\otimes 2}$ by stating tht any diagram in which strands from distinct tensor factors intersect contributes 0 to the bilinear form:

$$(w \otimes x, y \otimes z) = (w, y)(x, z).$$

Let I be the radical of the bilinear form in $N\Lambda^q$. In [3], the authors proved the following statements for any q.

1. Adjointness of multiplication and comultiplication for all x,y_1, y_2 in $N\Lambda^q$:

$$(y_1 \otimes y_2, \Delta(x)) = (y_1 y_2, x) \tag{5.2}$$

2. I is a q-bialgebra ideal in $N\Lambda^q$:

$$I N\Lambda^q = N\Lambda^q I = I \tag{5.3}$$

$$\Delta(I) \subset I \otimes N\Lambda^q + N\Lambda^q \otimes I \tag{5.4}$$

5.2 The Elementary Symmetric Functions

Define elements $e_k \in N\Lambda^q$ by $e_k = 0$ for $k < 0$, $e_0 = 1$, and

$$\sum_{i=0}^{k} (-1)^i q^{\binom{i}{2}} e_i h_{k-i} = 0 \text{ for } k \geq 1. \tag{5.5}$$

Or, equivalently,

$$e_n = q^{-\binom{n}{2}} \sum_{\alpha \vDash n} (-1)^{\ell(\alpha)-n} h_\alpha. \tag{5.6}$$

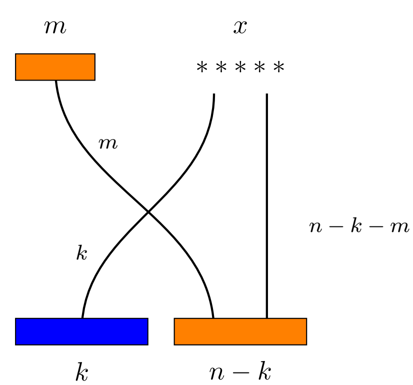
Lemma 5.1.

1. The coproduct of an elementary function is given by $\Delta(e_n) = \sum_{k=0}^{n} e_k \otimes e_{n-k}$.
2. If $\lambda \vDash n$, then $(h_\lambda, e_n) = \begin{cases} 1 & \text{if } \lambda = (1, \dots, 1) \\ 0 & \text{otherwise.} \end{cases}$

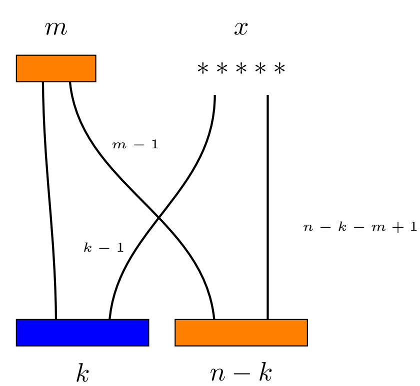
Proof. We begin by demonstrating (2), from which (1) will follow. To show (2), it suffices to show that

$$(h_m x, e_n) = \begin{cases} (x, e_{n-1}) & \text{if } m = 1 \\ 0 & \text{otherwise} \end{cases}$$

We will utilize a strong induction on n in order to find $(h_m x, e_k h_{n-k})$. The base cases $n = 0, 1$ are easy to show. There are two cases to consider by the inductive hypothesis applied to $k < n$. Either there is a strand connecting h_m and e_k, or there is not. Just as we used an orange platform to denote h_n, we will use a blue platform to denote e_k. The rules of the diagrammatic notation are the same for the blue platforms as they are for the orange platforms.



If there is not a strand connecting h_m and e_k, the configuration contributes $q^{km}(x, e_k h_{n-k-m})$.



If a stand connects h_m and e_k, this configuration contributes $q^{(k-1)(m-1)}(x, e_{k-1}h_{n-k-m+1})$. We have thus shown that $(h_m x, e_k h_{n-k}) = q^{km}(x, e_k h_{n-k-m}) + q^{(k-1)(m-1)}(x, e_{k-1}h_{n-k-m+1})$. Now we are equipped to consider $(h_m x, e_k)$.

$$\begin{aligned}(-1)^{n+1}q^{\binom{n}{2}}(h_m x, e_n) &= \sum_{k=0}^{n-1}(-1)^k q^{\binom{k}{2}}(h_m x, e_k h_{n-k}) \\ &= \sum_{k=0}^{n-1}(-1)^k q^{\binom{k}{2}+km}(x, e_k h_{n-k-m}) + \sum_{k=0}^{n-1}(-1)^k q^{\binom{k}{2}+(m-1)(k-1)}(x, e_{k-1}h_{n-k-m+1}) \\ &= \sum_{k=0}^{n-1}(-1)^k q^{\binom{k}{2}+km}(x, e_k h_{n-k-m}) + \sum_{k=0}^{n-2}(-1)^{k+1} q^{\binom{k+1}{2}+(m-1)(k)}(x, e_k h_{n-k-m}) \\ &= (-1)^{n-1}q^{\binom{n-1}{2}+nm}(x, e_{n-1}h_{1-m})\end{aligned}$$

Corresponding terms from the two sums cancel in pairs, since $q^{\binom{k}{2}+km} = q^{\binom{k+1}{2}+k(m-1)}$, leaving only the $k = n-1$ term in the first sum. The second statement of the lemma thus follows.

We will now use (2) to prove (1). This follows from the adjointness previously mentioned.

$$(\Delta(e_k), h_\lambda \otimes h_\mu) = (e_k, h_\lambda h_\mu) = \begin{cases} 1 & \lambda = (1^\ell), \mu = (1^p), \ell + p = k, \\ 0 & \text{otherwise.} \end{cases}$$

□

We now calculate the sign incurred when strands connect two blue (e_k) platforms.

$$\begin{aligned}(-1)^{n+1}q^{\binom{n}{2}}(e_n, e_n) &= \sum_{k=0}^{n-1}(-1)^k q^{\binom{k}{2}}(e_n, e_k h_{n-k}) \\ &= (-1)^{n-1}q^{\binom{n-1}{2}}(e_n, e_{n-1}h_1) \\ &= (-1)^{n-1}q^{\binom{n-1}{2}}(\Delta(e_n), e_{n-1}\otimes h_1) \\ &= (-1)^{n-1}q^{\binom{n-1}{2}}\sum_{k=0}^{n}(e_k \otimes e_{n-k}, e_{n-1}\otimes h_1) \\ &= (-1)^{n-1}q^{\binom{n-1}{2}}(e_{n-1}, e_{n-1})\end{aligned}$$

One may solve this recursion to find that $(e_n, e_n) = q^{-\binom{n}{2}}$.Here, the second equality follows from noting that at most one strand can connect h_{n-k} and e_n (so that $k = n-1$), the third equality follows from adjointness, and the fourth and fifth equalities follow from the diagrammatic considerations of the previous lemma.

To summarize the diagrammatics of the bilinear form thus developed:

- For each crossing, there is a factor of q in the bilinear form.
- If two blue platforms are connected by n strands, there is a factor of $q^{-\binom{n}{2}}$
- At most one strand can connect a blue platform to an orange one.

6 More on the q-Analogue of Noncommutative Symmetric Functions

Define $\mathrm{Sym}^q \cong N\Lambda^q/R$, where R is the radical of the bilinear form $(.,.)$.

Lemma 6.1. h_1^n is in the center of $N\Lambda^q$ if $q^n = 1$.

Proof. First, suppose q is a primitive n^{th} root of unity. Construct all ordered $k+1$-tuples of nonnegative integers that sum to $n-k$. Let R_{k+1}^{n-k} be the set of all such $k+1$-tuples. For any tuple $(a_1, a_2, \cdots, a_{k+1})$, let $|(a_1, a_2, \cdots, a_{k+1})|$ be the sum of the entries of the tuple.

For these tuples, $(a_1, a_2, \cdots, a_{k+1})$, define the map f as follows:

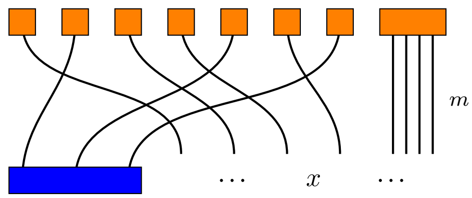
$$f(a_1, a_2, \cdots, a_{k+1}) = (ka_1, (k-1)a_2, (k-2)a_3, \cdots, a_k, 0)$$

Define

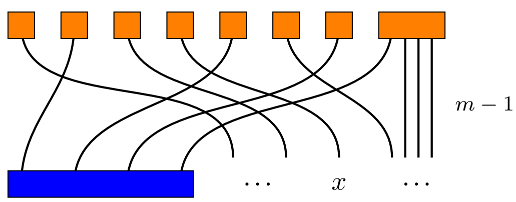
$$P(n,k) = \sum_{R_{k+1}^{n-k}} q^{|f(a_1,a_2,\cdots,a_{k+1})|}$$

Example 6.2.

$$P(7,2) = 1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + 3q^6 + 2q^7 + 2q^8 + q^9 + q^{10}$$

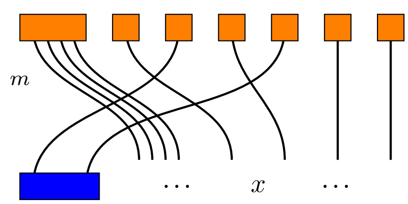


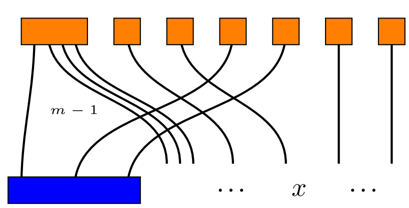
Consider the above diagram, representative of $(h_1^n h_m, e_k x)$. In the diagram, $n = 7$ and $m = 3$. The three strands from e_3 "split" the seven h_1's into groups of 1, 2, 1, and 0. This is a 3+1-tuple that sums to $7-3 = n-k = 4$. Numbering the h_1's from left to right, note that the first h_1 contributes q^k intersections, the third and fourth h_1's contribute q^{k-1} intersections, and so on. In general, the diagrams in which no strand connects h_m and e_k contribute $P(n,k)(h_1^{n-k}h_m, x)$ to $(h_1^n h_m, e_k x)$.



If a strand connects e_k to h_m, then it intersects the other $n-(k-1)$ strands connecting some h_1 to x, contributing a factor of q^{n-k+1}.The other intersections contribute $P(n, k-1)$. Putting this case and the previous case together,

$$(h_1^n h_m, e_k x) = P(n,k)(h_1^{n-k}h_m, x) + q^{n-k+1}P(n,k-1)(h_1^{n-k+1}h_{m-1}, x) \tag{6.1}$$





Similarly, the above two diagrams show that

$$(h_m h_1^n, e_k x) = q^{mk}P(n,k)(h_m h_1^{n-k}, x) + q^{(m-1)(k-1)}P(n,k-1)(h_{m-1}h_1^{n-k+1}, x) \tag{6.2}$$

Now, consider the case when $k = n+1$. In this case, there is only one diagram for the bilinear form, and it can be shown that

$$\begin{cases} (h_1^n h_m, e_{n+1}x) = (h_{m-1}, x) \\ (h_m h_1^n, e_{n+1}x) = q^{n(m-1)}(h_{m-1}, x) \end{cases}$$

which are equal since $q^n = 1$. Now, if $k \leq n$, we claim that $P(n,k) = 0$ for all $n \neq k$. This follows from the fact that $q^n = 1$, that $q^{n-\ell)} \neq 1$ for $\ell \in (1, 2, 3, \cdots, n-1)$, and the fact that

$$P(n,k) = \binom{n}{k}_q$$

The above statement follows from a bijection establishing $P(n,k)$ as the Gaussian binomial coefficient $\binom{n}{k}_q$. It is known that the coefficient of q^j in $\binom{n}{k}_q$ is the number of partitions of j into k or fewer parts, with each part less than or equal to k. $P(n,k)$ yields the same result since f takes every $k+1$-tuple to a $k+1$-tuple with last term 0. Each term must be less than or equal to $n-k$ since we have imposed that the sum of all the terms is $n-k$.

We substitute $P(n,k) = 0$ in (6.1) and (6.2) to find that both products $(h_1^n h_m, e_k x)$ and $(h_m h_1^n, e_k x)$ are 0 unless $n = k$ or $n = k-1$ (already addressed). If $n = k$, then

$$\begin{cases} (h_1^n h_m, e_n x) = (h_m, x) + qP(n, n-1)(h_1 h_{m-1}, x) \\ (h_m h_1^n, e_n x) = q^{nm}(h_m, x) + q^{(m-1)(n-1)} P(n, n-1)(h_{m-1} h_1, x) \end{cases}$$

Since $q^{mn} = 1$ and $P(n, n-1) = 0$, the above two expressions are equal. We therefore have the desired result when q is a primitive root of unity. By using some basic number theory and the recursive property of the Gaussian polynomials:

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q$$

one may extend the result to any root of unity. □

Other relations remain difficult to find. To illustrate the complexity of relations for $q^2 \neq 1$, consider the following relation obtained computationally, for $q^3 = 1$:

$$\begin{aligned} v_1 &= h_{11211} + h_{12111} + h_{21111} \\ v_2 &= h_{1122} - 2h_{1221} + 3h_{2112} + h_{2211} \\ v_3 &= 2h_{1131} - 2h_{114} + 2h_{1311} - 2h_{141} + 3h_{222} + 2h_{1113} - 2h_{411} \\ v_1 &+ q^2 v_2 + q v_3 = 0 \end{aligned}$$

7 Development of the q-nilHecke Algebra

We work in the $\mathbb{Z}$-graded, q-braided setting throughout. Let $\mathbb{C}$ be a commutative ring and let $q \in \mathbb{C}^\times$ be a unit. If V, W are graded $\mathbb{C}$-modules and $v \in V$, $w \in W$ are homogeneous, the braiding is the "q-twist":

$$\begin{aligned} \tau_q : &V \otimes W \to W \otimes V \\ &v \otimes w \mapsto q^{|v||w|} w \otimes v, \end{aligned} \tag{7.1}$$

where $|\cdot|$ is the degree function. By q-algebra we mean an algebra object in the category of graded $\Bbbk$-modules equipped with this braided monoidal structure; likewise for q-bialgebras, q-Hopf algebras, and so forth.

Definition 7.1. The q-algebra Pol_n^q is defined to be

$$\mathrm{Pol}_n^q = \Bbbk\langle x_1, \ldots, x_n \rangle / (x_j x_i - q x_i x_j \text{ if } i < j), \tag{7.2}$$

where $|x_i| = 1$ for $i = 1, \ldots, n$.

Note that $\mathrm{Pol}_n^q \cong \otimes_{i=1}^n \mathrm{Pol}_1^q$. There are two interesting subalgebras of Pol_n^q that can be thought of q-analogues of the symmetric polynomials. Define the k-th *elementary q-symmetric polynomial* to be

$$e_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1} \cdots x_{i_n}$$

and define the k-th *twisted elementary q-symmetric polynomial* to be

$$\widetilde{e}_k(x_1, \dots, x_n) = \sum_{1 \le i_1 < \dots < i_k \le n} \widetilde{x}_{i_1} \cdots \widetilde{x}_{i_n},$$

where $\widetilde{x}_j = q^{j-1} x_j$.

Definition 7.2. The q-algebra of *q-symmetric polynomials* in n variables, denoted Λ_n^q, is the subalgebra of Pol_n^q generated by $e_1, \dots, e_n$. Likewise for the *twisted q-symmetric polynomials*, $\widetilde{\Lambda}_n^q$, and $\widetilde{e}_1, \dots, \widetilde{e}_n$.

The type A braid group on n strands acts on Pol_n^q by setting

$$s_i(x_j) = \begin{cases} qx_{i+1} & j = i \\ q^{-1}x_i & j = i+1 \\ qx_j & j > i+1 \\ q^{-1}x_j & j < i \end{cases} \tag{7.3}$$

and extending multiplicatively.

Definition 7.3. For $i = 1, \dots . n-1$, the i-th *q-divided difference operator* ∂_i is the linear operator $\mathbb{C}\langle x_1, \dots, x_n \rangle \to \mathbb{C}\langle x_1, \dots, x_n \rangle$ defined by

$$\partial_i(x_j) = \begin{cases} q & j = i \\ -1 & j = i+1 \\ 0 & j \neq i, i+1 \end{cases} \tag{7.4}$$

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g). \tag{7.5}$$

Lemma 7.4. For every i and every $j < k$,

$$\partial_i(x_k x_j - q x_j x_k) = 0. \tag{7.6}$$

Thus ∂_i descends to an operator $\mathrm{Pol}_n^q \to \mathrm{Pol}_n^q$.

Proof. Since $\partial_i(x_j) = 0$ for $j > i+1$, one may reduce the lemma to having to prove:

$$\partial_1(x_2 x_1 - q x_1 x_2) = 0$$
$$\partial_1(x_3 x_1 - q x_1 x_3) = 0$$
$$\partial_1(x_3 x_2 - q x_2 x_3) = 0$$

The above statements are straightforward to check using the Leibniz Rule. □

Lemma 7.5.

$$\partial_i(x_i^k) = \sum_{j=0}^{k-1} q^{jk-2j-j^2+k} x_i^j x_{i+1}^{k-1-j} \tag{7.7}$$

$$\partial_i(x_{i+1}^k) = -\sum_{j=0}^{k-1} q^{-j} x_i^j x_{i+1}^{k-1-j} \tag{7.8}$$

Proof. We induct on k. The base cases follow from the definition of the ∂_i. □

By Lemma 7.5,

$$d_i(x_i^{nk} + x_{i+1}^{nk}) \neq 0 \text{ if } q^n = 1. \tag{7.9}$$

Lemma 7.6. For every $i = 1, \dots, n-1$ and every k,

$$\partial_i(\widetilde{e}_k) = 0. \tag{7.10}$$

Hence $\widetilde{\Lambda}_n^q \subseteq \bigcap_{i=1}^{n-1} \ker(\partial_i)$.

Proof. We can express $\widetilde{e}_k$ as

$$e_k = \sum_{\substack{|\underline{J}|=k \\ i,i+1\notin \underline{J}}} \widetilde{x}_{\underline{J}} + \sum_{\substack{|\underline{J}|=k-1 \\ i,i+1\notin \underline{J}}} q^{f(\underline{J},i,k)} \widetilde{x}_{\underline{J}}(x_i + qx_{i+1}) + \sum_{\substack{|\underline{J}|=k-2 \\ i,i+1\notin \underline{J}}} q^{g(\underline{J},i,k)} \widetilde{x}_{\underline{J}} x_i x_{i+1}, \tag{7.11}$$

for certain $\mathbb{Z}$-valued functions f, g. The result then follows from the easy calculation

$$\partial_i(x_i + qx_{i+1}) = \partial_i(x_i x_{i+1}) = 0 \tag{7.12}$$

and the Leibniz rule. □

Lemma 7.7. The following relations hold among the operators ∂_i and x_i (left multiplication by x_i):

$$\partial_i^2 = 0 \tag{7.13}$$
$$\partial_j\partial_i - q\partial_i\partial_j = 0 \text{ for } j > i+1 \tag{7.14}$$
$$x_j x_i = q x_i x_j \text{ for } i < j \tag{7.15}$$
$$\partial_i x_j - q x_j \partial_i = 0 \text{ for } j > i+1 \tag{7.16}$$
$$q\partial_i x_j - x_j \partial_i = 0 \text{ for } j < i \tag{7.17}$$
$$\partial_i x_i - q x_{i+1}\partial_i = q \tag{7.18}$$
$$x_i\partial_i - q\partial_i x_{i+1} = q. \tag{7.19}$$

Proof. To show that $\partial_i^2 = 0$, note that we can reduce to $i = 1$ and proceed by induction. Since $\partial_i(1) = 0$, the base case follows. Suppose that $\partial_i^2(f) = 0$. Then, note that

$$\partial_1^2(x_1 f) = \partial_1(qf + qx_2\partial_1(f)) = q\partial_1(f) - q\partial_1(f) + x_1\partial_1^2(f) = 0$$
$$\partial_1^2(x_2 f) = \partial_1(-f + q^{-1}x_1\partial_1(f)) = -\partial_1(f) + \partial_1(f) + x_2\partial_1^2(f) = 0$$
$$\partial_1^2(x_3 f) = \partial_1(qx_3\partial_1(f)) = q^2 x_3\partial_1^2(f) = 0$$

which completes the proof of the first statement in the lemma.

Statement 7.16 follows by definition. Statements 7.19, and 7.19 follow from a suitable application of the Leibniz Rule.

$$\partial_i(x_i f) = qf + qx_{i+1}\partial_i(f) \quad \partial_i(x_{i+1}f) = -f + q^{-1}x_i\partial_i(f)$$

Statements 7.17 and 7.18 also follow from a suitable application of the Leibniz Rule.

$$\partial_i(x_j f) = qx_j\partial_i(f) \text{ if } j > i+1 \quad \partial_i(x_j f) = q^{-1}x_j\partial_i(f) \text{ if } j < i$$

Statement 7.15 follows from an inductive argument. We can reduce to $i = 1$ and $j = 3$. Suppose that $\partial_j\partial_i = q\partial_i\partial_j$ if $j > i+1$. Then,

$$\partial_3\partial_1(x_1 f) - q\partial_1\partial_3(x_1 f) = (q\partial_3(f) + x_2\partial_3\partial_1(f)) - q(\partial_3(f) + x_2\partial_1\partial_3(f)) = 0$$
$$\partial_3\partial_1(x_2 f) - q\partial_1\partial_3(x_2 f) = (-\partial_3(f) + q^{-2}x_1\partial_3\partial_1(f)) - q(-q^{-1}\partial_3(f) + q^{-2}x_1\partial_1\partial_3(f)) = 0$$
$$\partial_3\partial_1(x_3 f) - q\partial_1\partial_3(x_3 f) = (q^2\partial_1(f) + q^2x_4\partial_3\partial_1(f)) - q(q\partial_1(f) + q^2x_4\partial_1\partial_3(f)) = 0$$
$$\partial_3\partial_1(x_4 f) - q\partial_1\partial_3(x_4 f) = (-q\partial_1(f) + x_3\partial_3\partial_1(f)) - q(-\partial_1(f) + x_3\partial_1\partial_3(f)) = 0$$
$$\partial_3\partial_1(x_5 f) - q\partial_1\partial_3(x_5 f) = q^2x_5\partial_3\partial_1(f) - q(q^2x_5\partial_1\partial_3(f)) = 0$$

thereby completing the induction. □

Lemma 7.8. $\partial_i\partial_{i+1}\partial_i\partial_{i+1}\partial_i\partial_{i+1} + \partial_{i+1}\partial_i\partial_{i+1}\partial_i\partial_{i+1}\partial_i = 0$

Proof. We utilize an inductive argument; reduce to $i = 1$ and assume that the braid relation holds true for some function f. Then, we check that the braid relation is true for $x_1 f$, $x_2 f$, $x_3 f$, and $x_4 f$ (since the behavior of $x_j f$ for $j \geq 4$ is the same).

For brevity, we will show the argument for $x_2 f$ only:

$$\partial_1\partial_2(x_2f) = q\partial_1(f) + q^2x_3\partial_1\partial_2(f) \qquad \partial_2\partial_1(x_2f) = -\partial_2(f) + q^{-2}x_1\partial_2\partial_1(f)$$
$$\partial_{212}(x_2f) = q\partial_2\partial_1(f) - q^2\partial_1\partial_2(f) + qx_2\partial_{212}(f) \qquad \partial_{121}(x_2f) = -\partial_1\partial_2(f) + q^{-1}\partial_2\partial_1(f) + q^{-1}\partial_{121}(f)$$
$$\partial_{1212}(x_2f) = q\partial_{121}(f) - q\partial_{212}(f) + x_1\partial_{1212}(f) \qquad \partial_{2121}(x_2f) = -\partial_{212}(f) + \partial_{121}(f) + x_3\partial_{2121}(f)$$
$$\partial_{21212}(x_2f) = q\partial_{2121}(f) + q^{-1}x_2\partial_{21212}(f) \qquad \partial_{12121}(x_2f) = -\partial_{1212}(f) + qx_3\partial 12121(f)$$
$$\partial_{121212}(x_2f) = q\partial_{12121}(f) + \partial_{21212}(f) + x_2\partial_{121212}(f) \qquad \partial_{212121}(x_2f) = -q\partial_{12121}(f) - \partial_{21212}(f) + x_2\partial_{121212}(f)$$

and the braid relation for x_2f follows from the inductive hypothesis. □

Remark 7.9. In this paper, we have discussed elementary symmetric functions in two contexts, but we can relate the two. Note that $N\Lambda^q$ is graded Hopf dual to a subalgebra $Q\Lambda^q$ of q-power series. It follows that Sym^q is a subalgebra of q-power series as well, so we can interpret elementary functions as q-power series. The map g from q-power series to q-polynomials in finitely many variables (say n) is given by setting $x_j = 0$ for $j > n$. The elementary functions from the previous section are the images of the elementary functions in this section through the map g .

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Dunkl operators have been an important tool in problems of non-commutative harmonic analysis, representation theory, algebraic and symplectic geometry, and combinatorics. They give rise to Cherednik algebras, an important feature of which is the existence of an action of the group SL(2), including in particular a Fourier transform. The corresponding action of the Lie algebra sl(2) is inner. Recently, a super version of the theory of symmetric polynomials, divided difference operators, etc has been developed. An "odd" version of Dunkl operators has been proposed, but it doesn't lead to an sl(2) triple as in the classical case.

In this project, the author constructs an sl(2) triple by suitably modifying the definition of Dunkl operators. This is potentially an important development, opening the way to an appropriate version of super Cherednik algebras and of super orthogonal polynomials, and possibly of quantization in the super geometry setting. Going into those topics would probably raise the project to the level of a PhD Thesis, and for a high school student the level this paper is very competitive. In this project, the author has displayed his abilities to become a strong and original research mathematician.

At the same time, we want to stress that the presentation of this paper (as distinct from the paper itself) should be reworked completely to satisfy the requirements of the competition. In the present form, a lot of topics are mentioned which are not directly relevant to the subject matter, and it is rather unlikely that the author would stand any questioning on those irrelevant topics (indeed, with the scope that wide, it could happen only on a PhD Thesis defense). Hence it would be wise to scale down _most_ of the presentation, mention motivation
and potential applications only in one section, and even then only the most relevant.

11-Evalucation Form

As to the other paper (no. 11), it is a very nice one, but I feel that it is not as good as no. 13. The results obtained in the paper are related Ellis' research. But I am not sure how important these results are.

YHMA Evaluation Form -- Regional Competition

Instruction: Please fill in all sections. This form is to help the organizers to communicate your assessments and rationales to others in the evaluation process.

Project Title	q-Symmetric Polynomials and nilHecke Algebras				
Evaluation level Choose one:	Referee Report	Regional Committee		Regional Presentation	
Selection Criteria (check one in each area below)	Very strong	Strong	Modest	Weak	Not Applicable
Mathematical Contents (1, 4, 5)		Yes			
Creativity, Originality (2, 3)		Yes			
Scholarship, Presentation (7)					Yes
Demonstrated Teamwork (8)			Yes		
Impact outside Math (9)		Yes			
COMMENTS					
Comments related to Criteria 1,4,5	PLEASE USE SEPARATE PARAGRAPH TO ELABORATE ON YOUR RATING FULLY The current paper under reviewing is written in a professional mathematical language. The authors used many mathematical tools such as Dunkl operators, (nil)-Hecke algebras, Spin Hecke algebras, symmetric functions and some representations theory. Also the authors have a good ability on developing a theorem and giving a mathematical proof. I will say that this paper is professional in the math area although it has some typos and need some improvements. Also I want to say something about the team work. I can see that the mentor did a very good job. He introduced a lot of modern mathematics to the author and it is not easy.				
Comments related to Criteria 2,3	~~PLEASE USE SEPARATE PARAGRAPH TO ELABORATE ON YOUR RATING FULLY~~ The authors gave several new constructions. For example, in section 4, 5. These constructions are known but the author use them gave many new results such as a sl_2 structure among these operators and in section 7, lots off new relations are founded. Unfortunately, the authors did not get some interesting and important results although there should be some based on what they found. But after all, this paper contains many original and creative ideas.				
Overall Recommendation For Presentation	Highly Competitive Yes	Perhaps Competitive	Not Competitive		

11--YHMA Evaluation Form -- Regional Competition

Instruction: Please fill in all sections. This form is to help the organizers to communicate your assessments and rationales to others in the evaluation process.

Project Title	q-Symmetric polynomials and nilHecke algebras				
Evaluation level Choose one:	Referee Report √	Regional Committee		Regional Presentation	
Selection Criteria (check one in each area below)	Very strong	Strong	Modest	Weak	Not Applicable
Mathematical Contents (1, 4, 5)		√			
Creativity, Originality (2, 3)		√			
Scholarship, Presentation (7)					
Demonstrated Teamwork (8)					
Impact outside Math (9)					
COMMENTS					
Comments related to Criteria 1,4,5	PLEASE USE SEPARATE PARAGRAPH TO ELABORATE ON YOUR RATING FULLY				
Comments related to Criteria 2,3	PLEASE USE SEPARATE PARAGRAPH TO ELABORATE ON YOUR RATING FULLY				
Overall Recommendation For Presentation	Highly Competitive	Perhaps Competitive √	Not Competitive		

Report on
"q-Symmetric polynomials and nilHecke algebras"

by Ritesh Ragavender

Given a positive integer n, consider the polynomial ring $R_n = \mathbb{C}[x_1, \dots, x_n]$ in n variables $x_1, \dots, x_n$ over the complex number field $\mathbb{C}$. The symmetric group $\mathfrak{S}_n$ acts on R_n by permuting the variables x_i. The fixed-point ring $\Lambda_n := R_n^{\mathfrak{S}_n}$ consists of symmetric polynomials. By taking the inverse limit of Λ_n via the maps $\Lambda_n \twoheadrightarrow \Lambda_{n-1}, x_n \mapsto 0$, we obtain the ring Λ of symmetric functions which admits several interesting bases with important applications in enumerative combinatorics. The theory of symmetric functions plays a fundamental role in several areas of mathematics and physics, including particularly the representation theory of symmetric groups and general linear groups; see [7].

For each $1 \leq i < n$, the (even) divided difference operator $\partial_i : R_n \to R_n$ is defined by

$$\partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}},$$

where $f \in R_n$, and $s_i(f)$ is obtained from f by swapping x_i and x_{i+1}. The operators ∂_i together with the operators of multiplication by x_i generate the nilHecke ring NH_n which is isomorphic to the matrix algebra of size $n! \times n!$ with coefficients in the ring Λ_n of symmetric polynomials. It turns out that the ring NH_n plays a central role in the theory of categorification of quantum groups.

On the other hand, Dunkl [1] introduced the operators $\eta_i : R_n \to R_n$ $(1 \leq i \leq n)$ by setting

$$\eta_i = \frac{\partial}{\partial x_i} + \alpha \sum_{k \neq i} \partial_{i,k},$$

where $\alpha \in \mathbb{C}$, $\frac{\partial}{\partial x_i}$ is the partial derivative with respect to x_i, and $\partial_{i,k}$ denotes the (even) divided difference operator $(1 - s_{i,k})/(x_i - x_k)$. Here $s_{i,k}$ acts a polynomial in R_n by swapping x_i and x_k. The operators η_i have various properties, e.g., they commute with each other $(\eta_i \eta_j = \eta_j \eta_i)$ and satisfy

$$\eta_i(fg) = \frac{\partial}{\partial x_i}(f)g + f\eta_i(g) \quad \text{and} \quad \eta_i x_j + x_i \eta_j = \eta_j x_i + x_j \eta_i$$

for $1 \leq i, j \leq n$, where $f, g \in R_n$. The Dunkl operators have been widely studied in the literature. For example, by setting operators

$$r^2 = \sum_{i=1}^{n} x_i^2, \; E = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + \frac{\mu}{2}, \; \text{and} \; \Delta = \sum_{i=1}^{n} \eta_i^2,$$

Heckman [5] showed that r^2, E, and Δ satisfy the defining relations of the Lie algebra $\mathfrak{sl}_2$.

A noncommutative theory of symmetric functions has been developed by Gelfand–Krob–Lascoux–Leclerc–Retakh–Thibon [4], based on the notion of quasi-determinant. The noncommutative analogs of symmetric polynomials have been also studied. In the noncommutative case, the variables x_i in general do not commute.

Recently, Ellis and Khovanov [2] studied a q-analogue of the standard bilinear form on the commutative ring of symmetric functions via introducing the notion of a q-Hopf algebra. The $q = -1$ case leads to a $\mathbb{Z}$-graded Hopf superalgebra, called the algebra of odd symmetric functions. They then described counterparts of elementary and complete symmetric functions, power sums, Schur functions, and combinatorial interpretations of associated change of basis relations. In [3], by further introducing the odd divided difference operators, Ellis, Khovanov, and Lauda defined an odd version of the nilHecke algebra, called the odd nilHecke algebra, and developed an odd analogue of the thick diagrammatic calculus for nilHecke algebras. They obtained a Morita equivalence between odd nilHecke algebras and the rings of odd symmetric functions in finitely many variables. Moreover, like the even counterparts, they proved that odd nilHecke algebras categorify the positive half of quantum $\mathfrak{sl}_2$.

The present paper is mainly based on [2, 3], and it deals with odd Dunkl operators, q-symmetric polynomials and q-nilHecke algebras. The author establishes a connection between odd Dunkl operators and odd nilHecke algebras, and introduces a variant of odd Dunkl operators which is used to construct operators that generate the Lie algebra $\mathfrak{sl}_2$. Using diagrammatic techniques, the author gives certain relations for q-symmetric polynomials. The author also defines q-analogues of divided difference operators and describes their properties.

In the following we are going to give a detailed explanation of the main contents and results of the present paper.

Sections 1 and 2 are introduction and background, respectively.

In Sections 3 and 4, the author works over the quotient ring R'_n of the free algebra $\mathbb{C}\langle x_1, \ldots, x_n\rangle$ by the ideal generated by $x_ix_j + x_jx_i$ for $1 \leq i < j \leq n$. Subsection 3.1 recalls from [3] the definition of the odd divided difference operator $\partial_i : R'_n \to R'_n$ for $1 \leq i < n$. It is defined by

$$\partial_i(x_j) = \begin{cases} 1, & \text{if } j = i, \text{ or } j+1; \\ 0, & \text{otherwise,} \end{cases}$$

$$\partial_i(fg) = \partial_i(f)g + (-1)^{\deg(f)} s_i(f)\partial_i(g),$$

where $f, g \in R'_n$, f is homogeneous with degree $\deg(f)$, and $s_i(f)$ is obtained from f by swapping x_i and x_{i+1}. As in the even case, one can define the odd divided difference operator $\partial_{i,k}$ for $i \neq k$ in terms of the transposition $s_{i,k}$. Thus, ∂_i is understood as $\partial_{i,i+1}$. In [2], the odd nilHecke algebra is defined to be the subalgebra generated by ∂_i $(1 \leq i < n)$ and the operators of multiplication by x_j $(1 \leq j \leq n)$. In Subsection 3.2, the author defines the -1-shift operator

$$\tau_i : R'_n \longrightarrow R'_n,\ f(x_1, \ldots, x_i, \ldots, x_n) \longmapsto f(x_1, \ldots, -x_i, \ldots, x_n)$$

and set for $k \neq i$,

$$r_{i,k} = \partial_{i,k} s_{i,k}.$$

In particular, set $r_i = r_{i,i+1}$ for $1 \leq i < n$. Then the author shows in Lemma 3.6 that the operators $r_{i,k}$ satisfy relations similar to those for the odd divided difference operators $\partial_{i,k}$. With the help of the formulas in Lemmas 3.7 and 3.8, the author finally shows that the odd Dunkl operators η_i^{odd} defined by Khongsap and Wang [6] can be expressed as

$$\eta_i^{\text{odd}} = t\delta_i + u \sum_{k \neq i} r_{i,k},$$

where $t, u \in \mathbb{C}$ and $\delta_i = (2x_i)^{-1}(1 - \tau_i)$.

In Section 4, the author defines operators $p_i : R_n' \to R_n'$ $(1 \leq i \leq n)$ by setting

$$p_i(x_1^{a_1} \cdots x_i^{a_i} \cdots x_n^{a_n}) = a_i(-1)^{a_1+\cdots+a_{i-1}} x_1^{a_1} \cdots x_i^{a_i-1} \cdots x_n^{a_n},$$

and then put

$$D_i = tp_i + u \sum_{k \neq i} r_{i,k}.$$

Clearly, the D_i are a modification of the odd Dunkl operators η_i^{odd}. Similar to the even case, the author defines r^2, the Euler operator E and odd Dunkl Laplacian operator Δ:

$$r^2 = \frac{1}{2t} \sum_{i=1}^{n} x_i^2,$$

$$E = \sum_{i=1}^{n} x_i p_i + \frac{n}{2} + \frac{u}{2} \sum_{k \neq i} s_{i,k},$$

$$\Delta = -\frac{1}{2t} \sum_{i=1}^{n} D_i^2.$$

The rest of this section is devoted to proving the relations

$$[E, r^2] = 2r^2, \ [E, \Delta] - 2\Delta, \text{ and } [r^2, \Delta] = E;$$

see Theorems 4.7 and 4.9. In other words, the three operators r^2, E and Δ give a realization of the Lie algebra $\mathfrak{sl}_2$ as in the even case [5]. It is also indicated in Remark 4.10 that if one uses the odd Dunkl operators η_i^{odd} instead of D_i to define the Dunkl Laplacian, then the three operators obtained generate an abelian Lie algebra rather than $\mathfrak{sl}_2$.

The final three sections, Sections 5, 6, and 7, deal with q-analogue of symmetric polynomials, as well as q-nilHecke algebras. Subsection 5.1 begins with the definition of a q-bialgebra $N\Lambda^q$ introduced in [2]. Let q be a nonzero complex number and let $N\Lambda^q$ be the free associative $\mathbb{Z}$-graded $\mathbb{C}$-algebra with generators h_m for $m \geq 1$, where $\deg h_m = m$ (For convenience, set $h_0 = 1$ and $h_m = 0$ for $h < 0$). Define a comultiplication Δ on $N\Lambda^q$ by setting

$$\Delta(h_m) = \sum_{i=0}^{m} h_i \otimes h_{m-i}$$

and a counit ϵ by setting $\epsilon(x) = 0$ for all homogeneous x with $\deg(x) > 0$. Then $N\Lambda^q$ becomes a q-bialgebra with the multiplication on $(N\Lambda^q)^{\otimes 2}$ given by

$$(w \otimes x)(y \otimes z) = q^{\deg(x)\deg(y)}(wx \otimes xz)$$

for homogeneous elements $w, x, y, z \in N\Lambda^q$. In [2, (2.1)], Ellis and Khovanov defined a bilinear form $(-,-) : N\Lambda^q \times N\Lambda^q \to \mathbb{C}$ which satisfies

$$(x_1 \otimes x_2, \Delta(y)) = (x_1 x_2, y).$$

Moreover, the radical I of the bilinear form is a q-bialgebra ideal of $N\Lambda^q$, i.e.,

$$IN\Lambda^q = N\Lambda^q I = I \text{ and } \Delta(I) \subset I \otimes N\Lambda^q + N\Lambda^q \otimes I.$$

In Subsection 5.2, the author defines elementary symmetric functions e_m in $N\Lambda_q$ by setting $e_0 = 1$ and

$$\sum_{i=0}^{m} (-1)^i q^{\binom{i}{2}} e_i h_{m-i} \text{ for } m \geq 1.$$

Lemma 5.1 states that

(1) $\Delta(e_m) = \sum_{i=0}^{m} e_i \otimes e_{m-i}$,

(2) If $\lambda = (\lambda_1, \ldots, \lambda_t)$ is a composition of m, then

$$(h_\lambda, e_m) = \begin{cases} 1, & \text{if } \lambda = (1, \ldots, 1; \\ 0, & \text{otherwise,} \end{cases}$$

where $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_t}$.

The lemma is a q-analogue of [2, Proposition 2.5], as well as its proof.

The entire Section 6 presents a proof of the fact that if $q^n = 1$, then h_1^n lies in the center of $N\Lambda^q$ (Lemma 6.1).

Section 7 deals with the q-algebra

$$\mathrm{Pol}_n^q = \mathbb{C}\langle x_1, \ldots, x_n \rangle / (x_j x_i - q x_i x_j : i < j),$$

where $q \in \mathbb{C}$ and all x_i have degree 1. The author defines q-analogues of elementary symmetric polynomials: the k-th elementary q-symmetric polynomial

$$e_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k} \quad (1 \leq k \leq n)$$

and the k-th twisted elementary q-symmetric polynomial

$$\widetilde{e}_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \widetilde{x}_{i_1} \cdots \widetilde{x}_{i_k},$$

where $\widetilde{x}_i = q^{i-1}x_i$. The $\widetilde{e}_k$ are q-analogue of odd elementary symmetric polynomials defined in [3, (2.21)]. The subalgebra generated by $e_1, \ldots, e_n$ (resp. $\widetilde{e}_1, \ldots, \widetilde{e}_n$) is denoted by Λ_n^q (resp. $\widetilde{\Lambda}_n^q$). The author defines the $\mathfrak{S}_n$-action on Pol_n^q by

$$s_i(x_j) = \begin{cases} qx_{i+1}, & \text{if } j = i; \\ q^{-1}x_i, & \text{if } j = i+1; \\ qx_j, & \text{if } j > i+1; \\ q^{-1}x_j, & \text{if } j < i. \end{cases}$$

Further, define the q-divided difference operator ∂_i $(1 \leq i < n)$ on Pol_n^q by

$$\partial_i(x_j) = \begin{cases} q, & \text{if } j = i; \\ -1, & \text{if } j = i+1; \\ 0, & \text{otherwise,} \end{cases}$$

$$\partial_i(fg) = \partial_i(f)g + s_i(f)g.$$

Lemma 7.1 shows that $\partial_i(\widetilde{e}_k) = 0$ for all i, k. This implies that

$$\widetilde{\Lambda}_n^q \subset \bigcap_{i=1}^{n-1} \mathrm{Ker}(\partial_i).$$

In Lemmas 7.7 and 7.8, the author obtains relations among the operators ∂_i and x_j (as left multiplication by x_j) which are q-analogues of those in [3, Proposition 2.1].

In conclusion, the results obtained in the present paper are interesting and seem to be new, but they are not surprising because most of them are analogues of results in the literature. For examples, the results in Section 4 are the odd counterparts of those obtained in [5]. Lemma 5.1 is a q-analogue of [2, Proposition 2.5]; Lemma 7.4, Lemma 7.5 and Lemma 7.7 are, respectively, q-analogues of (2.5), (2.6), Proposition 2.1 in [3].

The main new idea in the present paper may be the introduction of the operator D_i in (4.2) obtained from the odd Dunkl operator η_i^{odd} in (3.25) by substituting p_i for δ_i. With this modification, the author is able to generalize the result in [5] to the odd case. Many other proofs are modifications of certain proofs given in [2, 3].

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