Estimate of the electric potential outside a bounded charged conductor

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Abstract

Starting from an evident physical phenomenon, which states that in a three dimensional world, the distribution of the electric potential resulted from a charged conductor diminishes in a regular decaying rate to zero when the distance reaches infinity regardless of the shape of the conductor, this paper endeavors to generalize this case in an accurate mathematical way, and furthermore, some fascinating results of the same phenomenon in an higher or lower dimension are obtained along the way of my discovery.

To summarize the main point of this paper in a practical term other than the sophisticated and intangible mathematical terms, I could claim that for any charged conductor that is regular enough, the distribution of the electric potential far away from the conductor will not alter much when you transform the physical shape of the conductor, for instance, when you squeeze it or punch it.

The technique applied by this paper is purely calculus and with certain knowledge of partial differential equations, especially the Laplacian equations, I could arrive at the results mentioned above.

In order to clarify any potential misunderstandings of the significance and originality of this result, I hereby claim that all my work is based on the knowledge of analysis and partial differential equations that has been established long ago, what I did is simply to apply the method in an original way to explain a universal phenomenon, and therefore, some generalization of it.

Notice

Before the argument of this paper there are a few notice and assumptions we need to make in order to convert a physical phenomenon into a pure mathematical problem.

If we let u denote the electric potential, then laplacian u is the density of the charge.

In this system we will consider three things as self-evident:

First of all, there is no charge outside the conductor;

Second of all, the conductor is an equipotential body;

Third of all, the electric potential decays to zero at infinity. This will give rise to the Dirichlet problem in our main result.

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1 Introduction

In this paper, we consider a bounded charged conductor Ω in $\mathbf{R}^n$, where boundary, denoted as $\partial\Omega$, is assumed to be regular enough so that the Dirichlet problem is solvable in $B_R(0)\backslash\overline{\Omega}$; for instance, it suffices to suppose that $\partial\Omega$ satisfies an exterior cone condition.

We will give an estimate on the electic potential outside Ω and the rate of decay, by using the maximum principle of harmonic function; as a corollary, we will give a proof of the uniqueness of distribution of the electric potential.

2 Preliminaries

Definition (2.1). *The boundary of Ω is said to satisfy an exterior cone condition, if $\forall \xi \in \partial\Omega$, there exists a finite circular cone K, with vertex ξ satisfying $\overline{K} \cap \overline{\Omega} = \xi$.*

Theorem (2.2). *Dirichlet problem is solvable for any domain Ω satisfying an exterior cone condition.*

Theorem (2.3). *Weak Maximum Principle: Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $\Delta u \geq 0 (\leq 0)$. Then, provided Ω is bounded*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u, \quad (\inf_{\Omega} u = \inf_{\partial\Omega} u)$$

Theorem (2.4). *Strong Maximum Principle: Let* $\Delta u \geq 0$ (≤ 0) *in* Ω *and suppose there exists a point* $y \in \Omega$ *for which* $u(y) = \sup_\Omega u$ $(\inf_\Omega u)$. *Then* u *is constant.*

Theorem (2.5). *Let* $\{u_n\}$ *be a monotone increasing sequence of harmonic functions in a domain* Ω *and suppose that for some point* $y \in \Omega$, *the sequence* $\{u_n(y)\}$ *is bounded. Then the sequence converges uniformly on any bounded subdomain* $\Omega' \subset\subset \Omega$ *to a harmonic function.*

3 Main Results

Theorem (3.1). Ω *is a bounded domain whose closure contained in* $B_1(0) \subseteq \mathbf{R}^n$ $(n \geq 3)$. $\partial\Omega$ *satisfies exterior cone condition; There exists a function* u *on* $\mathbf{R}^n \backslash \overline{\Omega}$ *satisfying the following:*

$$\begin{aligned} u|_{\partial\Omega} &= 1 && (3.2) \\ \lim_{|x|\to\infty} u(x) &= 0 && (3.3) \\ \Delta u &= 0 \text{ in } \mathbf{R}^n \backslash \overline{\Omega} && (3.4) \end{aligned}$$

Furthermore, the rate of decaying of u *is comparable to* $|x|^{2-n}$; *i.e.* $\exists$ C_1 *and* C_2 *such that*

$$\frac{C_1}{|x|^{n-2}} \leq u(x) \leq \frac{C_2}{|x|^{n-2}} \quad \text{as } |x| \to \infty.$$

Remark 1. *One could view* Ω *as a charged conductor with electric potential equals to 1, while there is no charge outside* Ω. *Then the electric potential decays in the same rate as* $|x|^{2-n}$, *no matter how the shape of* Ω *looks like, given a certain regularity.*

4 The uniqueness

As a corollary of the main theorem, we immediately obtain the uniqueness of the solution u.

Corollary 1. *The function* u *satisfying conditions (3.2)* $\sim$ *(3.4) is unique.*

PROOF: Existence is given by theorem (3.1).

Let v be any function that satisfies (3.2) $\sim$ (3.4); consider $u - v$; $\forall y \in \mathbf{R}^n \backslash \overline{\Omega}$ and $\forall \epsilon > 0$.

Since both u and v satisfy (3.3) $\exists N > |y|$ such that on $\partial B_N(0)$, $|u| + |v| \leq \epsilon$.

We know $u - v = 0$ on $\partial\Omega$ by (3.2).

Then on $B_N(0)\backslash\overline{\Omega}$, $|u - v| \leq \epsilon$.

By maximum and minimum principle, $|u - v|\Big|_{(y)} \leq \epsilon$.

Since ϵ is arbitrary $|u - v|\Big|_{(y)} = 0$; Since y is arbitrary, $u \equiv v$ in $\mathbf{R}^n \backslash \overline{\Omega}$; therefore u is unique.

5 Proof of Theorem (3.1)

PROOF: For each positive integer N, define u_N to be the solution of

$$\begin{cases} \Delta u_N = 0, & \text{in } B_N(0)\backslash\overline{\Omega} \\ u_N = 1, & \text{on } \partial\Omega \\ u_N = 0, & \text{on } \partial B_N(0) \end{cases}$$

By the exterior cone condition u_N exists uniquely.

Notice by Strong Maximum Principle, for each $k > 0$, $u_k > 0$ in $B_{k-1}(0)\backslash\overline{\Omega}$, therefore $u_k > u_{k-1}$ on $\partial(B_{k-1}(0)\backslash\overline{\Omega})$.

By Maximum Principle again, $u_k > u_{k-1}$ in $B_{k-1}(0)\backslash\overline{\Omega}$.

As a matter of fact, for any compact subset K in $\mathbf{R}^n\backslash\overline{\Omega}$, $\{u_N\}$ is monotone increasing sequence in K.

By (2.5), we have $\{u_N\}$ converge to a harmonic function u uniformly on any compact subset of $\mathbf{R}^n\backslash\overline{\Omega}$.

We define, for each integer N, v_N to be the solution of

$$\begin{cases} \Delta v_N = 0, & \text{in } B_N(0)\backslash\overline{B_1(0)} \\ v_N = 1, & \text{on } \partial B_1(0) \\ v_N = 0, & \text{on } \partial B_N(0) \end{cases}$$

Indeed,

$$v_N(x) = \frac{N^{n-2}}{N^{n-2}-1} \cdot \frac{1}{|x|^{n-2}} - \frac{1}{N^{n-2}-1}.$$

By Strong Maximum Principle , $u_N\Big|_{\partial B_1(0)} < 1 = v_N\Big|_{\partial B_1(0)}$.

Since $u_N\Big|_{\partial B_N(0)} = v_N\Big|_{\partial B_N(0)} = 0$, by Strong Maximum Principle , $u_N < v_N$ in $B_N(0)\backslash\overline{B_1(0)}$.

Let $\epsilon_0 = \min u_2(y) \quad y \in \partial B_1(0)$, by Strong Maximum Principle, $\epsilon_0 > 0$;

Consider $\epsilon_0 v_N = \lambda_n$, which solves:

$$\begin{cases} \Delta\lambda_N = 0, & \text{in } B_N(0)\backslash\overline{B_1(0)} \\ \lambda_N = 0, & \text{on } \partial B_N(0) \\ \lambda_N = \epsilon_0, & \text{on } \partial B_1(0) \end{cases}$$

By Maximum Principle, $u_N \geq \lambda_N = \epsilon_0 v_N$ in $B_N(0)\backslash\overline{B_1(0)}$.

To sum up, we obtain that

$$\frac{\epsilon_0 N^{n-2}}{N^{n-2}-1} \cdot \frac{1}{|x|^{n-2}} - \frac{\epsilon_0}{N^{n-2}-1} = \epsilon_0 v_N \leq u_N$$

$$\text{while} \quad u_N < v_N = \frac{N^{n-2}}{N^{n-2}-1} \cdot \frac{1}{|x|^{n-2}} - \frac{1}{N^{n-2}-1}$$

Let $N \to \infty$,

$$\frac{\epsilon_0}{|x|^{n-2}} \leq u \leq \frac{1}{|x|^{n-2}}$$

especially $u \to 0$ as $|x| \to \infty$.

6 The 2-dimension case.

When $n = 2$, the result turns out to be quite the opposite; indeed, u doesn't decay as $|x| \to \infty$; actually we have the following stronger result:

Theorem. *Let Ω satisfy the conditions in (3.1). Let u be a positive harmonic function in $\mathbf{R}^2\backslash\overline{\Omega}$, then* $\inf_{x\in\mathbf{R}^2\backslash\overline{\Omega}} |u| > 0$.

Proof: Suppose not exist $R_N \to +\infty$ and $\epsilon_N \to 0$, such that

$$\inf_{y\in\partial B_{R_N}(0)} u(y) = \epsilon_N \to 0.$$

Define v_N to be the solution of

$$\begin{cases} \Delta v_N = 0, & \text{in } B_{R_N}\backslash\overline{B_1} \\ v_N\Big|_{\partial B_1} = \inf_{\partial B_1} u = \epsilon_0 > 0 \\ v_N\Big|_{\partial B_{R_N}} = \inf_{\partial B_{R_N}} u = \epsilon_N \end{cases}$$

Solve this equation, we obtain

$$v_N = \frac{\epsilon_N - \epsilon_1}{\log R_N}\log|x| + \epsilon_0.$$

Then $\forall B_R(0)$, choose N large enough such that $R_N > R$ from N on. Since $v_N \leq u$ on $\partial(B_R(0)\backslash\overline{B_1(0)})$ by maximum principle, $v_N \leq u$ on $B_R\backslash\overline{B_1}$. Let $N \to \infty$, we obtain

$$u \geq \epsilon_0 \quad \text{on} \quad B_R\backslash\overline{B_1}.$$

Since this holds for all R, this contradicts our assumption that

$$\inf_{y\in\mathbf{R}^n\backslash\overline{\Omega}} |u(y)| = 0.$$

References

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