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论文题目：若干新的 Littlewood 型不等式

Some New Littlewood-Type Inequalities

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Abstract: In 1998, Cheng et al. [On a problem of Littlewood, Math Practice Theory 28 (1998) 314–319] proved: Let $p,q \geq 1, r > 0, r(p-1) \leq 2(q-1), \alpha = [(p-1)(q+r)+p^2+1]/(p+1), \beta = (2q+2r+p-1)/(p+1), \delta = (q+r-1)/(p+q+r)$. For any nonnegative sequence $\{a_n\}_{n=1}^{N}$, we have for any $n \geq 1$,

$$\sum_{n=1}^{N} a_n^p \sum_{i=1}^{n} a_i^q A_i^r \leq 2^\delta \sum_{n=1}^{N} a_n^\alpha A_n^\beta. \tag{$*$}$$

In 2015, Agarwal et al. [Dynamic Littlewood-type inequalities, Proc. Amer. Math. Soc. 143 (2015) 667–677] and Saker et al. [Littlewood and Bennett Inequalities on Time Scales, Mediterr. J. Math. 12 (3) (2015) 605–619] gave Littlewood's inequality and related results on time scales. In particular, they gave integral Littlewood's inequality: Let $t_0 \in \mathbb{R}$, $p,\ q \geq 1$. Assume $a : \mathbb{R} \to [0,\infty)$ is continuous and define $A(t) := \int_{t_0}^{t} a(s)\mathrm{d}s, \widetilde{B}(t) := \int_{t}^{\infty} a^{1+p/q}(s)\mathrm{d}s$, then

$$\int_{t_0}^{\infty} a^p(t)A^q(t)\widetilde{B}^q(t)\mathrm{d}t \leq \left(\frac{2pq-q}{p+q}\right)^q \int_{t_0}^{\infty} a^{2p}(t)A^{2q}(t)\mathrm{d}t. \tag{$**$}$$

But they did not prove that the constant $\left(\frac{2pq-q}{p+q}\right)^q$ in the above inequality is best possible.

In this paper, firstly, we introduce a new parameter and employ Hölder's inequality and integration by parts to extend ($**$), and use strong skills to prove the best possible constant. Secondly, by introducing the proper parameters, applying Jensen's inequality and Hölder's inequality, we present an improved inequality of ($*$) and a more general one. Also, some new inequalities are obtained under the conditions of monotonically nonincreasing sequence. Thirdly, we give some analogues of Littlewood's inequalities. Finally, further discussions are given for future research.

Keywords: Littlewood's inequality, Jensen's inequality, Hölder's inequality, best possible constant.

若干新的 Littlewood 型不等式

摘要: 1998 年, 成礼智等在《关于 Littlewood 的一个问题》(《数学的实践与认识》第 28 卷第 4 期, 314-319 页) 中证明了: 设 $p,q \geq 1, r > 0, r(p-1) \leq 2(q-1)$, $\alpha = [(p-1)(q+r)+p^2+1]/(p+1)$, $\beta = (2q+2r+p-1)/(p+1)$, $\delta = (q+r-1)/(p+q+r)$. 对任意非负数列 $\{a_n\}_{n=1}^N$, 当 $n \geq 1$ 时有

$$\sum_{n=1}^{N} a_n^p \sum_{i=1}^{n} a_i^q A_i^r \leq 2^\delta \sum_{n=1}^{N} a_n^\alpha A_n^\beta. \tag{$*$}$$

2015 年, Agarwal 等在《动态 Littlewood 型不等式》(Proc. Amer. Math. Soc. 143 (2015) 667–677) 和 Saker 等在《时标上的 Littlewood 不等式和 Bennett 不等式》(Mediterr. J. Math. 12 (3) (2015) 605–619) 给出了时标上的 Littlewood 型不等式和相关不等式. 特别地, 他们给出 Littlewood 积分不等式: 设 $t_0 \in \mathbb{R}$, p, $q \geq 1$. $a : \mathbb{R} \to [0, \infty)$ 为连续函数, 定义 $A(t) := \int_{t_0}^t a(s)\mathrm{d}s$, $\widetilde{B}(t) := \int_t^\infty a^{1+p/q}(s)\mathrm{d}s$, 则

$$\int_{t_0}^\infty a^p(t)A^q(t)\widetilde{B}^q(t)\mathrm{d}t \leq \left(\frac{2pq-q}{p+q}\right)^q \int_{t_0}^\infty a^{2p}(t)A^{2q}(t)\mathrm{d}t. \tag{$**$}$$

但他们并没有证明不等式的常数因子 $\left(\frac{2pq-q}{p+q}\right)^q$ 是最佳的.

本论文中, 首先, 我们引入一个新的参数、运用 Hölder 不等式和分部积分法推广不等式 $(**)$, 并利用很强的技巧证明常数因子是最佳的. 其次, 通过引进适当的参数, 运用 Jensen 不等式和 Hölder 不等式, 得出不等式 $(*)$ 的改进和一个更一般的不等式. 同时, 在数列单调非增条件下还可以得到一些新的不等式. 再次, 我们还给出 Littlewood 不等式的类似形式. 最后, 为进一步研究做更深入的讨论.

Keywords: Littlewood 不等式, Jensen 不等式, Hölder 不等式, 最佳常数因子.

§1. Introduction

In 1967, Littlewood [1] presented several remarkable open problems concerning inequalities for infinite series which seem to be "very far from simple". One of his problems asks whether an absolute constant K exists such that the following inequalities hold for any non-negative sequence $\{a_n\}$ with $A_n = \sum_{k=1}^{n} a_k$:

$$\sum_{n=1}^{\infty} a_n A_n^2 \left(\sum_{k=n}^{\infty} a_k^{3/2} \right)^2 \leq K \sum_{n=1}^{\infty} a_n^2 A_n^4, \tag{1}$$

$$\sum_{n=1}^{\infty} a_n^3 \sum_{k=1}^{n} a_k^2 A_k \leq K \sum_{n=1}^{N} a_n^4 A_n^2. \tag{2}$$

As it was pointed out by Littlewood, (1), (2) and related inequalities, have a close connection to the theory of orthogonal series. The importance of (1) and (2) comes from the fact that they appear to be new "elementary inequalities", and also that they may be applied to obtain a result on orthogonal functions which is proved in [1]. The theory of general orthogonal series originated at the turn of the century as a natural generalization, based on Lebesgue integration, of the theory of trigonometric series.

An answer to Littlewood's above question was published in 1987 by Bennett [2, 3] who showed that (1) holds when $K = 4$ and (2) holds when $K = 3/2$. Actually, Bennett proved the following much more general result:

$$\sum_{n=1}^{\infty} a_n^p A_n^q \left(\sum_{m=n}^{\infty} a_m^{1+\frac{p}{q}} \right)^r \leq \left(\frac{p(q+r)-q}{p} \right)^r \sum_{n=1}^{\infty} \left(a_n^p A_n^q \right)^{1+\frac{r}{q}}, \tag{3}$$

where $p, q, r \geq 1$ and $\{a_n\}$ are non-negative sequence. The special case $p = 1, q = r = 2$ leads to $K = 4$ of (1), $p = 2, q = r = 1$ leads to $K = 3/2$ of (2) by interchanging the order of summation.

It remains an open problem to determine the best possible constants in (1) and (2). In 1996, Alzer [4] improved (3) for a special case: Let $a_1, a_2 \cdots, a_N$ be nonnegative real numbers such that $a_1 \leq a_2 \leq \cdots \leq a_N$. If $p \geq 1, q, r > 0$ are real numbers such that $d = [p(q+r)-q]/p \geq k$, where $k \geq 1$ is an integer, then

$$\sum_{n=1}^{N} a_n^p A_n^q \left(\sum_{m=n}^{N} a_m^{1+\frac{p}{q}} \right)^r \leq \prod_{i=0}^{k-1} (d-i)^{r/k} \sum_{n=1}^{N} \left(a_n^p A_n^q \right)^{1+\frac{r}{q}}. \tag{4}$$

In 1998, under additional conditions, Cheng et al. [5] discussed (3) with a best constant: If $\{a_n\}_{n=1}^{N}$ is a nondecresing sequence, $p, q > 0, 0 \leq r \leq 1$ and $p(q+r) \geq p+q$, then

$$\sum_{n=1}^{N} a_n^p A_n^q \left(\sum_{m=n}^{N} a_m^{1+\frac{p}{q}} \right)^r \leq 1 \cdot \sum_{n=1}^{N} \left(a_n^p A_n^q \right)^{1+\frac{r}{q}}, \tag{5}$$

where the constant 1 is best possible. At the same time, (3) was partially improved by them: Let $p, q \geq 1, r > 0, r(p-1) \leq 2(q-1), \alpha = \left[(p-1)(q+r) + p^2 + 1\right]/(p+1), \beta = (2q+2r+p-1)/(p+1), \delta = (q+r-1)/(p+q+r)$. For any nonnegative sequence $\{a_n\}_{n=1}^N$, we have for any $n \geq 1$,

$$\sum_{n=1}^{N} a_n^p \sum_{i=1}^{n} a_i^q A_i^r \leq 2^\delta \sum_{n=1}^{N} a_n^\alpha A_n^\beta. \tag{6}$$

A special case $p = 3, q = 2, r = 1$ of the above inequality reduces to

$$\sum_{n=1}^{N} a_n^3 \sum_{i=1}^{n} a_i^2 A_i \leq \sqrt[3]{2} \sum_{n=1}^{N} a_n^4 A_n^2, \tag{7}$$

where $\sqrt[3]{2} = 1.2599\ldots < 3/2$.

In 2015, Agarwal et al. [6] and Saker et al. [7] gave Littlewood's inequality and related results on time scales. In particular, they gave integral Littlewood's inequality: Let $t_0 \in \mathbb{R}$, p, $q \geq 1$. Assume $a : \mathbb{R} \to [0, \infty)$ is continuous and define $A(t) := \int_{t_0}^{t} a(s)\mathrm{d}s, \widetilde{B}(t) := \int_{t}^{\infty} a^{1+p/q}(s)\mathrm{d}s$, then

$$\int_{t_0}^{\infty} a^p(t) A^q(t) \widetilde{B}^q(t)\mathrm{d}t \leq \left(\frac{2pq - q}{p + q} \right)^q \int_{t_0}^{\infty} a^{2p}(t) A^{2q}(t)\mathrm{d}t. \tag{8}$$

But they did not prove that the constant $\left(\frac{2pq-q}{p+q}\right)^q$ in the above inequality is best possible. For more information about related inequalities of Littlewood, we refer the reader to [8]–[11].

In this paper, firstly, we introduce a new parameter and employ Hölder's inequality and integration by parts to extend (8), and use strong skills to prove the best possible constant. Secondly, by introducing the proper parameters, applying Jensen's inequality and Hölder's inequality, we present an improved inequality of (6) and a more general one. Also, some new inequalities are obtained under the conditions of monotonically nonincreasing sequence. Thirdly, we give some analogues of Littlewood's inequalities. Finally, further discussions are given for future research.

§2. Basic Lemmas

Definition 1 (See [12]) *A function f is convex on an interval $[a,b]$ if for any two points x_1 and x_2 in $[a,b]$ and any λ* where $0<\lambda<1$,

$$f[\lambda x_1+(1-\lambda)x_2]\le \lambda f(x_1)+(1-\lambda)f(x_2).$$

A function f is said to be concave on an interval $[a,b]$ if the function $-f$ is convex on that interval.

Lemma 1 (Jensen's inequality, see [12]) *If $\lambda_1,\cdots,\lambda_n$ are nonnegative real numbers such that $\sum_{i=1}^n \lambda_i=1$, and f is convex, then Jensen's inequality can be stated as*

$$f\left(\sum_{i=1}^n \lambda_i x_i\right)\le \sum_{i=1}^n \lambda_i f(x_i)$$

for any $x_1,\cdots,x_n$ in the domain of f. If f is concave, then the inequality reverses, giving

$$f\left(\sum_{i=1}^n \lambda_i x_i\right)\ge \sum_{i=1}^n \lambda_i f(x_i). \tag{9}$$

Lemma 2 (Hölder's inequality, see [12, 13]) *Let $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$. Then Hölder's inequality for integrals states that*

$$\int_a^b |f(x)g(x)|\mathrm{d}x\le \left(\int_a^b |f(x)|^p\mathrm{d}x\right)^{1/p}\left(\int_a^b |g(x)|^q\mathrm{d}x\right)^{1/q}.$$

with equality when $|g(x)|=c|f(x)|^{p-1}$. Similarly, Hölder's inequality for sums states that

$$\sum_{k=1}^n |a_kb_k|\le \left(\sum_{k=1}^n |a_k|^p\right)^{1/p}\left(\sum_{k=1}^n |b_k|^q\right)^{1/q}.$$

with equality when $|b_k|=c|a_k|^{p-1}$. If $0<p<1$, then the inequality reverses.

In what follows, for convenience, we set $A_n=\sum_{k=1}^n a_k$.

Lemma 3. (See [14]) *Let $p<0$. For any non-negative sequence $\{a_n\}$ with $a_1>0$, we have for any $n\ge 1$,*

$$\sum_{k=n}^{\infty} a_kA_k^{p-1}\le \left(1-\frac{1}{p}\right)A_n^p.$$

Lemma 4. *Let $p<0$. For any non-negative sequence $\{a_n\}$ with $a_n\ge a_{n+1}$, we have for any $n\ge 1$,*

$$\sum_{k=n}^{\infty} a_kA_k^{p-1}\le \left(\frac{1}{n}-\frac{1}{p}\right)A_n^p. \tag{10}$$

Proof. We start with the inequality $x^p-px+p-1\ge 0$. By setting $x=A_{k-1}/A_k$ for $k\ge 2$, we obtain

$$A_{k-1}^{p} - pA_{k-1}A_{k}^{p-1} + (p-1)A_{k}^{p} \geq 0.$$

Replacing A_{k-1} in the middle term of the left-hand side expression above by $A_k - a_k$ and simplifying, we obtain

$$A_{k-1}^{p} - A_{k}^{p} \geq -pa_k A_{k}^{p-1}.$$

Upon summing, we obtain

$$\sum_{k=n+1}^{\infty} a_k A_k^{p-1} \leq -\frac{1}{p} A_n^p.$$

In view of $a_k \geq a_{k+1}$, we have $a_k \leq \frac{A_k}{k}$. Hence

$$\sum_{k=n}^{\infty} a_k A_k^{p-1} = a_n A_n^{p-1} + \sum_{k=n+1}^{\infty} a_k A_k^{p-1} \leq \left(\frac{1}{n} - \frac{1}{p}\right) A_n^p.$$

The proof is complete. □

Remark 1. Since $\frac{1}{n} \leq 1$, we get Lemma 3 from Lemma 4 without the monotonicity of $\{a_n\}$.

§3. Integral Case

In this section, we extend (8) and get the best possible constant.

Theorem 1. *If* $a(t) \geq 0, p \geq 1, q, r > 0, (pq+pr-q)/p > 1, A(t) := \int_{t_0}^{t} a(s)\mathrm{d}s, \Lambda(t) := \int_{t}^{\infty} a^{1+p/q}(s)\mathrm{d}s$, *then the following inequality holds:*

$$\int_{t_0}^{\infty} a^p(t)A^q(t)\Lambda^r(t)\mathrm{d}t \leq \left(\frac{pq+pr-q}{p+q}\right)^r \int_{t_0}^{\infty} \left(a^p(t)A^q(t)\right)^{1+\frac{r}{q}} \mathrm{d}t, \tag{11}$$

where the constant $\left(\frac{pq+pr-q}{p+q}\right)^r$ *in the above inequality is best possible.*

Proof. (i) The case $p > 1$.

The left-hand side of (11) may be rewritten as

$$L := \int_{t_0}^{\infty} b^p(t)\Lambda^r(t)\mathrm{d}t,$$

where $b(t) := a(t)A^{q/p}(t)$. Let $x = p(p-1)(q+r)/(pq+pr-q), u = (pq+pr-q)/\left[q(p-1)\right], w = (pq+pr-q)/(pr), (1/u)+(1/w) = 1$. Noting $u > 1, (p-x)w = 1, xu = p\left(1+\frac{r}{q}\right)$, applying Hölder's inequality with indices u and w, we have

$$\begin{aligned} L &= \int_{t_0}^{\infty} b^x(t)\left(b^{p-x}(t)\Lambda^r(t)\right)\mathrm{d}t \\ &\leq \left(\int_{t_0}^{\infty} b^{p\left(1+\frac{r}{q}\right)}(t)\mathrm{d}t\right)^{1/u}\left(\int_{t_0}^{\infty} b(t)\Lambda^{rw}(t)\mathrm{d}t\right)^{1/w}. \end{aligned} \tag{12}$$

Since

$$\int_{t_0}^{t} b(s)\mathrm{d}s = \int_{t_0}^{t} a(s)A^{q/p}(s)\mathrm{d}s = \int_{t_0}^{t} A^{q/p}(s)\mathrm{d}A(s) = \frac{p}{p+q}A^{1+\frac{q}{p}}(t),$$

integrating by parts gives

$$\begin{aligned} &\int_{t_0}^{\infty} b(t)\Lambda^{rw}(t)\mathrm{d}t \\ =\ & \int_{t_0}^{\infty} \Lambda^{rw}(t)\mathrm{d}\left(\int_{t_0}^{t} b(s)\mathrm{d}s\right) \\ =\ & \frac{p}{p+q}\int_{t_0}^{\infty} \Lambda^{rw}(t)\mathrm{d}A^{1+\frac{q}{p}}(t) \\ =\ & \frac{p}{p+q}\left(A^{1+\frac{q}{p}}(t)\Lambda^{rw}(t)\big|_{t_0}^{\infty} + rw\int_{t_0}^{\infty} A^{1+\frac{q}{p}}(t)\Lambda^{rw-1}(t)a^{1+\frac{p}{q}}(s)\mathrm{d}t\right) \\ =\ & \frac{prw}{p+q}\int_{t_0}^{\infty} b^{1+\frac{p}{q}}(t)\Lambda^{rw-1}(t)\mathrm{d}t. \end{aligned}$$

Let $y = p^2(q+r)/\left[q(pq+pr-q)\right], \alpha = (pq+pr-q)/p, \beta = (pq+pr-q)/(pq+pr-p-q), (1/\alpha)+(1/\beta) = 1$. Noting $\alpha > 1, y\alpha = p\left(1+\frac{r}{q}\right), (1+\frac{p}{q}-y)\beta = 1, \alpha = rw = (rw-1)\beta$, applying Hölder's inequality with indices α and β, we obtain

$$\begin{aligned}
&\int_{t_0}^{\infty} b(t)\Lambda^{rw}(t)\mathrm{d}t \\
= \;& \frac{prw}{p+q}\int_{t_0}^{\infty} b^{1+\frac{p}{q}}(t)\Lambda^{rw-1}(t)\mathrm{d}t \\
= \;& \frac{prw}{p+q}\int_{t_0}^{\infty} b^{y}(t)\left(b^{1+\frac{p}{q}-y}(t)\Lambda^{rw-1}(t)\right)\mathrm{d}t \\
\le \;& \frac{prw}{p+q}\left(\int_{t_0}^{\infty} b^{y\alpha}(t)\mathrm{d}t\right)^{1/\alpha}\left(\int_{t_0}^{\infty} b^{(1+\frac{p}{q}-y)\beta}(t)\Lambda^{(rw-1)\beta}(t)\mathrm{d}t\right)^{1/\beta} \\
= \;& \frac{pq+pr-q}{p+q}\left(\int_{t_0}^{\infty} b^{p\left(1+\frac{r}{q}\right)}(t)\mathrm{d}t\right)^{1/\alpha}\left(\int_{t_0}^{\infty} b(t)\Lambda^{rw}(t)\mathrm{d}t\right)^{1/\beta}.
\end{aligned}$$

Hence

$$\int_{t_0}^{\infty} b(t)\Lambda^{rw}(t)\mathrm{d}t \le \left(\frac{pq+pr-q}{p+q}\right)^{\alpha}\int_{t_0}^{\infty} b^{p\left(1+\frac{r}{q}\right)}(t)\mathrm{d}t.$$

Substituting it into (12) yields

$$L \le \left(\frac{pq+pr-q}{p+q}\right)^{r}\int_{t_0}^{\infty} b^{p\left(1+\frac{r}{q}\right)}(t)\mathrm{d}t.$$

Thus (11) holds.

Here we just see that the constant $\left(\frac{pq+pr-q}{p+q}\right)^{r}$ is best possible in the case of $p = 2, q = r = 1$. In other words, (11) reduces to

$$\begin{aligned}
&\int_{t_0}^{\infty} a^2(t)\left(\int_{t_0}^{t} a(s)\mathrm{d}s\right)\left(\int_{t}^{\infty} a^3(s)\mathrm{d}s\right)\mathrm{d}t \\
&\le 1\cdot\int_{t_0}^{\infty} a^4(t)\left(\int_{t_0}^{t} a(s)\mathrm{d}s\right)^2\mathrm{d}t. \qquad (13)
\end{aligned}$$

Taking $a(t) = t^{-\frac{1}{2}-\theta}(\theta > 0)$ and $t_0 = 1$, elementary calculating gives

$$\begin{aligned}
&\int_{t_0}^{\infty} a^2(t)\left(\int_{t_0}^{t} a(s)\mathrm{d}s\right)\left(\int_{t}^{\infty} a^3(s)\mathrm{d}s\right)\mathrm{d}t \\
= \;& \int_{1}^{\infty} t^{2(-\frac{1}{2}-\theta)}\left(\int_{1}^{t} s^{-\frac{1}{2}-\theta}\mathrm{d}s\right)\left(\int_{t}^{\infty} s^{3(-\frac{1}{2}-\theta)}\mathrm{d}s\right)\mathrm{d}t \\
= \;& \frac{2}{3\theta(1+16\theta+60\theta^2)},
\end{aligned}$$

$$\int_{t_0}^{\infty} a^4(t)\left(\int_{t_0}^{t} a(s)\mathrm{d}s\right)^2\mathrm{d}t = \int_{1}^{\infty} t^{4(-\frac{1}{2}-\theta)}\left(\int_{1}^{t} s^{-\frac{1}{2}-\theta}\mathrm{d}s\right)^2\mathrm{d}t$$

$$= \frac{2}{3\theta + 42\theta^2 + 120\theta^3}.$$

Since $\lim\limits_{\theta\to 0^+} \frac{2}{3\theta(1+16\theta+60\theta^2)} \Big/ \frac{2}{3\theta+42\theta^2+120\theta^3} = \lim\limits_{\theta\to 0^+} \frac{1+4\theta}{1+6\theta} = 1$, the constant 1 in (13) is best possible.

(ii) The case $p = 1$. Its proof is similar to that of (i) by applying Hölder's inequality only once, hence we omit the details. □

Remark 2. (11) degenerates into (8) when $r = q$. Therefore we generalize (8) to (11) by introducing a new parameter r. Furthermore, we prove that the constant $\left(\frac{pq+pr-q}{p+q}\right)^r$ in (11) is best possible.

§4. Series Case

In this section, by introducing the proper parameters, applying Jensen's inequality and Hölder's inequality, we present an improved inequality of (6) and a more general one. Also, some new inequalities are obtained under the conditions of monotonically nonincreasing sequence.

Theorem 2. *Let* $p, q > 1, r > (q-1)/(p-1), \alpha = p+q-1, \beta = 1+r, \delta = (q-1)/(p+q-2)$. *For any non-negative sequence* $\{a_n\}$, *we have for any* $n \geq 1$,

$$\sum_{n=1}^{N} a_n^p \sum_{i=1}^{n} a_i^q A_i^r \leq \left(\frac{pr-r}{pr-r-q+1}\right)^{\delta} \sum_{n=1}^{N} a_n^{\alpha} A_n^{\beta}. \tag{14}$$

Proof. Set $x = (p-1)/(p+q-2), y = r(p+q-2)/(q-1), z = -2+p+q, \lambda = (pr-q-r+1)/(q-1)$. It is obvious that $0 < \delta < 1, \alpha x + \delta = p, \beta x - \lambda\delta = 1, z\delta + 1 = q, y\delta = r, x + \delta = 1, 1 + z = \alpha, y - \lambda = \beta$. Using Jensen's inequality gives

$$\begin{aligned} \sum_{n=1}^{N} a_n^p \sum_{i=1}^{n} a_i^q A_i^r &= \sum_{n=1}^{N} \left(a_n^{\alpha} A_n^{\beta}\right)^x \left(\frac{a_n}{A_n^{\lambda}}\right)^{\delta} \sum_{i=1}^{n} \frac{a_i}{A_n} \left(a_i^z A_i^y\right)^{\delta} \\ &\leq \sum_{n=1}^{N} \left(a_n^{\alpha} A_n^{\beta}\right)^x \left(\frac{a_n}{A_n^{\lambda}}\right)^{\delta} \sum_{i=1}^{n} \left(\frac{a_i^{1+z} A_i^y}{A_n}\right)^{\delta} \\ &= \sum_{n=1}^{N} \left(a_n^{\alpha} A_n^{\beta}\right)^x \sum_{i=1}^{n} \left(\frac{a_n a_i^{1+z} A_i^y}{A_n^{1+\lambda}}\right)^{\delta}. \end{aligned}$$

Applying Hölder's inequality gives

$$\begin{aligned} &\sum_{n=1}^{N} \left(a_n^{\alpha} A_n^{\beta}\right)^x \sum_{i=1}^{n} \left(\frac{a_n a_i^{1+z} A_i^y}{A_n^{1+\lambda}}\right)^{\delta} \\ \leq &\left(\sum_{n=1}^{N} a_n^{\alpha} A_n^{\beta}\right)^x \left(\sum_{n=1}^{N} \frac{a_n}{A_n^{1+\lambda}} \sum_{i=1}^{n} a_i^{1+z} A_i^y\right)^{\delta}. \end{aligned} \tag{15}$$

On the other hand, interchanging the order of summation, by Lemma 3, we get

$$\begin{aligned} \sum_{n=1}^{N} \frac{a_n}{A_n^{1+\lambda}} \sum_{i=1}^{n} a_i^{1+z} A_i^y &= \sum_{n=1}^{N} a_n^{1+z} A_n^y \sum_{m=n}^{N} \frac{a_m}{A_m^{1+\lambda}} \\ &\leq \left(1 + \frac{1}{\lambda}\right) \sum_{n=1}^{N} a_n^{1+z} A_n^{y-\lambda} \\ &= \frac{pr-r}{pr-r-q+1} \sum_{n=1}^{N} a_n^{\alpha} A_n^{\beta}. \end{aligned}$$

Hence

$$\sum_{n=1}^{N} a_n^p \sum_{i=1}^{n} a_i^q A_i^r \leq \left(\frac{pr-r}{pr-r-q+1}\right)^{\delta} \sum_{n=1}^{N} a_n^{\alpha} A_n^{\beta}.$$

The proof is complete. □

Remark 3. (i) In particular, setting $p = 3, q = 2$ and $r = 1$ in Theorem 2 gives

$$\sum_{n=1}^{N} a_n^3 \sum_{i=1}^{n} a_i^2 A_i \leq \sqrt[3]{2} \sum_{n=1}^{N} a_n^4 A_n^2,$$

which is (7).

(ii) (14) and (6) have the same constant $2^{(q-1)/(p+q-2)}$ when $(pr-r)/(pr-r-q+1) = 2$. In fact, we get $r = 2(q-1)/(p-1)$ from $(pr-r)/(pr-r-q+1) = 2$, then the constant of (14) is $\left(\frac{pr-r}{pr-r-q+1}\right)^{\delta} = 2^{(q-1)/(p+q-2)}$. At the same time, the constant of (6) is $2^{(q+r-1)/(p+q+r)} = 2^{(q-1)/(p+q-2)}$.

(iii) If $p = 3, q = 2, r = 3/2$, then the constant of (14) is $\left(\frac{pr-r}{pr-r-q+1}\right)^{\delta} = \sqrt[3]{3/2} = 1.1447\ldots$, the constant of (6) is $2^{(q+r-1)/(p+q+r)} = 2^{5/13} = 1.3055\ldots > \sqrt[3]{3/2}$. Hence, to some extent, (14) is an improvement of (6).

Theorem 3. *Let $p, q > 1, r > (q-1)/(p-1), \alpha = p+q-1, \beta = 1+r, \delta = (q-1)/(p+q-2)$. For any non-negative sequence $\{a_n\}$ with $a_1 \geq a_2 \geq a_3 \geq \cdots$, we have for any $n \geq 1$,*

$$\sum_{n=1}^{N} a_n^p \sum_{i=1}^{n} a_i^q A_i^r \leq \left(\sum_{n=1}^{N} a_n^{\alpha} A_n^{\beta}\right)^{1-\delta} \left[\sum_{n=1}^{N} \left(\frac{1}{n} + \frac{q-1}{pr-q-r+1}\right) a_n^{\alpha} A_n^{\beta}\right]^{\delta}. \qquad (16)$$

Proof. Set $x = (p-1)/(p+q-2), y = r(p+q-2)/(q-1), z = p+q-2, \lambda = (pr-q-r1)/(q-1)$. It is obvious that $\alpha x + \delta = p, \beta x - \lambda\delta = 1, z\delta + 1 = q, y\delta = r, x + \delta = 1, 1 + z = \alpha, y - \lambda = \beta$. Using Jensen's inequality gives

$$\begin{aligned}
\sum_{n=1}^{N} a_n^p \sum_{i=1}^{n} a_i^q A_i^r &= \sum_{n=1}^{N} \left(a_n^{\alpha} A_n^{\beta}\right)^x \left(\frac{a_n}{A_n^{\lambda}}\right)^{\delta} \sum_{i=1}^{n} \frac{a_i}{A_n} \left(a_i^z A_i^y\right)^{\delta} \\
&\leq \sum_{n=1}^{N} \left(a_n^{\alpha} A_n^{\beta}\right)^x \left(\frac{a_n}{A_n^{\lambda}}\right)^{\delta} \sum_{i=1}^{n} \left(\frac{a_i^{1+z} A_i^y}{A_n}\right)^{\delta} \\
&= \sum_{n=1}^{N} \left(a_n^{\alpha} A_n^{\beta}\right)^x \sum_{i=1}^{n} \left(\frac{a_n a_i^{1+z} A_i^y}{A_n^{1+\lambda}}\right)^{\delta}.
\end{aligned}$$

In view of $x + \delta = 1, 0 < \delta < 1$, applying Hölder's inequality gives

$$\sum_{n=1}^{N} \left(a_n^{\alpha} A_n^{\beta}\right)^x \sum_{i=1}^{n} \left(\frac{a_n a_i^{1+z} A_i^y}{A_n^{1+\lambda}}\right)^{\delta}$$

$$\leq \left(\sum_{n=1}^{N} a_n^{\alpha} A_n^{\beta}\right)^{x} \left(\sum_{n=1}^{N} \frac{a_n}{A_n^{1+\lambda}} \sum_{i=1}^{n} a_i^{1+z} A_i^{y}\right)^{\delta}. \tag{17}$$

On the other hand, interchanging the order of summation and by Lemma 4, we get

$$\begin{aligned}
\sum_{n=1}^{N} \frac{a_n}{A_n^{1+\lambda}} \sum_{i=1}^{n} a_i^{1+z} A_i^{y} &= \sum_{n=1}^{N} a_n^{1+z} A_n^{y} \sum_{m=n}^{N} \frac{a_m}{A_m^{1+\lambda}} \\
&= \sum_{n=1}^{N} a_n^{1+z} A_n^{y} \sum_{m=n}^{N} \frac{a_m}{A_m^{1+\lambda}} \\
&\leq \left(\frac{1}{n} + \frac{1}{\lambda}\right) \sum_{n=1}^{N} a_n^{1+z} A_n^{y-\lambda} \\
&= \left(\frac{1}{n} + \frac{q-1}{pr-q-r+1}\right) \sum_{n=1}^{N} a_n^{\alpha} A_n^{\beta}.
\end{aligned}$$

Hence

$$\sum_{n=1}^{N} a_n^{p} \sum_{i=1}^{n} a_i^{q} A_i^{r} \leq \left(\sum_{n=1}^{N} a_n^{\alpha} A_n^{\beta}\right)^{1-\delta} \left[\sum_{n=1}^{N} \left(\frac{1}{n} + \frac{q-1}{pr-q-r+1}\right) a_n^{\alpha} A_n^{\beta}\right]^{\delta},$$

which concludes the proof. □

Remark 4. (i) Since $\frac{1}{n} \leq 1$, we get Theorem 2 from Theorem 3 without the monotonicity of $\{a_n\}$.

(ii) In particular, setting $p = 3, q = 2$ and $r = 1$ in Theorem 3 gives

$$\sum_{n=1}^{N} a_n^{3} \sum_{i=1}^{n} a_i^{2} A_i \leq \left(\sum_{n=1}^{N} a_n^{4} A_n^{2}\right)^{2/3} \left[\sum_{n=1}^{N} \left(\frac{1}{n} + 1\right) a_n^{4} A_n^{2}\right]^{1/3}. \tag{18}$$

Next we extend Theorem 2 as follows.

Theorem 4. *Let $0 \leq r \leq 1, q > 1-r, p > q/(q+r-1), \alpha = p(q+r)/q, \beta = q+r, \delta = (pq + pr - p - q)/(pq + pr - q)$. For any non-negative sequence $\{a_n\}$, we have for any $n \geq 1$,*

$$\sum_{n=1}^{N} a_n^{p} A_n^{q} \left(\sum_{m=n}^{N} a_m^{1+\frac{p}{q}}\right)^{r} \leq \left[\frac{p(q+r-1)}{q}\right]^{\delta r} \sum_{n=1}^{N} (a_n^{p} A_n^{q})^{1+\frac{r}{q}}. \tag{19}$$

Proof. Set $x = p/(pq + pr - q), y = (q + r - 1)(pq + pr - q)/(pq + pr - p - q), z = (pq + pr - q)/q, \lambda = q/(pq + pr - p - q)$. It is obvious that $\alpha x + \delta = 1 + \frac{p}{q}, \beta x - \lambda\delta = 1, z\delta + 1 = p(q+r-1)/q, y\delta = r, x+\delta = 1, 1+z = \alpha, y-\lambda = \beta$. Using Jensen's inequality and interchanging the order of summation, we have

$$\begin{aligned}\frac{\sum_{n=1}^{N} a_n^p A_n^q \left(\sum_{m=n}^{N} a_m^{1+\frac{p}{q}}\right)^r}{\sum_{i=1}^{N} (a_i^p A_i^q)^{1+\frac{r}{q}}} &= \sum_{n=1}^{N} \frac{(a_n^p A_n^q)^{1+\frac{r}{q}}}{\sum_{i=1}^{N} (a_i^p A_i^q)^{1+\frac{r}{q}}} \left(\frac{\sum_{m=n}^{N} a_m^{1+\frac{p}{q}}}{a_n^{\frac{p}{q}} A_n}\right)^r \\ &\le \left(\sum_{n=1}^{N} \frac{(a_n^p A_n^q)^{1+\frac{r}{q}}}{\sum_{i=1}^{N} (a_i^p A_i^q)^{1+\frac{r}{q}}} \cdot \frac{\sum_{m=n}^{N} a_m^{1+\frac{p}{q}}}{a_n^{\frac{p}{q}} A_n}\right)^r \\ &= \left(\frac{\sum_{n=1}^{N} a_n^{1+\frac{p}{q}} \sum_{m=1}^{n} \left(a_m^{\frac{p}{q}} A_m\right)^{q+r-1}}{\sum_{i=1}^{N} (a_i^p A_i^q)^{1+\frac{r}{q}}}\right)^r.\end{aligned}$$

Applying Theorem 2 gives

$$\begin{aligned}&\sum_{n=1}^{N} a_n^{1+\frac{p}{q}} \sum_{m=1}^{n} \left(a_m^{\frac{p}{q}} A_m\right)^{q+r-1} \\ \le\; &\left[\frac{p(q+r-1)}{q}\right]^{\delta} \sum_{i=1}^{N} (a_i^p A_i^q)^{1+\frac{r}{q}}.\end{aligned} \tag{20}$$

Thus (19) is valid. □

Remark 5. For $p = 2, q = r = 1$ in (19), we also get (7).

Theorem 5. *Let* $0 \le r \le 1, q > 1 - r, p > q/(q + r - 1), \alpha = p(q + r)/q, \beta = q + r, \delta = (pq + pr - p - q)/(pq + pr - q)$. *For any non-negative sequence* $\{a_n\}$ *with* $a_1 \ge a_2 \ge a_3 \ge \cdots$, *we have for any* $n \ge 1$,

$$\begin{aligned}\sum_{n=1}^{N} a_n^p A_n^q \left(\sum_{m=n}^{N} a_m^{1+\frac{p}{q}}\right)^r \le &\left[\sum_{n=1}^{N} (a_n^p A_n^q)^{1+\frac{r}{q}}\right]^{1-\delta r} \\ &\times \left[\sum_{n=1}^{N} \left(\frac{1}{n} + \frac{pq + pr - p - q}{q}\right) (a_n^p A_n^q)^{1+\frac{r}{q}}\right]^{\delta r}.\end{aligned} \tag{21}$$

Proof. Set $x = p/(pq + pr - q), y = (q + r - 1)(pq + pr - q)/(pq + pr - p - q), z = (pq + pr - q)/q, \lambda = q/(pq + pr - p - q)$. It is obvious that $\alpha x + \delta = 1 + p/q, \beta x - \lambda\delta = 1, z\delta + 1 = p(q + r - 1)/q, y\delta = r, x + \delta = 1, 1 + z = \alpha, y - \lambda = \beta$. Using Jensen's inequality and interchanging the order of summation, we have

$$\begin{aligned}\frac{\sum_{n=1}^{N} a_n^p A_n^q \left(\sum_{m=n}^{N} a_m^{1+\frac{p}{q}}\right)^r}{\sum_{i=1}^{N} (a_i^p A_i^q)^{1+\frac{r}{q}}} &= \sum_{n=1}^{N} \frac{(a_n^p A_n^q)^{1+\frac{r}{q}}}{\sum_{i=1}^{N} (a_i^p A_i^q)^{1+\frac{r}{q}}} \left(\frac{\sum_{m=n}^{N} a_m^{1+\frac{p}{q}}}{a_n^{\frac{p}{q}} A_n}\right)^r \\ &\le \left(\sum_{n=1}^{N} \frac{(a_n^p A_n^q)^{1+\frac{r}{q}}}{\sum_{i=1}^{N} (a_i^p A_i^q)^{1+\frac{r}{q}}} \cdot \frac{\sum_{m=n}^{N} a_m^{1+\frac{p}{q}}}{a_n^{\frac{p}{q}} A_n}\right)^r \\ &= \left(\frac{\sum_{n=1}^{N} a_n^{1+\frac{p}{q}} \sum_{m=1}^{n} \left(a_m^{\frac{p}{q}} A_m\right)^{q+r-1}}{\sum_{i=1}^{N} (a_i^p A_i^q)^{1+\frac{r}{q}}}\right)^r.\end{aligned}$$

Applying Theorem 3 gives

$$\begin{aligned}
&\sum_{n=1}^{N} a_n^{1+\frac{p}{q}} \sum_{m=1}^{n} \left(a_m^{\frac{p}{q}} A_m\right)^{q+r-1} \\
\leq \quad & \left[\sum_{i=1}^{N} (a_i^p A_i^q)^{1+\frac{r}{q}}\right]^{1-\delta} \left[\sum_{i=1}^{N} \left(\frac{1}{i} + \frac{pq+pr-p-q}{q}\right) (a_i^p A_i^q)^{1+\frac{r}{q}}\right]^{\delta}. \qquad (22)
\end{aligned}$$

Then (21) holds. □

§5. Analogues of Littlewood's Inequalities

In this section, we give some analogues of Littlewood's inequalities.

Theorem 6. *Let* $p, q \geq 1, r > 0, r(p-1) \leq 2(q-1), \alpha = [(p-1)(q+r)+p^2+1]/(p+1), \beta = (2q+2r+p-1)/(p+1), \delta = (q+r-1)/(p+q+r)$. *For any non-negative sequence* $\{a_n\}_{n\in\mathbb{N}}$ *and* $a_0 = A_0 > 0, a_1 > 0, A_n = \sum_{k=1}^n a_k$, *we have for any* $n \geq 1$,

$$\sum_{n=1}^N a_n^p \frac{A_{n-1}}{A_n} \sum_{i=1}^n a_i^q A_{i-1}^r \leq \sum_{n=1}^N a_n^\alpha A_{n-1}^\beta. \tag{23}$$

Proof. Set $x = (p+1)/(p+q+r), y = (p+q+r)r/(q+r-1), z = (q-1)(p+q+r)/(q+r-1)$. It is obvious that $\alpha x + \delta = p, \beta x - \delta = 1, z\delta + 1 = q, y\delta = r, x + \delta = 1, z - \alpha = \beta - y$. Using Jensen's inequality gives

$$\begin{aligned}
\sum_{n=1}^N a_n^p \frac{A_{n-1}}{A_n} \sum_{i=1}^n a_i^q A_{i-1}^r &= \sum_{n=1}^N \left(a_n^\alpha A_{n-1}^\beta\right)^x \left(\frac{a_n}{A_{n-1}}\right)^\delta \sum_{i=1}^n \frac{a_i}{A_n}\left(a_i^z A_{i-1}^y\right)^\delta \\
&\leq \sum_{n=1}^N \left(a_n^\alpha A_{n-1}^\beta\right)^x \left(\frac{a_n}{A_{n-1}}\right)^\delta \left(\sum_{i=1}^n \frac{a_i^{1+z} A_{i-1}^y}{A_n}\right)^\delta \\
&= \sum_{n=1}^N \left(a_n^\alpha A_{n-1}^\beta\right)^x \sum_{i=1}^n \left(\frac{a_n a_i^{1+z} A_{i-1}^y}{A_{n-1} A_n}\right)^\delta.
\end{aligned}$$

In view of $x + \delta = 1, 0 < \delta < 1$, applying Hölder's inequality gives

$$\begin{aligned}
&\sum_{n=1}^N \left(a_n^\alpha A_{n-1}^\beta\right)^x \sum_{i=1}^n \left(\frac{a_n a_i^{1+z} A_{i-1}^y}{A_{n-1} A_n}\right)^\delta \\
\leq\ &\left(\sum_{n=1}^N a_n^\alpha A_{n-1}^\beta\right)^x \left(\sum_{n=1}^N \frac{a_n}{A_{n-1} A_n} \sum_{i=1}^n a_i^{1+z} A_{i-1}^y\right)^\delta.
\end{aligned}$$

On the other hand, interchanging the order of summation, we get

$$\begin{aligned}
\sum_{n=1}^N \frac{a_n}{A_{n-1} A_n} \sum_{i=1}^n a_i^{1+z} A_{i-1}^y &= \sum_{n=1}^N a_n^{1+z} A_{n-1}^y \sum_{m=n}^N \frac{a_m}{A_{m-1} A_m} \\
&= \sum_{n=1}^N a_n^{1+z} A_{n-1}^y \sum_{m=n}^N \frac{A_m - A_{m-1}}{A_{m-1} A_m} \\
&\leq \sum_{n=1}^N a_n^{1+z} A_{n-1}^{y-1}.
\end{aligned}$$

Since $1 + z - \alpha = \beta - y + 1 \geq 0$ from $r(p-1) \leq 2(q-1)$, it follows that

$$\sum_{n=1}^N a_n^{1+z} A_{n-1}^{y-1} \leq \sum_{n=1}^N a_n^\alpha A_{n-1}^\beta.$$

The proof is complete. □

§6. Further Discussions

In this paper, we've extended or improved some discrete inequalities formulated by Littlewood. The integral case with best possible constant and related inequalities are presented. The methods include Hölder's inequality, integration by parts and Jensen's inequality, and the skills are very strong.

In this section, we list some problems for further research.

(I) Numerical calculations indicate that the constant factor $\left(\frac{pr-r}{pr-r-q+1}\right)^{\delta}$ in Theorem 2 is not best possible (see [2]). Is it possible to replace the constant with a smaller one? Maybe we can try to modify Lemmas 3 and 4 to get a more accurate estimate. Noting inequality (5), Theorem 3, and combining with mathematical software, a monotonically decreasing sequence $\{a_n\}$ will reflect the best of the constant factor. As to what form of $\{a_n\}$ taken, the constant factor is the best possible? we will focus on it in the future.

(II) We can discuss the reverse inequalities, high dimensional inequalities and others. Hölder's inequality and Jensen's inequality remain fundamental tools.

(III) We have not discussed the improvements of inequality (1) in this article. Gao [14] has done some work on it. We expect to get some beautiful and simple inequalities by improving it or doing some related work.

(IV) We can discuss Littlewood-type inequalities on time scales. The theory of time scales was introduced by Hilger [15] in 1988 in order to unify continuous and discrete analysis (see also [16]). The study of dynamic inequalities on time scales has received a lot of attention in the literature.

(V) Littlewood's inequality has applications on the general theory of orthogonal series. We will seek more applications of the Littlewood-type inequalities.

参考文献

[1] J. E. Littlewood, Some new inequalities and unsolved problems, In: Shisha, O. (Ed.) Inequalities, Academic Press, New York, (1967) 151–162.

[2] G. Bennett, An inequality suggested by Littlewood, Proc. Amer. Math. Soc. 100 (1987) 474–476.

[3] G. Bennett, Some elementary inequalities, Quart. J. Math. 2 (1987) 401–425.

[4] H. Alzer, On a problem of Littlewood, J. Math. Anal. Appl. 199 (1996) 403–408.

[5] L. Z. Cheng, M. L. Tang, Z. X. Zhou, On a problem of Littlewood, Math Practice Theory 28 (1998) 314–319.

[6] R. Agarwal, M. Bohner, S. Saker, Dynamic Littlewood-type inequalities, Proc. Amer. Math. Soc. 143 (2015) 667–677.

[7] S. Saker, D. O'Regan, R. Agarwal, Littlewood and Bennett Inequalities on time scales, Mediterr. J. Math. 12 (3) (2015) 605–619.

[8] E. T. Copson, Note on series of positive terms, J. London Math. Soc. 3 (1928) 49–51.

[9] L. Persson, N. Samko, What should have happened if Hardy had discovered this, J. Inequal. Appl. 29 (2011) 1–11.

[10] B. He, Q. R. Wang, A multiple Hilbert-type discrete inequality with a new kernel and best possible constant factor, J. Math. Anal. Appl. 431 (2015) 889–902.

[11] R. Agarwal, D. O'Regan, S. Saker, Dynamic inequalities on time scales, Springer International Publishing, Switzerland, 2014.

[12] J. C. Kuang, Applied inequalities, Shangdong Science Technic Press, Jinan, 2004.

[13] G. H. Hardy, J. E. Littlewood, G. Pólya, Inequalities, Cambridge University Press, Cambridge, 1952.

[14] P. Gao, On an inequality suggested by Littlewood, J. Inequal. Appl. 5 (2011) 1–10.

[15] S. Hilger, EinMaßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, PhD. thesis, Universität Würzburg, 1988.

[16] S. Hilger, Analysis on measure chains – a unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18–56.