Probability of an Acute Triangle in the Two-dimensional Spaces of Constant Curvature

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Abstract

The interest in the statistical theory of shape has arisen since Kendall found that the metric geometry of spaces is precisely the required tool for the systematic comparison and classification of various shapes in 1980s. The statistical theory of shape is widely used in the fields such as quantum physics, biology, and medical science.

This paper concerns the probability of acute triangles on the spaces of constant curvature. We prove the following results:

1. On the unit sphere S^2, the probability for a triangle formed by choosing three points at random to be an acute one is $\frac{1}{16}$.

2. On Poincaré disc D, the probability for a triangle formed by choosing three points at random to be an acute one is $\frac{5}{8}$.

The paper involves many disciplines, including probability, geometry. The highlight is the idea of reducing the given probability problem to a question of solid geometry.

Key words: Random triangle, Geometric probability, Non-Euclidean geometry, Spaces of constant curvature, Rigidity theorem.

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§1 Introduction

The interest in the statistical theory of shape has arisen since Kendall found that the metric geometry of spaces is precisely the required tool for the systematic comparison and classification of various shapes in 1980s. The statistical theory of shape is widely used in the fields such as quantum physics, biology and medical science. ([1] ,[2] ,[3] ,[4] ,[6]).

The statistical theory of shape can be traced back from 1893 when Charles Dodgson (Lewis Carroll) proposed the following question.

Question: Find the probability that a triangle formed by choosing three points at random on an infinite plane would have an obtuse triangle.

In [5], by introducing the Cartesian coordinates of the three points, S. Portnoy argued that the set of triangles can be identified with the six-dimensional Euclidean space R^6. And the set T_O of obtuse triangles is a double cone. Also, he claimed that the requirement of taking three points "at random in the plane" can be understood as the induced probability distribution in R^6 being spherically symmetric. Hence, the conditional distribution given the distance from the origin is uniform on the appropriate sphere. S. Portnoy proved the probability of forming obtuse triangle is $\frac{3}{4}$.

This paper concerns the probability of an acute triangles in the two dimensional spaces of constant curvature. By choosing the values of interior angles as coordinates, the set of triangles can be identified with a region S in the three-dimensional Euclidean space R^3. Also, the requirement of taking three points at random in the two-dimensional spaces of constant curvature is understood as the point in the set S uniformly distributed. In this way, we can compute the probability of an acute triangle in the two- dimensional spaces of constant curvature. In particular, in the Euclidean case, we obtain the same result as S. Portnoy.

§2 Preliminary and Main results

It is well known that Euclidean geometry is based on five postulates. The Euclid's fifth postulate, called "the parallel postulate", can be expressed as follow:

The parallel postulate: There is at least one line L and at least one point P not on L, such that one line can be drawn through P coplanar with but not meeting L.

During a long period of time, people attempted to prove that the parallel postulate could be deduced from the other four postulates and found that the parallel postulate is equivalent to the fact that the sum of interior angles of a triangle equals π (represented by radian). In nineteenth century, Gauss, Bolyai, Lobachevsky found that the parallel postulate was independent of the other four postulates. By replacing the fifth postulate with one of the following two postulates while keeping the other four postulates unchanged, the spherical geometry and hyperbolic geometry may be established respectively.

The parallel postulate in spherical geometry: Given a line L and a point P not on it, there is no line can be drawn through the point P which is parallel to the given line L (that is, all lines through the point P intersect with the given line L).

The parallel postulate in hyperbolic geometry: There is at least one line L and at least one point P, not on L, such that two lines can be drawn through P coplanar with but not meeting L.

The unit sphere in three-dimensional Euclidean space

$$S^2 = \{(x, y, z) \in R^3 \,|\, x^2 + y^2 + z^2 = 1\}$$

can be regarded as a model of spherical geometry, where lines are defined as great circles (As the shortest distance between two points in a sphere is the inferior great circular arc which is analogue to the fact in Euclidean geometry that the shortest distance between two points is a line segment.)

Definition 1: Let *A*, *B*, *C* be three points in the sphere which are not on the same great circle. The side *AB* of the spherical triangle ΔABC is defined to be the inferior great circular arc joining *A* and *B*. The angle $\angle\ A$ with vertice *A* is defined to be the angle formed by the tangent vectors *AX* and *AY* of the sides *AB* and *AC* respectively. See

Figure 1.

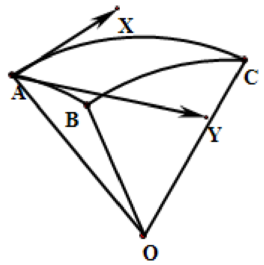

Figure 1

Definition 2: The spherical triangle ΔABC is called an acute triangle if $\angle A$、$\angle B$、$\angle C$ are all acute angles.

From the definition of dihedral angle, it is easy to see that $\angle XAY$ is the plane angle <*B*-*OA*-*C*> of the dihedral angle *B*-*OA*-*C*,

i.e. $\angle A$=<*B*-*OA*-*C*>.

Similarly, $\angle B$=<*C*-*OB*-*A*>, $\angle C$=<*A*-*OC*-*B*> .

Let ΔABC be a spherical triangle in S^2 and let $\angle A=\alpha, \angle B=\beta, \angle C=\gamma$ $(0<\alpha,\beta,\gamma<\pi)$.

Set $\alpha'=\pi-\alpha, \beta'=\pi-\beta, \gamma'=\pi-\gamma$.

i.e. α', β', γ' are the exterior angle of ΔABC. Then by the special case of Gauss-Bonnet Theorem (Theorem 2.10 in [7],) we have

$$\alpha'+\beta'+\gamma'+S_{\Delta ABC}=2\pi \Leftrightarrow \alpha'+\beta'+\gamma'=2\pi-S_{\Delta ABC}<2\pi \tag{1}$$

where $S_{\Delta ABC}$ is the area of the triangle ΔABC.

Gauss-Bonnet theorem suggests that the spherical triangle is, up to an isometry of the sphere, uniquely determined by its angles. Actually we have the following rigidity theorem:

Rigidity theorem: if α, β, γ satisfy the inequalities

$$\alpha+\beta+\gamma>\pi(\Leftrightarrow \alpha'+\beta'+\gamma'<2\pi) \tag{2}$$

$$-\alpha+\beta+\gamma<\pi(\Leftrightarrow \beta'+\gamma'>\alpha') \tag{3}$$

$$\alpha-\beta+\gamma<\pi(\Leftrightarrow \alpha'+\gamma'>\beta') \tag{4}$$

$$\alpha+\beta-\gamma<\pi(\Leftrightarrow \alpha'+\beta'>\gamma') \tag{5}$$

Then up to an isometry of the sphere, there exists a unique triangle ΔABC with α, β, γ as its interior angles.

The above Theorem is stated on page 62, [7], and a proof is indicated on page 66.

For hyperbolic geometry, the unit disc in complex-plane C

$$D = \{z \in C \| z| < 1\} = \{(x, y) \in R^2 | x^2 + y^2 < 1\}$$

can be regarded as its model, where "lines" are the diameters of D and arcs of circles in D that are orthogonal to the unit circle $\partial D = \{z \in D \| z| = 1\}$. The model is called Poincaré disc.

Definition 3: Given three points A,B,C in the Poincaré disc which are not on the same line, the side AB of the hyperbolic triangle ΔABC is defined to be the arc of circle in D joining A and B and orthogonal to the unit circle $\partial D = \{z \in D \| z| = 1\}$. The angle $\angle A$ with vertice A is defined to be the angle formed by the tangent vector AX and AY of the sides AB and AC respectively. See Figure 2.

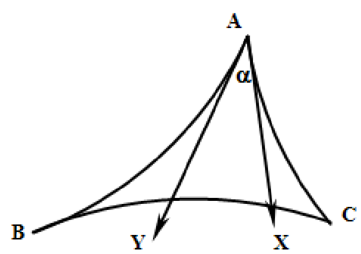

Figure 2

Definition 4: The hyperbolic triangle ΔABC is called an acute triangle if $\angle A$、$\angle B$、$\angle C$ are all acute angles.

From the special case of Gauss-Bonnet theorem (Theorem 2.10, [7]), we know that,

$$(\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) - S_{\Delta ABC} = 2\pi$$

Hence

$$\alpha + \beta + \gamma = \pi - S_{\Delta ABC} < \pi \tag{6}$$

Gauss-Bonnet theorem suggests that the hyperbolic triangle is, up to an isometry of the Poincaré disc, uniquely determined by its angles. Actually we have the following rigidity theorem:

Rigidity theorem: If α, β, γ satisfy the inequalities,

$$\begin{cases}\alpha+\beta+\gamma<\pi \\ 0<\alpha,\beta,\gamma<\pi\end{cases} \tag{7}$$

Then up to an isometry of the Poincare disc, there exists a unique triangle ΔABC with α, β, γ as its interior angles.

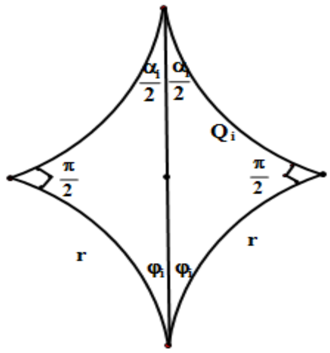

Figure 3

Note: The above Theorem is a special case of Theorem 2.8 in [7], where the existence is proved and the proof of uniqueness is indicated on page 66. For reader's convenience, we give a detailed proof here.

Proof : For existence, it is sufficient to look for the desired triangle in the class of triangles admitting an inscribed circle. For positive number $r>0,\alpha_i>0,i=1,2,3$, consider the quadrilateral $Q_i, i=1,2,3$ as in figure 3:

We need only to prove that there exists $r>0$ such that

$$\sum_{i=1}^{3}\varphi_i=\pi \tag{8}$$

In fact, if such an *r>0* can be found, then the problem is solved: one can simply lay the quadrilaterals Q_1,Q_2,Q_3 one beside the other, successively joining them along the sides equal to *r*, then the resulting triangle is the desired one.

Note that when *r* is small enough, Q_i, i=1, 2, 3 can be approximately regarded as the figure in the Euclidean plane. Therefore as $r\to 0+$, $\left|\varphi_i-(\frac{\pi}{2}-\frac{\alpha_i}{2})\right|\to 0$, which implies that $\sum_{i=1}^{3}\varphi_i\to\sum_{i=1}^{3}(\frac{\pi}{2}-\frac{\alpha_i}{2})=\frac{1}{2}(3\pi-\sum_{i=1}^{3}\alpha_i)>\pi$.

On the other hand, as $r\to\infty$, $\sum_{i=1}^{3}\varphi_i \to 0$.

Since $\sum_{i=1}^{3}\varphi_i$ is a continuous function of r, according to the intermediate value theorem of continuous function, there exists r satisfying $\sum_{i=1}^{3}\varphi_i = \pi$.

This finishes the proof of existence.

The uniqueness in rigidity theorem can be proved as follow:

By the dual cosine theorem of hyperbolic geometry,

$$\cos\alpha = -\cos\beta\cdot\cos\gamma + \sin\beta\cdot\sin\gamma\cdot\cosh a,$$

We have

$$\cosh a = \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma}.$$

Here a is the length of side BC in ΔABC, also $\cosh a = \frac{e^a + e^{-a}}{2}$.

Thus $\cosh a$ is determined by α, β, γ.

Now consider the function $f(x) = \frac{e^x + e^{-x}}{2}$, we have $f'(x) = \frac{e^x - e^{-x}}{2}$. When $x > 0$, $f'(x) > 0$, hence $f(x)$ is monotonic increasing in $(0, +\infty)$. As a consequence, a is uniquely determined by $\cosh a$.

Similarly, b, c is uniquely determined. Therefore sides a, b, c of hyperbolic triangle can be uniquely determined by α, β, γ.

Next we prove the triangle is uniquely determined up to isometry. By Cosine Law in hyperbolic geometry:

$$\cosh a = \cosh b\cdot\cosh c - \sinh b\cdot\sinh c\cdot\cos\alpha \tag{9}$$

We have, for fixed b, $c>0$, $\cosh a$ is monotonically increasing. Hence for fixed a, b, $c>0$, there exists a unique angle α satisfying (9). As a consequence, ΔABC is uniquely determined up to isometry.

The uniqueness is proved.

From the viewpoint of differential geometry , Euclidean plane E^2 , unit sphere S^2 in three-dimensional Euclidean space and the Poincaré disc (endowed with suitable metric) have Gaussian curvature 0, +1， -1 respectively. So they are generally called two-dimensional spaces of constant curvature.

The following special case of Toponogov's triangle comparison theorem in Riemannian geometry is intuitively clear.

Toponogov's triangle comparison theorem (Special Case)

Given a，b，$c>0$, Let T_1, T_2, T_3 be the triangles with side lengths a，b，c in S^2, E^2 and D respectively, and let A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3. be the corresponding angles, then we have

$$A_1>A_2>A_3$$

$$B_1>B_2>B_3$$

$$C_1>C_2>C_3$$

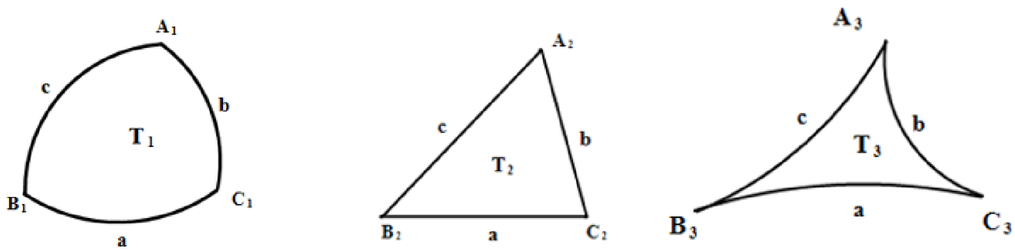

Figure 4

By the above theorem, it is reasonable to claim the following.

The probability of acute triangles in the Euclidean plane E^2 is greater than that of acute triangles in unit Sphere S^2, while less than that of acute triangles in Poincaré disc.

We verify the above claim and calculate the corresponding probability of an acute triangle. More precisely, we have the following:

Main Results

1. On the unit sphere S^2, the probability for a triangle formed by choosing three points at random to be an acute one is $\frac{1}{16}$.

2. On Poincaré disc D, the probability for a triangle formed by choosing three

points at random to be an acute one is $\frac{5}{8}$.

§3 Proof of the Main Results

3.1 Consider first the Euclidean plane E_0^2, in this case the Gaussian curvature K=0.

Assuming ΔABC is a random triangle in E_0^2, and $\angle A=\alpha$, $\angle B=\beta$, $\angle C=\gamma$ ($0<\alpha$, β, $\gamma<\pi$), then

$$\begin{cases}\alpha+\beta+\gamma=\pi \\ 0<\alpha,\beta,\gamma<\pi\end{cases} \tag{10}$$

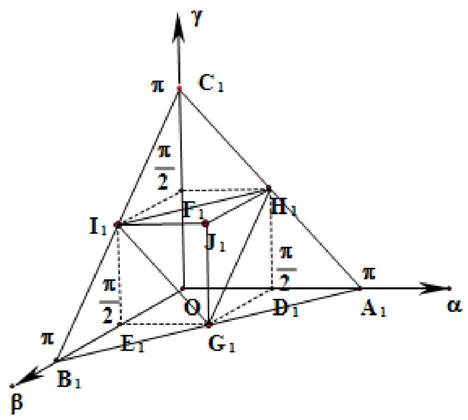

Figure 5

Taking α, β, γ as Cartesian coordinates, as shown in figure 5. Since ΔABC is randomly chosen on E_0^2, we may assume the points with coordinates (α, β, γ) is uniformly distributed in the region determined by (10), which corresponds to the set of Euclidean triangles.

The necessary and sufficient condition for ΔABC to be an acute triangle is:

$$\begin{cases}\alpha+\beta+\gamma=\pi \\ 0<\alpha,\beta,\gamma<\dfrac{\pi}{2}\end{cases} \tag{11}$$

(11) corresponds to $\Delta G_1H_1I_1$ in figure 5, which is obtained by cutting the cube $OD_1G_1E_1$-$F_1H_1J_1I_1$ by the triangle $\Delta A_1B_1C_1$. Since α, β, γ obeys uniform distribution in $\Delta A_1B_1C_1$, therefore,

$$\text{P}(\Delta ABC \text{ is an acute triangle}) = \frac{S_{\Delta G_1H_1I_1}}{S_{\Delta A_1B_1C_1}} = \frac{1}{4}.$$

3.2 Next, we consider the unit Sphere S^2_1; in this case the Gaussian curvature K=1.

As in Section 2, let α',β',γ' be the exterior angles of the triangle $\triangle ABC$.

Since $0<\alpha, \beta, \gamma<\pi$, we have $0<\alpha', \beta', \gamma'<\pi$.

Taking α', β', γ' as Cartesian coordinates, as in figure 6:

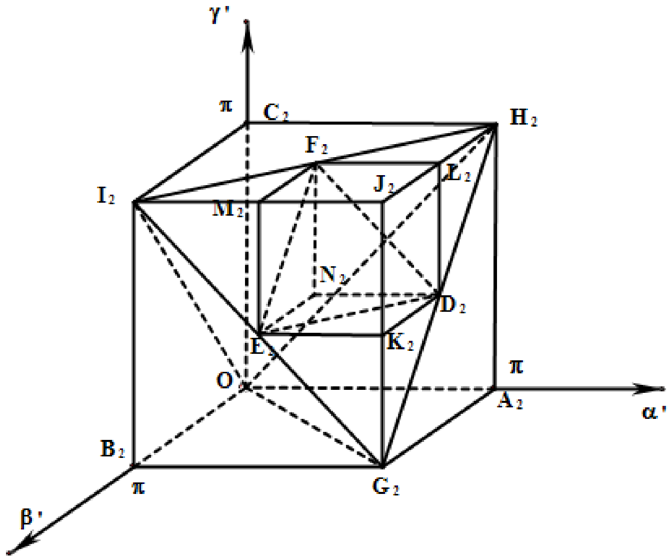

Figure 6

where D_2, E_2, F_2 are midpoints of G_2H_2, G_2I_2, H_2I_2 respectively.

Since ΔABC is randomly chosen on S^2_1, we may assume the points with coordinates (α, β, γ) is uniformly distributed in the region determined by (2)-(5), which corresponds to the set of spherical triangles.

The region determined by (2)~(5) is the interior of the tetrahedron $O\text{-}I_2G_2H_2$, which can be obtained from the cube $OA_2G_2B_2\text{-}C_2H_2J_2I_2$ by eliminating the tetrahedra $C_2\text{-}OI_2H_2$, $A_2\text{-}OG_2H_2$, $B_2\text{-}OG_2I_2$, $J_2\text{-}I_2G_2H_2$, as shown in Figure 6.

Hence the volume of the tetrahedron $O\text{-}I_2G_2H_2$ can be computed as

$$
\begin{aligned}
&V_{O-I_2G_2H_2} = V_{OA_2G_2B_2-C_2H_2J_2I_2} - (V_{A_2-OG_2H_2} + V_{B_2-OG_2I_2} + V_{C_2-OI_2H_2} + V_{J_2-I_2G_2H_2}) \\
&= \pi^3 - 4\times\frac{1}{3}\times\frac{\pi^2}{2}\times\pi \\
&= \frac{\pi^3}{3}
\end{aligned}
$$

Note that the triangle ΔABC is an acute one if and only if $\frac{\pi}{2}<\alpha',\beta',\gamma'<\pi$. The region determined by (2)-(5) and the condition $\frac{\pi}{2}<\alpha',\beta',\gamma'<\pi$ is the intersection of the cube $N_2D_2K_2E_2$-$F_2L_2J_2M_2$ and the tetrahedron O-$I_2G_2H_2$, which is also a tetrahedron N_2-$D_2E_2F_2$. The volume of the tetrahedron N_2-$D_2E_2F_2$ is

$$V_{N_2-D_2E_2F_2}=\frac{1}{3}\times\frac{(\frac{\pi}{2})^2}{2}\times\frac{\pi}{2}=\frac{\pi^3}{48}$$

and

$$\text{P}(\Delta ABC \text{ is an acute triangle})=\frac{V_{N_2-D_2E_2F_2}}{V_{O-I_2G_2H_2}}=\frac{1}{16}.$$

3.3 Finally, we consider the Poincaré disc D, in this case the Gaussian curvature K= -1

Let ΔABC be a random triangle on Π^2, and let $\angle A=\alpha$, $\angle B=\beta$, $\angle C=\gamma$ $(0<\alpha, \beta, \gamma<\pi)$. Taking α,β,γ as Cartesian coordinates, as shown in figure 7:

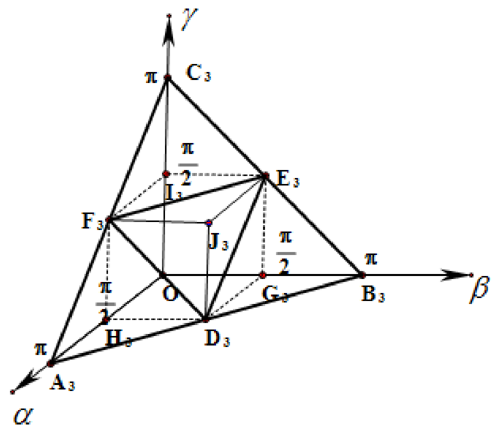

Figure 7

In figure 7, the region determined by (7) is the interior of tetrahedron O-$A_3B_3C_3$.

Note that the triangle ΔABC is an acute one if and only if $0<\alpha,\beta,\gamma<\frac{\pi}{2}$.

The region determined by (7) and the condition $0<\alpha,\beta,\gamma<\frac{\pi}{2}$ is the intersection T of the cube $OG_3D_3H_3$-$I_3E_3J_3F_3$ and the tetrahedron O-$A_3B_3C_3$. Note that T can be obtained from the cube $OG_3D_3H_3$-$I_3E_3J_3F_3$ by eliminating the tetrahedron J_3-$E_3D_3F_3$. The volumes of the tetrahedron O-$A_3B_3C_3$ and T can be computed as:

$$V_{O-A_3B_3C_3}=\frac{1}{3}\times\frac{\pi^2}{2}\times\pi=\frac{\pi^3}{6},$$

$$V_T=\left(\frac{\pi}{2}\right)^3-\frac{1}{3}\times\frac{\left(\frac{\pi}{2}\right)^2}{2}\times\frac{\pi}{2}=\frac{5\pi^3}{48}$$

and

$$P(\triangle ABC \text{ is an acute triangle})=\frac{V_T}{V_{O-A_3B_3C_3}}=\frac{5}{8}.$$

Thus the theorem is proved.

§4 Research Prospective

In this paper, we only investigate the probability of an acute triangle in the two-dimensional spaces of constant curvature. It is natural to continue our research on surfaces of variable curvature. For instance, we may study the probability of an acute triangle on the paraboloid of revolution $z=x^2+y^2$.

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