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Title： The Magic Points in the Triangle

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Catalog

1.Abstract

I heard the concept of the ”equal-sum point” in a math summer camps and found it very interesting. After searching on the Internet,I found that the ”equal-sum point” was associated with Soddy point. Soddy point is a magic point in plane geometry, which was found by British physicist, chemist Frederick.Soddy. Some research about the Soddy point has been done in foreign countries, but the properties are not comprehensive. In China, Teacher HuasongHuang raised the concept of the “equal-sum point” and the “equal-difference point” and drew some properties, I found that these two points were very similar to the Soddy point, but they did not link the two points with Soddy point. In this article ,I will connect the Soddy point with the “equal-sum point” and the “equal-difference point” and study their properties more deeply and find some new discoveries.

Key words: Soddy point , Soddy circle ,equal-sum point, equal-difference point, properties

2 Soddy Point

2.1Definition

Given a triangle △ABC and there exit three circles ⊙A（s-a）,⊙B（s-b）,⊙C （s-c）

（$s=\frac{1}{2}(a+b+c)$）which are mutually tangent, there are in general two other circles

with touch these three.
Reciprocally, we may wonder if, given any triangle ABC, there are three circles centered in A, B, C and mutually tangent. The answer is "yes".
The 4th circle is defined as the **Soddy circle** in the triangle ABC, and its center is the Soddy point。(image1)

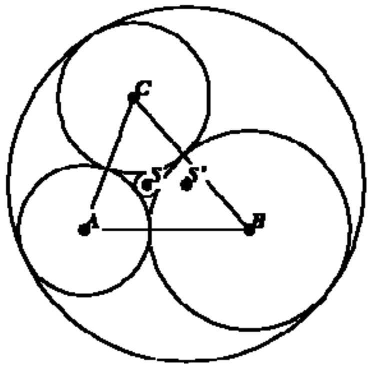

image1

2.2Property

Property1 Suppose the radius of the Soddy circle is r_4, ⊙A is r_1, ⊙B is r_2, ⊙C is r_3, so

$$r_4^{\pm}=\frac{r_1r_2r_3}{r_2r_3+r_1(r_2+r_3)\pm 2\sqrt{r_1r_2r_3(r_1+r_2+r_3)}}.$$

Property2 Suppose P is the outer Soddy point, so the three Ceva lines through P will divide the triangle into three circumscribed quadrilateral of a circle. (image2)Name the three circles ⊙I_1,⊙I_2,⊙I_3.I is the inner center of △ABC. Conduct three vertical lines from I to each side. So the three vertical lines are internal common tangent of⊙I_1, ⊙I_2,⊙I_3.(image 3)

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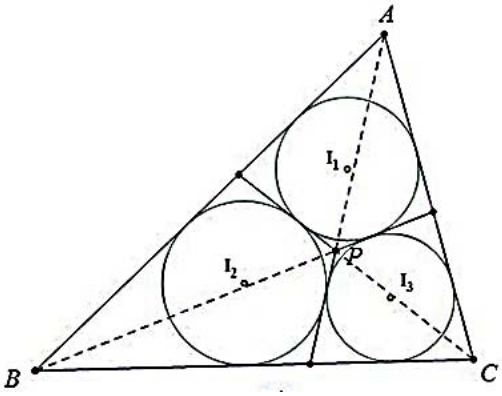

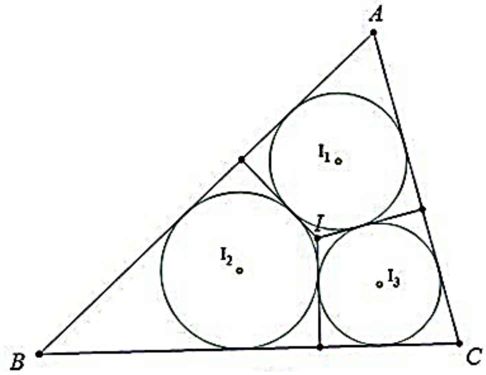

image2 image3

Property3 Suppose S' is the outer Soddy point of △ABC, so line S'A、S'B、BC、CA have a tangent circle.（image4）

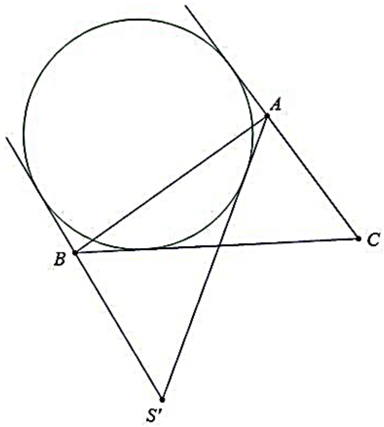

image4

Property4 The pedal circle of the Soddy point is tangent to the inscribed circle of △ABC.（image5）

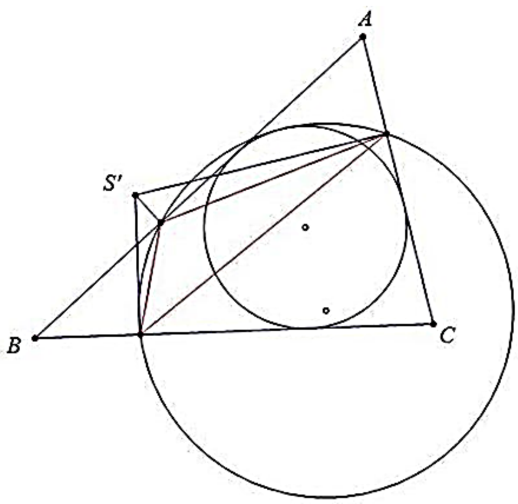

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Image5

Property5 The four Soddy lines and Euler lines have a intersection point.(image6)

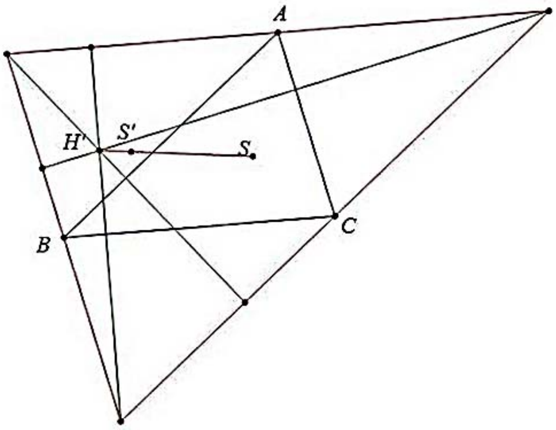

Image6

Property6 The radical axis of each pair of Soddy circles is the Gergonne line ,（Correspondingly）Soddy line and Gergonne line are vertical.(image7)

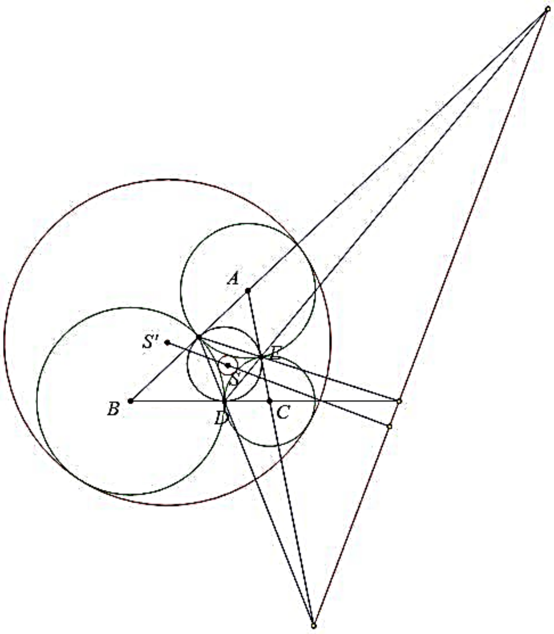

Image7

Property7 The inner Soddy point and the outer Soddy point are the two intersection points of the hyperbola with foci A and B, the hyperbola with foci B and C and the hyperbola with foci A and C（image8）

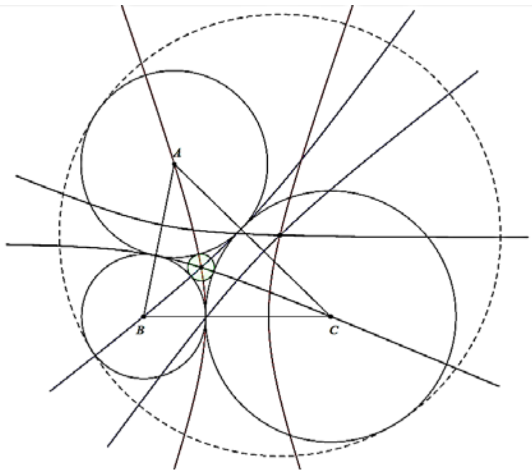

Image8

3 “equal-difference point”

3.1 Definition

In the plane of $\triangle ABC$, the point P which meet the condition that |PA-a|=|PB-b|=|PC-c|(a ,b ,c are the subtenses of∠A ,∠B,∠C)is called the “equal-difference point” of△ABC. Obviously, it includes two situations:（1）PA-a=PB-b=PC-c;（2）PA-a=b-PB =PC-c or PA-a=PB-b=PC or a-PA=PB-b=PC-c.

3.2Property

Situation（1）

Obviously, P in (1) is the outer Soddy point for $\triangle ABC$ .This P doesn’t always exit.
This only if the outer Soddy circle (P) surrounds the circles (A) (B) (C).
If the smaller of circles (A) (B) (C) is so small that (A) (B) (C) touch (P) externally, there is no outer Soddy point. The critical value is when the outer Soddy circle is a straight line : $UV^2 = AB^2 - (BU-AV)^2 = (r_A + r_B)^2 - (r_A - r_B)^2$,hence $UV=2\sqrt{RA*RB}$,$VW=2\sqrt{RA*RC}$, then UW = UV + VW gives a condition for existence of the outer Soddy point : $\frac{1}{\sqrt{RA}}<\frac{1}{\sqrt{RB}}+\frac{1}{\sqrt{RC}}$ or also a+b+c<4R+r (with r the inradius and R the circumradius). △ABC is divided into three isoperimetric triangles by the connection of P with three vertexes of △ABC.(image9)

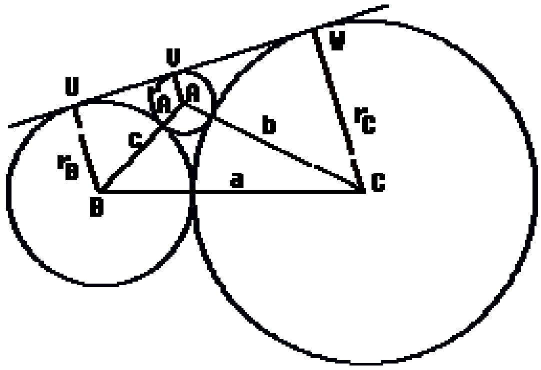

Image9

Situation（2）

In the situation (2), assume without loss of generality that PA-a=b-PB=PC-c. Suppose the escribed circle outside b tangent BC at E , tangent AC at F, tangent AB at D .Construct ⊙A（AD）,⊙B（BD）, ⊙C（CE）. ⊙A,⊙B,⊙C are mutually tangent .Obviously,⊙A，⊙B，⊙C have two tangent circles，name them ⊙P_1,⊙P_2.It is easy to prove that :$P_1A-a=r_1+r_A+r_C-r_B=b-P_1B=P_1C-c$,so P_1meet situation（2）,.Similarly,P_2 meet situation (2).So there are two points which meet the condition that PA-a=b-PB=PC-c.$P_1$is in the area surrounded by AC,BC and the extension of BA（name this area

K_b ）,we call it the first “equal-difference point” of △ABC.（image10）

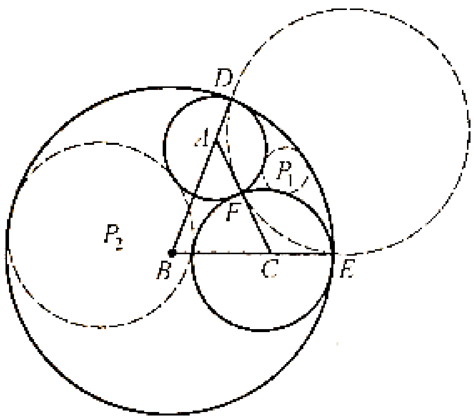

image10

Lemma △ABC only has three circles ⊙O1，⊙O2，⊙O3 which are mutually tangent，and internally tangent with the escribed circle ⊙I outside b of △ABC at the points of tangency of ⊙I on three sides of the triangle.

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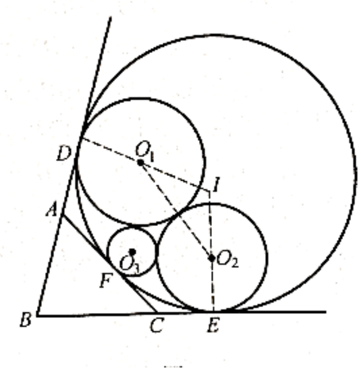

image11

Proof：(image11)Suppose ⊙I(r) is the escribed circle outside b. It is tangent with AB、BC、AC at D、E、F. Suppose ⊙O1(r1),⊙O2(r2),⊙O3(r3) are internally tangent with ⊙I at D、E、F ,and they are mutually tangent, let $\cot\frac{A}{2}=u$，$\cot\frac{B}{2}=v$，$\cot\frac{C}{2}=w$

In △IO1O2 $(r-r_1)^2+(r-r_2)^2+2(r-r_1)(r_1-r_2)\cos B=(r_1+r_2)^2$ (*)

Simplification $r^2-(r_1+r_2)r-r_1r_2v^2=0$ (1)

Similarly $r^2-(r_2+r_3)r-r_2r_3w^2=0$ (2)

$r^2-(r_1+r_3)r-r_1r_3u^2=0$ (3)

With (2),(3)

$r_1=r(r-r_3)(r+r_3u^2)^{-1}$

$r_2=r(r-r_3)(r+r_3w^2)^{-1}$

After substitution we have:

$(1+v^2)(2rr_3-r_2)+r_3^2[(u+w)^2+(uw-1)^2-(1+v^2)]=0$

With cotangent formula:

$(uw-1)=(u+w)\cot\frac{A+C}{2}=(u+w)v$

After substitution we have:

$[(u+w)^2-1]r_3^2+2rr_3-r^2=0$

The solution is $r_3=\frac{r}{1+u+w}$(4)

Similarly $r_1=\frac{r}{1+u-v}$(5)，$r_2=\frac{r}{1-w-v}$(6)

So there are only three circles meet the condition.

In turn, suppose the escribed circle ⊙I(r) outside b of △ABC tangent the three sides of triangle at D、E、F, construct ⊙$O_1(r_1)$、⊙$O_2(r_2)$、⊙$O_3(r_3)$ which tangent ⊙I at D、E、F, let r_1,r_2,r_3 meet the conditions (4),(5),(6).For r_1,r_2,r_3 are the solutions of (1),(2),(3),so r_1,r_2 must meet(*),that is $O_1O_2=r_1+r_2$.So ⊙O_1 and⊙O_2 are externally-tangent. Similarly,⊙$O_3$and⊙O_1,⊙O_2 are both externally-tangent .So there exit three circles meet the condition. Proved.

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Property1 In the area K_b outside b of △ABC,there is only one "equal-difference point" that meets the condition "PA-a=b-PB=PC-c". This point is the intersection point of three common tangents of the circles which are mutually tangent, and internally tangent with the escribed circle ⊙I outside b of △ABC at the points of tangency of ⊙I on three sides of the triangle. (image12)

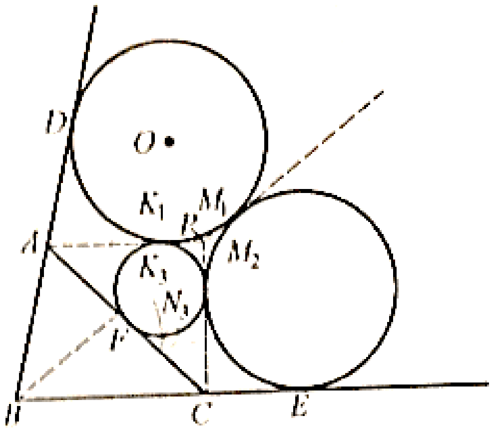

image12

Proof: Suppose the escribed circle $\odot O_1$ outside PA of △ABP, the escribed circle $\odot O_2$ outside PC of △PBC and the inscribed circle $\odot O_3$ of △PAC tangent the three lines of the triangle at D、M_1、K_1，E、M_2、N_2，F、N_3、K_3.PA-a=PA-BE+CE=PA-BM_2+CE=PA-PB-PM_2+CN_2=PA-PB+PC-2PM_2=PA-PB+PC-2PN_2.

Similarly :PC-c=PA-PB+PC-2PM_1=PA-PB+PC-2PK_1

b-PB=AK_3+CN_3-PB=PA-PK_3+PC-PN_3-PB=PA-PB+PC-2PK_3=PA-PB+PC-2PN_3

∵PA-a=b-PB=PC-c

∴PM_1=PM_2,PK_1=PK_3,PN_2=PN_3

So $M_1$and M_2，$K_1$and K_3，N_2 and N3are coincident. $O_1M_1 \perp PB$，$PB \perp O_2M_1$，so O_1、M_1、O_2 are collinear.$O_1O_2=O_1M_1+O_2M_2$.So $\odot O_1$,$\odot O_2$ are externally-tangent. Similarly,$\odot O_3$and$\odot O_1$,$\odot O_2$ are both externally-tangent, PA,PB,PC are internal common tangents. Otherwise, BE=BM_2=BD，CE=CN_2=CF，AD=AK_1=AF

So D、E、F are the three points of tangency of the escribed circle ⊙I(r) outside b of △ABC. Similarly, we can proved that ⊙I and$\odot O_1$,$\odot O_2$,$\odot O_3$ are all internally tangent. Proved.

Property2 If P is the First "equal-difference point" of △ABC ,so A is the First "equal-difference point" of △PBC ,B is the First "equal-difference point" of △PAC, C is the First "equal-difference point" of △PAB.

Property3 B is the outer Soddy point of △AP_2C、△AP_1C.

Property 4 (image13) the escribed circle $\odot O_1$ （r1）outside PA of △ABP, the escribed circle $\odot O_2$（r2） outside PC of △PBC and the inscribed circle $\odot O_3$（r3） of △PAC are mutually tangent, and internally tangent with the escribed circle ⊙I （r）outside b of △ABC at the points of tangency of ⊙I on three sides of the triangle. PA，PB，PC，AB，AC，BC are the common tangents of the circles. If ⊙I is tangent AB,BC,AC at D,E,F, $\frac{1}{r_3}=\frac{1}{r}+\frac{1}{AF}+\frac{1}{CF}$,

$\frac{1}{r_1}=\frac{1}{r}+\frac{1}{AD}-\frac{1}{BD}$, $\frac{1}{r_2}=\frac{1}{r}+\frac{1}{CE}-\frac{1}{BE}$。

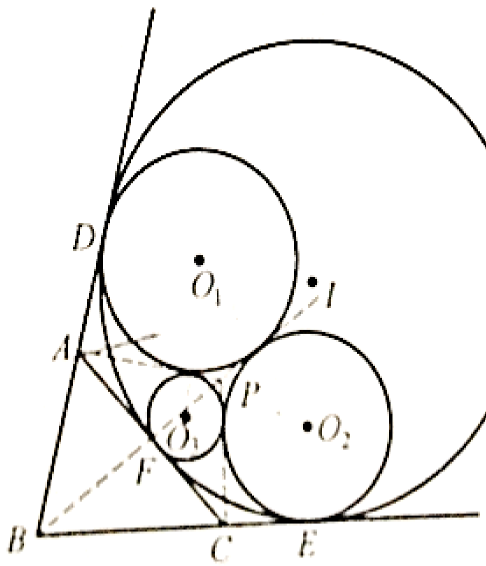

image13

Proof: The first half can be proved easily with property1.With lemma we can get that $r_1=\frac{r}{1+u-v}$,

$r_2=\frac{r}{1+w-v}$, $r_3=\frac{r}{1+u+w}$.Otherwise, $u=\cot\frac{A}{2}=\cot\angle AIF=\frac{r}{AF}$,$v=\tan\frac{B}{2}=\frac{r}{BE}=\frac{r}{BD}$,$w=\cot\frac{C}{2}=\cot\angle CIF=\frac{r}{CF}$,so

we easily prove the second half.

Property5 Given the escribed circle ⊙O1 outside PA of △PAB, the escribed circle ⊙O2 outside PC of △PBC ,the Inscribed circle ⊙O3 of △PAC, so P is the inner center of △$O_1O_2O_3$,This property can be easily proved with Property 4.

Property6 D is on the extension line of BA,E is on the extension line of BC .Name " ∠CAP,∠PAD,∠ABP,∠PBC,∠PCE,∠PCA"" θ_1、 θ_2、 θ_3、 θ_4、 θ_5、 θ_6",so

$\sin(\frac{\theta_1}{2})\sin(\frac{\theta_3}{2})\sin(\frac{\theta_5}{2})=\sin(\frac{\theta_2}{2})\sin(\frac{\theta_4}{2})\sin(\frac{\theta_6}{2})$

$\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_3}{2}\right)\cos\left(\frac{\theta_5}{2}\right)=\cos\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_4}{2}\right)\cos\left(\frac{\theta_6}{2}\right)$

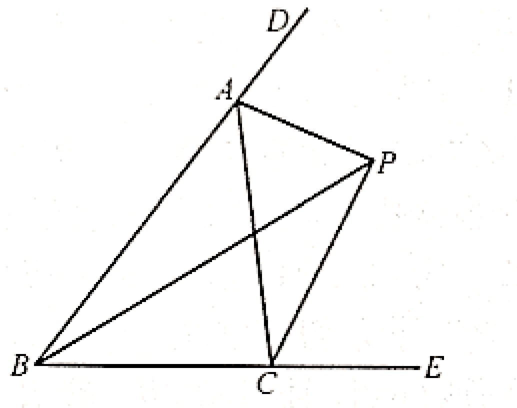

image14

Proof：According to image 14，with sine theorem and we have PAsin θ_1 =PCsin θ_6,PBsin θ_3 =PAsin θ_2,PCsin θ_5 =PBsin θ_4

∴sin $\theta_1$sin $\theta_3$sin θ_2=sin $\theta_2$sin $\theta_4$sin θ_6　（1）

According to property 4,the escribed circle ⊙O_1（r_1）tangent PA of △PAB, the escribed circle ⊙O_2（r2）tangent PC of △PBC and the inscribed circle ⊙O_3（r_3）of △PAC are mutually tangent ,so we can get it easily that

$$\tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_3}{2}\right)\tan\left(\frac{\theta_5}{2}\right)=\tan\left(\frac{\theta_2}{2}\right)\tan\left(\frac{\theta_4}{2}\right)\tan\left(\frac{\theta_6}{2}\right)\quad(2)$$

（1）*（2）$\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_3}{2}\right)\sin\left(\frac{\theta_5}{2}\right)=\sin\left(\frac{\theta_2}{2}\right)\sin\left(\frac{\theta_4}{2}\right)\sin\left(\frac{\theta_6}{2}\right)$

（1）/（2）得 $\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_3}{2}\right)\cos\left(\frac{\theta_5}{2}\right)=\cos\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_4}{2}\right)\cos\left(\frac{\theta_6}{2}\right)$

Property7 Name the symmetry point of P about the midpoint of BC, the midpoint of AB, the midpoint of AC" A_1,C_1,B_1".So BCB_1C_1，ACA_1C_1，$ABA_1B_1$are the parallelograms whose vertexes are on the hyperbola with foci A and A_1, the hyperbola with foci B and B_1,the hyperbola with foci C and C_1.These three hyperbolae have the same center and real axis, so they are externally-tangent with the circle whose diameter is the real axis.

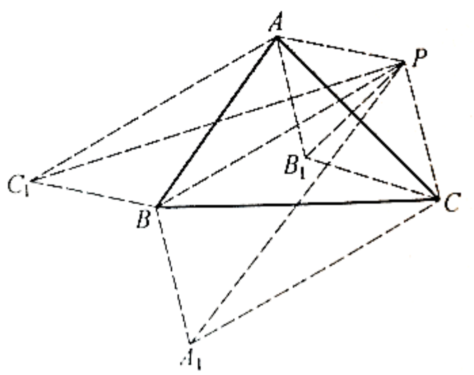

image15

Proof：image15

∵ CB_1、BC_1 and PA are parallel and equal
∴ $BCB_1C_1$is a parallelogram
Similarly ACA_1C_1、ABA_1B_1 are both parallelograms，and the midpoint of AA_1，BB_1，CC_1 are coincident.
∴ $|BA_1-BA|=|B_1A_1-B_1A|=|PC-AB|$
 $|C_1A_1-C_1A|=|CA_1-CA|=|CA-PB|$
∵ $|PC-AB|=|CA-PB|=2a'$
So BCB_1C_1，ACA_1C_1，$ABA_1B_1$are the parallelograms whose vertexes are on the hyperbola with foci A and A_1, the hyperbola with foci B and B_1,the hyperbola with foci C and C_1.
∴ So BCB_1C_1 is the parallelogram whose vertexes are on the hyperbola with foci A and A_1.
Similarly, ACA_1C_1，$ABA_1B_1$are the parallelograms whose vertexes are the hyperbola with foci B and B_1,the hyperbola with foci C and C_1.Obviously, these three hyperbolae have the same center and real axis, so they are externally-tangent with the circle whose diameteris the real axis.

4”equal-sum point”

4.1Definition

In the plane of △ABC ,the point P which meet the conditions that PA+a=PB+b=PC+c is called the ”equal-sum point” of △ABC.

4.2Property

Obviously, the ”equal-sum point” of △ABC is the inner Soddy point.
Lemma △ABC only has three circles ⊙O1，⊙O2，⊙O3 which are mutually tangent，and internally tangent with the inscribed circle⊙I of △ABC at the points of tangency of ⊙I on three sides of the triangle.

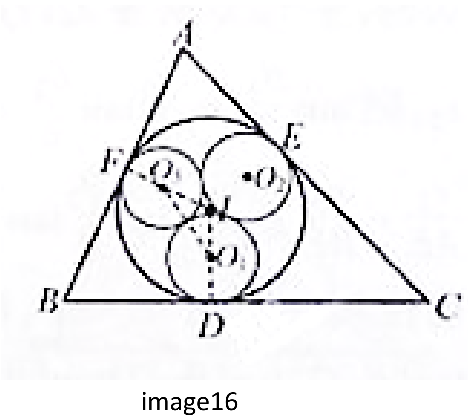

image16

Proof：(image16)The inscribed circle⊙I（r）of △ABC tangent the three sides of the triangle at D、E、F.
Suppose ⊙O1(r1),⊙O2(r2),⊙O3(r3) are internally tangent with ⊙I at D、E、F.Let tan

（A/2）=u，tan（B/2）=v，tan（C/2）=w

$(r-r_1)^2+(r-r_3)^2+2(r-r_1)(r-r_3)\cos B=(r_1+r_3)^2$ (*)

Simplification $r^2-r(r_1+r_3)=r_1r_3u^2$ (1)

Similarly $r^2-r(r_2+r_3)=r_2r_3w^2$ (2)

$r^2-r(r_1+r_2)=r_1r_2v^2$ (3)

With(1)、(3)$r2=\frac{r(r-r_1)}{r+r_1v^2}$,$r3=\frac{r(r-r_1)}{r+r_1w^2}$

After substitution we have:

$r_1^2(v^2+w^2+v^2w^2-u^2)+(2rr_1-r^2)(1+u^2)=0$

That is$r_1^2[(v+w)^2+(vw-1)^2-(1+u^2)]+(2rr_1-r^2)(1+u^2)=0$

The solution is $r_1=\frac{r}{v+w+1}$

Similarly,$r_2=\frac{r}{u+w+1}$,$r_3=\frac{r}{u+v+1}$

So there are only three circles meet the condition.

In turn, suppose the inscribed circle ⊙I(r) tangent the three sides of triangle at D、E、F, construct ⊙$O_1(r_1)$、⊙$O_2(r_2)$、⊙$O_3(r_3)$ which tangent ⊙I at D、E、F, let r_1,r_2,r_3 meet the conditions. For r_1,r_2,r_3 are the solutions of (1),(2),(3),so r_1,r_2 must meet(*),that is $O_1O_2=r_1+r_2$.So ⊙O_1 and⊙O_2 are externally-tangent. Similarly,⊙$O_3$and⊙O_1,⊙O_2 are both externally-tangent .So there exit three circles meet the condition. Proved.

Property1 If P is the "equal-sum point" of △ABC, so this point is the intersection point of three common tangents of the circles which are mutually tangent，and internally tangent with the inscribed circle ⊙I of △ABC.

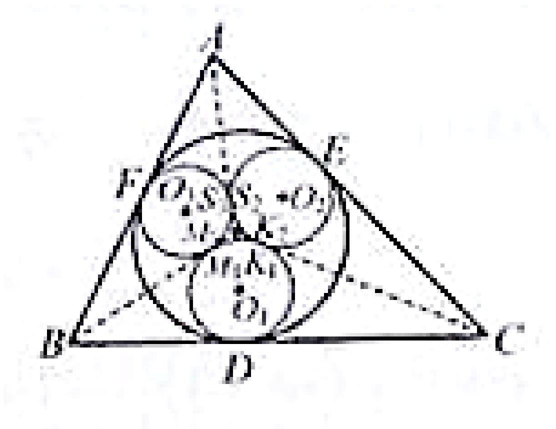

image17

Proof:(image17) Construct the inscribed circles of △PBC、△PCA、△PAB: ⊙O1（r1）、⊙O2（r2）、⊙O3（r3）、⊙I（r）.⊙O_1 tangent PB、BC、PC at M_1、D、K_1,⊙O_2 tangent PC、CA、PA at K_2、E、S_2,⊙O_3 tangent PA、AB、PB at S_3、F、M_3.Let PA=x,PB=y,PC=z.

$x+a=x+BD+DC=x+BM_1+CK_1=x+y+z-2PM_1=x+y+z-2PK_1$

Similarly $y+b=x+y+z-2PK_2=x+y+z-2PS_2$

z+AB=x+y+z-2PS_3=x+y+z-2PM_3

For P is the "equal-sum point" of △ABC

$PS_2=PS_3$, $PM_1=PM_2$, $PK_1=PK_3$

∴S_2 and S_3, M_1 and M_2,K_1 and K_3 are coincident

∴⊙O_1、⊙O_2、⊙O_3 are mutually tangent. PA、PB、PC are the internal common tangent s. AE=AS_3=AF, BD=BM_1=BF, CD=CK_2=CE

∴D、E、F are the points of tangency of ⊙I on three sides of △ABC.

For ID⊥BC, O_1D⊥BC,I、O_1、D are collinear.

So IO_1=ID-O_1D

Similarly,⊙O_2、⊙O_3 is internally tangent with⊙I at E、F.

Proved.

Property2 If P is the "equal-sum point" of △ABC ,so A is the "equal-sum point" of △PBC ,B is the "equal-sum point" of △PAC, C is the "equal-sum point" of △PAB.

Property3 If P is the "equal-sum point" of △ABC, the symmetry point of P about BC,CA,AB is "*A',B',C'*", so quadrilateral ABA'C,BCB'A,CAC'B have inscribed circles.

Property4 If P is the "equal-sum point" of △ABC ,the inscribed circles of △PBC、△PCA、△PAB、△ABC are ⊙O1（r1）、⊙O2（r2）、⊙O3（r3）、⊙I（r）.⊙O1,⊙O2,⊙O3 are mutually tangent，and internally tangent with the inscribed circle⊙I of △ABC at the points of tangency of ⊙I on three sides of the triangle.

PA、PB、PC、AB、AC、BC are the common tangents of the circles. If ⊙I is tangent BC at D、AB at E、AC at F, so（△ is the area of △ABC ,p is the half perimeter of △ABC）

$$\frac{1}{r1}=\frac{1}{r}+\frac{1}{BD}+\frac{1}{CD}=\frac{\Delta}{p}+\frac{a}{(p-b)(p-c)}$$

$$\frac{1}{r2}=\frac{1}{r}+\frac{1}{CE}+\frac{1}{AE}=\frac{\Delta}{p}+\frac{b}{(p-c)(P-a)}$$

$$\frac{1}{r3}=\frac{1}{r}+\frac{1}{AF}+\frac{1}{BF}=\frac{\Delta}{p}+\frac{c}{(p-a)(p-b)}$$

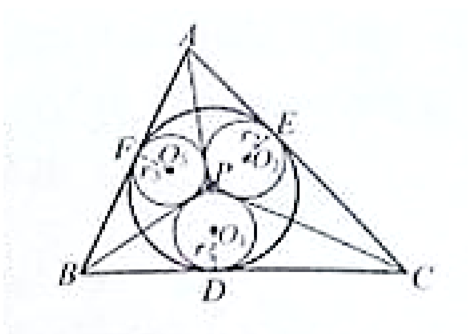

image18

Proof：(image18) The first half can be proved easily with property1.With lemma we can get that $r_1=\frac{r}{1+u-v}$, $r_2=\frac{r}{1+w-v}$, $r_3=\frac{r}{1+u+w}$.,so we can easily prove the second half.

Property5 If P is the "equal-sum point" of △ABC, the inscribed circles of △PBC、△PCA、

△PAB、△ABC are ⊙O1、⊙O2、⊙O3、⊙I, so P is the incenter of $\triangle O_1O_2O_3$,and the points of tangency of ⊙O_1、⊙O_2、⊙O_3"S、M、K" are the points of tangency of the inscribed circle ⊙P of $\triangle O_1O_2O_3$ on three sides.

Property6 Suppose the centers of the circles which are mutually tangent, and internally tangent with the inscribed circle ⊙I of △ABC are O_1,O_2,O_3 ,so the "equal-sum point" P is the outer Soddy point of $\triangle O_1O_2O_3$.

Property7 If P is the "equal-sum point" of △ABC. Name " ∠CAP,∠PAD,∠ABP,∠PBC, ∠PCE,∠PCA"" θ_1、θ_2、θ_3、θ_4、θ_5、θ_6",so

$$\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_3}{2}\right)\sin\left(\frac{\theta_5}{2}\right)=\sin\left(\frac{\theta_2}{2}\right)\sin\left(\frac{\theta_4}{2}\right)\sin\left(\frac{\theta_6}{2}\right)$$

$$\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_3}{2}\right)\cos\left(\frac{\theta_5}{2}\right)=\cos\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_4}{2}\right)\cos\left(\frac{\theta_6}{2}\right)$$

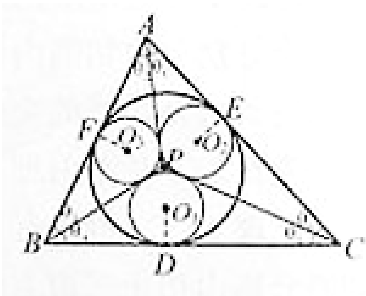

Image19

Proof: (image19) We can easily get it through sine theorem that $\sin\theta_1\sin\theta_3\sin\theta_5=\sin\theta_2\sin\theta_4\sin\theta_6$ (1)

Construct the inscribed circles of △PBC、△PAB、△PAC, the radii are r1、r2、r3.

We can easily get that $\tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_3}{2}\right)\tan\left(\frac{\theta_5}{2}\right)=\frac{r2*r3*r1}{AE*BF*CD}$

$\tan\left(\frac{\theta_2}{2}\right)\tan\left(\frac{\theta_4}{2}\right)\tan\left(\frac{\theta_6}{2}\right)=\frac{r3*r1*r2}{AF*BD*CF}$

∵AE=AF,BE=BD,CD=CF

∴$\tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_3}{2}\right)\tan\left(\frac{\theta_5}{2}\right)=\tan\left(\frac{\theta_2}{2}\right)\tan\left(\frac{\theta_4}{2}\right)\tan\left(\frac{\theta_6}{2}\right)$ (2)

(1) * (2) $\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_3}{2}\right)\sin\left(\frac{\theta_5}{2}\right)=\sin\left(\frac{\theta_2}{2}\right)\sin\left(\frac{\theta_4}{2}\right)\sin\left(\frac{\theta_6}{2}\right)$

(1) / (2) $\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_3}{2}\right)\cos\left(\frac{\theta_5}{2}\right)=\cos\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_4}{2}\right)\cos\left(\frac{\theta_6}{2}\right)$

Property8 Name the symmetry point of P about the midpoint of BC, the midpoint of AB, the midpoint of AC" A_1,C_1,B_1".So BCB_1C_1, ACA_1C_1, $ABA_1B_1$are the parallelograms whose vertexes are on the ellipse with foci A and A_1, the ellipse with foci B and B_1,the ellipse with foci C and C_1.These three ellipses have the same center and major axis, so they are externally-tangent with the circle whose diameter is the major axis.

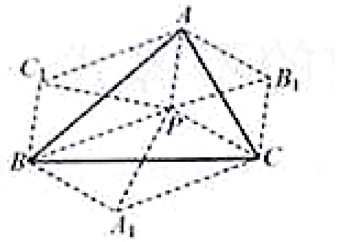

image20

Proof：(image20)∵$BA_1 // PC // AB_1$，$BA_1=PC=AB_1$

∴$ABA_1B_1$is a parallelogram

Similarly ACA_1C_1、BCB_1C_1 are both parallelograms，and the midpoint of AA_1，BB_1，CC_1 are coincident.

∵The distance from B、C_1、B_1、C to A、A_1 are equal

∴So BCB_1C_1 is the parallelogramwhose vertexes are on the ellipse with foci A and A_1,others can be similarly proved.

Proved.

The definition of "equal-sum Line"

Given△ABC ,if P meets the condition that PA+a=PB+b, the locus of P is a line. If AC=BC, the locus of P is a straight line. If AC≠BC, P the locus of P is a curve. We call this line the **"equal-sum Line".**

The property of the "equal-sum line"

Property1 In△ABC, there exit **one and only "equal-sum line"** which meets the condition that PA+a=PB+b .

Property2 In △ABC, the **"equal-sum line"** which meets the condition that PA+a=PB+b is between the angle bisector of ∠C and the vertical line of AB.

Property3 In △ABC ,AC≠BC，the **"equal-sum line"** which meets the condition that PA+a=PB+b is CD，CE is the angle bisector of ∠C，so CE is the tangent line of CD.

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