Computing the phases in the view of the moon

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Introduction

Japan's lunar probe satellite "Kaguya", China's lunar probe satellite "Chang' e-1 lunar probe" left the earth on September 14th, 2007 and October 24th, 2007, began the exploration to the moon. The launching of the China's lunar satellite is the proud to the global Chinese. However, at the same time as we are proud of the event, we must admit that the pictures published which were sent back by Japanese are clearer and more beautiful than Chinese. While we are saying that the Japan has a better technology, maybe we also need to say that the Japanese can hold better chance than Chinese when taking the pictures. The following two pictures were sent back by the satellite. Picture one is sent back by Japan's satellite, Picture two is sent back by China's.

It can be seen that there is a long way for China if we want to become stronger at science technology. We also hope that we can have some contribution in the prosperity of China now and in the future.

Abridgement:

During the lunar exploration of Chinese Chang'E satellite series, the functional relationship between the phase of the earth in the view of the moon (the earth's moon phases of fullness and loss) and the emergence time of the solar eclipses is constructed by the mathematical calculation, which is used to extrapolate the situation of the recent years. Besides, the best time and location on the moon for shooting the phase of the earth and eclipses is calculated.

Text

1. Talk about the sun from the law of gravity

Everything in the universe has mutual interaction. The great Mathematician, Physicist Newton has revealed the mystery of mutual interaction between objects in the universe by the law of universal gravitation. The law of gravity:

$$F = G\frac{m_1 m_2}{r^2} \qquad (1)$$

F ---------- Universal gravitation between objects

G ---------- Gravitational constant

m_1 ---------- Mass of gravitation of object one

m_2 ---------- Mass of gravitation of object two

r ---------- The distance between objects.

So we know, gravitation:

$$\begin{cases} F \propto m_1 \\ F \propto m_2 \\ F \propto \dfrac{1}{r^2} \end{cases}$$

Therefore, the law of universal gravitation can be used to calculate the mutual interaction among the sun, the earth, and the moon. From the table, it can be seen:

Gravitational constant: $G = 6.67\times10^{-11} m^3 / kg \cdot s^2$

Mass of the earth: $M_{地} = 5.974\times10^{24} kg$

Mass of the sun $M_{日} = 1.989\times10^{30} kg$

Mass of the moon $M_{月} = 7.36\times10^{22} kg$

The average distance between the sun and the earth $r_1 = 1.5\times10^{11} m$

The average distance between the earth and the moon $r_2 = 3.8\times10^{8} m$

From the calculation, it can be obtained that the interaction force between the sun and the earth is:

$$F_{SE} = G\frac{M_E M_S}{r_1^2} = 3.522\times10^{22} N \text{ ....(2)}$$

Since the moon is moving around the earth, the distance between the sun and the moon can approximately be seen as:

$r_1 \pm r_2 = (1500 \pm 3.8)\times10^8$ B $1.5\times10^{11} m$ ....

....(3)

Using the same method, we can calculate the mutual interaction force between the sun and the moon:

$$F_{EM} = G\frac{M_M M_S}{r_1^2} = 4.340\times10^{20} N \text{ ....(4)}$$

According to Newton's another great achievement: Newton's Second Law, the acceleration of the sun under this big amount of interaction force can be calculated. Newton's Second Law, as follows:

$F = ma$ ....(5)

m ---------- the mass of the object

a ---------- the acceleration of the object

F ---------- the force acting on the object

As followed, it is easy to calculate the acceleration of the sun which caused by the universal gravitation between the moon and the sun, the earth and the sun:

the acceleration of the sun caused by the earth:

$$a_1 = \frac{F_{SE}}{M_S} = 1.77\times10^{-8} m/s^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

the acceleration of the sun caused by the moon:

$$a_2 = \frac{F_{ME}}{M_S} = 2.18\times10^{-10} m/s^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

In order to let people understand the two acceleration more figuratively, we can take a example. If one object starts from static state, move under these two speeds each for one hour. We can calculate the displacement as follows by using the kinematics equation:

$$x = \frac{1}{2}at^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

x ---------- the displacement of the object
a ---------- the acceleration of the object
t ---------- the moving time of the object

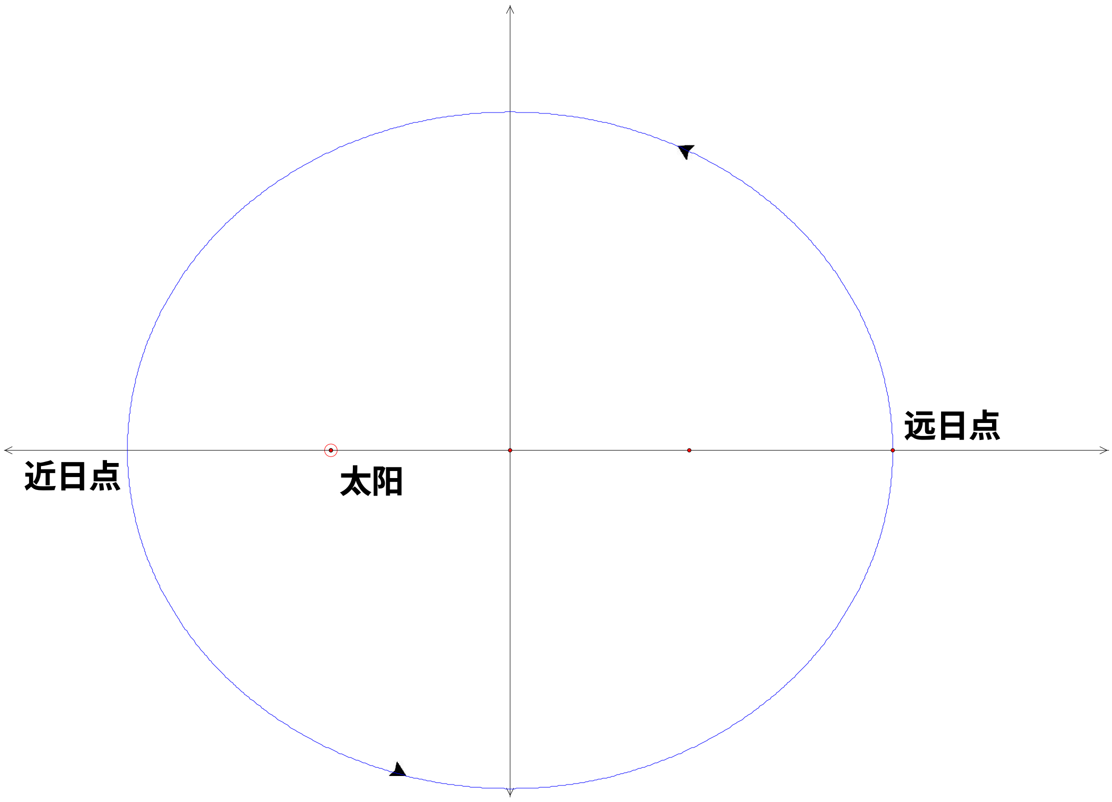
Insert the data (t=1h): $x_1 = \frac{1}{2}a_1t^2 = 1.15\times10^{-1} m$

$$x_2 = \frac{1}{2}a_2t^2 = 1.41\times10^{-3} m \ldots\ldots\ldots(9)$$

We can have a conclusion: since the mass of the sun is much greater than the mass of the moon and the earth, and the earth, the moon and the sun are far away from each other, they make that the earth and the moon have little influence on the moving of the sun. Upon that we can see the sun as static state (inertial frame) and do the calculation.

2. The equation of the orbit of the centroid of moon and earth

We can see the centroid of moon and earth as an object which is moving around the sun, make it as the particle and do the calculation. As the picture shows below (the blue ellipse is the orbit of the centroid of moon and earth, the red spot is the location of the sun)



According to the Kepler's first law, all the planets are moving around the sun in an ellipse, the sun is on the focus of the ellipse: it can be educed that the blue orbit of the centroid of moon and earth is an ellipse, and the red spot—sun is on the focus of the ellipse. On the plane which is formed by the orbit of the centroid of moon and earth and the sun, using the center of the elliptical orbit of the centroid of moon and earth as the original point, the link of the perihelion, the aphelion and the sun as x-axis to build a Cartesian plane coordinates. Therefore, the equation of the elliptical orbit can be written as:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (10)$$

a ---------- the semi-major axis of elliptical orbit of the centroid of the earth and the moon moving around the sun

b ---------- the semi-minor axis of elliptical orbit of the centroid of the earth and the moon moving around the sun

Since the sun is on one of the focuses on the ellipse, the coordinates can be written as (-c, 0), c is the focal length of the ellipse. Since from the elliptical equation $a^2-c^2=b^2$, the coordinates of the sun is $(-\sqrt{a^2-b^2},0)$.

In order to confirm the equation of the elliptical orbit that is the routine of the centroid of the moon and earth moving around the sun. From the table, the longest distance and the shortest distance from the earth to the sun are $1.521\times10^{11}m$ and $1.471\times10^{11}m$. Therefore, the linear equations with two variables:

$$\begin{cases} a-c=1.4710\times10^{11} \\ a+c=1.5210\times10^{11} \end{cases} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (11)$$

So:

$$\begin{cases} a=1.4960\times10^{11} \\ c=0.0250\times10^{11} \end{cases}$$

According to above $a^2-c^2=b^2$, it can be known that

$$b=\sqrt{a^2-c^2}\ \mathrm{B}\, 1.4958\times10^{11} m \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (12)$$

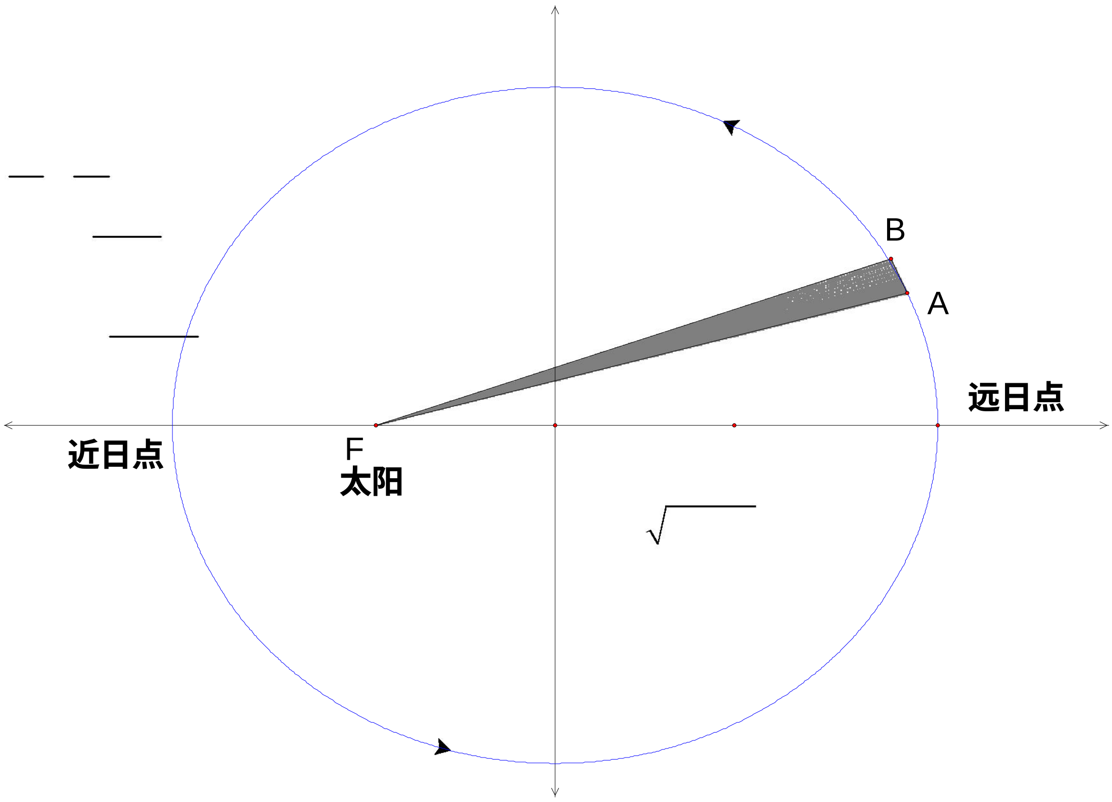
So the elliptical equation is

$$\frac{x^2}{2.2380\times10^{22}}+\frac{y^2}{2.2374\times10^{22}}=1 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (13)$$

The coordinates of the sun is F $(-0.0250\times10^{11}, 0)$

3. The angular moon momentum of the centroid of moon and earth

From the Kepler's second law, to any planet, the link of the planet and the sun (called radial vector) sweep the same area during the same amount of time. As the picture shows below:



During a very short time Δt, the planet move from location A to location B. It can be obtained that the area which is swept by the radial vector of the planet as a triangle because the time is very short. So:

The area of the triangle:

$$\Delta S = \frac{1}{2} \cdot AF \cdot AB \cdot \sin \angle BAF \quad \text{......(14)}$$

Both sides divided by Δt, so

$$\frac{\Delta S}{\Delta t} = \frac{1}{2} \cdot AF \cdot \frac{AB}{\Delta t} \cdot \sin \angle BAF \quad \text{......(15)}$$

When $\Delta t \to 0$, ΔS is the area that is swept by the radial vector, $\frac{AB}{\Delta t}$ is the instantaneous speed when the planet is at location A.

Kepler's second law indicates that the radial vector of the planet sweeps the same area during the unit time, i.e. $\frac{\Delta S}{\Delta t}$ is a constant, all of $\sin \angle BAF \cdot v \cdot AF$ are constants.

Let radial vector $AF = r$, the angle between the speed and the radial vector $\angle BAF = \alpha$, and the mass of the planet. So, the constant angular momentum is $J = m \cdot \frac{\Delta S}{\Delta t} = r \cdot mv \cdot \sin\alpha$ ......(16)

The area which is swept by the radical vector is just the area of the ellipse when the centroid of moon and earth moves around the sun one loop. And from the Kepler's second law, it can be seen that $\frac{\Delta S}{\Delta t}$ is a constant, so the value of $\frac{\Delta S}{\Delta t}$ is the area of the ellipse divided by the cycle time of the centroid of moon and sun moves one lap.

According to the formula of area of ellipse, the area of the ellipse can be obtained: $S = \pi \cdot a \cdot b = 7.0300 \times 10^{22} m^2$.

According to the table, the sidereal year is T=365.25636=3.1558×10^7 s

So,

$$\frac{\Delta S}{\Delta t} = \frac{S}{T} = 2.2276 \times 10^{15} m^2 / s \quad \text{......(17)}$$

Furthermore,

$$J_{EM} = (M_E + M_M) \times \frac{\Delta S}{\Delta t} = 1.3472 \times 10^{40} kg \cdot m^2 / s \quad \text{......(18)}$$

The value of angular momentum will be used later in the article.

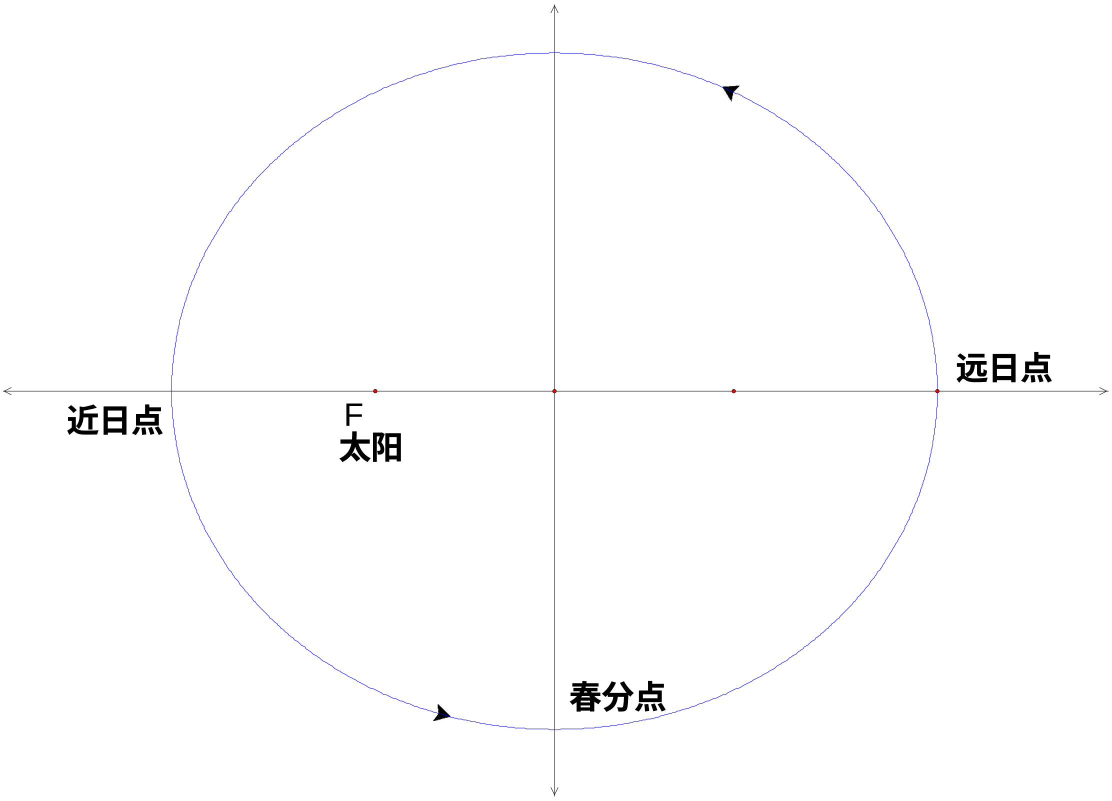
4. The calculation of the initial setting of the centroid of moon and earth

From the astronomical ephemeris, at 12 o'clock on July 1st, 2007, the locations of the sun, the earth and the moon on the J2000 inertial coordinate system are:

the location of the sun $(-2.429 \times 10^{10}, 1.377 \times 10^{11}, 5.972 \times 10^{10})$

the location of the moon $(1.272\times10^{8}, -3.213\times10^{8}, -1.671\times10^{8})$

In the J2000 inertial coordinate, the barycenter of earth is seen as the original point, equatorial planes are seen as the x, y planes, the ecliptic obliquity which is a dihedral angle formed by equatorial plane and the plane where the orbit of the centroid of moon and earth is at is $23°26'21''$. X-axis is pointing to the location of spring equinox in the x, y plane, which means the orthographic projection of x-axis on ecliptic plane in J2000 inertial coordinate system passes through the spring equinox. As the graph shows bellow:



For the future convenient calculation, the J2000 inertial coordinate system is been transferred to a rectangular coordinate system. In this new system, the center of the elliptical orbit is seen as the original point, the link of perihelion and aphelion is seen as the x-axis, the link of spring equinox and center of the ellipse is seen as the y-axis.

Let the J2000 inertial coordinate system be $\mathrm{II}[O'; e'_1, e'_2, e'_3]$, the inertial coordinate system above be $\mathrm{I}[O; e_1, e_2, e_3]$

and let

$$\begin{cases} e_1' = a_{11}e_1 + a_{12}e_2 + a_{13}e_3 \\ e_2' = a_{21}e_1 + a_{22}e_2 + a_{23}e_3 \\ e_3' = a_{31}e_1 + a_{32}e_2 + a_{33}e_3 \end{cases} \qquad \text{...} (19)$$

The coordinate of a random point P in I is (x, y, z), the coordinate of P in II is (x', y', z').

Let the equation of this ellipse in I be $\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ z = 0 \end{cases}$ ....(20)

Since the earth is moving on the elliptical orbit, so point

$O'(a', b', 0) \in \{(x, y, z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 且 z = 0\}$ (the coordinate in I)

It can be obtained that the equations transferred by the system is:

$$\begin{cases} x = a' + a_{11}x' + a_{21}y' + a_{31}z' \\ y = b' + a_{12}x' + a_{22}y' + a_{32}z' \\ z = 0 + a_{13}x' + a_{23}y' + a_{33}z' \end{cases} \text{..(21)}$$

From $O'(a', b', 0)$ is on the ellipse, it can be obtained: $\frac{a'^2}{a^2} + \frac{b'^2}{b^2} = 1$ ------------ 『1』

Since the dihedral angle of the x、y planes in I、II $\theta_0 = 23°26'21''$, which means that θ_0 is the normal vector of two planes,

So $\cos\theta_0 = \frac{e_3' g e_3}{|e'_3||e_3|} = a_{33}$ ---------------- 『2』

Since the positive photograph of x-axis in II which in on the x、 y planes in I passes through $A(0, -b, 0)$, it can be obtained $\frac{0 - a'}{a_{11}} = \frac{-b - b'}{a_{12}}$ ------------------------ 『3』

Since the sun is on the focus of ellipse $F(-c, 0, 0)$ in I, and the coordinate of F in II is (x_0, y_0, z_0), so we can get the equation system:

$-c = a' + a_{11}x_0 + a_{21}y_0 + a_{31}z_0$ ---------- 『4』

$0 = b' + a_{12}x_0 + a_{22}y_0 + a_{32}z_0$ ------------ 『5』

$0 = 0 + a_{13}x_0 + a_{23}y_0 + a_{33}z_0$ -------------- 『6』

And the from the base vectors are perpendicular to each other and the length is 1, so:

$a_{11}^2 + a_{12}^2 + a_{13}^2 = 1$ ------------------- 『7』

$a_{21}^2 + a_{22}^2 + a_{23}^2 = 1$ ------------------- 『8』

$a_{31}^2 + a_{32}^2 + a_{33}^2 = 1$ -------------------- 『9』

$a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0$ --------------------- 『10』

$a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0$ ---------------------- 『11』

$a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} = 0$ ----------------------- 『12』

So, there is a ， as the solution would be very complicated, we will make it more convenient.

Since the earth is the original point in II, the coordinate of the sun is $(-2.429\times10^{10}, 1.377\times10^{11}, 5.972\times10^{10})$, and according to the formula of the distance between two points in rectangular coordinate system, the distance between the sun and the earth is:

$$d_{地日} = \sqrt{(-2.429\times10^{10})^2 + (1.377\times10^{11})^2 + (5.972\times10^{10})^2} = 1.52045\times10^{11} m \text{ ....(22)}$$

In the coordinate system I, since the sun is on the elliptical orbit, let the coordinate of the earth is $(x, y, 0)$, satisfy the equation of the elliptical orbit:

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \\ z = 0 \end{cases} \text{ ....(20)}$$

The coordinate of the sun in I is $F(-c, 0, 0)$, the distance between the earth and the sun is d_{ES}, so it can be obtained

$$d_{ES} = \sqrt{(x+c)^2 + y^2} \text{ ....(23)}$$

Combine the two equations and a equation system is formed:

$$\begin{cases} d_{ES} = \sqrt{(x+c)^2 + y^2} \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \end{cases} \text{ ....(24)}$$

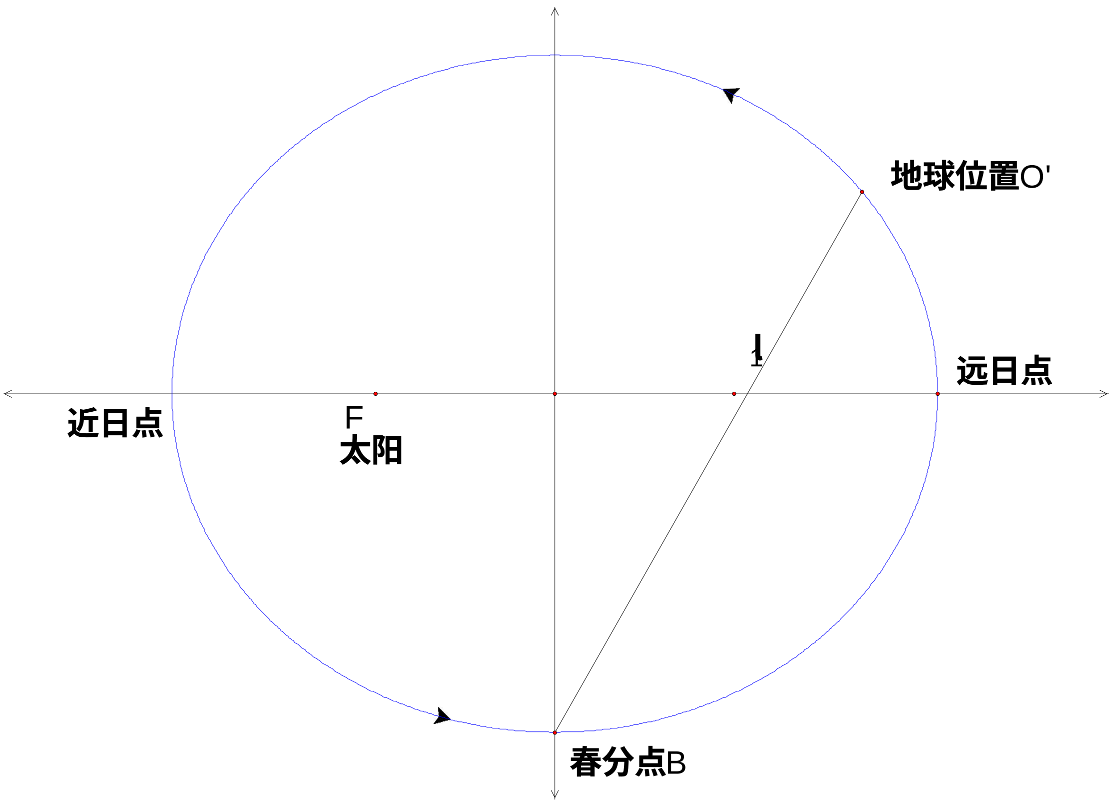
Substitute the data into the system:

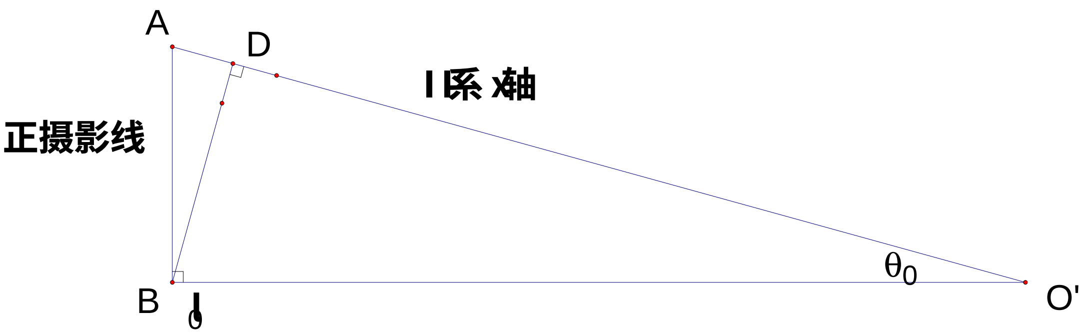
$$\begin{cases} 1.52045\times10^{11} = \sqrt{(x + 0.0250\times10^{11})^2 + y^2} \\ \dfrac{x^2}{(1.4960\times10^{11})^2} + \dfrac{y^2}{(1.4958\times10^{11})^2} = 1 \end{cases} \text{ ....(25)}$$

Solve:

$$\begin{cases} x = 1.0452\times 10^{11} \\ y = 1.0800\times 10^{11} \end{cases} \begin{cases} x = 1.0452\times 10^{11} \\ y = -1.0800\times 10^{11} \end{cases} \begin{cases} x = -1.0702\times 10^{11} \\ y = 1.1042\times 10^{11} \end{cases} \begin{cases} x = -1.0702\times 10^{11} \\ y = -1.1042\times 10^{11} \end{cases}$$

Since it was July at the time, just passed the summer, the correct solution should be $\begin{cases} x = 1.0452\times 10^{11} \\ y = 1.0800\times 10^{11} \end{cases}$. A result can be obtained: at twelve o'clock on July 1st, 2007, the position of the earth in coordinate system I is $\begin{cases} x = 1.0452\times 10^{11} \\ y = 1.0800\times 10^{11} \end{cases}$.





As the graph shows above, according to the formula of the distance between two points, it can be calculated $O'B=\sqrt{x^2+(y+b)^2}=2.7798\times10^{11}m$.

So:

$$AO'=\frac{O'B}{\cos\theta_0}=3.0298\times10^{11}m \quad\text{..........(26)}$$

$$BD=BO'\sin\theta_0=1.1057\times10^{11}m \quad\text{..........(27)}$$

$$O'D=BO'\cos\theta_0=2.5504\times10^{11}m \quad\text{..........(28)}$$

AO' is x-axis, $BD\perp AO'$, so the coordinate of B in system II is $B(2.5504\times10^{11},0,-1.1057\times10^{11})$.

The coordinates of three points F,B,O' in the ecliptic plane is already been known:

$F(-2.429\times10^{10},1.377\times10^{11},5.972\times10^{10})$

$B(2.5504\times10^{11},0,-1.1057\times10^{11})$

$O'(0,0,0)$

So the equation of the ecliptic plane in II is

$$1.1057x-0.9111y+2.5504z=0 \quad\text{..........(29)}$$

The normal vector of the ecliptic plane in II is: $\mathbf{n}=(1.1057,-0.9111,2.5504)$.

Since it has been known that the coordinate of the moon in II is $M(1.272\times10^{8},-3.213\times10^{8},-1.671\times10^{8})$, so the vector

$\mathbf{O'M}=(1.272,-3.213,-1.671)$ and also

$$\mathbf{O'M}\cdot\mathbf{n}=|\mathbf{O'M}|\times|\mathbf{n}|\times\cos<\mathbf{n},\mathbf{O'M}> \quad\text{..........(30)}$$

Solve and get

$$\cos<\mathbf{n},\mathbf{O'M}>=6.42\times10^{-3} \quad\text{..........(31)}$$

So, the angle between the plane and the line formed by the vector $\mathbf{O'M}$ and the ecliptic plane is $\theta=\frac{\pi}{2}-<\mathbf{n},\mathbf{O'M}>=0°22'4''$.

Therefore, it can be seen that the angle between the link of the earth and the moon and the ecliptic plane is very small. From the table, it also can be seen that the dihedral angle formed by the ecliptic plane and the plane of Moon's path is 1.5424 degree, it can be seen as one plane.

In the coordinate system II, $\mathbf{FO'} = (0.2429, -1.377, 0.5972)$, so

$$\cos < \mathbf{FO'}, \mathbf{O'M} >= \frac{\mathbf{FO'} \cdot \mathbf{O'M}}{|\mathbf{FO'}| \times |\mathbf{O'M}|} = 0.3316 . \quad \text{.....................................(32)}$$

Therefore, $< \mathbf{FO'}, \mathbf{O'M} >= 70°38'5''$

The distance between the earth and the moon is

$$d_{地月} = \sqrt{(1.272 \times 10^8)^2 + (3.213 \times 10^8)^2 + (1.671 \times 10^8)^2} = 3.838 \times 10^8 \quad \text{.................(33)}$$

To the coordinate system I, $O'(1.0452 \times 10^{11}, 1.0800 \times 10^{11})$, $F(-0.0250 \times 10^{11}, 0)$, so the rate of slope $O'F$ is 1.0092. And because the angle between FO' and $O'M$ is $70°38'5''$, it can be obtained that the rate of slope of $O'M$ is

$$\frac{1.0092 + \tan 70°38''5'}{1 - 1.0092 \tan 70°38''5'} = -2.0597 \quad \text{...(34)}$$

From $d_{ES} = 3.838 \times 10^8$, it can be seen that the distance between the earth and the moon is $3.838 \times 10^8 m$ at twelve o'clock on Jul 1st, 2007.

In the coordinate system I, since the rate of slope of $O'M$ is -2.0597, $O'M = 3.838 \times 10^8 m$, $O'(1.0452 \times 10^{11}, 1.0800 \times 10^{11})$, it can be obtained that the coordinate of M is $M(1.0445 \times 10^{11}, 1.0815 \times 10^{11})$. According to the table, the distance of perigee of the moon is $3.633 \times 10^8 m$, the distance of apogee of the moon is $4.055 \times 10^8 m$, so it can be seen that the orbit of the moon moving around the earth is an ellipse.

Let the semi-major axis, semi-minor axis be a_M and b_S,

so:

$$\begin{cases} b_M^2 = a_M^2 + c_M^2 \\ a_M - c_M = 3.633 \times 10^8 \\ c_M + a_M = 4.055 \times 10^8 \end{cases} \quad \text{...(35)}$$

Solve:

$$\begin{cases} a_M = 3.844 \times 10^8 \\ c_M = 0.211 \times 10^8 \\ b_M = 3.838 \times 10^8 \end{cases} \quad \text{...(36)}$$

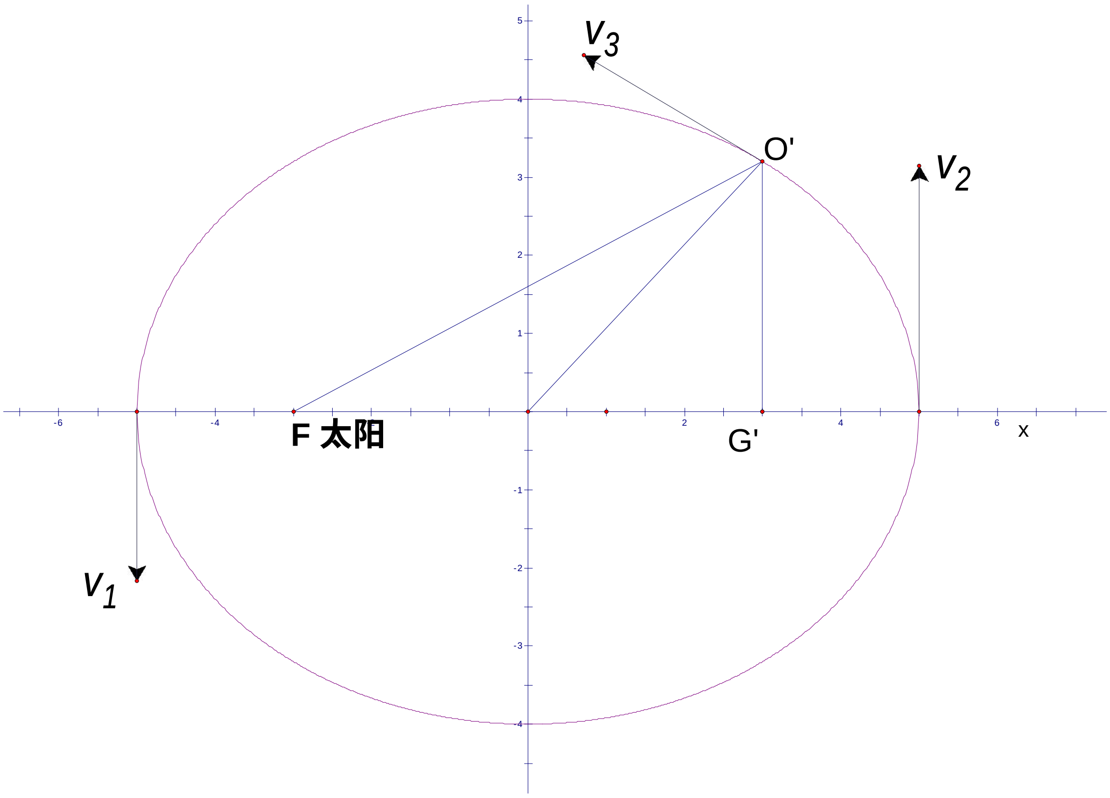
Therefore, there is conclusion: in the coordinate system I, which seen the plane of elliptical orbit of the earth as x、 y plane and the center of the ellipse as the original point, at twelve o'clock on July 1st, 2007, the coordinate of the earth is $O'(1.0452\times10^{11}, 1.0800\times10^{11})$, the coordinate of the moon is $M(1.0445\times10^{11}, 1.0815\times10^{11})$, the coordinate of the sun is $F(-0.0250\times10^{11}, 0)$. The orbits of the moon moving around earth are the semi-major axis and semi-minor axis of the ellipse which are $3.844\times10^{8}m$ and $3.838\times10^{8}m$.

5. the relationship between the position of the centroid of earth and moon and the time of the centroid

Define the rate of the area swept by the link of the centroid of earth and moon and the center of the sun (the focal point of the ellipse) and the time cost be the areal velocity φ_1, according to the Kelper's second law: To any planet, the line joining the planet to the Sun (called radical vector) sweeps out equal areas in equal intervals of time.

$$\varphi_1 = \frac{J_{EM}}{M_E + M_M} = \frac{1.3472\times10^{40}}{5.974\times10^{24} + 7.36\times10^{22}} = 2.228\times10^{15} m^2/s \text{ ....(37)}$$

The centroid of earth and moon starts moving from point $O'(x_b, y_b)$; deduce the distance passed by the link of the centroid and the sun to the original point $O'(x_b, y_b)$ at any time.



An equation can be deduced for the elliptical orbit of the earth:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Leftrightarrow y=\pm\frac{b}{a}\sqrt{a^2-x^2} \quad \ldots\ldots(38)$$

If only think about half of the ellipse:

$$y=\frac{b}{a}\sqrt{a^2-x^2} \quad \ldots\ldots(39)$$

Integral:

$$\int ydx=\frac{b}{a}[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin(\frac{x}{a})]+C=F(x) \quad \ldots\ldots(40)$$

also $F(-a)=0$, so $C=0$, furthermore

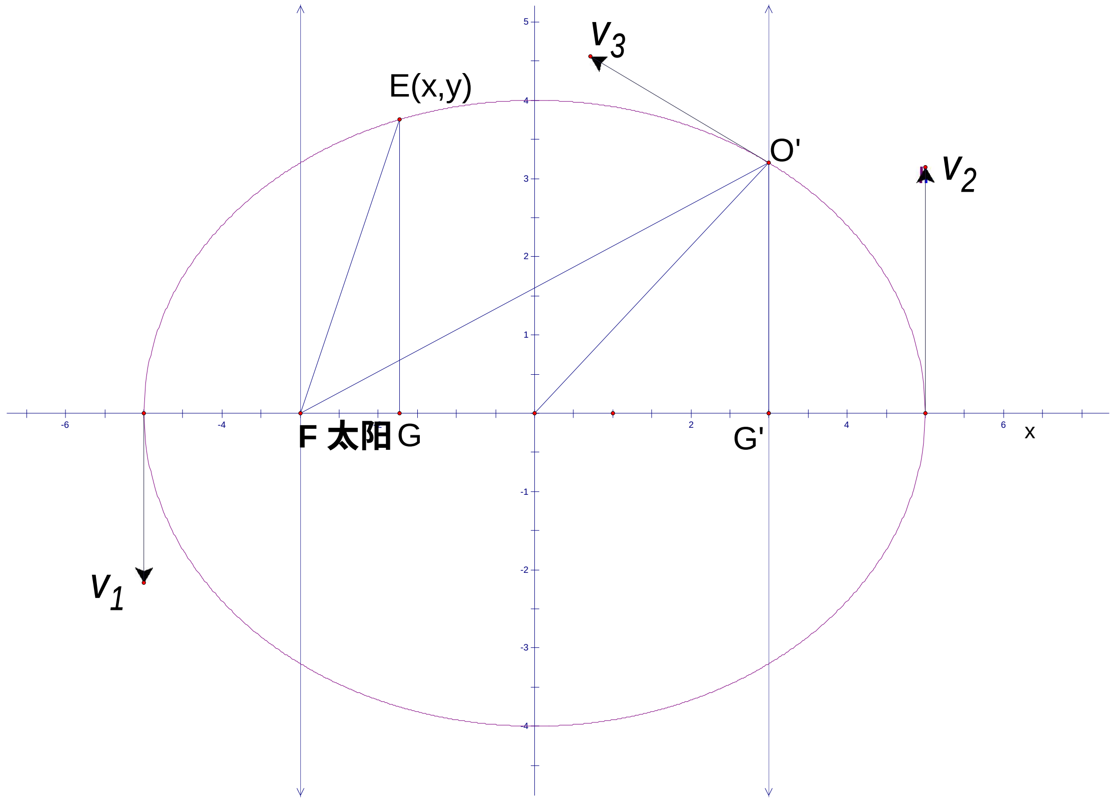
$$\int ydx=\frac{b}{a}[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin(\frac{x}{a})] \quad \ldots\ldots(41)$$

Substitute in the data:

$$\int ydx=\frac{7479}{7480}[\frac{x}{2}\sqrt{2.2380\times10^{22}-x^2}+1.1190\times10^{22}\arcsin(\frac{x}{1.4960\times10^{11}})] \quad \ldots\ldots(42)$$

Now, the discussion for different situations according to the positions of the earth:

When the earth is on the right side of the horizontal position of the sun, and on the left side of the horizontal position of $O'(x_b, y_b)$, as the graph shows bellow, the earth moves to point E. Draw a perpendicular line to the x-axis which is passing through point E and intersects x-axis at point G.



The area swept by the radical vector is the area of the figure which formed by points F, O', E, let the area be S

After a simple cut--complement process, it can be deduced:

$$S=\int_{x}^{x_{初}} ydx + S_{VFGE} - S_{VO'FG} \qquad (43)$$

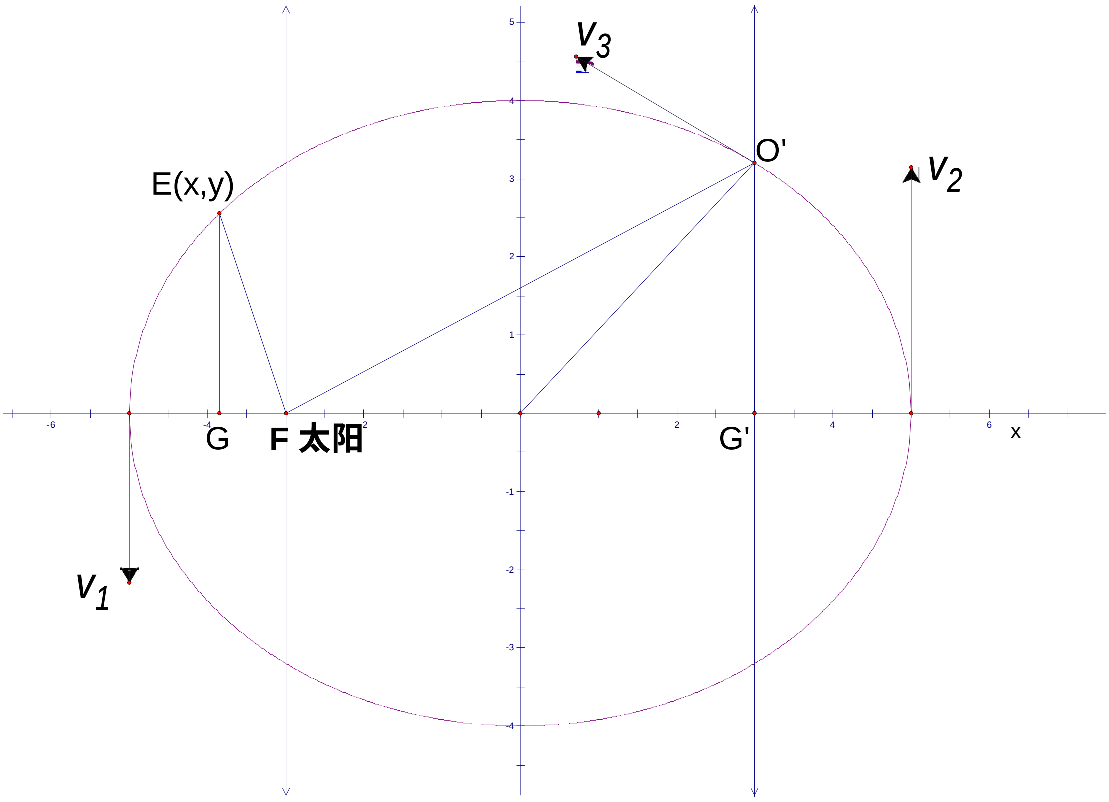
Substitute the data:

$$S = 0.8468\times10^{22} - \frac{7479}{7480}\left[\frac{x}{2}\sqrt{2.2380\times10^{22}-x^2} + 1.1190\times10^{22}\arcsin\left(\frac{x}{1.4960\times10^{11}}\right)\right] + \frac{1}{2}(x+0.0250\times10^{11})y \qquad (44)$$

$$S = 0.8468\times10^{22} - \frac{7479}{7480}\left[\frac{x}{2}\sqrt{2.2380\times10^{22}-x^2} + 1.1190\times10^{22}\arcsin\left(\frac{x}{1.4960\times10^{11}}\right)\right] + \frac{7479}{14960}(x+0.0250\times10^{11})\sqrt{2.2380\times10^{22}-x^2} \qquad (45)$$

$$S = 0.8468\times10^{22} + 0.0125\times10^{11}\times\sqrt{2.2380\times10^{22}-x^2} - 1.1189\times10^{22}\arcsin\left(\frac{x}{1.4960\times10^{11}}\right) \qquad (46)$$

When the earth is at the second quadrant and on the left side of the horizontal position of the sun, as the graph shows bellow:

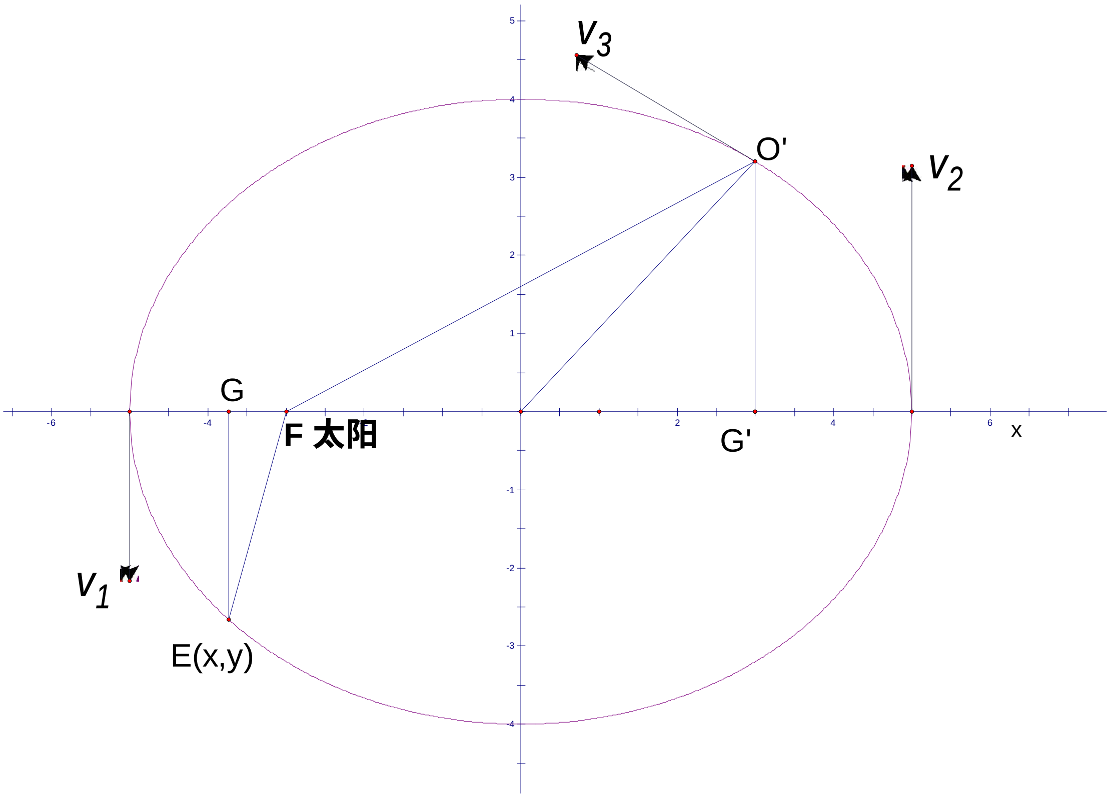


For the same reason: after a cut--complement process, it can be deduced that the area swept by the radical vector is $S = \int_{x}^{x_b} y dx - S_{VFGE} - S_{VO'FG}$.

$$S = 0.8468 \times 10^{22} + 0.0125 \times 10^{11} \times \sqrt{2.2380 \times 10^{22} - x^2} - 1.1189 \times 10^{22} \arcsin(\frac{x}{1.4960 \times 10^{11}})$$

………………………………………………………………………………………...(47)

When E is at the third or forth quadrant, as the graph shows bellow:

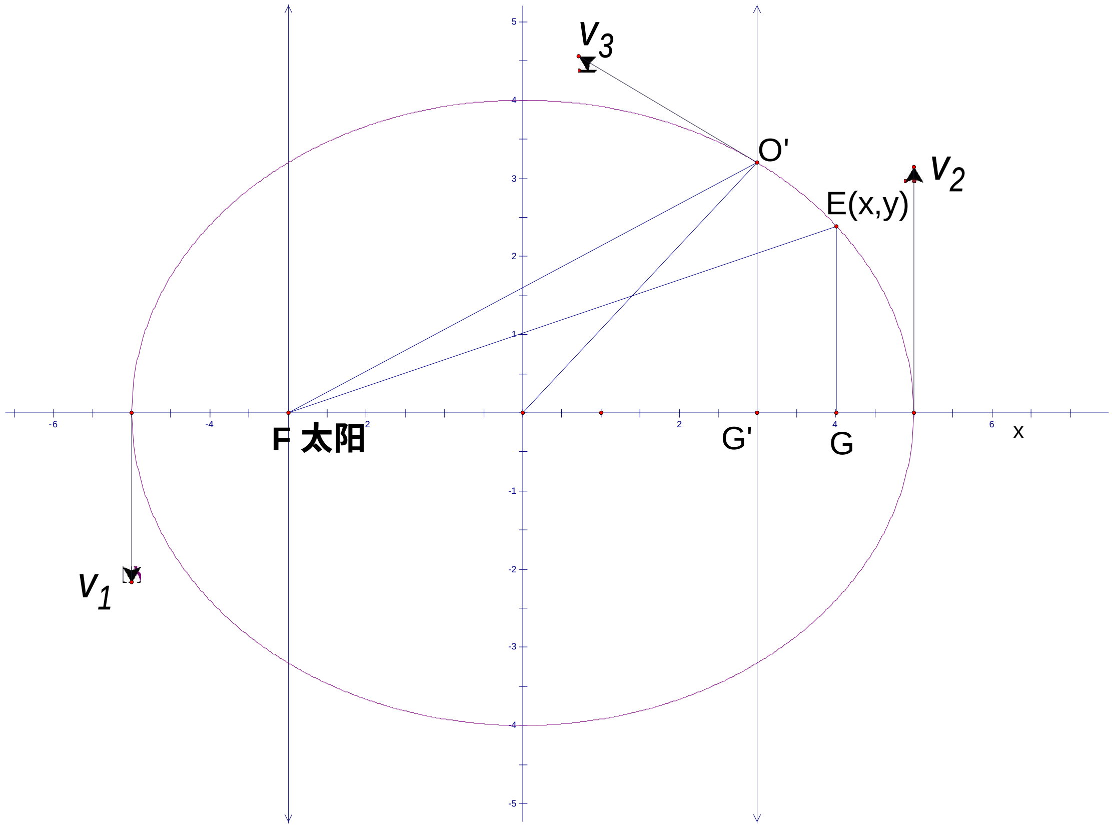


$$S=\int_{-a}^{x_b} ydx+\int_{-a}^{x} ydx-S_{VO'G'F}+\frac{1}{2}\cdot FG\cdot y \text{ ....(48)}$$

$$S=4.3619\times10^{22}+1.1189\times10^{22}\arcsin(\frac{x}{1.4960\times10^{11}})-0.0125\times10^{11}\sqrt{2.2380\times10^{22}-x^2}$$

....(49)

When E is at the first quadrant and on the right side of the horizontal position of $O'(x_{初}, y_{初})$, as the graph shows bellow:



For the same reason and after the cut--complement process:

$$S = \pi ab - (\int_{x_b}^{x} ydx - S_{VFGE} + S_{VO'G'F}) = \pi ab - \int_{x_b}^{x} ydx + S_{VFGE} - S_{VO'G'F} \quad \text{……………..(50)}$$

$$S = 7.8771\times10^{22} - 1.1189\times10^{22}\arcsin(\frac{x}{1.4960\times10^{11}}) + 0.0125\times10^{11}\sqrt{2.2380\times10^{22} - x^2}$$

……………………………………………………………...……………………(51)

So a piecewise function can be deduced:

$$S = \begin{cases} 0.8468\times10^{22} + 0.0125\times10^{11}\times\sqrt{2.2380\times10^{22} - x^2} - 1.1189\times10^{22}\arcsin(\frac{x}{1.4960\times10^{11}}) \\ 4.3619\times10^{22} - 0.0125\times10^{11}\times\sqrt{2.2380\times10^{22} - x^2} + 1.1189\times10^{22}\arcsin(\frac{x}{1.4960\times10^{11}}) \\ 7.8771\times10^{22} + 0.0125\times10^{11}\times\sqrt{2.2380\times10^{22} - x^2} - 1.1189\times10^{22}\arcsin(\frac{x}{1.4960\times10^{11}}) \end{cases}$$

…………………………………………….....…………………………………(52)

$$y = \pm\frac{7479}{7480}\sqrt{2.2380\times10^{22} - x^2}$$

The domains are $\begin{cases} -1.4960\times10^{11} \le x < 1.0452\times10^{11}, y \ge 0 \\ -1.4960\times10^{11} < x < 1.4960\times10^{11}, y < 0 \\ 1.0452\times10^{11} \le x < 1.4960\times10^{11}, y \ge 0 \end{cases}$

The time t taken by earth for it to get to any point E on the elliptical track can be calculated by using the area S swept by the radical vector divided by the areal velocity

φ_1 of the earth's revolution around the sun.

So $t = \dfrac{S}{\varphi_1}$

Substitute into the data:

$$t = \begin{cases} 0.3801\times10^7 + 5.6104\times10^{-7}\times\sqrt{2.2380\times10^{22} - x^2} - 0.5022\times10^7 \arcsin(\dfrac{x}{1.4960\times10^{11}}) \\ 1.9578\times10^7 - 5.6104\times10^{-7}\times\sqrt{2.2380\times10^{22} - x^2} + 0.5022\times10^7 \arcsin(\dfrac{x}{1.4960\times10^{11}}) \\ 3.5355\times10^7 + 5.6104\times10^{-7}\times\sqrt{2.2380\times10^{22} - x^2} - 0.5022\times10^7 \arcsin(\dfrac{x}{1.4960\times10^{11}}) \end{cases}$$

……………………………………………………………………………………………………..(53)

The domains which are the same as the last one are:

$$\begin{cases} -1.4960\times10^{11} \le x < 1.0452\times10^{11}, y \ge 0 \\ -1.4960\times10^{11} < x < 1.4960\times10^{11}, y < 0 \\ 1.0452\times10^{11} \le x < 1.4960\times10^{11}, y \ge 0 \end{cases}$$

The relationship between the link of the sun and the earth (the radical vector of the earth) and the position of the earth can be deduced.

From the knowledge of plane geometry, it can be seen that the slope $k = \dfrac{\Delta y}{\Delta x}$, and in previous the coordinate of the sun is calculated which is $(-c, 0)$.

So, there is

$$k = \frac{y}{x+c} \quad \text{……………………………………………………………....………………….(54)}$$

Using $y = \pm\dfrac{b}{a}\sqrt{a^2 - x^2}$, it can be obtained that:

$$k = \begin{cases} \dfrac{a\sqrt{a^2 - x^2}}{b(x+c)} \\ -\dfrac{a\sqrt{a^2 - x^2}}{b(x+c)} \end{cases}$$

$$\begin{cases} y \ge 0 \\ y < 0 \end{cases} \quad \text{………………………………………………………………………………….(55)}$$

Square the two sides and after an arrangement:

$$(b^2k^2 + a^2)x^2 + 2b^2ck^2x + b^2c^2k^2 - a^4 = 0 \quad \text{………………………...………………(56)}$$

Using the quadratic formula to solve and can get:

$$x=\frac{-b^2ck^2\pm\sqrt{a^2b^4k^2+a^6}}{b^2k^2+a^2} \qquad (57)$$

$$x=\frac{0.0559k^2\pm\sqrt{25.0671k^2+11.2096}}{2.2374k^2+2.2380}\times10^{11} \qquad (58)$$

Sub $x=\frac{0.0559k^2\pm\sqrt{25.0671k^2+11.2096}}{2.2374k^2+2.2380}\times10^{11}$ into the equation of t and x, then,

$$t=\begin{cases} 0.3801\times10^7+5.6104\times10^4\times\sqrt{2.2380-\frac{3.12481\times10^{-3}k^4+25.0671k^2+11.2096\pm0.1118\sqrt{25.0671k^2+11.2096}}{5.0060k^4+10.01460k^2+5.0086}}-0.5022\times10^7\arcsin(\frac{0.0559k^2\pm\sqrt{25.0671k^2+11.2096}}{3.347150k^2+3.348048}) \\ 1.9578\times10^7-5.6104\times10^4\times\sqrt{2.2380-\frac{3.12481\times10^{-3}k^4+25.0671k^2+11.2096\pm0.1118\sqrt{25.0671k^2+11.2096}}{5.0060k^4+10.01460k^2+5.0086}}+0.5022\times10^7\arcsin(\frac{0.0559k^2\pm\sqrt{25.0671k^2+11.2096}}{3.347150k^2+3.348048}) \\ 3.5355\times10^7+5.6104\times10^4\times\sqrt{2.2380-\frac{3.12481\times10^{-3}k^4+25.0671k^2+11.2096\pm0.1118\sqrt{25.0671k^2+11.2096}}{5.0060k^4+10.01460k^2+5.0086}}-0.5022\times10^7\arcsin(\frac{0.0559k^2\pm\sqrt{25.0671k^2+11.2096}}{3.347150k^2+3.348048}) \end{cases}$$

……………………………………………………………………………………(59)

6. The calculation for the slope of the link of the moon and the sun

In the previous discussion, a conclusion was deduced. The conclusion is that the orbit for the moon moving around the sun is an ellipse with the semi-major axis that is $3.844\times10^8 m$, and the semi-minor axis that is $3.838\times10^8 m$.

$$.\begin{cases} a_{月}=3.844\times10^8 \\ c_{月}=0.211\times10^8 \\ b_{月}=3.838\times10^8 \end{cases}$$

The areal velocity of the moon moving around the earth can be easily deduced by using the same method for the areal velocity of the earth moving around the sun.

$$\varphi_2=\frac{\pi a_M b_M}{T_M} \qquad (60)$$

Looking up to the table and get that the revolution period of the moon is 27 days 7 hours 43 minutes 11.559 seconds. After a conversion, it can be obtained that $T_{月}=2.36\times10^6 s$. The areal velocity of the revolution of the moon moving around the earth is $\varphi_2=1.9639\times10^{11} m^2/s$. The moving momentary rate of the perigee and the apogee can be calculated which are $2.9759\times10^{-6} rad/s$、$2.3887\times10^{-6} rad/s$.

Therefore, it can be deduced that the difference of the angular velocity of the moon is very small, so it can be calculated as an approximate calculation, seeing the moving of the moon as a uniform speed circular motion. So, the angular velocity can be calculated: $\omega_M = \dfrac{2\pi}{T_M} = 2.662\times10^{-6}\, rad/s$ ……………………………………….(61)

The slope of the link of the earth and the moon $k' = \tan\alpha$, α is the inclination of the link. If the inclination of the centroid of earth and moon is α_0 at twelve o'clock on July 1st, 2007, then the value of k' can be expressed as $k' = \tan(\alpha_0 + \omega_M t)$ at any time……………………………………………………………………………(62)

Substitute into the data:

$$k' = \tan(1.2328 + 2.662\times10^{-6} t) \quad \text{……………………………………………….(63)}$$

Therefore,

$$t = \frac{\arctan k' - 1.2328}{2.662\times10^{-6}} \quad \text{…………………………………………………….(64)}$$

7. Solve the equation system by put the equations together

It can be known that the condition for the eclipse happen is the earth, the moon and the sun are on one line, it also can be seen as that on one moment and $k = k'$, observe on the moon, eclipse of the earth or solar eclipse can be seen. So we can get an equation system:

$$t = \begin{cases} 0.3801\times10^7 + 5.6104\times10^4 \times \sqrt{2.2380 - \dfrac{3.12481\times10^{-3}k^4 + 25.0671k^2 + 11.2096 \pm 0.1118\sqrt{25.0671k^2 + 11.2096}}{5.0060k^4 + 10.01460k^2 + 5.0086}} - 0.5022\times10^7 \arcsin\left(\dfrac{0.0559k^2 \pm \sqrt{25.0671k^2 + 11.2096}}{3.347150k^2 + 3.348048}\right) \\ 1.9578\times10^7 - 5.6104\times10^4 \times \sqrt{2.2380 - \dfrac{3.12481\times10^{-3}k^4 + 25.0671k^2 + 11.2096 \pm 0.1118\sqrt{25.0671k^2 + 11.2096}}{5.0060k^4 + 10.01460k^2 + 5.0086}} + 0.5022\times10^7 \arcsin\left(\dfrac{0.0559k^2 \pm \sqrt{25.0671k^2 + 11.2096}}{3.347150k^2 + 3.348048}\right) \\ 3.5355\times10^7 + 5.6104\times10^4 \times \sqrt{2.2380 - \dfrac{3.12481\times10^{-3}k^4 + 25.0671k^2 + 11.2096 \pm 0.1118\sqrt{25.0671k^2 + 11.2096}}{5.0060k^4 + 10.01460k^2 + 5.0086}} - 0.5022\times10^7 \arcsin\left(\dfrac{0.0559k^2 \pm \sqrt{25.0671k^2 + 11.2096}}{3.347150k^2 + 3.348048}\right) \end{cases}$$

……………………………………………………………………………………..(65)

Calculate as

$$t = \frac{\arctan k' - 1.2328}{2.662\times10^{-6}} \quad \text{…………………………………………………….(66)}$$

Since it's very complicated, it can be calculated by using a computer program. First, create a function which has k as the independent variable:

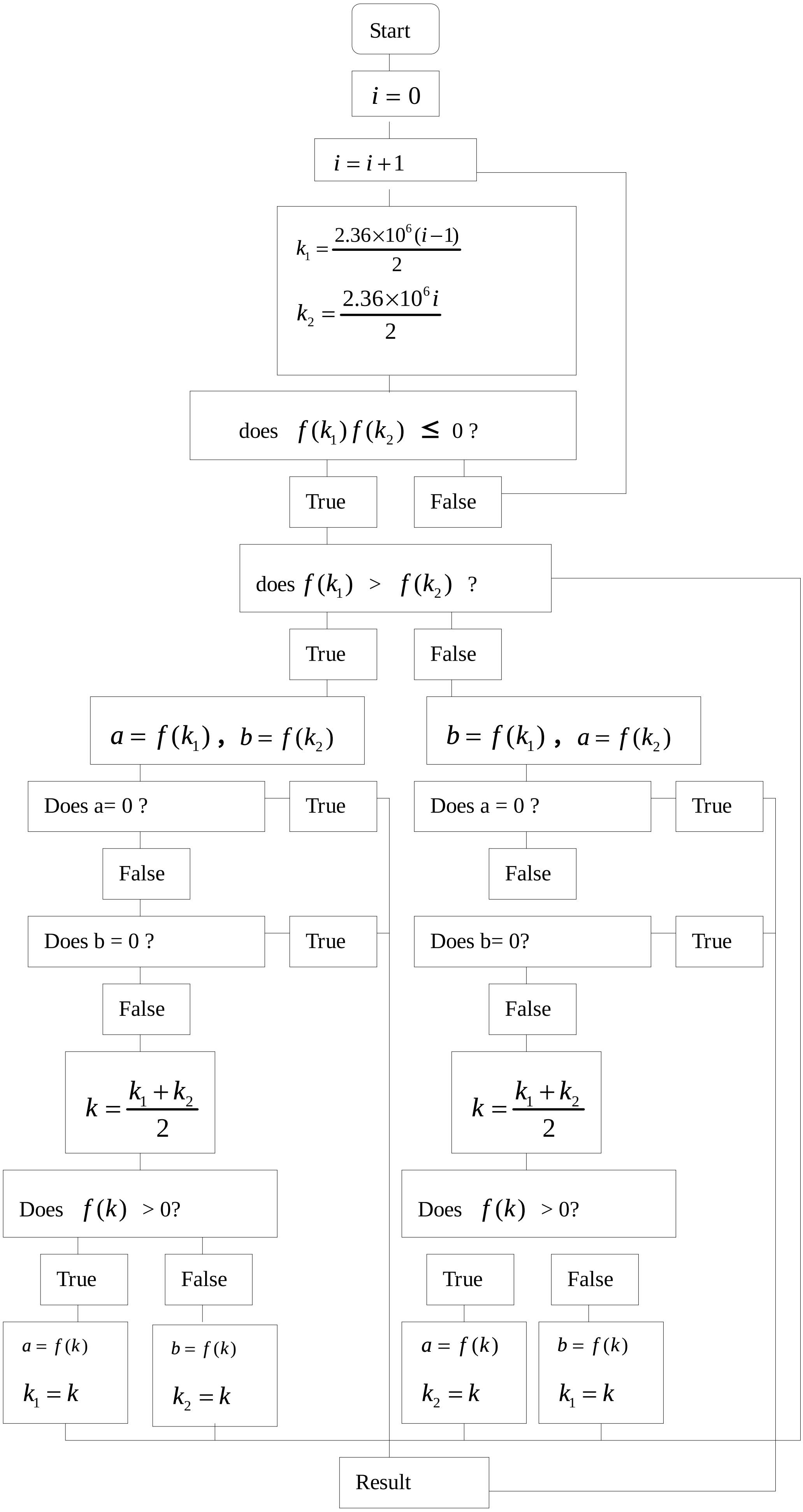
$$f_1(k) = 0.3801\times10^7 + 5.6104\times10^4 \times \sqrt{2.2380 - \frac{3.12481\times10^{-3}k^4 + 25.0671k^2 + 11.2096 \pm 0.1118\sqrt{25.0671k^2 + 11.2096}}{5.0060k^4 + 10.01460k^2 + 5.0086}} - 0.5022\times10^7 \arcsin\left(\frac{0.0559k^2 \pm \sqrt{25.0671k^2 + 11.2096}}{3.347150k^2 + 3.348048}\right) - \frac{\arctan k' - 1.2328}{2.662\times10^{-6}}$$

$$f_2(k) = 1.9578\times10^7 - 5.6104\times10^4 \times \sqrt{2.2380 - \frac{3.12481\times10^{-3}k^4 + 25.0671k^2 + 11.2096 \pm 0.1118\sqrt{25.0671k^2 + 11.2096}}{5.0060k^4 + 10.01460k^2 + 5.0086}} + 0.5022\times10^7 \arcsin\left(\frac{0.0559k^2 \pm \sqrt{25.0671k^2 + 11.2096}}{3.347150k^2 + 3.348048}\right) - \frac{\arctan k' - 1.2328}{2.662\times10^{-6}}$$

$$f_3(k) = 3.5355\times10^7 + 5.6104\times10^4 \times \sqrt{2.2380 - \frac{3.12481\times10^{-3}k^4 + 25.0671k^2 + 11.2096 \pm 0.1118\sqrt{25.0671k^2 + 11.2096}}{5.0060k^4 + 10.01460k^2 + 5.0086}} - 0.5022\times10^7 \arcsin\left(\frac{0.0559k^2 \pm \sqrt{25.0671k^2 + 11.2096}}{3.347150k^2 + 3.348048}\right) - \frac{\arctan k' - 1.2328}{2.662\times10^{-6}}$$

……………………………………………………………………………………..(67)

The program flowchart is shown as bellow:


Start
$i = 0$
$i = i + 1$
$k_1 = \frac{2.36\times10^6(i-1)}{2}$
$k_2 = \frac{2.36\times10^6 i}{2}$
does $f(k_1)f(k_2) \leq 0$?
True
False
does $f(k_1) > f(k_2)$?
True
False
$a = f(k_1)$, $b = f(k_2)$
$b = f(k_1)$, $a = f(k_2)$
Does a= 0 ?
True
Does a = 0 ?
True
False
False
Does b = 0 ?
True
Does b= 0?
True
False
False
$k = \frac{k_1 + k_2}{2}$
$k = \frac{k_1 + k_2}{2}$
Does $f(k) > 0$?
Does $f(k) > 0$?
True
False
True
False
$a = f(k)$
$k_1 = k$
$b = f(k)$
$k_2 = k$
$a = f(k)$
$k_2 = k$
$b = f(k)$
$k_1 = k$
Result

Conclusion:

We deduced the formula of Three-Object-One-line among the earth, moon and the sun with mathematics throughing the existing physical knowledge, and we also provided the program flowchart for computers. This result would help Chinese lunar exploration workers to find better photographic opportunity for even more exquisite cosmic image.

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