THE EFFECT OF CLASSICAL MODELS ON GEOMETRIC MODELS AND SOME CONCLUTIONS ON IT

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INTRODUCTION

Our essay begins with a simple and smart probabilistic question given by Martin Gardner. He pointed out in [1] that:” Dividing ten gold coins and ten silver coins into two containers which look the same outside, then randomly choose one of the containers and take out one coin from the chosen container, the probability of taking out a gold coin may change in turns of the change of distribution.”

Then it continues the thought of him: How large or how small can the probability be? What if they are not coins but powder? What if there are more than 2 containers?

The questions above then lead to some interesting conclusions. For example: In the situation of the ‘powder’ and ‘unlimited containers’, with a particular distribution, we can get either gold or silver ones be chosen 100%.

In this way, we went on studying and successfully understood this question more deeply. The reason of the change of probability can not only be explained by ‘different sample space and sample point’ (according to [3] and [4]) but also be explained by ‘THE EFFECT OF CLASSICAL MODELS ON GEOMETRIC MODELS’, which was the root theory we got from the question.

According to this theory, we realized that this question can be a useful mathematics model by recognizing gold→A and silver→not A. And that’s what the essay talks about in the example of the application: an explanation of a phenomenon discovered in [2], a supposal and a proof (though it’s not so good looking). And with the thought that the theory is relevant to reality, the rest of it talks about some transforms of the conclusions mentioned.

And that’s about all that this essay is about.

1. Abstract:

Our paper begins with a simple but beautiful subject given by Martin Gardner (Dividing ten gold coins and ten silver coins into two containers which look the same outside, then randomly choose one of the containers and take out one coin from the chosen container, Is the probability of getting golden coin becomes higher when the distributive method changes). And we change and consider the problem from some other points of view.

1. **Researching the original subject’s probability.**
2. **Discussing the minimal unit of distributing golden coins.**
3. **Researching the number of containers.**
4. **Researching the minimal and maximal probability of getting golden coin which is the function expression of containers’ number.**

As a result, we will understand the subject more deeply. The essence of the subject tells us: the reason of probability’s changing can not only be interpreted by “Sample Space difference and Sample Point difference”, but also “Classical Probability’s infection to Geometry Probability”.

According to this theory, we can model this subject: Gold→A and Silver→not A. Then we use the model in our example, successfully interpreting what J·BERTRAND had found. And we also get a conjecture and prove it. Further, we did some transformation of the original formulation according to the actual situation.

This is our paper’s main content.

2. Discussion of one subject

Original Subject: Dividing ten gold coins and ten silver coins into two containers which look the same outside, then randomly choose one of the containers and take out one coin from the chosen container, Is the probability of getting golden coin becomes higher when the distributive method changes? How can we distribute the coins to maximize the probability? —Martin Gardner《Bathsheba's Wisdom》

The author mentioned that: one container contains one golden coin and the other contains the rest coins, then the probability is 1/2*(1/1+9/19) = 14/19, which is higher than average distributed. This told us that changing distribution can make probability change. However, the author did not mention the second question.

Thinking: How to distribute to maximize the probability?

2.1 Researching the probability of the original subject

Question 1: How to distribute n gold coins and m silver coins to 2 opacity containers and randomly take one coin from either container to maximize the probability of getting gold coin? What about the minimal?

2.1.1 The probability of getting gold coin is maximal:

Analyze: The number of gold and silver coins remain unchanged. The coin's number of one container determines one distributing method. After understanding infection to gold coin's probability caused by changing silver coins number in containers, we will do classified discussion.

Resolution: according to our analysis, distribute a(a≤n/2) gold coins and b(b≤m) silver coins to one container, and the other container contains n-a gold coins and m-b silver coins. We set the probability of getting gold coin is P. So we have 3 kinds of method to distribute the coins:

Case 1: if a=1, b=0, then $P1=\frac{1}{2}(1+\frac{n-1}{n+m-1})$

Case 2: if a≠1, b=0, then $P2=\frac{1}{2}(1+\frac{n-a}{n+m-a})$

Case 3: if a≠1, b≠0, then $P3=\frac{1}{2}(\frac{a}{a+b}+\frac{n-a}{n+m-a-b})$

Now we will prove P1>P2>P3:

$$\because \frac{n-1}{n+m-1}=\frac{a-1}{\frac{n+m-1}{n-1}(a-1)}$$

$$\therefore \frac{n-1}{n+m-1}=\frac{n-a}{n+m-1-\frac{n+m-1}{n-1}(a-1)}$$ …… (Componendo)

and:

$$\frac{n+m-1}{n-1}>1$$

$\therefore \quad \frac{n+m-1}{n-1}(a-1) > a-1$

$\therefore \quad P1 = \frac{1}{2} + \frac{1}{2} \cdot \frac{n-a}{n+m-1-\frac{n+m-1}{n-1}(a-1)} > \frac{1}{2} + \frac{1}{2} \cdot \frac{n-1}{n+m-1-(a-1)} = P2$

and:

$$
\begin{aligned}
P2 - P3 &= \frac{1}{2}[(1-\frac{a}{a+b}) - (\frac{n-a}{n+m-a-b} - \frac{n-a}{n+m-a})] \\
&= \frac{1}{2}(\frac{b}{a+b} - \frac{b(n-a)}{[(n+m-a)-b](n+m-a)}) \\
&= \frac{b}{2} \times \frac{(n+m-a-b)(n+m-a)-(n-a)(a+b)}{(a+b)(n+m-a-b)(n+m-a)}
\end{aligned}
$$

$\because \quad m-b \geq 0 \Rightarrow n+m-a-b \geq n-a$

$\begin{cases} a \leq \frac{n}{2} \\ m \geq b \end{cases} \Rightarrow n+m-a \geq a+b$

$$\therefore \textit{The original function} \geq \frac{b}{2} \times \frac{(n-a)[(n+m-a)-(a+b)]}{(a+b)(n+m-a-b)(n+m-a)} \geq 0$$

(if and only if a=n/2 and b=m, the original function = 0)

So P1>P2≥P3, if and only if a=n/2 and b=m, the “=” holds.

Now we can say: a=1, b=0 is the best solution.

2.1.2 The probability of getting gold coin is minimal:

Analysis: Because the coin we get is either gold or silver, P_gold + P_silver = 1, so when the probability of getting silver coin is maximal, the probability of getting gold coin is minimal.

Solution: since $P\_gold(\max) = \frac{1}{2}(1+\frac{n-1}{n+m-1})$, so we can get $p\_silver(\max) = \frac{1}{2}(1+\frac{m-1}{n+m-1})$. And because P_gold + P_silver = 1, so $P\_gold(\min) = \frac{1}{2} \cdot \frac{n-1}{n+m-1}$.

Conclusion 1:

$$\frac{1}{2} \cdot \frac{n}{n+m-1} \leq P = \frac{1}{2} \cdot (\frac{a}{a+b} + \frac{n-a}{n+m-a-b}) \leq \frac{1}{2}(1+\frac{n-1}{n+m-1})$$

If and only if a=1, b=0, the “=” holds on the right side of the inequality; a=0, b=1, the “=” holds on the left side. (m, n ∈ Z+, a, b∈Z and n≥a≥0, m≥b≥0) i.e. if one container only contains one gold(silver) coin, the probability is maximal.

Thinking: The conclusion is acquired when the number of coin is integer. What about the case when the number is not integrated?

2.2 Discussion if the number of distributed items is not integrated

Question 2: distributing n grams of gold powder and m grams of silver powder. What is the mean maximum of gold powder quality in one gram of powder? What about k grams? (n ,m ≥2 or n ,m≥2k, assuming the gold and silver powder density are the same, and can be divided unlimitedly).

Analyze: the powder can be divided unlimitedly, which is not limited in integer scope. However considering the solution of question 1, we know that the result remains if n, m≥1. So we consider the n grams of gold powder and m grams of silver powder as n/k gold coins and m/k silver coins which each coin is k gram heavy. According conclusion 1:

$$\frac{1}{2}\cdot\frac{n}{n+m-k}\le P_{gold}\le\frac{1}{2}(1+\frac{n-k}{n+m-k})\qquad(0<k\le n)$$

Solution: we consider the n grams of gold powder and m grams of silver powder as n/k gold coins and m/k silver coins which each coin is k gram heavy. Use the same method as conclusion 1, we can easily get:

$$P_{gold}=\frac{1}{2}(\frac{a}{a+b}+\frac{\frac{n}{k}-a}{\frac{n}{k}+\frac{m}{k}-a-b})=\frac{1}{2}(\frac{ak}{ak+bk}+\frac{n-ak}{n+m-ak-bk})$$

Related with conclusion 1, we have $\frac{1}{2}\cdot\frac{n}{n+m-k}\le P_{gold}\le\frac{1}{2}(1+\frac{n-k}{n+m-k})$

Conclusion 2:

$$\frac{1}{2}\cdot\frac{n}{n+m-k}\le P=\frac{1}{2}(\frac{a}{a+b}+\frac{n-ak}{n+m-ak-bk})\le\frac{1}{2}(1+\frac{n-k}{n+m-k})$$

(k, m, n are constants, $k\in R^{+},\quad m,n\ge 2k,\quad m\ge b, n\ge a$)

If and only if a=0, b=k, the “=” holds on the left side of the inequality; a=k, b=0, the “=” holds on the right side.

Thinking: What is the influence of the number of containers on the probability and distributed method?

2.3 Researching the number of containers]

Question 3: distributing n grams of gold powder and m grams of silver powder to t containers, what is the mean maximum of gold powder quality in k grams of powder? （$t\in Z^{+}$ *and* $n,m\ge t*k$）

Analyze: Having solved the case t=2, we will use induction.

Conjecture: We conjecture, t≥2, t-1 containers contain k grams of gold powder, and the rest is enclosed the last container, this case we get maximal value; t-1 containers contain k grams of silver powder, and the rest is enclosed the last container, this case we get minimal value. For the probability of getting gold coin P_t, we have:

$$\frac{1}{t}\cdot\frac{n}{n+m-(t-1)k}\le P_t\le\frac{1}{t}[(t-1)+\frac{n-(t-1)k}{n+m-(t-1)k}]$$

Prove:

When t=2, $P=\frac{1}{2}(\frac{a}{a+b}+\frac{n-ak}{n+m-ak-bk})$, according conclusion 2, the conjecture is true.

When t>2, we assume the conjecture is true if $t=t^{'}-1$. Then we set $t=t^{'}$:

We separate $t^{'}$ containers into two parts A and B. A contains $t^{'}-1$ containers and B contains one.

Suppose the qualities of gold and silver powers contained in the part B are a, b which each unit is k gram. And the qualities of gold and silver powers contained in the part A are n-a, m-b which each unit is k gram. (a,b∈N)

Because the probability of randomly selecting part A and B is $\frac{1}{t^{'}}$ and $\frac{t^{'}-1}{t^{'}}$, so:

$$
\begin{aligned}
P_{t`} &= \frac{1}{t`}\cdot\frac{a}{a+b}+\frac{t`-1}{t`}\cdot\frac{1}{t`-1}\cdot P_{t`-1} \\
&\le \frac{1}{t`}\cdot\frac{a}{a+b}+\frac{t`-1}{t`}\cdot\frac{1}{t`-1}[t`-2+\frac{n-ak-(t`-2)k}{n+m-ak-bk-(t`-2)k}] \\
&= \frac{1}{t`}\cdot[\frac{a}{a+b}+\frac{n-ak-(t`-2)k}{n+m-ak-bk-(t`-2)k}+t`-2] \\
&\le \frac{1}{t`}[1+\frac{n-(t`-1)k}{n+m-(t`-1)k}+t`-2]
\end{aligned}
$$

(according to conclusion 1, we take $n-(t^{'}+2)k$ as n)

$$
=\frac{1}{t`}[t`-1+\frac{n-(t`-1)k}{n+m-(t`-1)k}]
$$

By the same token:

$$
\begin{aligned}
P_{t^{'}} &= \frac{1}{t^{'}}\cdot\frac{a}{a+b}+\frac{t^{'}-1}{t^{'}}\cdot\frac{1}{t^{'}-1}\cdot P_{t^{'}-1} \\
&\ge \frac{1}{t^{'}}\cdot\frac{a}{a+b}+\frac{t^{'}-1}{t^{'}}\cdot\frac{1}{t^{'}-1}\cdot\frac{n-ak}{n+m-ak-bk-(t^{'}+2)k} \\
&\ge \frac{1}{t^{'}}\cdot\frac{n}{n+m-(t^{'}-1)k} \\
&= \frac{1}{t^{'}}\cdot[\frac{a}{a+b}+\frac{n-ak}{n+m-ak-bk-(t^{'}+2)k}] \\
&= \frac{1}{t^{'}}\cdot\frac{n}{n+m-(t^{'}-1)k}
\end{aligned}
$$

(according to conclusion 1, we take $m-(t^{'}+2)k$ as m)

$$
\therefore\quad \frac{1}{t^{'}}\cdot\frac{n}{n+m-(t^{'}-1)k}\le P_{t^{'}}\le\frac{1}{t^{'}}[(t^{'}-1)+\frac{n-(t^{'}-1)k}{n+m-(t^{'}-1)k}]\qquad (m,n,a,b\in N, and\ m\ge b, n\ge a)
$$

If and only if a=0, b=k, the “=” holds on the left side of the inequality; a=k, b=0, the “=” holds on the right side.

Hence, for $t=t^{'}$, the conjecture still holds.

Therefore, for all t∈N, the conjecture holds.

Conclusion 3: $\frac{1}{t}\cdot\frac{n}{n+m-(t-1)k}\le P_t\le\frac{1}{t}[t-1+\frac{n-(t-1)k}{n+m-(t-1)k}],\quad (m,n\ge k*t)$

If and only if t-1 containers contain k grams of gold powder, and the rest is enclosed the last container, this case we get maximal value; t-1 containers contain k grams of silver powder, and the rest is enclosed the last container, this case we get minimal value;

2.4 Research the maximal and minimal probability of pitching gold coins through function

Question 4: Distributing n grams of gold powder and m grams of silver powder to x containers, and taking k grams of powder from them, what is the range of ratio of the gold powder’s average quality to that of selected powder? $(x\in N, 2\le x\le\frac{n}{x} and \frac{m}{k})$

Solution: Let f(x) be the maximal probability of getting gold coin and g(x) be the minimal probability.

According conclusion 3, we can easily get:

$$f(x)=\frac{1}{x}[x-1+\frac{n-(x-1)k}{n+m-(x-1)k}] \qquad g(x)=\frac{1}{x}\cdot\frac{n}{n+m-(x-1)k}$$

$$\geq\frac{x-1}{x} \qquad\qquad \leq\frac{1}{x}$$

When $x\to\infty,(k\to 0)$:

$$\lim_{x\to\infty} f(x)=\frac{1}{x}[x-1+\frac{n-(x-1)k}{n+m-(x-1)k}] \qquad \lim_{x\to\infty} g(x)=\frac{1}{x}\cdot\frac{n}{n+m-(x-1)k}$$

$$\geq\frac{x-1}{x}=1 \qquad\qquad \leq\frac{1}{x}=0$$

By the principle of probability, $f(x)\leq 1$, $0\leq g(x)$, we get:

Conclusion 4: In terms of the function of the maximal number of getting gold coins, minimal number of getting gold coins and the number of containers, we have:

$$when\ k\to 0,\ x\to\infty, \quad f(x)=1,\ g(x)=0$$

3. The essence and application of the conclusion

Consider: Why the probability changes so great due to different distributing methods, while the number of gold and silver coins remains unchanged? We find the reason is that "Randomly choose a container" is a standard classical probability model. No matter how many things the container contains, the probability of choosing one container is the same. This changes the original geometrical characteristic which is the more the matter is, the higher probability to choose it. And this makes the probability to getting one grain of rice the same as one bag of rice.

Refer to reference [2] and [3], they explain this situation beginning with Sample Space and the change of Sample Point. This is a very universal explanation.

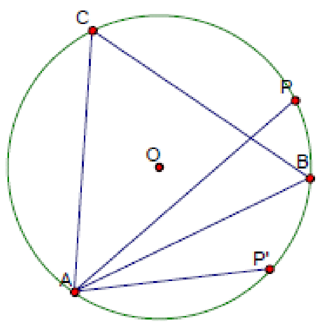
However, this paper is better in application on explaining the influence the classical probability model to collection probability model.

3.1 "Bayesian Paradox"

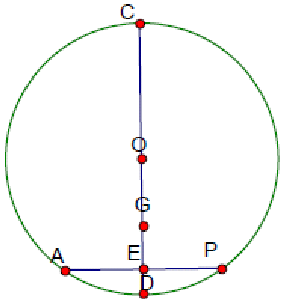
Original Problem: Drawing a chord in a circle whose radius is 1. What is the probability if the length of the chord is longer than $\sqrt{3}$? ---- [2]

[3] gives us some methods, we cite here for deep discussion.

Method 1: The original problem can be regarded as "There is a point A on the circle, now pick another point P on the circle, what is the probability if the inferior arc AP is bigger than 1/3?" We can get the solution through picture.



Method 2: through drawing and analyzing picture, the original problem can be regarded as “Picking a point E on the specific radius OD, what is the probability if OE<1/2?” We can easily get: 1/2.

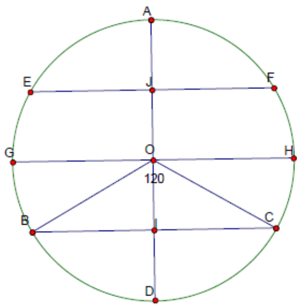


Reference [3] analyses why the probability changes: it is caused by that the sample point and sample space is different between the two methods.

However, we have another interpretation: it is affected by classical probability model.

3.2 The influence of classical probability model

If point A is fixed, the position of P determines if the length of AP is longer than $\sqrt{3}$.



As is shown in the picture, if point P is on the inferior arc BC, AP> $\sqrt{3}$; if point P is on superior arc, AP< $\sqrt{3}$. Because length of inferior arc/ length of superior arc = 1:2, the probability that point P in on inferior arc BC is 1/3.

Another method is drawing a chord through one radius’s vertical line and this method changes the probability that the length of chord AP is longer than $\sqrt{3}$ like distributing coins. If diameter OD is fixed, the length of the set of the points which can be distributed to chords whose length is longer than $\sqrt{3}$ is 1/3 of the perimeter of the circle, i.e. inferior arc BE + superior arc CF = 120 = 1/3 * the length of circle.

In order to relate the changing of probability with the influence of classical probability model to geometrical probability model, we do some analogies’: the points that are on circle and can make chords longer than $\sqrt{3}$ are gold coins; the other points that are on circle and can not make chords longer than $\sqrt{3}$ are silver coins; the four segments on diameter are four containers. After this, we consider picking points on radius as that container AJ has 120 silver coins, container JO has 60 gold coins, container OI has 60 gold coins, and container ID has 120 silver coins.

Consequently, the probability of getting gold coins changes from 1/3 to $\frac{2}{4}\times 1+\frac{2}{4}\times 0=\frac{1}{2}$

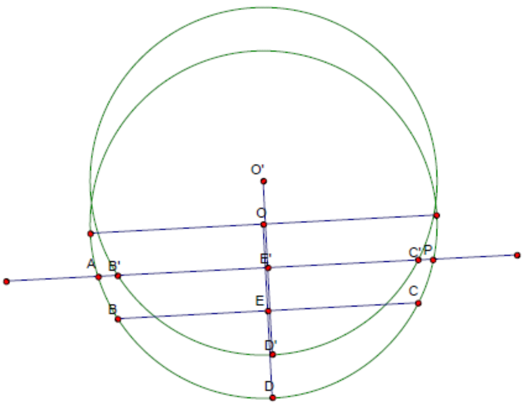
This is the function of the container which is a typical object in classical probability model. We conjecture: if the situation in conclusion 4 also exists in this problem?

3.3 The conjecture from theory and its imperfect proof

One conjecture: We can find one way to draw a chord to make the probability that the chord is longer than $\sqrt{3}$ is 1.

Analyze: To achieve this, we require that $\lim\limits_{the\ number\ of\ containers \to \infty}$ and the points on circle that make chords shorter than $\sqrt{3}$ are in less containers.

Solution: The following is the drawing method.



Drawing a circle O, OD is an arbitrary radius, E is the center point of OD, drawing BC⊥OD through E, B,C are two points on circle O. D' is arbitrary point on segment DE. O', D', B', C', E' are the points generated by O, D, B, C, E translating vector DD' parallelly. A, P are joint of the line B'C' and circle O.

When we choose a point D', i.e. drawing a chord form two points which are joints of the inferior arc B'C' and circle O, the probability that the chord is longer than $\sqrt{3}$ is 1. We prove it as following:

Proof: By observing, we can get easily, if and only if D=D', AP may be shorter than $\sqrt{3}$.

Then the probability that AP> $\sqrt{3}$ is $\dfrac{number\ of\ \ po\text{int}\ s\ on\ OD-1}{number\ of\ \ po\text{int}\ s\ on\ OD}=\dfrac{\infty-1}{\infty}=1$.

Thinking: This is a little wild scheme of drawing a chord, and is only drawn for construction. However in [3], the method 2, i.e. make vertical line from radius, looks better. The methods distributing "gold coins" to man-made containers are various. Is there one natural way to satisfy the conjecture?

This is our last problem. Because of limited time and capability, we do not find a "Perfect Proof".

Corollary: The situation that the influence of classical probability model to geometrical probability model can change instance A's probability is as following:

$$\frac{n}{m}\times\frac{1}{t}\le P_A\le(t-1+\frac{n}{m})\times\frac{1}{t}\le 1$$

(When t→+∞, PA=1; n, m is the ratio of probability of instance A happens to non A in original geometrical probability model)

*Note: This corollary is not precise, please refer to 4: Theorization and ….

4. Theorization and Corollary

Thinking: Since our conclusion has applied value, it is necessary to contact the formulation with actual situation.

4.1 The transformation based on natural number

Sometimes the sample points which we distribute in original geometrical probability model can not be divided unlimitedly, so it is essential to transform the function whose parameters are all natural number:
k=1, the conclusion 3 becomes:

Deformation 1:

$$\frac{1}{t}\cdot\frac{n}{n+m-t+1}\le P_t\le\frac{1}{t}(t-1+\frac{n-t+1}{n+m-t+1})\quad (m,\ n,\ t\in Z^+,\ m,\ n\geqslant t)$$

If and only if t-1 containers have one unit of sample points A and the left points are enclosed in the last container, we get the maximal value; t-1 containers have one unit of sample points of non A, and the left points are enclosed in the last container, we get the minimal value.

Because we can not divide the sample points unlimitedly, the containers' number trends to n or m. Then we change the conclusion to function:

Deformation 2:

$$f(x)=1+\frac{m}{x^2-(n+m+1)x}\quad g(x)=\frac{n}{(n+m+1)x-x^2}\qquad (x\in Z^+,\ x\leqslant m,n)$$

and

$$f(x)=1+\frac{m}{(x-\frac{n+m+1}{2})^2-\frac{(n+m+1)^2}{4}}\quad g(x)=\frac{n}{-(x-\frac{n+m+1}{2})^2+\frac{(n+m+1)^2}{4}}$$

Therefore: 1. n≠m: when x is the positive integer closest to $\frac{n+m+1}{2}$, f(x) is maximal and g(x) is minimal.
2. n=m: when x=m, f(x) is maximal and g(x) is minimal .

4.1 The transformation based on real number

In this case, sample points in original geometrical probability model can be divided unlimitedly (such as gold and silver powder, points), when k→0.

Conclusion 3 can be changed to:

Deformation 3:

$$\frac{1}{t}\cdot\frac{n}{n+m}\le P_t\le\frac{1}{t}(t-1+\frac{n}{n+m})\qquad (m,\ n\in R^+,\ t\in Z^+)$$

If and only if t-1 containers have one unit of sample points A and the left points are enclosed in the last container, we get the maximal value; t-1 containers have one unit of sample points of non A, and the left points are enclosed in the last container, we get the minimal value.

5. Summarize

In this paper, we deeply research one problem related to probability, from simple to complicated, from special to common, from known to unknown, and have more profound understanding with the essence. Further using original standpoint, we did practical explanation to the situation. We abstract the influence of classical probability model to geometrical probability model and apply it, hence get some achievement. In the end, for applying, we did some transformation to the original formulation to make it easy to use.

Because of limitation of time, ability and experience, the paper stops here.

At last, I will appreciate our hierophant, Martin Gardner who finds out the wonderful problem, Doc. Bertrand, Doc. Kolmogorov who inspires our thought, Doc. Shan Dun and all organizations which give us this

chance.

Reference:

[1]: Martin·Gardner [USA], 《Bathsheba's Wisdom》World Book Publishing Company, 2004-5.

[2]: J·Bertand [France], 《Computing of Probability》1889.

[3]: Shan Dun [China], 《Olympic Series of Mathematics—Probability and Expectation》2005.

[4]: Kolmogorov [Russia], 《The Axiom Basis of Probability》1933.