THE RESEARCH ON THE DIFFICULTY GRADE AND PIECING SKILLS OF TANGRAM

七巧板的难度分级及拼凑技巧研究

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Abstract: Tangram is one of the oldest puzzle games in China. It can make up volatile and wonderful patterns. To master the game, it would be ideal if the players can go through several complexity levels from the easiest to the most complicated. However, tangram today does not have the difficulty grade. Only a similar puzzle game—Japanese jigsaw puzzle has the difficulty grade which is not very reasonable. And the grading process is not provided. Therefore, we try to provide the theoretical basis of dividing the difficulty grades by means of probability of the patterns. We hope we can make some complements to this ancient game. Among the complicated influential factors, we sum up three key elements: the number of answers，the length of side, the basic figures. Based on these points, we have devised three plans, the procedural thinking of computer, the length-of-side principle, division principle, to calculate the probability. Then, we made comprehensive analysis on the probability. Finally, we got the difficulty grade of some classical tangram patterns. Moreover, we summarize some skills of piecing together the patterns.

摘要： 七巧板(Tangram)是中国最古老的益智游戏之一，它所能组成的图案千变万

化. 作为益智游戏，理应有从简到繁、由浅入深的过程.而如今市面上的七巧板并没有经过难度分级，只有与其相似的日本智力板[1]有难度分级图，但其等级划分并不是很合理，且没有展示其分级的过程.因此，我们试图通过研究图案拼凑正确的概率，来为部分经典的七巧图案的进行难度分级，以完善这个古老而青春的游戏.考虑到图案拼凑过程中受到影响因素较复杂，我们抽象概括出三个关键性的因素：图案正确解答的数目、边长和基本形状.从这三个不同的角度，分别以纯计算机"思维"、边长、分割图形（基本图形）为基本原则，设定了对应的三种计算概率的方案.通过对得到的概率进行综合分析，得出了部分经典的七巧图案的难度分级及七巧板图案难度的一般规律.通过对一系列图案的研究，我们总结出拼凑七巧板图案的技巧.

Key words: Tangram Piecing together Patterns Probability Difficulty Grade

Key words' explanation

Tangram（七巧板）：One of the traditional puzzles in China

Pattern（图案）：The graphics consist of all 7 pieces of tangram. Figures are parts of patterns.

Piecing together（拼凑）：The process of putting up tangram pieces into patterns

1. use probability to grade the patterns of tangram

Tangram is one of the most ancient puzzle games in China and it is widely spread around the world. We are into the various patterns of tangram and indulge in the endless fancy. When we grow up, urged by rational thoughts, we raise the questions: can we make use of our knowledge about math to show the difficulty levels about the tangram? Can we summarize the skills of piecing together the patterns?

In order to solve these problems, we have lots of consideration and discussion.

Linking to what we have learnt in class about algorithm, probability and statistics, we educe that the difficulty levels of tangram patterns should be transformed into the probability of scrabbling the patterns up correctly. With a view to researching step by step, we plan to add visual thinking manners, which is unique to human brains, to our plans gradually. Therefore we decided on three plans to calculate the probability according to the three elements: the procedural thought of computer, the length of side, the basic figures. These three plans are set in light of different angles, different ways of thinking and different influencing factors.

Plan 1: the procedural thinking of computer

The plan simulates the procedural thinking of a computer without considering any effect of human brains. We abstract the process of piecing as an algorithm, and consider every pattern to be seven blanks, without taking the shape of the patterns or the length of the sides into account. Each piece of the tangram is numbered according to different shapes (Illustration 1.1) and every figure is numbered according to different angles (Illustration 1.2). (This mark will also apply to other parts of the essay.) Special note: to simplify the research process, the patterns we choose can be figured out by rotating through specific angles (45°, 90°, 135°, 180°). Patterns which rotate through unspecific angles can also be done by similar methods. We can get the answer to the pattern when all the figures and angles match what the procedure has set.

We use the following formula to calculate the probability of each step:

$$\text{the probability of each step} = \frac{\text{The number of correct ways of piecing}}{\text{The number of possible ways of piecing}}$$

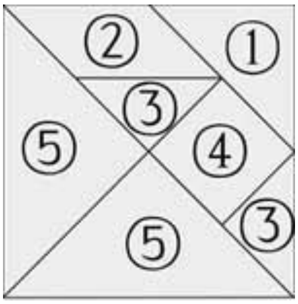


Illustration 1.1
Numbers of pieces

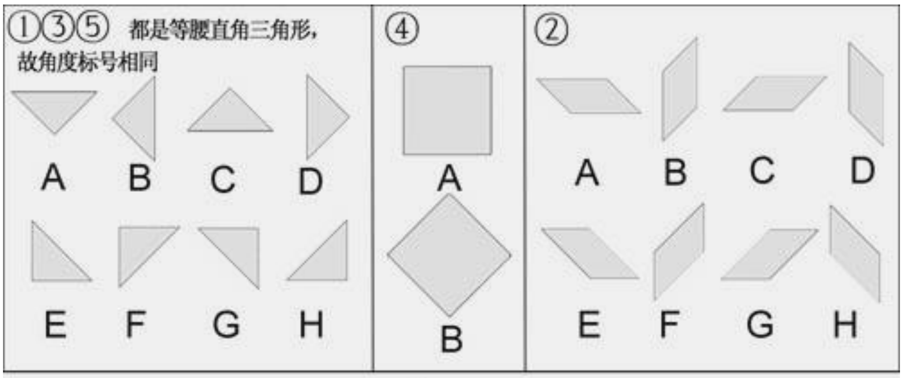


Illustration 1.2
Numbers of angles

1. Example Pattern One: the calculation of Square’s probability

Answer 1 : ① G+ ② A+ ③ A+ ④ B+ ③ B+ ⑤ D+ ⑤ C

(Illustration1.3)

(The order which is from top to bottom, from right to left, from outside to inside is set down.)

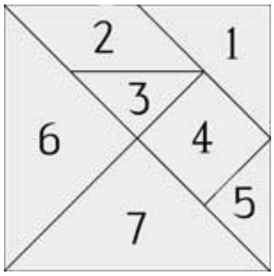


Illustration1.3
One of the correct answers of square

Process:

Step 1: put ①G to blank 1;

$$P(1)=\frac{1}{C_7^1 \cdot C_8^1}$$

(Notes: C_7^1 is the number of ways of choosing any single piece from all, C_8^1 is the number of ways of choosing any single angle from all, after choosing piece ①. ‘1’ is the only answer for blank one. Choosing pieces and angles are independent events. So if we set choosing pieces as event A and choosing angles as event B, the product of P(A) and P(B) equals to the probability of event A and B happen simultaneously, that is P(AB)=P(A)P(B).)

Step 2: put ②A to blank 2;

$$P(2)=\frac{1}{C_6^1 \cdot C_8^1}$$

Step 3: put ③A to blank 3;

$$P(3)=\frac{C_2^1}{C_5^1 \cdot C_8^1}$$

Step 4: put ④B to blank 4;

$$P(4)=\frac{1}{C_4^1 \cdot C_2^1}$$

Step 5: put ③B to blank 5;

$$P(5)=\frac{1}{C_3^1 \cdot C_8^1}$$

Step 6: put ⑤D to blank 6;

$$P(6)=\frac{C_2^1}{C_2^1 \cdot C_8^1}$$

Step 7: put ⑤C to blank 7;

$$P(7)=\frac{1}{C_1^1 \cdot C_8^1}$$

To sum up: the probability of square: $P_0 = P(1)\times P(2)\times P(3)\times P(4)\times P(5)\times P(6)\times P(7) = \frac{1}{660602880} \approx 1.51\times10^{-9}$

As square is a central symmetry and axial symmetry geometric figure so there are 8 correct answers to it. (Illustration 1.4) .

Therefore, the probability of square is $P=P_0\times C_8^1=\frac{1}{82575360}\approx 1.21\times10^{-8}$

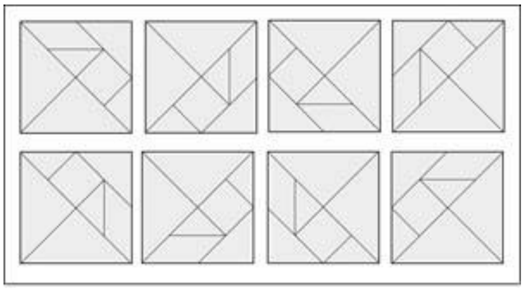


Illustration1.4
All correct answers of square

2. Example Pattern Two: the calculation of Hollow Square's probability

Answer 1: ⑤ G+ ② E+ ③ E+ ④ A+ ⑤ E+ ① A+ ③ H (Illustration1.5)

Since the process of calculation is similar to the ones above, we remove the details to appendix 1.1.

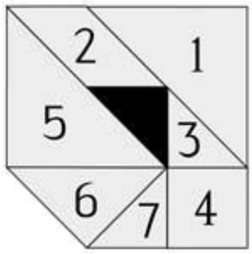


Illustration1.5
One of the correct answers of Hollow Square

The probability of Hollow Square is $P_0 = P(1) \times P(2) \times P(3) \times P(4) \times P(5) \times P(6) \times P(7) = \frac{1}{660602880} \approx 1.51 \times 10^{-9}$

As Hollow Square is an axial symmetry geometric figure so there are 8 correct answers to it. (Illustration 1.6) So, the probability of Hollow Square is $P = P_0 \times C_8^1 = \frac{1}{82575360} \approx 1.21 \times 10^{-8}$

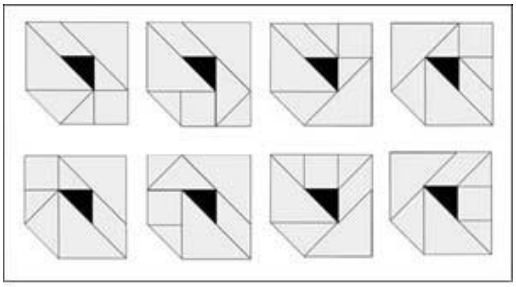


Illustration1.6
All correct answers of Hollow Square

3. Example Pattern Three: the calculation of probability of 'a kid with a hat'

Answer: ①F+④B+⑤D+②B+⑤H+③F+③E (Illustration1.7)

Since the process of calculation is similar to the ones above, we remove the details to appendix 1.2 (table 4).

As there is only one answer to 'a kid with a hat', its probability is $P = P_0 \times C_1^1 = \frac{1}{660602880} \approx 1.51 \times 10^{-9}$

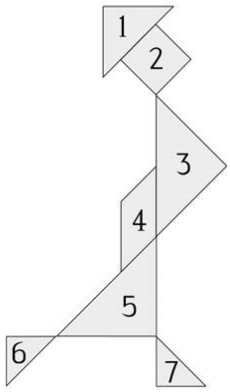


Illustration1.7
The correct answers of 'a kid with a hat'

We calculate the probability of 20 classic patterns according to this plan. Details will be shown later in 'Analyze and Sum-up'.

We come to the conclusion: In plan one, with the pure procedural thinking of computer, the probability of patterns with more answers is larger than that of those with fewer answers. That is to say, in this plan, the more answers the patterns have, the easier the patterns are.

From this process we find that the number of the correct answers to a pattern exerts effects on its difficulty level, which will be shown later. Obviously, this plan, which excludes all influences of human brain and uses the method of exhaustion to piece the patterns together like decoding，does not correspond with practical operation. So the probability calculated above can not precisely represent the difficulty of the patterns.

Plan 2: length-of-side principle

The plan translates two-dimensional area which is invariable to one-dimensional side. According to the ratio, we work out the length of the side. We take this side as the criterion of calculating probability, and exclude any other effects including the shape of tangram piece. There are only 4 sizes of side length of tangram pieces (the side length of the square in tangram pieces is set as 1). For example, there are 16 ways of piecing the length $2\sqrt{2}$. Given profile of the pattern, players are required to piece the pattern together according to the length of side.

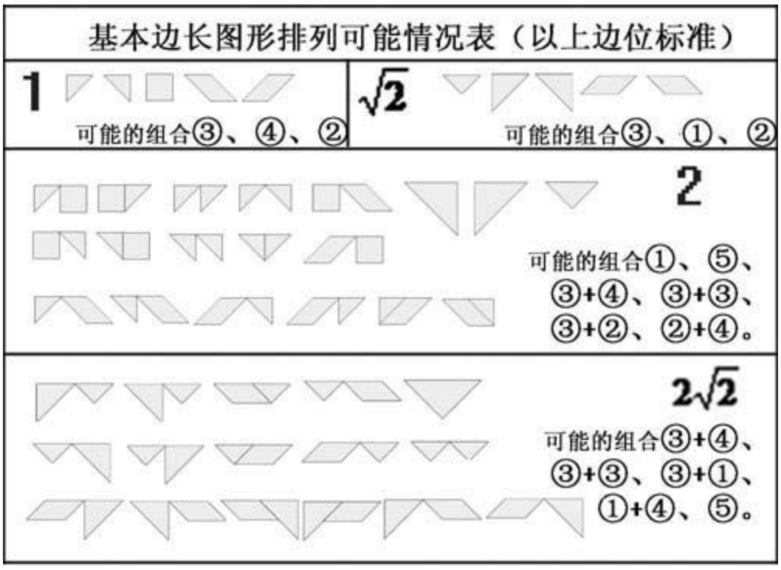


Illustration 2.1
The chart of the possible piecing ways of basic length of sides

As the option of the latter side is based on the former one, we should use conditional probability in calculation. So the pieces which have been chosen should be eliminated. In addition, choosing the side in a different order, we will get different probability of the patterns. In order to standardize the procedure of calculation, we fix a choosing order: from top to bottom, from right to left, from outside to inside.

We use the following formula to calculate the probability of each step:

$$\text{the probability of each step} = \frac{\text{The number of correct pieces combination}}{\text{The number of possible pieces combination}}$$

The correct answers of step 1, step 2⋯step n are set down as event A_1, A_2⋯A_n respectively. According to the formula of conditional probability: $P(A|B)=\frac{P(AB)}{P(B)}$, we know that $P(AB)=P(A|B)\cdot P(B)$. So, in accordance with a correct way of piecing, the probability of piecing a pattern together is:

$$P_X = P(A_1A_2A_3\cdots A_N)$$
$$=P(A_1A_2A_3\cdots A_{N-1})\cdot P(A_N|A_1A_2A_3\cdots A_{N-1})$$
$$=P(A_1A_2A_3\cdots A_{N-2})\cdot P(A_{N-1}|A_1A_2A_3\cdots A_{N-2})\cdot P(A_N|A_1A_2A_3\cdots A_{N-1})$$
$$\vdots$$
$$=P(A_1)\cdot P(A_2|A_1)P(A_3|A_1A_2)\cdots P(A_N|A_1A_2A_3\cdots A_{N-1})$$

Because there are many correct answers to patterns, we add up all the probability of correct piecing to represent the final one, that is $P_0 = P_1+P_2+P_3+\cdots+P_x$

1. Example Pattern One: the calculation of Square's probability

Answer 1: ①G+②A+③A+④B+③B+⑤D+⑤C（Illustration 2.2)

Process:

The side length of the square in tangram is set as 1, so its

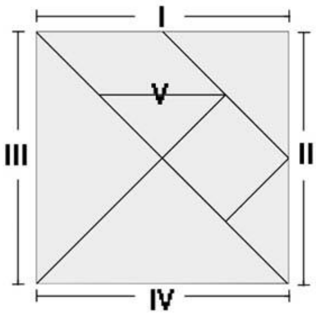


Illustration2.2
One of the square's choosing order

area is 1and the total area of the tangram is 8，then we can work out that the length of square is $2\sqrt{2}$.

Step 1: fill up side I of which length is $2\sqrt{2}$, the possible ways are shown in illustration 2.1，as for answer 1, ①G+②A is the only one that matches the side. So the probability is P(I)= $P(A_1)=\frac{1}{C_{16}^{1}}$, and the profile goes to image ㈠ in illustration 2.3.

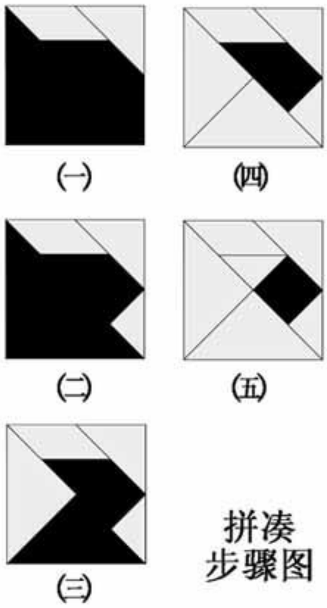


Illustration2.3
Piecing procedure
of square

Step 2: fill up side II of which length is $\sqrt{2}$. As for answer 1, ①G+②A has been chosen in step 1, ③B is the only one that matches the side. So the probability is P(II)= $P(A_2|A_1)=\frac{1}{C_1^1}$, and the profile goes to image ㈡ in illustration 2.3.

Step 3: fill up side III of which length is $2\sqrt{2}$. As for answer 1, ⑤D is the only one that matches the side. There are two pieces of ⑤, so the probability is P(III) $=P(A_3|A_1A_2)=\frac{2}{C_2^1}$, and the profile goes to image ㈢ in illustration 2.3.

Step 4: fill up side IV of which length is $2\sqrt{2}$. As for answer 1, ⑤D is the only one that matches the side and there is only one piece of ⑤ left, so the probability is P(IV) $=P(A_4|A_1A_2A_3)=\frac{1}{C_1^1}$, and the profile goes to image ㈣ in illustration 2.3.

Step 5: fill up side V of which length is 1. ④A and ③A are the possible answers. As for answer 1, ③A is the only one that matches the side, so the probability is P(V) $= P\left(A_5 \middle| A_1 A_2 \cdots A_4\right) = \frac{1}{C_2^1}$, and the profile goes to image ㈤ in illustration 2.3.

Step 6: Use ④B to fill the last blank. The probability is P(VI) $= P\left(A_6 \middle| A_1 A_2 \cdots A_5\right) = \frac{1}{C_1^1}$

To sum up: the probability of square's answer 1 is : $P_1 = P(I) \times P(II) \times P(III) \times P(IV) \times P(V) \times P(VI) = \frac{1}{32} \approx 3.13 \times 10^{-2}$

Likewise,

Answer 2： ⑤A+⑤B+③D+④B+③C+②A+①G (Illustration 2.4)

Since the process of calculation is similar to the ones above, we remove the details to appendix 1.3.

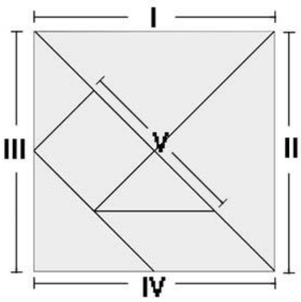


Illustration2.4
One of the square's choosing order

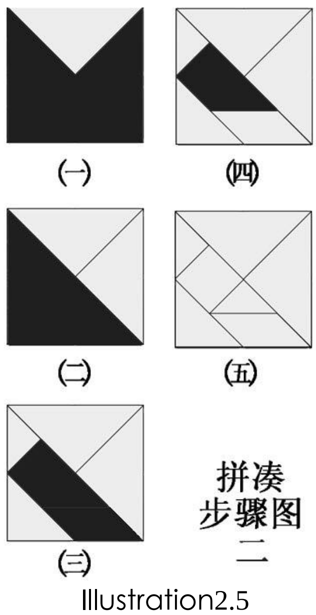


Illustration2.5
Piecing procedure of square II

Square has 8 correct answers; others are available in Illustration 2.6.

According to the calculation principles, we work out other probability:

$P_1=\frac{1}{32}\approx 3.13\times 10^{-2}$

$P_2=\frac{1}{46080}\approx 2.17\times 10^{-5}$；

$P_3=\frac{1}{512}\approx 1.95\times 10^{-3}$；

$P_4=\frac{1}{11520}\approx 8.68\times 10^{-5}$；

$P_5=\frac{1}{144}\approx 6.94\times 10^{-3}$；

$P_6=\frac{1}{15360}\approx 6.51\times 10^{-5}$；

$P_7=\frac{1}{720}\approx 1.39\times 10^{-3}$；

$P_8=\frac{1}{32}\approx 3.13\times 10^{-2}$.

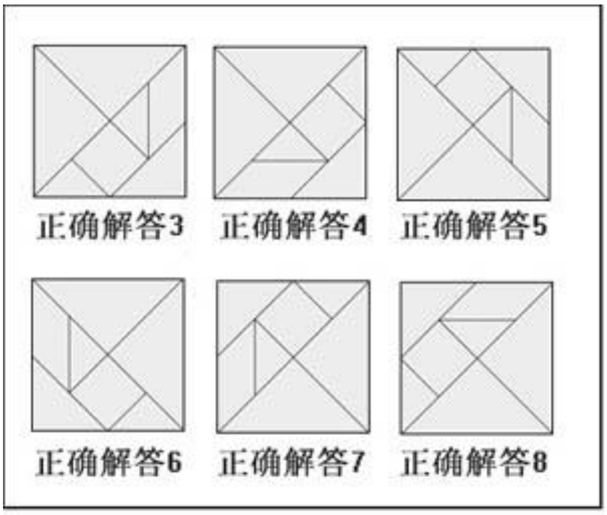


Illustration 2.6
Other correct answers of square

And the final probability of square is:

$$P_0=P_1+P_2+P_3+P_4+P_5+P_6+P_7+P_8=\frac{1681}{23040}\approx 0.07296$$

2. Example Pattern Two: the calculation of Hollow Square's probability

Answer 1 : ⑤G+②E+④A +⑤E+①A +③H+③E (Illustration 2.6)

Process:

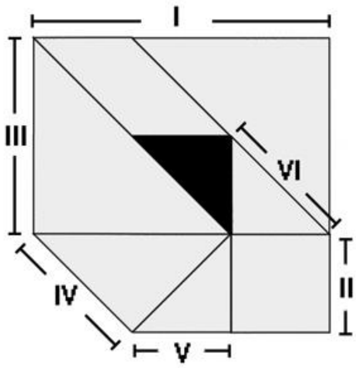


Illustration 2.6
One of the hollow square's choosing order

Step 1: fill up side I of which length is 3, the possible answers are ①A＋③G、①A＋③F、③G＋①A、①A＋④A、④A＋①A、②A＋①E、②A+①G、①A +②G、①A +②E、②E+③G+③F、②E+③G+③G、②G+③F+③F、②G+③G+③G、②G+③G+③F、②G+③F+③G、③F+③F+②E、③F+③F+②G、③G+③G+②E、③F+③G+②E、③G+③F+②E、③G+③F+②G、③G+②E+③G、③F＋②E+③G、③F+②G+③F、③F+②G+③G、③F+④A+③F、③F+④A+③G、③G+④A+③G、③G+④A+③F、③F +③F+④A、③F +③G+④A、③G +③F+④A、③G +③G+④A、④A+③F+③F、④A+③F+③G、④A+③G+③G、④A+③G+③F、⑤

G+④A、⑤F+④A、④A+⑤G、④A+⑤F、⑤G+②E、⑤F+②E、⑤F+②G、②E+⑤G、②G+⑤G、②G+⑤F、⑤G+③G、⑤F+③F、⑤G+③F、⑤F+③G、③F+⑤F、③G+⑤G、③F+⑤G、③G+⑤F、②G+④A+③G、②G+④A+③F、②E+③G+④A、②G+③G+④A、②G +③F+④A、④A+②E+③G、④A +③G+②E、④A+③F+②E、④A +③F+②G、③F+②G +④A、③F+④A+②E、③G+④A+②E (68 ways in all). But as for answer 1, ②E+⑤G is the only one that matches the side, so the probability is P(I)= $P(A_1)=\frac{1}{C_{68}^{1}}$ and the profile goes to image ㈠ in illustration 2.7.

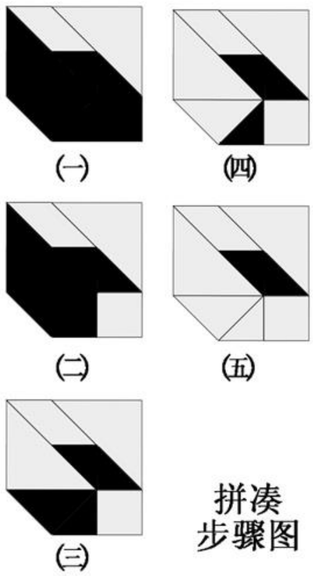


Illustration2.7
Piecing procedure of Hollow Square

Step 2: fill up side II of which length is 1. ③B and ④A are possible answers , but as for answer 1, ④A is the only one that matches, so the probability is P(II)= $P(A_2|A_1)=\frac{1}{C_2^1}$, and the profile goes to image ㈡ in illustration 2.7.

Step 3: fill up side III of which length is 2. ⑤E、⑤F、①D、③E + ③E、③F + ③F、③E + ③F and ③F + ③E are possible answers , but as for answer 1, ⑤E is the only one that matches, so the probability is P(III) = $P(A_3|A_1A_2)=\frac{1}{C_7^1}$, and the profile goes to image ㈢ in illustration 2.7.

Step 4: fill up side IV of which length is $\sqrt{2}$. ①A and ③G are possible answers , but as for answer 1, ①A is the only one that matches, so the probability is P(IV) = $P(A_4|A_1A_2A_3)=\frac{1}{C_2^1}$, and the profile goes to image ㈣ in illustration 2.7.

Step 5: fill up side IV of which length is1. ③H and ③E are possible answers , but as for answer 1, ③H is the only one that matches, so the probability is P(V) $= P\left(A_5 \middle| A_1 A_2 \cdots A_4\right) = \frac{1}{C_2^1}$, and the profile goes to image ㈤ in illustration 2.7.

Step 6: Use ③E to fill the last blank. The probability is P(VI) = $\frac{1}{C_1^1}$

To sum up: the probability of Hollow Square's answer1 is: $P_1 = P(I) \times P(II) \times P(III) \times P(IV) \times P(V) \times P(VI) = \frac{1}{3808} \approx 2.63 \times 10^{-4}$

Likewise,

Answer 2: ⑤G+③E+④A+②E+③F+①B+⑤E (Illustration 2.8)

Since the process of calculation is similar to the ones above, we remove the details to appendix 1.4.

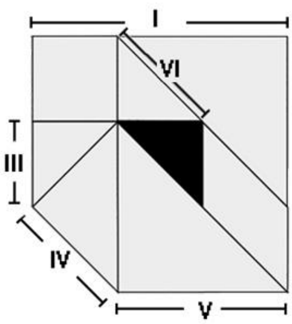


Illustration 2.8
One of the Hollow Square's choosing order

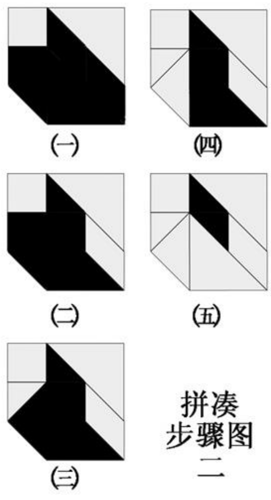


Illustration2.9
Piecing procedure of Hollow Square II

Hollow Square has 8 correct answers; others are available in Illustration 2.10.

According to the calculation principles, we work out other probability:

$P_1 = \frac{1}{3808} \approx 2.63 \times 10^{-4}$;

$P_2 = \frac{1}{3264} \approx 3.06\times10^{-4}$；

$P_3 = \frac{1}{25704} \approx 3.89\times10^{-5}$；

$P_4 = \frac{1}{3672} \approx 2.72\times10^{-4}$；

$P_5 = \frac{1}{10608} \approx 9.43\times10^{-5}$；

$P_6 = \frac{1}{18360} \approx 5.45\times10^{-5}$；

$P_7 = \frac{1}{4080} \approx 2.45\times10^{-4}$；

$P_8 = \frac{1}{8160} \approx 1.23\times10^{-4}$；

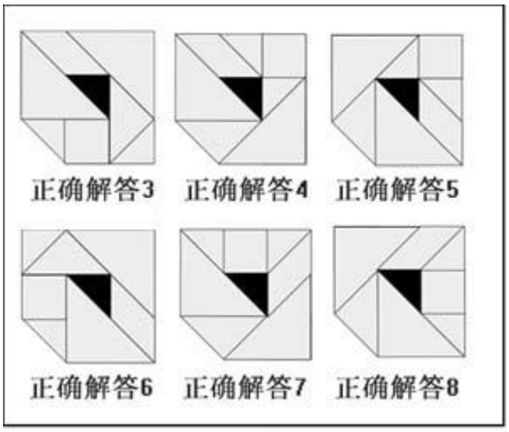


Illustration 2.10
Other correct answers of
Hollow Square

And the final probability of Hollow Square is : $P_0 = P_1+P_2+P_3+P_4+P_5+P_6+P_7+P_8 \approx 1.40\times10^{-3}$

3. Example Pattern Three: the calculation of probability of ‘a kid with a hat’

Answer: ①F+④B+⑤D+②B+⑤H+③F+③E（Illustration 2.11）

Since the process of calculation is similar to the ones above, we remove the details to appendix 1.5.

The probability of ‘a kid with a hat’ is P_0= P(I)×P(II)×P(III)×P(IV)×P(V)×P(VI) $= \frac{1}{8640} \approx 1.16\times10^{-4}$

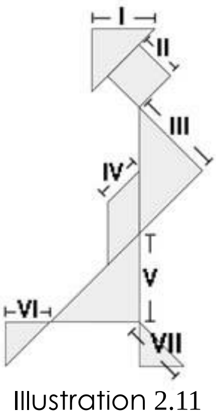


Illustration 2.11
Choosing order of ‘a kid with a hat’

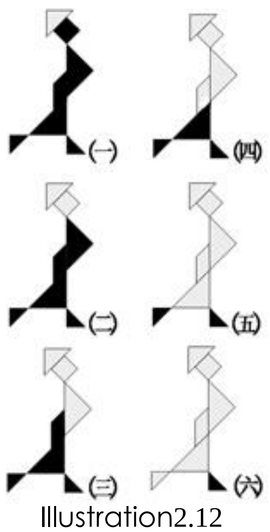


Illustration2.12
Piecing procedure of ‘a kid with a hat’

We calculate the probability of 20 classic patterns according to this plan. Details will be shown in later discussion in ‘Analyze and Sum-up’ (table 4).

We come to the conclusion: in plan two—length-of-side principle, the probability of regular patterns is larger, which means they are easier, for with this principle, the more regular the patterns are, the more cases will be eliminated.

However, this principle hasn’t taken all kinds of thoughts of human brains into consideration, so the probability calculated above can not precisely represent the difficulty of the patterns.

Plan 3: Division principle (basic-figure principle)

During practical operation, we find out that, facing a new pattern, players will, firstly, come to think about what basic figures the pattern consist of, and then carry some experiment. Therefore we design a third plan to highly simulate human thinking.

Using equivalent transformation, we translate piecing-up to dividing. That is to say, we transform piecing process to basic figure chart and standardize the dividing process according to the principle we set. Players divide the pattern entirely into several basic figures or use the basic figures to piece together. Every step of dividing should skive the pattern into two parts, and at least one part contains no more than three pieces, and it is possible for ordinary people to remember basic figures with no more than three pieces. Therefore we define that the basic figures are

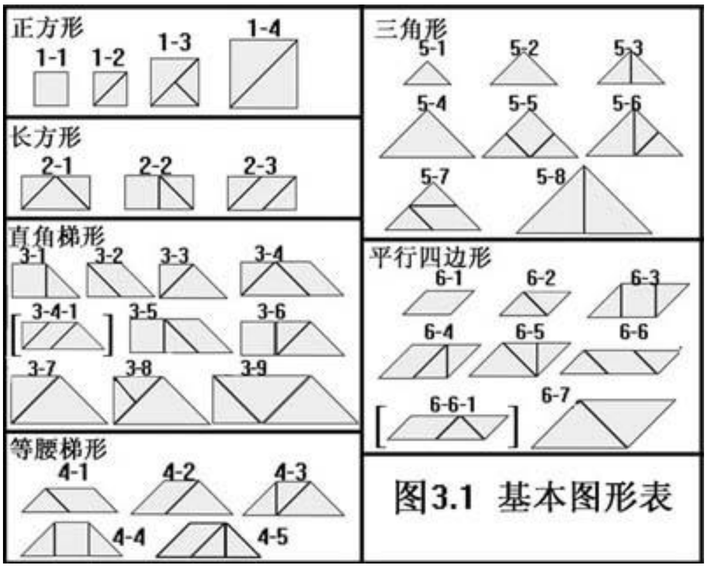


Illustration 3.1
The chart of basic figures

made up of no-more-than-three-piece regular figures including rectangle, triangle and so on (Illustration 3.1 is the chart of basic figures.)

Given profile of the pattern, players will fill in the pattern with basic figures until working out the correct answer.

In order to standardize the calculating procedure, we set following principles:

1. Dividing principle: each cut ought to cut off at least one basic figure; otherwise, it should not be considered a way of dividing.

2. Selecting figures principle: after each dividing, we choose basic figures from the chart only in light of the size and the sort of piece, without considering the symmetry. For example, figure 3-4 and figure 3-4-1 are of the same size and of the same pieces, but they are partly symmetrical. So we do not distinguish these two figures during the calculation.

3. No repetition principle: after consulting the chart, basic figures which include piece that has been used should be eliminated automatically.

4. Priority principle: study one part cut off at a time. Priorities are assigned according to how small the measurement is. If two parts are of the same size, we should discuss them respectively.

5. Cut-off principle: the first step of dividing is cutting off figures without common side, and we define that the probability of this step is one.

6. Possibility principle: as there are only 4 sizes of side length of tangram pieces (1、$\sqrt{2}$、2、$2\sqrt{2}$) and the combinations of sides are finite, we ignore those impossible combinations and therefore do not list them as possible dividing.(For example: side of $\sqrt{2}$ can not be divide into $\sqrt{2}-1$and1, so this dividing is eliminated)

Besides, we use the following formula to calculate the probability of the correct piecing in each step. (A= the number of correctly selected basic figures; B= the number of probably ways of cutting; C= the number of probably selected basic figures)

$$\text{Every step's probability} = \frac{A}{B \times C}$$

1. Example Pattern One: the calculation of Square’s probability

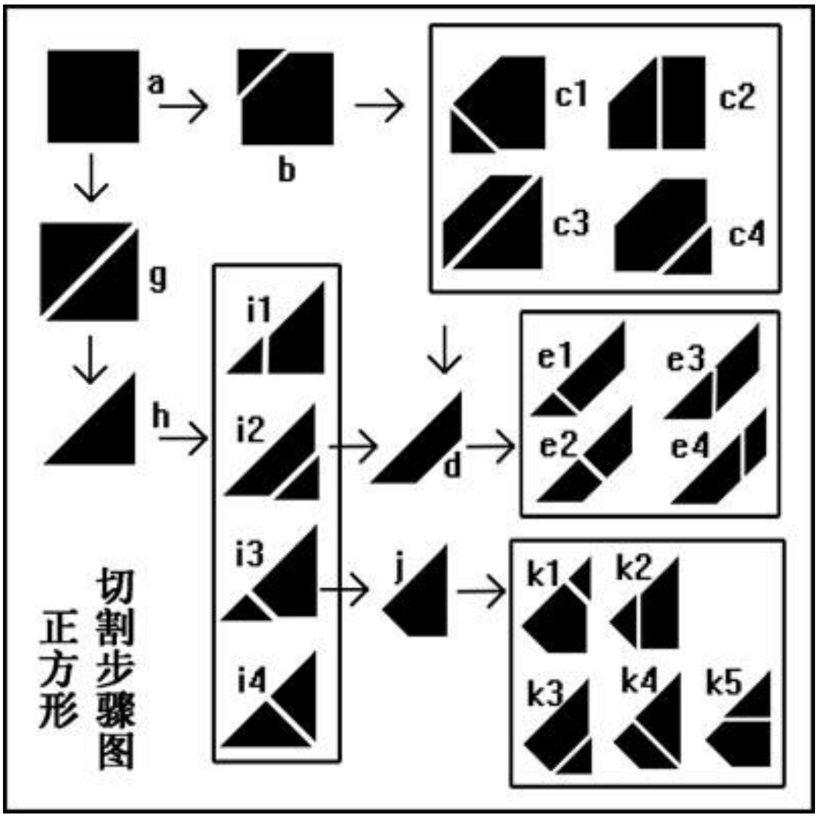


Illustration 3.2
The cutting procedure of square

Process: (the following marks are all from Illustration 3.2)

Starting from a, there are altogether 7 pieces of tangram for us to choose from. Considering that each side of square is $2\sqrt{2}$, we refer to the chart of basic figures, and find it can be cut in two ways, b and g. Because both these two cuttings are correct, we need to discuss them respectively.

Case 1: b is selected.

Step 1: The probability of selecting b is $\frac{1}{2}$. After selecting b, the triangle cut off is a basic figure, so we consult the chart and find out it can be pieced in two ways, 5-2 and 5-3, in which only basic figure 5-2 is the correct way of piecing.

Then the probability of selecting 5-2 is $\frac{1}{2}$.

So in this step, we have P（a-b）$=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$。

Step 2: After cutting off basic figure 5-2, we have only 6 pieces of tangram left. Then it can be cut in four ways, which are c1, c2, c3 and c4. And c3 is the correct one. So the probability of selecting c3 is $\frac{1}{4}$.

So in this step, we have P（b-c）$=\frac{1}{4}$。

Step 3: Basic figure 5-8 is the only way of piecing of the big triangle cut off in c3, So we have P（c-d）=1。

Step 4: When it comes to d, there are only 4 pieces of tangram left. And can be cut in four ways, e1, e2, e3 and e4, among which e1, e2 and e4 are the correct ones. e1 consists of 5-1 and 3-5, so the probability is 1; e2 consists of 3-1 and 3-2, the probability is 1; the parallelogram cut off of e4 can be pieced in two ways, basic figure 6-1 and 6-2, in which 6-1 is the correct one, so the probability is $\frac{1}{2}$, and the remaining trapezoid is basic figure 4-4, whose selecting probability is 1.

So in this step, we have P（d-e）$=\frac{1}{4}\times1+\frac{1}{4}\times1+\frac{1}{4}\times\frac{1}{2}=\frac{5}{8}$

Thus, we have finished the cutting procedure of case 1 (a-b-c-d-e), and we calculate that: P（a-b-c-d-e）= P（a-b）×P（b-c）×P（c-d）×P（d-e）$=\frac{5}{128}$

Case 2: g is selected.

Step 1: The probability of selecting g is $\frac{1}{2}$. After selecting g, the triangle cut off is a basic figure, so we consult the chart and find out only one way of piecing, 5-8, whose selecting probability is 1.

So P（a-g）$=\frac{1}{2}$。

Step 2: After basic figure 5-8 is cut off from g, there are 5 pieces of tangram left, then we have h. h can be cut in four ways, i1, i2, i3 and i4, among which i2 and i3 are the correct ones.

Case 2 – 1: i2 is selected.

The probability of selecting i2 is $\frac{1}{4}$. The triangle cut off in i2 is a basic figure, so we consult the chart and find it can be pieced in two ways, 5-2 and 5-3, in which only 5-2 is correct, and the probability is $\frac{1}{2}$. Then we have d.

So P（g-h-i2-d）$=\frac{1}{4}\times\frac{1}{2}=\frac{1}{8}$, thus we know P（d-e）$=\frac{1}{4}\times 1+\frac{1}{4}\times 1+\frac{1}{4}\times\frac{1}{2}=\frac{5}{8}$.

Then P（a-g-h-i2-d-e）=P（a-g）× P（g-h-i2-d）×P（d-e）$=\frac{1}{2}\times\frac{1}{8}\times\frac{5}{8}=\frac{5}{128}$,

Case 2 – 1 (a-g-h-i2-d-e) is finished.

Case 2 – 2: i3 is selected.

The probability of selecting i3 is $\frac{1}{4}$. After cutting off the small triangle, it only consists of basic figure 5-1, so its probability is 1. Then we have j.

So P（i3-j）=1

When it comes to j, there are only 4 pieces of tangram left. j can be cut in five ways, k1, k2, k3, k4 and k5. The only correct one is k3, so the probability of selecting k3 is $\frac{1}{5}$. k3 consists of only basic figure 5-2 and 3-5, so the probability is 1.

We have P（j-k）$=\frac{1}{5}\times 1=\frac{1}{5}$

Then P（a-g-h-i3-j-k）=P（a-g）× P（g-h-i3）×P（i3-j）×P（j-k）$=\frac{1}{2}\times\frac{1}{4}\times\frac{1}{5}=\frac{1}{40}$,

Case 2 – 2 (a-g-h-i3-j-k) is finished.

Thus the cutting procedure of square is finished, and we have:

P = P（a-b-c-d-e）+P（a-g-h-i2-d-e）+P（a-g-h-i3-j-k）$=\frac{33}{320}$

2. Example Pattern Two: the calculation of Hollow Square’s probability

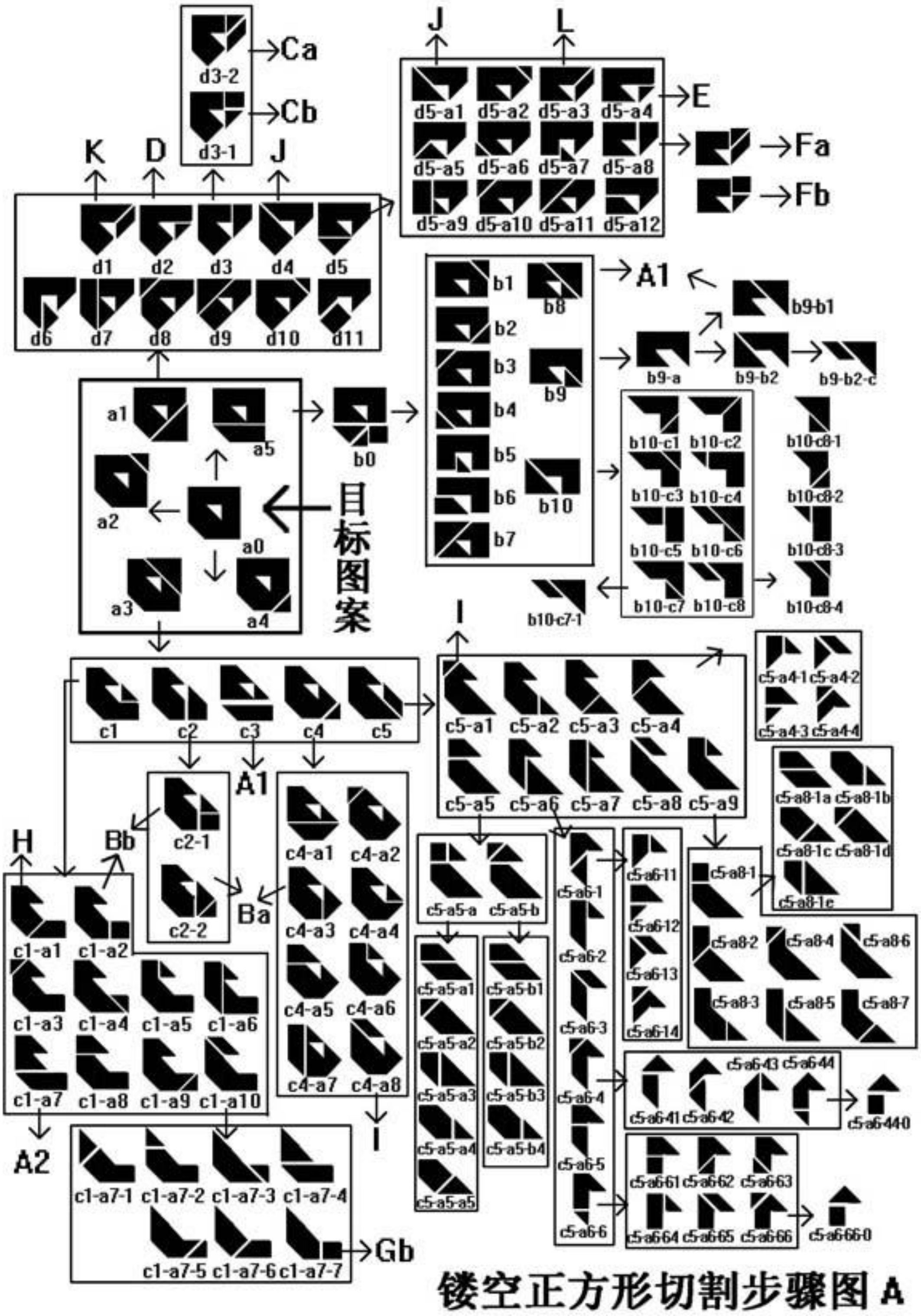


Illustration 3.3
Chart A of the cutting
procedure of Hollow Square

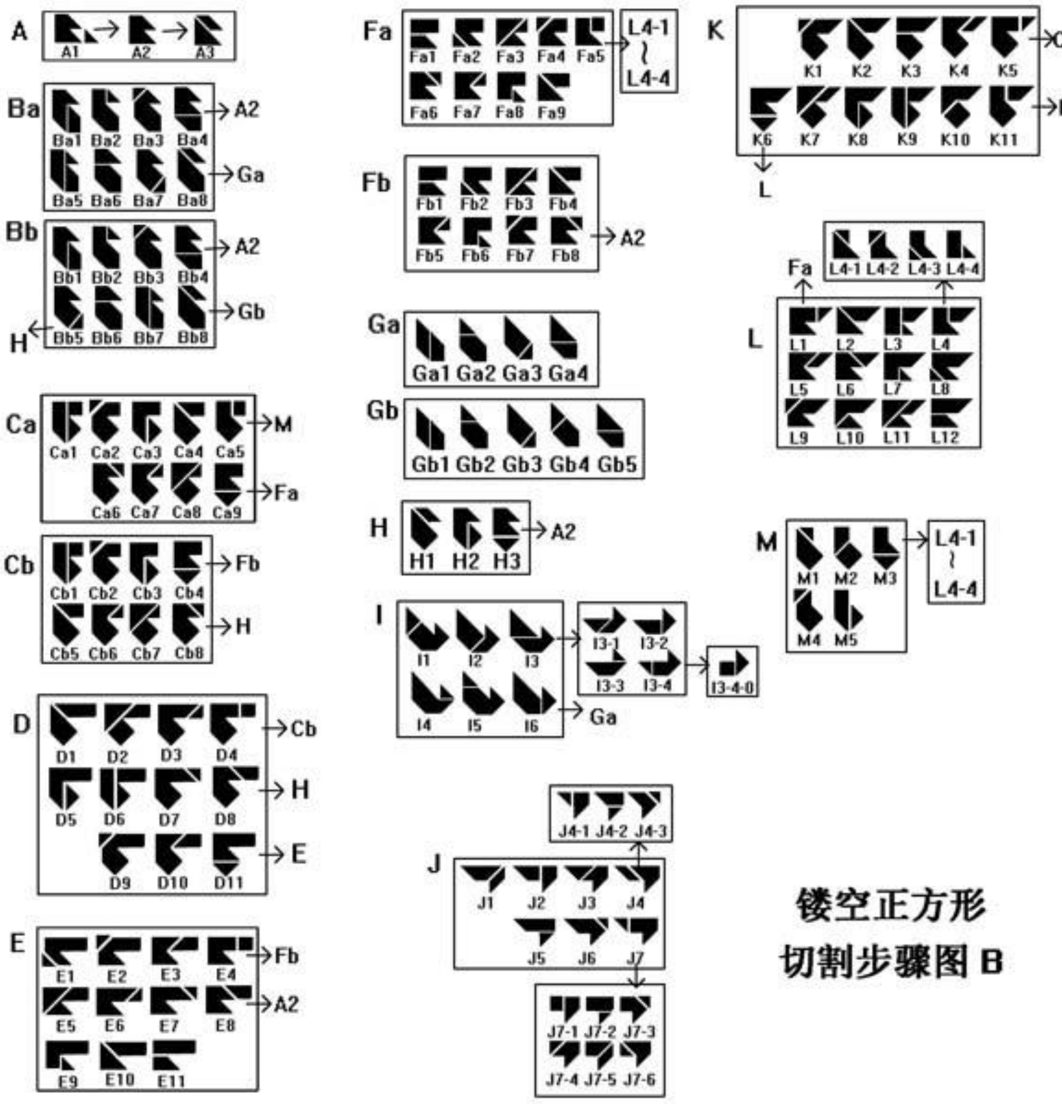


Illustration 3.4
Chart B of the cutting procedure
of Hollow Square

Process:

(The following marks are all from Illustration 3.3 and Illustration 3.4)

Starting from a0, there are altogether 7 pieces of tangram for us to choose from. We refer to the chart of basic figures, and find it can be cut in five ways, a1, a2, a3, a4 and a5, among which a1, a3 and a5 are correct, so we need to discuss them respectively.

Case 1

Step 1: The probability of selecting a1 is $\frac{1}{5}$. After selecting b, the triangle cut off is a basic figure, so we consult the chart and find it can be pieced in four ways, 5-4, 5-5, 5-6 and 5-7 among which only basic figure 5-4 is the correct way of piecing. Then the probability of selecting 5-4 is $\frac{1}{4}$. Then we have d

So in this step, we have P（a0-a1-d）$=\frac{1}{5}\times\frac{1}{4}=\frac{1}{20}$

Step 2: After cutting off basic figure 5-4, we have only 6 pieces of tangram left. Then it can be cut in eleven ways, that is d1~d11. And d1, d2, d3, d4 and d5 are the correct ones.

Case 1 – 1: d1 is selected.

The probability of selecting d1 is $\frac{1}{11}$. The parallelogram cut off in d1 is a basic figure, so we consult the chart and find it can be pieced in two ways, 6-1 and 6-2, in which only 6-1 is correct, and the probability is $\frac{1}{2}$. Then we have k.

So P（d-d1-k）$=\frac{1}{11}\times\frac{1}{2}=\frac{1}{22}$

We find k can be cut in eleven ways, which are k1~k11, among which k2, k5, k6 and k11 are correct, so we need to discuss them respectively.

The probability of selecting k2 is $\frac{1}{11}$. The trapezoid cut off in k2 can be pieced in two ways, 4-3 and 4-4, in which only 4-4 is correct, and the probability is $\frac{1}{2}$. The big remaining trapezoid can only be pieced in one way, 3-7, so its probability of being correctly pieced is1.

Then P（k-k2）$=\frac{1}{11}\times\frac{1}{2}=\frac{1}{22}$

Thus, case (a0-a1-d-d1-k-k2) is finished,

P(a0-a1-d-d1-k-k2~)=P(a0-a1-d)×P(d-d1-k)×P(k-k2~)$=\frac{1}{20}\times\frac{1}{22}\times\frac{1}{22}=\frac{1}{9680}$

The probability of selecting k5 is $\frac{1}{11}$. The triangle cut off in k5 can only be pieced in one way, 5-1, so the probability is1. The remaining part is Ca, which can be cut in nine ways, Ca-1~Ca-9, among which Ca-4, Ca-5 and Ca-9 are correct.

The probability of selecting Ca-4 is $\frac{1}{9}$. The trapezoid cut off in Ca-4 can be pieced in two ways, 3-1 and 3-3, in which only 3-1 is correct, so the probability is $\frac{1}{2}$, and the probability of the remaining part is1.

The probability of selecting Ca-5 is $\frac{1}{9}$. The square cut off in Ca-5 can only be pieced in one way, 1-1, so the probability is1; the remaining part is M, which can be cut in five ways, M1~M5, among which M1 and M3 are correct. The probability of selecting M1 is $\frac{1}{5}$, and both two remaining parts are basic figures, so the probability is1. The probability of selecting M3 is $\frac{1}{5}$, and the probability of its being correctly cut into L1~L4 and pieced is $\frac{1}{4}$.

The probability of selecting Ca-9 is $\frac{1}{9}$. The square cut off in Ca-5 can only be pieced in one way, 5-2, so the probability is1; the remaining part is Fa, which can be cut in nine ways, Fa1~Fa9, among which Fa5 and Fa9 are correct. The probability of selecting Fa5 is $\frac{1}{9}$, and the probability of its being correctly cut into L1~L4 and pieced is $\frac{1}{4}$. The probability of selecting Fa9 is $\frac{1}{9}$, and the probability of its being correctly cut and pieced is1.

Thus, case (a0-a1-d-d1-k-k5) is finished. We have:

$$P(k\text{-}k5\sim)=\frac{1}{11}\times[\frac{1}{9}\times\frac{1}{2}\times1+\frac{1}{9}\times1\times(\frac{1}{5}\times1+\frac{1}{5}\times\frac{1}{4})+\frac{1}{9}\times1(\frac{1}{9}\times\frac{1}{4}+\frac{1}{9}\times1)]=\frac{8}{891}$$

Likewise, we calculated that:

$$P(k\text{-}k6\sim)=\frac{1}{11}\times[\frac{1}{12}\times1\times\frac{1}{4}+\frac{1}{12}\times1\times(\frac{1}{9}\times1+\frac{1}{9}\times\frac{1}{4})+\frac{1}{12}]=\frac{7}{432}$$

$$P(k\text{-}k11\sim)=\frac{1}{11}\times\frac{1}{2}\times(\frac{1}{5}\times1+\frac{1}{5}\times1\times\frac{1}{4})=\frac{1}{88}$$

To sum up:

$$P(a0\text{-}a1\text{-}d\text{-}d1\text{-}k\sim)=P(a0\text{-}a1\text{-}d)\times P(d\text{-}d1\text{-}k)\times[(P(k\text{-}k2\sim)+P(k\text{-}k5\sim)+P(k\text{-}k6\sim)+P(k\text{-}k11\sim)]$$

$$=\frac{1}{20}\times\frac{1}{22}\times(\frac{1}{22}+\frac{8}{891}+\frac{7}{432}+\frac{1}{88})=1.863649\times10^{-4}$$

Likewise, we have:

Case 1 – 2 (a0-a1-d-d2-D~) : d2 is selected.

(We use the same arithmetic to calculate, so we omit parts of analyzing)

So we have:

P（a0-a1-d-d2-D）$=\frac{1}{20}\times\frac{1}{11}=\frac{1}{220}$

P（D-D1~）$=\frac{1}{11}\times\frac{1}{3}\times 1=\frac{1}{33}$

P（D-D3~）$=\frac{1}{11}\times[\frac{1}{11}\times 1+\frac{1}{11}\times 1\times(\frac{1}{8}+\frac{1}{8})+\frac{1}{11}\times\frac{1}{2}\times 1]=\frac{7}{484}$

P（D-D4~）$=\frac{1}{11}\times 1\times[\frac{1}{8}\times\frac{1}{2}\times 1+\frac{1}{8}\times 1\times(\frac{1}{8}+\frac{1}{8})+\frac{1}{8}\times 1\times\frac{2}{3}]=\frac{17}{1056}$

P（D-D8~）$=\frac{1}{11}\times\frac{1}{2}\times\frac{2}{3}=\frac{1}{33}$

P（a0-a1-d-d2-D~）= P（a0-a1-d-d2-D）×[P（D-D1~）+P（D-D3~）+P（D-D4~）+P（D-D8~）]

$$=\frac{1}{220}\times(\frac{1}{33}+\frac{7}{484}+\frac{17}{1056}+\frac{1}{33})=\frac{353}{851840}$$

Because the calculating process is similar to those above, we remove the details of counting process to appendix i-Vi:

Likewise:

Case 1 – 3（a0-a1-d-d3~）: d3 is selected.

So we have:

$$P（a0\text{-}a1\text{-}d\text{-}d3\sim）=\frac{1}{220}\times(\frac{17}{288}+\frac{8}{243})=\frac{13}{31104}$$

Case 1 – 4（a0-a1-d-d4~）: d4 is selected.

So we have:

$$P（a0\text{-}a1\text{-}d\text{-}d4\sim）=\frac{1}{220}\times\frac{1}{2}\times(\frac{1}{14}+\frac{1}{7}+\frac{1}{21}+\frac{1}{7}+\frac{1}{21})=\frac{19}{18480}$$

Case 1 – 5（a0-a1-d-d5~）: d5 is selected.

So we have:

$$P（a0\text{-}a1\text{-}d\text{-}d5\sim）=\frac{1}{220}\times(\frac{19}{3024}+\frac{41}{20736}+\frac{7}{1056}+\frac{7}{864})=1.0450\times 10^{-4}$$

Thus, case 1 is finished. We have the probability of case 1:

P (a0-a1~) = P（a0-a1-d-d1 ~) +P (a0-a1-d-d2 ~) +P (a0-a1-d-d3~) +P (a0-a1-d-d4~)

+P（a0-a1-d-d5~）

= 2.15×10^{-3}

Case 2:

From a0 to c, we have P（a0-a3-c）$=\frac{1}{5}\times\frac{1}{4}=\frac{1}{20}$

Starting from c, altogether there are 5 cases.

Case 2 – 1（c-c1~）: c1 is selected.

So we have:

P（c-c1~）$=\frac{919}{50400}$

Case 2 – 2（c-c2~）: c2 is selected.

So we have:

P（c-c2~）$=\frac{139}{7200}$

Case 2 – 3（c-c3~）: c3 is selected.

So we have:

P（c-c3~）$=\frac{1}{5}\times1\times1=\frac{1}{5}$

Case 2 – 4（c-c4~）: c4 is selected.

So we have:

P（c-c4~）$=\frac{7}{1280}$

Case 2 – 5（c-c5~）: c5 is selected.

So we have:

P（c-c1~）$=\frac{8639}{680400}$

Thus, case 2 is finished. We have the probability of case 2:

P（a0-a3~）= P（a0-a3-c） [P（c-c1~）+P（c-c2~）+P（c-c3~）+P（c-c4~）+P（c-c5~）]

=0.1279

Likewise, we have the probability of case 3: P（a0-a5~）$=\frac{73}{4800}$

To sum up, the probability of correctly piecing of Hollow Square is:

P（a0~）=P（a0-a1~）+P（a0-a3~）+P（a0-a5~）=0.0301

3. Example Pattern Three: the calculation of probability of ‘a kid with a hat’

Process: (The following marks are all from Illustration 3.5)

Starting from a, there are altogether 7 pieces of tangram for us to choose from.

Step 1: We separate all the cut-offs. Then we get b. (this is based on principle 5)

Thus: P（a-b）=1

Step 2: Considering the area of the ‘kid’ s ‘feet’ is the smallest, we study the ‘feet’ first and discover that they consist of basic figure 5-1. So the probability of piecing the ‘feet’ together correctly is 1. Then we come to the ‘head’. Since there are only 5 pieces of tangram left, we find that c is the only way of cutting, and its probability is1. Now we get d.

Thus: P（b-c-d）=1

Step 3: There are 3 pieces of tangram left. So d can be cut in two ways, e1 and e2, and both are correct. We have to discuss them respectively.

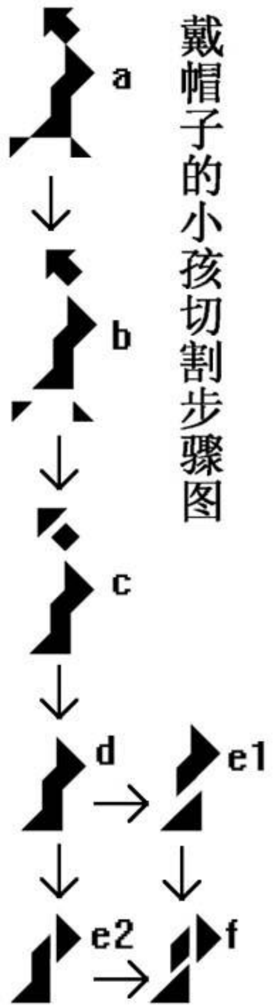


Illustration 3.5
The cutting procedure of ‘a kid with a hat’

Case 1: e1 is selected.

The probability of selecting e1 is $\frac{1}{2}$. The big triangle cut off in e1 can only be pieced in one way, 5-4, so the probability is1; the remaining part consists of 5-4 and 6-1, so f is the only way of cutting, the probability of which is1.

So we have: P（d-e1-f）= $\frac{1}{C_2^1 \cdot C_1^1}$ = $\frac{1}{2}$

Thus, case 1 (a-b-c-d-e1-f) is finished.

P（a-b-c-d-e1-f）=P（a-b）×P（b-c-d）×P（d-e1-f）=1×1×$\frac{1}{2}$=$\frac{1}{2}$；

Case 2: e2 is selected.

The probability of selecting e2 is $\frac{1}{2}$. The big triangle cut off in e1 can only be pieced in one way, 5-4, so the probability is1; the remaining part consists of 5-4 and 6-1, so f is the only way of cutting, the probability of which is1.

So we have: P（d-e2-f）= $\frac{1}{C_2^1 \cdot C_1^1}$ = $\frac{1}{2}$

Thus, case 2 (a-b-c-d-e2-f) is finished.

P（a-b-c-d-e2-f）=P（a-b）×P（b-c-d）×P（d-e2-f）=1×1×$\frac{1}{2}$=$\frac{1}{2}$；

To sum up, the probability of correctly piecing of ‘a kid with a hat’ is:

P=P（a-b-c-d-e1-f）+ P（a-b-c-d-e2-f）=$\frac{1}{2}$ +$\frac{1}{2}$=1

We calculate the probability of 20 classic patterns according to this plan. Details will be shown in ‘Analyze and Sum-up’ (table 4) that follows.

We come to this conclusion: With plan three—Division principle (basic-figure principle)，the probability of regular patterns and vivid patterns with many cut-offs is larger, which means they are easier.

Thanks to the limit of sides and the symmetry of the patterns, regular patterns, especially the regular polygons, have fewer ways of being divided, many of which are correct. So the probability of piecing them up is large. Likewise, thanks to the distinct dividing and dissymmetry of patterns, the ways to divide irregular patterns, especially vivid patterns, are obvious and invariable, so the probability of piecing them up is also large. Because hollow patterns have more edges and angles and most of them are not central symmetry, they have various ways of being divided and relatively fewer correct ways. Therefore, the probability of correctly piecing up hollow patterns calculated in this plan is very small.

Same as the side choosing in plan two, dividing method is also based on people’s customary thinking and sense of patterns. We have taken all the ways of dividing into account, while in practical operation lots of impossible ways will be directly left out by human brain. Therefore the probability of correctly piecing up the patterns calculated in this plan does not exactly match the patterns’ actual difficulty. But apparently, plan three is more similar to human’s thinking and is therefore of much more reference value than plan two and one.

With more and more influential elements added, we find the design of plan and the calculation of the probability tougher and tougher, which is the obstacle to our further precise calculation.

2. Testing and Comparing

With further research, we find that the more human thinking is involved, the more distinct the differences between patterns are. But the thinking involved is only tip of the iceberg and the calculation data can not exactly reflect the difficulty of the patterns. We attempted to put into practice the theory based on the calculation in designing a tangram test in accordance with data on fingertips, aiming to record the time needed for finishing the task for future analysis. We conducted the first round of test on two groups, equally divided among 50 individuals of different age levels yet without previous experiences on tangram. Each individual is asked to work on the 8 patterns selected from the 20 classic ones, with the first group working on the patterns numbered 1 to 8 in turn while the second group 1-6-8-7 in turn. The purpose of using these 8 patterns includes verifying and perfecting our theory, determining the effect that the orders as well as similarity of the patterns have on the difficulty of the task.

Table 1 reveals the testing result (specific data for individuals is available in appendix 2)

To eliminate the influences of differences between individuals on the result, we apply the following formula to regulate the relative difficulty.

$$\text{Relative difficulty} = \frac{\text{The time needed in piecing one pattern}}{\text{The time needed in piecing all patterns}}$$

Table 1

	the time needed in piecing one pattern(unit: s)									relative difficulty							
Patterns										一	二	三	四	五	六	七	八
number 1～25																	
average	20.9	396	350	147	133	65.5	368	330		0.013	0.187	0.184	0.086	0.081	0.04	0.211	0.199
difficulty	8	1	3	5	6	7	2	4		8	3	4	5	6	7	1	2
mid-range	20	236	175	118	96	39	288	222		0.012	0.128	0.108	0.067	0.074	0.036	0.163	0.155
difficulty	8	2	4	5	6	7	1	3		8	3	4	6	5	7	1	2
number 26～50																	
average	14.2	159	133	68.8	60.3	29.6	165	140		0.019	0.196	0.19	0.064	0.066	0.043	0.244	0.178
difficulty	8	2	3	6	5	7	1	4		8	2	3	6	5	7	1	4
mid-range	14	119	89.5	61.5	51	23	145	113		0.012	0.178	0.173	0.044	0.071	0.034	0.217	0.107
difficulty	8	2	3	6	5	7	1	4		8	2	3	6	5	7	1	4
number 1～50																	
average	25.8	419	386	132	127	71	456	345		0.016	0.192	0.187	0.075	0.073	0.041	0.228	0.189
difficulty	8	2	3	5	6	7	1	4		8	2	4	5	6	7	1	3
mid-range	21.5	284	192	99	101	56	353	209		0.012	0.153	0.113	0.062	0.071	0.035	0.173	0.13
difficulty	8	2	4	6	5	7	1	3		8	2	4	6	5	7	1	3

Comparing the data of pattern 7 and 8, we find that, as for similar patterns, the order of piecing the patterns together does not exert too much effect on the difficulty of the pattern. While the way of thinking that the person is born with plays an important role in it. For most patterns, the testing results are in accordance with the theoretical ones, except the discord in pattern 2, 4, 6. The fact that among the three patterns, No.4&6 both belong to hollow patterns leads to the working hypothesis that the discord comes from the order of the patterns assigned to the participant has particularly big influence on the hollow patterns. Thus we conducted another experiment, using No.4&2.

Table 2 reveals the testing result：

Table 2

numbers	2	4		relative difficulty	
patterns			Unit:s	2	4
1	573	110	pattern 2 first	0. 8389	0. 1611
2	667	129		0. 8379	0. 1621
3	800	142		0. 8493	0. 1507
4	1718	110		0. 9398	0. 0602
5	230	100		0. 697	0. 303
6	585	30		0. 9512	0. 0488
7	401	439		0. 4774	0. 5226
8	973	116		0. 8935	0. 1065
average	743. 38	147		0. 8106	0. 1894
mid-range	626	113		0. 8441	0. 1559

numbers	2	4		relative difficulty	
patterns			Unit:s	2	4
9	969	167	pattern 4 first	0. 853	0. 147
10	467	495		0. 4854	0. 5146
11	423	131		0. 7635	0. 2365
12	106	118		0. 4732	0. 5268
13	839	81		0. 912	0. 088
14	21	101		0. 1721	0. 8279
15	682	88		0. 8857	0. 1143
16	318	663		0. 3242	0. 6758
average	478. 13	230. 5		0. 6086	0. 3914
mid-range	445	124. 5		0. 6245	0. 3755

Final Statistic

average	610. 75	188. 75		0. 7096	0. 2904
mid-range	579	117		0. 8384	0. 1616

Due to the data in table 2, we discover that although the order of piecing patterns has an effect on the difficulty grades, it is not a determinant. The difficulty grades of Hollow Square and Square that is educed from the test still commit an error to the grades based on calculation.

After the survey on each participant, we realize that the participants are inevitably with casualness. Since human brain is much advanced, it can eliminate some impossible ways of cutting and piecing, and this is what we can not realize in the three plans. Taking into consideration that Hollow Square and Square are the representatives of hollow patterns and regular patterns, through analogism we educe that regular patterns are much more difficult than hollow patterns. In order to make our conclusion much closer to the reality, we need to add a group of parameter to these two kinds of patterns to adjust their difficulty levels when we establish the final chart of difficulty grades of tangram.

3. Analysis and Sum-up:

Following are the main ways of thinking illustrated in the three plans.

Plan one: exhaustion, try every piece at random

Plan two: refer to permutation and combination of side and area before piecing the figures together

Plan three: observe the shape, divide the pattern and then piece them together.

All these three ways of thinking will be illustrated in everyone’s piecing procedure. So when we come to the grading of tangram, all these results should be taken into account. We decide to add weight to the probability calculated in three plans in light of their effect on the piecing procedure.

Considering plan three reflects human intelligence most effectively, we deem plan three exerts greatest effect during piecing procedure, plan two hypo and plan one is of the least effect. So we estimate the weight of each plan accordingly. We compute plan three contributes 50%, plan two 30% and plan one 20%.

Explanation for the data in table 3: since the data calculated according to these plans represents the probability of correct piecing, the larger the probability is, the easier the pattern is. However, difficulty of each pattern got from the test is based on the proportion of time, so the less time it takes the smaller proportion it generates. The smaller the number of difficulty is, the easier it is.

Table 3

numbers	patterns	PLAN ONE		PLAN TWO		PLAN THREE		TEST	
		probability ($\times 10^{-8}$)	difficulty (fome easy to difficult)	probability $\times 10^{-4}$	difficulty (fome easy to difficult)	probability	difficulty (fome easy to difficult)	relative difficulty	difficulty (fome easy to difficult)
1		1. 21	2	729. 6	1	0. 103124	2	0. 1915	7
2		1. 82	1	10. 7	4	0. 026452	5	0. 1866	5
6		1. 21	2	14	3	0. 0301	3	0. 0751	4
7		0. 908	4	52. 7	2	0. 008299	8	0. 0412	2
8		0. 606	5	1. 97	7	0. 011488	7	0. 0735	3
14		0. 151	6	6. 94	5	0. 02925	4	0. 1886	6
15		0. 151	6	0. 0992	8	0. 019794	6	0. 2276	8
19		0. 151	6	1. 16	6	1	1	0. 0159	1

We grade the patterns in light of three plans and the result of test respectively. Since the plans only take human thinking into consideration, the grading of a particular plan does not exactly agree with that of the test. So we compare the three grade tables with that of the test and manage to design the weight of each plan according to the similarity of the grades from calculation and test.

As is shown above, from table 3, we can figure out the following grading:

(each square represents a difficulty level. Numbers in the boxes are the number of the pattern. The difficulty level is getting higher from left to right. In each level, patterns are arranged in numerical order)

The grading of plan 1:	2	1、6	7、8	14、15、19
The grading of plan 2:	1	2、6、7	8、14、19	15
The grading of plan 3:	19	1、6	2、8、14、15	7
The grading of test:	19	6、7、8	1、2、14	15

Obviously, the difficulty level of 6 and 15 of plan 1 agree with that of the test. The difficulty level of 6, 7, 14 and 15 of plan 2 agree with that of the test. The difficulty level of 19, 6, 2 and 14 of plan 3 agree with that of the test.

Also, we define: $\text{Similarity}=\dfrac{M}{N}$

M= the number of patterns that agree, in each column, with the previous distribution of theoretical grading in each plan.

N= the total number of the patterns that agree, in each column, with the previous distribution of theoretical grading in all plans.

Therefore, we work out that each plan’s similarity to test data is 0.2, 0.4 and 0.4. Apparently, this result is very similar to what we have predicted. So we decide the weight of each plan is 0.2，0.4，0.4。

The weighted difficulty level:

19	1、6	2、14	7、8、15

From the grading above, we can see that the weighted grading is in accordance with that of the test, except pattern 1, 7, 8 (square and hollow patterns) which need to be adjusted. This verifies that the weights we set are rational. Since it’ s necessary to adjust regular and hollow patterns, we apply parameter to the grading process.

Referring to the method of working out coefficient in physics, we continuously adjust the parameter until the weighted grade matches the test grade to the greatest extent.

And finally we figure out the most accordant parameters.

The final probability of regular patterns= weighted probability×0.07

The final probability of hollow patterns= weighted probability×3. 3

Then we figure out the difficulty levels after regarding these eight patterns with the parameters:

Difficulty levels after adjusting:

19	6、7、8	1、2、14	15

Difficulty levels of test:

19	6、7、8	1、2、14	15

Apparently, the grade after adjusting with parameters is the same with the grade of test. So the parameters we set are rational.

So far, the grade of the calculation exactly matches that of the test. Making use of the weight and parameters above, we can employ quantitative grading to any tangram patterns.

We work out the probability of the 20 patterns according to the result of the three plans. Results are shown in table 4:

Table 4

numbers	patterns	PLAN ONE		PLAN TWO		PLAN THREE		WEIGHTED		ADD PARAMETER		The final difficulty
		probability $\times 10^{-8}$	difficulty	probability $\times 10^{-4}$	difficulty	probability	difficulty	probability	difficulty	probability	difficulty	
1		1.21	2	729.6	2	0.1031	7	0.0704	4	0.0049	16	16
2		1.82	2	10.7	11	0.0265	14	0.011	14	0.0008	18	18
3		2.42	1	39.4	6	0.1364	5	0.0561	6	0.0039	17	17
4		1.21	2	1280	1	0.0822	8	0.0841	3	0.0059	14	14
5		1.82	2	46.4	5	0.0218	16	0.0106	15	0.0007	19	19
6		1.21	2	14	10	0.0301	12	0.0126	11	0.0416	6	6
7		0.908	10	52.7	4	0.0083	20	0.0054	19	0.0179	9	9
8		0.606	11	1.97	18	0.0115	19	0.0047	20	0.0154	10	10
9		1.21	2	30.6	7	0.0665	10	0.0278	10	0.0278	8	8
10		0.303	12	4.09	15	0.0135	18	0.0056	18	0.0056	15	15
11		0.303	12	7.4	13	0.0304	11	0.0125	12	0.0125	11	11
12		1.21	2	14.4	9	0.0223	15	0.0095	16	0.0095	13	13
13		1.21	2	84.5	3	0.0764	9	0.0339	9	0.0339	7	7
14		0.151	14	6.94	14	0.0293	13	0.012	13	0.012	12	12
15		0.151	14	0.0992	20	0.0198	17	0.0079	17	0.0006	20	20

4. Difficulty Grade Chart

16		0.151	14	0.585	19	0.1111	6	0.0445	8	0.0445	5	5
17		0.151	14	8.33	12	0.2275	2	0.0913	2	0.0913	2	2
18		0.151	14	22.2	8	0.1481	3	0.0601	5	0.0601	3	3
19		0.151	14	1.16	16	1	1	0.4001	1	0.4001	1	1
20		0.151	14	2.38	17	0.1389	4	0.0557	7	0.0557	4	4

Based on table 4, we devise a difficulty grade chart. (Illustration 4.1)

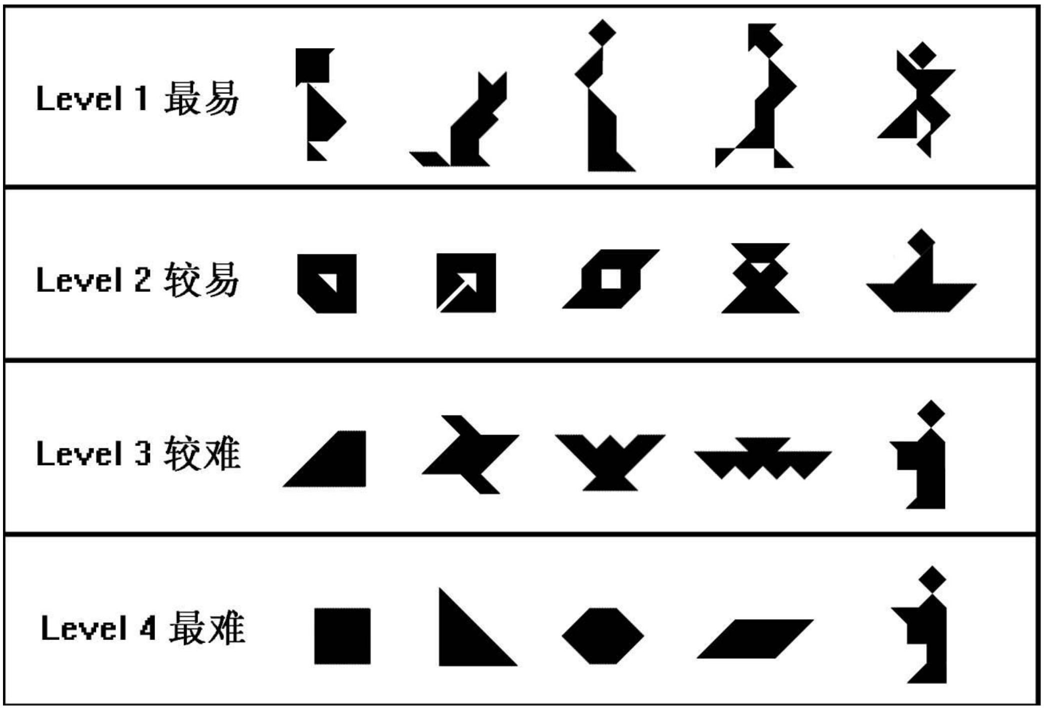


Illustration 4.1
The difficulty grade chart

Level 1 easiest 16、17、18、19、20 (relative difficulty: 0.000~0.005)
Level 2 easy 6 、7 、8 、9 、13 (relative difficulty: 0.005~0.015)
Level 3 difficult 4 、10、11、12、14 (relative difficulty: 0.015~0.045)
Level 4 most difficult 1 、2 、3 、5 、15 (relative difficulty: 0.045~1.000)

According to this, we summarize the general rule of piecing difficulty of tangram patterns: the more regular the patterns are, the greater the difficulty it is. (That is: the difficulty of vivid patterns < the difficulty of hollow patterns< the difficulty of regular patterns)

5. Sum-up of tangram skills

1. When there are cut-offs, participants should separate it and piece from the

smaller part. When the patterns are regular, participants should fix the biggest piece first.

2. As for vivid patterns (irregular patterns with many sides), plan three should be a priority, for after dividing the pattern into several parts, participants can easily piece every part together. As for regular patterns (patterns with few sides) and hollow patterns, plan two is more applicable, for participants can easily work out the answer in this principle.

3. Participants should apply the method of elimination through the whole process of piecing, trying not to leave out or reuse any piece. Think twice before piecing.

6. The Experience of Research

During the research, we three work together, bringing our strong suit into play. It is this cooperation that leads our way to accomplishing this essay. This essay records the whole process of our research. Passing through haze, we carry out a series of analysis and calculation. Then we test and analyze the theoretical result. Finally, with confusion cleared, we become more acquainted with tangram.

In the process, we use mathematical insight to identify questions and transformation thought to analyze them. We apply math knowledge we have learnt in class, such as independent event, conditional event, algorithm, exhaustion, to our analysis and solution to our question. This definitely deepens our comprehension and application of math.

In addition, we have been totally exposed to the fascination of math. This research is not only an achievement in math, but it also brings about the strength of our capability, the sublimation of our perseverance and merits and the broadening of our thinking.

7. Expectation

We can patent the plan and the grades we carried out in the future. Tangram with reasonable grades will definitely be beneficial to the development of teenagers’ and children’s intelligence, which is the guarantee of being a craze among the customers.

Furthermore, we can carry out bigger scale of test and calculation, summing up more skills. We will eventually perfect tangram and evoke the vigor of this traditional puzzle.

Above all, the method of the difficulty grades research in tangram can be generalized to the grading in other puzzles.

Appendix:

Appendix 1.1:

2. Example Pattern Two: the calculation of Hollow Square’s probability

Answer 1: ⑤G+②E+③E+④A+⑤E+①A+③H (Illustration1.5)

Process:

Step 1: put ⑤G to blank 1;

P (1) = $\frac{C_2^1}{C_7^1 \cdot C_8^1}$

Step 2: put ②E to blank 2;

P (2) = $\frac{1}{C_6^1 \cdot C_8^1}$

Step 3: put ③E to blank 3;

P (3) = $\frac{C_2^1}{C_5^1 \cdot C_8^1}$

Step 4: put ④A to blank 4;

P (4) = $\frac{1}{C_4^1 \cdot C_2^1}$

Step 5: put ⑤E to blank 5;

P (5) = $\frac{1}{C_3^1 \cdot C_8^1}$

Step 6: put ①A to blank 6;

P (6) = $\frac{1}{C_2^1 \cdot C_8^1}$

Step 7: put ③H to blank 7;

P (7) = $\frac{1}{C_1^1 \cdot C_8^1}$

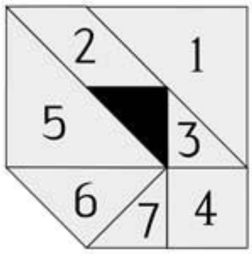


Illustration1.5
One of the correct answers of Hollow Square

To sum up: the probability of Hollow Square is $P_0 = P(1) \times P(2) \times P(3) \times P(4) \times P(5) \times P(6) \times P(7) = \frac{1}{660602880} \approx 1.51 \times 10^{-9}$

As Hollow Square is an axial symmetry geometric figure so there are 8 correct answers of it. (Illustration 1.6) So , the probability of Hollow Square is

P=P$_0$× $C_8^1 = \frac{1}{82575360} \approx 1.21\times10^{-8}$

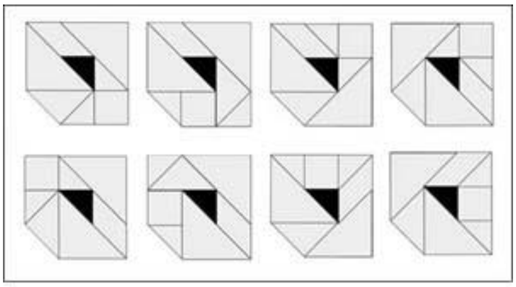


Illustration1.6
All correct answers of
Hollow Square

Appendix 1.2:

3. Example Pattern Three: the calculation of probability of ‘a kid with a hat’

Answer: ①F+④B+⑤D+②B+⑤H+③F+③E (Illustration1.7)

Process:

Step 1: put ①F to blank 1;

P (1) = $\frac{1}{C_7^1 \cdot C_8^1}$

Step 2: put ④B to blank 2;

P (2) = $\frac{1}{C_6^1 \cdot C_2^1}$

Step 3: put ⑤D to blank 3;

P (3) = $\frac{C_2^1}{C_5^1 \cdot C_8^1}$

Step 4: put ②B to blank 4;

P (4) = $\frac{1}{C_4^1 \cdot C_8^1}$

Step 5: put ⑤H to blank 5;

P (5) = $\frac{1}{C_3^1 \cdot C_8^1}$

Step 6: put ③F to blank 6;

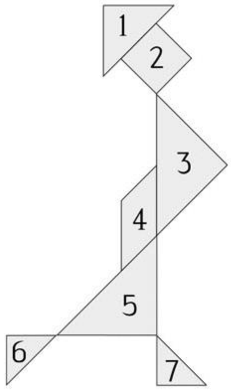


Illustration 1.7
The correct answers of
‘a kid with a hat’

P (6) = $\frac{C_2^1}{C_2^1 \cdot C_8^1}$

Step 7: put ③E to blank 7;

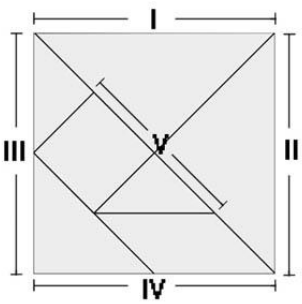
P (7) = $\frac{1}{C_1^1 \cdot C_8^1}$

To sum up: the probability of 'a kid with a hat' is $P_0 = P(1)\times P(2)\times P(3)\times P(4)\times P(5)\times P(6)\times P(7) = \frac{1}{660602880} \approx 1.51\times 10^{-9}$

As there's only one answer for 'a kid with a hat', the probability of Hollow Square is $P = P_0 \times C_1^1 = \frac{1}{660602880} \approx 1.51\times 10^{-9}$

Appendix 1.3:

1. Example Pattern One: the calculation of Square's probability



Answer 2: ⑤A+⑤B+③D+④B+③C+②A+①G (Illustration 2.4)

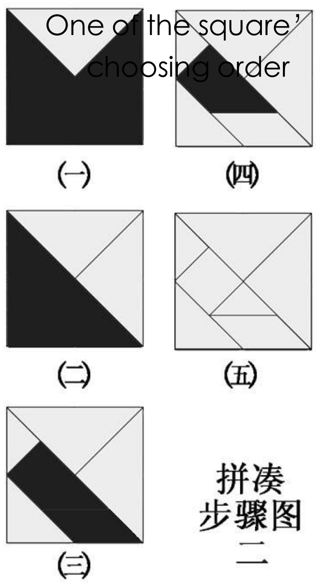
Process:

Step 1: fill up side I of which length is $2\sqrt{2}$, the possible ways are shown in illustration 2.1, as for answer 2, ⑤A is the only one that matches the side.

So the probability is P(I)= $P(A_1)$= $\frac{1}{C_{16}^1}$, and the profile goes to image ㈠ in illustration 2.5.

Step 2: fill up side II of which length is $2\sqrt{2}$. As for answer 2, ⑤B is the only one that matches the side.

Illustration 2.4
One of the square's choosing order



So the probability is P(II)= $P(A_2|A_1)=\frac{1}{C_{16}^1}$, and the profile goes to image ㈡ in illustration 2.5.

Step 3: fill up side III of which length is $2\sqrt{2}$. As for answer 2, ③D+①G is the only one that matches the side since two ⑤ pieces have been chosen , so the probability is P(III) = $P(A_3|A_1A_2)=\frac{1}{C_{15}^1}$, and the profile goes to image ㈢ in illustration 2.5.

Illustration2.5
Piecing procedure
of square II

Step 4: fill up side IV of which length is $\sqrt{2}$. ③C, ②A and ②C are possible answers. As for answer 2, ②A is the only one that matches. So the probability is P(IV) = $P(A_4|A_1A_2A_3)=\frac{1}{C_3^1}$, and the profile goes to image ㈣ in illustration 2.5.

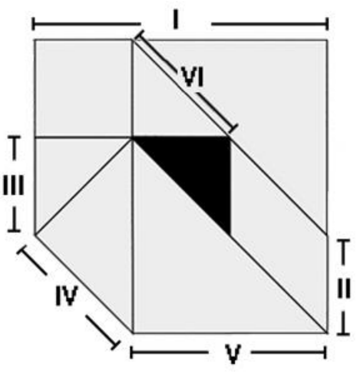
Step 5: fill up side V of which length is 1. ④B+③C, ④B+③D, ③C+④B, and ③D+④B are the possible answers. As for answer 2, ④B+③C is the only one that matches the side, so the probability is P(V) = $P(A_5|A_1A_2\cdots A_4)=\frac{1}{C_4^1}$, and the profile goes to image ㈤ in illustration 2.5.

To sum up: the probability of Hollow Square is $P_2 = P(1)\times P(2)\times P(3)\times P(4)\times P(5)\times P(6)\times P(7) = \frac{1}{46080}\approx 2.17\times 10^{-5}$

Appendix 1.4:

Example Pattern Two: Hollow Square

Answer 2 : ⑤ G+ ③ E+ ④ A+ ② E+ ③ F+ ① B+ ⑤ E



(Illustration 2.8)

Process:

Step 1: fill up side I of which length is 3，the possible answers are ①A＋③G、①A＋③F、③G＋①A、①A＋④A、④A＋①A、②A＋①E、②A+①G、①A +②G、①A +②E、②E+③G+③F、②E+③G+③G、②G+③F+③F、②G+③G+③G、②G+③G+②F、②G+③F+③G、③F+③F+②E、③F+③F+②G、③G+③G+②E、③F+③G+②E、③G+③F+②E、③G+③F+②G、③G+②E+③G、③F＋②E+③G、③F+②G+③F、③F+②G+③G、③F+④A+③F、③F+④A+③G、③G+④A+③G、③G+④A+③F、③F +③F+④A、③F +③G+④A、③G +③F+④A、③G +③G+④A、④A+③F+③F、④A+③F+③G、④A+③G+③G、④A+③G+③F、⑤G+④A、⑤F+④A、④A+⑤G、④A+⑤F、⑤G+②E、⑤F+②E、⑤F+②G、②E+⑤G、②G+⑤G、②G+⑤F、⑤G+③G、⑤F+③F、⑤G+③F、⑤F+③G、③F+⑤F、③G+⑤G、③F+⑤G、③G+⑤F、②G+④A+③G、②G+④A+③F、②E+③G+④A、②G+③G+④A、②G +③F+④A、④A+②E+③G、④A +③G+②E、④A+③F+②E、④A +③F+②G、③F+②G +④A、③F+④A+②E、③G+④A+②E（68 ways in all）. But as for answer 2,④A+⑤G is the only one that matches the side, so the probability is P(I)= $P(A_1)=\frac{1}{C_{68}^{1}}$ and the profile goes to image ㈠ in illustration 2.9.

Illustration 2.8 One of the hollow square’s choosing order

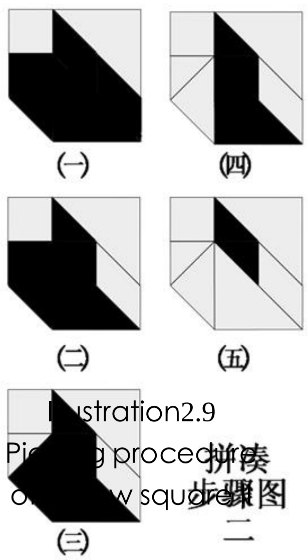


Illustration2.9 Picking procedure of hollow square 拼凑步骤图二

Step 2: fill up side II of which length is 1. ②F, ②H, ③G, and ③H are possible answers , but as for answer 2, ②H is the only one that matches , so the probability is P(II)= $P(A_2|A_1)=\frac{1}{C_{4}^{1}}$, and the profile goes to image ㈡ in illustration 2.9.

Step 3: fill up side III of which length is 2. ③E and ③F are possible answers , but as for answer 2, ③F is the only one that matches , so the probability is P(III)

= $P(A_3|A_1A_2)$ = $\frac{1}{C_2^1}$, and the profile goes to image ㈢ in illustration 2.9.

Step 4: fill up side IV of which length is 1. ③G, ①B and ①A are possible answers , but as for answer 2, ①B is the only one that matches , so the probability is P(IV) = $P(A_4|A_1A_2A_3)$ = $\frac{1}{C_3^1}$, and the profile goes to image ㈣ in illustration 2.9.

Step 5: fill up side V of which length is 2. ⑤H and ⑤E are possible answers , but as for answer 2, ⑤E is the only one that matches , so the probability is P(V) = $P(A_5|A_1A_2\cdots A_4)$ = $\frac{1}{C_2^1}$, and the profile goes to image ㈤ in illustration 2.9.

Step 6: Use ③E to fill the last blank. The probability is P(VI) = $P(A_6|A_1A_2\cdots A_5)$ = $\frac{1}{C_1^1}$

To sum up: the probability of Hollow Square's answer 1 is: P_2= P(I)×P(II)×P(III)×P(IV)×P(V)×P(VI) = $\frac{1}{3264}\approx 3.06\times 10^{-4}$

Appendix1.5:

3. Example Pattern Three: the calculation of probability of 'a kid with a hat'

Answer: ①F+④B+⑤D+②B+⑤H+③F+③E（Illustration 2.11）

Step 1: fill up side I of which length is $\sqrt{2}$, ①F, ①G, ②A,

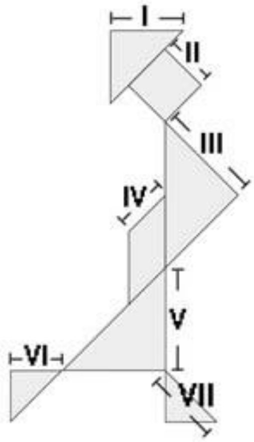


Illustration 2.11

②C and ③A are the possible answers. As for the correct answer, ①F is the only one that matches the side. So the probability is P(I)= $P(A_1)=\frac{1}{C_5^1}$, and the profile goes to image (一) in illustration 2.12.

Step 2: fill up side II of which length is 1. ④B, ③C and ③D are the possible answers. As for the correct answer, ④B is the only one that matches the side. So the probability is P(II)= $P(A_2|A_1)=\frac{1}{C_3^1}$, and the profile goes to image (二) in illustration 2.12.

Step 3: fill up side III of which length is 2. ③C+③D, ③D+③C, ③D+③D,③C+③C, ⑤D, ⑤C, ③D+②D, ②D +③D, ③C+②A, ②A+③C, ②A +③D and ③C+②D are the possible answers. As for the correct answer, ⑤D is the only one that matches the side. So the probability is P(III) $=P(A_3|A_1A_2)=\frac{1}{C_{12}^1}$, and the profile goes to image (三) in illustration 2.12.

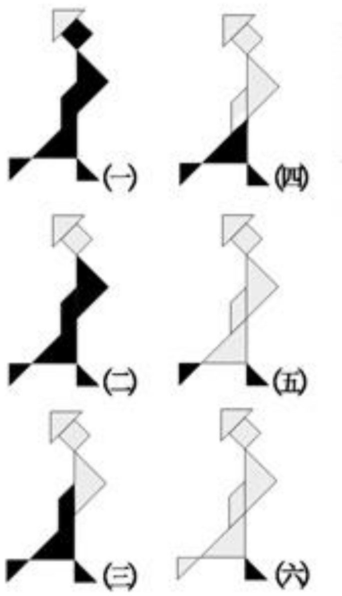


Illustration2.12
Piecing procedure of 'a kid with a hat'

Step 4: fill up side IV of which length is 1. ②B, ③B, ③C and ②C are the possible answers. As for the correct answer, ②B is the only one that matches the side. So the probability is P(IV) $=P(A_4|A_1A_2A_3)=\frac{1}{C_4^1}$, and the profile goes to image (四) in illustration 2.12.

Step 5: fill up side V of which length is 2. ③G+③H, ③H+③G, ③H+③H, ③G+③G, ⑤G and ⑤H are the possible answers. As for the correct answer, ⑤H is the only one that matches the side. So the probability is P(V) = $P\left(A_5 \middle| A_1 A_2 \cdots A_4\right) = \frac{1}{C_6^1}$, and the profile goes to image ㈤ in illustration 2.12.

Step 6: fill up side VI of which length is 1. ③G and ③F are the possible answers. As for the correct answer, ③F is the only one that matches the side. So the probability is P(VI) = $P\left(A_6 \middle| A_1 A_2 \cdots A_5\right) = \frac{1}{C_2^1}$, and the profile goes to image ㈥ in illustration 2.12.

Step 7: Use ③E to fill the last blank. The probability is P(VII) = $P\left(A_7 \middle| A_1 A_2 \cdots A_6\right) = \frac{1}{C_1^1}$

To sum up: the probability of ‘a kid with a hat’ is P_0= P(I)×P(II)×P(III)×P(IV)×P(V)×P(VI) = $\frac{1}{8640} \approx 1.16 \times 10^{-4}$

Appendix 1.6:

2. Example Pattern Two: the calculation of Hollow Square’s probability

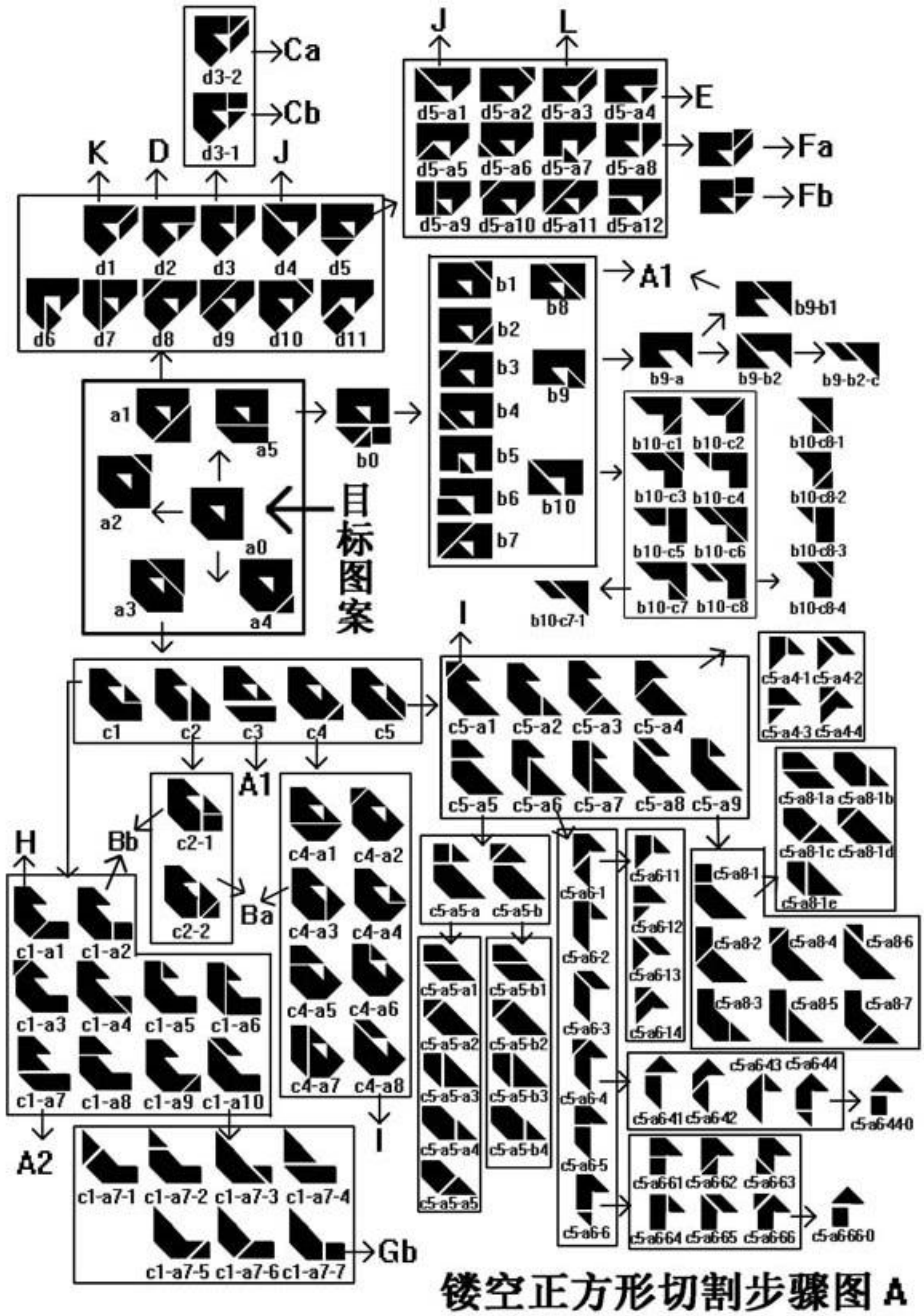


Illustration 3.3
Chart A of the cutting
procedure of hollow square

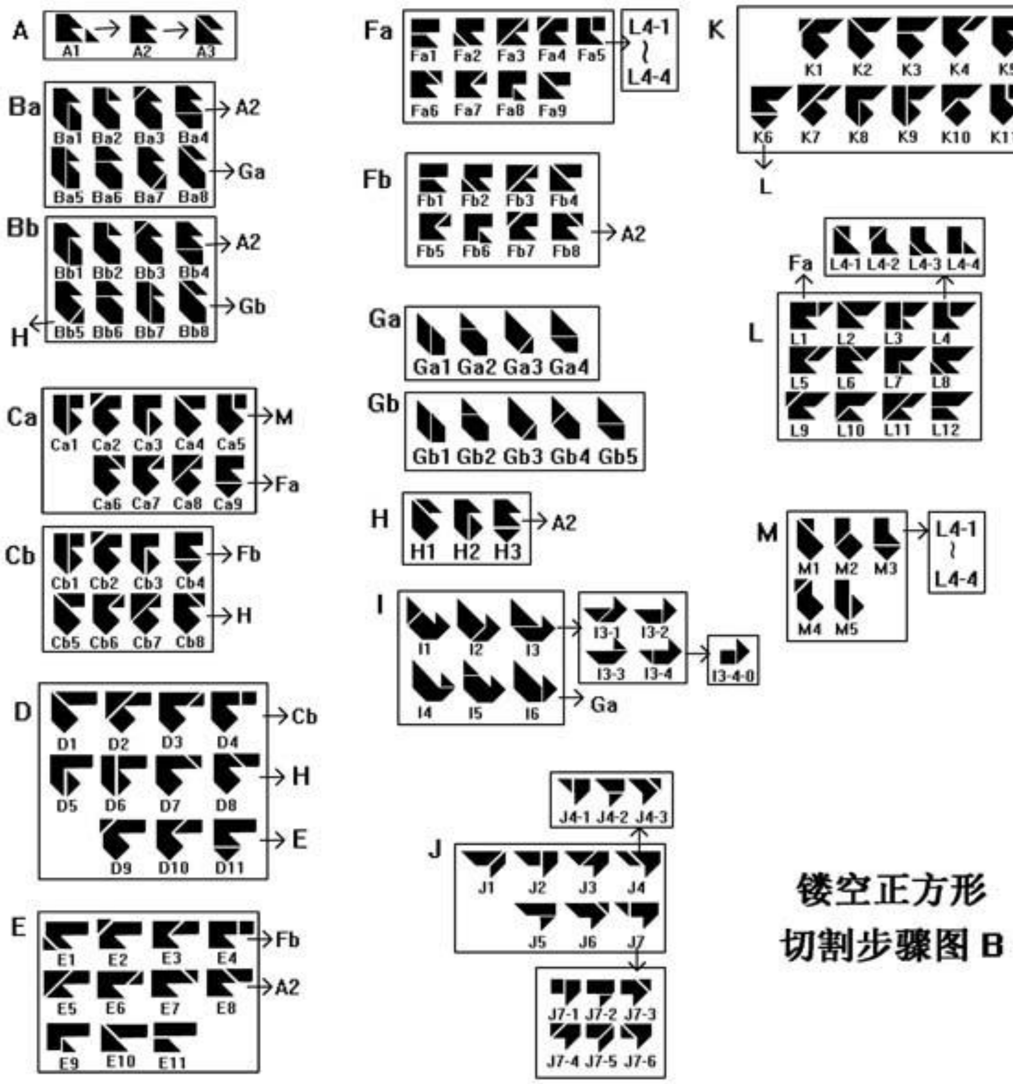


Illustration 3.4
Chart B of the cutting procedure
of Hollow Square

Process: (The following marks are all from Illustration 3.3 and Illustration 3.4)

Starting from a0, there are altogether 7 pieces of tangram for us to choose from. We refer to the chart of basic figures, and find it can be cut in five ways, a1, a2, a3, a4 and a5, among which a1, a3 and a5 are correct, so we need to discuss them respectively.

Case 1

Step 1: The probability of selecting a1 is $\frac{1}{5}$. After selecting b, the triangle cut off is a basic figure, so we consult the chart and find it can be pieced in four ways, 5-4, 5-5, 5-6 and 5-7 among which only basic figure 5-4 is the correct way of piecing. Then the probability of selecting 5-4 is $\frac{1}{4}$. Then we have d

So in this step, we have P（a0-a1-d）$=\frac{1}{5}\times\frac{1}{4}=\frac{1}{20}$

Step 2：After cutting off basic figure 5-4, we have only 6 pieces of tangram left. Then it can be cut in eleven ways, that is d1~d11. And d1, d2, d3, d4 and d5 are the correct ones.

Case 1 – 1: d1 is selected.

The probability of selecting d1 is $\frac{1}{11}$. The parallelogram cut off in d1 is a basic figure, so we consult the chart and find it can be pieced in two ways, 6-1 and 6-2, in which only 6-1 is correct, and the probability is $\frac{1}{2}$. Then we have k.

So P（d-d1-k）$=\frac{1}{11}\times\frac{1}{2}=\frac{1}{22}$

We find k can be cut in eleven ways, which are k1~k11, among which k2, k5, k6 and k11 are correct, so we need to discuss them respectively.

The probability of selecting k2 is $\frac{1}{11}$. The trapezoid cut off in k2 can be pieced in two ways, 4-3 and 4-4, in which only 4-4 is correct, and the probability is $\frac{1}{2}$. The big remaining trapezoid can only be pieced in one way, 3-7, so its probability of being correctly pieced is 1.

Then P（k-k2）$=\frac{1}{11}\times\frac{1}{2}=\frac{1}{22}$

Thus, case (a0-a1-d-d1-k-k2) is finished,

P(a0-a1-d-d1-k-k2~)=P(a0-a1-d)×P(d-d1-k)×P(k-k2~)$=\frac{1}{20}\times\frac{1}{22}\times\frac{1}{22}=\frac{1}{9680}$

The probability of selecting k5 is $\frac{1}{11}$. The triangle cut off in k5 can only be pieced in one way, 5-1, so the probability is1. The remaining part is Ca, which can be cut in nine ways, Ca-1~Ca-9, among which Ca-4, Ca-5 and Ca-9 are correct.

The probability of selecting Ca-4 is $\frac{1}{9}$. The trapezoid cut off in Ca-4 can be pieced in two ways, 3-1 and 3-3, in which only 3-1 is correct, so the probability is $\frac{1}{2}$, and the probability of the remaining part is 1.

The probability of selecting Ca-5 is $\frac{1}{9}$. The square cut off in Ca-5 can only be pieced in one way, 1-1, so the probability is1; the remaining part is M, which can be cut in five ways, M1~M5, among which M1 and M3 are correct. The probability of selecting M1 is $\frac{1}{5}$, and both two remaining parts are basic figures, so the probability is1. The probability of selecting M3 is $\frac{1}{5}$, and the probability of its being correctly cut into L1~L4 and pieced is $\frac{1}{4}$.

The probability of selecting Ca-9 is $\frac{1}{9}$. The square cut off in Ca-5 can only be pieced in one way, 5-2, so the probability is1; the remaining part is Fa, which can be cut in nine ways, Fa1~Fa9, among which Fa5 and Fa9 are correct. The probability of selecting Fa5 is $\frac{1}{9}$, and the probability of its being correctly cut into L1~L4 and pieced is $\frac{1}{4}$. The probability of selecting Fa9 is $\frac{1}{9}$, and the probability of its being correctly cut and pieced is1.

Thus, case (a0-a1-d-d1-k-k5) is finished. We have:

$$P\,(k\text{-}k5\sim) = \frac{1}{11}\times[\frac{1}{9}\times\frac{1}{2}\times1+\frac{1}{9}\times1\times(\frac{1}{5}\times1+\frac{1}{5}\times\frac{1}{4})+\frac{1}{9}\times1(\frac{1}{9}\times\frac{1}{4}+\frac{1}{9}\times1)] = \frac{8}{891}$$

Likewise, we calculated that:

$$P\,(k\text{-}k6\sim) = \frac{1}{11}\times[\frac{1}{12}\times1\times\frac{1}{4}+\frac{1}{12}\times1\times(\frac{1}{9}\times1+\frac{1}{9}\times\frac{1}{4})+\frac{1}{12}] = \frac{7}{432}$$

$$P\,(k\text{-}k11\sim) = \frac{1}{11}\times\frac{1}{2}\times(\frac{1}{5}\times1+\frac{1}{5}\times1\times\frac{1}{4}) = \frac{1}{88}$$

To sum up:

$$P\,(a0\text{-}a1\text{-}d\text{-}d1\text{-}k\sim) = P\,(a0\text{-}a1\text{-}d)\times P\,(d\text{-}d1\text{-}k)\times[\,(P\,(k\text{-}k2\sim)+P\,(k\text{-}k5\sim)+P\,(k\text{-}k6\sim)+P\,(k\text{-}k11\sim)\,]$$

$$= \frac{1}{20}\times\frac{1}{22}\times(\frac{1}{22}+\frac{8}{891}+\frac{7}{432}+\frac{1}{88}) = 1.863649\times10^{-4}$$

Likewise, we have:

Case 1 – 2 (a0-a1-d-d2-D~) : d2 is selected.

(We use the same arithmetic to calculate, so we omit parts of analyzing)

So we have:

P（a0-a1-d-d2-D）$=\frac{1}{20}\times\frac{1}{11}=\frac{1}{220}$

P（D-D1~）$=\frac{1}{11}\times\frac{1}{3}\times 1=\frac{1}{33}$

P（D-D3~）$=\frac{1}{11}\times[\frac{1}{11}\times 1+\frac{1}{11}\times 1\times(\frac{1}{8}+\frac{1}{8})+\frac{1}{11}\times\frac{1}{2}\times 1]=\frac{7}{484}$

P（D-D4~）$=\frac{1}{11}\times 1\times[\frac{1}{8}\times\frac{1}{2}\times 1+\frac{1}{8}\times 1\times(\frac{1}{8}+\frac{1}{8})+\frac{1}{8}\times 1\times\frac{2}{3}]=\frac{17}{1056}$

P（D-D8~）$=\frac{1}{11}\times\frac{1}{2}\times\frac{2}{3}=\frac{1}{33}$

P（a0-a1-d-d2-D~）= P（a0-a1-d-d2-D）×[P（D-D1~）+P（D-D3~）+P（D-D4~）+P（D-D8~）]

$$=\frac{1}{220}\times(\frac{1}{33}+\frac{7}{484}+\frac{17}{1056}+\frac{1}{33})=\frac{353}{851840}$$

Likewise:

Case 1 – 3（a0-a1-d-d3~）: d3 is selected.

So we have:

P（a0-a1-d-d3）$=\frac{1}{20}\times\frac{1}{11}=\frac{1}{220}$

Then we calculate P（d3-d3-1~）:

P（d3-d3-1-Cb~）$=\frac{1}{3}\times[\frac{1}{8}\times\frac{1}{2}\times 1+\frac{1}{8}\times 1\times(\frac{1}{8}+\frac{1}{8})+\frac{1}{8}\times\frac{2}{3}]=\frac{17}{288}$

Then we calculate P（d3-d3-2~）:

P（d3-d3-2-Ca~）$=\frac{1}{3}\times[\frac{1}{9}\times\frac{1}{2}\times 1+\frac{1}{9}\times(\frac{1}{5}+\frac{1}{5}\times\frac{1}{4})+\frac{1}{9}\times(\frac{1}{9}\times\frac{1}{4}+\frac{1}{9})]=\frac{8}{243}$

Thus, P (a0-a1-d-d3 ~）= P（a0-a1-d-d3）×[P（d3-d3-1~）+P（d3-d3-2~）]

$$=\frac{1}{220}\times(\frac{17}{288}+\frac{8}{243})=\frac{13}{31104}$$

Case 1 – 4（a0-a1-d-d4~）: d4 is selected.

So we have:

P（a0-a1-d-d4）$=\frac{1}{20}\times\frac{1}{11}=\frac{1}{220}$

P（d4-J）$=\frac{1}{2}$

Then we calculate:

P（J-J1~）$=\frac{1}{7}\times\frac{1}{2}\times 1=\frac{1}{14}$

P（J-J2~）$=\frac{1}{7}\times 1=\frac{1}{7}$

P（J-J4~）$=\frac{1}{7}\times\frac{1}{2}\times\frac{2}{3}=\frac{1}{21}$

P（J-J5~）$=\frac{1}{7}\times 1=\frac{1}{7}$

P（J-J7~）$=\frac{1}{7}\times(\frac{1}{6}+\frac{1}{6})=\frac{1}{21}$

Thus, P（a0-a1-d-d4~）= P（a0-a1-d-d4）×P（d4-J）×[P（J-J1~）+P（J-J2~）+P（J-J4）+P（J-J5~）+P（J-J7~）]

$$= \frac{1}{220}\times\frac{1}{2}\times(\frac{1}{14}+\frac{1}{7}+\frac{1}{21}+\frac{1}{7}+\frac{1}{21})=\frac{19}{18480}$$

Case 1 – 5（a0-a1-d-d5~）: d5 is selected.

So we have:

P（a0-a1-d-d5）$=\frac{1}{20}\times\frac{1}{11}=\frac{1}{220}$

P（d5-d5-a1~）$=\frac{1}{2}\times\frac{1}{12}\times\frac{1}{3}\times[\frac{1}{7}\times\frac{1}{2}\times 1+\frac{1}{7}\times 1+\frac{1}{7}\times\frac{1}{2}\times\frac{2}{3}+\frac{1}{7}\times 1+\frac{1}{7}\times(\frac{1}{6}+\frac{1}{6})]$

$=\frac{19}{3024}$

P（d5-d5-a3 ~）$=\frac{1}{2}\times\frac{1}{12}\times\frac{1}{2}\times[\frac{1}{12}\times(\frac{1}{9}\times\frac{1}{4}+\frac{1}{9})+\frac{1}{12}\times(\frac{1}{2}\times 1+\frac{1}{2}\times\frac{1}{2})+\frac{1}{12}\times\frac{1}{4}]$

$=\frac{41}{20736}$

P（d5-d5-a4~）$=\frac{1}{2}\times\frac{1}{12}\times[\frac{1}{11}\times(\frac{1}{8}+\frac{1}{8})+\frac{1}{11}\times\frac{1}{2}\times 1+\frac{1}{11}\times 1]=\frac{7}{1056}$

P（d5-d5-a8~）$=\frac{1}{2}\times\frac{1}{12}\times[\frac{1}{2}\times(\frac{1}{8}+\frac{1}{8})+\frac{1}{2}\times(\frac{1}{9}\times\frac{1}{4}+\frac{1}{9})]=\frac{7}{864}$

P (a0-a1-d-d5~) = P (a0-a1-d-d5) ×[P (d5-d5-a1~) +P (d5-d5-a3~) +P (d5-d5-a4~) +P（d5-d5-a8~）]

$$= \frac{1}{220}\times(\frac{19}{3024}+\frac{41}{20736}+\frac{7}{1056}+\frac{7}{864}) =1.0450\times 10^{-4}$$

P（a0-a1-d-d5~）$= \frac{1}{220}\times(\frac{19}{3024}+\frac{41}{20736}+\frac{7}{1056}+\frac{7}{864}) =1.0450\times 10^{-4}$

Thus, case 1 is finished. We have the probability of case 1:

P (a0-a1~) = P (a0-a1-d-d1 ~) +P (a0-a1-d-d2 ~) +P (a0-a1-d-d3~) +P (a0-a1-d-d4~) +P (a0-a1-d-d5~)
= 2.15×10^{-3}

Case 2:

From a0 to c, we have P (a0-a3-c) $=\frac{1}{5}\times\frac{1}{4}=\frac{1}{20}$

Starting from c, altogether there are 5 cases.

Case 2 – 1 (c-c1~) : c1 is selected.

So we have:

P (c-c1~) $=\frac{1}{5}\times1\times[\frac{1}{10}\times\frac{1}{3}\times\frac{2}{3}+\frac{1}{10}\times1\times(\frac{1}{8}\times1/2+\frac{1}{8}\times1\times\frac{2}{3}+\frac{1}{8}\times1\times\frac{2}{5})+\frac{1}{10}\times\frac{1}{3}+\frac{1}{10}\times(\frac{1}{7}+\frac{1}{7}\times\frac{1}{2}+\frac{1}{7}\times\frac{2}{5})]$
$=\frac{919}{50400}$

Case 2 – 2 (c-c2~) : c2 is selected.

So we have:

P (c-c2 ~) $=\frac{1}{5}\times[\frac{1}{3}\times(\frac{1}{8}\times\frac{1}{2}+\frac{1}{8}\times1\times\frac{2}{3}+\frac{1}{8}\times1\times\frac{2}{5})+\frac{1}{3}\times(\frac{1}{8}\times\frac{1}{2}+\frac{1}{8}\times\frac{1}{4})]$
$=\frac{139}{7200}$

Case 2 – 3 (c-c3~) : c3 is selected.

So we have:

P (c-c3~) $=\frac{1}{5}\times1\times1=\frac{1}{5}$

Case 2 – 4 (c-c4~) : c4 is selected.

So we ha ve:

P (c-c4 ~) $=\frac{1}{5}\times1\times[\frac{1}{8}\times(\frac{1}{8}\times\frac{1}{2}+\frac{1}{8}\times\frac{1}{4})+\frac{1}{8}\times(\frac{1}{6}\times\frac{1}{2}+\frac{1}{6}\times\frac{1}{4})]$
$=\frac{7}{1280}$

Case 2 – 5 (c-c5~) : c5 is selected.

So we have:

We calculated P (c-c5~) :

$P\text{（c-c5-c5-a）} = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}$

$P\text{（c5-a-c5-a1\textasciitilde）} = \frac{1}{9} \times (\frac{1}{6} \times \frac{1}{2} + \frac{1}{6} \times \frac{1}{4}) = \frac{1}{72}$

$P\text{（c5-a-c5-a5\textasciitilde）} = \frac{1}{9} \times [\frac{1}{2} \times (\frac{1}{5} + \frac{1}{5}) + \frac{1}{2} \times \frac{1}{4}] = \frac{13}{360}$

$P\text{（c5-a-c5-a6\textasciitilde）} = \frac{1}{9} \times \frac{1}{3} \times [\frac{1}{6} \times \frac{1}{2} \times (\frac{1}{4} + \frac{1}{4}) + \frac{1}{6} \times 1 + \frac{1}{6} \times (\frac{1}{4} + \frac{1}{4}) + \frac{1}{6} + \frac{1}{6} \times (\frac{1}{6} + \frac{1}{6})] = \frac{37}{1944}$

$P\text{（c5-a-c5-a4\textasciitilde）} = \frac{1}{9} \times \frac{1}{2} \times (\frac{1}{4} + \frac{1}{4}) = \frac{1}{36}$

$P\text{（c5-a-c5-a9\textasciitilde）} = \frac{1}{9} \times (\frac{1}{7} \times \frac{2}{5} + \frac{1}{7} \times \frac{1}{2} + \frac{1}{7}) = \frac{19}{630}$

P（c-c5~) = P（c-c5-c5-a）×[P（c5-a-c5-a1~) +P（c5-a-c5-a5~) +P（c5-a-c5-a6~) +P（c5-a-c5-a4~）+P（c5-a-c5-a9~）]

$= \frac{8639}{680400}$

Thus, case 2 is finished. We have the probability of case 2:

P (a0-a3~) = P (a0-a3-c) [P (c-c1~) +P (c-c2~) +P (c-c3~) +P (c-c4~) +P (c-c5~)]

$= \frac{1}{20} \times (\frac{919}{50400} + \frac{139}{7200} + \frac{1}{5} + \frac{7}{1280} + \frac{8639}{680400}) = 0.1279$

Case 3:

$P\text{（a0-a5-b0）} = \frac{1}{5} \times \frac{1}{3} = \frac{1}{15}$

$P\text{（b0-b8\textasciitilde）} = \frac{1}{10} \times 1 \times 1 = \frac{1}{10}$

$P\text{（b0-b9\textasciitilde）} = \frac{1}{10} \times 1 \times (\frac{1}{2} \times 1 + \frac{1}{2} \times 1) = \frac{1}{10}$

$P\text{（b0-b10）} = \frac{1}{10} \times 1 \times (\frac{1}{8} \times 1 \times 1 + \frac{1}{8} + \frac{1}{8} \times 1 \times \frac{1}{4}) = \frac{9}{320}$

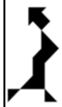
Thus, P（a0-a5~）= P（a0-a5-b0）×[P（b0-b8~）+P（b0-b9~）+P（b0-b10~）]

$= \frac{1}{15} \times \frac{1}{10} + \frac{1}{10} + \frac{9}{320}) = \frac{73}{4800}$

To sum up, the probability of correctly piecing of Hollow Square is:

P（a0~）=P（a0-a1~）+P（a0-a3~）+P（a0-a5~）=0.0301

Appendix 2:

numbers	1	2	3	4	5	6	7	8		Relative Difficulty							
Patterns									Unit: s	1	2	3	4	5	6	7	8
1	10	155	39	47	93	21	33	304	Sevenfirst	0.01425	0.2208	0.05556	0.06695	0.13248	0.02991	0.04701	0.43305
2	20	271	167	235	61	138	596	273		0.01136	0.15389	0.09483	0.13345	0.03464	0.07836	0.33844	0.15503
3	26	236	798	100	356	39	56	684		0.01133	0.10283	0.34771	0.04357	0.15512	0.01699	0.0244	0.29804
4	16	70	35	86	83	30	364	144		0.01932	0.08454	0.04227	0.10386	0.10024	0.03623	0.43961	0.17391
5	16	587	47	29	96	40	140	36		0.01615	0.59233	0.04743	0.02926	0.09687	0.04036	0.14127	0.03633
6	24	101	556	48	50	21	13	374		0.02022	0.08509	0.46841	0.04044	0.04212	0.01769	0.01095	0.31508
7	26	1711	283	203	56	23	121	1230		0.00712	0.46838	0.07747	0.05557	0.01533	0.0063	0.03312	0.33671
8	16	343	64	126	212	24	191	135		0.0144	0.30873	0.05761	0.11341	0.19082	0.0216	0.17192	0.12151
9	29	1437	310	222	142	38	1149	89		0.00849	0.42067	0.09075	0.06499	0.04157	0.01112	0.33636	0.02605
10	25	401	453	402	105	134	355	300		0.01149	0.18437	0.20828	0.18483	0.04828	0.06161	0.16322	0.13793
11	21	21	56	196	77	111	925	53		0.01438	0.01438	0.03836	0.13425	0.05274	0.07603	0.63356	0.0363
12	30	25	175	67	56	75	350	842		0.01852	0.01543	0.10802	0.04136	0.03457	0.0463	0.21605	0.51975
13	15	238	1320	58	76	21	21	106		0.00809	0.1283	0.71159	0.03127	0.04097	0.01132	0.01132	0.05714
14	27	67	124	60	140	20	1710	85		0.01209	0.03	0.05553	0.02687	0.0627	0.00896	0.76579	0.03807
15	18	47	60	169	152	38	98	566		0.01568	0.04094	0.05226	0.14721	0.1324	0.0331	0.08537	0.49303
16	16	24	634	181	79	66	55	693		0.00915	0.01373	0.3627	0.10355	0.04519	0.03776	0.03146	0.39645
17	11	518	1322	61	49	32	411	420		0.0039	0.18343	0.46813	0.0216	0.01735	0.01133	0.14554	0.14873
18	27	31	116	63	53	34	365	128		0.03305	0.03794	0.14198	0.07711	0.06487	0.04162	0.44676	0.15667
19	35	975	250	118	196	56	570	43		0.0156	0.43469	0.11146	0.05261	0.08738	0.02497	0.25412	0.01917
20	20	1267	109	429	223	195	288	200		0.00732	0.46393	0.03991	0.15709	0.08166	0.0714	0.10546	0.07323
21	17	572	221	46	368	27	651	138		0.00833	0.28039	0.10833	0.02255	0.18039	0.01324	0.31912	0.06765
22	22	62	34	324	122	65	347	673		0.01334	0.0376	0.02062	0.19648	0.07398	0.03942	0.21043	0.40813
23	20	122	665	100	165	141	93	366		0.01196	0.07297	0.39773	0.05981	0.09868	0.08433	0.05562	0.2189
24	20	600	821	155	240	162	93	139		0.00897	0.26906	0.36816	0.06951	0.10762	0.07265	0.0417	0.06233
25	16	22	102	150	80	86	216	222		0.0179	0.02461	0.11409	0.16779	0.08949	0.0962	0.24161	0.24832
26	32	432	832	70	172	135	36	468	Eightfirst	0.0147	0.19844	0.38218	0.03215	0.07901	0.06201	0.01654	0.21497
27	29	352	442	24	250	147	445	166		0.01563	0.18976	0.23827	0.01294	0.13477	0.07925	0.23989	0.08949
28	30	262	294	33	120	40	482	213		0.02035	0.17775	0.19946	0.02239	0.08141	0.02714	0.327	0.1445
29	17	58	29	142	97	25	21	350		0.023	0.07848	0.03924	0.19215	0.13126	0.03383	0.02842	0.47361
30	18	1050	93	78	150	33	57	541		0.00891	0.5198	0.04604	0.03861	0.07426	0.01634	0.02822	0.26782
31	30	272	483	207	146	159	566	176		0.01471	0.1334	0.23688	0.10152	0.0716	0.07798	0.27759	0.08632
32	26	119	208	95	166	45	111	431		0.02165	0.09908	0.17319	0.0791	0.13822	0.03747	0.09242	0.35887

numbers	1	2	3	4	5	6	7	8		Relative Difficulty							
patterns									Unit:s	1	2	3	4	5	6	7	8
34	35	1601	64	153	311	31	1292	197	Eighth first	0.0095	0.43458	0.01737	0.04153	0.08442	0.00841	0.35071	0.05347
35	25	65	56	61	34	28	1740	116		0.01176	0.03059	0.02635	0.02871	0.016	0.01318	0.81882	0.05459
36	23	295	261	55	31	119	973	183		0.01186	0.15206	0.13454	0.02835	0.01598	0.06134	0.50155	0.09433
37	36	646	142	98	90	143	370	694		0.01622	0.29112	0.06399	0.04416	0.04056	0.06444	0.16674	0.31275
38	20	420	173	212	232	50	300	508		0.01044	0.21932	0.09034	0.1107	0.12115	0.02611	0.15666	0.26527
39	13	194	580	60	134	23	557	113		0.00777	0.11589	0.34648	0.03584	0.08005	0.01374	0.33274	0.0675
40	37	784	625	32	27	81	1218	373		0.01165	0.24677	0.19673	0.01007	0.0085	0.0255	0.38338	0.11741
41	21	42	134	79	77	82	382	33		0.02471	0.04941	0.15765	0.09294	0.09059	0.09647	0.44941	0.03882
42	18	409	459	101	123	122	115	173		0.01184	0.26908	0.30197	0.06645	0.08092	0.08026	0.07566	0.11382
43	25	139	612	188	139	73	183	1071		0.01029	0.0572	0.25185	0.07737	0.0572	0.03004	0.07531	0.44074
44	79	445	146	71	160	70	1851	85		0.02718	0.15308	0.05022	0.02442	0.05504	0.02408	0.63674	0.02924
45	79	365	70	32	41	52	40	75		0.10477	0.48408	0.09284	0.04244	0.05438	0.06897	0.05305	0.09947
46	21	512	1065	203	55	92	432	89		0.00851	0.20737	0.43135	0.08222	0.02228	0.03726	0.17497	0.03605
47	21	31	134	89	37	46	127	39		0.04008	0.05916	0.25573	0.16985	0.07061	0.08779	0.24237	0.07443
48	16	795	156	279	53	63	896	704		0.0054	0.2684	0.05267	0.09419	0.01789	0.02127	0.3025	0.23768
49	19	354	2060	48	95	134	425	376		0.00541	0.10083	0.58673	0.01367	0.02706	0.03817	0.12105	0.10709
50	26	159	152	282	182	64	103	1604		0.01011	0.06182	0.0591	0.10964	0.07076	0.02488	0.04005	0.62364

Appendix 3:

Japanese jigsaw puzzle：Invented in Japan, it is used to test intelligence. Made of a 200mm*35mm narrow board divided into 4 pieces (shown as the illustration), the puzzle set 9 grades including infancy level, baby level, preprimary level, primary level, junior level, senior level, academy level, college level and doctor level.

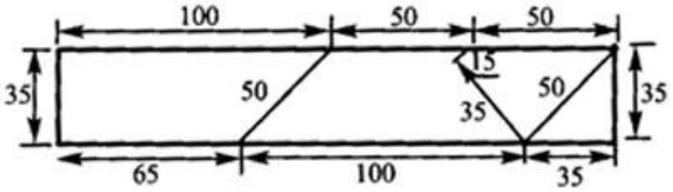


Illustration.1 Japanese jigsaw puzzle

Quote from <好玩的数学之七巧板、九连环与华容道——中国古典智力游戏三绝>

Reference:

[1]. Name of the book: <好玩的数学之七巧板、九连环与华容道——中国古典智力游戏三绝>.

Author：Zhang jingzhong, Wu heling

Publisher: Science Publisher

Publishing time: 2004 October

[2] Name of the book: Standard mathematic text book (Optional 2-3)

Publisher: Jiangsu Educational Publisher

Publishing time: 2006 June