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COMPACTIFICATIONS OF SYMMETRIC SPACES

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Abstract

Compactifications of symmetric spaces have been constructed by different methods for various applications. One application is to provide the so-called rational boundary components which can be used to compactify locally symmetric spaces. In this paper, we construct many compactifications of symmetric spaces using a uniform method, which is motivated by the Borel-Serre compactification of locally symmetric spaces. Besides unifying compactifications of both symmetric and locally symmetric spaces, this uniform construction allows one to compare and relate easily different compactifications, to extend the group action continuously to boundaries of compactifications, and to clarify the structure of the boundaries.

1. Introduction

Let $X = G/K$ be a symmetric space of noncompact type. Compactifications of X arise from many different sources and have been studied extensively (see [**GJT**], [**BJ1**], [**Os**] and the references there). There are two types of compactifications depending on whether one copy of X, always assumed to be open, is dense or not. For example, the Satake compactifications, Furstenberg compactifications, the conic (or geodesic) compactification, the Martin compactification, and the Karpelevic compactification belong to the first type, while the Oshima compactification and the Oshima-Sekiguchi compactification belong to the second type. In this paper, we will only study the case where X is dense.

All these compactifications were constructed by different methods and motivated by different applications. For example, the Satake compactifications are obtained by embedding symmetric spaces X into the space of positive definite Hermitian matrices as a totally geodesic submanifold, which is in turn embedded into the (compact) real projective space of Hermitian matrices; on the other hand, the Furstenberg compactifications are defined by embedding X into the space of probability

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measures on the Furstenberg boundaries. The Martin compactification was motivated by potential theory, and the ideal boundary points are determined by the asymptotic behaviors of the Green function. On the other hand, the conic (or geodesic) compactification $X \cup X(\infty)$ and the Karpelevic compactification $\overline{X}^K$ are defined in terms of equivalence classes of geodesics with respect to various relations. In all these compactifications, parabolic subgroups play an important role in describing the geometry at infinity.

In this paper, we propose a new, uniform approach, called the *attachment method* in [**BJ1**], to construct most of the known compactifications of symmetric spaces by making direct use of parabolic subgroups. Briefly, the compactifications of X are obtained by attaching boundary components associated with parabolic subgroups, and the topology is also described in terms of the Langlands decomposition of parabolic subgroups. A basic feature of this method is that it uses reduction theory for real parabolic subgroups; in particular, separation property of Siegel sets and strong separation of generalized Siegel sets, and hence relates compactifications of symmetric spaces closely to compactifications of locally symmetric spaces. In fact, this method is suggested by the compactification of locally symmetric spaces in [**BS**].

This method allows one to show easily that the G-action on X extends continuously to the compactifications of X and to describe explicitly neighborhoods of boundary points and sequences of interior points converging to them. Explicit descriptions of neighborhoods of boundary points are important for applications (see [**Zu**] and the references there), but they do not seem to be available in literature for the Satake and the Furstenberg compactifications; in particular, the non-maximal Satake compactifications. As explained above, this procedure is closely related to compactifications of locally symmetric spaces, and it seems conceivable that the method in §§4, 5 can be modified to give more explicit descriptions than those in [**Zu**] of neighborhoods of boundary points of the Satake compactifications of locally symmetric spaces.

Since compactifications of X are obtained by adding boundary faces associated with real parabolic subgroups, this procedure relates the boundary of the compactifications of X to the spherical Tits building of X, a point of view emphasized in [**GJT**]. A basic geometric construction in [**GJT**] is the dual cell compactification $X \cup \Delta^*(X)$, which is isomorphic to the maximal Satake compactification. The dual cell compactification $X \cup \Delta^*(X)$ is constructed by gluing together polyhedral compactifications of maximal totally geodesic flat submanifolds in X passing through a fixed basepoint in X and plays an important role in identifying the Martin compactification of X, one of the main results of [**GJT**]. Briefly, for each such flat, the Weyl chambers and their faces form a polyhedral cone decomposition and their dual cell complex gives the boundary at infinity; and the K-action glues the compactification of all such flats to give the compactification $X \cup \Delta^*(X)$. Because of the nature of the construction, the continuous extension of the G-action to the dual cell compactification $X \cup \Delta^*(X)$ is not clear. On the other hand, from the construction in this paper, the continuous extension of the G-action follows easily. This difficulty of extending the G-action to the dual cell compactification is one of the motivations of this paper.

The organization of this paper is as follows. In §2, we recall Siegel sets and generalized Siegel sets of real parabolic subgroups, and their separation property for different parabolic subgroups. In §3, we outline a general approach to compactifications of symmetric spaces. One key point is to define boundary components for real parabolic subgroups and to attach them by means of the horospherical decomposition. In §4, we construct the maximal Satake compactification using this method. §§5, 6, 7, 8 are respectively devoted to the non-maximal Satake compactifications, the conic (or geodesic) compactification $X \cup X(\infty)$, the Martin compactification and the Karpelevic compactification.

Some of the results in this paper have been announced in [**BJ1**]. This paper is mainly written up by the second author, who will bear the primary responsibility for it.

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Conventions. In this paper, for any $x, y \in G$, define

$$x^y = y^{-1}xy, \quad {}^y x = yxy^{-1}.$$

The same notation applies when x is replaced by a subset of G. For two sets A, B, $A \subset B$ means that A is a proper subset of B; and $A \subseteq B$ means that A is a subset of B and could be equal to B. A reference to an equation is to one in the same section unless indicated otherwise.

2. Parabolic subgroups and Siegel sets

In this section, we introduce some basic facts about (real) parabolic subgroups and their Siegel sets.

Let G be an adjoint connected semisimple Lie group, $K \subset G$ a maximal compact subgroup, and $X = G/K$ the associated symmetric space of non-compact type. We remark that the adjoint assumption is used in the proof of Proposition 2.4. Since every symmetric space X of noncompact type is a quotient of such an adjoint group G, there is no loss of generality in assuming this.

Let $x_0 = K \in X$ be the fixed basepoint. Then for every (real and proper) parabolic subgroup P of G, there is a Langlands decomposition

$$P = N_P A_P M_P,$$

where N_P is the unipotent radical of P, $A_P M_P$ is the unique Levi component stable under the Cartan involution associated with K, and A_P is the split component. In fact, the map

$$N_P \times A_P \times M_P \to P, \quad (n, a, m) \mapsto nam$$

is a diffeomorphism and the right multiplication by P is given by

$$n_0 a_0 m_0 (n, a, m) = (n_0 \, {}^{a_0 m_0} n, a_0 a, m_0 m). \tag{1}$$

Let $K_P = M_P \cap K$. Then K_P is a maximal compact subgroup of M_P, and

$$X_P = M_P / K_P$$

is a symmetric space of noncompact type of lower dimension, called the *boundary symmetric space* associated with P.

Since $G = PK$ and $K_P = K \cap P$, the Langlands decomposition of P gives a horospherical decomposition of X:

$$X = N_P \times A_P \times X_P,$$

$$(n, a, mK_P) \in N_P \times A_P \times X_P \mapsto namK \in X, \tag{2}$$

and the map is an analytic diffeomorphism. We note that this diffeomorphism depends on the choice of the basepoint x_0 and will be denoted by

$$\mu_0 : N_P \times A_P \times X_P \to X \tag{3}$$

as in [**Bo2**, 4.1] if the basepoint x_0 needs to be specified. This map μ_0 is equivariant with respect to the following P-action on $N_P \times A_P \times X_P$:

$$n_0 a_0 m_0 (n, a, z) = (n_0 \, {}^{a_0 m_0} n, a_0 a, m_0 z).$$

In the following, for $(n, a, z) \in N_P \times A_P \times X_P$, the point $\mu_0(n, a, z)$ in X is also denoted by (n, a, z) or naz for simplicity.

The group G acts on the set of parabolic subgroups by conjugation, and the subgroup K preserves the Langlands decomposition of these parabolic subgroups and the induced horospherical decompositions of X. Specifically, for any $k \in K$ and any parabolic subgroup P,

$$N_{{}^k P} = {}^k N_P, \quad M_{{}^k P} = {}^k M_P, \quad A_{{}^k P} = {}^k A_P, \tag{4}$$

and hence the Langlands decomposition of ${}^k P$ is given by

$${}^k P = {}^k N_P \times {}^k A_P \times {}^k M_P.$$

To describe the K-action on the horospherical coordinate decomposition, for any $z = mK_P \in X_P$, $k \in K$, define

$$k \cdot z = {}^k m \ {}^k K_P \in X_{^kP}. \tag{5}$$

Note that ${}^k K_P = K_{^kP}$. Under this action k maps X_P to $X_{^kP}$. Then for $(n, a, z) \in N_P \times A_P \times X_P = X$, $k \in K$, the point $k\mu_0(n, a, t) = k \cdot (n, a, z)$ has horospherical coordinates with respect to ${}^k P$,

$$k \cdot (n, a, z) = ({}^k n, {}^k a, kz) \in N_{^kP} \times A_{^kP} \times X_{^kP}. \tag{6}$$

The reason is that K fixes the basepoint x_0 and these components are defined with respect to x_0. On the other hand, conjugation by elements outside K does not preserve the horospherical decomposition with respect to the basepoint x_0.

For a pair of parabolic subgroups P, Q with $P \subset Q$, P determines a unique parabolic subgroup P' of M_Q such that

$$X_{P'} = X_P, \quad N_P = N_Q N_{P'} = N_Q \rtimes N_{P'}, \quad A_P = A_Q A_{P'} = A_Q \times A_{P'}, \tag{7}$$

where the split component $A_{P'}$ and the boundary symmetric space $X_{P'}$ are defined with respect to the basepoint $x_0 = K_Q$ in $X_Q = M_Q/K_Q$. Conversely, every parabolic subgroup of M_Q is of this form P' for some parabolic subgroup P of G contained in Q.

Let $\Phi(P, A_P)$ be the set of roots of the adjoint action of $\mathfrak{a}_P$ on the Lie algebra $\mathfrak{n}_P$, and $\Delta = \Delta(P, A_P)$ be the subset of simple roots in $\Phi(P, A_P)$. We will also view them as characters of A_P defined by $a^\alpha = \exp \alpha(\log a)$. For any $t > 0$, let

$$A_{P,t} = \{a \in A_P \mid a^\alpha > t, \alpha \in \Delta\}. \tag{8}$$

Then $A_P^+ = A_{P,1}$ is the positive chamber, and its image in the Lie algebra $\mathfrak{a}_P$ is

$$\mathfrak{a}_P^+ = \{H \in \mathfrak{a}_P \mid \exp H \in A_P^+\} = \{H \in \mathfrak{a}_P \mid \alpha(H) > 0, \alpha \in \Delta\}.$$

Define

$$\mathfrak{a}_P(\infty) = \{H \in \mathfrak{a}_P \mid \|H\| = 1\}, \tag{9}$$

the unit sphere to be identified with the sphere at infinity of $\mathfrak{a}_P$, and

$$\mathfrak{a}_P^+(\infty) = \mathfrak{a}_P^+ \cap \mathfrak{a}_P(\infty),$$

an open simplex.

Then for any $k \in K$,

$$A_{^kP,t} = {}^k A_{P,t}, \quad \mathfrak{a}_{^kP}^+(\infty) = ad(k)\mathfrak{a}_P^+(\infty).$$

For any subset $I \subset \Delta(P, A_P)$, there is a unique parabolic subgroup P_I containing P such that

$$A_{P_I} = \{a \in A_P \mid a^\alpha = 1, \alpha \in I\},$$

and $\Delta(P_I, A_{P_I})$ is the set of restrictions to A_{P_I} of the elements of $\Delta(P, A_P) - I$. When $I = \emptyset$, $P_I = P$; when $I_1 \subset I_2$, $P_{I_1} \subset P_{I_2}$. Any parabolic subgroup containing P is of this form. If P is a minimal parabolic subgroup P_0, the parabolic subgroups $P_{0,I}$ containing it are called *standard parabolic subgroups.*

For simplicity, in the following, we denote the subset $\Delta(P, A_P) - I$ also by $\Delta(P_I, A_P)$ to indicate its relation to P_I (i.e., to convey intuitively that the roots in $\Delta(P, A_P) - I$ are in the directions of P_I):

$$\Delta(P_I, A_P) = \Delta(P, A_P) - I. \tag{10}$$

For each P_I, let $\mathfrak{a}_P^I$ be the orthogonal complement of $\mathfrak{a}_{P_I}$ in $\mathfrak{a}_P$ with respect to the Killing form. Then

$$\mathfrak{a}_P = \mathfrak{a}_{P_I} \oplus \mathfrak{a}_P^I. \tag{11}$$

Let P' be the unique parabolic subgroup in M_{P_I} corresponding to P in Equation (7). Then its split component with respect to the basepoint x_0

$$\mathfrak{a}_{P'} = \mathfrak{a}_P^I. \tag{12}$$

Let $\overline{\mathfrak{a}_P^+(\infty)}$ be the closure of $\mathfrak{a}_P^+(\infty)$ in $\mathfrak{a}_P(\infty)$, a closed simplex. Then each $\mathfrak{a}_{P_I}^+(\infty)$ is a simplicial (open) face of $\overline{\mathfrak{a}_P^+(\infty)}$, and

$$\overline{\mathfrak{a}_P^+(\infty)} = \mathfrak{a}_P^+(\infty) \cup \coprod_{I \neq \emptyset} \mathfrak{a}_{P_I}^+(\infty), \tag{13}$$

where $I \subset \Delta(P, A_P)$.

For bounded sets $U \subset N_P, V \subset X_P$ and $t > 0$, the set

$$U \times A_{P,t} \times V \tag{14}$$

is identified with the subset $\mu_0(U \times A_{P,t} \times V)$ of X through the horospherical decomposition of X and called a *Siegel set* in X associated with the parabolic subgroup P.

An important property of Siegel sets is the following *separation property.*

Proposition 2.1. *Let P_1, P_2 be two parabolic subgroups of G and $\mathcal{S}_i = U_i \times A_{P,t_i} \times V_i$ be a Siegel set for P_i $(i = 1, 2)$. If $P_1 \neq P_2$ and $t_i \gg 0$, then*

$$\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset.$$

Proof. This is a special case of [**Bo1**, Proposition 12.6]. In fact, let P be a fixed minimal parabolic subgroup. Then P_1, P_2 are conjugate to standard parabolic subgroups P_{I_1}, P_{I_2} containing P,

$$P_1 = {}^{k_1}P_{I_1}, \quad P_2 = {}^{k_2}P_{I_2},$$

for some $k_1, k_2 \in K$. If for all $t_i > 0$,

$$U_1 \times A_{P_1,t_1} \times V_1 \cap U_2 \times A_{P_2,t_2} \times V_2 \neq \emptyset,$$

then [**Bo1**, Proposition 12.6] implies that $I_1 = I_2$ and $k_1 k_2^{-1} \in P_{I_1}$. This implies that $P_1 = P_2$. q.e.d.

A special case of this proposition concerns rational parabolic subgroups and their Siegel sets. This separation property for rational parabolic subgroups plays an important role in reduction theory for arithmetic subgroups and compactifications of locally symmetric spaces (see [**BJ2**]). For compactifications of symmetric spaces, we need stronger separation properties.

Proposition 2.2. *Let $P_1, P_2, \mathcal{S}_1, \mathcal{S}_2$ be as in Proposition 2.1 and let C be a compact neighborhood of the identity element in K. Assume that $P_1^k \neq P_2$ for every $k \in C$. Then there exists $t_0 > 0$ such that $k\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ for all $k \in C$ if $t_1, t_2 \geq t_0$.*

This proposition follows from the even stronger separation property of Proposition 2.4 below, which plays a crucial role in this paper.

Let $B(,)$ be the Killing form on $\mathfrak{g}$, θ the Cartan involution on $\mathfrak{g}$ associated with K. Then

$$\langle X, Y \rangle = -B(X, \theta Y), \quad X, Y \in \mathfrak{g},$$

defines an inner product on $\mathfrak{g}$ and hence a Riemannian metric on G and N_P. Let $B_{N_P}(\varepsilon)$ be the ball in N_P of radius ε with center the identity element.

For a bounded set V in X_P and $\varepsilon > 0$, $t > 0$, define

$$S_{\varepsilon,t,V} = S_{P,\varepsilon,t,V} = \Big\{(n, a, z) \in N_P \times A_P \times X_P = X \mid z \in V, a \in A_{P,t}, n^a \in B_{N_P}(\varepsilon)\Big\}. \tag{15}$$

We shall call $S_{P,\varepsilon,t,V}$ a *generalized Siegel set* associated with P, and P will be omitted when it is clear.

Lemma 2.3. *For any bounded set $U \subset N_P$ and $\varepsilon > 0$, when $t \gg 0$,*

$$U \times A_{P,t} \times V \subset S_{\varepsilon,t,V}.$$

Proof. Since the action of $A_{P,t}^{-1}$ by conjugation on N_P shrinks N_P towards the identity element as $t \to +\infty$, it is clear that for any bounded set $U \subset N_P$ and $\varepsilon > 0$, when $t \gg 0$, $a \in A_{P,t}$,

$$U^a \subset B_{N_P}(\varepsilon),$$

and the lemma follows. q.e.d.

On the other hand, $S_{\varepsilon,t,V}$ is not contained in the union of countably infinitely many Siegel sets defined above. In fact, for any strictly increasing sequence $t_j \to +\infty$ and a sequence of bounded sets $U_j \subset N_P$ with $\cup_{j=1}^{\infty} U_j = N_P$, we claim that

$$S_{\varepsilon,t,V} \not\subset \cup_{j=1}^{n} U_j \times A_{P,t_j} \times V.$$

In fact,

$$S_{\varepsilon,t,V} = \cup_{a\in A_{P,t}} {}^a B_{N_P}(\varepsilon) \times \{a\} \times V.$$

For every j such that $t_{j+1} > t_j$, there is an unbounded sequence $a_k \in A_{P,t_j} \setminus A_{P,t_{j+1}}$. Fix such a j and a sequence a_k. Then ${}^{a_k}B_{N_P}(\varepsilon)$ is not bounded, and hence

$${}^{a_k}B_{N_P}(\varepsilon) \times \{a_k\} \times V \not\subset \cup_{l=1}^{j} U_l \times A_{P,t_l} \times V.$$

On the other hand, since $a_k \notin A_{P,t_l}$ for all $l \geq j+1$,

$${}^{a_k}B_{N_P}(\varepsilon) \times \{a_k\} \times V \not\subset U_l \times A_{P,t_l} \times V,$$

and the claim follows.

To cover $S_{\varepsilon,t,V}$, we need to define Siegel sets slightly differently. For any $T \in A_P$, define

$$A_{P,T} = \{a \in A_P \mid a^\alpha > T^\alpha, \alpha \in \Delta(P, A_P)\}, \tag{16}$$

and

$$S_{\varepsilon,T,V} = S_{P,\varepsilon,T,V} = \Big\{(n,a,z) \in N_P \times A_P \times X_P = X \mid z \in V, a \in A_{P,T}, n^a \in B_{N_P}(\varepsilon)\Big\}. \tag{17}$$

Siegel sets of the form $U \times A_{P,T} \times V$ are needed for the precise reduction theory of arithmetic subgroups (see [**Sap**] for more details) and will also be used in §5 to describe the topology of nonmaximal Satake compactifications; and an analogue of Lemma 2.3 holds for them. Then there exist sequences $T_j \in A_{P,t}$ and bounded sets $U_j \subset N_P$ such that

$$S_{\varepsilon,t,V} \subset \cup_{j=1}^{\infty} U_j \times A_{P,T_j} \times V.$$

In fact, T_j could be any sequence in $A_{P,t}$ such that every point of $A_{P,t}$ belongs to a δ-neighborhood of some T_j, where δ is independent of j.

Proposition 2.4. *For any two distinct parabolic subgroups P, P' and generalized Siegel sets $S_{\varepsilon,t,V}$, $S_{\varepsilon,t,V'}$ associated with them, and a compact neighborhood C of the identity element in K such that for every $k \in C$, ${}^kP \neq P'$, if $t \gg 0$ and ε is sufficiently small, then for all $k \in C$,*

$$kS_{\varepsilon,t,V} \cap S_{\varepsilon,t,V'} = \emptyset.$$

Proof. Let $\tau : G \to PSL(n,\mathbb{C})$ be a faithful irreducible projective representation whose highest weight is generic. Since G is semisimple of adjoint type, such a representation exists. Choose an inner product on $\mathbb{C}^n$ such that $\tau(\theta(g)) = (\tau(g)^*)^{-1}$, where θ is the Cartan involution on G associated with K, and $A \to (A^*)^{-1}$ is the Cartan involution on $PSL(n,\mathbb{C})$ associated with $PSU(n)$. Then $\tau(K) \subset PSU(n)$. Let $M_{n\times n}$ be the vector space of complex $n \times n$ matrices, and $P_{\mathbb{C}}(M_{n\times n})$ the associated projective space. Composed with the map $PSL(n,\mathbb{C}) \to P_{\mathbb{C}}(M_{n\times n})$, τ induces an embedding

$$i_\tau : G \to P_{\mathbb{C}}(M_{n\times n}).$$

For each Siegel set $S_{\varepsilon,t,V}$ in X associated with P, its inverse image in G under the map $G \to X = G/K, g \mapsto gx_0$, is $\{(n,a,m) \in N_P \times A_P \times M_P K = G \mid m \in VK, a \in A_{P,t}, n^a \in B_{N_P}(\varepsilon)\}$ and denoted by $S_{\varepsilon,t,V}K$.

We claim that the images $i_\tau(kS_{\varepsilon,t,V}K)$ and $i_\tau(S_{\varepsilon,t,V'}K)$ are disjoint for all $k \in C$ under the above assumptions.

Let P_0 be a minimal parabolic subgroup contained in P. Then $P = P_{0,I}$ for a subset $I \subset \Delta(P_0, A_{P_0})$. Let

$$\mathbb{C}^n = V_{\mu_1} \oplus \cdots \oplus V_{\mu_k}$$

be the weight space decomposition under the action of A_{P_0}. Let μ_τ be the highest weight of τ with respect to the positive chamber $\mathfrak{a}^+_{P_0}$. Then each weight μ_i is of the form

$$\mu_i = \mu_\tau - \sum_{\alpha\in\Delta} c_\alpha \alpha,$$

where $c_\alpha \geq 0$. The subset $\{\alpha \in \Delta \mid c_\alpha \neq 0\}$ is called the support of μ_i, and denoted by $\mathrm{Supp}(\mu_i)$. For $P = P_{0,I}$, let V_P be the sum of all weight spaces V_{μ_i} whose support $\mathrm{Supp}(\mu_i)$ is contained in I. Since τ is generic, V_P is nontrivial. In fact, $P_{0,I}$ leaves V_P invariant and is equal to the stabilizer of V_P in G, and the representation of M_P on V_P is a multiple of an irreducible, faithful one, and hence τ induces an embedding $\tau_P : M_P \to PSL(V_P)$. The group $PSL(V_P)$ can be canonically embedded into $P_{\mathbb{C}}(M_{n\times n})$ by extending each matrix in $PSL(V_P)$ to act as the zero linear transformation on the orthogonal complement of V_P. Under this identification, for every $A \in PSL(V_P)$,

$$A(\mathbb{C}^n) = V_P.$$

Denote the composed embedding $M_P \to PSL(V_P) \hookrightarrow P_{\mathbb{C}}(M_{n\times n})$ also by τ_P,

$$\tau_P : M_P \to P_{\mathbb{C}}(M_{n\times n}).$$

Similarly, for P', we get a subspace $V_{P'}$ invariant under P' and hence under $M_{P'}$, a subset $PSL(V_{P'})$ in $P_{\mathbb{C}}(M_{n\times n})$, and an embedding

$$\tau_{P'} : M_{P'} \to PSL(V_{P'}) \subset P_{\mathbb{C}}(M_{n\times n}).$$

For any $k \in C$, ${}^kP \neq P'$, and hence

$$V_{{}^kP} \neq V_{P'}.$$

Since for any $m \in M_P$, $m' \in M_{P'}$, and any $g \in G$,

$$\tau_P(m)\tau(g)(\mathbb{C}^n) = \tau_P(m)(\mathbb{C}^n) = V_P,$$

$$\tau_{P'}(m')\tau(g)(\mathbb{C}^n) = \tau_{P'}(m')(\mathbb{C}^n) = V_{P'},$$

it follows that for any $g, g' \in G$, $m \in M_P, m' \in M_{P'}$, and $k \in C$,

$$\tau(k)\tau_P(m)\tau(g) \neq \tau_{P'}(m')\tau(g'). \tag{18}$$

If the claim is false, then there exists a sequence g_j in G such that

$$g_j \in k_j S_{\varepsilon_j,t_j,V}K \cap S_{\varepsilon_j,t_j,V'}K,$$

where $k_j \in C$, $\varepsilon_j \to 0$, $t_j \to +\infty$. Since $g_j \in k_j S_{\varepsilon_j,t_j,V} K$, g_j can be written as

$$g_j = k_j n_j a_j m_j c_j,$$

where $n_j \in N_P$, $a_j \in A_{P,t_j}$, $m_j \in M_P$, and $c_j \in K$ satisfy (1) for all $\alpha \in \Delta(P, A_P)$, $a_j^\alpha \to +\infty$, (2) $n_j^{a_j} \to e$, (3) $m_j \in V$. By passing to a subsequence, we can assume that $k_j \to k_\infty \in C$, m_j converges to some $m_\infty \in M_P$, and c_j converges to some c_∞ in K.

By choosing suitable coordinates, we can assume that for $a \in A_{P_0}$, $\tau(a)$ is a diagonal matrix,

$$\tau(a) = \mathrm{diag}\,(a^{\mu_1}, \dots, a^{\mu_n}),$$

where the weights μ_i with support contained in I are $\mu_1, \dots, \mu_l$ for some $l \geq 1$, and μ_1 is the highest weight μ_τ. Since τ is faithful and I is proper, $l < n$. Recall that $P = P_{0,I}$, and

$$A_P = \{a \in A_{P_0} \mid a^\alpha = 1, \alpha \in I\}.$$

Then

$$\begin{aligned}\tau(a_j) &= \mathrm{diag}\left(a_j^{\mu_1}, \dots, a_j^{\mu_l}, a_j^{\mu_{l+1}}, \dots, a_j^{\mu_n}\right)\\ &= \mathrm{diag}\left(a_j^{\mu_\tau}, \dots, a_j^{\mu_\tau}, a_j^{\mu_\tau - \sum_\alpha c_{l+1,\alpha}\alpha}, \dots, a_j^{\mu_\tau - \sum_\alpha c_{n,\alpha}\alpha}\right),\end{aligned}$$

where for each $j \in \{l+1, \dots, n\}$, there exists at least one $\alpha \in \Delta - I$ such that $c_{j,\alpha} > 0$. Then as $j \to +\infty$, the image of $\tau(a_j)$ in $P_{\mathbb{C}}(M_{n\times n})$

$$\begin{aligned}i_\tau(a_j) = &\left[\mathrm{diag}\left(1, \dots, 1, a_j^{-\sum_\alpha c_{l+1,\alpha}\alpha}, \dots, a_j^{-\sum_\alpha c_{n,\alpha}\alpha}\right)\right],\\ &\to [\mathrm{diag}\,(1, \dots, 1, 0, \dots, 0)],\end{aligned}$$

where the image of an element $A \in M_{n\times n} \setminus \{0\}$ in $P_{\mathbb{C}}(M_{n\times n})$ is denoted by $[A]$. This implies that

$$\begin{aligned}i_\tau(g_j) &= \tau(k_j) i_\tau(a_j) \tau(n_j^{a_j}) \tau(m_j) \tau(c_j)\\ &\to \tau(k_\infty)[\mathrm{diag}\,(1, \dots, 1, 0, \dots, 0)]\tau(m_\infty)\tau(c_\infty)\\ &= \tau(k_\infty)\tau_P(m_\infty)\tau(c_\infty),\end{aligned}$$

since $k_j \to k_\infty$, $n_j^{a_j} \to e$, $m_j \to m_\infty$, $c_j \to c_\infty$, and the image of $\mathbb{C}^n$ under $\mathrm{diag}\,(1, \dots, 1, 0, \dots, 0)$ is equal to V_P. Using $g_j \in S_{\varepsilon_j,t_j,V'} K$, we can similarly prove that

$$i_\tau(g_j) \to \tau_{P'}(m'_\infty)\tau(c'_\infty)$$

for some $m'_\infty \in M_{P'}$ and $c'_\infty \in K$. This contradicts Equation (18), and the claim and hence the proposition is proved. q.e.d.

Remark 2.5. It seems that the proof of [**Bo1**, Proposition 12.6] does not apply here. Assume that P, P' are both minimal. Then there exists an element $g \in G$, $g \notin P$ such that $P' = {}^gP$. In the proof of [**Bo1**, Proposition 12.6], g is written in the Bruhat decomposition

$uwzv$, where $w \in W(\mathfrak{g}, \mathfrak{a}_P)$, $w \neq id$, $u, v \in N_P$, $z \in A_P$. For each fixed g, the components u, v, z are bounded. This is an important step in the proof. If w is equal to the element w_0 of longest length, then for a sufficiently small neighborhood C of g in G (or K), every $g' \in C$ is of the form $u'w_0z'v'$ with the same Weyl group element W_0 and the components u', v', z' are uniformly bounded, and the same proof works. On the other hand, if w is not equal to w_0, then any neighborhood C of g contains elements g' of the form $u'w_0z'v'$ whose components u', z', v' are not uniformly bounded, and the method in [**Bo1**, Proposition 12.6] does not apply directly. The reason for the unboundedness of the components is that $N_P w_0$ is mapped to an open dense subset of G/P.

Remark 2.6. The above proof of Proposition 2.4 was suggested by the Hausdorff property of the maximal Satake compactification $\overline{X}^S_{\max}$. In fact, Proposition 2.4 follows from the Hausdorff property of $\overline{X}^S_{\max}$, by computations similar to those in the proof of Proposition 4.7. But the point here is to prove this separation property without using any compactification, so that it can be used to construct other compactifications.

Proposition 2.4 gives the separation property for different parabolic subgroups. For the same parabolic subgroup P, separation of Siegel sets for disjoint neighborhoods in X_P is proved in Proposition 4.1 below.

As mentioned earlier, the separation property and the finiteness property of Siegel sets for rational parabolic subgroups is a crucial result in the reduction theory of arithmetic subgroups of algebraic groups (see [**Bo1**], [**BJ2**]) and plays an important role in compactifications of locally symmetric spaces $\Gamma\backslash X$. One of the main points of this paper is that the above (stronger) separation property of the generalized Siegel sets for real parabolic subgroups will play a similar role in compactifications of X.

3. An intrinsic approach to compactifications

In this section, we propose an uniform, intrinsic approach to compactifications of X, suggested by the method in [**BS**] to compactify locally symmetric spaces. In the terminology in [**BJ1**], this method is called the *attachment method*, in contrast to the *embedding method* for the Satake, Furstenberg compactifications.

It consists of three steps:

1) Choose a suitable collection of parabolic subgroups of G.
2) For every parabolic subgroup P in the collection, define a boundary face (or component) $e(P)$ by making use of the Langlands decomposition of P and its refinements.
3) Attach the boundary face $e(P)$ to X via the horospherical decomposition of X to obtain $X \cup \coprod_P e(P)$, and show that the induced

topology on $X \cup \coprod_P e(P)$ is compact and Hausdorff, and the G-action on X extends continuously to the compactification.

All the known compactifications can be constructed this way by varying the choices of the collection of parabolic subgroups and their boundary faces. In fact, the maximal Satake compactification $\overline{X}^S_{\max}$, the conic compactification $X \cup X(\infty)$, the Martin compactification $X \cup \partial_\lambda X$, and the Karpelevic compactification $\overline{X}^K$ will be obtained by choosing the full collection of parabolic subgroups. On the other hand, for the non-maximal Satake compactifications, we can specify a sub-collection of parabolic subgroups according to a dominant weight vector.

There are several general features of this approach which will become clearer later.

1) It gives an explicit description of neighborhoods of boundary points in the compactifications of X and sequences of interior points converging to them, which clarifies the structure of the compactifications and is also useful for applications (see [**Zu**]). In [**Sa**] and other works [**GJT**], the G-orbits in the Satake compactifications $\overline{X}^S_\tau$ and convergent sequences in a maximal totally geodesic flat submanifold in X through the basepoint x_0 are fully described, but there does not seem to be explicit descriptions of neighborhoods of the boundary points in $\overline{X}^S_\tau$.
2) It relates compactifications of symmetric spaces X directly to compactifications of locally symmetric spaces $\Gamma\backslash X$; in fact, for locally symmetric spaces, the method of [**BS**] modified in [**BJ2**] consists of similar steps by considering only boundary faces associated with rational parabolic subgroups instead of all real parabolic subgroups for symmetric spaces, and both constructions depend on the reduction theory; in particular, separation property of Siegel sets in Propositions 2.1 and 2.4.
3) By decomposing the boundary into boundary faces associated with parabolic subgroups, its relation to the spherical Tits building of G becomes transparent. (We note that the spherical Tits building is an infinite simplicial complex with one simplex for each real parabolic subgroup whose dimension is equal to the parabolic rank minus 1, and the face inclusion relation is opposite to the inclusion relation for parabolic subgroups.) It can be seen below that for many known compactifications, the boundary faces are cells, and hence the whole boundary of the compactifications is a cell complex parametrized by the Tits building, a fact emphasized in [**GJT**].
4) By treating all the compactifications of X uniformly, relations between them can easily be determined by comparing their boundary faces.

5) Due to the gluing procedure using the horospherical decomposition, the extension of the G-action to the compactifications can be obtained easily. In [**GJT**], the extension of the G-action to the dual cell compactification $X \cup \Delta^*(X)$ is obtained through identification with the Martin compactification, rather than directly. This fact is one of the motivations of this paper.
6) Due to the definition of the topology at infinity, one difficulty is to show the Hausdorff property of the topology. This will follow from the strong separation property of the generalized Siegel sets in Proposition 2.4 and Proposition 4.1.

Remark 3.1. Since there are continuous families of real parabolic subgroups, we also need to put a topology on the set of boundary faces, for example to measure whether points on different boundary faces of conjugate parabolic subgroups are close to each other, while such a problem does not arise for compactifications of locally symmetric spaces. Using the parametrization of boundary faces by the spherical Tits building, we can use the topological Tits building in [**BuS**] to topologize the set of boundary faces. In this sense, the compactifications in this paper are more closely related to the topological Tits building than the usual Tits building.

4. The maximal Satake compactification

In this section, we follow the general method outlined in §3 to construct a compactification $\overline{X}_{\max}$ which will turn out to be isomorphic to the maximal Satake compactification $\overline{X}^S_{\max}$.

For $\overline{X}_{\max}$, we use the whole collection of parabolic subgroups. For every parabolic subgroup P, define its boundary face by

$$e(P) = X_P,$$

the boundary symmetric space defined in Equation (2) in §2. Let

$$\overline{X}_{\max} = X \cup \coprod_P X_P.$$

By Equations (4, 5) in §2, the K-action on parabolic subgroups preserves the Langlands, horospherical decomposition, and hence K acts on $\overline{X}_{\max}$ as follows: for $k \in K$, $z = mK_P \in X_P$,

$$k \cdot z = {}^k m \in X_{{}^k P}.$$

The topology of $\overline{X}_{\max}$ is defined as follows. First we note that X and X_P have a topology defined by the invariant metric. We need to define convergence of sequences of interior points in X to boundary points and convergence of sequences of boundary points:

1) For a boundary face X_P and a point $z_\infty \in X_P$, a unbounded sequence y_j in X converges to z_∞ if and only if y_j can be written in

the form $y_j = k_j n_j a_j z_j$, where $k_j \in K, n_j \in N_P, a_j \in A_P, z_j \in X_P$ such that

(a) $k_j \to e$, where e is the identity element.
(b) For all $\alpha \in \Phi(P, A_P)$, $a_j^\alpha \to +\infty$.
(c) $n_j^{a_j} \to e$.
(d) $z_j \to z_\infty$.

2) Let Q be a parabolic subgroup containing P. For a sequence $k_j \in K$ with $k_j \to e$, and a sequence $y_j \in X_Q$, the sequence $k_j y_j \in X_{k_j Q}$ converges to $z_\infty \in X_P$ if the following conditions are satisfied. Let P' be the unique parabolic subgroup in M_Q that corresponds to P as in Equation (7) in §2, and write $X_Q = N_{P'} \times A_{P'} \times X_{P'}$. The sequence y_j can be written as $y_j = k'_j n'_j a'_j z'_j$, where $k'_j \in K_Q, n'_j \in N_{P'}, a'_j \in A_{P'}, z'_j \in X_{P'} = X_P$ satisfy the same condition as part (1) above when K, N_P, A_P, X_P are replaced by $K_Q, N_{P'}$, $A_{P'}$, $X_{P'}$. Note that if $Q = P$, then $P' = M_Q$, and $N_{P'}, A_{P'}$ are trivial.

These are special convergent sequences, and combinations of them give general convergent sequences. By a combination of these special sequences, we mean a sequence $\{y_j\}$, $j \in \mathbb{N}$, and a splitting $\mathbb{N} = A_1 \coprod \cdots \coprod A_s$ such that for each infinite A_i, the corresponding subsequence $y_j, j \in A_i$, is a sequence of type either 1 or 2. It can be shown easily that these convergent sequences satisfy the conditions in [**JM**, §6]. In fact, the main condition to check is the double sequence condition and this condition is satisfied by double sequences of either type 1 or type 2 above, and hence by general double sequences. Therefore these convergent sequences define a unique topology on $\overline{X}_{\max}$. In fact, a neighborhood system of boundary points can be given explicitly.

For every parabolic subgroup P, let P_I, $I \subset \Delta(P, A_P)$, be all the parabolic subgroups containing P. For every P_I, X_{P_I} contains X_P as a boundary face. For any point $z \in X_P$, let V be a neighborhood of z in X_P. For $\varepsilon > 0, t > 0$, let $S_{\varepsilon,t,V}$ be the generalized Siegel set in X defined in Equation (15) in §2, and let $S^I_{\varepsilon,t,V}$ be the generalized Siegel set of X_{P_I} associated to the parabolic subgroup P' in M_{P_I} as in Equation (7) in §2. Let C be a (compact) neighborhood of e in K. Then the union

$$C\left(S_{\varepsilon,t,V} \cup \coprod_{I \subset \Delta} S^I_{\varepsilon,t,V}\right)$$

is a neighborhood of z in $\overline{X}_{\max}$. For sequences of $\varepsilon_i \to 0, t_i \to +\infty$, a basis V_i of neighborhoods of z in X_P and a basis of compact neighborhoods C_j of e in K, the above union forms a countable basis of the neighborhoods of z in $\overline{X}_{\max}$.

It can be checked easily that these neighborhoods define a topology on $\overline{X}^{\max}$ whose convergent sequences are exactly those given above.

When a point $y_j \in X$ is written in the form $k_j n_j a_j z_j$ with $k_j \in K, n_j \in N_P, a_j \in A_P, z_j \in X_P$, none of these factors is unique, since $X = N_P \times A_P \times X_P$ and the extra K-factor causes non-uniqueness. Then a natural question is the uniqueness of the limit of a convergent sequence y_j in $\overline{X}_{\max}$, or equivalently, whether the topology on $\overline{X}_{\max}$ is Hausdorff.

Proposition 4.1. *For a parabolic subgroup P and two different boundary points $z, z' \in X_P$, let V, V' be compact neighborhoods of z, z' with $V \cap V' = \emptyset$. If ε is sufficiently small, t is sufficiently large and C is a sufficiently small compact neighborhood of e in K, then for all $k, k' \in C$, the generalized Siegel sets $kS_{\varepsilon,t,V}, k'S_{\varepsilon,t,V'}$ are disjoint.*

Proof. We prove this proposition by contradiction. If not, then for all $\varepsilon > 0, t > 0$ and any neighborhood C of e in K,

$$kS_{\varepsilon,t,V} \cap k'S_{\varepsilon,t,V'} \neq \emptyset,$$

for some $k, k' \in C$. Therefore, there exist sequences $k_j, k'_j \in K$, $n_j, n'_j \in N_P$, $a_j, a'_j \in A_P$, $m_j \in VK_P, m'_j \in V'K_P$ such that

1) $k_j, k'_j \to e$,
2) $n_j^{a_j} \to e,\ n_j'^{a'_j} \to e$,
3) For all $\alpha \in \Delta(P, A_P)$, $a_j^\alpha,\ a_j'^\alpha \to +\infty$,
4) $k_j n_j a_j m_j K = k'_j n'_j a'_j m'_j K$.

Since $VK_P, V'K_P$ are compact, after passing to a subsequence, we can assume that both m_j and m'_j converge. Denote their limits by m_∞, m'_∞. By assumption, $VK_P \cap V'K_P = \emptyset$, and hence

$$m_\infty K \neq m'_\infty K.$$

We claim that the conditions (1), (2) and (3) together with $m_\infty K \neq m'_\infty K$ contradict the condition (4).

As in the proof of Proposition 2.4, let $\tau : G \to PSL(n, \mathbb{C})$ be a faithful representation whose highest weight μ_τ is generic and $\tau(\theta(g)) = (\tau(g)^*)^{-1}$, where θ is the Cartan involution associated with K. Let $\mathcal{H}_n$ be the real vector space of $n \times n$ Hermitian matrices and $P(\mathcal{H}_n)$ the associated projective space. Then τ defines an embedding

$$i_\tau : G/K \to \mathcal{P}(\mathcal{H}_n), \quad gK \mapsto [\tau(g)\tau^*(g)],$$

where $[\tau(g)\tau^*(g)]$ denotes the line determined by $\tau(g)\tau^*(g)$. We will prove the claim by determining the limits of $i_\tau(k_j n_j a_j m_j)$ and $i_\tau(k'_j n'_j a'_j m'_j)$.

Let P_0 be a minimal parabolic subgroup contained in P. Then $P = P_{0,I}$ for a unique subset $I \subset \Delta(P_0, A_{P_0})$. As in the proof of Proposition 2.4, we can assume that for $a \in A_{P_0}$, $\tau(a)$ is diagonal,

$$\tau(a) = \operatorname{diag}(a^{\mu_1}, \ldots, a^{\mu_n}),$$

and the weights $\mu_1, \ldots, \mu_l$ are the weights whose supports are contained in I. Then

$$\begin{aligned}(1)\quad & i_\tau(k_j n_j a_j m_j)\\ &= [\tau(k_j)\tau(a_j)\tau(n_j^{a_j})\tau(m_j)\tau(m_j)^*\tau(n_j^{a_j})^*\tau(a_j)^*\tau(k_j)^*]\\ &\to [\mathrm{diag}\,(1,\ldots,1,0,\ldots,0)\tau(m_\infty)\tau(m_\infty)^*\mathrm{diag}\,(1,\ldots,1,0\ldots,0)^*]\\ &= [\tau_P(m_\infty)\tau_P(m_\infty)^*],\end{aligned}$$

where

$$\begin{aligned}\tau_P : M_P \to PSL(V_P) \hookrightarrow P_{\mathbb{C}}(M_{n\times n}),\\ m \mapsto [\mathrm{diag}\,(1,\ldots,1,0,\ldots,0)\tau(m)],\end{aligned}$$

is the map in the proof of Proposition 2.4. Since τ_P is a faithful representation,

$$\tau_P\tau_P^* : X_P \to P(\mathcal{H}_n), \quad mK_P \mapsto [\tau_P(m)\tau_P(m)^*]$$

is an embedding.

Similarly, we get

$$(2)\qquad i_\tau(k_j' n_j' a_j' m_j') \to [\tau_P(m_\infty')\tau_P(m_\infty')^*].$$

Since $m_\infty, m_\infty' \in M_P$ and $m_\infty K \neq m_\infty' K$, we get

$$[\tau_P(m_\infty)\tau_P(m_\infty)^*] \neq [\tau_P(m_\infty')\tau_P(m_\infty')^*].$$

Then the condition (4) implies that Equation (2) contradicts Equation (1) and the claim is proved. q.e.d.

As mentioned earlier, Proposition 2.4 concerns separation of general Siegel sets associated with different parabolic subgroups, while Proposition 4.1 here concerns Siegel sets associated with different points on the same boundary face. They are both needed below.

Proposition 4.2. *The topology on $\overline{X}_{\max}$ is Hausdorff.*

Proof. We need to show that every pair of different points $x_1, x_2 \in \overline{X}_{\max}$ admit disjoint neighborhoods. This is clearly the case when at least one of x_1, x_2 belongs to X. Assume that both belong to the boundary and let P_1, P_2 be the parabolic subgroups such that $x_1 \in X_{P_1}, x_2 \in X_{P_2}$. There are two cases to consider: $P_1 = P_2$ or not.

For the second case, let C be a sufficiently small compact neighborhood of e in K such that for $k_1, k_2 \in C$,

$$^{k_1}P_1 \neq {}^{k_2}P_2.$$

Then $C(S_{\varepsilon,t,V_i} \cup \coprod_I S^I_{\varepsilon,t,V_i})$ is a neighborhood of x_i.

Proposition 2.4 implies that

$$CS_{\varepsilon,t,V_1} \cap CS_{\varepsilon,t,V_2} = \emptyset.$$

For all pairs of I_1, I_2, $k_1, k_2 \in C$, either

$$ {}^{k_1}P_{1,I_1} \neq {}^{k_2}P_{2,I_2}, $$

and hence

$$ k_1 S^{I_1}_{\varepsilon,t,V_1} \cap k_2 S^{I_2}_{\varepsilon,t,V_2} = \emptyset, $$

or

$$ {}^{k_1}P_{1,I_1} = {}^{k_2}P_{2,I_2}. $$

In the latter case, $P_1, ({}^{k_2}P_2)^{k_1}$ are contained in P_{1,I_1} and correspond to two different parabolic subgroups of $M_{P_{1,I_1}}$. As in the case above for general Siegel sets in X, we get

$$ k_1 S^{I_1}_{\varepsilon,t,V_1} \cap k_2 S^{I_2}_{\varepsilon,t,V_2} = \emptyset. $$

This implies that the two neighborhoods are disjoint.

In the first case, $P_1 = P_2$. Since $x_1 \neq x_2$, we can choose compact neighborhoods V_1, V_2 in X_{P_1} such that $V_1 \cap V_2 = \emptyset$. Then Proposition 4.1 together with similar arguments as above imply x_1, x_2 admit disjoint neighborhoods. This completes the proof of this proposition. q.e.d.

Proposition 4.3. *The topological space $\overline{X}_{\max}$ is compact and contains X as a dense open subset.*

Proof. Let P_0 be a minimal parabolic subgroup, and $P_{0,I}$, $I \subset \Delta = \Delta(P_0, A_{P_0})$, be all the standard parabolic subgroups. Then

$$ \overline{X}_{\max} = X \cup \coprod_{I \subset \Delta} K X_{P_{0,I}}. $$

Since K is compact, it suffices to show that every sequence in X and $X_{P_{0,I}}$ has a convergent subsequence. First, we consider a sequence in X. If y_j is bounded, it clearly has a convergent subsequence in X. Otherwise, writing $y_j = k_j a_j x_0$, $k_j \in K, a_j \in \overline{A^+_{P_0}}$, we can assume, by replacing by a subsequence, that the components of y_j satisfy the conditions:

1) $k_j \to k_\infty$ for some $k_\infty \in K$,
2) there exists a subset $I \subset \Delta(P_0, A_{P_0})$ such that for $\alpha \in \Delta - I$, $\alpha(\log a_j) \to +\infty$, while for $\beta \in I$, $\beta(\log a_j)$ converges to a finite number.

Decompose

$$ \log a_j = H_{I,j} + H^I_j, \quad H_{I,j} \in \mathfrak{a}_{P_{0,I}}, H^I_j \in \mathfrak{a}^I_{P_0}. $$

Since $\Delta(P_0, A_{P_0}) - I$ restricts to $\Delta(P_{0,I}, A_{P_{0,I}})$, it follows from the definition that $k_j^{-1} y_j = a_j x_0$ converges to $e^{H^I_\infty} x_0 \in X_{P_{0,I}}$ in $\overline{X}_{\max}$, where x_0 also denotes the basepoint $K_{P_{0,I}}$ in $X_{P_{0,I}}$, and H^I_∞ is the unique vector in $\mathfrak{a}^I_{P_0}$ such that for all $\beta \in I$, $\beta(H^I_\infty) = \lim_{j \to +\infty} \beta(\log a_j)$. Together with the action of K on parabolic subgroups and the Langlands decomposition in Equation (6) in §2, this implies that $y_j = (k_j k_\infty^{-1})^{\ k_\infty} a_j x_0$ converges to a point in $X_{({}^{k_\infty}P_{0,I})}$ in $\overline{X}_{\max}$.

For a sequence in $X_{P_{0,I}}$, we can similarly use the Cartan decomposition $X_{P_{0,I}} = K_{P_{0,I}} \exp \overline{\mathfrak{a}_{P_0}^{I,+}} x_0$ to extract a convergent subsequence in $\overline{X}_{\max}$. q.e.d.

Proposition 4.4. *The action of G on X extends to a continuous action on $\overline{X}_{\max}$.*

Proof. First we define a G-action on the boundary $\partial \overline{X}_{\max} = \coprod_P X_P$, then show that this gives a continuous G-action on $\overline{X}_{\max}$.

For $g \in G$ and a boundary point $z \in X_P$, write

$$g = kman,$$

where $k \in K, m \in M_K, a \in A_P, n \in N_P$. Define

$$g \cdot z = k \cdot (mz) \in X_{{}^kP},$$

where $k \cdot (mz)$ is defined in Equation (5) in §2. We note that k, m are determined up to a factor in K_P, but km is uniquely determined by g, and hence this action is well-defined. As pointed out earlier, under this action, $k \cdot X_P = X_{{}^kP}$.

To prove the continuity of this G-action, we first show that if $g_j \to g_\infty$ in G and a sequence $y_j \in X$ converging to $z_\infty \in X_P$, then $g_j y_j \to g_\infty z_\infty$.

By definition, y_j can be written in the form $y_j = k_j n_j a_j z_j$ such that (1) $k_j \in K$, $k_j \to e$, (2) $a_j \in A_P$, and for all $\alpha \in \Phi(P, A_P)$, $\alpha(\log a_j) \to +\infty$, (3) $n_j \in N_P$, $n_j^{a_j} \to e$, and (4) $z_j \in X_P$, $z_j \to z_\infty$. Write

$$g_j k_j = k'_j m'_j a'_j n'_j,$$

where $k'_j \in K, m'_j \in M_P$, $a'_j \in A_P$, $n'_j \in N_P$. Then a'_j, n'_j are uniquely determined by $g_j k_j$ and bounded, and $k'_j m'_j$ converges to the KM_P-component of g. By choosing suitable factors in K_P, we can assume that $k'_j \to k$, and $m'_j \to m$, where $g = kman$ as above. Since

$$g_j y_j = k'_j m'_j a'_j n'_j n_j a_j z_j = k'_j\ {}^{m'_j a'_j}(n'_j n_j) a'_j a_j m'_j z_j,$$

and

$$(n'_j n_j)^{a'_j a_j} \to e, \quad m'_j \to m,$$

it follows from the definition of convergence of sequences that $g_j y_j$ converges to $k \cdot (m z_\infty) \in X_{{}^kP}$ in $\overline{X}_{\max}$, which is equal to $g \cdot z_\infty$ as defined above.

The same proof works for a sequence y_j in X_Q for any parabolic subgroup $Q \supset P$. A general sequence in $\overline{X}_{\max}$ follows from combinations of these two cases, and the continuity of this extended G action on $\overline{X}_{\max}$ is proved. q.e.d.

Remark 4.5. From the above proof it is clear that the N_P-factor in the definition of convergence to boundary points is crucial to the continuous extension of the G-action. In [**GJT**], convergence to boundary points in the dual cell compactification $X \cup \Delta^*(X)$ is defined in terms of the Cartan decomposition $X = K\overline{A^+}x_0$. Because of this difference, the extension of the G-action to $X \cup \Delta^*(X)$ is not easy. In fact, the continuity of the G-action is not proved directly there. As mentioned earlier, this is one of the motivations of this paper.

Next we identify this compactification with the maximal Satake compactification of X. We first recall the Satake compactifications. As mentioned in §2, G is assumed to be an adjoint semisimple Lie group in this paper; otherwise the faithful projective representations below need to be replaced by locally faithful representations.

As in the proof of Proposition 2.4, for every faithful projective representation $\tau : G \to PSL(n, \mathbb{C})$ satisfying

$$\tau(\theta(g)) = (\tau(g)^*)^{-1}, \quad g \in G,$$

there is an embedding

$$\tau : X \to PSL(n, \mathbb{C})/PSU(n), \quad gK \mapsto \tau(g)PSU(n),$$

which is in fact a totally geodesic embedding (see [**Sa**]). Let $\mathcal{H}_n$ be the real vector space of $n \times n$ Hermitian matrices, and $P(\mathcal{H}_n)$ be the real projective space. Then τ induces an embedding

$$i_\tau : X \to P(\mathcal{H}_n), \quad gK \to [\tau(g)\tau(g)^*],$$

where $[A]$ represents the line in $P(\mathcal{H}_n)$ determined by A. The closure of $i_\tau(X)$ in $P(\mathcal{H}_n)$ is the Satake compactification associated with τ and denoted by $\overline{X}^S_\tau$.

Let P_0 be a minimal parabolic subgroup, and $\mu_\tau \in \overline{\mathfrak{a}^+_{P_0}(\infty)}$ the highest weight of the representation τ. Then it is shown in [**Sa**] that as a topological G-space, $\overline{X}^S_\tau$ only depends on the degeneracy of μ_τ, i.e., the Weyl chamber face which contains μ_τ as an interior point. The set of Satake compactifications is partially ordered. The following fact was first proved in [**Zu**, Proposition 2.11], though it was expected and understood earlier by others. We will obtain another proof in Remark 5.8 below, which gives an explicit surjective map between the two compactifications.

Proposition 4.6. *For two Satake compactifications $\overline{X}^S_{\tau_1}, \overline{X}^S_{\tau_2}$, let $\mathfrak{a}^+_{P_{0,I_1}}, \mathfrak{a}^+_{P_{0,I_2}}$ be the Weyl chamber faces containing the highest weights $\mu_{\tau_1}, \mu_{\tau_2}$ as interior points respectively. If $\mathfrak{a}^+_{P_{0,I_2}}$ contains $\mathfrak{a}^+_{P_{0,I_1}}$ as a face in its closure, i.e., μ_{τ_2} is more regular than μ_{τ_1}, then the identity map on X extends to a continuous surjective G-equivariant map*

$$\overline{X}^S_{\tau_2} \to \overline{X}^S_{\tau_1}.$$

When μ_τ is generic, i.e., belongs to the positive chamber $\mathfrak{a}_P^+$, $\overline{X}_\tau^S$ is the unique maximal Satake compactification, denoted by $\overline{X}_{\max}^S$.

In [**Fu**], Furstenberg defined compactifications of X by embedding X into the space of probability measures $\mathcal{M}(G/P)$ on G/P, where P is a parabolic subgroup. Specifically, since $G = KP$, there is a unique K-invariant probability measure ν_P on G/P. Define a map

$$i_P : X \to \mathcal{M}(G/P), \quad gK \mapsto g^*\nu_P.$$

Since ν_P is K-invariant, this map is well-defined. Assume that P is a standard parabolic subgroup containing P_0 and does not contain any simple factor of G. Then i_P is an embedding, and the closure of $i_P(X)$ in $\mathcal{M}(G/P)$ is a Furstenberg compactification. Moore showed in [**Mo**] that this compactification is isomorphic to the Satake compactification $\overline{X}_\tau^S$ whose highest weight μ_τ belongs to the interior of $\mathfrak{a}_P^+$. The above condition on P is equivalent to the condition that μ_τ is connected to every connected component of $\Delta(P_0, A_{P_0})$, which is satisfied for every faithful projective representation τ. In particular, for the minimal parabolic subgroup P_0, the associated Furstenberg compactification $\overline{i_{P_0}(X)}$ is isomorphic to the maximal Satake compactification $\overline{X}_{\max}^S$. Due to this connection, the Satake compactifications are also called *Satake-Furstenberg compactifications.*

Proposition 4.7. *For any Satake compactification $\overline{X}_\tau^S$, the identity map on X extends to a continuous G-equivariant surjective map $\overline{X}_{\max} \to \overline{X}_\tau^S$.*

Proof. Since every boundary point of $\overline{X}_{\max}$ is the limit of a sequence of points in X, by [**GJT**, Lemma 3.28], it suffices to show that for any unbounded sequence y_j in X which converges in $\overline{X}_{\max}$, then y_j also converges in $\overline{X}_\tau^S$. By definition, there exists a parabolic subgroup P such that y_j can be written as $y_j = k_j n_j a_j m_j K_P$, where $k_j \in K, n_j \in N_P, a_j \in A_P$, $m_j K_P \in X_P$ satisfy the conditions: (1) $k_j \to e$, (2) $n_j^{a_j} \to e$, (3) for all $\alpha \in \Phi(P, A_P)$, $\alpha(\log a_j) \to +\infty$, (4) $m_j K_P$ converges to $m_\infty K_P$ for some m_∞. Then under the map $i_\tau : X \to P(\mathcal{H}_n)$,

$$\begin{aligned} i_\tau(y_j) &= [\tau(k_j n_j a_j m_j)\tau(k_j n_j a_j m_j)^*] \\ &= [\tau(k_j)\tau(a_j)\tau(n_j^{a_j})\tau(m_j)\tau(m_j)^*\tau(n_j^{a_j})^*\tau(a_j)\tau(k_j)^*]. \end{aligned}$$

Let P_0 be a minimal parabolic subgroup contained in P. Write $P = P_I$. As in the proof of Proposition 2.4 (or 4.1), we can assume, with respect to a suitable basis, that $\tau(a_j) = \mathrm{diag}\,(a_j^{\mu_1}, \dots, a_j^{\mu_n})$ and that $\mu_1, \dots, \mu_l$ are all the wrights whose supports are contained in I. Then as $j \to +\infty$,

$$[\mathrm{diag}\,(a_j^{\mu_1}, \dots, a_j^{\mu_n})] \to [\mathrm{diag}\,(1, \dots, 1, 0, \dots, 0)];$$

and hence

$$i_\tau(y_j) \to [\mathrm{diag}\,(1,\dots,1,0,\dots,0)\tau(m_\infty)\tau(m_\infty)^*\mathrm{diag}\,(1,\dots,1,0,\dots,0)^*].$$

q.e.d.

Proposition 4.8. *For the maximal Satake compactification $\overline{X}^S_{\max}$, the map $\overline{X}_{\max} \to \overline{X}^S_{\max}$ in Proposition 4.7 is a homeomorphism.*

Proof. Since both $\overline{X}_{\max}$ and $\overline{X}^S_{\max}$ are compact and Hausdorff, it suffices to show that the map $\overline{X}_{\max} \to \overline{X}^S_{\max}$ is injective. It was shown in [**Sa**] that $\overline{X}^S_{\max} = X \cup \coprod_P X_P$. By the proof of the previous proposition, a sequence $y_j = k_j n_j a_j m_j K_P$ in X satisfying the conditions above with $m_j K_P \to m_\infty K_P$ in $\overline{X}_{\max}$ converges to the same limit as the sequence $a_j m_j K_P$. Under the above identification of $\overline{X}^S_{\max}$, $a_j m_j K_P$ converges to $m_\infty K_P \in X_P$ in $\overline{X}^S_{\max}$. This implies that the map $\overline{X}_{\max} \to \overline{X}^S_{\max}$ is the identity map under the identification $\overline{X}_{\max} = X \cup \coprod_P X_P = \overline{X}^S_{\max}$, and hence is injective. q.e.d.

Recall that the dual cell compactification $X \cup \Delta^*(X)$ in [**GJT**] is constructed via the Cartan decomposition. Specifically, let $A^+ = A^+_{P_0}$ be a positive chamber, where P_0 is a minimal parabolic subgroup. Then a sequence y_j in X converges in $X \cup \Delta^*(X)$ if and only if y_j admits a decomposition $y_j = k_j a_j x_0$, where $k_j \in K, a_j \in \overline{A^+}$ satisfy the following conditions:

1) k_j converges to some k_∞,
2) there exists a subset I of $\Delta(P_0, A_{P_0})$ such that for $\alpha \in I$, a_j^α converges to a finite number, while for $\alpha \in \Delta - I$, $a_j^\alpha \to +\infty$.

Proposition 4.9. *The identity map on X extends to a continuous map from $X \cup \Delta^*(X)$ to $\overline{X}_{\max}$, which is a bijection and hence a homeomorphism.*

Proof. For an unbounded sequence $y_j = k_j a_j x_0$ in X that converges in $X \cup \Delta^*(X)$, let I be the subset of Δ in the above definition. Write

$$a_j = a_{j,I} a_j^I, \quad \log a_{j,I} \in \mathfrak{a}_{P_0,I}, \quad \log a_j^I \in \mathfrak{a}^I_{P_0}.$$

Then for $\alpha \in I$, $\alpha(\log a_j^I) = \alpha(\log a_j)$, and hence a_j^I has a limit a_∞^I in $\mathfrak{a}^I_{P_0}$. This also implies that for $\alpha \in \Delta - I$, $\alpha(\log a_{j,I}) \to +\infty$. Since $y_j = k_j a_{j,I} a_j^I x_0$ and $a_j^I x_0 \in X_{P_{0,I}}$, it is clear that y_j converges to $k_\infty a_\infty^I K_P \in k_\infty X_{P_{0,I}} = X_{k_\infty P_{0,I}}$ in $\overline{X}_{\max}$. This implies that there is a well-defined map from $X \cup \Delta^*(X)$ to $\overline{X}_{\max}$ which restricts to the identity on X. Since $X \cup \Delta^*(X)$ is a metrizable space, it can be shown as in [**GJT**, Lemma 3.28] that this map is continuous, and hence is automatically surjective. By [**GJT**, Proposition 3.44],

$$X \cup \Delta^*(X) = X \cup \coprod_P X_P,$$

and the above sequence y_j also converges to $k_\infty a_\infty^I K_P \in k_\infty X_{P_0,I}$. Therefore, the map $\overline{X}_{\max} \to X \cup \Delta^*(X)$ is bijective and hence is a homeomorphism. q.e.d.

5. Non-maximal Satake compactifications

In the previous section, we constructed the maximal Satake compactification following the general method in §3. In this section, we use it to construct non-maximal Satake compactifications.

For $\overline{X}^S_{\max}$ in §4, we used the whole collection of parabolic subgroups. In this section, we choose a sub-collection of parabolic subgroups and attach only boundary faces associated with them at infinity. Let P_0 be a minimal parabolic subgroup, and $\mu \in \overline{\mathfrak{a}^{*+}_{P_0}}$, a dominant weight. For each such μ, we will construct a compactification $\overline{X}_\mu$. Before defining the boundary components, we need to choose the collection of parabolic subgroups and to refine the horospherical decomposition of X which are needed to attach the boundary components at infinity.

A subset $I \subset \Delta(P_0, A_{P_0})$ is called *μ-connected* if the union $I \cup \{\mu\}$ is connected, i.e., it can not be written as a disjoint union $I_1 \coprod I_2$ such that elements in I_1 are perpendicular to elements in I_2 with respect to a positive definite inner product on $\mathfrak{a}^*_{P_0}$ invariant under the Weyl group.

A standard parabolic subgroup $P_{0,I}$ is called μ-connected if I is μ-connected. For any standard parabolic subgroup $P_{0,J}$, let P_{0,I_J} be the unique maximal one among all the μ-connected standard parabolic subgroups contained in $P_{0,J}$, i.e., I_J is the largest μ-connected subset contained in J. Then P_{0,I_J} is called a μ-reduction of $P_{0,J}$. In general, a parabolic subgroup P is called μ-connected if P is conjugate to a μ-connected standard parabolic subgroup, and for every parabolic subgroup Q, there are maximal subgroups P among the collection of all the μ-connected parabolic subgroups contained in Q. Such parabolic subgroups P in Q are not unique. In fact, for any $g \in Q \setminus P$, ${}^gP$ is also a maximal μ-connected subgroup in this collection, but ${}^gP \neq P$; all such μ-connected parabolic subgroups P are conjugate by elements in Q, and called the μ-connected reductions of Q.

On the other hand, for every μ-connected parabolic subgroup P, there is a unique maximal subgroup Q among all the parabolic subgroups which contain P as a μ-connected reduction. Such a Q is called the μ-connected saturation of P, denoted by $S_\mu(P)$. A parabolic subgroup Q is called *μ-saturated* if it is equal to the μ-saturation of a μ-connected parabolic subgroup P.

For each dominant weight μ as above, we will construct a compactification of X which is isomorphic to a Satake compactification. Instead of the whole collection of parabolic subgroups, we choose the sub-collection of μ-saturated parabolic subgroups Q.

For each such Q, let P_0 be a minimal parabolic subgroup contained in Q. Then $Q = P_{0,J}$, $J \subset \Delta(P_0, A_{P_0})$. Let $I \subset J$ be the maximal μ-connected subset as above, and $I^\perp$ the complement of I in J, which is orthogonal to I and μ. Then I spans a sub-root system Σ_I, and $I^\perp$ spans $\Sigma_{I^\perp}$. Let $\mathfrak{g}_I$ be the Lie subalgebra generated by $\sum_{\alpha\in\Sigma_I}\mathfrak{g}_\alpha$, $\mathfrak{g}_{I^\perp}$ by $\sum_{\alpha\in\Sigma_{I^\perp}}\mathfrak{g}_\alpha$; and G_I, $G_{I^\perp}$ the corresponding subgroups in G. Then $K\cap G_I$, $K \cap G_{I^\perp}$ are maximal compact subgroups in G_I and $G_{I^\perp}$ respectively. Define

$$X_I = G_I/K\cap G_I = X_{P_{0,I}}, \quad X_{I^\perp} = G_{I^\perp}/K\cap G_{I^\perp} = X_{P_{0,I^\perp}}.$$

Then

$$X_Q = X_I \times X_{I^\perp} = X_{P_{0,I}} \times X_{P_{0,I^\perp}}. \tag{1}$$

In fact, $M_Q = G_I G_{I^\perp} M'$, where M' is the identity component of the intersection of M_{P_0} with the centralizer of $G_I G_I^\perp$.

Define the boundary component $e(Q)$ by

$$e(Q) = X_I.$$

To extend the group action to the compactification and to show that X_I and the splitting in Equation (1) do not depend on the choice of the minimal parabolic subgroup $P_0 \subset Q$, we realize X_I as a different quotient. Define

$$\mathcal{Z}(e(Q)) = N_Q A_Q G_{I^\perp} M_0.$$

Then X_I can be canonically identified with $Q/\mathcal{Z}(e(Q))K_Q$, where $K_Q = K \cap Q$, by the map

$$g(K\cap G_I) \in X_I \to g\mathcal{Z}(e(Q))K_Q \in Q/\mathcal{Z}(e(Q))K_Q.$$

Since all minimal parabolic subgroups P_0 contained in Q are conjugate under K_Q, $\mathcal{Z}(e(Q))$ is independent of the choice of P_0. In fact, it turns out to be the centralizer of $e(Q)$ defined below. Similarly, $X_{I^\perp}$ is well-defined, and hence the splitting in Equation (1) is independent of the choice of P_0.

As mentioned earlier, the μ-connected reductions P of Q are not unique. On the other hand, for any such P, X_P can be canonically identified with X_I and is hence independent of the choice of P.

To define the topology of the compactification, we need the following lemma.

Lemma 5.1. *Let Q be a μ-saturated parabolic subgroup, P_0 a minimal parabolic subgroup contained in Q, and $P_{0,I}$ the μ-connected reduction* *of Q as above. For any parabolic subgroup R satisfying $P_{0,I} \subseteq R \subseteq Q$, write $R = P_{0,J'}$ and $J' = I \cup I'$, where I' is perpendicular to I. Let*

$$X_R = X_I \times X_{P_{0,I'}}$$

be the decomposition similar to that in Equation (1). Then the horospherical decomposition of X with respect to R

$$X = N_R \times A_R \times X_R$$

can be refined to

$$X = N_R \times A_R \times X_I \times X_{P_{0,I'}}. \tag{2}$$

Proof. When $R = Q$, the decomposition $X_Q = X_I \times X_{P_{0,I'}}$ was described above. The general case is similar. q.e.d.

As commented earlier in the case $R = Q$, the decomposition $X_R = X_I \times X_{P_{0,I'}}$ is independent of the choice of the minimal parabolic subgroup P_0. In the following, $X_{P_{0,I'}}$ is denoted by $X_{I'}$ and the decomposition in Equation (2) is written as

$$X = N_R \times A_R \times X_I \times X_{I'}. \tag{3}$$

Now we are ready to define the compactification $\overline{X}_\mu$ of X by attaching only the boundary components associated with μ-saturated parabolic subgroups. Define

$$\begin{aligned}\overline{X}_\mu = X \cup \coprod_{\mu\text{-saturated } Q} e(Q) &= X \cup \coprod_{\mu\text{-saturated } Q} X_I \\ &= X \cup \coprod_{\mu\text{-saturated } Q} X_{P(Q)},\end{aligned}$$

where $P(Q)$ is a μ-connected reduction of Q.

We can define a G-action on $\overline{X}_\mu$ as follows. For a μ-saturated parabolic subgroup Q, a point

$$z \in e(Q) = Q/\mathcal{Z}(e(Q))K_Q,$$

and an element $g \in G$, write $z = h\mathcal{Z}(e(Q))K_Q$, and $g = kq$, where $k \in K$, $q \in Q$. Then

$$qh\mathcal{Z}(e(Q))K_Q \in e(Q), \quad k \cdot (qh\mathcal{Z}(e(Q))K_Q) \in e({}^kQ) = e({}^gQ).$$

Though the decomposition $g = kq$ is not unique, the point

$$k \cdot (qh\mathcal{Z}(e(Q))K_Q) \in e({}^kQ)$$

is well-defined. Define the group action by

$$g \cdot z = k \cdot (qh\mathcal{Z}(e(Q))K_Q) \in e({}^gQ).$$

With respect to the above action of G, the group Q is the normalizer of $e(Q)$, i.e., the set of $g \in G$ such that $g \cdot e(Q) = e(Q)$ and is often denoted by $\mathcal{N}(e(Q))$, and the subgroup $\mathcal{Z}(e(Q))$ defined above is the centralizer of $e(Q)$,

$$\mathcal{Z}(e(Q)) = \{g \in G \mid g \cdot z = z \quad \text{for all} \quad z \in e(Q)\}.$$

It is a normal subgroup of $\mathcal{N}(e(Q))$ and we have

$$\mathcal{N}(e(Q)) = G_I \cdot \mathcal{Z}(e(Q)),$$

where $G_I \cap \mathcal{Z}(e(Q))$ is finite.

A topology on $\overline{X}_\mu$ is given as follows:

1) For a μ-saturated parabolic subgroup Q, let $P_0 \subseteq Q$ be a minimal parabolic subgroup, and $P_{0,I}$ be a μ-connected reduction of Q. Then an unbounded sequence y_j in X converges to a boundary point $z_\infty \in e(Q) = X_I$ if there exists a parabolic subgroup R that is contained in Q and contains $P_{0,I}$ such that in the refined horospherical decomposition $X = N_R \times A_R \times X_I \times X_{I'}$ in Equation (3), y_j can be written in the form $y_j = k_j n_j a_j z_j z'_j$ such that the factors $k_j \in K$, $n_j \in N_R$, $a_j \in A_R$, $z_j \in X_I$, $z'_j \in X_{I'}$ satisfy the following conditions:
   (a) the image of k_j in the quotient $K/K \cap \mathcal{Z}(e(Q))$ converges to the identity coset,
   (b) for $\alpha \in \Delta(P_0, A_{P_0}) \setminus (I \cup I^\perp) = \Delta(Q, A_{P_0})$ (see Equation (10) in §2 for the definition of $\Delta(Q, A_{P_0})$ and related sets below), $a_j^\alpha \to +\infty$, while for $\alpha \in I^\perp \setminus I' = \Delta(R, A_{P_0}) - \Delta(Q, A_{P_0})$, a_j^α is bounded from below.
   (c) $n_j^{a_j} \to e$,
   (d) $z_j \to z_\infty$,
   (e) z'_j is bounded.
2) For a pair of μ-saturated parabolic subgroups Q_1 and Q_2 such that a μ-connected reduction of Q_1 is contained in a μ-connected reduction of Q_2, let Q'_1 be the unique parabolic subgroup in M_{Q_2} determined by $Q_1 \cap Q_2$. For a sequence $k_j \in K$ whose image in $K/K \cap \mathcal{Z}(e(Q_1)$ converges to the identity coset, and a sequence y_j in $e(Q_2) = X_{I(Q_2)}$, the sequence $k_j y_j$ in $\overline{X}_\mu$ converges to $z_\infty \in e(Q_1) = X_{I(Q_1)}$ if y_j satisfies the same condition as in part (1) above when G is replaced by the subgroup $G_{I(Q_2)}$ of M_{Q_2} whose symmetric spaces of maximal compact subgroups is $X_{I(Q_2)}$, and Q by $Q'_1 \cap G_{I(Q_2)}$.

These are special convergent sequences, and their (finite) combinations give general convergent sequences. We note that all the μ-connected reductions of Q are conjugate under $\mathcal{Z}(e(Q))$. Hence in (1), it does not matter which μ-connected reduction is used, and we can fix a minimal parabolic subgroup $P_0 \subseteq Q$ and the μ-connected reduction $P_{0,I}$ of Q. Then there are only finitely many parabolic subgroups R with $P_{0,I} \subseteq R \subseteq Q$.

Note also that in §2 (Equation 10), for any R containing P_0, $\Delta(R, A_R)$ is identified with the subset $\Delta(R, A_{P_0})$ of $\Delta(P_0, A_{P_0})$. Then $\Delta(P_0, A_{P_0})\setminus (I \cup I^{\perp})$ is equal to $\Delta(Q, A_{P_0})$, and $I^{\perp} \setminus I'$ is equal to $\Delta(R, A_{P_0}) \setminus \Delta(Q, A_{P_0})$ as above in the condition 1.(b), which says roughly that the component a_j has to go to $+\infty$ in the direction of Q but is only bounded below in the other directions. To relate this condition better to convergence of sequences in $\overline{X}^S_{\max}$, we note that one consequence of condition 1.(b) is that for any subsequence $y_{j'}$ of y_j, there is a further subsequence $y_{j''}$ for which there exists a parabolic subgroup R', $R \subseteq R' \subseteq Q$, such that with respect to R', all conditions 1.(a), (c), (d) and (e) are satisfied by this sequence $y_{j''}$, and condition 1.(b) is replaced by a stronger condition 1.(b'): *For* $\alpha \in \Delta(R', A_{P_0})$, $a^{\alpha}_{j''} \to +\infty$, *and for* $\Delta(P_0, A_{P_0}) \setminus \Delta(R', A_{P_0})$, $a^{\alpha}_{j''}$ *is bounded.* The reason for using the weaker condition 1.(b) instead of 1.(b') is that a sequence y_j may split into infinitely many such subsequences $y_{j''}$ satisfying condition 1.(b'), and hence we could not use combinations of only finitely many special sequences to get general convergent sequences.

The definition of the convergent sequences is motivated by the fact that the maximal Satake compactification $\overline{X}^S_{\max}$ of X dominates all non-maximal Satake compatifications and the characterizations of convergent sequences of $\overline{X}^S_{\max}$ in §4, together with the observation that the action by elements in the centralizer $\mathcal{Z}(e(Q))$ will not change the convergence of sequences to points in $e(Q)$.

Proposition 5.2. *The topology on* $\overline{X}_{\mu}$ *defined above is Hausdorff.*

Proof. It suffices to show that if an unbounded sequence y_j in X converges in $\overline{X}_{\mu}$, then it has a unique limit. Suppose y_j has two different limits $z_{1,\infty} \in e(Q_1)$, $z_{2,\infty} \in e(Q_2)$, where Q_1, Q_2 are two μ-saturated parabolic subgroups. By passing to a subsequence, we can assume that there exist two parabolic subgroups R_1, R_2, $R_1 \subseteq Q_1$, $R_2 \subseteq Q_2$, such that y_j satisfies the condition (1) in the definition of convergent sequences with respect to both R_1 and R_2. Let $P_{0,1}$ (resp. $P_{0,2}$) be a minimal parabolic subgroup contained in R_1 (resp. R_2) as above. By passing to a further subsequence, which is still denoted by y_j for convenience, we can assume that there exist parabolic subgroups R'_1, R'_2: $Q_1 \supseteq R'_1 \supseteq R_1$, $Q_2 \supseteq R'_2 \supseteq R_2$ such that in the condition with respect to R_1, $a^{\alpha}_j \to +\infty$ if and only if $\alpha \in \Delta(R'_1, A_{R'_1})$, and for $\alpha \in \Delta(R_1, A_{P_0}) - \Delta(R'_1, A_{P_0})$, a^{α}_j is bounded. The same conditions hold for R'_2. This implies that

$$y_j \in C_j k_{1,\infty} S_{R'_1, \varepsilon_j, t_j, V_1} \cap C_j k_{2,\infty} S_{R'_2, \varepsilon_j, t_j, V_2}, \tag{4}$$

where $C_j \subset K$ is a sequence of compact neighborhoods of e converging to e, $k_{i,\infty} \in K \cap \mathcal{Z}(e(Q_i))$, $i = 1, 2$, $\varepsilon_j \to 0$, $t_j \to +\infty$, and V_1, V_2 are bounded sets in $X_{R'_1}$ and $X_{R'_2}$ respectively. Specifically, V_1 is the product of a compact neighborhood of $z_{1,\infty}$ in $e(Q_1) = X_{I_1}$ and a bounded set in $X_{I_1'}$, where $X_{R_1'} = X_{I_1} \times X_{I_1'}$ as in Condition (1), and V_2 is given similarly. By replacing R_i' by ${}^{k_{i,\infty}}R_i'$, we can assume that $k_{i,\infty} = e$ for $i = 1, 2$. If $R_1' \neq R_2'$, Equation (4) contradicts the separation property of general Siegel sets for R_1', R_2' in Proposition 2.4. If $R_1' = R_2'$, we can take V_1, V_2 to be disjoint since $z_{1,\infty} \neq z_{2,\infty}$. Then Equation (4) contradicts the separation property in Proposition 4.1. These contradictions show that y_j must have a unique limit in $\overline{X}_\mu$. q.e.d.

For the sake of explicitness, we also describe neighborhoods of boundary points explicitly. For any μ-saturated parabolic subgroup Q and a boundary point $z_\infty \in e(Q)$, let V be a neighborhood of z_∞ in $e(Q)$.

Fix a minimal parabolic subgroup P_0 contained in Q. Let $\mathfrak{a}_{P_0}$ be the Lie algebra of its split component A_{P_0}, W be the Weyl group of $\mathfrak{a}_{P_0}$, and $T \in \mathfrak{a}_{P_0}^+$ be a regular vector. The convex hull of the Weyl group orbit $W \cdot T$ is a convex polytope Σ_T in $\mathfrak{a}_{P_0}$, whose faces are in one-to-one correspondence with parabolic subgroups R whose split component $\mathfrak{a}_R$ is contained in $\mathfrak{a}_{P_0}$. Denote the closed face of Σ corresponding to R by σ_R.

Then

$$\mathfrak{a}_{P_0} = \Sigma_T \cup \coprod_R (\sigma_R + \mathfrak{a}_R^+), \tag{5}$$

where $\mathfrak{a}_R \subseteq \mathfrak{a}_{P_0}$ as above. Define

$$\mathfrak{a}_{\mu,Q,T} = \cup_{P \subseteq R \subseteq Q} \ (\sigma_R + \mathfrak{a}_R^+), \tag{6}$$

where P ranges over all the μ-connected reductions of Q with $\mathfrak{a}_P \subseteq \mathfrak{a}_{P_0}$, and for each such P, R ranges over all the parabolic subgroups lying between P and Q. Then for any such pair P and R, $\mathfrak{a}_{\mu,Q,T} \cap \mathfrak{a}_P$ is a connected open subset of $\mathfrak{a}_P$; when $R \neq P$, it has the property that when a point moves out to infinity along $\mathfrak{a}_R$ in the direction of the positive chamber, its distance to the boundary of $\mathfrak{a}_{\mu,Q,T} \cap \mathfrak{a}_P$ goes to infinity.

More precisely, write $R = P_J$, $J \neq \emptyset$. Recall from Equations (10) and (11) in §2 that $\mathfrak{a}_P = \mathfrak{a}_{P_J} \oplus \mathfrak{a}_P^J$ and $\mathfrak{a}_P^J$ is the split component of the parabolic subgroup of M_{P_J} corresponding to P. Then for a sequence $H_j \in \mathfrak{a}_R^+$ with $\alpha(H_j) \to +\infty$ for all $\alpha \in \Delta(R, A_R)$ and any bounded set $\Omega \subset \overline{\mathfrak{a}_P^{J,+}}$, when $j \gg 1$,

$$(H_j + \Omega) \subset \mathfrak{a}_{\mu,Q,T} \cap \mathfrak{a}_P. \tag{7}$$

The positive chamber $\mathfrak{a}_R^+$ intersects the face σ_R at a unique point T_R, and

$$\exp(\mathfrak{a}_R^+ \cap (\sigma_R + \mathfrak{a}_R^+)) = A_{R,\exp T_R} = \exp(\mathfrak{a}_R^+ + T_R),$$

the shifted chamber defined in Equation (16) in §2. The face σ_R is contained in the shift by T_R of the orthogonal complement of $\mathfrak{a}_R$ in $\mathfrak{a}_{P_0}$, and $K_R \exp \sigma_R \cdot x_0$ is a codimension 0 set in X_R. Denote the image of

$K_R \exp \sigma_R \cdot x_0$ in $X_{I'}$ under the projection $X_R = X_I \times X_{I'} \to X_{I'}$ by $W_{R,T}$.

For each such parabolic subgroup R which is contained in Q and contains a μ-connected reduction P with $\mathfrak{a}_P \subseteq \mathfrak{a}_{P_0}$, define

$$(8) \quad S_{R,\varepsilon,e^{T_R},V\times W_{R,T}} = \Big\{(n,a,z,z') \in N_R \times A_R \times X_I \times X_{I'} \mid \\ a \in A_{R,e^{T_R}}, n^a \in B_{N_R}(\varepsilon), z \in V, z' \in W_{R,T}\Big\},$$

a generalized Siegel set in X associated with R defined in Equation (17) in §2. Then

$$(9) \qquad \exp \mathfrak{a}_{\mu,Q,T} \cdot x_0 \subset \cup_{P\subseteq R\subseteq Q} \ S_{R,\varepsilon,e^{T_R},V\times W_{R,T}},$$

where P ranges over all the μ-connected reductions of Q with $\mathfrak{a}_P \subseteq \mathfrak{a}_{P_0}$, and for each such P, R ranges over all the parabolic subgroups lying between P and Q.

Define

$$(10) \qquad S^{\mu}_{\varepsilon,T,V} = \cup_{P\subseteq R\subseteq Q} \ S_{R,\varepsilon,e^{T_R},V\times W_{R,T}},$$

where P ranges over all μ-connected reductions of Q with $\mathfrak{a}_P \subseteq \mathfrak{a}_{P_0}$, and for any such P, R ranges over all parabolic subgroups lying between P and Q. It is important to note that there are only finitely many parabolic subgroups R in the above union.

For each μ-saturated parabolic subgroup Q' such that one of its μ-connected reductions contains a μ-connected reduction P of Q with $\mathfrak{a}_P \subseteq \mathfrak{a}_{P_0}$, we get a similar set $S^{Q',\mu}_{\varepsilon,T,V}$ in $e(Q')$.

For a compact neighborhood C of the identity coset in $K/K\cap\mathcal{Z}(e(Q))$, let $\tilde{C}$ be the inverse image in K of C for the map $K \to K/K\cap\mathcal{Z}(e(Q))$. Then

$$(11) \qquad \tilde{C}(S^{\mu}_{\varepsilon,T,V} \cup_{\mu-\text{saturated} Q'} S^{Q',\mu}_{\varepsilon,T,V})$$

is a neighborhood of z_∞ in $\overline{X}_\mu$.

Proposition 5.3. *With the above notation, for a basis of neighborhoods C_i of the identity coset in $K/K\cap\mathcal{Z}(e(Q))$, a sequence of points $T_i \in \mathfrak{a}^+_{P_0}$ with $\alpha(T_i) \to +\infty$ for all $\alpha \in \Delta(P_0, A_{P_0})$, a sequence $\varepsilon_i \to 0$, and a basis V_i of neighborhoods of z_∞ in $e(Q)$, the associated sets in Equation* (11) *form a neighborhood basis of z_∞ in $\overline{X}_\mu$ with respect to the topology defined by the convergent sequences above.*

Proof. We first show that any unbounded sequence y_j in X converges to $z_\infty \in e(Q)$ if and only if for any $\tilde{C}S^{\mu}_{\varepsilon,T,V}$, when $j \gg 1$, $y_j \in \tilde{C}S^{\mu}_{\varepsilon,T,V}$. If $y_j \to z_\infty$, then by definition, it is a combination of a finite number of sequences of the type in Condition (1). Since each of these sequences can be handled in the same way, we can assume for simplicity that there exist a μ-connected reduction P of Q with $\mathfrak{a}_P \subseteq \mathfrak{a}_{P_0}$ and a parabolic subgroup R, $P \subseteq R \subseteq Q$, such that the condition (1) is satisfied with respect to R. It suffices to show that for any subsequence $y_{j'}$, there is a further subsequence $y_{j''}$ such that $y_{j''} \in \tilde{C}S^{\mu}_{\varepsilon,T,V}$ when $j \gg 1$. By the comments on condition (1) of the definition of convergent sequences (before Proposition 5.2), there exist a parabolic subgroup R', $R \subseteq R' \subseteq Q$, a subsequence $y_{j''}$, which is denoted by y_j for simplicity, and a sequence $C_j \subset K/K \cap \mathcal{Z}(e(Q))$ of neighborhoods of the identity coset shrinking to it, a sequence V_j of neighborhoods of z_∞ shrinking to z_∞, $\varepsilon_j \to 0$, a sequence $T_j \in A_{R'}$ with $T_j^{\alpha} \to +\infty$ for all $\alpha \in \Delta(R', A_{R'})$, and a bounded set $W \subset X_{I'}$ such that for $j \gg 1$,

$$y_j = k_j n_j a_j z_j z_j', \quad k_j \in \tilde{C}_j, \quad (n_j, a_j, z_j, z_j') \in S_{R',\varepsilon_j,T_j,V_j\times W}.$$

When $R' = P$, I' is empty and $X_{I'}$ is reduced to one point, and hence $W \subseteq W_{R',T}$. This implies that for $j \gg 0$,

$$y_j \in \tilde{C}S_{R',\varepsilon,\exp T_{R'},V\times W_{R',T}}$$

and hence

$$y_j \in \tilde{C}S^{\mu}_{\varepsilon,T,V}.$$

Otherwise, $R' \neq P$. Write $z_j' = k_j' \exp H_j' x_0$, where $k_j' \in K_{P_{0,I'}}$, and $H_j' \in \overline{\mathfrak{a}_{P_0}^{I',+}}$. (Recall from Equation (11) in §2 that $\mathfrak{a}_0^{I'}$ is the split component of the minimal parabolic subgroup of $M_{P_{0,I'}}$ corresponding to P_0.) Since W is bounded and H_j' is bounded, Equation (7) implies that when $j \gg 1$,

$$\log a_j + H_j' \in \mathfrak{a}_{\mu,Q,T} \cap \mathfrak{a}_P,$$

and hence by Equation (9),

$$y_j \in \tilde{C}S^{\mu}_{\varepsilon,T,V}.$$

Conversely, suppose that for a sequence $T_j \in \mathfrak{a}_{P_0}$ with $\alpha(T_j) \to +\infty$ for all $\alpha \in \Delta(P_0, A_{P_0})$, $\varepsilon_j \to 0$, C_j shrinks to the identity coset, and V_j is a sequence of neighborhoods of z_∞ shrinking to z_∞,

$$y_j \in \tilde{C}_j S^{\mu}_{\varepsilon_j,T_j,V_j}.$$

It can be shown that we can assume without loss of generality that $\alpha(T_j)$ is monotonically increasing for all α, $\varepsilon_j \to 0$ monotonically, C_j shrinks to the identity coset monotonically, and that V_j shrinks to $\{z_\infty\}$ monotonically.

Since there are only finitely many R in Equation (10), we can assume without loss of generality that there exist a μ-connected reduction P of Q with $\mathfrak{a}_P \subseteq \mathfrak{a}_{P_0}$ and a parabolic subgroup R, $P \subseteq R \subseteq Q$, such that

$$y_j \in \tilde{C}_j S_{R,\varepsilon_j,\,\exp T_{j,R},V_j\times W_{R,T_j}}.$$

When $j \to +\infty$, W_{R,T_j} is expanding, and the sequence y_j may not satisfy the conditions 1.(b) and 1.(e) with respect to R.

We need to divide $S_{R,\varepsilon_j,\exp T_{j,R},V_j\times W_{R,T_j}}$ into finitely many regions and hence to show that the sequence y_j can be split into finitely many subsequences $\{y_{j,l}\}$, $l=1,\dots,m$, such that for every $\{y_{j,l}\}$, we can find a parabolic subgroup R', $P\subseteq R'\subseteq Q$, so that the sequence $y_{j,l}$ satisfies condition (1) with respect to R' in the definition of convergent sequences.

Since $\alpha(T_j)$ is monotonically increasing for all $\alpha\in\Delta(P_0,A_{P_0})$, it follows from the definition in Equation (6) that for any pair j,k with $j\le k$,

$$\mathfrak{a}_{\mu,Q,T_j}\supseteq\mathfrak{a}_{\mu,Q,T_k}. \tag{12}$$

Using the above equation (12), by a compactness argument and the method of proof by contradiction, we can show that there exists a sequence $\tilde\varepsilon_j\to0$ such that for $k\ge j$,

$$S^{\mu}_{\tilde\varepsilon_j,T_{j,V_j}}\supseteq S^{\mu}_{\varepsilon_k,T_{k,V_k}}. \tag{13}$$

By Equation (10), $S^{\mu}_{\varepsilon_k,T_{k,V_k}}$ is a union of pieces $S_{R,\varepsilon_k,e^{T_{k,R}},V_k\times W_{R,T_k}}$. It follows that, more generally, for any R, for any $k\ge j$,

$$\bigcup_{P\subseteq R'\subseteq R}S_{R',\tilde\varepsilon_j,e^{T_{j,R'}},V_j\times W_{R',T_j}}\supseteq S_{R,\varepsilon_k,e^{T_{k,R}},V_k\times W_{R,T_k}}, \tag{14}$$

where P ranges over all μ-connected reductions of Q with $\mathfrak{a}_P\subseteq\mathfrak{a}_{P_0}$ and R' ranges over all parabolic subgroups lying between P and Q.

Now let $R'_1,\dots,R'_m$ be all the parabolic subgroups which satisfy $P\subseteq R'_k\subseteq R$ for some parabolic subgroup P which is a μ-connected reduction of Q with $\mathfrak{a}_P\subseteq\mathfrak{a}_{P_0}$. We are going to define a splitting of $\{y_j\}$ into subsequences $\{y_{j,k}\}$, $k=1,\dots,m$, such that each $\{y_{j,k}\}$ satisfies condition (1) with respect to R'_k. Specifically, let $\{y_{j,k}\}$ be the subsequence of those y_j which satisfy

$$y_j\in\tilde C_1 S_{R'_k,\tilde\varepsilon_1,e^{T_{1,R'_k}},V_1\times W_{R'_k,T_1}};$$

in other words, include those y_j which lie in the R'_k-piece of the first neighborhood $S^{\mu}_{\tilde\varepsilon_1,T_{1,V_1}}$. Note that by Equation (14), such a finite decomposition is possible. By Equation (14) again, such a y_j lies in the $\tilde R$-piece of the j^{th} neighborhood for some $\tilde R\supseteq R'_k$.

Next we show that for each $k=1,\dots,m$, the subsequence $\{y_{j,k}\}$ satisfies condition (1) with respect to R'_k. The definition of $\{y_{j,k}\}$ and the above observation show that the only conditions to check are those in 1.(b) on the a_j^{α}. The condition that for $\alpha\in I^{\perp}-I'$, a_j^{α} is bounded from below follows from $y_{j,k}$ being in the R'_k-piece of the first neighborhood above. The condition that for $\alpha\in\Delta(P_0,A_{P_0})-(I\cup I^{\perp})$, $a_j^{\alpha}\to+\infty$ follows from the fact that the lower bound of every $\alpha\in\Delta(P_0,A_{P_0})-(I\cup I^{\perp})=\Delta(Q,A_{P_0})$ (see Equation 10 in §2) on $\mathfrak{a}_{\mu,Q,T_j}(\subseteq\mathfrak{a}_{P_0})$ goes to $+\infty$ and the fact that for any pair of parabolic subgroups $\tilde{R}, R$ with $\tilde{R} \supseteq R$, the lower bound of α on $\sigma_{\tilde{R}} + \mathfrak{a}^+_{\tilde{R}}$ is greater than or equal to the lower bound of α on $\sigma_R + \mathfrak{a}^+_R$. This latter inequality follows from the fact that $\sigma_{\tilde{R}} + \mathfrak{a}^+_{\tilde{R}}$ and $\sigma_R + \mathfrak{a}^+_R$ are pieces of the complement of the convex polytope Σ_T contained in $\mathfrak{a}_{P_0}$ (see Equation (5)), and Σ_T is the intersection of half spaces defined by $\alpha(H) \leq \alpha(T)$, $\alpha \in \Delta(P_0, A_{P_0})$, and their images under the Weyl group.

Similar arguments show that for any μ-saturated parabolic subgroup Q' such that one of its μ-connected reductions contains a μ-connected reduction of Q, an unbounded sequence y_j in $e(Q')$ converges to $z_\infty \in e(Q)$ if and only if for any $\tilde{C}S^{Q',\mu}_{\varepsilon,T,V}$, $y_j \in \tilde{C}S^{Q',\mu}_{\varepsilon,T,V}$ when $j \gg 1$. This completes the proof of the proposition. q.e.d.

The construction of the neighborhoods described in the above proposition is motivated by a result in [**Ca**] and [**Ji**]. In fact, it was shown in [**Ca**] and [**Ji**] that the closure of the flat $\mathfrak{a}_{P_0} = \mathfrak{a}_{P_0}x_0$ in the Satake compactification $\overline{X}^S_\tau$ is canonically homeomorphic to the closure of the convex hull of the Weyl group orbit $W\mu_\tau$ of the highest weight μ_τ. When μ_τ is generic, we can take $T = \mu_\tau$, and the closure of the convex hull is Σ_T given in Equation (5). For non-generic μ_τ, there is a collapsing from Σ_T to the convex hull of $W\mu_\tau$, and all the faces σ_R for $P \subseteq R \subseteq Q$ collapse to the face σ_P. The domain $S^\mu_{\varepsilon,T,V}$ was suggested by this consideration.

Proposition 5.4. *For any two dominant weights $\mu_1, \mu_2 \in \overline{\mathfrak{a}^{*+}_{P_0}}$, if μ_2 is more regular than μ_1, i.e., if $\mu_i \in \mathfrak{a}^{*+}_{P_0,I_i}$ and $I_2 \subseteq I_1$, then there exists a continuous surjective map from $\overline{X}_{\mu_2} \to \overline{X}_{\mu_1}$. If μ_1, μ_2 belong to the same Weyl chamber face, then $\overline{X}_{\mu_1}$ is homeomorphic to $\overline{X}_{\mu_2}$. For any $\overline{X}_\mu$, there is a continuous surjective map from $\overline{X}_{\max}$ to $\overline{X}_\mu$.*

Proof. Since μ_2 is more regular than μ_1, every μ_1-connected parabolic subgroup is also μ_2-connected, and every subset of Δ perpendicular to μ_2 is also perpendicular to μ_1. This implies that for any μ_1-connected parabolic subgroup P, its μ_2-saturation Q_{μ_2} is contained in its μ_1-saturation Q_{μ_1}.

For every μ_2-saturated parabolic subgroup Q, let P_{μ_2} be a μ_2-connected reduction, and P_{μ_1} a μ_1-connected reduction contained in P_{μ_2}. Then P_{μ_1} is also a maximal μ_1-connected parabolic subgroup contained in P_{μ_2}. In Lemma 5.1 and Equation (3), for $R = P_{\mu_2}$, the decomposition of $X_R = X_I \times X_{P_{0,I'}}$ gives

$$X_{P_{\mu_2}} = X_{P_{\mu_1}} \times X_{P_{0,I'}}.$$

Let Q_{μ_1} be the μ_1-saturation of P_{μ_1}. Then $Q_{\mu_1} \subseteq Q$.

Define a map $\pi : \overline{X}_{\mu_2} \to \overline{X}_{\mu_1}$ such that it is equal to the identity on X, and on the boundary component $e(Q) = X_{P_{\mu_2}}$, a point $(z, z') \in$

$X_{P_{\mu_1}} \times X_{P_{0,I'}} = X_{P_{\mu_2}}$ is mapped to $z \in X_{P_{\mu_1}} = e(Q_{\mu_1})$. Clearly, π is surjective.

Since the convergence in $\overline{X}_{\mu_1}$ is determined in terms of the refined horospherical coordinates decomposition, using $A_{Q_{\mu_1}} \subseteq A_Q$, and $X_{P_{\mu_2}} = X_{P_{\mu_1}} \times X_{P_{0,I'}}$, it can be checked easily that any unbounded sequence y_j in X that converges to $(z, z') \in X_{P_{\mu_1}} \times X_{P_{0,I'}} = X_{P_{\mu_2}} \subset \overline{X}_{\mu_2}$ also converges to $z \in X_{P_{\mu_1}} \subset \overline{X}_{\mu_1}$. By [**GJT**, Lemma 3.28], this proves that the map π is continuous. q.e.d.

Proposition 5.5. *For any dominant weight $\mu \in \overline{\mathfrak{a}_{P_0}^{*+}}$, $\overline{X}_\mu$ is compact.*

Proof. It was shown in Proposition 4.3 that $\overline{X}_{\max}$ is compact. By Proposition 5.4, $\overline{X}_\mu$ is the image of a compact set under a continuous map and hence compact. q.e.d.

Proposition 5.6. *The G-action on X extends to a continuous action on $\overline{X}_\mu$.*

Proof. The proof is similar to that of Proposition 4.4. It uses the more refined decomposition in Lemma 5.1, instead of the horospherical and Langlands decompositions. q.e.d.

Proposition 5.7. *Let τ be a faithful projective representation, $\tau : G \to PGL(n, \mathbb{C})$, whose highest weight μ_τ belongs to the same Weyl chamber face as μ. Then the Satake compactification $\overline{X}^S_\tau$ is isomorphic to $\overline{X}_\mu$.*

Proof. We first show that for any unbounded sequence y_j in X, if it converges in $\overline{X}_\mu$, then it also converges in $\overline{X}^S_\tau$. For simplicity, we can assume that μ is equal to μ_τ.

It is known that every weight μ_i of τ has support equal to a μ-connected subset, i.e., $\mu_i = \mu_\tau - \sum_{\alpha \in \Delta} c_{i,\alpha}\alpha$ and $\mathrm{Supp}(\mu_i) = \{\alpha \in \Delta(P_0, A_{P_0}) \mid c_{i,\alpha} > 0\}$ is equal to a μ-connected subset. Conversely, every μ-connected subset is equal to the support of a weight of τ.

Let $P = P_{0,I}$ be a μ-connected parabolic subgroup, and $Q = S_\mu(P)$ its μ-connected saturation. For any point $z_\infty = m_\infty K_P \in X_P$, let $y_j = g_j x_0$ be a sequence in X converging to z_∞, i.e., there exists a parabolic subgroup R satisfying $P \subseteq R \subseteq Q$ such that with respect to the refined horospherical decomposition of X determined by R in Equation (3), $y_j = k_j n_j a_j m_j m'_j x_0$, the components satisfy (1) $k_j \in K$, $k_j \to k_\infty \in K \cap \mathcal{Z}(e(Q))$, (2) $a_j \in A_R$, for all $\alpha \in \Delta(Q, A_Q)$, $\alpha(\log a_j) \to +\infty$, and for $\alpha \in \Delta(R, A_R) - \Delta(Q, A_Q)$, $\alpha(\log a_j)$ is bounded from below, (3) $n_j \in N_R$, $n_j^{a_j} \to e$, (4) $m_j \in M_P$, $m_j \to m_\infty$, (5) $m'_j \in M_{P_{0,I'}}$ is bounded.

Then we have

$$
\begin{aligned}
&i_\tau(y_j)\\
&= [\tau(g_j)\tau(g_j)^*]\\
&= [\tau(k_j)\tau(a_j)\tau(n_j^{a_j})\tau(m_j)\tau(m'_j)\tau(m'_j)^*\tau(m_j)^*\tau(n_j^{a_j})\tau(a_j)^*\tau(k_j)^*],
\end{aligned}
$$

which has the same limit as the sequence

$$
\begin{aligned}
&\tau(k_j m_j)[\tau(a_j)\tau(m'_j)\tau(m'_j)^*\tau(a_j)^*]\tau(k_j m_j)^*\\
&= \tau(k_\infty m_\infty)[\tau(a_j)\tau(m'_j)\tau(m'_j)\tau(a_j)^*]\tau(k_\infty m_\infty)^*,
\end{aligned}
$$

if the latter converges. To determine the limit of this sequence, order the weights $\mu_1, \dots, \mu_n$ (with multiplicity) of τ so that the weights with support contained in I are $\mu_1 = \mu_\tau, \dots, \mu_l$, and the rest of the weights are $\mu_{l+1}, \dots, \mu_n$.

Note that $P_{0,I}$ is a μ-reduction of Q. By definition, for any $j \geq l+1$, $\mathrm{Supp}(\mu_j)$ is μ-connected and hence contains at least one root $\alpha \in \Delta(Q, A_Q) = \Delta(P_0, A_{P_0}) - (I \cup I^\perp)$, which satisfies $\alpha(\log a_j) \to +\infty$. For other roots in $\mathrm{Supp}(\mu_j)$, $\alpha(\log a_j)$ is bounded from below. Then as in the proof of Proposition 2.4,

$$
[\tau(a_j)] \to [\mathrm{diag}\,(1, \dots, 1, 0, \dots, 0)],
$$

where the first l entries are equal to 1. On the other hand, write the Cartan decomposition $m'_j = k'_j a'_j k''_j$ for the elements in $M_{P_{0,I'}}$. Since m'_j is bounded, we can assume, by passing to a subsequence if necessary, that all the components converge, i.e., $k'_j \to k'_\infty$, $a'_j \to a'_\infty$, and $k''_j \to k''_\infty$. Since the roots in I' are the simple roots of $M_{P_{0,I'}}$ with respect to $\mathfrak{a}_{P_0}^{I'}$ and perpendicular to I, this implies that for $\alpha \in I$, $\alpha(\log a'_j) = 0$, and hence for $\mu = \mu_1, \dots, \mu_l$, $(a'_j)^\mu = (a'_j)^{\mu_\tau}$. It follows that

$$
\begin{aligned}
&[\tau(a'_j)] \to [\mathrm{diag}\,(1, \dots, 1, (a'_\infty)^{\mu_{l+1}-\mu_\tau}, \dots, (a'_\infty)^{\mu_n-\mu_\tau})],\\
&[\tau(a_j)\tau(a'_j)\tau(a'_j)^*\tau(a_j)^*] \to [\mathrm{diag}\,(1, \dots, 1, 0, \dots, 0)],
\end{aligned}
$$

and

$$
\begin{aligned}
&[\tau(a_j)\tau(m'_j)\tau(m'_j)^*\tau(a_j)^*]\\
&= [\tau(a_j)\tau(k'_j)\tau(a'_j)\tau(k''_j)\tau(k''_j)^*\tau(a'_j)^*\tau(k'_j)^*\tau(a_j)^*]\\
&= \tau(k'_j)[\tau(a_j)\tau(a'_j)\tau(a'_j)^*\tau(a_j)^*]\tau(k'_j)^*\\
&\to \tau(k'_\infty)[\mathrm{diag}\,(1, \dots, 1, 0, \dots, 0)]\tau(k'_\infty)^*,
\end{aligned}
$$

since $\tau(k''_j)\tau(k''_j)^* = id$, and k'_j commutes with a_j. Because k'_j commutes with a_j, $\tau(k'_\infty)$ commutes with $\mathrm{diag}\,(1, \dots, 1, 0, \dots, 0)$, and hence

$$
\begin{aligned}
&\tau(k'_\infty)[\mathrm{diag}\,(1, \dots, 1, 0, \dots, 0)]\tau(k'_\infty)^*\\
&= [\mathrm{diag}\,(1, \dots, 1, 0, \dots, 0)]\tau(k'_\infty)\tau(k'_\infty)^*\\
&= [\mathrm{diag}\,(1, \dots, 1, 0, \dots, 0)].
\end{aligned}
$$

Since k_∞ also commutes with a_j and m_∞, it implies that

$$\begin{aligned} i_\tau(y_j) &\to \tau(k_\infty)\tau(m_\infty)[\mathrm{diag}\,(1,\dots,1,0,\dots,0)]\tau(m_\infty)^*\tau(k_\infty)^* \\ &= \tau(m_\infty)[\mathrm{diag}\,(1,\dots,1,0,\dots,0)]\tau(m_\infty)^*, \end{aligned}$$

and hence y_j converges in the Satake compactification $\overline{X}^S_\tau$.

This gives a well-defined map

$$\varphi : \overline{X}_\mu \to \overline{X}^S_\tau.$$

By [**GJT**, Lemma 3.28], this map φ is continuous.

In [**Sa**], it is shown that

$$\overline{X}^S_\tau = X \cup G \left(\coprod_{I \text{ is } \mu\text{-connected}} X_{P_{0,I}} \right),$$

where P_0 is a minimal parabolic subgroup, $I \subset \Delta(P_0, A_{P_0})$, and that $\overline{X}^S_\tau$ can be identified with $X \cup \coprod' X_{P(Q)}$, where the union is over μ_τ-saturated parabolic subgroups Q and P is a μ-connected reduction of Q. Under this identification, the map φ is the identity map. Since both $\overline{X}^S_\tau$ and $\overline{X}_\mu$ are compact and Hausdorff, φ is a homeomorphism. q.e.d.

Remark 5.8. Propositions 5.4 and 5.7 give a more explicit proof of Proposition 4.6. In [**Zu**], Proposition 4.6 was proved by comparing the closures of a flat in these compactifications of X. On the other hand, the proof here gives an explicit map and its surjectivity is immediate.

Remark 5.9. Another application of the construction of $\overline{X}_\mu$ in this section is that it gives an explicit description of neighborhoods of the boundary points.

6. The conic compactification

The symmetric space X is a complete Riemannian manifold of non-positive curvature and hence admits a compactification $X \cup X(\infty)$ whose boundary is the set of equivalence classes of geodesics. In [**GJT**], this compactification is called the conic compactification. It will also be called the geodesic compactification below. In this section, we construct $X \cup X(\infty)$ by following the approach in §3. This construction is less direct than the geometric definition, but is needed for the Martin compactification in §7.

As in the compactification $\overline{X}^S_{\max}$, we use the whole collection of parabolic subgroups. For every parabolic subgroup P, define its boundary face to be

$$e(P) = \mathfrak{a}^+_P(\infty).$$

Define

$$\overline{X}^c = X \cup \coprod_P \mathfrak{a}^+_P(\infty).$$

We note that the K-action on the set of parabolic subgroups preserves the Langlands decomposition with respect to $x_0 = K$ (see Equations (4, 5) in §2) and hence K acts on $\overline{X}^c$ as follows: for $k \in K$, $H \in \mathfrak{a}_P^+(\infty)$,

$$k \cdot H = Ad(k)H \in \mathfrak{a}_{^kP}^+(\infty).$$

The space $\overline{X}^c$ is given the following topology.

1) An unbounded sequence y_j in X converges to $H_\infty \in \mathfrak{a}_P^+(\infty)$ if and only if y_j can be written as $y_j = k_j n_j a_j z_j$ with $k_j \in K, n_j \in N_P, a_j \in A_P, z_j \in X_P$ satisfying the conditions:
   (a) $k_j \to e$,
   (b) for all $\alpha \in \Delta(P, A_P)$, $\alpha(\log a_j) \to +\infty$ and $\log a_j / \|\log a_j\| \to H_\infty$,
   (c) $n_j^{a_j} \to e$,
   (d) $d(z_j, x_o)/\|\log a_j\| \to 0$, where $x_0 \in X_P$ is the basepoint K_P, and $d(z_j, x_0)$ is the Riemannian distance on X_P for the invariant metric.
2) For a sequence $k_j \in K$, $k_j \to e$, and a parabolic subgroup Q contained in P and a sequence $H_j \in \mathfrak{a}_Q^+(\infty)$, $k_j H_j$ converges to $H_\infty \in \mathfrak{a}_P^+(\infty)$ if and only if $H_j \to H_\infty \in \overline{\mathfrak{a}_Q^+(\infty)}$. (Note that $\mathfrak{a}_P^+(\infty) \subset \overline{\mathfrak{a}_Q^+(\infty)}$.)

Combinations of these two special types of convergent sequences give general convergent sequences.

Neighborhoods of boundary points can be given as follows. For any parabolic subgroup P, let P_0 be a minimal parabolic subgroup contained in P. Then $\mathfrak{a}_P^+(\infty) \subset \overline{\mathfrak{a}_{P_0}^+(\infty)}$. For $H \in \mathfrak{a}_P^+(\infty)$ and $\varepsilon > 0$, let

$$U_{\varepsilon,H} = \{H' \in \overline{\mathfrak{a}_{P_0}^+(\infty)} \mid \|H' - H\| < \varepsilon\},$$

a neighborhood of H in $\overline{\mathfrak{a}_{P_0}^+(\infty)}$. For $t > 0$, let

$$V_{\varepsilon,t,H} = \{(n, a, z) \in N_P \times A_P \times X_P = X$$

$$\mid a \in A_{P,t},\ \log a/\|\log a\| \in U_{\varepsilon,H},\ n^a \in B_{N_P}(\varepsilon), d(z, x_0)/\|\log a\| < \varepsilon\}.$$

Let C be a compact neighborhood of e in K. Then the set

$$CV_{\varepsilon,t,H} \cup CU_{\varepsilon,H}$$

is a neighborhood of H in $\overline{X}^c$.

For sequences $\varepsilon_i \to 0$, $t_i \to +\infty$, and a basis C_i of neighborhoods of e in K, $C_i V_{\varepsilon_i,t_i,H} \cup CU_{\varepsilon_i,H}$ forms a basis of neighborhoods of H in $\overline{X}^c$.

Proposition 6.1. *The space $\overline{X}^c$ is a compact Hausdorff space.*

Proof. To prove that $\overline{X}^c$ is compact, we note that

$$\begin{aligned}\overline{X}^c &= K\left(\exp \overline{\mathfrak{a}_{P_0}^+} x_0 \cup \coprod_{\emptyset \subseteq I \subset \Delta} \mathfrak{a}_{P_{0,I}}^+(\infty)\right) \\ &= K\left(\exp \overline{\mathfrak{a}_{P_0}^+} x_0 \cup \overline{\mathfrak{a}_{P_0}^+(\infty)}\right),\end{aligned}$$

where P_0 is a minimal parabolic subgroup. Since K and $\overline{\mathfrak{a}_{P_0}^+(\infty)}$ are compact, it suffices to show that every unbounded sequence of the form $\exp H_j x_0$, $H_j \in \overline{\mathfrak{a}_{P_0}^+}$, has a convergent subsequence. Replacing by a subsequence, we can assume that $H_j/\|H_j\|$ converges to $H_\infty \in \mathfrak{a}_{P_{0,I}}^+(\infty)$ for some I. By decomposing

$$H_j = H_{j,I} + H_j^I, \quad H_{j,I} \in \mathfrak{a}_{P_0,I},\ H_j^I \in \mathfrak{a}_{P_0}^I,$$

it follows immediately that $\exp H_j x_0 = \exp H_{j,I}(\exp H_j^I x_0)$ converges to H_∞ in $\overline{X}^c$.

To prove the Hausdorff property, let $H_1, H_2 \in \overline{X}^c$ be two distinct points. Clearly they admit disjoint neighborhoods if at least one of them belongs to X. Assume that $H_i \in \mathfrak{a}_{P_i}^+(\infty)$ for some parabolic subgroups P_1, P_2.

First consider the case that $P_1 = P_2$. Let P_0 be a minimal parabolic subgroup contained in P_1, and $U_{\varepsilon,1}, U_{\varepsilon,2}$ be two neighborhoods of H_1, H_2 in $\overline{\mathfrak{a}_{P_0}^+(\infty)}$ with $\overline{U_{\varepsilon,1}} \cap \overline{U_{\varepsilon,2}} = \emptyset$. Let C be a small compact neighborhood of e in K such that for all $k_1, k_2 \in C$, $k_1 U_{\varepsilon,1} \cap k_2 U_{\varepsilon,2} = \emptyset$. We claim that the neighborhoods $CV_{\varepsilon,t,H_1} \cup CU_{\varepsilon,1}$, $CV_{\varepsilon,t,H_2} \cup CU_{\varepsilon,2}$ are disjoint when $t \gg 0$, ε and C are sufficiently small.

By the choice of C,

$$CU_{\varepsilon,1} \cap CU_{\varepsilon,2} = \emptyset.$$

We need to show that

$$CV_{\varepsilon,t,H_1} \cap CV_{\varepsilon,t,H_2} = \emptyset.$$

If not, there exist sequences $\varepsilon_j \to 0, t_j \to +\infty$, $C_j \to e$ such that

$$C_j V_{\varepsilon_j,t_j,H_1} \cap C_j V_{\varepsilon_j,t_j,H_2} \neq \emptyset.$$

Let $y_j \in C_j V_{\varepsilon_j,t_j,H_1} \cap C_j V_{\varepsilon_j,t_j,H_2}$. Since $y_j \in CV_{\varepsilon_j,t_j,H_1}$, y_j can be written as $y_j = k_j n_j a_j z_j$ with the components $k_j \in K, n_j \in N_{P_1}$, $a_j \in A_{P_1}$, $z_j \in X_{P_1}$ satisfying (1) $k_j \to e$, (2) $\|\log a_j\| \to +\infty$, $\log a_j/\|\log a_j\| \to H_1$, (3) $n_j^{a_j} \to e$, (4) $d(z_j, x_0)/\|\log a_j\| \to 0$. Similarly, y_j can be written as $y_j = k_j' n_j' a_j' z_j'$ with k_j', n_j', a_j', z_j', satisfying similar properties and $\log a_j'/\|\log a_j'\| \to H_2$. We note that

$$d(y_j, x_o) = (1 + o(1))\|\log a_j\| = (1 + o(1))\|\log a_j'\|,$$

hence

$$\|\log a_j'\| = (1 + o(1))\|\log a_j\|;$$

and

$$\begin{aligned} d(y_j, k_j a_j x_0) &= d(k_j n_j a_j z_j, k_j a_j x_0) = d(n_j a_j z_j, a_j x_0) \\ &= d(n_j^{a_j} z_j, x_0) = o(1)\|\log a_j\|, \\ d(y_j, k'_j a'_j x_0) &= o(1)\|\log a'_j\|. \end{aligned}$$

Since X is simply connected, nonpositively curved, $H_1 \neq H_2$, and $k_j a_j x_0, k'_j a'_j x_0$ lie on two geodesics from x_0 with a uniform separation of angle between them, comparison with the flat space gives

$$d(k_j a_j x_0, k'_j a'_j x_0) \geq c_0 \|\log a_j\|$$

for some positive constant c_0. This contradicts with the inequality

$$d(k_j a_j x_0, k'_j a'_j x_0) \leq d(y_j, k_j a_j x_0) + d(y_j, k'_j a'_j x_0) = o(1)\|\log a_j\|.$$

The claim is proved.

The case $P_1 \neq P_2$ can be proved similarly. In fact, for suitable neighborhoods $U_{\varepsilon,H_1}, U_{\varepsilon,H_2}$ and a neighborhood C of e in K such that for all $k_1, k_2 \in C$,

$$k_1 U_{\varepsilon,H_1} \cap k_2 U_{\varepsilon,H_2} = \emptyset,$$

the same proof works. q.e.d.

Proposition 6.2. *The G-action on X extends to a continuous action on $\overline{X}^c$.*

Proof. For $g \in G$ and $H \in \mathfrak{a}_P^+(\infty)$, write $g = kp$ with $k \in K$ and $p \in P$. Define

$$g \cdot H = Ad(k) H \in \mathfrak{a}_{{}^kP}.$$

Since k is uniquely determined up to a factor in K_P and K_P commutes with A_P, this action is well-defined. Clearly, P fixes $\mathfrak{a}_P^+(\infty)$.

To show that it is continuous, by [**GJT**, Lemma 3.28], it suffices to show that for any unbounded sequence y_j in X, if y_j converges to H_∞ in $\overline{X}^c$, then gy_j converges to gH_∞. By definition, y_j can be written as $y_j = k_j n_j a_j z_j$ with (1) $k_j \in K$, $k_j \to e$, (2) $a_j \in A_P$, for all $\alpha \in \Delta(P, A_P)$, $\alpha(\log a_j) \to +\infty$, $\log a_j / \|\log a_j\| \to H_\infty$, (3) $n_j \in N_P$, $n_j^{a_j} \to e$, and (4) $z_j \in X_P$, $d(z_j, x_0)/\|\log a_j\| \to 0$. Write

$$g k_j = k'_j m'_j a'_j n'_j,$$

where $k'_j \in K$, $m'_j \in M_P$, $n'_j \in N_P$ and $a'_j \in A_P$. Then m'_j, n'_j, a'_j are bounded, and ${}^{k'_j}P$ converges to ${}^kP$. Since

$$g y_j = k'_j m'_j a'_j n'_j n_j a_j z_j = k'_j \, {}^{m'_j a'_j}(n'_j n_j) a'_j a_j (m'_j z_j),$$

it is clear that gy_j converges to $Ad(k) H_\infty \in \mathfrak{a}_{{}^kP}^+(\infty)$, which is equal to gH_∞ by definition. q.e.d.

Proposition 6.3. *The identity map on X extends to a continuous map $\overline{X}^c \to X \cup X(\infty)$, and this map is a homeomorphism.*

Proof. First we recall the construction of $X \cup X(\infty)$ (see [**GJT**], [**BGS**]). Two unit speed geodesics γ_1, γ_2 in X are defined to be equivalent if $\limsup_{t\to+\infty} d(\gamma_1(t), \gamma_2(t)) < +\infty$. Let $X(\infty)$ be the set of equivalence classes of geodesics. $X(\infty)$ can be identified with the unit sphere in the tangent space $T_{x_0}X$ and is endowed with the topology of the latter. The topology of $X \cup X(\infty)$ is defined such that an unbounded sequence y_j in X converges to a geodesic class $[\gamma]$ if the geodesic passing through x_0 and y_j converges to a geodesic in $[\gamma]$.

To prove that the identity map extends to a continuous map $\overline{X}^c \to X \cup X(\infty)$, by [**GJT**, Lemma 3.28], it suffices to prove that if an unbounded sequence in X converges in $\overline{X}^c$, then it also converges in $X \cup X(\infty)$. For an unbounded sequence y_j in X which converges to $H_\infty \in \mathfrak{a}_P^+(\infty)$ in $\overline{X}^c$, y_j can be written as $y_j = k_j n_j a_j z_j$ where the components satisfy (1) $k_j \in K, k_j \to e$, (2) $\|\log a_j\| \to +\infty$, $\log a_j/\|\log a_j\| \to H_\infty$, (3) $n_j^{a_j} \to e$, (4) $d(z_j, x_0)/\|\log a_j\| \to 0$.

Clearly, the geodesic passing through $a_j x_0$ and x_0 converges to the geodesic $\exp tH_\infty x_0$. Since $k_j \to e$, the geodesic passing through $k_j a_j x_0$ and x_0 also converges to $\exp tH_\infty x_0$. We claim that the geodesic passing through y_j and x_0 also converges to $\exp tH_\infty x_0$.

Since

$$d(k_j n_j a_j z_j, k_j n_j a_j x_0) = d(z_j, x_0),$$

and hence

$$d(k_j n_j a_j z_j, k_j n_j a_j x_0)/\|\log a_j\| \to 0,$$

comparison with the Euclidean space shows that the two sequences y_j and $k_j n_j a_j x_0$ will converge to the same limit if $k_j n_j a_j x_0$ converges in $X \cup X(\infty)$. Since

$$d(k_j n_j a_j x_0, k_j a_j x_0) = d(n_j^{a_j} x_0, x_0) \to 0$$

and the geodesic passing through $k_j a_j x_0$ and x_0 clearly converges to the geodesic $\exp tH_\infty x_0$, it follows that $k_j n_j a_j x_0$ converges to $H_\infty \in X \cup X(\infty)$.

To show that this extended continuous map is a homeomorphism, it suffices to prove that it is bijective, since $\overline{X}^c$ and $X \cup X(\infty)$ are both compact and Hausdorff. We note that $\coprod_P \mathfrak{a}_P^+(\infty)$ can be identified with the unit sphere in the tangent space $T_{x_0}X$ (see [**GJT**, Chap. III]), and hence under this identification, the map $\overline{X}^c \to X \cup X(\infty)$ becomes the identity map and is bijective. q.e.d.

7. The Martin compactification

For any complete Riemannian manifold X, there is a family of Martin compactifications parametrized by $\lambda \leq \lambda_0(X)$, where $\lambda_0(X)$ is the bottom of the spectrum of the Laplace operator of X. For each λ, denote the corresponding Martin compactification by $X \cup \partial_\lambda X$. For more details about Martin compactifications, see [**GJT**] and [**Ta**].

When X is a symmetric space of noncompact type, the Martin compactifications are completely determined in [**GJT**]. In fact, it is shown that for $\lambda = \lambda_0(X)$, $X \cup \partial_\lambda X$ is isomorphic to the maximal Satake compactification $\overline{X}^S_{\max}$, and for all $\lambda < \lambda_0(X)$, $X \cup \partial_\lambda X$ are the same and equal to the least common refinement $\overline{X}^S_{\max} \bigvee X \cup X(\infty)$ of $\overline{X}^S_{\max}$ and $X \cup X(\infty)$. In the following, we always assume that $\lambda < \lambda_0(X)$, and call $X \cup \partial_\lambda X$ the Martin compactification. In this section, we follow the method in §3 and construct a compactification $\overline{X}^M$, which will turn out to be isomorphic to the Martin compactification $X \cup \partial_\lambda X$.

For every parabolic subgroup P, define its boundary face to be

$$e(P) = \overline{\mathfrak{a}_P^+(\infty)} \times X_P.$$

Define

$$\overline{X}^M = X \cup \coprod_P \overline{\mathfrak{a}_P^+(\infty)} \times X_P,$$

where P runs over all parabolic subgroups. We note that the K-action on the set of parabolic subgroups preserves the Langlands, horospherical decompositions with respect to $x_0 = K$ (see Equations (4, 5) in §2) and hence K acts on $\overline{X}^M$ as follows: for $k \in K$, $(H, z) \in \overline{\mathfrak{a}_P^+(\infty)} \times X_P = e(P)$,

$$k \cdot (H, z) = (Ad(k)H, k \cdot z) \in \overline{\mathfrak{a}_{{}^kP}^+(\infty)} \times X_{{}^kP}.$$

A topology on $\overline{X}^M$ is given as follows:

1) For a boundary point $(H_\infty, z_\infty) \in \overline{\mathfrak{a}_P^+(\infty)} \times X_P$, an unbounded sequence y_j in X converges to (H_∞, z_∞) if y_j can be written in the form $y_j = k_j n_j a_j z_j$ with the components $k_j \in K$, $n_j \in N_P$, $a_j \in A_P$, $z_j \in X_P$ satisfying the conditions:
   (a) $k_j \to e$,
   (b) for all $\alpha \in \Delta(P, A_P)$, $\alpha(\log a_j) \to +\infty$, $\log a_j / \| \log a_j \| \to H_\infty$,
   (c) $n_j^{a_j} \to e$,
   (d) $z_j \to z_\infty$.
2) For a pair of parabolic subgroups P, Q, $P \subset Q$, let P' be the unique parabolic subgroup of M_Q determined by P. For a sequence $k_j \in K$ with $k_j \to e$ and a sequence $y_j = (H_j, z_j) \in \overline{\mathfrak{a}_Q^+(\infty)} \times X_Q$, the sequence $k_j y_j$ converges to $(H_\infty, z_\infty) \in \overline{\mathfrak{a}_P^+(\infty)} \times X_P$ if $H_\infty \in \overline{\mathfrak{a}_Q^+(\infty)}$, $H_j \to H_\infty$, and z_j can be written in the form $z_j = k'_j n'_j a'_j z'_j$, where $k'_j \in K_Q$, $n'_j \in N_{P'}$, $a'_j \in A_{P'}$, $z'_j \in X_{P'}$ satisfy the same condition as part (1) above when the pair (G, P)

is replaced by (M_Q, P') except (b) is replaced by (b'): for all $\alpha \in \Delta(P', A_{P'})$, $\alpha(\log a'_j) \to +\infty$, and $\log a'_j/\|\log a'_j\| \to H_\infty$.

These are special convergent sequences, and their combinations give the general convergent sequences.

Remark 7.1. As mentioned above, $X \cup \partial_\lambda X$ is the least common refinement $\overline{X}^S_{\max} \bigvee X \cup X(\infty)$, but its boundary faces are not products $\mathfrak{a}_P^+(\infty) \times X_P$ of the boundary faces X_P and $\mathfrak{a}_P^+(\infty)$ of $\overline{X}^S_{\max}$ and $X \cup X(\infty)$. The reason is that the fibers over X_P must be compact and hence equal to a compactification $\overline{\mathfrak{a}^+(\infty)}$ of $\mathfrak{a}^+(\infty)$.

Remark 7.2. In the previous sections for the Satake compactifications and the conic (or geodesic) compactification, for a pair of parabolic subgroups P, Q, $P \subset Q$, their boundary faces $e(P), e(Q)$ satisfy either $e(P) \subset \overline{e(Q)}$ or $e(Q) \subset \overline{e(P)}$. For $\overline{X}^M$, neither inclusion is true. It will be shown in the next section that the same phenomenon occurs for the Karpelevic compactification.

Neighborhoods of boundary points can be given explicitly as follows. For a parabolic subgroup P and a point $(H, z) \in \overline{\mathfrak{a}_P^+(\infty)} \times X_P$, there are two cases to consider: $H \in \mathfrak{a}_P^+(\infty)$ or not.

In the first case, let U be a neighborhood of H in $\mathfrak{a}_P^+(\infty)$ and V a neighborhood of z in X_P. Let

$$S^M_{\varepsilon,t,U,V} = \Big\{(n, a, z) \in N_P \times A_P \times X_P = X \mid \\ a \in A_{P,t}, \log a/\|\log a\| \in U, n^a \in B_{N_P}(\varepsilon), z \in V\Big\}.$$

For a neighborhood C of e in K, the set

$$C(S^M_{\varepsilon,t,U,V} \cup U \times V)$$

is a neighborhood of (H, z) in $\overline{X}^M$. The reason is that (H, z) is contained in the closure of other boundary components only when $H \in \partial\overline{\mathfrak{a}_P^+(\infty)}$.

In the second case, $H \in \partial\overline{\mathfrak{a}_P^+(\infty)}$. Let Q be the unique parabolic subgroup containing P such that H is contained in $\mathfrak{a}_Q^+(\infty)$. Let $Q = P_J$. Then P_I with $I \subseteq J$ are all the parabolic subgroups containing P such that $H \in \overline{\mathfrak{a}_{P_I}^+(\infty)}$, and X_P is a boundary symmetric space of X_{P_I}. Let $S^I_{\varepsilon,t,V}$ be the generalized Siegel set in X_{P_I} associated with P' as defined in Equation (15) in §2, where V is a bounded neighborhood of z. Let U be a neighborhood of H in $\overline{\mathfrak{a}_P^+(\infty)}$. Then

$$(\overline{\mathfrak{a}_{P_I}^+(\infty)} \cap U) \times S^I_{\varepsilon,t,V}$$

is the intersection of a neighborhood of (H, z) with the boundary face $\overline{\mathfrak{a}^+_{P_I}(\infty)} \times X_{P_I}$. Let C be a neighborhood of e in K. Then

$$C\left(S^M_{\varepsilon,t,U,V} \cup \coprod_{I\subseteq J} (\overline{\mathfrak{a}^+_{P_I}(\infty)} \cap U) \times S^I_{\varepsilon,t,V}\right)$$

is a neighborhood of (H, z) in $\overline{X}^M$.

Proposition 7.3. *The topology on* $\overline{X}^M$ *is Hausdorff.*

Proof. We need to show that every pair of distinct points $x_1, x_2 \in \overline{X}^M$ admit disjoint neighborhoods. If at least one of them belongs to X, it is clear. Assume that they both lie on the boundary, $x_i = (H_i, z_i) \in \overline{\mathfrak{a}^+_{P_i}(\infty)} \times X_{P_i}$ for a pair of parabolic subgroups P_1, P_2. There are two cases to consider: $P_1 = P_2$ or not.

In the first case, $(H_1, z_1), (H_2, z_2) \in \overline{\mathfrak{a}^+_{P_1}(\infty)} \times X_{P_1}$. If $z_1 \neq z_2$, existence of the disjoint neighborhoods follows from the corresponding results for $\overline{X}_{\max}$ in Proposition 4.2. If $z_1 = z_2$, then $H_1 \neq H_2$, and the existence of disjoint neighborhoods follows from the similar result of $\overline{X}^c$ in Proposition 6.1.

In the second case, $P_1 \neq P_2$, existence of the disjoint neighborhoods follows similarly from the results for $\overline{X}_{\max}$ and $\overline{X}^c$. q.e.d.

Proposition 7.4. *The* G*-action on* X *extends to a continuous action on* $\overline{X}^M$*.*

Proof. First we define a G-action on the boundary of $\overline{X}^M$. For $(H, z) \in \overline{\mathfrak{a}^+_P(\infty)} \times X_P$, $g \in G$, write $g = kman$, $k \in K, m \in M_P, a \in A_P$, $n \in N_P$. Define

$$g \cdot (H, z) = (Ad(k)H, k \cdot mz) \in \overline{\mathfrak{a}^+_{{}^kP}(\infty)} \times X_{{}^kP},$$

where $k \cdot X_P$ is defined in Equation (5) in §2. To show that this is a continuous extension of the G-action on X, by [**GJT**, Lemma 3.28], it suffices to show that if an unbounded sequence y_j in X converges to a boundary point (H, z), then gy_j converges to $g \cdot (H, z)$.

By definition, y_j can be written in the form $y_j = k_j n_j a_j z_j$, where $k_j \in K$, $n_j \in N_P$, $a_j \in A_P$, $z_j \in X_P$ satisfy (1) $k_j \to e$, (2) for all $\alpha \in \Delta(P, A_P)$, $\alpha(\log a_j) \to +\infty$, $\log a_j/\|\log a_j\| \to H$, (3) $n_j^{a_j} \to e$, (4) $z_j \to z$. Write

$$gk_j = k'_j m'_j a'_j n'_j,$$

where $k'_j \in K$, $m'_j \in M_P$, $n'_j \in N_P$, $a'_j \in A_P$. Then n'_j, a'_j are bounded and $k'_j m'_j \to km$. The components k'_j, m'_j are not uniquely determined, but determined up to an element in K_P. By choosing this element suitably, we can assume that k'_j, m'_j converge to k, m respectively. Then

$$\begin{aligned} gy_j = k'_j m'_j a'_j n'_j n_j a_j z_j &= k'_j \ {}^{m'_j a'_j}(n'_j n_j) \ \ a'_j a_j m'_j z_j \\ &= {}^{k'_j m'_j a'_j}(n'_j n_j) \ {}^{k'_j}(a'_j a_j) \ (k_j{}' m'_j z_j) \\ &= (k'_j k^{-1}) \ {}^{k m'_j a'_j}(n'_j n_j) \ {}^{k}(a'_j a_j)(k m'_j z_j). \end{aligned}$$

From the last expression it can be checked easily that the conditions for convergence in $\overline{X}^M$ are satisfied, and gy_j converges to $(Ad(k)H, kmz) \in \overline{\mathfrak{a}^+_{{}^kP}(\infty)} \times X_{{}^kP}$. q.e.d.

Proposition 7.5. *The space $\overline{X}^M$ is compact.*

Proof. We need to show that every sequence in $\overline{X}^M$ has a convergent subsequence. Since X is dense and every point in $\overline{X}^M$ is the limit of a sequence of points in X, it suffices to consider sequences y_j in X.

If y_j is bounded, it clearly has a convergent subsequence. Otherwise, we can assume that y_j goes to infinity. Using the Cartan decomposition $X = K \exp \overline{\mathfrak{a}^+_{P_0}} x_0$, $y_j = k_j \exp H_j x_0$ and replacing H_j by a subsequence, we can assume that

1) k_j converges to some $k \in K$,
2) there exists a subset $I \subset \Delta(P_0, A_{P_0})$, such that for $\alpha \in I$, $\alpha(H_j)$ converges to a finite number, while for $\alpha \in \Delta - I$, $\alpha(H_j) \to +\infty$.
3) $H_j / \|H_j\| \to H_\infty \in \overline{\mathfrak{a}^+_{P_{0,I}}(\infty)}$.

Writing $H_j = H_{j,I} + H^I_j$, where $H_{j,I} \in \mathfrak{a}_{P_{0,I}}$, $H^I_j \in \mathfrak{a}^I_{P_0}$, $\exp H^I_j x_0$ converges to a point $z_\infty \in X_{P_{0,I}}$, and $H_{j,I}/\|H_{j,I}\| \to H_\infty$. From this, it is clear that $y_j = k_j \exp H_{j,I} H^I_j x_0$ converges to $k(H_\infty, z_\infty) \in \overline{\mathfrak{a}^+_{{}^kP_{0,I}}(\infty)} \times X_{{}^kP_{0,I}}$ in $\overline{X}^M$. q.e.d.

Proposition 7.6. *$\overline{X}^M$ is isomorphic to the least common refinement $\overline{X}_{\max} \bigvee \overline{X}^c$ of $\overline{X}_{\max}$ and $\overline{X}^c$.*

Proof. By [**GJT**, Lemma 3.28], it suffices to show that an unbounded sequence y_j in X converges in $\overline{X}^M$ if and only if y_j converges in both $\overline{X}_{\max}$ and $X \cup X(\infty)$.

If y_j in X converges in $\overline{X}^M$ to $(H, z) \in \overline{\mathfrak{a}^+_P(\infty)} \times X_P$, then it can be written in the form $y_j = k_j n_j a_j z_j$ with $k_j \in K$, $n_j \in N_P$, $a_j \in A_P$ and $z_j \in X_P$ satisfying (1) $k_j \to e$, (2) $\alpha(\log a_j) \to +\infty$, $\alpha \in \Delta(P, A_P)$, $\log a_j / \|\log a_j\| \to H$, (3) $z_j \to z$. Since these conditions are stronger than the convergence conditions of $\overline{X}_{\max}$, it is clear that y_j converges in $\overline{X}_{\max}$ to $z \in X_P$. On the other hand, let P_I be the unique parabolic subgroup containing P such that $\mathfrak{a}^+_P(\infty)$ contains H as an interior point. By decomposing $\log a_j$ according to $\mathfrak{a}_P = \mathfrak{a}_{P,I} \oplus \mathfrak{a}^I_P$, it can be seen easily that y_j converges to $H \in \mathfrak{a}^+_{P_I}(\infty)$ in $\overline{X}^c$.

Conversely, suppose that y_j converges in both $\overline{X}_{\max}$ and $\overline{X}^c$. Let $z \in X_P$ be the limit in $\overline{X}_{\max}$. Then y_j can be written as $y_j = k_j n_j a_j z_j$ with the components satisfying similar conditions as above except (2) is replaced by (2'): $\alpha(\log a_j) \to +\infty$, $\alpha \in \Delta(P, A_P)$. We claim that the second part of (2) is also satisfied, i.e., $\log a_j/\|\log a_j\| \to H$ for some $H \in \overline{\mathfrak{a}_P^+(\infty)}$. In fact, since y_j converges in $\overline{X}^c$, the proof of Proposition 6.1 shows that $a_j x_0$ also converges in $\overline{X}^c$. This implies that $\log a_j/\|\log a_j\|$ converges in $\mathfrak{a}_P(\infty)$ and the limit clearly belongs to $\overline{\mathfrak{a}_P^+(\infty)}$. Since all the conditions (1)–(4) are satisfied, y_j converges in $\overline{X}^M$ to $(H, z) \in \overline{\mathfrak{a}_P^+(\infty)} \times X_P$. q.e.d.

Corollary 7.7. *The compactification $\overline{X}^M$ is isomorphic to the Martin compactification $X \cup \partial_\lambda X$, $\lambda < \lambda_0(X)$.*

Proof. It was proved in [**GJT**] that $X \cup \partial_\lambda X$, $\lambda < \lambda_0(X)$, is isomorphic to the least common refinement $\overline{X}^S_{\max} \bigvee X \cup X(\infty)$. By Proposition 4.8, $\overline{X}^S_{\max} \cong \overline{X}_{\max}$, and by Proposition 6.3, $\overline{X}^c = X \cup X(\infty)$. Then the corollary follows from the previous proposition. q.e.d.

8. The Karpelevic compactification

Karpelevic [**Ka**] defined a compactification $\overline{X}^K$ of X using structures of geodesics; in particular, he refined equivalence relations on them. The original definition is given by induction on the rank and is quite involved. A simplified, more direct approach is given in [**GJT**] by first identifying the closure of a flat. But in the latter approach, the continuous extension of the G-action is not clear. In this section, we follow the general approach in §3 to construct a compactification of X admitting a continuous G-action, to be denoted by $\overline{X}_K$, which will turn out to be isomorphic to the Karpelevic compactification $\overline{X}^K$. Consequently, we obtain another non-inductive explicit description of the Karpelevic topology.

The boundary faces of $\overline{X}_K$ are refinements of the boundary faces of $\overline{X}^M$. Recall that for any parabolic subgroup P, $\Delta = \Delta(P, A_P)$ is the set of simple roots in $\Phi(P, A_P)$. For a pair of $J, J' \subset \Delta(P, A_P)$, $J \subset J'$, let

(1) $$\mathfrak{a}_J^{J'} = \mathfrak{a}_{P_J} \cap \mathfrak{a}_P^{J'}.$$

The restriction of the roots in $J' - J$ yields a homeomorphism $\mathfrak{a}_J^{J'} \cong \mathbb{R}^{J'-J}$. Define

(2) $$\mathfrak{a}_J^{J',+}(\infty) = \{H \in \mathfrak{a}_J^{J'} \mid \|H\| = 1, \alpha(H) > 0, \alpha \in J' - J\}.$$

For any ordered partition

$$\Sigma : I_1 \cup \cdots \cup I_k = \Delta,$$

let $J_i = I_i \cup \cdots \cup I_k$, $1 \le i \le k$, $J_{k+1} = \emptyset$, be the induced decreasing filtration. Define

$$\mathfrak{a}_P^{\Sigma,+}(\infty) = \mathfrak{a}_{J_2}^{+}(\infty) \times \mathfrak{a}_{J_3}^{J_2,+}(\infty) \times \cdots \times \mathfrak{a}_{J_{k+1}}^{J_k,+}(\infty).$$

Note that $\mathfrak{a}_{J_{k+1}}^{J_k,+}(\infty) = \mathfrak{a}_P^{J_k,+}(\infty)$. If we use the improper parabolic subgroup $P_\Delta = G$, $\mathfrak{a}_{J_2}^{+}(\infty) = \mathfrak{a}_{P_{J_2}}^{+}(\infty)$ can be identified with $\mathfrak{a}_{J_2}^{J_1,+}(\infty)$. When Σ is the trivial partition consisting of only Δ, $\mathfrak{a}_P^{\Sigma,+}(\infty) = \mathfrak{a}_P^{+}(\infty)$. Other pieces are blow-ups of the boundary of $\mathfrak{a}_P^{+}(\infty)$ in as shown in Proposition 8.1 below.

Define

$$\overline{\mathfrak{a}}_P^{K,+}(\infty) = \coprod_{\Sigma} \mathfrak{a}_P^{\Sigma,+}(\infty), \tag{3}$$

where Σ runs over all the partitions of Δ.

The space is given the following topology:

1) For every partition Σ, $\mathfrak{a}_P^{\Sigma,+}(\infty)$ is given the product topology.
2) For two partitions Σ, Σ', $\mathfrak{a}_P^{\Sigma,+}(\infty)$ is contained in the closure of $\mathfrak{a}_P^{\Sigma',+}(\infty)$ if and only if Σ is a refinement of Σ', i.e., every part in Σ' is union of parts of Σ. Specifically, the convergence of a sequence of points in $\mathfrak{a}_P^{\Sigma',+}(\infty)$ to limits in $\mathfrak{a}_P^{\Sigma,+}(\infty)$ is given as follows. Assume $\Sigma : I_1 \cup \cdots \cup I_k$, $\Sigma' : I'_1 \cup \cdots \cup I'_{k'}$. For any part I'_m in Σ', write $I'_m = I_{n_1} \cup \cdots \cup I_{n_s}$, where the indexes $n_1, \dots, n_s$ are strictly increasing. Then it suffices to describe how a sequence in $\mathfrak{a}_{J'_m+1}^{J'_m,+}(\infty)$ converges to a limit in $\mathfrak{a}_{J_{n_1}+1}^{J_{n_1},+}(\infty) \times \cdots \times \mathfrak{a}_{J_{n_s}+1}^{J_{n_s},+}(\infty)$. Let H_j be a sequence in $\mathfrak{a}_{J'_m+1}^{J'_m,+}(\infty)$, and $(H_{n_1,\infty}, \dots, H_{n_s,\infty}) \in \mathfrak{a}_{J_{n_1}+1}^{J_{n_1},+}(\infty) \times \cdots \times \mathfrak{a}_{J_{n_s}+1}^{J_{n_s},+}(\infty)$. Then H_j converges to $(H_{n_1,\infty}, \dots, H_{n_s,\infty})$ if and only if the following conditions are satisfied:
   (a) For $\alpha \in I_{n_1}$, $\alpha(H_j) \to \alpha(H_{n_1,\infty})$, in particular, $\alpha(H_j) \not\to 0$.
   (b) For $\alpha \in I'_m - I_{n_1}$, $\alpha(H_j) \to 0$.
   (c) For $\alpha \in I_{n_a}$, $\beta \in I_{n_b}$, $a < b$,
   $$\frac{\beta(H_j)}{\alpha(H_j)} \to 0.$$
   (d) For $\alpha, \beta \in I_{n_a}$, $1 \le a \le s$,
   $$\frac{\beta(H_j)}{\alpha(H_j)} \to \frac{\beta(H_{n_a,\infty})}{\alpha(H_{n_a,\infty})}.$$

For $\varepsilon > 0$, define a subset $U_\varepsilon^{\Sigma'}(H_\infty)$ in $\mathfrak{a}_P^{\Sigma',+}(\infty)$ as follows:

(4)

$$\begin{aligned} U_\varepsilon^{\Sigma'}(H_\infty) = \Big\{ &(H_1, \dots, H_{k'}) \in \mathfrak{a}_P^{\Sigma',+}(\infty) \mid \\ &\text{for } 1 \le m \le k', I'_m = I_{n_1} \cup \cdots \cup I_{n_s}, \\ &(1) \text{ for } \alpha \in I_{n_1}, |\alpha(H_m) - \alpha(H_{n_1,\infty})| < \varepsilon, \\ &(2) \text{ for } \alpha \in I'_m - I_{n_1}, |\alpha(H_m)| < \varepsilon, \\ &(3) \text{ for } \alpha \in I_{n_a}, \beta \in I_{n_b}, a < b, \left|\frac{\beta(H_m)}{\alpha(H_m)}\right| < \varepsilon, \\ &(4) \text{ for } \alpha, \beta \in I_{n_a}, 1 \le a \le s, \left|\frac{\beta(H_m)}{\alpha(H_m)} - \frac{\beta(H_{n_a,\infty})}{\alpha(H_{n_a,\infty})}\right| < \varepsilon \Big\}. \end{aligned}$$

Proposition 8.1. *There exists a continuous surjective map* $\pi : \overline{\mathfrak{a}}_P^{K,+}(\infty) \to \overline{\mathfrak{a}_P^+(\infty)}$. *This map is a homeomorphism if and only if* $\dim A_P \le 2$.

Proof. Recall that

$$\overline{\mathfrak{a}_P^+(\infty)} = \coprod_{I \subset \Delta} \mathfrak{a}_{P_I}^+(\infty).$$

For each partition $\Sigma : I_1 \cup \cdots \cup I_k = \Delta$, define a map by projecting to the first factor:

$$\pi : \mathfrak{a}_P^{\Sigma,+}(\infty) = \mathfrak{a}_{J_2}^+(\infty) \times \mathfrak{a}_{J_3}^{J_2,+}(\infty) \times \cdots \times \mathfrak{a}_{J_{k+1}}^{J_k,+}(\infty) \to \mathfrak{a}_{J_2}^+(\infty) = \mathfrak{a}_{P_{J_2}}^+(\infty),$$

$$\pi(H_{1,\infty}, \dots, H_{k,\infty}) = H_{1,\infty}$$

where $J_i = I_i \cup \cdots \cup I_k$ as above. This gives a surjective map

$$\pi : \overline{\mathfrak{a}}_P^{K,+}(\infty) \to \overline{\mathfrak{a}_P^+(\infty)}.$$

If $H_j \in \mathfrak{a}_P^+(\infty)$ converges to $(H_{1,\infty}, \dots, H_{k,\infty}) \in \mathfrak{a}_P^{\Sigma,+}(\infty)$, it is clear from the description of the topology of $\overline{\mathfrak{a}}_P^{K,+}(\infty)$ that in particular conditions (a) and (b) H_j converges to $H_{1,\infty}$ in $\overline{\mathfrak{a}_P^+(\infty)}$. Since $\mathfrak{a}_P^+(\infty)$ is dense in $\overline{\mathfrak{a}}_P^{K,+}(\infty)$ and hence every point in $\overline{\mathfrak{a}}_P^{K,+}(\infty)$ is the limit of a sequence of points in $\mathfrak{a}_P^+(\infty)$, this proves the continuity of π.

When $\dim A_P = 1$, this map π is clearly bijective. When $\dim A_P = 2$, there are only two nontrivial ordered partitions, $\Sigma_1 : I_1 \cup I_2$, $\Sigma_2 : I_2 \cup I_1$ of Δ, and each of $\mathfrak{a}_P^{\Sigma_1,+}(\infty)$, $\mathfrak{a}_P^{\Sigma_2,+}(\infty)$ consists of one point, corresponding to the two end points of the 1-simplex $\mathfrak{a}_P^+(\infty)$, and hence the map π is bijective. On the other hand, if $\dim A_P \ge 3$, there are nontrivial ordered partitions $\Sigma : \Delta = I_1 \cup I_2$ where $|I_1| = 1, |I_2| \ge 2$. For such a Σ, $\mathfrak{a}_P^{\Sigma,+}(\infty)$ has positive dimension and is mapped to the zero dimensional space $\mathfrak{a}_{J_2}^+(\infty)$, and hence π is not injective. q.e.d.

For each parabolic subgroup P, define its boundary face to be

$$e(P) = \overline{\mathfrak{a}}_P^{K,+}(\infty) \times X_P.$$

Define

$$\overline{X}_K = X \cup \coprod_P \overline{\mathfrak{a}}_P^{K,+}(\infty) \times X_P,$$

where P runs over all parabolic subgroups. By Equations (4), (5) in §2, the K-action on parabolic subgroups preserves the Langlands decomposition, and hence K acts on $\overline{X}^K$ as follows: for $k \in K$, $(H, z) \in \overline{\mathfrak{a}}_P^{K,+}(\infty) \times X_P$,

$$k \cdot (H, z) = (Ad(H), k \cdot z) \in \overline{\mathfrak{a}}_{^kP}^{K,+}(\infty) \times X_{^kP},$$

where for $H = (H_1, \dots, H_j) \in \mathfrak{a}_P^{\Sigma,+}(\infty) \subset \overline{\mathfrak{a}}_P^{K,+}(\infty)$ and $\Sigma : I_1 \cup \dots \cup I_j$,

$$Ad(k)H = (Ad(k)H_1, \dots, Ad(k)H_j)) \in \mathfrak{a}_{^kP}^{\Sigma,+}(\infty),$$

where Σ induces an ordered partition of $\Delta(^kP, A_{^kP})$ by $Ad(k) : \mathfrak{a}_P \to \mathfrak{a}_{^kP}$, which is denoted by Σ also in the above equation.

Before defining a topology on $\overline{X}_K$, we need to define a topology of $\mathfrak{a}_P \cup \overline{\mathfrak{a}}_P^{K,+}(\infty)$. Given an ordered partition $\Sigma : I_1 \cup \dots \cup I_k$ and a point $H_\infty = (H_{1,\infty} \dots, H_{k,\infty}) \in \mathfrak{a}_P^{\Sigma,+}(\infty)$, an unbounded sequence $H_j \in \mathfrak{a}_P$ converges to $(H_{1,\infty} \dots, H_{k,\infty})$ if and only if

1) For all $\alpha \in \Delta$, $\alpha(H_j) \to +\infty$.
2) For every pair $m < n$, $\alpha \in I_m$, $\beta \in I_n$, $\beta(H_j)/\alpha(H_j) \to 0$.
3) For every m, $\alpha, \beta \in I_m$, $\beta(H_j)/\alpha(H_j) \to \beta(H_{m,\infty})/\alpha(H_{m,\infty})$.

Neighborhoods of boundary points in $\mathfrak{a}_P \cup \overline{\mathfrak{a}}_P^{K,+}(\infty)$ can be given explicitly. For any $H_\infty = (H_{1,\infty} \dots, H_{k,\infty}) \in \mathfrak{a}_P^{\Sigma,+}(\infty)$, and $\varepsilon > 0$, define

(5)

$$U_\varepsilon^X(H_\infty) = \Big\{ H \in \mathfrak{a}_P \mid$$

$$(1) \text{ for } \alpha \in \Delta, \alpha(H) > \frac{1}{\varepsilon},$$

$$(2) \text{ for every pair } m < n, \alpha \in I_m, \beta \in I_n, |\beta(H)/\alpha(H)| < \varepsilon,$$

$$(3) \text{ for every } m, \alpha, \beta \in I_m, |\beta(H)/\alpha(H) - \beta(H_{m,\infty})/\alpha(H_{m,\infty})| < \varepsilon \Big\}.$$

Combining with the set $U_\varepsilon^{\Sigma'}(H_\infty)$ in Equation (4), set

$$U_\varepsilon(H_\infty) = U_\varepsilon^X(H_\infty) \cup \cup_{\Sigma'} U_\varepsilon^{\Sigma'}(H_\infty), \tag{6}$$

where Σ' runs over all the ordered partitions for which Σ is a refinement. This is a neighborhood of H_∞ in $\mathfrak{a}_P \cup \overline{\mathfrak{a}}_P^{K,+}(\infty)$.

The space $\overline{X}_K$ is given the following topology:

1) An unbounded sequence y_j in X converges to $(H_\infty, z_\infty) \in \overline{\mathfrak{a}}_P^{K,+}(\infty) \times X_P$ if y_j can be written as $y_j = k_j n_j a_j z_j$ with $k_j \in K, n_j \in N_P$, $a_j \in A_P$, $z_j \in X_P$ satisfying the conditions:
   (a) $k_j \to e$.
   (b) $\log a_j \to H_\infty$ in $\mathfrak{a}_P \cup \overline{\mathfrak{a}}_P^{K,+}(\infty)$ in the topology described above.
   (c) $n_j^{a_j} \to e$.
   (d) $z_j \to z_\infty$.
2) Let Q be a parabolic subgroup containing P, and P' be the parabolic subgroup in M_Q corresponding to P. We note that a partition Σ_Q of $\Delta(Q, A_Q)$ and a partition of $\Sigma^{P'}$ of $\Delta(P', A_{P'})$ combine to form a partition $\Sigma_P = \Sigma_Q \cup \Sigma^{P'}$ of $\Delta(P, A_P)$, where the roots in $\Delta(Q, A_Q)$ and $\Delta(P', A_{P'})$ are identified with the roots in $\Delta(P, A_P)$ whose restrictions are equal to them.

   For a sequence $k_j \in K$ with $k_j \to e$, and a sequence $y_j = (H_j, z_j)$ in $\overline{\mathfrak{a}}_Q^{K,+}(\infty) \times X_Q$, the sequence $k_j y_j$ converges to $(H_\infty, z_\infty) \in \overline{\mathfrak{a}}_P^{K,+}(\infty) \times X_P$ if and only if z_j can be written as $z_j = k'_j n'_j a'_j z'_j$ with $k'_j \in K_Q, n'_j \in N_{P'}$, $a'_j \in A_{P'}$, $z'_j \in X_{P'}$, and these components and H_j satisfy the conditions:
   (a) There exists a partition Σ of the form $\Sigma = \Sigma_Q \cup \Sigma^{P'}$, i.e., a combination of two partitions $\Sigma_Q, \Sigma^{P'}$, such that $H_\infty \in \mathfrak{a}_P^{\Sigma,+}$. Write $H_\infty = (H_{\infty,Q}, H'_\infty) \in \mathfrak{a}_Q^{\Sigma_Q,+}(\infty) \times \mathfrak{a}_{P'}^{\Sigma^{P'},+}(\infty)$. The components k'_j, n'_j, a'_j, z'_j satisfy the same condition as in part (1) above when the pair X, P is replaced by X_Q, P' and the limit by $(H'_\infty, z_\infty) \in \overline{\mathfrak{a}}_{P'}^{K,+}(\infty) \times X_{P'}$.
   (b) $(H_j, H'_\infty) \to H_\infty$ in $\mathfrak{a}_P \cup \overline{\mathfrak{a}}_P^{K,+}(\infty)$.

Neighborhood systems of boundary points can be described as follows. For a point $(H_\infty, z_\infty) \in \overline{\mathfrak{a}}_P^{K,+}(\infty) \times X_P$, let V be a bounded neighborhood of z_∞ in X_P, and $U_\varepsilon = U_\varepsilon(H_\infty)$ a neighborhood of H_∞ in $\mathfrak{a}_P \cup \overline{\mathfrak{a}}_P^{K,+}(\infty)$ defined in Equation (6) above. For $\varepsilon > 0, t > 0$, define

$$(7) \quad S^K_{\varepsilon,t,V} = \Big\{(n, a, z) \in N_P \times A_P \times X_P = X \mid \log a \in U_\varepsilon, a \in A_{P,t}, n^a \in B_{N_P}(\varepsilon), z \in V\Big\}.$$

Let Σ be the partition of $\Delta(P, A_P)$ such that $H_\infty \in \mathfrak{a}_P^{\Sigma,+}(\infty)$. For a parabolic subgroup P_I containing P, if a partition Σ is of the form $\Sigma_{P_I} \cup \Sigma^{P'}$ in the above notation, we call P_I a Σ-admissible parabolic subgroup. For example, when $\Sigma = \Delta$, the only Σ-admissible parabolic subgroup is P.

For each Σ-admissible parabolic subgroup P_I, write $H_\infty = (H_{\infty,I}, H^I_\infty) \in \overline{\mathfrak{a}}_{P_I}^{K,+}(\infty) \times \overline{\mathfrak{a}}_{P'}^{K,+}(\infty)$. For $\varepsilon > 0$, let $U_{I,\varepsilon} = U_\varepsilon(H_{\infty,I}) \cap \overline{\mathfrak{a}}_{P_I}^{K,+}(\infty)$, a neighborhood of $H_{\infty,I}$ in $\overline{\mathfrak{a}}_{P_I}^{K,+}(\infty)$. (Recall that $U_\varepsilon(H_{\infty,I})$ is a neighborhood of $H_{\infty,I}$ in $\mathfrak{a}_{P_I} \cup \overline{\mathfrak{a}}_{P_I}^{K,+}(\infty)$ as defined in Equation (6) above). Let $S_{\varepsilon,t,V}^{K,I}$ be the corresponding neighborhood of H_∞^I in X_{P_I} defined as in Equation (7). For a neighborhood C of e in K, the set

$$C\left(S_{\varepsilon,t,V}^{K} \cup \coprod_{\Sigma-\text{admissible } P_I} U_{I,\varepsilon} \times S_{\varepsilon,t,V}^{K,I}\right) \tag{8}$$

is a neighborhood of (H_∞, z_∞) in $\overline{X}_K$.

Proposition 8.2. *The topology on $\overline{X}_K$ is Hausdorff.*

Proof. We need to show that any two distinct points x_1, x_2 in $\overline{X}_K$ admit disjoint neighborhoods. This is clearly the case when at least one of them belongs to X. Assume that $x_i = (H_i, z_i) \in \overline{\mathfrak{a}}_{P_i}^{K,+}(\infty) \times X_{P_i}$. There are two cases to consider: $P_1 = P_2$ or not.

In the first case, if $z_1 \neq z_2$, it follows from the proof of Proposition 7.3 (or rather the corresponding result for $\overline{X}_{\max}$ in Proposition 4.2) that when $t \gg 0$, ε is sufficiently small, C is a sufficiently small neighborhood of e, and V_i a sufficiently small neighborhood of z_i, $i = 1, 2$, the neighborhoods $C(S_{\varepsilon,t,V_i}^{K} \cup \coprod U_{I_i,\varepsilon_j} \times S_{\varepsilon,t,V_i}^{K,I})$ of x_i are disjoint. On the other hand, if $z_1 = z_2$, then $H_1 \neq H_2$. We claim that the same conclusion holds.

If not, there exists a sequence y_j in the intersection of $C_j(S_{\varepsilon_j,t_j,V_{i,j}}^{K} \cup \coprod U_{I_i,\varepsilon_j} \times S_{\varepsilon_j,t_j,V_{i,j}}^{K,I})$ for sequences $\varepsilon_j \to 0$, $t_j \to +\infty$, C_j that shrinks to e, and $V_{i,j}$ shrinks to z_i, $i = 1, 2$.

Assume first $y_j \in X$. Then $y_j = k_j n_j a_j z_j$ with $k_j \in K$, $n_j \in N_P$, $a_j \in A_P$, $z_j \in X_P$ satisfying the conditions (1) $k_j \to e$, (2) $\log a_j \to H_1 \in \overline{\mathfrak{a}}_P^{K,+}(\infty)$, (3) $n_j^{a_j} \to e$, (4) $z_j \to z_1$. Similarly, $y_j = k'_j n'_j a'_j z'_j$ with the components satisfying the same condition except $\log a'_j \to H_2$, $z'_j \to z_2$. Since $H_1 \neq H_2$,

$$\|\log a_j - \log a'_j\| \to +\infty. \tag{9}$$

Since

$$d(k_j n_j a_j z_j, k_j a_j z_j) = d(n_j^{a_j} z_j, z_j) \to 0$$

and

$$d(k_j a_j z_j, k_j a_j x_0) = d(z_j, x_0)$$

is bounded, it follows that

$$d(k_j a_j x_0, k'_j a'_j x_0) \leq c$$

for some constant c. By [**AJ**, Lemma 2.1.2],

$$d(k_j a_j x_0, k'_j a'_j x_0) \geq \|\log a_j - \log a'_j\|,$$

and hence

$$\| \log a_j - \log a'_j \| \le c.$$

This contradicts Equation (9), and the claim is proved. The case where y_j belongs to the boundary of $\overline{X}_K$ can be handled similarly.

In the second case, $P_1 \neq P_2$, and we can use the fact that $S^K_{\varepsilon,t,V}$ is contained in the generalized Siegel set $S_{\varepsilon,t,V}$ defined in Equation (15) in §2 and the separation result in Proposition 2.4 to prove that x_1, x_2 admit disjoint neighborhoods. q.e.d.

Proposition 8.3. *The G-action on X extends to a continuous action on $\overline{X}_K$.*

Proof. For any $g \in G$ and $(H, z) \in \overline{\mathfrak{a}}^{K,+}_P(\infty) \times X_P$, write $g = kman$ with $k \in K, m \in M_P$, $a \in A_P$, $n \in N_P$. Define

$$g \cdot (H, z) = (Ad(k)H, k \cdot mz) \in \overline{\mathfrak{a}}^{K,+}_{{}^kP}(\infty) \times X_{{}^kP},$$

where k canonically identifies $\overline{\mathfrak{a}}^{K,+}_P(\infty)$ with $\overline{\mathfrak{a}}^{K,+}_{{}^kP}(\infty)$, and the K-action on X_P is defined in Equation (5) in §2. This defines an extended action of G on $\overline{X}_K$. Arguments similar to those in the proof of Proposition 7.4 show that this extended action is continuous. q.e.d.

Proposition 8.4. *The space $\overline{X}_K$ is compact.*

Proof. Since $X = K \exp \overline{\mathfrak{a}^+_{P_0}} x_0$, $\overline{X}_K = K \overline{\exp \overline{\mathfrak{a}^+_{P_0}} x_0}$, where P_0 is a minimal parabolic subgroup, and $\overline{\exp \overline{\mathfrak{a}^+_{P_0}} x_0}$ is the closure of $\exp \overline{\mathfrak{a}^+_{P_0}} x_0$ in $\overline{X}_K$. Since K is compact, it suffices to prove the compactness of $\overline{\exp \overline{\mathfrak{a}^+_{P_0}} x_0}$, which follows easily from the definition. In fact, for any unbounded sequence $H_j \in \mathfrak{a}^+_{P_0}$, there exists an ordered partition Σ' : $I_1 \cup \dots \cup I_k \cup J = \Delta$ of $\Delta(P_0, A_{P_0})$, where J could be empty, such that, after replacing by a subsequence, H_j satisfies the conditions:

1) For all $\alpha \in J$, $\alpha(H_j)$ converges to a finite limit.
2) For all $\alpha \notin J$, $\alpha(H_j) \to +\infty$.
3) For $\alpha, \beta \in I_m$, $\alpha(H_j)/\beta(H_j)$ converges to a finite positive number.
4) For $\alpha \in I_m, \beta \in I_n$, $m < n$, $\alpha(H_j)/\beta(H_j) \to +\infty$.

Then it follows from definition that $\exp H_j x_0$ converges to a point in $\mathfrak{a}^{\Sigma,+}_{P_J}(\infty) \times X_{P_J} \subset \overline{\mathfrak{a}}^{K,+}_{P_J}(\infty) \times X_{P_J}$, where Σ is the partition $I_1 \cup \dots \cup I_k$ of $\Delta \setminus J = \Delta(P_{0,J}, A_{P_{0,J}})$. q.e.d.

Proposition 8.5. *The identity map on X extends to a continuous surjective map $\pi : \overline{X}_K \to \overline{X}^c$, and for every point $H \in \mathfrak{a}^+_P(\infty)$, the fiber $\pi^{-1}(H)$ is equal to $\overline{(X_P)}_K$; in particular,*

$$\overline{X}_K = X \cup \coprod_P \mathfrak{a}^+_P(\infty) \times \overline{(X_P)}_K,$$

where P runs over all parabolic subgroups.

Proof. For any unbounded sequence y_j in X, if it converges to $(H_\infty, z) \in \overline{\mathfrak{a}}_P^{K,+}$ in $\overline{X}_K$, then it follows from the definitions of convergence of sequences that y_j converges to $\pi(H_\infty)$ in $\overline{X}^c$, where π is the map in Proposition 8.1. By [**GJT**, Lemma 3.28], this defines a continuous map

$$\pi : \overline{X}_K \to \overline{X}^c = X \cup X(\infty).$$

For any point $H \in \partial \overline{X}^c = \coprod_Q \mathfrak{a}_Q^+(\infty)$, let Q be the unique parabolic subgroup such that $H \in \mathfrak{a}_Q^+(\infty)$. For any parabolic subgroup P contained in Q, let P' be the corresponding parabolic subgroup in M_Q. Let $J \subset \Delta(P, A_P)$ such that $Q = P_J$. For any partition $\Sigma : I_1 \cup \cdots \cup I_k$ of $\Delta(P, A_P)$ satisfying $I_1 = \Delta - J$, Σ induces a partition $\Sigma' : I_2 \cup \cdots \cup I_k$ of $\Delta(P', A_{P'})$. For every point $H' \in \mathfrak{a}_{P'}^{\Sigma',+}(\infty)$, $z \in X_P$, then $(H, H') \in \mathfrak{a}_P^{\Sigma,+}(\infty)$, $((H, H'), z) \in \overline{\mathfrak{a}}_P^{K,+}(\infty) \times X_P$, and the fiber $\pi^{-1}(H)$ consists of the union

$$\cup_{P \subseteq Q} \cup_\Sigma \{((H, H'), z) \mid H' \in \mathfrak{a}_{P'}^{\Sigma',+}(\infty), z \in X_P\}$$

where for each $P \subseteq Q$, write $Q = P_J$ as above, the second union is over all the partitions Σ with $I_1 = \Delta - J$. This set can be identified with

$$\cup_{P' \subseteq M_Q} \cup_{\Sigma'} \mathfrak{a}_{P'}^{\Sigma',+}(\infty) \times X_{P'} = X_Q \cup \cup_{P' \subset M_Q} \overline{\mathfrak{a}}_{P'}^{K,+}(\infty) \times X_{P'},$$

which is equal to $\overline{(X_Q)_K}$ by definition.

q.e.d.

Proposition 8.6. *The identity map on X extends to a continuous map $\overline{X}_K \to \overline{X}^M$, and this map is a homeomorphism if and only if the rank of X is less than or equal to 2.*

Proof. It is clear from the definitions that if an unbounded sequence y_j in X converges to $(H_\infty, z_\infty) \in \overline{\mathfrak{a}}_P^{K,+}(\infty) \times X_P$ in $\overline{X}_K$, then y_j also converges in $\overline{X}^M$ to $(\pi(H_\infty), z_\infty) \in \overline{\mathfrak{a}_P^+(\infty)} \times X_P$, where π is the map defined in Proposition 8.1. By [**GJT**, Lemma 3.28], this defines a continuous map $\overline{X}_K \to \overline{X}^M$. By Proposition 8.1, this map is bijective if and only if $rk(X) \leq 2$. In this case, it is a homeomorphism. q.e.d.

Proposition 8.7. *Let $\overline{X}^K$ be the Karpelevic compactification, and $\overline{X}_K$ the compactification defined in this section. Then the identity map on X extends to a homeomorphism $\chi : \overline{X}^K \to \overline{X}_K$.*

Proof. The Karpelevic compactification $\overline{X}^K$ is defined in [**Ka**, §13] and the construction is fairly complicated. The original definition of the Karpelevic compactification $\overline{X}^K$ is also recalled in [**GJT**, Chap. V] using notations similar to those in this paper. Briefly, $\overline{X}^K$ is defined inductively on the rank of X. When $rk(X) = 1$, $\overline{X}^K$ is defined to be $X \cup X(\infty) = \overline{X}^c$. When $rk(X) > 1$, for any $H \in X(\infty) = \coprod_P \mathfrak{a}_P^+(\infty)$, let P be a unique parabolic subgroup such that $H \in \mathfrak{a}_P^+(\infty)$, i.e., P is the stabilizer of H in G. Let $X_H = X_P$. Then X_H is a symmetric space of rank strictly less than $rk(X)$. By induction, we can assume that $\overline{X_H}^K$ is defined already. Then define

$$\overline{X}^K = X \cup \coprod_{H \in X(\infty)} \overline{X_H}^K.$$

Neighborhoods of boundary points are also defined inductively.

To prove the proposition, by [**GJT**, Lemma 3.28], it suffices to prove that any unbounded sequence y_j in X that converges in $\overline{X}^K$ also converges in $\overline{X}_K$, and hence we only need to describe the intersection with X of neighborhoods of boundary points in $\overline{X}^K$ which is given in [**Ka**, §13.8], [**GJT**, §5.5] and show that it is contained in the intersection with X of the neighborhoods of boundary points in $\overline{X}_K$.

For a point $(H_\infty, z_\infty) \in \overline{\mathfrak{a}}_P^{K,+}(\infty) \times X_P$, let $H_\infty = (H_{\infty,1}, \dots, H_{\infty,k}) \in \mathfrak{a}_P^{\Sigma,+}(\infty)$, where $\Sigma : I_1 \cup \cdots \cup I_k$ is an ordered partition of $\Delta(P, A_P)$. The subset $J_2 = I_2 \cup \cdots \cup I_k$ determines a parabolic subgroup P_{J_2} containing P. Let P' be the unique parabolic subgroup in $M_{P_{J_2}}$ corresponding to P as in Equation (7) in §2. Then $I_2 \cup \cdots \cup I_k$ can be identified with $\Delta(P', A_{P'})$ and gives an ordered partition Σ' of the latter. Hence (H_∞, z_∞) defines a boundary point $(H'_\infty, z_\infty) \in \overline{(X_{P_{J_2}})}_K$, where $H'_\infty = (H_{\infty,2}, \dots, H_{\infty,k})$, $(H'_\infty, z_\infty) \in \overline{\mathfrak{a}}_{P'}^{K,+}(\infty) \times X_{P'}$.

For any point $x \in X$, the directed geodesic from x_0 to x is denoted by $\overline{x_0, x}$, and the geodesic issued from x_0 with direction $H_{\infty,1}$ by $\overline{x_0, H_{\infty,1}}$. Since X is simply connected and nonpositively curved, the geodesic $\overline{x_0, x}$ is unique. Denote the angle between two such geodesics at x_0 by $\angle\, \overline{x_0, x}, \overline{x_0, H_{\infty,1}}$. Let W be the intersection with $X_{P_{J_2}}$ of a neighborhood of (H'_∞, z_∞) in $\overline{(X_{P_{J_2}})}^K$. Identify $A_{P_{J_2}} \times X_{P_{J_2}}$ with a subset of X by $(a, z) \in A_{P_{J_2}} \times X_{P_{J_2}} \mapsto az \in X$. For $\varepsilon > 0, t > 0$, define

$$\hat{S}^K_{\varepsilon,t,W} = \Big\{(a, z) \in A_{P_{J_2}} \times X_{P_{J_2}} \mid \\ d(x_0, az) > t, \angle\, \overline{x_0, az}, \overline{x_0, H_{\infty,1}} < \varepsilon, z \in W\Big\}.$$

Let D be a neighborhood of e in G. Then $D\hat{S}^K_{\varepsilon,t,W}$ is the intersection with X of a neighborhood of (H_∞, z_∞) in $\overline{X}^K$.

In the following, we use the induction on the rank of X to prove $\overline{X}^K = \overline{X}_K$. When $rk(X) = 1$, both $\overline{X}^K$ and $\overline{X}_K$ are isomorphic to $X \cup X(\infty)$, and hence $\overline{X}^K \cong \overline{X}_K$. Since $rk(X_{P_{J_2}}) < rk(X)$, we can assume by induction that

$$\overline{(X_{P_{J_2}})}^K = \overline{(X_{P_{J_2}})}_K.$$

Hence, W can be taken to be a subset in $X_{P_{J_2}}$ of the form

$$C' S^K_{\varepsilon',t',V'} = C'\Big\{(n', a', z') \in N_{P'} \times A_{P'} \times X_{P'} \mid \log a' \in U_{\varepsilon'}, a' \in A_{P',t'}, {n'}^{a'} \in B_{N_{P'}}(\varepsilon'), z' \in V'\Big\}, \tag{10}$$

where V' is a bounded neighborhood of z_∞ in $X_{P'} = X_P$, $U_{\varepsilon'} = U_{\varepsilon'}(H'_\infty)$, C' a neighborhood of e in $K_{P_{J_2}}$, and other sets are defined similarly to those in Equation (7).

For any $CS^K_{\varepsilon,t,V}$, we show that if ε'', W, D are sufficiently small and $t'' \gg 0$, then

$$D\hat{S}^K_{\varepsilon'',t'',W} \subset CS^K_{\varepsilon,t,V}.$$

We first show that $\hat{S}^K_{\varepsilon'',t'',W}$ is contained in $CS^K_{\varepsilon,t,V}$. We need to show that for any $(a,z) \in \hat{S}^K_{\varepsilon'',t'',W}$, there exists $k \in K$ such that $k^{-1} \in C$, and $k(a,z) \in S^K_{\varepsilon,t,V}$. Write the horospherical decomposition with respect to P:

$$kaz = (n_P, a_P, z_P) \in N_P \times A_P \times X_P. \tag{11}$$

Then we need to check the following conditions: (a) $a_P \in A_{P,t}$, (b) $\log a_P \in U_\varepsilon$, (c) $n_P^{a_P} \in B_{N_P}(\varepsilon)$, (d) $z_P \in V$.

Since W is of the form in Equation (10) and $z \in W$, we can write $z = k'n'a'z'$, where $k' \in C' \subset K_{P_{J_2}}$, $n' \in N_{P'}$, $a' \in A_{P'}$, $z' \in V'$, and $(n', a', z') \in S^K_{\varepsilon',t',V'}$. When W is sufficiently small, $k' \in C$. Take $k = {k'}^{-1}$ in Equation (11). Then

$$kaz = akz = an'a'z' = n'aa'z',$$

where we have used the fact that $a \in A_{P_{J_2}}$ commutes with $k, n' \in M_{P_{J_2}}$. Hence

$$n_P = n', \quad a_P = aa', \quad z_P = z'.$$

For condition (a), we note that for $\alpha \in J_2$, $a^\alpha = 1$, and hence

$$a_P^\alpha = {a'}^\alpha \geq t'$$

since $a' \in A_{P',t'}$, and hence $a_P^\alpha \to +\infty$ as W shrinks to (H_∞, z_∞). We need to show that for $\alpha \in I_1 = \Delta(P_{J_2}, A_{P_{J_2}})$, $a_P^\alpha \to +\infty$ as well when $\varepsilon'' \to 0$, $t'' \to +\infty$, and W shrinks to (H'_∞, z_∞). By the definition of $\hat{S}^K_{\varepsilon'',t'',W}$,

$$\angle \overline{x_0, az},\ \overline{x_0, H_{\infty,1}} < \varepsilon''.$$

Since $k \to e$ as W shrinks to (H'_∞, z_∞),

$$\begin{aligned} \angle \overline{x_0, n_P a_P z_P},\ \overline{x_0, H_{\infty,1}} &= \angle \overline{x_0, kaz},\ \overline{x_0, H_{\infty,1}} \\ &= \angle k(\overline{x_0, az}),\ \overline{x_0, H_{\infty,1}} \to 0 \end{aligned}$$

as W shrinks to (H'_∞, z_∞) and $\varepsilon'' \to 0$, where $k(\overline{x_0, az})$ is the geodesic through x_0 obtained from $\overline{x_0, az}$ under the action (or rotation) of k. We note that

$$\angle\, \overline{x_0, a_P x_0},\ \overline{x_0, H_{\infty,1}} \leq \angle\, \overline{x_0, n_P a_P z_P},\ \overline{x_0, H_{\infty,1}}.$$

The reason is that in the decomposition $X = N_P \times A_P \times X_P$, $A_P \cong A_P x_0$ is a totally geodesic submanifold in the simply connected, nonpositively curved manifold X, the geodesic $\overline{x_0, H_{\infty,1}} \subset A_P$, and picking out the A_P-component a_P is the orthogonal projection to A_P. Note that there exist positive constants c_1, c_2 such that

$$\begin{aligned} c_1 \| \log a_P / \| \log a_P \| - H_{\infty,1} \| &\leq \angle\, \overline{x_0, a_P x_0},\ \overline{x_0, H_{\infty,1}} \\ &\leq c_2 \| \log a_P / \| \log a_P \| - H_{\infty,1} \|, \end{aligned}$$

and hence

$$\| \log a_P / \| \log a_P \| - H_{\infty,1} \| \to 0 \tag{12}$$

as W shrinks to (H'_∞, z_∞) and $\varepsilon'' \to 0$. Since $H_{\infty,1} \in \mathfrak{a}^+_{P_{J_2}}(\infty)$, for $\alpha \in I_1$,

$$\frac{\alpha(\log a_P)}{\| \log a_P \|} \to \frac{\alpha(H_{\infty,1})}{\| H_{\infty,1} \|} > 0. \tag{13}$$

To use this result to prove $a_P^\alpha \to +\infty$, $\alpha \in I_1$, we need that $\| \log a_P \| \to +\infty$ as $t'' \to +\infty$. Consider the triangle with vertexes x_0, $a_P x_0$ and $kza = n_P a_P z_P$. Since $z_P \in V'$, $d(z_P, x_0)$ is bounded. Since

$$n_P^{a_P} = n'^{a_P} = n'^{aa'} = n'^{a'} \in B_{N_{P'}}(\varepsilon') \tag{14}$$

is bounded,

$$d(kaz, a_P x_0) = d(n_P a_P z_P, a_P x_0) = d(n_P^{a_P} z_P, x_0) = d(n'^{a'} z_P, x_0)$$

is also bounded. Then

$$\begin{aligned} d(a_P x_0, x_0) &\geq d(kaz, x_0) - d(kaz, a_P x_0) \\ &= d(az, x_0) - d(kaz, a_P x_0) \\ &\geq t'' - d(kaz, a_P x_0) \end{aligned}$$

goes to infinity as $t'' \to +\infty$, and hence

$$\| \log a_P \| \to +\infty.$$

Combined with Equation (13), it implies that as $\varepsilon'' \to 0$, W shrinks to (H'_∞, z_∞) and $t'' \to +\infty$, for $\alpha \in I_1$,

$$a_P^\alpha \to +\infty.$$

Hence the condition (a) is satisfied.

For condition (b), we need to check the conditions in Equation (5). Condition (1) follows from condition (a) above. Equation (12) implies that for every pair $\alpha \in I_1$, $\beta \in J_2$, as $\varepsilon'' \to 0$, $t'' \to +\infty$ and W shrinks to (H'_∞, z_∞),

$$\frac{\beta(\log a_P)}{\alpha(\log a_P)} \to \frac{\beta(H_{\infty,1})}{\alpha(H_{\infty,1})} = \frac{0}{\alpha(H_{\infty,1})} = 0.$$

For $\alpha, \beta \in J_2$, $\alpha(\log a_P) = \alpha(\log a')$, $\beta(\log a_P) = \beta(\log a')$, and hence

$$\frac{\beta(\log a_P)}{\alpha(\log a_P)} = \frac{\beta(\log a')}{\alpha(\log a')}.$$

Since $\log a' \in U_{\varepsilon'}(H'_\infty)$, the other conditions in Equation (5) are also satisfied.

To show that the condition (c) is satisfied, we note as in Equation (14) that

$$n_P^{a_P} = n'^{a'} \in B_{N_{P'}}(\varepsilon').$$

Since $N_P = N_{P_{J_2}} N_{P'}$,

$$n_P^{a_P} \in B_{N_P}(\varepsilon)$$

when $\varepsilon' < \varepsilon$. Condition (d) that $z_P \in V$ is clearly satisfied when $V' \subset V$. This proves that when ε'', W are sufficiently small and $t'' \gg 0$, $\hat{S}^K_{\varepsilon'',t'',W} \subset CS^K_{\varepsilon,t,V}$.

Next we prove that when D is also sufficiently small,

$$D\hat{S}^K_{\varepsilon'',t'',W} \subset CS^K_{\varepsilon,t,V}. \tag{15}$$

In fact, the only remaining problem is that D is a neighborhood of e in G instead of in K as C is. To overcome this problem, we note that for any $g_j \to e$, we can write

$$g_j = k_j m_j a_j n_j, \quad k_j \in K, m_j \in M_P, a_j \in A_P, n_j \in N_P$$

such that

$$k_j, m_j, a_j, n_j \to e.$$

Then the inclusion in Equation (15) follows from Equation (1) in §2 and the inclusion $\hat{S}^K_{\varepsilon'',t'',W} \subset CS^K_{\varepsilon,t,V}$ is proved above.

As pointed out at the beginning of the proof, the inclusion in Equation (15) implies that any unbounded sequence y_j in X which converges in $\overline{X}^K$ also converges in $\overline{X}_K$. By [**GJT**, Lemma 3.28], this implies that the identity map on X extends to a continuous map

$$\chi : \overline{X}^K \to \overline{X}_K.$$

As mentioned earlier, by induction on the rank, for any $H \in X(\infty)$,

$$\overline{X_H}^K = \overline{(X_H)}_K.$$

By Proposition 8.5,

$$\overline{X}^K = X \cup \coprod_{H \in X(\infty)} \overline{X_H}^K = X \cup \coprod_{H \in X(\infty)} \overline{(X_H)}_K = \overline{X}_K.$$

This implies that χ is bijective. Since both $\overline{X}^K$ and $\overline{X}_K$ are compact and Hausdorff, χ is a homeomorphism.

Remark 8.8. A corollary of the identification of the topologies of $\overline{X}^K$ and $\overline{X}_K$ in the above proposition is that the two collections of subsets $D\hat{S}^K_{\varepsilon,t,W}$, $CS^K_{\varepsilon,t,V}$ of X are co-final when $\varepsilon \to 0$, $t \to +\infty$, and V ranges over a neighborhood basis of z_∞ in X_P, and W over the intersection with $X_{P_{J_2}}$ of a neighborhood basis of (H'_∞, z_∞) in $\overline{X_{P_{J_2}}}^K$. It does not seem to be easy to prove this co-finality directly. The reason is that there is an extra factor n_P in $S^K_{\varepsilon,t,V}$ which does not appear in $D\hat{S}^K_{\varepsilon,t,W}$. Since $n_P^{a_P} \in B_{N_P}(\varepsilon)$ and $n_P a_P = a_P n_P^{a_P}$, it is intuitively clear and basically implied by the above co-finality that this factor n_P (or rather $n_P^{a_P}$) can be absorbed into C, i.e., $n_P a_P z_P = k a'_P z'_P$, where $k \in C$ and a'_P, z'_P satisfy similar conditions to those satisfied by a_P, z_P; but it is not obvious how to do this, because $a_P n_P^{a_P} \neq n_P^{a_P} a_P$. The issue is how to relate the two basic decompositions: the Cartan decomposition and the Iwasawa (or Langlands, horospherical) decomposition. This is also the basic difficulty in [**GJT**, Chap. III] to show that the G-action on X extends continuously to the dual-cell compactification $X \cup \Delta^*(X)$ as mentioned in the introduction.

q.e.d.

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