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Criterions on Equality of Symmetric Inequalities(II)

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Abstract

In this paper, we continues to investigate the properties of symmetric inequalities. We study the necessary condition for the extremum to be attained in a semi-algebraic system. From which, we partly solve the problem of the further generalization of *Jensen* inequality which the author brought up in the previous article. Namely, by using the sign of the $(n+1)^{th}$ derivative while fixing the value of the sum of power of n variables, we get a dimension-descending method. Further, we prove that the necessary and sufficient condition for the symmetric inequality of degree m with n variables to hold on R_n^+ is that it holds when the number of nonzero variables does not exceed $\max\{1, [\frac{m}{2}]\}$.

Key words: symmetric inequality, dimension-descending method, Jensen inequality

Contents

1 Introduction

Symmetric inequality has become a research focus of inequality [1;2]. The dimension-descending method is a good example of the thought of mathematics mechanization: reduction of the difficulty in quality to the complexity in quantity [3]. Over the past year, some dimension-descending methods are developed in the National Basic Research Program(973 Program) [4;5]. We have studied the criterion on equality and its proof of a class of symmetric inequalities in the previous paper [6]. In this article, we study some properties of symmetric inequalities and try to partially solve the problem and respond to the prospect mentioned in the previous article, that is we obtain a new dimension-descending method which is the further generalization of Jensen inequality. We also apply this result to our proof in the case of symmetric inequality.

2 Notation

For the rigor and convenient of the redaction,we begin with introducing some definitions in common use.

Definition 2.1. *Let R be the real field, R^n the n-dimensional real vector space.*

$$R^n_+ = [0, +\infty)^n; R^n_{++} = (0, +\infty)^n;$$

Definition 2.2. *A function of the form $f : R_{++} \to R, f(x) = \sum_{i=1}^{n} a_i x^{\alpha_i}$, where $m \in N, a_i \in R, a_i \neq 0, \alpha_i \in R$ and α_i's pairwise different, is called a generalized polynomial.*

Definition 2.3. *A polynomial $f(x_1, x_2, \ldots, x_n)$ is said to be symmetric ,if*

$$f(x_1, x_2, \ldots, x_n) = f(\sigma(x_1, x_2, \ldots, x_n))$$

for all $\sigma \in S_n$, where S_n is the symmetric group of degree n.

Definition 2.4. *Let $\sigma_{(n,1)}, \sigma_{(n,2)}, \ldots, \sigma_{(n,n)}$ be the n elementary symmetric polynomials of $(x_1, x_2, \ldots, x_n)$.*

Definition 2.5. *Let $n, k \in \mathbb{N}$, $s_{(n,k)}$ stands for the sum of k^{th} power of $\boldsymbol{x_n} = (x_1, x_2, \ldots, x_n)$,namely*

$$s_{(n,k)}(\boldsymbol{x_n}) = \sum_{i=1}^{n} x_i^k \qquad (k = 1, 2, \ldots)$$

Definition 2.6. *For any $\boldsymbol{x_n} = (x_1, x_2, \ldots, x_n) \in R^n_+$, we note*

$$v(\boldsymbol{x_n}) = |\{x_j | j = 1, 2 \ldots, n\}| \, , v(\boldsymbol{x_n})^* = |\{x_j | x_j \neq 0, j = 1, 2 \ldots, n\}|$$

Definition 2.7. *Let $A = [a_1, a_2, \ldots, a_n]$,$a_i \neq 0 (i = 1, 2, \ldots, n)$ be a real sequence and $sgn(x)$ the sign function. Let C_A be the number of -1 in the sequence $[sgn(a_1 a_2), sgn(a_2 a_3), \ldots, sgn(a_{n-1} a_n)]$. We call C_A the sign-changing number of A.*

3 Lemmas

Lemma 3.1. *Any polynomial of degree m with n variables is uniquely expressed as a polynomial of $s_{(n,1)}, s_{(n,2)}, \dots, s_{(n,d)}$,where $d = \min\{n, m\}$.*

Proof *It suffice to notice that*

$$k!\sigma_{(n,k)} = \begin{vmatrix} s_{(n,1)} & 1 & 0 & \cdots & 0 \\ s_{(n,2)} & s_{(n,1)} & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{(n,k)} & s_{(n,k-1)} & s_{(n,k-2)} & \cdots & s_{(n,1)} \end{vmatrix} \quad (k = 1, 2, \dots, n)$$

The lemma follows from the fact that any polynomial of degree m with n variables is uniquely represented as a polynomial of $\sigma_{(n,1)}, \sigma_{(n,2)}, \dots, \sigma_{(n,d)}$,$d = \min\{n, m\}$. □

Lemma 3.2. [5] *Let $a_1, a_2, \dots, a_n$, $\alpha_1, \alpha_2, \dots, \alpha_n$ be real constants,$\alpha_1 < \alpha_2 < \dots < \alpha_n$,$a_i \neq 0 (i = 1, 2, \dots, n)$, Z_f be the number of positive roots of $f(x) = \sum_{i=1}^{n} a_i x^{\alpha_i}$, and C_A the sign-changing number of $[a_1, a_2, \dots, a_n]$. Then $Z_f \leq C_f$.*

Lemma 3.3. *Let $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ satisfying $\alpha_1 < \alpha_2 < \dots < \alpha_n$, $0 < x_1 < x_2 \dots < x_n$. Then we have*

$$D_n = \begin{vmatrix} x_1^{\alpha_1} & x_2^{\alpha_1} & \dots & x_n^{\alpha_1} \\ x_1^{\alpha_2} & x_2^{\alpha_2} & \dots & x_n^{\alpha_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\alpha_n} & x_2^{\alpha_n} & \dots & x_n^{\alpha_n} \end{vmatrix} > 0$$

Proof *We prove firstly that D_n never vanishes.*

We prove it by reductio ad absurdum. Suppose $(x_i^{\alpha_1}, x_i^{\alpha_2}, \dots, x_i^{\alpha_n}) (i = 1, 2, \dots, n)$ linearly dependent. Then there exist $a_1, a_2, \dots, a_n$ not all zero, such that $a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + \dots + a_n x^{\alpha_n} = 0$ has n positive solution. $x = x_1, x_2, \dots, x_n$. then $Z_f \geq n$. On the other hand, it's obvious that the sign-changing number of the sequence $[a_1, a_2, \dots, a_n]$, $C_f \leq n - 1$. By lemma 3.2, $Z_f \leq C_f \leq n - 1$, which is contradiction! This shows that $D_n \neq 0$.

Next,we prove by induction that $D_n > 0$.

When $n = 1$,$D_1 = {x_1}^{\alpha_1} > 0$. Suppose that the proposition holds for $n = k$, namely $D_k > 0$.

Then for $n = k + 1$, we regard x_{k+1} as a variable. $D_{k+1} \neq 0$ when x_{k+1} is increasing in $(x_k, +\infty)$, so D_{k+1} has a definitive sign. When $x_{k+1} \to +\infty$, the sign of D_{k+1} is also that of the k^{th} principal minor determinant in order and hence is $D_{k+1} > 0$.

So the proposition holds for all $n \in \mathbb{N}$, and the lemma is proved. □

Lemma 3.4. *Consider the following semi-algebraic system*

$$\begin{cases} F_1(x_1, x_2, \dots, x_{n+1}) = {x_1}^{\alpha_1} + {x_2}^{\alpha_1} + \dots + {x_{n+1}}^{\alpha_1} - P_1 = 0 \\ F_2(x_1, x_2, \dots, x_{n+1}) = {x_1}^{\alpha_2} + {x_2}^{\alpha_2} + \dots + {x_{n+1}}^{\alpha_2} - P_2 = 0 \\ \qquad \cdots\cdots \\ F_n(x_1, x_2, \dots, x_{n+1}) = {x_1}^{\alpha_n} + {x_2}^{\alpha_n} + \dots + {x_{n+1}}^{\alpha_n} - P_n = 0 \\ 0 \leq x_1 \leq x_2 \leq \dots \leq x_{n+1} \end{cases} \tag{3.1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}, \alpha_1 < \alpha_2 < \ldots < \alpha_n$. If $\alpha_i = 0$ for some i,we define

$$x_1{}^{\alpha_i} + x_2{}^{\alpha_i} + \ldots + x_{n+1}{}^{\alpha_i} = x_1 x_2 \cdots x_{n+1}$$

Then all the solutions $(x_1, x_2, \ldots, x_{n+1})$ of (3.1) form a compact subset of R^{n+1}. Let $(y_1^0, y_2^0, \ldots, y_{n+1}^0)$ be a solution of (3.1), where $0 \le y_1^0 \le y_2^0 \le \ldots \le y_{n+1}^0$, and the equality holds in no more than one place.
(1) If $\alpha_i > 0 (i = 1, 2, \ldots, n)$,
then there exist $a, b, a \le y_1^0 \le b$,such that for any $x_1 \in (a, b)$
there exist $x_1, x_2, \ldots, x_{n+1}$ which satisfy (3.1), and $x_1 < x_2 < \ldots < x_{n+1}$.
if $a \neq 0$, then for $x_1 = a$,there exist $x_2, \ldots, x_{n+1}$ satisfying (3.1) with $x_{2i} = x_{2i+1}$ for some $i(2 \le 2i \le n)$.
For $x_1 = b$, there exist $x_2, \ldots, x_{n+1}$ satisfying (3.1) with $x_{2i} = x_{2i-1}$ for some $i(2 \le 2i \le n+1)$.
(2) If $\alpha_i \le 0$ for some i,
then there exist $a, b, a \le y_1^0 \le b$, such that for all $x_1 \in (a, b)$
there exist $x_1, x_2, \ldots, x_{n+1}$ satisfying (3.1), with $x_1 < x_2 < \ldots < x_{n+1}$.
For $x_1 = a$, there exist $x_2, \ldots, x_{n+1}$ satisfying (3.1) with $x_{2i} = x_{2i+1} (2 \le 2i \le n)$.
For $x_1 = b$, there exist $x_2, \ldots, x_{n+1}$ satisfying (3.1) with $x_{2i} = x_{2i-1} (2 \le 2i \le n+1)$.

Proof *Clearly, the set of solutions for any equation of the system (3.1) is compact in R^{n+1}, and the set of all x_i for which $0 \le x_1 \le x_2 \le \cdots \le x_{n+1}$ is a close subset of R^{n+1}. Their intersection is therefore a compact subset of R^{n+1}. We argue only for the case where $y_2^0 < y_3^0 \ldots < y_{n+1}^0$. The reasoning is similar in the other cases.*

Obviously, if there is a solution to (3.1) for which $x_1 \to 0$, we have $\alpha_i > 0$.If $y_1^0 = 0$, we take $a = 0$, then $\alpha_i > 0$.

If $y_1^0 > 0$,a positive solution of (3.1) with $x_2 < x_3 < \cdots < x_{n+1}$ is called required solution. It is clear that $(y_1^0, y_2^0, \ldots, y_{n+1}^0)$ is a required solution.

Taking $(y_1^0, y_2^0, \ldots, y_{n+1}^0)$ as an example , we prove firstly that for all required solutions, there exist $\rho > 0$,such that for all $x_1 \in (y_1^0 - \rho, y_1^0]$, there exist a corresponding required solution.

Consider the Jacobi determinant at point $(y_1^0, y_2^0, \ldots, y_{n+1}^0)$ of the function F_i of $x_2, x_3, \ldots, x_{n+1}$. If $\alpha_i \neq 0 (i = 1, 2, \ldots, n)$, we have

$$\frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(x_2, x_3, \ldots, x_{n+1})} = \prod_{i=1}^{n} \alpha_i \begin{vmatrix} x_2^{\alpha_1 - 1} & x_3^{\alpha_1 - 1} & \ldots & x_{n+1}^{\alpha_1 - 1} \\ x_2^{\alpha_2 - 1} & x_3^{\alpha_2 - 1} & \ldots & x_{n+1}^{\alpha_2 - 1} \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{\alpha_n - 1} & x_3^{\alpha_n - 1} & \ldots & x_{n+1}^{\alpha_n - 1} \end{vmatrix}$$

If there exist i, such that $\alpha_i = 0$, then

$$\frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(x_2, x_3, \ldots, x_{n+1})} = \prod_{\substack{j=1 \\ j \neq i}}^{n} \alpha_i \cdot \prod_{i=1}^{n+1} x_i \begin{vmatrix} x_2^{\alpha_1 - 1} & x_3^{\alpha_1 - 1} & \ldots & x_{n+1}^{\alpha_1 - 1} \\ x_2^{\alpha_2 - 1} & x_3^{\alpha_2 - 1} & \ldots & x_{n+1}^{\alpha_2 - 1} \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{\alpha_n - 1} & x_3^{\alpha_n - 1} & \ldots & x_{n+1}^{\alpha_n - 1} \end{vmatrix}$$

In all these two cases, we see by Lemma 3.3 that the determinant > 0.

As it doesn't change the result whether $\alpha_i = 0$ or not, in which follows we only treat with the case that $\alpha_i \neq 0$.

As the function $F_i (i = 1, 2, \ldots, n)$ is continuous and continuously differentiable for $|x_i - y_i^0| \leq \frac{y_1^0}{2} (i = 1, 2, \ldots, n+1)$, according to the implicit function theorem[7]*, there exists ρ, satisfying $\frac{y_1^0}{2} \geq \rho > 0$, such that for $x_1 \in (y_1^0 - \rho, y_1^0]$, an continuously derivable implicit function of vectorial value is uniquely determined by the system of equations*

$$\begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} x_2(x_1) \\ x_3(x_1) \\ \vdots \\ x_{n+1}(x_1) \end{pmatrix}$$

and then $x_1, x_2, \ldots, x_{n+1}$ is a required solution.

Thus, we see that the connected component with y_1^0 as the right endpoint, of the solution with the first coordinate x_1, must be an left hand half open interval (otherwise the solution should have been prolonged at the left endpoint). We set $(a, y_1^0]$ for this interval. Thus there exist a corresponding required solution for $x_1 \in (a, y_1^0]$, while there doesn't exist a required solution for $x_1 = a$.

On the one hand, we take the sequence $\{x_{(1,m)}\}$,$x_{(1,i)} = a + \frac{1}{i}$. Since $\lim_{n\to+\infty} x_{(1,n)} = a$, there must be a positive integer N such that for all $m > N$, $x_{(1,m)} \in (a, y_1^0)$.

On the other hand, as is similar to the proof of existence of ρ, we see clearly that for $x_1 \in (a, y_1^0)$, there exist unique implicit functions $x_2, x_3, \ldots, x_{n+1}$ of x_1 satisfying

$$\begin{pmatrix} \alpha_1 x_2^{\alpha_1-1} & \alpha_1 x_3^{\alpha_1-1} & \ldots & \alpha_1 x_{n+1}^{\alpha_1-1} \\ \alpha_2 x_2^{\alpha_2-1} & \alpha_2 x_3^{\alpha_2-1} & \ldots & \alpha_2 x_{n+1}^{\alpha_2-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n x_2^{\alpha_n-1} & \alpha_n x_3^{\alpha_n-1} & \ldots & \alpha_n x_{n+1}^{\alpha_n-1} \end{pmatrix} \begin{pmatrix} x_2'(x_1) \\ x_3'(x_1) \\ \vdots \\ x_{n+1}'(x_1) \end{pmatrix} = \begin{pmatrix} -\alpha_1 x_1^{\alpha_1-1} \\ -\alpha_2 x_1^{\alpha_2-1} \\ \vdots \\ -\alpha_n x_1^{\alpha_n-1} \end{pmatrix}$$

By Cramer rule,

$$x_i'(x_1) = \frac{D_i}{D_1}, i = 2, 3, \ldots, n+1$$

where

$$D_1 = \prod_{i=1}^{n} \alpha_i \begin{vmatrix} x_2^{\alpha_1-1} & x_3^{\alpha_1-1} & \cdots & x_{n+1}^{\alpha_1-1} \\ x_2^{\alpha_2-1} & x_3^{\alpha_2-1} & \cdots & x_{n+1}^{\alpha_2-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{\alpha_n-1} & x_3^{\alpha_n-1} & \cdots & x_{n+1}^{\alpha_n-1} \end{vmatrix}$$

$$\begin{aligned} D_i =& \prod_{j=1}^{n} \alpha_j \begin{vmatrix} x_2^{\alpha_1-1} & \cdots & x_{i-1}^{\alpha_1-1} & -x_1^{\alpha_1-1} & x_{i+1}^{\alpha_1-1} & \cdots & x_{n+1}^{\alpha_1-1} \\ x_2^{\alpha_2-1} & \cdots & x_{i-1}^{\alpha_2-1} & -x_1^{\alpha_2-1} & x_{i+1}^{\alpha_2-1} & \cdots & x_{n+1}^{\alpha_2-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_2^{\alpha_n-1} & \cdots & x_{i-1}^{\alpha_n-1} & -x_1^{\alpha_n-1} & x_{i+1}^{\alpha_n-1} & \cdots & x_{n+1}^{\alpha_n-1} \end{vmatrix} \\ =& (-1)^{i-1} \prod_{j=1}^{n} \alpha_j \begin{vmatrix} x_1^{\alpha_1-1} & x_2^{\alpha_1-1} & \cdots & x_{i-1}^{\alpha_1-1} & x_{i+1}^{\alpha_1-1} & \cdots & x_{n+1}^{\alpha_1-1} \\ x_1^{\alpha_2-1} & x_2^{\alpha_2-1} & \cdots & x_{i-1}^{\alpha_2-1} & x_{i+1}^{\alpha_2-1} & \cdots & x_{n+1}^{\alpha_2-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_1^{\alpha_n-1} & x_2^{\alpha_n-1} & \cdots & x_{i-1}^{\alpha_n-1} & x_{i+1}^{\alpha_n-1} & \cdots & x_{n+1}^{\alpha_n-1} \end{vmatrix} \end{aligned}$$

Then from Lemma 3.3, $sgn(D_i) = (-1)^{i-1}(i = 1, 2, \ldots, n+1)$.
Thus for $x_1 \in (a, y_1^0)$, $x_{2k+1}(x_1)$ is strictly increasing, and $x_{2k}(x_1)$ strictly decreasing.
That is to say, for the sequence $\{x_{(1,m)}\}$, when $x_{(1,m)} \in (a, y_1^0)$, there exist a corresponding sequence $\{x_{(i,m)}\}(i = 2, 3, \ldots, n+1)$ such that $x_{(i,m)}(i = 1, 2, \ldots, n+1)$ is a required solution with $x_{(2k+1,m)}$ strictly decreasing and $x_{(2k,m)}(x_1)$ strictly increasing. Further more, x_i has the lower bound 0, and upper bound P_1. Thus $x_{(i,k)}$ is monotone on k and bounded, hence convergent.
Let $\lim\limits_{k\to+\infty} x_{(i,k)} = y_i$, then

$$P_1 = \lim_{k\to+\infty}(x_{(1,k)}^{\alpha_1} + x_{(2,k)}^{\alpha_1} + \cdots + x_{(n+1,k)}^{\alpha_1}) = a^{\alpha_1} + y_2^{\alpha_1} + \cdots + y_{n+1}^{\alpha_1}$$
$$P_2 = \lim_{k\to+\infty}(x_{(1,k)}^{\alpha_2} + x_{(2,k)}^{\alpha_2} + \cdots + x_{(n+1,k)}^{\alpha_2}) = a^{\alpha_2} + y_2^{\alpha_2} + \cdots + y_{n+1}^{\alpha_2}$$
$$\cdots\cdots$$
$$P_n = \lim_{k\to+\infty}(x_{(1,k)}^{\alpha_n} + x_{(2,k)}^{\alpha_n} + \cdots + x_{(n+1,k)}^{\alpha_n}) = a^{\alpha_n} + y_2^{\alpha_n} + \cdots + y_{n+1}^{\alpha_n}$$

Namely, $(a, y_{2,1}, \ldots, y_{n+1})$ is also a solution of (3.1).
If there are two equal numbers among y_i, they must be $y_{2i} = y_{2i+1}$ for some $i, 2 \le 2i \le n$. Then a satisfies the requirement.
If $y_i's$ pairwise different, then we must have $a = 0$. Otherwise, for $x_1 = a$, we would get a required solution, which leads to a contradiction !

If $y_1^0 = y_2^0$, we take $b = y_1^0$. Otherwise, we can prove in a similar way that there exist b, such that for $x_1 \in [y_1^0, b)$, there exist $x_2, x_3, \ldots, x_{n+1}$ satisfying (3.1), and $x_1 < x_2 < \cdots < x_{n+1}$. When $x_1 = b$, there exist $x_2 \le x_3 \le \cdots \le x_{n+1}$ satisfying (3.1) with $x_{2i} = x_{2i-1}$ for some $i(1 \le 2i \le n+1)$.

The lemma is proved. □

Lemma 3.5. [8] *Suppose that $h_1(x), h_2(x), \ldots, h_{n-1}(x)$ satisfying*

$$\begin{vmatrix} h_1(x) & h_1'(x) & \cdots & h_1^{(i-1)}(x) \\ h_2(x) & h_2'(x) & \cdots & h_2^{(i-1)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ h_i(x) & h_i'(x) & \cdots & h_i^{(i-1)}(x) \end{vmatrix} > 0 \quad (i = 1, 2, \ldots, n-1)$$

Let $f(x)$ be an arbitrary function, we term the determinant

$$\begin{vmatrix} h_1(x) & h_1'(x) & \cdots & h_1^{(n-1)}(x) \\ h_2(x) & h_2'(x) & \cdots & h_2^{(n-1)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ h_{n-1}(x) & h_{n-1}'(x) & \cdots & h_{n-1}^{(n-1)}(x) \\ f(x) & f'(x) & \cdots & f^{(n-1)}(x) \end{vmatrix} = W(x)$$

the Wronskian determinant of the system of function. If $x_1 < x_2 < \cdots < x_n$, then there exist a number ξ, $x_1 < \xi < x_n$, such that

$$sgn\begin{vmatrix} h_1(x_1) & h_1(x_2) & \ldots & h_1(x_n) \\ h_2(x_1) & h_2(x_2) & \ldots & h_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ h_{n-1}(x_1) & h_{n-1}(x_2) & \ldots & h_{n-1}(x_n) \\ f(x_1) & f(x_2) & \ldots & f(x_n) \end{vmatrix} = sgn(W(\xi))$$

4 Main Result

Theorem 4.1. *Suppose that* $0 \le x_1 \le x_2 \le \cdots \le x_{n+1} (n \in \mathbb{N})$, *and the equality holds in no more than one place. Let* $f(x_1; x_2, \ldots, x_{n+1})$ *be a function of* $x_1, x_2, \ldots, x_{n+1}$, *symmetric with respect to* $x_2, x_3, \ldots, x_{n+1}$. *We note* $f(x_1; x_2, \ldots, x_{n+1})$ *simply as* $f(x_1)$. *Analogously, we define* $f(x_i)(i = 2, 3, \ldots, n+1)$. *Let* $\alpha_i \in R$ *be given reals, satisfying* $\alpha_1 < \alpha_2 < \ldots < \alpha_n$.
Fixing the following system of generalized polynomials, with the value of F_i *invariant*

$$\begin{cases} F_1(x_1, x_2, \ldots, x_{n+1}) = {x_1}^{\alpha_1} + {x_2}^{\alpha_1} + \ldots + {x_{n+1}}^{\alpha_1} \\ F_2(x_1, x_2, \ldots, x_{n+1}) = {x_1}^{\alpha_2} + {x_2}^{\alpha_2} + \ldots + {x_{n+1}}^{\alpha_2} \\ \qquad \cdots \\ F_n(x_1, x_2, \ldots, x_{n+1}) = {x_1}^{\alpha_n} + {x_2}^{\alpha_n} + \ldots + {x_{n+1}}^{\alpha_n} \end{cases}$$

As a function of x_i, *if* $f(x)$ *is* $(n+1)^{th}$ *derivable on* $(0, +\infty)$, *and continuous on* $[0, +\infty)$. *Let*

$$F(\boldsymbol{x_{n+1}}) = f(x_1) + f(x_2) + \cdots + f(x_{n+1})$$

and $W(x)$ *the Wronskian determinant of the system of functions* $(x^{\alpha_1 - 1}, x^{\alpha_2 - 1}, \ldots, x^{\alpha_n - 1}, f'(x))$.
(1) *If* $(-1)^n W(x) \ge 0$, *there exists* $\boldsymbol{x_{n+1}}$ *with* $x_{2i-1} = x_{2i}$ *for some* $i(2 \le 2i \le n+1)$, *at which point* F *attains its maximum, and there exists* $\boldsymbol{x_{n+1}}$ *with* $x_{2i} = x_{2i+1}$ *for some* $i(2 \le 2i \le n+1)$ *or* $x_1 = 0$, *at which point* F *attains its minimum.*
(2) *If* $(-1)^n W(x) \le 0$, *there exists* $\boldsymbol{x_{n+1}}$ *with* $x_{2i-1} = x_{2i}$ *for some* $i(2 \le 2i \le n+1)$, *at which point* F *attains its minimum, and there exists* $\boldsymbol{x_{n+1}}$ *with* $x_{2i} = x_{2i+1}$ *for some* $i(2 \le 2i \le n+1)$ *or* $x_1 = 0$, *at which point* F *attains its maximum.*

Proof *We prove only* (1) *for the case that* $x_2 < x_3 \cdots < x_{n+1}$, *in the other cases, the proof is similar.*

By Lemma 3.4, there exist a, b, $a \le x_1 \le b$. *For* $x_1 \in [a, b]$, F *is considered as a function* $F(x_1)$ *of* x_1. *For* $x_1 \in (a, b)$, $x_1 < x_2 < \cdots < x_{n+1}$ *and* $F(x_1)$ *is derivable. So we have*

$$F'(x_1) = \sum_{i=1}^{n+1} \frac{\partial x_i(x_1)}{\partial x_1} \cdot f'(x_i) = \frac{\sum\limits_{i=1}^{n+1} f'(x_i) D_i}{D_1}$$

$$= \frac{\begin{vmatrix} f'(x_1) & f'(x_2) & \ldots & f'(x_{n+1}) \\ x_1^{\alpha_1 - 1} & x_2^{\alpha_1 - 1} & \ldots & x_{n+1}^{\alpha_1 - 1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\alpha_n - 1} & x_2^{\alpha_n - 1} & \ldots & x_{n+1}^{\alpha_n - 1} \end{vmatrix}}{D_1}$$

$$= (-1)^n \frac{\begin{vmatrix} x_1^{\alpha_1 - 1} & x_2^{\alpha_1 - 1} & \ldots & x_{n+1}^{\alpha_1 - 1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\alpha_n - 1} & x_2^{\alpha_n - 1} & \ldots & x_{n+1}^{\alpha_n - 1} \\ f'(x_1) & f'(x_2) & \ldots & f'(x_{n+1}) \end{vmatrix}}{D_1}$$

We note $(x^k)^{(m)}$ as the m^{th} derivative of x^k. By Lemma 3.3 we have

$$\begin{vmatrix} (x^{\alpha_1-1})^{(0)} & (x^{\alpha_1-1})^{(1)} & \cdots & (x^{\alpha_1-1})^{(i)} \\ (x^{\alpha_2-1})^{(0)} & (x^{\alpha_2-1})^{(1)} & \cdots & (x^{\alpha_2-1})^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ (x^{\alpha_i-1})^{(0)} & (x^{\alpha_i-1})^{(1)} & \cdots & (x^{\alpha_i-1})^{(i)} \end{vmatrix}$$

$$= x^{(\sum_{j=1}^{i}\alpha_j - \frac{i(i+1)}{2})} \begin{vmatrix} 1 & \alpha_1 - 1 & \cdots & \prod_{j=1}^{i-1}(\alpha_1 - j) \\ 1 & \alpha_2 - 1 & \cdots & \prod_{j=1}^{i-1}(\alpha_2 - j) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_i - 1 & \cdots & \prod_{j=1}^{i-1}(\alpha_i - j) \end{vmatrix}$$

$$= x^{(\sum_{j=1}^{i}\alpha_j - \frac{i(i+1)}{2})} \begin{vmatrix} 1 & \alpha_1 - 1 & \cdots & (\alpha_1 - 1)^{i-1} \\ 1 & \alpha_2 - 1 & \cdots & (\alpha_2 - 1)^{i-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_i - 1 & \cdots & (\alpha_i - 1)^{i-1} \end{vmatrix} > 0 \quad (i = 1, 2, \ldots, n)$$

As $D_1 > 0$, we see from Lemma 3.5 that there exist ξ, $x_1 < \xi < x_n$, such that

$$(-1)^n sgn \begin{vmatrix} x_1^{\alpha_1-1} & x_2^{\alpha_1-1} & \ldots & x_{n+1}^{\alpha_1-1} \\ x_1^{\alpha_2-1} & x_2^{\alpha_2-1} & \ldots & x_{n+1}^{\alpha_2-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\alpha_n-1} & x_2^{\alpha_n-1} & \ldots & x_{n+1}^{\alpha_n-1} \\ f'(x_1) & f'(x_2) & \ldots & f'(x_{n+1}) \end{vmatrix} = (-1)^n sgn(W(\xi))$$

As we are in case (1), $(-1)^n W(x) \geq 0$, namely $F'(x_1) \geq 0$. Thus $F(x_1)$ is increasing on (a, b).
Since F is continuous on $[a, b]$, there exists $\boldsymbol{x_{n+1}}$ with $x_{2i-1} = x_{2i}$ for some $i(2 \leq 2i \leq n+1)$, at which point F attains its maximum, and there exists $\boldsymbol{x_{n+1}}$ with $x_{2i} = x_{2i+1}$ for some $i(2 \leq 2i \leq n+1)$ or $x_1 = 0$, at which point F attains its minimum.

The theorem is proved. □

Putting $n = 2$ in Theorem 4.1, we have

Corollary 4.2. *Suppose that $0 \leq x \leq y \leq z$. Let $f(x; y, z)$ be a function of x, y, z, symmetric with respect to y and z. We note $f(x; y, z)$ simply as $f(x)$. And we similarly define $f(y), f(z)$. $\alpha_1 < \alpha_2$. We fix the value of $x^{\alpha_1} + y^{\alpha_1} + z^{\alpha_1}$,$x^{\alpha_2} + y^{\alpha_2} + z^{\alpha_2}$. As a function of x, if $f(x)$ is 3^{rd} derivable on $(0, +\infty)$, and continuous on $[0, +\infty)$, let $F(x, y, z) = f(x) + f(y) + f(z)$,*

$$\begin{vmatrix} x^{\alpha_1-1} & (\alpha_1 - 1)x^{\alpha_1-2} & (\alpha_1 - 1)(\alpha_1 - 2)x^{\alpha_1-3} \\ x^{\alpha_2-1} & (\alpha_2 - 1)x^{\alpha_2-2} & (\alpha_2 - 1)(\alpha_2 - 2)x^{\alpha_2-3} \\ f'(x) & f''(x) & f'''(x) \end{vmatrix} = W(x)$$

(1) if $W(x) \geq 0$, F attains its maximum when $x = y \leq z$, and its minimum when $x \leq y = z$ or $x = 0$.
(2) if $W(x) \leq 0$, F attains its minimum when $x = y \leq z$, and its maximum when $x \leq y = z$ or $x = 0$.

Remark *In particular, let $g(x^{m-1}) = f'(x)$,$\alpha_1 = 1, \alpha_2 = m(m > 1)$ or $\alpha_1 = m, \alpha_2 = 1(m < 1)$ in Corollary 4.2, we obtain Theorem 3[6] in the previous article.*

Theorem 4.3. *Let $f(x_1; x_2, \ldots, x_m)$ be a function of $x_1, x_2, \ldots, x_m$, symmetric with respect to $x_2, x_3, \ldots, x_m$, we note $f(x_1; x_2, \ldots, x_m)$ simply as $f(x_1)$. And we similarly define $f(x_i)(i = 2, 3, \ldots, m)$. Fixing the value of $s_{(m,i)}(i = 1, 2, \ldots, n; m \geq n+1)$, if $f(x_i)$ is a function of x_i, $(n+1)^{th}$ derivable on $(0, +\infty)$, and continuous on $[0, +\infty)$, let*

$$F(\boldsymbol{x_m}) = f(x_1) + f(x_2) + \cdots + f(x_m)$$

(1) If $(-1)^n f^{(n+1)}(x) \geq 0$, there exists $\boldsymbol{x_m}$ with $v(\boldsymbol{x_m})^* \leq n-1$ or at least $m-n+1$ variables to be equal, at which point F attains its maximum, and there exists $\boldsymbol{x_m}$ with $v(\boldsymbol{x_m})^* \leq n-1$ or d variables to be zero and at least $m-n+1-d$ positive variables to be equal, at which point F attains its minimum.
(2) If $(-1)^n f^{(n+1)}(x) \leq 0$, there exists $\boldsymbol{x_m}$ with $v(\boldsymbol{x_m})^* \leq n-1$ or at least $m-n+1$ variables to be equal, at which point F attains its minimum, and there exists $\boldsymbol{x_m}$ with $v(\boldsymbol{x_m})^* \leq n-1$ or d variables to be zero and at least $m-n+1-d$ positive variables to be equal, at which point F attains its maximum.

Proof *We prove only for the minimum of F in (1), in the other cases the proof is similar.*

As $s_{(m,i)}(i = 1, 2, \ldots, n; m \geq n+1)$ is invariant, $(x_1, x_2, \ldots, x_m)$ is a compact subset of R_+^m. Since F is continuous on R_+^m, the minimum of F must exist.

When F attains its minimum at x_i, we can suppose without loss of generality that $x_1 \leq x_2 \leq \ldots \leq x_{n+1}$. Let $m-n-1$ variables $x_{n+2}, x_{n+3}, \ldots, x_m$ be fixed, we see from Theorem 4.1 that, if $v(\boldsymbol{x_{n+1}})^* \geq n$, there exist $\boldsymbol{x_{n+1}}$ with $x_{2i-1} = x_{2i}$ for some i or $x_1 = 0$, at which point F attains the minimum. So we know there exists $\boldsymbol{x_m}$ with $v(\boldsymbol{x_m})^* \leq n-1$ or d variables to be zero and at least $m-n+1-d$ positive variables to be equal, at which point F attains its minimum.

The theorem is proved. □

Corollary 4.4. *Let $f(x_1; x_2, \ldots, x_m)$ be a function of $x_1, x_2, \ldots, x_m$, symmetric with respect to $x_2, x_3, \ldots, x_m$. We note $f(x_1; x_2, \ldots, x_m)$ simply as $f(x_1)$. Fixing the value of $s_{(m,i)}(i = 1, 2, \ldots, n; m \geq n+1)$, if $f(x_i)$ is a function of x_i, $(n+1)^{th}$ derivable on $(0, +\infty)$, and continuous on $[0, +\infty)$, let*

$$F(\boldsymbol{x_m}) = f(x_1) + f(x_2) + \cdots + f(x_m)$$

(1) If $(-1)^n f^{(n+1)}(x) \geq 0$, there exists $\boldsymbol{x_m}$ with $v(\boldsymbol{x_m}) \leq n$, at which point F attains its maximum, and there exists $\boldsymbol{x_m}$ with $v(\boldsymbol{x_m})^* \leq n$ at which point F attains its minimum.
(2) If $(-1)^n f^{(n+1)}(x) \leq 0$, there exists $\boldsymbol{x_m}$ with $v(\boldsymbol{x_m}) \leq n$, at which point F attains its minimum, and there exists $\boldsymbol{x_m}$ with $v(\boldsymbol{x_m})^* \leq n$ at which point F attains its maximum.

Remark *The Jensen inequality consist of fixing the value of $s_{(m,1)}$, and adjusting the m variables by observing the sign of $f''(x)$ until that $v(x)^* \leq 1$. As we are now able to adjust the variables by observing the sign of $f^{(n+1)}(x)$ until that $v(x)^* \leq n$, all along with the value of $s_{(m,i)}(i = 1, 2, \ldots, n; m \geq n+1)$ fixed, this could be regarded as a generalization of Jensen inequality (in fact, this is just the Jensen inequality when $n = 1$). It also answers Question 7.2 raised in the previous article[6].*

Corollary 4.5. *Let $p \in \mathbb{R}, p \neq 1, 2, \ldots, n, x_i \geq 0 (i = 1, 2, \ldots, m)$ and the value of $s_{(m,i)} (i = 1, 2, \ldots, n; m \geq n + 1)$ be fixed. Let*

$$g(m, p) = \sum_{i=1}^{m} x_i^p$$

There exists $\boldsymbol{x_m}$ with $v(\boldsymbol{x_m})^* \leq n$, at which point $g(m, p)$ attains its maximum, and there exists $\boldsymbol{x_m}$ with $v(\boldsymbol{x_m})^* \leq n$, at which point $g(m, p)$ attains its minimum.

Theorem 4.6. *A symmetric inequality with n variables of degree m, $F(\boldsymbol{x_n}) \geq 0$ holds on R_+^n if and only if it holds on $\{\boldsymbol{x_n} | \boldsymbol{x_n} \in R_+^n, v(\boldsymbol{x_n})^* \leq \max([\frac{m}{2}], 1)\}$.*

Proof *The 'only if' part is trivial. Let's prove the 'if' part.*

When $m \geq 2n$ or $m = 1$, the theorem holds obviously. Now we prove it for $2 \leq m \leq 2n - 1$.

Let $[\frac{m}{2}] = t$. We fix the value of $s_{(n,i)} (i = 1, 2, \ldots, t)$. Then $(x_1, x_2, \ldots, x_n)$ is a compact subset of R_+^n. As F is continuous on R_+^n, the minimum of F must exist.

When F attains its minimum, for any $t + 1$ ones among all the x_i's, say $x_1, x_2, \ldots, x_{t+1}$, we fix the value of $n - t - 1$ variables $x_{t+2}, \ldots, x_n$. According to Lemma 3.1, F is uniquely expressed as a polynomial of $s_{(t+1,i)} (i = 1, 2, \ldots, t + 1)$. As $s_{(t+1,i)} (i = 1, 2, \ldots, t)$ is invariant, $2(t + 1) > m$. Thus F is a function of one single variable $s_{(t+1,t+1)}$, and $\deg(F) \leq 1$. By Corollary 4.5, we see that there exists $\boldsymbol{x_{t+1}}$ with $v(\boldsymbol{x_{t+1}})^* \leq t$, at which point F attains its minimum.

The theorem is proved. □

5 Summary and Prospect

In this paper, we continue to investigate the properties of symmetric inequalities. By use of the monotonicity and the property of *Wronskian* determinant, we obtain a sufficient condition for the extremum of the function $F = \sum_{i=1}^{n+1} f(x_i)$ is attained when $x_i \geq 0$, and $\sum_{i=1}^{n+1} x_i^{\alpha_j} (j = 1, 2, \ldots, n)$ is a constant. This result partly solved the problem of further generalization of *Jensen* inequality. Namely, fixing the value of the sum of power of n variables, by dimension descent depending on the sign of order $n + 1$ derivative, we obtain a necessary and sufficient condition for $F(\boldsymbol{x_n}) \geq 0$, a symmetric inequality with n variables of degree m, to hold on R_+^n, that is, it holds when $\{\boldsymbol{x_n} | \boldsymbol{x_n} \in R_+^n, v(\boldsymbol{x_n})^* \leq \max([\frac{m}{2}], 1)$.
Besides, we still have the following problems worthy discussing:

1. In the condition of Lemma 3.4, 'if (3.1) has a solution $(y_1^0, y_2^0, \ldots, y_{n+1}^0)$, with $0 \leq y_1^0 \leq y_2^0 \leq \ldots \leq y_{n+1}^0$ where the equality holds in no more than one place', can we eliminate the additional demand 'where the equality holds in no more than one place'? If we can, shall we obtain the same result? If the answer is positive, the following Theorem 4.1, Theorem 4.3 will be improved. What's more, the semi-algebraic system studied in the lemmas, has it a solution connected?

2. The conclusion of Theorem 4.6, is it optimal?
Here are some special examples.
Taking the homogeneous inequality with 2 variables of degree 4

$$F(x,y) = (x^2 + y^2 - 2xy)(x^2 + y^2 - 4xy) \geq 0$$

$F(1,2) < 0$,namely it doesn't hold on R_+^2. However, when $v(\boldsymbol{x_2})^* \leq 1$, $F(1,1) \geq 0, F(1,0) \geq 0$: the inequality then does hold. So the result could not be improved in this situation.
Taking the homogeneous inequality with 3 variables of degree 6

$$F(x,y,z) = (\sigma_3 + \frac{2}{27}\sigma_1^3 - \frac{1}{3}\sigma_1\sigma_2)^2 - (\sigma_1^2 - \frac{36}{11}\sigma_2)^2(\frac{41}{10}\sigma_2 - \sigma_1^2) + \frac{9}{110}\sigma_2^3 \geq 0$$

$f(12,1,16) < 0$, namely it doesn't hold on R_+^3. However, when $v(\boldsymbol{x_3})^* \leq 2$, $f(1,1,x) \geq 0, f(x,1,0) \geq 0 (x \in R_+)$: the inequality then does hold. So the result could not be improved in this situation, either.
Can we find an example in each case to show that the conclusion of Theorem 4.6 could not be improved? Or can we find in contrary a counterexample?

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