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CHARACTERISTIC CLASSES OF FOLIATED SURFACE BUNDLES WITH AREA-PRESERVING HOLONOMY

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Abstract

Making use of the *extended* flux homomorphism defined in [**13**] on the group $\mathrm{Symp}\,\Sigma_g$ of symplectomorphisms of a closed oriented surface Σ_g of genus $g \geq 2$, we introduce new characteristic classes of foliated surface bundles with symplectic, equivalently area-preserving, total holonomy. These characteristic classes are stable with respect to g and we show that they are highly non-trivial. We also prove that the second homology of the group $\mathrm{Ham}\,\Sigma_g$ of Hamiltonian symplectomorphisms of Σ_g, equipped with the discrete topology, is very large for all $g \geq 2$.

1. Introduction

In this paper we study the homology of symplectomorphism groups of surfaces considered as discrete groups. We shall prove that certain homology groups are highly non-trivial by constructing characteristic classes of foliated surface bundles with area-preserving holonomy, and proving non-vanishing results for them.

Let Σ_g be a closed oriented surface of genus $g \geq 2$, and $\mathrm{Diff}_+\,\Sigma_g$ its group of orientation preserving selfdiffeomorphisms. We fix an area form ω on Σ_g, which, for dimension reasons, we can also think of as a symplectic form. We denote by $\mathrm{Symp}\,\Sigma_g$ the subgroup of $\mathrm{Diff}_+\,\Sigma_g$ preserving the form ω. The classifying space $\mathrm{BSymp}^\delta\,\Sigma_g$ for the group $\mathrm{Symp}\,\Sigma_g$ with the *discrete* topology is an Eilenberg-MacLane space $K(\mathrm{Symp}^\delta\,\Sigma_g, 1)$ which classifies foliated Σ_g-bundles with area-preserving total holonomy groups.

Our construction of characteristic classes proceeds as follows. Let $\mathrm{Symp}_0\,\Sigma_g$ be the identity component of $\mathrm{Symp}\,\Sigma_g$. A well-known theorem of Moser [**24**] concerning volume-preserving diffeomorphisms implies that the quotient $\mathrm{Symp}\,\Sigma_g/\,\mathrm{Symp}_0\,\Sigma_g$ can be naturally identified with the mapping class group $\mathcal{M}_g$, so that we have an extension

$$1 \longrightarrow \mathrm{Symp}_0\,\Sigma_g \longrightarrow \mathrm{Symp}\,\Sigma_g \overset{p}{\longrightarrow} \mathcal{M}_g \longrightarrow 1.$$

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There is a surjective homomorphism Flux : $\mathrm{Symp}_0\,\Sigma_g \to H^1(\Sigma_g;\mathbb{R})$, called the flux homomorphism. In [**13**] we proved that this homomorphism can be extended to a crossed homomorphism

$$\widetilde{\mathrm{Flux}}\colon \mathrm{Symp}\,\Sigma_g \longrightarrow H^1(\Sigma_g;\mathbb{R}),$$

which we call the *extended* flux homomorphism. This extension is essentially unique in the sense that the associated cohomology class

$$[\widetilde{\mathrm{Flux}}] \in H^1(\mathrm{Symp}^\delta\,\Sigma_g; H^1(\Sigma_g;\mathbb{R}))$$

with twisted coefficients is uniquely defined. Now we consider the powers

$$[\widetilde{\mathrm{Flux}}]^k \in H^k(\mathrm{Symp}^\delta\,\Sigma_g; H^1(\Sigma_g;\mathbb{R})^{\otimes k}) \quad (k = 2, 3, \ldots),$$

and apply $\mathcal{M}_g$-invariant homomorphisms

$$\lambda\colon H^1(\Sigma_g;\mathbb{R})^{\otimes k} \longrightarrow \mathbb{R}$$

to obtain cohomology classes

$$\lambda([\widetilde{\mathrm{Flux}}]^k) \in H^k(\mathrm{BSymp}^\delta\,\Sigma_g;\mathbb{R})$$

with constant coefficients. The usual cup-product pairing $H^1(\Sigma_g;\mathbb{R})^{\otimes 2} \to \mathbb{R}$ is the main example of such a homomorphism λ.

This method of constructing constant cohomology classes out of twisted ones was already used in [**23**] in the case of the mapping class group, where the Torelli group (respectively the Johnson homomorphism) played the role of $\mathrm{Symp}_0\,\Sigma_g$ (respectively of the flux homomorphism) here. In that case, it was proved in *loc. cit.* that all the Mumford–Morita–Miller classes can be obtained in this way. The precise formulae were given in [**11**, **12**], with the important implication that no other classes appear. In our context here, we can go further by enhancing the coefficients $\mathbb{R}$ to associated $\mathbb{Q}$-vector spaces which appear as the targets of various multiples of the *discontinuous* cup-product pairing

$$H^1(\Sigma_g;\mathbb{R}) \otimes_{\mathbb{Z}} H^1(\Sigma_g;\mathbb{R}) \longrightarrow S^2_{\mathbb{Q}}\mathbb{R},$$

where $S^2_{\mathbb{Q}}\mathbb{R}$ denotes the second symmetric power of $\mathbb{R}$ over $\mathbb{Q}$, see Section 2 for the details. In this way, we obtain many new characteristic classes in

$$H^*(\mathrm{Symp}^\delta\,\Sigma_g; S^*(S^2_{\mathbb{Q}}\mathbb{R})),$$

where

$$S^*(S^2_{\mathbb{Q}}\mathbb{R}) = \bigoplus_{k=1}^{\infty} S^k(S^2_{\mathbb{Q}}\mathbb{R})$$

denotes the symmetric algebra of $S^2_{\mathbb{Q}}\mathbb{R}$. On the other hand, we proved in [**13**] that any power

$$e_1^k \in H^{2k}(\mathrm{Symp}^\delta\,\Sigma_g;\mathbb{Q})$$

of the first Mumford–Morita–Miller class e_1 is non-trivial for $k \leq \frac{g}{3}$. Now we can consider the cup products of e_1^k with the new characteristic classes defined above. The main purpose of the present paper is to prove that these characteristic classes are all non-trivial in a suitable stable range.

The contents of this paper is as follows. In Section 2 precise statements of the main results are given. In Section 3 we study, in detail, the transverse symplectic class of foliated Σ_g-bundles with area-preserving total holonomy groups. In Section 4 we construct two kinds of *extended* flux homomorphisms for open surfaces $\Sigma_g^0 = \Sigma_g \setminus D^2$. We compare these extended flux homomorphisms with the one obtained in the case of closed surfaces. This is used in Section 5 to generalize our result on the second homology of the symplectomorphism group to the case of open surfaces. Sections 4 and 5 are the heart of this paper. Then in Section 6 we use the results of the previous sections to show the non-triviality of cup products of various characteristic classes, thus yielding proofs of the main results about the homology of symplectomorphism groups as discrete groups. In the final Section 7 we give definitions of yet more characteristic classes, other than the ones given in Section 2. We propose several conjectures and problems about them.

2. Statement of the main results

Consider the usual cup-product pairing

$$\iota \colon H^1(\Sigma_g;\mathbb{R}) \otimes H^1(\Sigma_g;\mathbb{R}) \longrightarrow \mathbb{R}$$

in cohomology, dual to the intersection pairing in homology. For simplicity, we denote $\iota(u,v)$ by $u \cdot v$, where $u, v \in H^1(\Sigma_g;\mathbb{R})$. We first lift this pairing as follows.

As in Section 1, let $S^2_{\mathbb{Q}}\mathbb{R}$ denote the second symmetric power of $\mathbb{R}$ over $\mathbb{Q}$. In other words, this is a vector space over $\mathbb{Q}$, consisting of the homogeneous polynomials of degree two generated by the elements of $\mathbb{R}$ considered as a vector space over $\mathbb{Q}$. For each element $a \in \mathbb{R}$, we denote by $\hat{a}$ the corresponding element in $S^1_{\mathbb{Q}}\mathbb{R}$. Thus any element in $S^2_{\mathbb{Q}}\mathbb{R}$ can be expressed as a finite sum

$$\hat{a}_1\hat{b}_1 + \cdots + \hat{a}_k\hat{b}_k$$

with $a_i, b_i \in \mathbb{R}$. We have a natural projection

$$S^2_{\mathbb{Q}}\mathbb{R} \longrightarrow \mathbb{R}$$

given by the correspondence $\hat{a} \mapsto a$ $(a \in \mathbb{R})$.

With this terminology we make the following definition:

Definition 1 (Discontinuous intersection pairing)**.** Define a pairing

$$\tilde{\iota} \colon H^1(\Sigma_g;\mathbb{R}) \times H^1(\Sigma_g;\mathbb{R}) \longrightarrow S^2_{\mathbb{Q}}\mathbb{R}$$

as follows. Choose a basis $x_1, \dots, x_{2g}$ of $H^1(\Sigma_g; \mathbb{Q})$. For any two elements, $u, v \in H^1(\Sigma_g; \mathbb{R})$, write

$$u = \sum_i a_i x_i, \quad v = \sum_i b_i x_i \quad (a_i, b_i \in \mathbb{R}).$$

Then we set

$$\tilde{\iota}(u,v) = \sum_{i,j} \iota(x_i, x_j) \hat{a}_i \hat{b}_j \in S^2_{\mathbb{Q}} \mathbb{R}.$$

Clearly $\tilde{\iota}$ followed by the projection $S^2_{\mathbb{Q}}\mathbb{R} \longrightarrow \mathbb{R}$ is nothing but the usual intersection or cup-product pairing ι. Henceforth we simply write $u \odot v$ for $\tilde{\iota}(u,v)$.

It is easy to see that $\tilde{\iota}$ is well defined independently of the choice of basis in $H^1(\Sigma_g; \mathbb{Q})$. We can see from the following proposition that $\tilde{\iota}$ enumerates all $\mathbb{Z}$-multilinear skew-symmetric pairings on $H^1(\Sigma_g; \mathbb{R})$ which are $\mathcal{M}_g$-invariant. Let $\Lambda^2_{\mathbb{Z}} H^1(\Sigma_g; \mathbb{R})$ denote the second exterior power, over $\mathbb{Z}$, of $H^1(\Sigma_g; \mathbb{R})$ considered as an abelian group, rather than as a vector space over $\mathbb{R}$. Also let $\left(\Lambda^2_{\mathbb{Z}} H^1(\Sigma_g; \mathbb{R})\right)_{\mathcal{M}_g}$ denote the abelian group of coinvariants of $\Lambda^2_{\mathbb{Z}} H^1(\Sigma_g; \mathbb{R})$ with respect to the natural action of $\mathcal{M}_g$.

Proposition 2. *There exists a canonical isomorphism*

$$\left(\Lambda^2_{\mathbb{Z}} H^1(\Sigma_g; \mathbb{R})\right)_{\mathcal{M}_g} \cong S^2_{\mathbb{Q}} \mathbb{R}$$

given by the correspondence

$$\left(\sum_i a_i u_i\right) \wedge \left(\sum_j b_j v_j\right) \longmapsto \sum_{i,j} \iota(u_i, v_j)\ \hat{a}_i \hat{b}_j,$$

where $a_i, b_j \in \mathbb{R}$, $u_i, v_j \in H^1(\Sigma_g; \mathbb{Q})$.

In order not to digress, we refer the reader to the appendix for a proof.

Before we can define some new cocycles on the group $\operatorname{Symp} \Sigma_g$, we have to recall some facts from [**13**]. The symplectomorphism group $\operatorname{Symp} \Sigma_g$ acts on its identity component by conjugation, and acts on $H^1(\Sigma_g; \mathbb{R})$ from the left via $\varphi(w) = (\varphi^{-1})^*(w)$. The flux homomorphism $\operatorname{Flux}\colon \operatorname{Symp}_0 \Sigma_g \to H^1(\Sigma_g; \mathbb{R})$ is equivariant with respect to these actions by Lemma 6 of [**13**]. Its extension $\widetilde{\operatorname{Flux}}\colon \operatorname{Symp} \Sigma_g \to H^1(\Sigma_g; \mathbb{R})$ is a crossed homomorphism for the above action in the sense that

$$\widetilde{\operatorname{Flux}}(\varphi\psi) = \widetilde{\operatorname{Flux}}(\varphi) + (\varphi^{-1})^* \widetilde{\operatorname{Flux}}(\psi). \tag{1}$$

Definition 3. Let $\varphi_1, \dots, \varphi_{2k} \in \operatorname{Symp} \Sigma_g$, and

$$\xi_i = ((\varphi_1 \dots \varphi_{i-1})^{-1})^* \widetilde{\operatorname{Flux}}(\varphi_i).$$

Define a $2k$-cocycle $\tilde{\alpha}^{(k)}$ with values in $S^k(S^2_{\mathbb{Q}}\mathbb{R})$ by

$$\tilde{\alpha}^{(k)}(\varphi_1,\dots,\varphi_{2k}) = \frac{1}{(2k)!}\sum_{\sigma\in\mathfrak{S}_{2k}} \operatorname{sgn}\sigma\ (\xi_{\sigma(1)}\odot\xi_{\sigma(2)})\dots(\xi_{\sigma(2k-1)}\odot\xi_{\sigma(2k)}) \in S^k(S^2_{\mathbb{Q}}\mathbb{R}),$$

where the sum is over permutations in the symmetric group $\mathfrak{S}_{2k}$.

That $\tilde{\alpha}^{(k)}$ is indeed a cocycle is easy to check by a standard argument in the theory of cohomology of groups using (1). Thus we have the corresponding cohomology classes

$$\tilde{\alpha}^{(k)} \in H^{2k}(\mathrm{Symp}^\delta\,\Sigma_g; S^k(S^2_{\mathbb{Q}}\mathbb{R})),$$

denoted by the same letters. If we apply the canonical projection $S^k(S^2_{\mathbb{Q}}\mathbb{R})\to\mathbb{R}$ to these classes, we obtain real cohomology classes

$$\alpha^k \in H^{2k}(\mathrm{Symp}^\delta\,\Sigma_g;\mathbb{R}),$$

which are the usual cup products of the first one $\alpha\in H^2(\mathrm{Symp}^\delta\,\Sigma_g;\mathbb{R})$. The refined classes $\tilde{\alpha}^{(k)}$ can be considered as a twisted version of *discontinuous invariants* in the sense of [**18**] arising from the flux homomorphism.

Now we can state our first main result.

Theorem 4. *For any $k\geq 1$ and $g\geq 3k$, the characteristic classes*

$$e_1^k, e_1^{k-1}\tilde{\alpha},\dots,e_1\tilde{\alpha}^{(k-1)},\tilde{\alpha}^{(k)}$$

induce a surjective homomorphism

$$H_{2k}(\mathrm{Symp}^\delta\,\Sigma_g;\mathbb{Z})\longrightarrow\mathbb{Z}\oplus S^2_{\mathbb{Q}}\mathbb{R}\oplus\cdots\oplus S^k(S^2_{\mathbb{Q}}\mathbb{R}).$$

For $k=1$ this is not hard to see, so we give the proof right away. For $k>1$ the proof is given in Section 6 below and requires the technical results developed in the body of this paper.

Consider the subgroup $\mathrm{Ham}\,\Sigma_g$ of $\mathrm{Symp}_0\,\Sigma_g$ consisting of all Hamiltonian symplectomorphisms of Σ_g. As is well known (see [**1**, **16**]), we have an extension

$$1\longrightarrow \mathrm{Ham}\,\Sigma_g\longrightarrow\mathrm{Symp}_0\,\Sigma_g\xrightarrow{\mathrm{Flux}} H^1(\Sigma_g;\mathbb{R})\longrightarrow 1. \tag{2}$$

This gives rise to a 5-term exact sequence in cohomology:

$$0\longrightarrow H^1(H^1(\Sigma_g;\mathbb{R})^\delta;\mathbb{Z})\xrightarrow{\mathrm{Flux}^*} H^1(\mathrm{Symp}_0^\delta\,\Sigma_g;\mathbb{Z})\longrightarrow H^1(\mathrm{Ham}^\delta\,\Sigma_g;\mathbb{Z})^{H^1_{\mathbb{R}}}$$
$$\longrightarrow H^2(H^1(\Sigma_g;\mathbb{R})^\delta;\mathbb{Z})\xrightarrow{\mathrm{Flux}^*} H^2(\mathrm{Symp}_0^\delta\,\Sigma_g;\mathbb{Z}),$$

where we have written $H^1_{\mathbb{R}}$ for $H^1(\Sigma_g;\mathbb{R})$. Now $\mathrm{Ham}\,\Sigma_g$ is a perfect group by a result of Thurston [**25**], see also Banyaga [**1**]. Therefore, Flux^* injects the second cohomology of $H^1(\Sigma_g;\mathbb{R})$ as a discrete group into that of $\mathrm{Symp}_0^\delta\,\Sigma_g$. By definition, the class $\tilde{\alpha}$ is the image of the class of $\tilde{\iota}$ under Flux^*. So $\tilde{\alpha}$ is nontrivial on $\mathrm{Symp}_0^\delta\,\Sigma_g$, and is defined on the whole $\mathrm{Symp}^\delta\,\Sigma_g$. We conclude that $\tilde{\alpha}$ defines a surjective homomorphism

$$H_2(\mathrm{Symp}^\delta\,\Sigma_g;\mathbb{Z}) \to S^2_{\mathbb{Q}}\mathbb{R},$$

for any $g \geq 2$. We already proved in [**13**] that the first Mumford–Morita–Miller class e_1 defines a surjection $H_2(\mathrm{Symp}^\delta\,\Sigma_g;\mathbb{Z}) \longrightarrow \mathbb{Z}$ for all $g \geq 3$. Clearly the two classes are linearly independent because $\tilde{\alpha}$ is nonzero on $\mathrm{Symp}^\delta_0\,\Sigma_g$, to which e_1 restricts trivially. This proves Theorem 4 in the easy case when $k = 1$.

We can restrict the homomorphism

$$\mathrm{Flux}^*\colon H^*(H^1(\Sigma_g;\mathbb{R})^\delta;\mathbb{R}) \longrightarrow H^*(\mathrm{Symp}^\delta_0\,\Sigma_g;\mathbb{R})$$

to the *continuous cohomology*

$$H^*_{ct}(H^1(\Sigma_g;\mathbb{R})^\delta;\mathbb{R}) \cong H_*(T^{2g};\mathbb{R}) \subset H^*(H^1(\Sigma_g;\mathbb{R})^\delta;\mathbb{R}),$$

see Section 3 for the precise definition. Thereby we obtain a ring homomorphism

$$\mathrm{Flux}^*\colon H_*(T^{2g};\mathbb{R}) \longrightarrow H^*(\mathrm{Symp}^\delta_0\,\Sigma_g;\mathbb{R}),$$

where $T^{2g} = K(\pi_1\Sigma_g, 1)$ is the Jacobian manifold of Σ_g, and the ring structure on the homology of T^{2g} is induced by the Pontrjagin product. Let $\omega_0 \in H_2(T^{2g};\mathbb{R})$ be the homology class represented by the dual of the standard symplectic form on T^{2g}. We decompose the $Sp(2g,\mathbb{R})$-module $H_k(T^{2g};\mathbb{R})$ into irreducible components. For this, consider the homomorphism

$$\omega_0 \wedge\ \colon H_{k-2}(T^{2g};\mathbb{R}) \longrightarrow H_k(T^{2g};\mathbb{R})$$

induced by the wedge product with ω_0. On the one hand, it is easy to see using Poincaré duality on T^{2g}, that the above homomorphism is surjective for any $k \geq g+1$. On the other hand, it is well-known (see [**4**]), that the kernel of the contraction homomorphism

$$C\colon H_k(T^{2g};\mathbb{Q}) \longrightarrow H_{k-2}(T^{2g};\mathbb{Q})$$

induced by the intersection pairing $H_2(T^{2g};\mathbb{Q}) \cong \Lambda^2 H_1(\Sigma_g;\mathbb{Q}) \to \mathbb{Q}$ is the irreducible representation of the algebraic group $Sp(2g,\mathbb{Q})$ corresponding to the Young diagram $[1^k]$ for any $k \leq g$. Let $[1^k]_{\mathbb{R}} = [1^k] \otimes \mathbb{R}$ denote the real form of this representation. Then we have a direct sum decomposition

$$H_k(T^{2g};\mathbb{R}) = [1^k]_{\mathbb{R}} \oplus \omega_0 \wedge H_{k-2}(T^{2g};\mathbb{R}) \quad (k \leq g). \tag{3}$$

Theorem 5. *The kernel of the homomorphism*

$$\mathrm{Flux}^*\colon H_*(T^{2g};\mathbb{R}) \longrightarrow H^*(\mathrm{Symp}^\delta_0\,\Sigma_g;\mathbb{R})$$

induced by the flux homomorphism is the ideal generated by the subspace $\omega_0 \wedge H_1(T^{2g};\mathbb{R}) \subset H_3(T^{2g};\mathbb{R})$*, and the image of this homomorphism can* *be described as*

$$\operatorname{Im}\operatorname{Flux}^* \cong \mathbb{R} \oplus \bigoplus_{k=1}^{g} [1^k]_{\mathbb{R}},$$

where $\mathbb{R}$ *denotes the image of the subspace of* $H_2(T^{2g};\mathbb{R})$ *spanned by* ω_0*.*

The group $H^1(\Sigma_g;\mathbb{R})$ acts on $\operatorname{Ham}\Sigma_g$ by outer automorphisms. In particular, it acts on the homology $H_*(\operatorname{Ham}^\delta\Sigma_g;\mathbb{Z})$ of the discrete group $\operatorname{Ham}^\delta\Sigma_g$ so that we can consider the coinvariants $H_*(\operatorname{Ham}^\delta\Sigma_g;\mathbb{Z})_{H^1_{\mathbb{R}}}$, where for simplicity we have written $H^1_{\mathbb{R}}$ instead of $H^1(\Sigma_g;\mathbb{R})$. The following result shows that the second homology group $H_2(\operatorname{Ham}^\delta\Sigma_g;\mathbb{Z})$ is highly non-trivial.

Theorem 6. *For any* $g \geq 2$*, there exists a natural injection*

$$H^1(\Sigma_g;\mathbb{R}) \subset H_2(\operatorname{Ham}^\delta\Sigma_g;\mathbb{Z})_{H^1_{\mathbb{R}}}.$$

3. The transverse symplectic class

Since Σ_g is an Eilenberg-MacLane space, the total space of the universal foliated Σ_g-bundle over the classifying space $\operatorname{BSymp}^\delta\Sigma_g$ is again a $K(\pi,1)$ space. Hence if we denote by $\operatorname{ESymp}^\delta\Sigma_g$ the fundamental group of this total space, then we obtain a short exact sequence

$$1 \longrightarrow \pi_1\Sigma_g \longrightarrow \operatorname{ESymp}^\delta\Sigma_g \longrightarrow \operatorname{Symp}^\delta\Sigma_g \longrightarrow 1 \tag{4}$$

and any cohomology class of the total space can be considered as an element in the group cohomology of $\operatorname{ESymp}^\delta\Sigma_g$. Now on the total space of any foliated Σ_g-bundle with total holonomy group contained in $\operatorname{Symp}\Sigma_g$ there is a closed 2-form $\tilde{\omega}$ which restricts to the symplectic form ω on each fiber. At the universal space level, the de Rham cohomology class of $\tilde{\omega}$ defines a class $v \in H^2(\operatorname{ESymp}^\delta;\mathbb{R})$ which we call the transverse symplectic class. We normalize the symplectic form ω on Σ_g so that its total area is equal to $2g-2$. It follows that the restriction of v to a fiber is the same as the negative of the Euler class $e \in H^2(\operatorname{ESymp}^\delta;\mathbb{R})$ of the vertical tangent bundle.

Let $\operatorname{ESymp}_0\Sigma_g$ denote the subgroup of $\operatorname{ESymp}^\delta\Sigma_g$ obtained by restricting the extension (4) to $\operatorname{Symp}_0\Sigma_g \subset \operatorname{Symp}\Sigma_g$. Since any foliated Σ_g-bundle with total holonomy in $\operatorname{Symp}_0\Sigma_g$ is trivial as a differentiable Σ_g-bundle, there exists an isomorphism

$$\operatorname{ESymp}_0\Sigma_g \cong \pi_1\Sigma_g \times \operatorname{Symp}_0\Sigma_g.$$

Henceforth we identify the above two groups. By the Künneth decomposition, we have an isomorphism

$$\begin{aligned} H^2(\operatorname{ESymp}^\delta_0\Sigma_g;\mathbb{R}) &\cong H^2(\Sigma_g;\mathbb{R}) \oplus \\ &\oplus \Big(H^1(\Sigma_g;\mathbb{R}) \otimes H^1(\operatorname{Symp}^\delta_0\Sigma_g;\mathbb{R})\Big) \oplus H^2(\operatorname{Symp}^\delta_0\Sigma_g;\mathbb{R}), \end{aligned} \tag{5}$$

where we identify $H^*(\pi_1\Sigma_g;\mathbb{R})$ with $H^*(\Sigma_g;\mathbb{R})$. Let $\mu \in H^2(\Sigma_g;\mathbb{Z})$ be the fundamental cohomology class of Σ_g. Clearly the Euler class $e \in H^2(\mathrm{ESymp}_0^\delta\,\Sigma_g;\mathbb{R})$ is equal to $(2-2g)\mu$. The flux homomorphism gives rise to an element

$$\begin{aligned}[\mathrm{Flux}] \in\ & \mathrm{Hom}_{\mathbb{Z}}(H_1(\mathrm{Symp}_0^\delta\,\Sigma_g;\mathbb{Z}), H^1(\Sigma_g;\mathbb{R}))\\ &\cong \mathrm{Hom}_{\mathbb{R}}(H_1(\mathrm{Symp}_0^\delta\,\Sigma_g;\mathbb{R}), H^1(\Sigma_g;\mathbb{R}))\\ &\cong H^1(\Sigma_g;\mathbb{R})\otimes H^1(\mathrm{Symp}_0^\delta\,\Sigma_g;\mathbb{R}),\end{aligned}$$

where the last isomorphism exists because $H^1(\Sigma_g;\mathbb{R})$ is finite dimensional. Choose a symplectic basis $x_1,\dots,x_g,\ y_1,\dots,y_g$ of $H_1(\Sigma_g;\mathbb{R})$ and denote by $x_1^*,\dots,x_g^*,\ y_1^*,\dots,y_g^*$ the dual basis of $H^1(\Sigma_g;\mathbb{R})$. Then Poincaré duality $H_1(\Sigma_g;\mathbb{R}) \cong H^1(\Sigma_g;\mathbb{R})$ is given by the correspondence $x_i \mapsto -y_i^*, y_i \mapsto x_i^*$. The element [Flux] can be described explicitly as

$$\text{(6)}\quad [\mathrm{Flux}] = \sum_{i=1}^{g}(x_i^*\otimes\tilde{x}_i + y_i^*\otimes\tilde{y}_i) \in H^1(\Sigma_g;\mathbb{R})\otimes H^1(\mathrm{Symp}_0^\delta\,\Sigma_g;\mathbb{R})$$

where $\tilde{x}_i, \tilde{y}_i \in H^1(\mathrm{Symp}_0^\delta\,\Sigma_g;\mathbb{R}) \cong \mathrm{Hom}(H_1(\mathrm{Symp}_0\,\Sigma_g;\mathbb{Z});\mathbb{R})$ is defined by the equality

$$\mathrm{Flux}(\varphi) = \sum_{i=1}^{g}(\tilde{x}_i(\varphi)x_i + \tilde{y}_i(\varphi)y_i) \quad (\varphi \in \mathrm{Symp}_0\,\Sigma_g).$$

The elements $\tilde{x}_i, \tilde{y}_i$ can be also interpreted as follows. The flux homomorphism induces a homomorphism in cohomology

$$\text{(7)}\qquad \mathrm{Flux}^*\colon H^*(H^1(\Sigma_g;\mathbb{R})^\delta;\mathbb{R}) \longrightarrow H^*(\mathrm{Symp}_0^\delta\,\Sigma_g;\mathbb{R})$$

where the domain

$$H^*(H^1(\Sigma_g;\mathbb{R})^\delta;\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda_{\mathbb{Z}}^*(H^1(\Sigma_g;\mathbb{R})),\mathbb{R})$$

is the cohomology group of $H^1(\Sigma_g;\mathbb{R})$ considered as a *discrete abelian group*, rather than as a vector space over $\mathbb{R}$, so that it is a very large group. Its *continuous part* is defined as

$$\begin{aligned}H_{ct}^*(H^1(\Sigma_g;\mathbb{R})^\delta;\mathbb{R}) &= \mathrm{Hom}_{\mathbb{R}}(\Lambda_{\mathbb{R}}^*(H^1(\Sigma_g;\mathbb{R})),\mathbb{R})\\ &\subset \mathrm{Hom}_{\mathbb{Z}}(\Lambda_{\mathbb{R}}^*(H^1(\Sigma_g;\mathbb{R})),\mathbb{R})\\ &\subset \mathrm{Hom}_{\mathbb{Z}}(\Lambda_{\mathbb{Z}}^*(H^1(\Sigma_g;\mathbb{R})),\mathbb{R}) \cong H^*(H^1(\Sigma_g;\mathbb{R})^\delta;\mathbb{R}),\end{aligned}$$

where the second inclusion is induced by the natural projection

$$\Lambda_{\mathbb{Z}}^*(H^1(\Sigma_g;\mathbb{R})) \longrightarrow \Lambda_{\mathbb{R}}^*(H^1(\Sigma_g;\mathbb{R})).$$

Denoting by $T^{2g} = K(\pi_1\Sigma_g, 1)$ the Jacobian torus of Σ_g, there is a canonical isomorphism

$$\Lambda_{\mathbb{R}}^* H^1(\Sigma_g;\mathbb{R}) \cong H^*(T^{2g};\mathbb{R}),$$

so that we can identify

$$H^*_{ct}(H^1(\Sigma_g;\mathbb{R})^\delta;\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(H^*(T^{2g};\mathbb{R});\mathbb{R}) \cong H_*(T^{2g};\mathbb{R}) \ .$$

Thus, by restricting the homomorphism Flux^* in (7) to the continuous cohomology, we obtain a homomorphism

$$\mathrm{Flux}^* \colon H_*(T^{2g};\mathbb{R}) \longrightarrow H^*(\mathrm{Symp}_0^\delta \Sigma_g;\mathbb{R}).$$

It is easy to see that under this homomorphism we have

$$\tilde{x}_i = \mathrm{Flux}^*(x_i) \ , \quad \tilde{y}_i = \mathrm{Flux}^*(y_i).$$

Let $\omega_0 \in \Lambda^2_{\mathbb{R}} H_1(\Sigma_g;\mathbb{R})$ be the symplectic class defined by

$$\omega_0 = \sum_{i=1}^{g} x_i \wedge y_i$$

and set

$$\tilde{\omega}_0 = \mathrm{Flux}^*(\omega_0) = \sum_{i=1}^{g} \tilde{x}_i \tilde{y}_i \in H^2(\mathrm{Symp}_0^\delta \Sigma_g;\mathbb{R}).$$

Lemma 7. *We have the equality*

$$[\mathrm{Flux}]^2 = -2\mu \otimes \tilde{\omega}_0 \in H^2(\Sigma_g;\mathbb{R}) \otimes H^2(\mathrm{Symp}_0^\delta \Sigma_g;\mathbb{R}).$$

Proof. A direct calculation using the expression (6) yields

$$[\mathrm{Flux}]^2 = -\sum_{i=1}^{g} (x_i^* y_i^* \otimes \tilde{x}_i \tilde{y}_i + y_i^* x_i^* \otimes \tilde{y}_i \tilde{x}_i).$$

Since $x_i^* y_i^* = -y_i^* x_i^* = \mu$, we obtain

$$[\mathrm{Flux}]^2 = -2\mu \otimes \sum_{i=1}^{g} \tilde{x}_i \tilde{y}_i = -2\mu \otimes \tilde{\omega}_0$$

as required. q.e.d.

Now we can completely determine the transverse symplectic class v of foliated Σ_g-bundles whose total holonomy groups are contained in the identity component $\mathrm{Symp}_0 \Sigma_g$ of $\mathrm{Symp}\, \Sigma_g$ as follows.

Proposition 8. *On the subgroup* $\mathrm{ESymp}_0^\delta \Sigma_g$ *the transverse symplectic class* $v \in H^2(\mathrm{ESymp}_0^\delta \Sigma_g;\mathbb{R})$ *is given by*

$$v = (2g-2)\mu + [\mathrm{Flux}] + \frac{1}{2g-2}\tilde{\omega}_0$$

under the isomorphism (5)*. Furthermore, the homomorphism*

$$\tilde{\omega}_0 \otimes H^1_{ct}(\mathrm{Symp}_0^\delta \Sigma_g;\mathbb{R}) \longrightarrow H^3(\mathrm{Symp}_0^\delta \Sigma_g;\mathbb{R})$$

induced by the cup product is trivial, where $H^1_{ct}(\mathrm{Symp}_0^\delta \Sigma_g;\mathbb{R}) \cong H_1(\Sigma_g;\mathbb{R})$ *denotes the subgroup of* $H^1(\mathrm{Symp}_0^\delta \Sigma_g;\mathbb{R})$ *generated by the continuous cohomology classes* $\tilde{x}_i, \tilde{y}_i$*. In particular,* $\tilde{\omega}_0^2 = 0$.

Proof. Since the restriction of v to each fiber is equal to the negative of that of the Euler class e by our normalization, v restricts to $(2g-2)\mu$ on the fiber. This gives the first component of the formula. The second component follows from Lemma 8 of [**13**]. Thus we can write

$$v = (2g-2)\mu + [\mathrm{Flux}] + \gamma \in H^*(\Sigma_g; \mathbb{R}) \otimes H^*(\mathrm{Symp}_0^\delta \Sigma_g; \mathbb{R})$$

for some $\gamma \in H^2(\mathrm{Symp}_0^\delta \Sigma_g; \mathbb{R})$. Now observe that $v^2 = 0$ because $\tilde{\omega}^2 = 0$. Also, because we have restricted to $\mathrm{Symp}_0 \Sigma_g$, we have $\mu^2 = 0$, $\mu[\mathrm{Flux}] = 0$. Hence we obtain

$$[\mathrm{Flux}]^2 + \gamma^2 + 2(2g-2)\mu\gamma + 2[\mathrm{Flux}]\gamma = 0.$$

It follows that

$$\begin{aligned} [\mathrm{Flux}]^2 + 2(2g-2)\mu\gamma &= 0 \\ [\mathrm{Flux}]\gamma &= 0 \\ \gamma^2 &= 0 \end{aligned}$$

because these three elements belong to different summands in the Künneth decomposition of $H^*(\Sigma_g; \mathbb{R}) \otimes H^*(\mathrm{Symp}_0^\delta; \mathbb{R})$. If we combine the first equality above and Lemma 7, then we can conclude that

$$\gamma = \frac{1}{2g-2}\tilde{\omega}_0. \tag{8}$$

This proves the first claim of the proposition. If we substitute (6) and (8) in the second equality above, then we see that $\tilde{\omega}_0 \tilde{x}_i = \tilde{\omega}_0 \tilde{y}_i = 0$ for any i, whence the second claim. Observe that the third equality $\gamma^2 = 0$, which is equivalent to $\tilde{\omega}_0^2 = 0$ by the above, is a consequence of the second claim. q.e.d.

Now we can calculate the restriction of the cocycle α defined in Section 2 to the identity component $\mathrm{Symp}_0 \Sigma_g$.

Proposition 9. *Let* $i\colon \mathrm{Symp}_0^\delta \Sigma_g \to \mathrm{Symp}^\delta \Sigma_g$ *be the inclusion. Then*

$$i^*\alpha = 2\tilde{\omega}_0 \in H^2(\mathrm{Symp}_0^\delta \Sigma_g; \mathbb{R})$$

and

$$i^*\alpha^2 = 0.$$

Proof. Let $\varphi, \psi \in \mathrm{Symp}_0 \Sigma_g$ be any two elements. Then, by the definition of α, see Definition 3 and the subsequent discussion, we have

$$\alpha(\varphi, \psi) = \iota(\mathrm{Flux}(\varphi), \mathrm{Flux}(\psi)).$$

By the definition of the cohomology classes $\tilde{x}_i, \tilde{y}_i$, we have

$$\mathrm{Flux}(\varphi) = \sum_{i=1}^{g} (\tilde{x}_i(\varphi)x_i + \tilde{y}_i(\varphi)y_i), \quad \mathrm{Flux}(\psi) = \sum_{i=1}^{g} (\tilde{x}_i(\psi)x_i + \tilde{y}_i(\psi)y_i).$$

Hence we obtain

$$\alpha(\varphi, \psi) = \sum_{i=1}^{g} \{\tilde{x}_i(\varphi)\tilde{y}_i(\psi) - \tilde{y}_i(\varphi)\tilde{x}_i(\psi)\}.$$

Using the Alexander–Whitney cup product, we also have

$$\tilde{\omega}_0(\varphi, \psi) = \sum_{i=1}^{g} \tilde{x}_i(\varphi)\tilde{y}_i(\psi) = -\sum_{i=1}^{g} \tilde{x}_i(\psi)\tilde{y}_i(\varphi).$$

Thus $\alpha = 2\tilde{\omega}_0$ as required. The last statement follows from Proposition 8. q.e.d.

We see from this proposition that $i^*\alpha^2 = 0$ in $H^4(\mathrm{Symp}_0^\delta \Sigma_g; \mathbb{R})$, while we will prove the non-triviality of $\alpha^2 \in H^4(\mathrm{Symp}^\delta \Sigma_g; \mathbb{R})$, which is a special case of Theorem 4. One could say that the non-triviality of α^2 is realized by an interaction of the two groups $\mathrm{Symp}_0 \Sigma_g$ and $\mathcal{M}_g$.

4. Extended flux homomorphisms for open surfaces

We consider the open surface $\Sigma_g^0 = \Sigma_g \setminus D$ obtained from Σ_g by removing a closed embedded disk $D \subset \Sigma_g$. Let $j\colon \Sigma_g^0 \to \Sigma_g$ be the inclusion. We denote by $\mathrm{Symp}^c \Sigma_g^0$ the symplectomorphism group of $(\Sigma_g^0, j^*\omega)$ with *compact supports*. Hence the group $\mathrm{Symp}^c \Sigma_g^0$ can be considered as a subgroup of $\mathrm{Symp}\,\Sigma_g$ with inclusion $j\colon \mathrm{Symp}^c \Sigma_g^0 \to \mathrm{Symp}\,\Sigma_g$. Let $\mathrm{Symp}_0^c \Sigma_g^0$ be the identity component of $\mathrm{Symp}^c \Sigma_g^0$. Clearly $\mathrm{Symp}_0^c \Sigma_g^0$ is a subgroup of $\mathrm{Symp}_0 \Sigma_g$.

Let $\mathcal{M}_{g,1}$ denote the mapping class group of Σ_g *relative to* the embedded disk $D^2 \subset \Sigma_g$, equivalently, the mapping class group of the compact surface $\overline{\Sigma_g^0}$ with boundary. We have a natural homomorphism $p\colon \mathrm{Symp}^c \Sigma_g^0 \to \mathcal{M}_{g,1}$ which is easily seen to be surjective. Moreover, Moser's theorem [**24**] adapted to the present case (see [**26**] for a general statement), implies that the kernel of this surjection is precisely the group $\mathrm{Symp}_0^c \Sigma_g^0$. We summarize the situation in the following diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Symp}_0^c \Sigma_g^0 & \xrightarrow{i^c} & \mathrm{Symp}^c \Sigma_g^0 & \xrightarrow{p} & \mathcal{M}_{g,1} & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle j_0} & & \Big\downarrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle q} & & \\
1 & \longrightarrow & \mathrm{Symp}_0 \Sigma_g & \xrightarrow{i} & \mathrm{Symp}\,\Sigma_g & \longrightarrow & \mathcal{M}_g & \longrightarrow & 1
\end{array} \tag{9}$$

where $q\colon \mathcal{M}_{g,1} \to \mathcal{M}_g$ denotes the natural projection.

The restriction of the flux homomorphism

$$\mathrm{Flux}\colon \mathrm{Symp}_0 \Sigma_g \longrightarrow H^1(\Sigma_g; \mathbb{R}) \tag{10}$$

to the subgroup $\mathrm{Symp}_0^c \Sigma_g^0$, denoted $j^*\,\mathrm{Flux}$, can be described as follows. The restriction $j^*\omega$ of the area form ω to the open surface Σ_g^0 is exact. Choose a 1-form λ such that $d\lambda = -j^*\omega$. Then, for any element $\varphi \in$

$\mathrm{Symp}_0^c\,\Sigma_g^0$, the 1-form $\lambda - \varphi^*\lambda$ is a closed form with compact support. Hence the corresponding de Rham cohomology class $[\lambda - \varphi^*\lambda]$, which can be shown to be independent of the choice of λ, is an element of the first cohomology group $H_c^1(\Sigma_g^0;\mathbb{R})$ of Σ_g^0 with compact support. It is easy to see that $H_c^1(\Sigma_g^0;\mathbb{R})$ is canonically isomorphic to $H^1(\Sigma_g;\mathbb{R})$, and that under this isomorphism

$$(j^*\,\mathrm{Flux})(\varphi) = [\lambda - \varphi^*\lambda] \in H_c^1(\Sigma_g^0;\mathbb{R}) \cong H^1(\Sigma_g;\mathbb{R}) \tag{11}$$

for all $\varphi \in \mathrm{Symp}_0^c\,\Sigma_g^0$, see Lemma 10.14 of [**16**]. We obtain the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Symp}_0^c\,\Sigma_g^0 & \xrightarrow{j^*\,\mathrm{Flux}} & H_c^1(\Sigma_g^0;\mathbb{R}) \\ \downarrow & & \downarrow \cong \\ \mathrm{Symp}_0\,\Sigma_g & \xrightarrow[\mathrm{Flux}]{} & H^1(\Sigma_g;\mathbb{R}). \end{array} \tag{12}$$

From now on we identify $H_c^1(\Sigma_g^0;\mathbb{R})$ with $H^1(\Sigma_g;\mathbb{R})$.

As was already mentioned in the Introduction, we proved in [**13**] that the flux homomorphism (10) can be extended to a crossed homomorphism

$$\widetilde{\mathrm{Flux}}\colon\ \mathrm{Symp}\,\Sigma_g \longrightarrow H^1(\Sigma_g;\mathbb{R})\ , \tag{13}$$

and that the extension is unique up to the addition of coboundaries. The restriction of such a crossed homomorphism to the subgroup $\mathrm{Symp}^c\,\Sigma_g^0 \subset \mathrm{Symp}\,\Sigma_g$, denoted $j^*\widetilde{\mathrm{Flux}}$, is of course an extension of the flux homomorphism $j^*\,\mathrm{Flux}$. However, we also have another extension of the same flux homomorphism $j^*\,\mathrm{Flux}$ to the group $\mathrm{Symp}^c\,\Sigma_g^0$ as follows.

Proposition 10. *The map*

$$\widetilde{\mathrm{Flux}}_c\colon\ \mathrm{Symp}^c\,\Sigma_g^0 \longrightarrow H_c^1(\Sigma_g^0;\mathbb{R}) \cong H^1(\Sigma_g;\mathbb{R})$$

defined by $\widetilde{\mathrm{Flux}}_c(\varphi) = [(\varphi^{-1})^*\lambda - \lambda] \in H_c^1(\Sigma_g^0;\mathbb{R})$ *is a crossed homomorphism which extends the flux homomorphism* $j^*\,\mathrm{Flux}$. *Its cohomology class* $[\widetilde{\mathrm{Flux}}_c] \in H^1(\mathrm{Symp}^c\,\Sigma_g^0; H^1(\Sigma_g;\mathbb{R}))$ *is uniquely determined independently of the choice of the* 1*-form* λ *such that* $d\lambda = -j^*\omega$.

Proof. First observe that, for any $\varphi \in \mathrm{Symp}_0^c\,\Sigma_g^0$, we have

$$[\lambda - \varphi^*\lambda] = (\varphi^{-1})^*[\lambda - \varphi^*\lambda] = [(\varphi^{-1})^*\lambda - \lambda]$$

because φ acts trivially on $H^1(\Sigma_g;\mathbb{R})$. Hence by (11) we have

$$\widetilde{\mathrm{Flux}}_c(\varphi) = (j^*\,\mathrm{Flux})(\varphi).$$

Next, for any two elements $\varphi, \psi \in \mathrm{Symp}^c\, \Sigma_g^0$, we have

$$\begin{aligned}\widetilde{\mathrm{Flux}}_c(\varphi\psi) &= [((\varphi\psi)^{-1})^*\lambda - \lambda]\\ &= [(\varphi^{-1})^*\lambda - \lambda + (\varphi^{-1})^*(\psi^{-1})^*\lambda - (\varphi^{-1})^*\lambda]\\ &= [(\varphi^{-1})^*\lambda - \lambda] + [(\varphi^{-1})^*((\psi^{-1})^*\lambda - \lambda)]\\ &= \widetilde{\mathrm{Flux}}_c(\varphi) + (\varphi^{-1})^*\widetilde{\mathrm{Flux}}_c(\psi).\end{aligned}$$

Therefore $\widetilde{\mathrm{Flux}}_c$ is a crossed homomorphism which extends $j^*\,\mathrm{Flux}$.

Finally, let λ' be another 1-form on Σ_g^0 satisfying $d\lambda' = -j^*\omega$, and let $\widetilde{\mathrm{Flux}}'_c$ be the corresponding crossed homomorphism. Then $a = \lambda' - \lambda$ is a closed 1-form defining a de Rham cohomology class $[a] \in H^1(\Sigma_g^0;\mathbb{R}) \cong H^1(\Sigma_g;\mathbb{R})$. Now

$$\begin{aligned}\widetilde{\mathrm{Flux}}'_c(\varphi) &= [(\varphi^{-1})^*\lambda' - \lambda']\\ &= [(\varphi^{-1})^*(\lambda + a) - (\lambda + a)]\\ &= [(\varphi^{-1})^*\lambda - \lambda] + [(\varphi^{-1})^*a - a]\\ &= \widetilde{\mathrm{Flux}}_c(\varphi) + (\varphi^{-1})^*[a] - [a] \in H^1(\Sigma_g;\mathbb{R}).\end{aligned}$$

This shows that the difference $\widetilde{\mathrm{Flux}}'_c - \widetilde{\mathrm{Flux}}_c$ is a coboundary, completing the proof of the proposition. q.e.d.

We have proved that the restriction $j^*\,\mathrm{Flux}$ of the flux homomorphism (10) to the subgroup $\mathrm{Symp}_0^c\,\Sigma_g^0$ has two extensions to the group $\mathrm{Symp}^c\,\Sigma_g^0$ as a crossed homomorphism. One is the restriction $j^*\widetilde{\mathrm{Flux}}$ of $\widetilde{\mathrm{Flux}}$, and the other is $\widetilde{\mathrm{Flux}}_c$. We will show that these two crossed homomorphisms are essentially different. More precisely, we will show that the difference of these two crossed homomorphisms can be expressed by an element of the cohomology group

$$H^1(\mathcal{M}_{g,1}; H^1(\Sigma_g;\mathbb{R})). \tag{14}$$

It was proved in [**20**] that $H^1(\mathcal{M}_{g,1}; H^1(\Sigma_g;\mathbb{Z}))$ is isomorphic to $\mathbb{Z}$ for all $g \geq 2$. It follows that the above group (14) is isomorphic to $\mathbb{R}$ and if

$$k\colon \mathcal{M}_{g,1} \longrightarrow H^1(\Sigma_g;\mathbb{Z})$$

is any crossed homomorphism whose cohomology class is a generator of $H^1(\mathcal{M}_{g,1}; H^1(\Sigma_g;\mathbb{Z}))$, then the associated crossed homomorphism

$$k_{\mathbb{R}}\colon \mathcal{M}_{g,1} \longrightarrow H^1(\Sigma_g;\mathbb{R})$$

represents the element $1 \in H^1(\mathcal{M}_{g,1}; H^1(\Sigma_g;\mathbb{R})) \cong \mathbb{R}$. Let $p^*k_{\mathbb{R}} \in H^1(\mathrm{Symp}^c\,\Sigma_g; H^1(\Sigma_g;\mathbb{R}))$ be the class induced from $k_{\mathbb{R}}$ by the projection $p\colon \mathrm{Symp}^c\,\Sigma_g \rightarrow \mathcal{M}_{g,1}$.

Theorem 11. *For any $g \geq 2$, there exists an isomorphism*

$$H^1(\operatorname{Symp}^c \Sigma_g^0; H^1(\Sigma_g; \mathbb{R})) \cong \mathbb{R} \oplus \operatorname{Hom}_{\mathbb{Z}}(\mathbb{R}, \mathbb{R})$$

such that $p^* k_{\mathbb{R}}$ and $[j^* \widetilde{\operatorname{Flux}}]$ represent the classes $1 \in \mathbb{R}$ and $\operatorname{id} \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{R}, \mathbb{R})$ respectively.

Proof. The top extension in (9) gives rise to an exact sequence

$$0 \longrightarrow H^1(\mathcal{M}_{g,1}; H^1(\Sigma_g; \mathbb{R})) \ \cong \mathbb{R} \stackrel{p^*}{\longrightarrow}$$
$$H^1(\operatorname{Symp}^c \Sigma_g; H^1(\Sigma_g; \mathbb{R})) \longrightarrow H^1(\operatorname{Symp}_0^c \Sigma_g; H^1(\Sigma_g; \mathbb{R}))^{\mathcal{M}_{g,1}} \longrightarrow \cdots .$$

It was proved in [**13**] that $j^* \operatorname{Flux}$ induces an isomorphism

$$j^* \operatorname{Flux} \colon H_1(\operatorname{Symp}_0^c \Sigma_g^0; \mathbb{Z}) \cong H_c^1(\Sigma_g^0; \mathbb{R}) = H^1(\Sigma_g; \mathbb{R}) \ .$$

Hence, by an argument similar to the one given for $\operatorname{Symp}_0 \Sigma_g$ in [**13**], we have an isomorphism

$$H^1(\operatorname{Symp}_0^c \Sigma_g^0; H^1(\Sigma_g; \mathbb{R}))^{\mathcal{M}_{g,1}} \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{R}, \mathbb{R}),$$

and clearly $j^*[\widetilde{\operatorname{Flux}}]$ corresponds to $\operatorname{id} \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{R}, \mathbb{R})$. The result follows from this. q.e.d.

Theorem 12. *We have the identity*

$$[\widetilde{\operatorname{Flux}_c}] = j^*[\widetilde{\operatorname{Flux}}] - p^* k_{\mathbb{R}}$$

in $H^1(\operatorname{Symp}^c \Sigma_g^0; H^1(\Sigma_g; \mathbb{R}))$.

Proof. Since both crossed homomorphisms $\widetilde{\operatorname{Flux}_c}$ and $j^*\widetilde{\operatorname{Flux}}$ are extensions of the flux homomorphism $j^* \operatorname{Flux}$, the proof of Theorem 11 implies that

$$[\widetilde{\operatorname{Flux}_c}] = j^*[\widetilde{\operatorname{Flux}}] + a\ p^* k_{\mathbb{R}}$$

for some constant $a \in \mathbb{R}$. Let $\mathcal{I}_{g,1} \subset \mathcal{M}_{g,1}$ denote the Torelli subgroup consisting of mapping classes which act trivially on homology. We set $\mathcal{I} \operatorname{Symp}^c \Sigma_g^0 = p^{-1}(\mathcal{I}_{g,1}) \subset \operatorname{Symp}^c \Sigma_g^0$. If we restrict the crossed homomorphims $\widetilde{\operatorname{Flux}_c}$ and $\widetilde{\operatorname{Flux}}$ to this subgroup $\mathcal{I} \operatorname{Symp}^c \Sigma_g^0$, then they become homomorphisms which depend only on the cohomology classes $[\widetilde{\operatorname{Flux}_c}]$ and $j^*[\widetilde{\operatorname{Flux}}]$, and not on the particular crossed homomorphisms representing these cohomology classes. This is because any crossed homomorphism which is a coboundary is trivial on $\mathcal{I} \operatorname{Symp}^c \Sigma_g^0$. It was proved in [**20**] that a generator of the group $H^1(\mathcal{M}_{g,1}; H^1(\Sigma_g; \mathbb{Z})) \cong \mathbb{Z}$ is characterized by the fact that the *Poincaré dual* of its value on a single non-trivial element $\varphi \in \mathcal{I}_{g,1}$ is equal to $\pm C\tau(\varphi)$, where $\tau \colon \mathcal{I}_{g,1} \to \Lambda^3 H_1(\Sigma_g; \mathbb{Z})$ denotes the (first) Johnson homomorphism and $C \colon \Lambda^3 H_1(\Sigma_g; \mathbb{Z}) \to H_1(\Sigma_g; \mathbb{Z})$ denotes the contraction. Hence we only have to compute the values of $\widetilde{\operatorname{Flux}_c}$ and $\widetilde{\operatorname{Flux}}$ on some particular element $\tilde{\varphi} \in \mathcal{I} \operatorname{Symp}^c \Sigma_g^0$. We choose such element as follows.

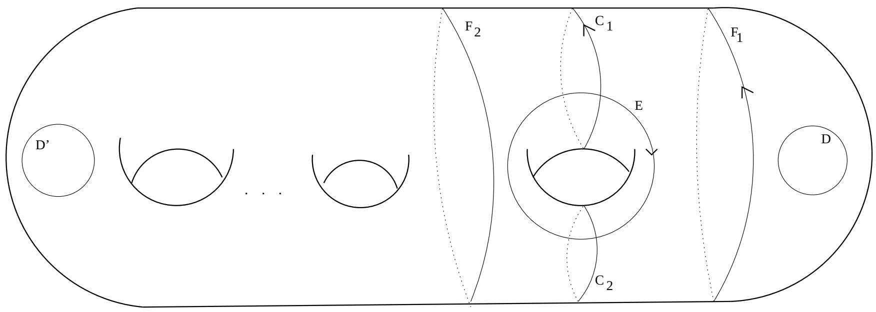

Figure 1.

Consider an embedded disk D' in $\Sigma_g^0 = \Sigma_g \setminus D$ as depicted in Figure 1 and set $\Sigma_g^{00} = \Sigma_g \setminus (D \cup D')$. We also consider two disjoint simple closed curves $C_1, C_2 \subset \Sigma_g^{00}$ as shown in Figure 1. Let τ_i $(i = 1, 2)$ denote the Dehn twist along C_i and choose lifts $\tilde{\tau}_i \in \mathrm{Symp}^c\,\Sigma_g^0$ of τ_i with supports in small neighborhoods of the C_i. Now we set

$$\tilde{\varphi} = \tilde{\tau}_1 \tilde{\tau}_2^{-1}.$$

Since $\varphi = \tau_1\tau_2^{-1}$ belongs to the Torelli group $\mathcal{I}_{g,1} \subset \mathcal{M}_{g,1}$, its lift $\tilde{\varphi}$ is an element of the subgroup $\mathcal{I}\,\mathrm{Symp}^c\,\Sigma_g^0$. We would like to compute the difference

$$\begin{aligned}(15)\qquad \widetilde{\mathrm{Flux}}_c(\tilde{\varphi}) - \widetilde{\mathrm{Flux}}(\tilde{\varphi}) &= a\; p^* k_{\mathbb{R}}(\tilde{\varphi}) \\ &= a\; k(\varphi) \in H^1_c(\Sigma_g^0; \mathbb{R}) \cong H^1(\Sigma_g; \mathbb{R}).\end{aligned}$$

Now we consider the long exact sequence

$$\begin{aligned}(16)\qquad 0 \longrightarrow H^1(\Sigma_g; \mathbb{R}) &\xrightarrow{i^*} H^1(\Sigma_g^{00}; \mathbb{R}) \longrightarrow \\ &H^2(\Sigma_g, \Sigma_g^{00}; \mathbb{R}) \cong \mathbb{R}^2 \longrightarrow H^2(\Sigma_g; \mathbb{R}) \longrightarrow 0\end{aligned}$$

of the pair $(\Sigma_g, \Sigma_g^{00})$. Also consider the following short exact sequences

$$(17)\qquad 0 \longrightarrow H^1_c(\Sigma_g^0; \mathbb{R}) \cong H^1(\Sigma_g; \mathbb{R}) \xrightarrow{j^*} H^1(\Sigma_g^{00}; \mathbb{R}) \longrightarrow \mathbb{R} \longrightarrow 0$$

$$0 \longrightarrow H^1_c(\Sigma_g \setminus D'; \mathbb{R}) \cong H^1(\Sigma_g; \mathbb{R}) \xrightarrow{(j')^*} H^1(\Sigma_g^{00}; \mathbb{R}) \longrightarrow \mathbb{R} \longrightarrow 0.$$

In view of the above exact sequences (16) and (17), we can determine the value of (15) in the group $H^1(\Sigma_g^{00}; \mathbb{R})$ because both groups $H^1_c(\Sigma_g^0; \mathbb{R})$ and $H^1(\Sigma_g; \mathbb{R})$ are embedded in it. Choose 1-forms λ and λ' on $\Sigma_g^0 = \Sigma_g \setminus D$ and $\Sigma_g \setminus D'$ respectively, such that

$$d\lambda = -j^*\omega, \quad d\lambda' = -(j')^*\omega$$

where $j\colon \Sigma_g^0 \to \Sigma_g$ and $j'\colon \Sigma_g \setminus D' \to \Sigma_g$ are the inclusions. Then $d(\lambda - \lambda') = 0$ so that $\nu = \lambda - \lambda'$ is a closed 1-form on Σ_g^{00}. We have the identity

$$\begin{aligned}(\tilde{\varphi}^{-1})^*\lambda - \lambda &= (\tilde{\varphi}^{-1})^*(\lambda' + \nu) - (\lambda' + \nu)\\ &= (\tilde{\varphi}^{-1})^*\lambda' - \lambda' + (\tilde{\varphi}^{-1})^*\nu - \nu.\end{aligned}$$

Hence we have the equality

$$[(\tilde{\varphi}^{-1})^*\lambda - \lambda] = [(\tilde{\varphi}^{-1})^*\lambda' - \lambda'] + [(\tilde{\varphi}^{-1})^*\nu - \nu] \tag{18}$$

in the group $H^1(\Sigma_g^{00};\mathbb{R})$. By the definition of $\widetilde{\mathrm{Flux}}_c$, we have

$$\widetilde{\mathrm{Flux}}_c(\tilde{\varphi}) = [(\tilde{\varphi}^{-1})^*\lambda - \lambda] \in H^1_c(\Sigma_g^0;\mathbb{R}) \subset H^1(\Sigma_g^{00};\mathbb{R}). \tag{19}$$

Next we compute $\widetilde{\mathrm{Flux}}(\tilde{\varphi})$. For this, observe that $\tilde{\varphi}$ is isotopic to the identity as an element of $\mathrm{Symp}^c(\Sigma_g \setminus D') \subset \mathrm{Symp}\,\Sigma_g$, although it is *not* isotopic to the identity as an element of $\mathrm{Symp}^c(\Sigma_g \setminus D)$. Hence $\tilde{\varphi} \in \mathrm{Symp}^c_0(\Sigma_g \setminus D')$ and $\widetilde{\mathrm{Flux}}(\tilde{\varphi}) = ((j')^*\,\mathrm{Flux})(\tilde{\varphi})$. By replacing D with D' in equation (11), we obtain

$$\begin{aligned}((j')^*\,\mathrm{Flux})(\tilde{\varphi}) &= [\lambda' - \tilde{\varphi}^*\lambda']\\ &=[(\tilde{\varphi}^{-1})^*\lambda' - \lambda'] \in H^1_c(\Sigma_g \setminus D';\mathbb{R}) \subset H^1(\Sigma_g^{00};\mathbb{R}).\end{aligned} \tag{20}$$

By combining the equations (15), (18), (19) and (20), we shall prove

$$[(\tilde{\varphi}^{-1})^*\nu - \nu] = ak(\tilde{\varphi}) = ak(\varphi) = aPD \circ C\tau(\varphi), \tag{21}$$

where PD denotes the Poincaré duality isomorphism.

Clearly $(\tilde{\varphi}^{-1})^*\nu - \nu$ is a closed 1-form on Σ_g whose support is contained in a neighborhood of $C_1 \cup C_2$. Hence the Poincaré dual of the de Rham cohomology class $[(\tilde{\varphi}^{-1})^*\nu - \nu] \in H^1(\Sigma_g;\mathbb{R})$ is a multiple of the homology class $[C_1] \in H_1(\Sigma_g;\mathbb{Z})$ represented by the simple closed curve C_1 with a fixed orientation depicted in Figure 1. However, according to Johnson [**9**], we have

$$\tau(\varphi) = (x_1 \wedge y_1 + \cdots + x_{g-1} \wedge y_{g-1}) \wedge [C_1]$$

where $x_1.y_1, \ldots, x_{g-1}, y_{g-1}$ is a symplectic basis of the homology group of the left hand subsurface of Σ_g obtained by cutting Σ_g along the simple closed curve F_2 depicted in Figure 1. It follows that

$$C\tau(\varphi) = 2(g-1)[C_1].$$

This checks the equality (21), for some constant a.

To determine this constant, it is enough to compute the value of the cohomology class $[(\tilde{\varphi}^{-1})^*\nu - \nu]$ on the homology class represented by the oriented simple closed curve E depicted in Figure 1. Observe that the homology class $\tilde{\varphi}^{-1}{}_*[E] - [E] \in H_1(\Sigma_g^{00};\mathbb{Z})$ can be represented by the oriented simple closed curve F_1 also depicted in Figure 1. Let $\widetilde{D}$ denote the right hand compact subsurface of Σ_g obtained by cutting along F_1. Thus $\widetilde{D}$ is diffeomorphic to a disk which contains the original embedded disk D in its interior. Also let $\Sigma = \Sigma_g \setminus \mathrm{Int}\tilde{D}$.

Now we compute

$$\begin{aligned}[(\tilde{\varphi}^{-1})_*\nu - \nu]([E]) &= \nu((\tilde{\varphi}^{-1})_*[E]) - \nu([E]) \\ &= \nu([F_1]) = \int_{F_1} \lambda - \lambda' \\ &= \int_{\partial\Sigma} \lambda - \int_{-\partial\widetilde{D}} \lambda' \\ &= \int_{\Sigma} d\lambda + \int_{\widetilde{D}} d\lambda' \\ &= \int_{\Sigma_g} -\omega = 2 - 2g.\end{aligned}$$

Thus we can conclude that $a = -1$, completing the proof. q.e.d.

5. The second homology group of $\mathrm{Symp}^{c,\delta}\,\Sigma_g^0$

In this section, we generalize the claim of Theorem 4 about the second homology, i. e. for $k = 1$, to the case of the open surface Σ_g^0. We have the cohomology class

$$j^*\tilde{\alpha} \in H^2(\mathrm{Symp}^{c,\delta}\,\Sigma_g^0; S^2_{\mathbb{Q}}\mathbb{R})$$

induced by the inclusion $j\colon \Sigma_g^0 \to \Sigma_g$.

Theorem 13. *The characteristic classes e_1 and $j^*\tilde{\alpha}$ induce a surjective homomorphism*

$$H_2(\mathrm{Symp}^{c,\delta}\,\Sigma_g^0; \mathbb{Z}) \longrightarrow \mathbb{Z} \oplus S^2_{\mathbb{Q}}\mathbb{R}$$

for any $g \geq 3$. For $g = 2$, the class $j^*\tilde{\alpha}$ induces a surjection

$$H_2(\mathrm{Symp}^{c,\delta}\,\Sigma_2^0; \mathbb{Z}) \longrightarrow S^2_{\mathbb{Q}}\mathbb{R}.$$

The proof of this theorem, which occupies the rest of this section, consists of a rather long and delicate argument. We first describe the reason why the easy proof of Theorem 4 in the case $k = 1$, which treated the case of closed surfaces, does not work for Theorem 13, thereby making the difficulty in the case of open surfaces explicit.

Consider the top extension

$$1 \longrightarrow \mathrm{Symp}_0^c\,\Sigma_g^0 \xrightarrow{i^c} \mathrm{Symp}^c\,\Sigma_g^0 \xrightarrow{p} \mathcal{M}_{g,1} \longrightarrow 1 \tag{22}$$

in the commutative diagram (9). We have the following proposition, contrasting with our discussion of the closed case in Section 2.

Proposition 14. *For any $g \geq 2$ the image of the homomorphism*

$$(j^*\,\mathrm{Flux})_*\colon H_2(\mathrm{Symp}_0^{c,\delta}\,\Sigma_g^0; \mathbb{Z}) \longrightarrow H_2(H^1(\Sigma_g; \mathbb{R})^\delta; \mathbb{Z}) \cong \Lambda^2_{\mathbb{Z}} H^1(\Sigma_g; \mathbb{R})$$

induced by the restriction $j^*\,\mathrm{Flux}$ of the flux homomorphism to the subgroup $\mathrm{Symp}_0^c\,\Sigma_g^0$ is equal to the kernel of the natural intersection pairing

$$\Lambda^2_{\mathbb{Z}} H^1(\Sigma_g; \mathbb{R}) \longrightarrow \mathbb{R}.$$

Proof. Recall that we have an extension

$$1 \longrightarrow \mathrm{Ham}^c\,\Sigma_g^0 \longrightarrow \mathrm{Symp}_0^c\,\Sigma_g^0 \stackrel{j^*\,\mathrm{Flux}}{\longrightarrow} H^1(\Sigma_g;\mathbb{R}) \longrightarrow 1, \tag{23}$$

where $\mathrm{Ham}^c\,\Sigma_g^0$ denotes the subgroup consisting of Hamiltonian symplectomorphisms with compact supports (see [**16**] as well as [**13**]). Also recall that there is a surjective homomorphism

$$\mathrm{Cal}\colon\ \mathrm{Ham}^c\,\Sigma_g^0 \longrightarrow \mathbb{R}$$

called the (second) Calabi homomorphism (see [**3**]). Banyaga [**1**] proved that the kernel of this homomorphism is perfect. Hence we have an isomorphism

$$H_1(\mathrm{Ham}^{c,\delta}\,\Sigma_g^0;\mathbb{Z}) \cong \mathbb{R}.$$

Now we consider the Hochschild–Serre exact sequence

$$\begin{aligned} H_2(\mathrm{Symp}_0^{c,\delta}\,\Sigma_g^0;\mathbb{Z}) &\stackrel{j^*\,\mathrm{Flux}_*}{\longrightarrow} H_2(H^1(\Sigma_g;\mathbb{R})^\delta;\mathbb{Z}) \\ &\stackrel{\partial}{\longrightarrow} H_1(\mathrm{Ham}^{c,\delta}\,\Sigma_g^0;\mathbb{Z})_{H^1_{\mathbb{R}}} \cong \mathbb{R} \\ &\longrightarrow H_1(\mathrm{Symp}_0^{c,\delta}\,\Sigma_g^0;\mathbb{Z}) \\ &\stackrel{j^*\,\mathrm{Flux}_*}{\longrightarrow} H_1(H^1(\Sigma_g;\mathbb{R})^\delta;\mathbb{Z}) \longrightarrow 0 \end{aligned}$$

of the group extension (23). We proved in [**13**] (Proposition 11 and Corollary 12) that the last homomorphism $j^*\,\mathrm{Flux}_*$ in the above sequence is an isomorphism, and that the boundary operator ∂ coincides with the intersection pairing. The result follows. q.e.d.

Corollary 15. *The restriction of $\alpha \in H^2(\mathrm{Symp}^\delta\,\Sigma_g;\mathbb{R})$ to the subgroup $\mathrm{Symp}_0^{c,\delta}\,\Sigma_g^0$ is trivial.*

Proof. This follows from Proposition 14 and the definition of the cohomology class α. q.e.d.

Thus, in order to prove the non-triviality of α on the group $\mathrm{Symp}^c\,\Sigma_g^0$, we must combine the roles of the two groups $\mathcal{M}_{g,1}$ and $\mathrm{Symp}_0^{c,\delta}\,\Sigma_g^0$. This contrasts sharply with the case of closed surfaces treated in Section 2.

Let $\{E^r_{p,q}\}$ be the Hochschild–Serre spectral sequence for the integral homology of the extension (22). This gives rise to two short exact sequences

$$\begin{aligned} 0 \longrightarrow \mathrm{Ker} \longrightarrow H_2(\mathrm{Symp}^{c,\delta}\,\Sigma_g^0;\mathbb{Z}) \longrightarrow E^\infty_{2,0} \longrightarrow 0 \\ 0 \longrightarrow E^\infty_{0,2} \longrightarrow \mathrm{Ker} \longrightarrow E^\infty_{1,1} \longrightarrow 0, \end{aligned} \tag{24}$$

where $E^\infty_{2,0} \subset H_2(\mathcal{M}_{g,1};\mathbb{Z})$ concerns the first Mumford-Morita-Miller class already discussed in [**13**]. Proposition 14 shows that the image of the map

$$E^\infty_{0,2} = \mathrm{Im}\left(H_2(\mathrm{Symp}_0^{c,\delta}\,\Sigma_g^0;\mathbb{Z}) \rightarrow H_2(\mathrm{Symp}^{c,\delta}\,\Sigma_g^0;\mathbb{Z})\right) \longrightarrow S^2_{\mathbb{Q}}\mathbb{R}$$

defined by the Kronecker product with $\tilde{\alpha}$ is precisely the kernel of the natural map $S^2_{\mathbb{Q}}\mathbb{R}\to\mathbb{R}$. It remains to determine the $E^\infty_{1,1}$-term. We have

$$E^2_{1,1} = H_1(\mathcal{M}_{g,1}; H_1(\mathrm{Symp}^{c,\delta}_0\,\Sigma^0_g;\mathbb{Z})) \cong H_1(\mathcal{M}_{g,1}; H_1(\Sigma_g;\mathbb{R})) \tag{25}$$

because, as was already mentioned above, it was proved in [**13**] that j^* Flux induces an isomorphism $H_1(\mathrm{Symp}^{c,\delta}_0\,\Sigma^0_g;\mathbb{Z}) \cong H^1(\Sigma_g;\mathbb{R})$.

We now recall a result which was essentially proved in [**20**]. Without repeating everything done in [**20**], we want to give a precise statement and proof of what is needed in the sequel. We use the Lickorish generators for the mapping class group $\mathcal{M}_{g,1}$, denoted λ_i, μ_i, ν_i as in Figure 1 of [**20**], and a symplectic basis $x_1,\dots,x_g,y_1,\dots,y_g$ of $H_1(\Sigma_g;\mathbb{Z})$. In particular

$$\nu_i(x_i) = x_i - y_i + y_{i+1}, \quad \mu_i(x_i) = x_i - y_i \tag{26}$$

which we record here for later use.

Proposition 16. *For any $g \geq 2$, we have an isomorphim*

$$H_1(\mathcal{M}_{g,1}; H_1(\Sigma_g;\mathbb{Z})) \cong \mathbb{Z}.$$

Furthermore, for any $i = 1,2,\dots,g-1$, the element

$$c = \nu_i \otimes x_i - \mu_i \otimes x_i + \mu_{i+1} \otimes x_{i+1}$$

is a 1*-cycle of $\mathcal{M}_{g,1}$ with twisted coefficients in $H_1(\Sigma_g;\mathbb{Z})$ and it represents a generator of the above infinite cyclic group.*

Proof. The group extension

$$1\longrightarrow\pi_1\Sigma_g\longrightarrow\mathcal{M}_{g,*}\longrightarrow\mathcal{M}_g\longrightarrow 1,$$

where $\mathcal{M}_{g,*}$ denotes the mapping class group of Σ_g *relative* to a basis point, yields the Hochschild–Serre exact sequence

$$H_2(\mathcal{M}_g;H)\longrightarrow(H\otimes H)_{\mathcal{M}_g}\longrightarrow H_1(\mathcal{M}_{g,*};H)\longrightarrow H_1(\mathcal{M}_g;H)\longrightarrow 0. \tag{27}$$

Here and henceforth H is a shorthand for $H_1(\Sigma_g;\mathbb{Z})$. It is easy to see that the intersection pairing induces an isomorphism $(H\otimes H)_{\mathcal{M}_g} \cong \mathbb{Z}$ and the element $x_1 \otimes y_1$, for example, represents a generator. It was proved in [**20**] that

$$H^1(\mathcal{M}_{g,*}; H^1(\Sigma_g;\mathbb{Z})) \cong Z, \quad H_1(\mathcal{M}_g; H^1(\Sigma_g;\mathbb{Z})) \cong \mathbb{Z}/(2g-2)\mathbb{Z}.$$

If we apply the crossed homomorphism $f\colon \mathcal{M}_{g,*}\to H^1(\Sigma_g;\mathbb{Z})$ given in the above cited paper, which detects a generator of the above infinite cyclic group, to the element $x_1 \otimes y_1$ considered as a cycle of $\mathcal{M}_{g,*}$ with coefficients in H, we obtain

$$f(x_1)(y_1) = 2 - 2g.$$

On the other hand, c is a cycle because

$$\partial c = x_i - \nu_i(x_i) - x_i + \mu_i(x_i) + x_{i+1} - \mu_{i+1}(x_{i+1}) = 0$$

by (26), and it was shown that

$$f(c) = f(\nu_i)(x_i) - f(\mu_i)(x_i) + f(\mu_{i+1})(x_{i+1}) = 1 \ , \tag{28}$$

see [**20**] for details. In view of the exact sequence (27), we can conclude that $H_1(\mathcal{M}_{g,*}; H) \cong \mathbb{Z}$. Finally, it is easy to deduce from the central group extension $0 \to \mathbb{Z} \to \mathcal{M}_{g,1} \to \mathcal{M}_{g,*} \to 1$ that we have an isomorphism $H_1(\mathcal{M}_{g,1}; H) \cong H_1(\mathcal{M}_{g,*}; H)$. This finishes the proof. q.e.d.

Going back to the $E^2_{1,1}$-term in (25), we have

$$E^2_{1,1} \cong H_1(\mathcal{M}_{g,1}; H \otimes \mathbb{R}) \cong \mathbb{R}$$

by Proposition 16. Now consider the exact sequence

$$E^2_{3,0} \cong H_3(\mathcal{M}_{g,1}; \mathbb{Z}) \stackrel{d^2}{\longrightarrow} E^2_{1,1} \cong \mathbb{R} \longrightarrow E^3_{1,1} = E^\infty_{1,1} \longrightarrow 0.$$

Harer [**7**] determined the third stable rational cohomology group

$$\lim_{g \to \infty} H^3(\mathcal{M}_{g,1}; \mathbb{Q})$$

of the mapping class group to be trivial[1] . It follows that $E^2_{3,0}$ is a finite group for all sufficiently large g. Hence we can conclude that $E^\infty_{1,1} \cong \mathbb{R}$ for such g. It is natural to expect that this $\mathbb{R}$ will recover the missing $\mathbb{R}$ in $E^\infty_{0,2}$ so that we obtain the surjectivity of the $\tilde{\alpha}$-factor in Theorem 13. It turns out that this is indeed the case, and below we shall give a proof of this fact which does not use Harer's result mentioned above. Before doing so, we have to prepare some general facts concerning group (co)homology in small degrees.

Consider a group extension

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1, \tag{29}$$

and suppose we are given a 1-cycle $c = \sum_i q_i \otimes u_i \in Z_1(Q; H_1(K))$ of the group Q with coefficients in the abelianization $H_1(K)$ of K considered as a natural Q-module, where $q_i \in Q$ and $u_i \in H_1(K)$.

Lemma 17. *For any choices of lifts $\tilde{q}_i \in G$ of the q_i and representatives $k_i \in K$ with $[k_i] = u_i$, the element*

$$\tilde{c} = \sum_i \left\{ (\tilde{q}_i, k_i) + (\tilde{q}_i k_i, \tilde{q}_i^{-1}) - (\tilde{q}_i, \tilde{q}_i^{-1}) - (\mathrm{id}, \mathrm{id}) \right\} + d$$

is a 2-cycle of G, where d is a 2-chain of the group K such that

$$\partial d = \sum_i \left\{ (\tilde{q}_i k_i \tilde{q}_i^{-1}) - (k_i) \right\} .$$

[1]There is now a final result on the stable cohomology of $\mathcal{M}_g$ due to Madsen and Weiss [**15**].

Furthermore, $p_*([\tilde{c}]) = 0 \in H_2(Q;\mathbb{Z})$*, where* $p\colon G \to Q$ *denotes the projection, and in the short exact sequence*

$$0 \longrightarrow E^{\infty}_{0,2} \longrightarrow \mathrm{Ker}\left(H_2(G;\mathbb{Z}) \xrightarrow{p_*} H_2(Q;\mathbb{Z})\right) \longrightarrow E^{\infty}_{1,1} \ \left(\cong H_1(Q;H_1(K))/d^2(E^2_{3,0})\right) \longrightarrow 0$$

arising from the Hochschild–Serre spectral sequence of (29) *the class* $[\tilde{c}] \in \mathrm{Ker}$ *is a lift of* $[c] \in H_1(Q;H_1(K))$.

Proof. Since c is a cycle by the assumption, we have

$$\partial c = \sum_i (q_i(u_i) - u_i) = 0 \in H_1(K).$$

It follows that there exists a 2-chain $d \in C_2(K;\mathbb{Z})$ with the property described in the statement of the lemma. Then a direct computation shows that $\partial\tilde{c} = 0$. Clearly $p_*(\tilde{c}) = 0$ and it is easy to check the rest of the required assertions. q.e.d.

The following can be proved by a standard argument in the cohomology theory of groups, see [**2**].

Lemma 18. *Let* G *be a group and* M *a* G*-module. Assume we have a* G*-invariant skew-symmetric bilinear pairing*

$$\iota\colon M \times M \longrightarrow A,$$

where A *is an abelian group with trivial* G*-action, and we are given two crossed homomorphisms*

$$f_i\colon G \longrightarrow M \quad (i = 1, 2)$$

so that $f_i(gh) = f_i(g) + g_* f_i(h)$ *for all* $g, h \in G$*. Then the assignment*

$$G \times G \ni (g,h) \longmapsto \iota(f_1(g), g_* f_2(h)) \in A,$$

which we denote by $f_1 \cdot f_2$*, is a 2-cocycle of* G *with values in* A *and its cohomology class in* $H^2(G;A)$ *depends only on the cohomology classes* $[f_i] \in H^1(G;M)$ *of the crossed homomorphisms* f_i*. Furthermore,* $f_2 \cdot f_1$ *is cohomologous to* $f_1 \cdot f_2$*, so that* $[f_2 \cdot f_1] = [f_1 \cdot f_2] \in H^2(G;A)$.

Now in the situation of Lemma 18, we consider the case where $G = \mathrm{Symp}^c\,\Sigma^0_g$, $M = H^1(\Sigma_g;\mathbb{R})$ and ι is the intersection pairing. Then we have three crossed homomorphisms

$$j^*\widetilde{\mathrm{Flux}},\ \widetilde{\mathrm{Flux}}_c,\ p^*k_{\mathbb{R}} : \mathrm{Symp}^c\,\Sigma^0_g \longrightarrow H^1(\Sigma_g;\mathbb{R})$$

and, by the definition of α, we have $j^*\alpha = [j^*\widetilde{\mathrm{Flux}} \cdot j^*\widetilde{\mathrm{Flux}}]$.

Proposition 19. *In the above notation, we have* $[\widetilde{\mathrm{Flux}}_c \cdot \widetilde{\mathrm{Flux}}_c] = 0$ *and*

$$j^*\alpha = 2[p^*k_{\mathbb{R}} \cdot \widetilde{\mathrm{Flux}}_c] - p^*e_1.$$

Proof. Define a map

$$\widetilde{\mathrm{Cal}}\colon \operatorname{Symp}^c \Sigma_g^0 \longrightarrow \mathbb{R}$$

by the formula

$$\widetilde{\mathrm{Cal}}(\varphi) = \int_{\Sigma_g^0} (\varphi^{-1})^* \lambda \wedge \lambda.$$

The restriction of this map to the subgroup $\operatorname{Ham}^c \Sigma_g^0$ is a homomorphism, which, suitably normalized, is called the (second) Calabi homomorphism (see [**3**]). In our previous paper [**13**], we examined how the restriction of $\widetilde{\mathrm{Cal}}$ to the subgroup $\operatorname{Symp}_0^c \Sigma_g^0$ fails to be a homomorphism. Here we extend this discussion to the whole group $\operatorname{Symp}^c \Sigma_g^0$. For any two elements $\varphi, \psi \in \operatorname{Symp}^c \Sigma_g^0$, we claim that

$$\widetilde{\mathrm{Cal}}(\varphi\psi) = \widetilde{\mathrm{Cal}}(\varphi) + \widetilde{\mathrm{Cal}}(\psi) + \widetilde{\mathrm{Flux}}_c(\varphi) \cdot (\varphi^{-1})^* \widetilde{\mathrm{Flux}}_c(\psi). \tag{30}$$

This is because

$$\begin{aligned}
\widetilde{\mathrm{Cal}}(\varphi\psi) &= \int_{\Sigma_g^0} ((\varphi\psi)^{-1})^* \lambda \wedge \lambda \\
&= \int_{\Sigma_g^0} (\varphi^{-1})^* ((\psi^{-1})^* \lambda - \lambda) \wedge \lambda + (\varphi^{-1})^* \lambda \wedge \lambda \\
&= \int_{\Sigma_g^0} (\varphi^{-1})^* \widetilde{\mathrm{Flux}}_c(\psi) \wedge ((\varphi^{-1})^* \lambda - \widetilde{\mathrm{Flux}}_c(\varphi)) + \widetilde{\mathrm{Cal}}(\varphi) \\
&= \widetilde{\mathrm{Flux}}_c(\varphi) \cdot (\varphi^{-1})^* \widetilde{\mathrm{Flux}}_c(\psi) + \int_{\Sigma_g^0} ((\psi^{-1})^* \lambda - \lambda) \wedge \lambda + \widetilde{\mathrm{Cal}}(\varphi) \\
&= \widetilde{\mathrm{Flux}}_c(\varphi) \cdot (\varphi^{-1})^* \widetilde{\mathrm{Flux}}_c(\psi) + \widetilde{\mathrm{Cal}}(\psi) + \widetilde{\mathrm{Cal}}(\varphi).
\end{aligned}$$

Equation (30) implies that the 2-cocycle $\widetilde{\mathrm{Flux}}_c \cdot \widetilde{\mathrm{Flux}}_c$ of the group $\operatorname{Symp}^c \Sigma_g^0$ is a coboundary. Hence $[\widetilde{\mathrm{Flux}}_c \cdot \widetilde{\mathrm{Flux}}_c] = 0$ as claimed.

The definition of α implies

$$j^*\alpha = [j^* \widetilde{\mathrm{Flux}} \cdot j^* \widetilde{\mathrm{Flux}}]$$

so that, by Theorem 12, we can write

$$j^*\alpha = [\widetilde{\mathrm{Flux}}_c \cdot \widetilde{\mathrm{Flux}}_c] + [\widetilde{\mathrm{Flux}}_c \cdot p^* k_{\mathbb{R}}] + [p^* k_{\mathbb{R}} \cdot \widetilde{\mathrm{Flux}}_c] + p^*[k_{\mathbb{R}} \cdot k_{\mathbb{R}}]. \tag{31}$$

By Lemma 18 we have $[\widetilde{\mathrm{Flux}}_c \cdot p^* k_{\mathbb{R}}] = [p^* k_{\mathbb{R}} \cdot \widetilde{\mathrm{Flux}}_c]$, and it was proved in [**21**] that $[k_{\mathbb{R}} \cdot k_{\mathbb{R}}] = -e_1$. If we substitute these relations in (31), we obtain the desired identity. q.e.d.

Proof of Theorem 13. In view of Proposition 14 and the discussion following Corollary 15, it suffices to show that there exist 2-cycles of the group $\operatorname{Symp}^c \Sigma_g^0$ such that the evaluations of $j^*\alpha$ on them have as values any real number. To construct such 2-cycles, we use the 1-cycle c

of $\mathcal{M}_{g,1}$ with coefficients in H described in Proposition 16, which represents a generator of $H_1(\mathcal{M}_{g,1}; H) \cong \mathbb{Z}$. In order to adapt to the situation here, we consider the dual cycle

$$c^* = \nu_i \otimes y_i^* - \mu_i \otimes y_i^* + \mu_{i+1} \otimes y_{i+1}^*$$

with coefficients in $H^1(\Sigma_g; \mathbb{Z})$, where the $x_1^*, \ldots, x_g^*, y_1^*, \ldots, y_g^*$ denote the dual basis of $H^1(\Sigma_g; \mathbb{Z})$. If $k\colon \mathcal{M}_{g,1} \to H^1(\Sigma_g; \mathbb{Z})$ is a crossed homomorphism which represents a generator of $H^1(\mathcal{M}_{g,1}; H^1(\Sigma_g; \mathbb{Z}))$, then we have

$$k(c^*) = k(\nu_i) \cdot y_i^* - k(\mu_i) \cdot y_i^* + k(\mu_{i+1}) \cdot y_{i+1}^* = 1 \ . \tag{32}$$

For any real number $r \in \mathbb{R}$, we consider the 1-cycle

$$c_r^* = \nu_i \otimes r y_i^* - \mu_i \otimes r y_i^* + \mu_{i+1} \otimes r y_{i+1}^*$$

of $\mathcal{M}_{g,1}$ with coefficients in $H^1(\Sigma_g; \mathbb{R})$. Now we apply Lemma 17 in the case where $G = \operatorname{Symp}^c \Sigma_g^0$, $K = \operatorname{Symp}_0^c \Sigma_g^0$, and $Q = \mathcal{M}_{g,1}$. We proved in [**13**] that the flux homomorphism induces an isomorphism

$$\operatorname{Flux}\colon H_1(\operatorname{Symp}_0^c \Sigma_g^0; \mathbb{Z}) \cong H^1(\Sigma_g; \mathbb{R}).$$

Therefore, c_r^* can be considered as a 1-cycle of $\mathcal{M}_{g,1}$ with coefficients in the abelianization of $\operatorname{Symp}_0^c \Sigma_g^0$. Hence, if we choose elements

$$\tilde{\nu}_i, \tilde{\mu}_i \in \operatorname{Symp}^c \Sigma_g^0, \quad \varphi_i^r \in \operatorname{Symp}_0^c \Sigma_g^0$$

such that

$$p(\tilde{\nu}_i) = \nu_i, \quad p(\tilde{\mu}_i) = \mu_i, \quad \operatorname{Flux}(\varphi_i^r) = \operatorname{Flux}_c(\varphi_i^r) = r y_i^*, \tag{33}$$

where $p\colon \operatorname{Symp}^c \Sigma_g^0 \to \mathcal{M}_{g,1}$ denotes the projection as before, then

$$\begin{aligned} \tilde{c}_r^* =& (\tilde{\nu}_i, \varphi_i^r) + (\tilde{\nu}_i \varphi_i^r, \tilde{\nu}_i^{-1}) - (\tilde{\nu}_i, \tilde{\nu}_i^{-1}) - (\mathrm{id}, \mathrm{id}) + \\ & (\tilde{\mu}_i, \varphi_i^r) + (\tilde{\mu}_i \varphi_i^r, \tilde{\mu}_i^{-1}) - (\tilde{\mu}_i, \tilde{\mu}_i^{-1}) - (\mathrm{id}, \mathrm{id}) + \\ & (\tilde{\mu}_{i+1}, \varphi_{i+1}^r) + (\tilde{\mu}_{i+1} \varphi_{i+1}^r, \tilde{\mu}_{i+1}^{-1}) - (\tilde{\mu}_{i+1}, \tilde{\mu}_{i+1}^{-1}) - (\mathrm{id}, \mathrm{id}) + d \end{aligned}$$

is a 2-cycle of $\operatorname{Symp}^c \Sigma_g^0$, where d is a 2-chain of the group $\operatorname{Symp}_0^c \Sigma_g^0$ such that

$$\partial d = (\tilde{\nu}_i \varphi_i^r \tilde{\nu}_i^{-1}) - (\varphi_i^r) + (\tilde{\mu}_i \varphi_i^r \tilde{\mu}_i^{-1}) - (\varphi_i^r) + (\tilde{\mu}_{i+1} \varphi_{i+1}^r \tilde{\mu}_{i+1}^{-1}) - (\varphi_{i+1}^r).$$

Now we claim that

$$j^* \alpha(\tilde{c}_r^*) = 2r, \tag{34}$$

which will finish the proof of the theorem. To show this, observe first that $p^* e_1(\tilde{c}_r^*) = 0$ because clearly $p_*(\tilde{c}_r^*) = 0$. Hence, by Proposition 19,

$$j^* \alpha(\tilde{c}_r^*) = 2[p^* k \cdot \widetilde{\operatorname{Flux}_c}](\tilde{c}_r^*).$$

Observe that

$$[p^* k \cdot \widetilde{\operatorname{Flux}_c}] \left((\tilde{\nu}_i \varphi_i^r, \tilde{\nu}_i^{-1}) - (\tilde{\nu}_i, \tilde{\nu}_i^{-1}) \right) = 0,$$

because $p^*k(\tilde{\nu}_i\varphi_i^r) = p^*k(\tilde{\nu}_i)$ and φ_i^r acts trivially on the homology of Σ_g^0. The same is true for two other similar terms. Since d is a 2-chain of $\mathrm{Symp}_0^c\,\Sigma_g^0$, the evaluation of $[p^*k\cdot\widetilde{\mathrm{Flux}_c}]$ on it vanishes. Keeping in mind equations (33) and (32), we can now conclude that

$$j^*\alpha(\tilde{c}_r^*) = 2[p^*k\cdot\widetilde{\mathrm{Flux}_c}](\tilde{c}_r^*) = 2rp^*k(\tilde{c}^*) = 2r.$$

This proves (34) and hence the theorem. q.e.d.

6. Proofs of the main results

In this section we give the proofs of the main results described in Section 2.

Proof of Theorem 4. Here we follow the argument of [**19**] and of our previous paper [**13**] to prove the non-triviality of the cup products $e_1^k\tilde{\alpha}^{(\ell)}$. For this we first observe that, similar to the class e_1, the class $\tilde{\alpha}$ is stable, with respect to g, and also that it is primitive in the following sense. For each k, consider the genus kg surface $\Sigma_{kg,1} = \Sigma_{kg}\setminus\mathrm{Int}\,D^2$ with one boundary component as the boundary connected sum

$$\Sigma_{kg,1} = \Sigma_{g,1}\ \natural\cdots\natural\ \Sigma_{g,1}$$

of k copies of $\Sigma_{g,1} = \Sigma_g\setminus\mathrm{Int}\,D^2$. This induces a homomorphism

$$f_k\colon\ \mathrm{Symp}^c\,\Sigma_g^0\times\cdots\times\mathrm{Symp}^c\,\Sigma_g^0\longrightarrow\mathrm{Symp}^c\,\Sigma_{kg}^0 \tag{35}$$

from the direct product of k copies of the group $\mathrm{Symp}^c\,\Sigma_g^0$ to $\mathrm{Symp}^c\,\Sigma_{kg}^0$. Under this homomorphism we have the equality

$$f_k^*(\tilde{\alpha}) = \tilde{\alpha}\times 1\times\cdots\times 1+\cdots+1\times\cdots\times 1\times\tilde{\alpha},$$

which follows easily from the definition of $\tilde{\alpha}$. Now we can combine Theorem 13 with the above property to show the required assertion in the theorem. This finishes the proof. q.e.d.

Proof of Theorem 5.
The fact that the ideal generated by $\omega_0\wedge H_1(T^{2g};\mathbb{R})$ is contained in the kernel of Flux^* has already been proved in Proposition 8. To show that $\mathrm{Ker}\,\mathrm{Flux}^*$ is precisely this ideal, we use the decomposition (3) of $H^k(T^{2g};\mathbb{R})$ into irreducible summands given in Section 2. It is easy to see that the quotient of this module divided by the ideal generated by $\omega_0\wedge H_1(T^{2g};\mathbb{R})\subset H_3(T^{2g};\mathbb{R})$ is precisely

$$\mathbb{R}\oplus\bigoplus_{k=1}^{g}[1^k]_{\mathbb{R}}.$$

The $\mathbb{R}$-summand in degree 2 corresponds to the class α and its non-triviality has already been shown in Theorem 4. Hence, to prove the assertion, it remains to show that $\mathrm{Flux}^*([1^k])$ is non-trivial for any $k\le g$. The highest weight vector of the irreducible representation

$[1^k]$ is $x_1 \wedge \cdots \wedge x_k$, where $x_1, \dots, x_g, y_1, \dots, y_g$ is a symplectic basis of $H_1(\Sigma_g; \mathbb{Z})$ as before. Now the definition of the flux homomorphism implies that, for any i, there exists an element $\varphi_i \in \mathrm{Symp}_0 \Sigma_g$ such that $\mathrm{Flux}(\varphi_i) = x_i^*$. Here we can choose the support of φ_i to be contained in an arbitrarily small neighbourhood of a simple closed curve which represents the homology class x_i. Then the k elements $\varphi_1, \dots, \varphi_k$ mutually commute because their supports are disjoint. Hence they form a cycle of $\mathrm{Symp}_0 \Sigma_g$ supported on a k-dimensional torus, and the cohomology class $\mathrm{Flux}^*(x_1 \wedge \cdots \wedge x_k) \in H^k(\mathrm{Symp}_0^\delta \Sigma_g; \mathbb{R})$ takes a non-zero value (namely 1) on this cycle. This completes the proof. q.e.d.

Proof of Theorem 6. We consider the extension

$$1 \longrightarrow \mathrm{Ham}\, \Sigma_g \longrightarrow \mathrm{Symp}_0 \Sigma_g \stackrel{\mathrm{Flux}}{\longrightarrow} H^1(\Sigma_g; \mathbb{R}) \longrightarrow 1.$$

Let $\{E^r_{p,q}\}$ be the Hochschild–Serre spectral sequence for its homology. Since $\mathrm{Ham}\, \Sigma_g$ is perfect by Thurston [**25**], see also [**1**], we have

$$E^2_{1,1} \cong H_1(H^1(\Sigma_g; \mathbb{R})^\delta; H_1(\mathrm{Ham}^\delta \Sigma_g; \mathbb{Z})) = 0.$$

Hence the differential $d^2 \colon E^2_{3,0} \to E^2_{1,1}$ vanishes, so that

$$E^3_{3,0} \cong E^2_{3,0} \cong H_3(H^1(\Sigma_g; \mathbb{R})^\delta; \mathbb{Z}) \cong \Lambda^3_{\mathbb{Z}} H^1(\Sigma_g; \mathbb{R}).$$

Similarly $E^2_{2,1} = 0$, so that the differential $d^2 \colon E^2_{2,1} \to E^2_{0,2}$ vanishes and

$$E^3_{0,2} \cong E^2_{0,2} \cong H_0(H^1(\Sigma_g; \mathbb{R})^\delta; H_2(\mathrm{Ham}^\delta \Sigma_g; \mathbb{Z})) \cong H_2(\mathrm{Ham}^\delta \Sigma_g; \mathbb{Z})_{H^1_{\mathbb{R}}}.$$

However, $E^\infty_{3,0} = E^4_{3,0}$ is equal to the image of the homomorphism

$$\mathrm{Flux}_* \colon H_3(\mathrm{Symp}_0^\delta \Sigma_g; \mathbb{Z}) \longrightarrow H_3(H^1(\Sigma_g; \mathbb{R})^\delta; \mathbb{Z}).$$

Now we can conclude that the exact sequence

$$E^\infty_{3,0} = E^4_{3,0} \longrightarrow E^3_{3,0} \longrightarrow E^3_{0,2} = E^2_{0,2}$$

yields an exact sequence

$$(36) \quad H_3(\mathrm{Symp}_0^\delta \Sigma_g; \mathbb{Z}) \longrightarrow H_3(H^1(\Sigma_g; \mathbb{R})^\delta; \mathbb{Z}) \longrightarrow H_2(\mathrm{Ham}^\delta \Sigma_g; \mathbb{Z})_{H^1_{\mathbb{R}}}.$$

By the same computation as above in the dual context of cohomology, we obtain an exact sequence

$$(37) \quad H^2(\mathrm{Ham}^\delta \Sigma_g; \mathbb{R})^{H^1_{\mathbb{R}}} \longrightarrow H^3(H^1(\Sigma_g : \mathbb{R})^\delta; \mathbb{R}) \stackrel{\mathrm{Flux}^*}{\longrightarrow} H^3(\mathrm{Symp}_0^\delta \Sigma_g; \mathbb{R}).$$

Proposition 8 (see also Theorem 5) implies that the continuous cohomology classes

$$\tilde{\omega}_0 \wedge H_1(\Sigma_g; \mathbb{R}) \subset H^3(H^1(\Sigma_g : \mathbb{R})^\delta; \mathbb{R})$$

vanish under the homomorphism Flux^* so that they can be lifted to elements of $H^2(\mathrm{Ham}^\delta \Sigma_g; \mathbb{R})^{H^1_{\mathbb{R}}}$ by (37). Now consider the cycles

$$\omega_0 \wedge H^1(\Sigma_g; \mathbb{R}) \subset \Lambda^3_{\mathbb{Z}} H^1(\Sigma_g; \mathbb{R}) \cong H_3(H^1(\Sigma_g; \mathbb{R})^\delta; \mathbb{Z}),$$

where $\omega_0 \in \Lambda^2_{\mathbb{Z}} H^1(\Sigma_g;\mathbb{R})$, and also their images in $H_2(\mathrm{Ham}^\delta\,\Sigma_g;\mathbb{Z})_{H^1_{\mathbb{R}}}$ in the exact sequence (36). If we consider the Kronecker products of these cycles with the above lifted cohomology classes, we can conclude that $\omega_0 \wedge H^1(\Sigma_g;\mathbb{R})$ maps injectively into $H_2(\mathrm{Ham}^\delta\,\Sigma_g;\mathbb{Z})_{H^1_{\mathbb{R}}}$. This completes the proof. q.e.d.

Remark 20. It would be interesting to obtain explicit group cocycles of $\mathrm{Ham}\,\Sigma_g$ which represent the above degree two cohomology classes. There should also be a relation to the work of Ismagilov [**8**]. We shall pursue this elsewhere.

7. Further discussion

As in Section 3, let $e, v \in H^2(\mathrm{ESymp}^\delta\,\Sigma_g;\mathbb{R})$ be the Euler class and the transverse symplectic class, respectively. By analogy with the definition $e_1 = \pi_*(e^2)$, where

$$\pi_*\colon H^4(\mathrm{ESymp}^\delta\,\Sigma_g;\mathbb{R}) \longrightarrow H^2(\mathrm{Symp}^\delta\,\Sigma_g;\mathbb{R})$$

denotes the integration over the fiber, we can define a cohomology class

$$v_1 \in H^2(\mathrm{Symp}^\delta\,\Sigma_g;\mathbb{R})$$

by setting $v_1 = \pi_*(ev)$. After we conjectured that v_1 is a linear combination of the two classes α and e_1, Kawazumi [**10**] kindly provided a proof. More precisely, he pointed out that the contraction formula, Theorem 6.2 of [**12**], can be adapted to the case of the cohomology class α of the group $\mathrm{ESymp}^\delta\,\Sigma_g$, and that the following equality holds:

$$\alpha = -\pi_*\left((e+v)^2\right) = -e_1 - 2v_1. \tag{38}$$

Since we know by [**13**] that $e^2 \neq 0$, we could also apply integration over the fiber to the cohomology class $e^2 v$ in order to obtain some more cohomology. However, $e^2 v$ vanishes in $H^6(\mathrm{ESymp}^\delta\,\Sigma_g;\mathbb{R})$ by the Bott vanishing theorem.

A more promising approach to find more cohomology for the symplectomorphism groups is the following. Consider the extension

$$1 \longrightarrow \mathrm{Symp}_0\,\Sigma_g \longrightarrow \mathrm{Symp}\,\Sigma_g \xrightarrow{p} \mathcal{M}_g \longrightarrow 1. \tag{39}$$

On the one hand, Theorem 5 shows that we have an injection

$$\bigoplus_{k=1}^{g} [1^k]_{\mathbb{R}} \subset H^*(\mathrm{Symp}_0^\delta\,\Sigma_g;\mathbb{R}). \tag{40}$$

On the other hand, Looijenga [**14**] determined the stable cohomology $H^*(\mathcal{M}_g;V)$ of the mapping class group with coefficients in any irreducible representation V of the algebraic group $Sp(2g;\mathbb{Q})$. In particular, the cohomology groups $H^*(\mathcal{M}_g;[1^k])$ are highly non-trivial. In the spectral sequence for the cohomology of the extension (39), there are many non-trivial E_2-terms

$$E_2^{p,q} = H^p(\mathcal{M}_g; H^q(\mathrm{Symp}_0^\delta \Sigma_g; \mathbb{R})).$$

For example, it was proved in [**22**] that $H^1(\mathcal{M}_g; [1^3]_\mathbb{R}) \cong \mathbb{R}$, and an explicit computation using Looijenga's formula shows $H^2(\mathcal{M}_g; [1^2]_\mathbb{R}) \cong \mathbb{R}$. Hence we have injections

$$\mathbb{R} \subset E_2^{1,3} = H^1(\mathcal{M}_g; H^3(\mathrm{Symp}_0^\delta \Sigma_g; \mathbb{R}))$$
$$\mathbb{R} \subset E_2^{2,2} = H^2(\mathcal{M}_g; H^2(\mathrm{Symp}_0^\delta \Sigma_g; \mathbb{R}))$$

for all sufficiently large g. It seems likely that these two copies of $\mathbb{R}$ survive to the E_∞ term, so that they define certain cohomology classes in $H^4(\mathrm{Symp}^\delta \Sigma_g; \mathbb{R})$. More generally, the summands (40) and the non-trivial cohomology groups $H^*(\mathcal{M}_g; [1^k]_\mathbb{R})$ should give rise to infinitely many cohomology classes in $H^*(\mathrm{Symp}^\delta \Sigma_g; \mathbb{R})$.

Problem 21. Prove that these cohomology classes are non-trivial.

Next, there are completely different candidates for possible classes in $H^*(\mathrm{Symp}^\delta \Sigma_g; \mathbb{R})$ coming from the cohomology of the Lie algebra V_n of formal Hamiltonian vector fields on $\mathbb{R}^{2n}$ first studied by Gelfand, Kalinin and Fuks in [**5**]. This Lie algebra V_n contains $\mathfrak{s}p(2n, \mathbb{R})$ as a subalgebra consisting of vector fields corresponding to linear symplectomorphisms. Let $\mathrm{B}\bar{\Gamma}_2^\omega$ be the Haefliger classifying space for the pseudogroup of local symplectomorphisms of $\mathbb{R}^2$ with respect to the standard symplectic form. Then there is a natural homomorphism

$$H_c^*(V_1; \mathfrak{sp}(2, \mathbb{R})) \longrightarrow H^*(\mathrm{B}\bar{\Gamma}_2^\omega; \mathbb{R}) \tag{41}$$

from the continuous cohomology of V_1 relative to the subalgebra $\mathfrak{sp}(2, \mathbb{R})$ to the real cohomology group of $\mathrm{B}\bar{\Gamma}_2^\omega$.

There is also an obvious continuous mapping

$$K(\mathrm{ESymp}^\delta \Sigma_g, 1) \longrightarrow \mathrm{B}\bar{\Gamma}_2^\omega$$

which classifies the transversely symplectic codimension 2 foliation on the classifying space for the group $\mathrm{ESymp}^\delta \Sigma_g$, that is the total space of the universal foliated Σ_g-bundle over $\mathrm{BSymp}^\delta \Sigma_g$. This induces homomorphisms

$$H^*(\mathrm{B}\bar{\Gamma}_2^\omega; \mathbb{R}) \longrightarrow H^*(\mathrm{ESymp}^\delta \Sigma_g; \mathbb{R}) \xrightarrow{\pi_*} H^{*-2}(\mathrm{Symp}^\delta \Sigma_g; \mathbb{R}), \tag{42}$$

where the last homomorphism is the integration along the fibre. Combining (41) and (42) we obtain a homomorphism

$$H_c^*(V_1; \mathfrak{sp}(2, \mathbb{R})) \longrightarrow H^{*-2}(\mathrm{Symp}^\delta \Sigma_g; \mathbb{R}). \tag{43}$$

Now, Gelfand, Kalinin and Fuks [**5**] found a new cohomology class in $H_c^7(V_1; \mathfrak{sp}(2, \mathbb{R}))$. Later, Metoki [**17**] extended their computation and found another exotic class in $H^9_c(V_1;\mathfrak{sp}(2,\mathbb{R}))$. It seems to be widely believed that there should exist infinitely many such exotic classes.

Problem 22. Study the cohomology classes in $H^*(\mathrm{Symp}^\delta\,\Sigma_g;\mathbb{R})$ induced from exotic classes in $H^{*+2}_c(V_1;\mathfrak{sp}(2,\mathbb{R}))$. In particular, prove that the two elements in $H^5(\mathrm{Symp}^\delta\,\Sigma_g;\mathbb{R})$ and $H^7(\mathrm{Symp}^\delta\,\Sigma_g;\mathbb{R})$ induced from the classes found by Gelfand, Kalinin and Fuks and by Metoki, are non-trivial.

Recall that Harer [**6**] proved that the homology groups of the mapping class groups $\mathcal{M}_g$ stabilize with respect to the genus g in a certain stable range. In view of the fact that all the characteristic classes we introduced in this paper are stable with respect to g, we would like to propose the following problem, although it appears to be beyond the range of available techniques at the moment:

Problem 23. Determine whether the homology groups of $\mathrm{Symp}^\delta\,\Sigma_g$ stabilize with respect to g, or not. In particular, is it true that

$$H_2(\mathrm{Symp}^\delta\,\Sigma_g;\mathbb{Z}) \cong \mathbb{Z}\oplus S^2_{\mathbb{Q}}\mathbb{R}$$

for all $g \geq 3$?

Appendix: Proof of Proposition 2

To prove Proposition 2, observe first that the second exterior power $\Lambda^2_{\mathbb{Z}}\,H^1(\Sigma_g;\mathbb{R})$ over $\mathbb{Z}$ is naturally isomorphic to the same power $\Lambda^2_{\mathbb{Q}}H^1(\Sigma_g;\mathbb{R})$ over $\mathbb{Q}$ because $H^1(\Sigma_g;\mathbb{R})$ is a uniquely divisible abelian group. Choose a Hamel basis a_λ $(\lambda\in A)$ of $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Then we can write

$$H^1(\Sigma_g;\mathbb{R}) = \sum_\lambda a_\lambda H^1(\Sigma_g;\mathbb{Q}).$$

Hence

$$\Lambda^2_{\mathbb{Q}}H^1(\Sigma_g;\mathbb{R}) = \sum_\lambda a_\lambda \Lambda^2_{\mathbb{Q}}H^1(\Sigma_g;\mathbb{Q}) \oplus \sum_{\lambda<\mu} a_\lambda H^1(\Sigma_g;\mathbb{Q})\otimes a_\mu H^1(\Sigma_g;\mathbb{Q}),$$

where we choose a total order in the index set A. Clearly, this is a decomposition of $\mathcal{M}_g$-modules. It is easy to see that the intersection pairing gives rise to an isomorphism

$$\left(\Lambda^2_{\mathbb{Q}}H^1(\Sigma_g;\mathbb{Q})\right)_{\mathcal{M}_g} \cong \mathbb{Q}.$$

We also have

$$\left(H^1(\Sigma_g;\mathbb{Q})\otimes H^1(\Sigma_g;\mathbb{Q})\right)_{\mathcal{M}_g} \cong \left(S^2H^1(\Sigma_g;\mathbb{Q})\oplus \Lambda^2H^1(\Sigma_g;\mathbb{Q})\right)_{\mathcal{M}_g} \cong \mathbb{Q}.$$

Here we have used the fact that

$$\left(S^2H^1(\Sigma_g;\mathbb{Q})\right)_{\mathcal{M}_g} = 0,$$

which is true because the action of $\mathcal{M}_g$ on $S^2H^1(\Sigma_g;\mathbb{Q})$ factors through that of the algebraic group $Sp(2g,\mathbb{Q})$, and $S^2H^1(\Sigma_g;\mathbb{Q})$ is a non-trivial irreducible $Sp(2g,\mathbb{Q})$-module.

Thus we obtain an isomorphism

$$\left(\Lambda^2_{\mathbb{Z}}H^1(\Sigma_g;\mathbb{R})\right)_{\mathcal{M}_g} \cong \left(\sum_{\lambda} a_\lambda \otimes a_\lambda \mathbb{Q}\right) \oplus \left(\sum_{\lambda<\mu} a_\lambda \otimes a_\mu \mathbb{Q}\right).$$

It is easy to see that the right-hand side of the above expression can be naturally identified with $S^2_{\mathbb{Q}}\mathbb{R}$. This completes the proof.

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