

SINGULAR PERTURBATIONS OF MEAN CURVATURE FLOW

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Abstract

We consider a regularization method for mean curvature flow of a submanifold of arbitrary codimension in the Euclidean space, through higher order equations.

We prove that the regularized problems converge to the mean curvature flow for all times before the first singularity.

1. Introduction

It is well known that a smooth compact submanifold of the Euclidean space, flowing by mean curvature, develops singularities in finite time. This is a common aspect of geometric evolutions, and motivates the study of the flow past singularities. Concerning the mean curvature motion, several notions of weak solutions have been proposed, after the pioneering work of Brakke [9], see for instance [1, 3, 4, 7, 8, 10, 11, 15, 20, 21, 22, 23, 27]. We recall that some of these solutions may differ, in particular in presence of the so-called fattening phenomenon (see for instance [6]).

Following a suggestion of Ennio De Giorgi in [12], we consider and study a regularization of mean curvature flow with a singular perturbation of higher order, which could lead to a new definition of generalized solution in any dimension and codimension.

Let us state our main result.

Let $\varphi : M \rightarrow \mathbb{R}^{n+m}$ be a smooth compact n -dimensional immersion in \mathbb{R}^{n+m} . For $k > [n/2] + 2$ (where $[n/2]$ denotes the integer part of $n/2$) and $\varepsilon > 0$ we consider the functional

$$\mathcal{G}_k^\varepsilon(\varphi) = \int_M \left(1 + \varepsilon |A^k(\varphi(p))|^2\right) d\mu(p),$$

where μ is the canonical volume measure associated with the metric g induced on M via the immersion φ . With A^k we denote the k -differential in \mathbb{R}^{n+m} of the function A^M given by

$$A^M(x) = \frac{|x|^2 - [d^M(x)]^2}{2},$$

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where $[d^M]^2$ is the square of the distance function from $\varphi(M)$, which is smooth in a neighborhood of a point of the submanifold without self-intersections. Since locally on M every immersion is an embedding, we can define A^k also at such points. More precisely, the tensor A^k is defined as

$$A_{i_1 \dots i_k}^k = \frac{\partial^k A^M}{\partial x_{i_1} \dots \partial x_{i_k}}$$

for every k -tuple of indexes $i_1, \dots, i_k \in \{1, \dots, n+m\}$.

For simplicity, in the following we always write A^k for $A^k \circ \varphi$.

We remark (see [14, Prop. 2.2 and Cor. 2.4]) that the tensors A^k and $\nabla^{k-3}B$, where B is the second fundamental form of M and ∇ is the covariant derivative associated with the induced metric g , are strictly related; hence, in a way the functional $\mathcal{G}_k^\varepsilon$ is a perturbation of the area functional by a term containing the squares of the high order derivatives of the curvatures of M .

By means of Theorem 4.5 and Theorem 5.9 in [2] and the results of [14], the *gradient flow* associated with the functional $\mathcal{G}_k^\varepsilon$ is given by the PDE system

$$(1.1) \quad \frac{\partial \varphi^\varepsilon}{\partial t} = H + 2\varepsilon k(-1)^k \overbrace{\left(\Delta^M \circ \Delta^M \circ \dots \circ \Delta^M \right)}^{(k-2)\text{-times}} H^\perp + \varepsilon \text{LOT}$$

where H is the mean curvature vector and LOT denotes terms of lower order in the curvature and its derivatives.

We can see then that (1.1) is a *singular perturbation* of the mean curvature flow, and coincides with it when $\varepsilon = 0$. In [14] (see also [13] and [26]) it is proved that for every $\varepsilon > 0$ the system in (1.1) admits a unique smooth solution defined *for all times*; we are then interested in the convergence to the mean curvature flow when $\varepsilon \rightarrow 0$.

Our main result is the following.

Theorem 1.1. *Let $\varphi_0 : M \rightarrow \mathbb{R}^{n+m}$ be a smooth immersion of a compact n -dimensional manifold without boundary. Let $T_{\text{sing}} > 0$ be the first singularity time of the mean curvature flow $\varphi : M \times [0, T_{\text{sing}}) \rightarrow \mathbb{R}^{n+m}$ of M . For any $\varepsilon > 0$ let $\varphi^\varepsilon : M \times [0, +\infty) \rightarrow \mathbb{R}^{n+m}$ be the flows associated with the functionals $\mathcal{G}_k^\varepsilon$, with $k > [n/2] + 2$, all starting from the same initial immersion φ_0 . Then the maps φ^ε converge locally in $C^\infty(M \times [0, T_{\text{sing}}))$ to the map φ , as $\varepsilon \rightarrow 0$.*

Example 1.2. In case of immersed plane curves $\gamma : S^1 \rightarrow \mathbb{R}^2$ ($n = m = 1$) the simplest choice is $k = 3$. Since it turns out that $|A^3|^2 = 3\kappa^2$, where κ is the curvature of γ , in this simple case the approximating functionals read as

$$\int_\gamma (1 + \varepsilon \kappa^2) ds$$

where s is the arclength parameter, and we have replaced 3ε with ε . The regularized system which approximates the curve shortening flow is then

$$(1.2) \quad \frac{\partial \gamma}{\partial t} = (\kappa - 2\varepsilon \partial_s^2 \kappa - \varepsilon \kappa^3) \nu,$$

where ν is a suitable choice of the normal unit vector to the curve.

The crucial point in order to prove Theorem 1.1 is to obtain ε -independent estimates of the curvature and its derivatives in order to gain sufficient compactness properties. We get these by computing the evolution equations satisfied by the L^2 norms of the derivatives of the second fundamental form of the flowing manifolds, and by estimating via Gagliardo–Nirenberg interpolation inequalities.

At the present moment we are not able to characterize the limit of the approximating flows after the first singularity, as the proof of Theorem 1.1 relies heavily on the smoothness of the mean curvature flow in the time interval of existence. Our goal would be to provide some limit flow defined *for all times*, thus providing a new weak definition of solution in any dimension and codimension.

We mention the simplest open problem in defining a limit flow after the first singularity.

It is well known (Gage–Hamilton [16, 17] and Huisken [19]) that a convex curve in the plane (or hypersurface in \mathbb{R}^{n+1}) moving by mean curvature shrinks to a point in finite time, becoming exponentially round. In this case we expect that the approximating flows converge (in a way to be made precise) to such point for every time after the extinction one.

The plan of the paper is the following. In Section 2 we give some notation and we recall the relations between the squared distance function and the second fundamental form and its covariant derivatives. In Section 3, in order to make the line of proof clearer, we work out in detail the ε -independent estimates in the simplest case of plane immersed curves; also in this special case, the result appears to be nontrivial. In Sections 4, 5 and 6 we consider the general case of a n -dimensional submanifold of \mathbb{R}^{n+m} . Section 7 is devoted to showing Theorem 1.1.

We remark here (but we will not discuss such an extension in this paper) that our method works in general for any geometric evolution of submanifolds in a Riemannian manifold till the first singularity time, even when the equations are of high order (like, for instance, in the Willmore flow, see [24, 25, 28]), choosing a regularizing term of appropriately high order.

Finally, it should be noted that, looking at the evolution equation (1.1), these perturbations of the mean curvature flow could be considered, in the framework of geometric evolution problems, as an analogue

of the so-called *vanishing viscosity method*. Indeed, we perturb the mean curvature flow equation with a regularizing higher order term multiplied by a small parameter $\varepsilon > 0$. The lower order terms, denoted by LOT, which appear in (1.1) are due to the fact that we actually perturb the area functional and not directly the evolution equation. However, the analogy with the classical viscosity method cannot be pushed too far. For instance, because of the condition $k > [n/2] + 2$, our regularized equations are of order not less than four (precisely at least four for evolving curves, at least six for evolving surfaces). Moreover, as the Laplacians appearing in equation (1.1) are relative to the induced metric, the system is quasilinear and the lower order terms are nonlinear (polynomial).

2. Notation and Preliminaries

We denote with e_1, \dots, e_{n+m} the canonical basis of \mathbb{R}^{n+m} and with $\langle \cdot, \cdot \rangle$ its standard scalar product.

We let $\varphi : M \rightarrow \mathbb{R}^{n+m}$ be a smooth, compact, n -dimensional, regularly immersed submanifold without boundary. Identifying (at least locally) M with its image $\varphi(M)$, we denote by $T_x M$, $N_x M \subset \mathbb{R}^{n+m}$ respectively, the tangent space and the normal space at $x \in M$.

The *distance function* d^M and the *squared distance function* η^M from M are given by

$$d^M(x) = \inf_{y \in M} |x - y| \quad \text{and} \quad \eta^M(x) = [d^M(x)]^2$$

for any $x \in \mathbb{R}^{n+m}$ (we will drop the superscript M when no ambiguity is possible). In this section we recall some facts from [2] and [14] about the distance function and the relations between the high derivatives of η^M and the second fundamental form of M .

When M is embedded, there exists an open neighborhood $\Omega \subset \mathbb{R}^{n+m}$ of M such that d^M is smooth in $\Omega \setminus M$ and η^M is smooth in the whole of Ω . If M is only immersed, at any point of M we consider the distance (we still use the symbols d^M and η^M for simplicity) from an embedded image of a suitable neighborhood of the point; in this case the regularity properties of d^M and η^M hold in a neighborhood (still denoted by Ω) of such an embedded image.

Clearly, $\eta^M(x) = 0$ and $\nabla \eta^M(x) = 0$ at every $x \in M$; moreover, for every $x \in \Omega$ we have that $x - \nabla \eta^M(x)/2$ is the *unique* point in M of minimum distance from x (the *projection* of x on M), that we denote with $\pi^M(x)$.

Another nice property of the squared distance is that, for every $x \in M$ the Hessian matrix $\nabla^2 \eta^M(x)$ is twice the matrix of orthogonal projection onto the normal space $N_x M$. We will denote respectively with X^M and

X^\perp the orthogonal projections of a vector X on the tangent and normal space of M .

Let $x \in M$ and $X, Y \in T_xM$; the vector valued second fundamental form of M at the point x is given by

$$B(X, Y) = \left(\frac{\partial Y}{\partial X} \right)^\perp,$$

where we extended locally the two vectors X, Y to tangent vector fields on M (the derivative is well defined since X is a tangent vector at x).

If $\{\nu_\alpha\}_{\alpha=1,\dots,m}$ is a local basis of the normal bundle we have

$$B(X, Y) = -\left\langle \frac{\partial \nu_\alpha(x)}{\partial X}, Y \right\rangle \nu_\alpha(x),$$

where here and throughout all the paper we use the convention of summing on repeated indexes.

We will see B as a bilinear map from $T_xM \times T_xM$ to \mathbb{R}^{n+m} ; hence, as a family of $n + m$ bilinear forms $B^k = \langle B, e_k \rangle : T_xM \times T_xM \rightarrow \mathbb{R}$. Moreover, we consider B acting also on vectors of \mathbb{R}^{n+m} , not necessarily tangent, by setting $B(V, W) = B(V^M, W^M) \in N_xM \subset \mathbb{R}^{n+m}$ for every pair $V, W \in \mathbb{R}^{n+m}$. With such a definition, $B_{ij}^k = \langle B(e_i, e_j), e_k \rangle$.

It is well known that B is a symmetric bilinear form and its trace is the *mean curvature* of components $H^k = B_{jj}^k$.

We now introduce the function

$$A^M(x) = \frac{|x|^2 - [d^M(x)]^2}{2},$$

smooth as η^M in the neighborhood Ω of M , and we set

$$A_{i_1 \dots i_k}^M(x) = \frac{\partial^k A^M(x)}{\partial x_{i_1} \dots \partial x_{i_k}}$$

for the derivatives of A^M at every point $x \in \Omega$.

The following proposition (see [2] for the proof) shows the first connection between the second fundamental form and the function A^M (or equivalently, the squared distance function).

Proposition 2.1. *The following relations hold.*

- For any $x \in \Omega$, the point $\nabla A^M(x)$ is the projection point $\pi^M(x)$.
- If $x \in M$, then $\nabla^2 A^M(x)$ is the matrix of orthogonal projection on T_xM .
- For every $x \in M$,

$$\begin{aligned} B_{ij}^k &= A_{ijs}^M(\delta_{ks} - A_{ks}^M), \\ A_{ijk}^M &= B_{ij}^k + B_{jk}^i + B_{ki}^j, \\ H^k &= A_{iik}^M. \end{aligned}$$

We define now the k -derivative tensor $A^k(x)$ acting on k -tuples of vectors $X_i \in \mathbb{R}^{n+m}$, where $X_i = X_i^j e_j$, as follows:

$$A^k(x)(X_1, \dots, X_k) = A_{i_1 \dots i_k}^M(x) X_1^{i_1} \dots X_k^{i_k};$$

notice that the tensors A^k are symmetric.

For notational simplicity, we drop the superscript M on A^k ; for the same reason, we also avoid to indicate the point $x \in M$ in the sequel.

The tensors A^k and $\nabla^{k-3}B$ are strictly related by a recurrence formula proved in [14, Prop. 2.2 and Cor. 2.4].

Remark 2.2. We underline here an important convention used in the paper. Due to the high codimension, we will work with several tensors (like normal vector fields or the second fundamental form B) taking values in \mathbb{R}^{n+m} ; these tensors will be considered as *families* of $n+m$ tensors with values in \mathbb{R} . With this convention, for instance, ∇B means that we are considering the family of covariant derivatives of the tensors B^1, \dots, B^{n+m} , one component at time. Unless otherwise specified, this convention will be used even also for *tangent* vector fields, that is, when X is a tangent vector field, ∇X is not the covariant derivative of X but the derivative of its components in the basis of \mathbb{R}^{n+m} .

Throughout the paper we write $T * S$, following Hamilton [18], to denote a tensor formed by contraction on some indexes of the tensors T and S using the coefficients g^{ij} .

If T_1, \dots, T_l are tensors (here l is not an index of the tensor T), with the symbol

$$\bigotimes_{i=1}^l T_i$$

we mean $T_1 * T_2 * \dots * T_l$.

Definition 2.3. We use the symbol $\mathfrak{p}^s(\nabla^l B)$ for a polynomial (with the $*$ -product) tensor with constant coefficients in the coordinate basis $\partial\varphi/\partial x_i$, the second fundamental form B and its derivatives up to the order l , such that each of its monomials is of the form

$$\bigotimes_{k=1}^N \nabla^{j_k} B \quad \text{with } 0 \leq j_k \leq l \text{ and } N \geq 1$$

or

$$\bigotimes_{k=1}^N \nabla^{j_k} B \frac{\partial\varphi}{\partial x_i} \quad \text{with } 0 \leq j_k \leq l \text{ and } N \geq 1$$

where, in both cases, the *rescaling order* s equals

$$s = \sum_{k=1}^N (j_k + 1).$$

We use instead the symbol $\mathfrak{q}^s(\nabla^l B)$ for a polynomial of the kind $\mathfrak{p}^s(\nabla^l B)$ such that the contraction with the metric is total, both in the covariant and in the \mathbb{R}^{n+m} -indexes.

As the contraction in the ambient space \mathbb{R}^{n+m} “cancels” all the basis elements $\partial\varphi/\partial x_i$ appearing in the formulas, it follows that every monomial of $\mathfrak{q}^s(\nabla^l B)$ has the form

$$\bigotimes_{k=1}^N \nabla^{j_k} B \quad \text{with } 0 \leq j_k \leq l \text{ and } \sum_{k=1}^N (j_k + 1) = s,$$

where the covariant indexes are all completely contracted with g^{ij} .

Remark 2.4. See the paper [26, Sect. 2] for more details on these polynomials and the geometric interpretation of the rescaling order. Notice that, differently from [26], here we need to consider in $\mathfrak{p}^s(\nabla^l B)$ monomials of two types, because of the codimension higher than one.

We advise the reader that the polynomials \mathfrak{p}^s and \mathfrak{q}^s may vary from line to line, and similarly the constants (usually indicated by C).

3. Evolving Plane Curves

Let $\gamma \in C^\infty(\mathbb{S}^1; \mathbb{R}^2)$ be a regular immersed closed curve in the plane \mathbb{R}^2 . Let $\tau = \gamma_x/|\gamma_x| = \gamma_s$ and $\nu = R\tau$ be respectively the tangent and the normal to the curve γ , where R is the counterclockwise rotation of $\pi/2$ in the plane, and $\gamma_x = \partial_x \gamma$.

We recall that $\partial_s = \partial_x/|\gamma_x|$ and

$$(3.1) \quad \partial_s \tau = \kappa \nu, \quad \partial_s \nu = -\kappa \tau$$

where κ is the curvature of γ . In the sequel we let $L = L(\gamma) = \int_\gamma 1 ds$ be the length of the curve γ .

Let us consider the functional

$$\mathcal{G}^\varepsilon(\gamma) = \int_\gamma (1 + \varepsilon \kappa^2) ds,$$

which is obtained from $\mathcal{G}_3^\varepsilon$ (with $n = m = 1$) by replacing 3ε with ε . Set

$$E^\varepsilon = -\kappa + 2\varepsilon \partial_s^2 \kappa + \varepsilon \kappa^3.$$

Then the gradient flow by \mathcal{G}^ε is given by a smooth map $\gamma : \mathbb{S}^1 \times [0, +\infty) \rightarrow \mathbb{R}^2$ which is an immersion for any $t \in [0, +\infty)$, equals a given immersion γ_0 at time $t = 0$, and satisfies

$$(3.2) \quad \partial_t \gamma = -E^\varepsilon \nu,$$

where $\partial_t = \frac{\partial}{\partial t}$. For notational simplicity, we omit the dependence of γ on ε .

Lemma 3.1. *We have*

$$\partial_s \partial_t \gamma = -(\partial_s E^\varepsilon) \nu + \kappa E^\varepsilon \tau,$$

in particular

$$(3.3) \quad \langle \partial_s \partial_t \gamma, \tau \rangle = \kappa E^\varepsilon.$$

Proof. It follows from equations (3.1) and the evolution equation (3.2). q.e.d.

Lemma 3.2. *Let γ be a smooth closed curve; then*

$$(3.4) \quad \frac{1}{L} \leq \frac{1}{4\pi^2} \int_\gamma \kappa^2 ds.$$

Proof. By Borsuk and Schwartz–Hölder inequalities we have

$$2\pi \leq \int_\gamma |\kappa| ds \leq \left(\int_\gamma \kappa^2 ds \right)^{1/2} L^{1/2}.$$

q.e.d.

Lemma 3.3. *The following commutation rule holds:*

$$(3.5) \quad \partial_t \partial_s = \partial_s \partial_t - \kappa E^\varepsilon \partial_s.$$

Proof. Observing that $\frac{\partial_t \partial_x}{|\gamma_x|} = \frac{\partial_x}{|\gamma_x|} \partial_t = \partial_s \partial_t$, we have

$$\begin{aligned} \partial_t \partial_s &= \partial_t \left(\frac{\partial_x}{|\gamma_x|} \right) = \frac{\partial_t \partial_x}{|\gamma_x|} - \frac{\langle \gamma_x, \partial_t \gamma_x \rangle \partial_x}{|\gamma_x|^3} \\ &= \frac{\partial_x}{|\gamma_x|} \partial_t - \left\langle \frac{\gamma_x}{|\gamma_x|}, \frac{\partial_t \gamma_x}{|\gamma_x|} \right\rangle \frac{\partial_x}{|\gamma_x|} \\ &= \partial_s \partial_t - \langle \tau, \partial_s \partial_t \gamma \rangle \partial_s. \end{aligned}$$

Then the commutation rule (3.5) follows from equation (3.3). q.e.d.

Lemma 3.4. *We have*

$$(3.6) \quad \partial_t \kappa = -\partial_s^2 E^\varepsilon - \kappa^2 E^\varepsilon = \partial_s^2 \kappa + \kappa^3 - 2\varepsilon \partial_s^4 \kappa - 6\varepsilon \kappa (\partial_s \kappa)^2 - 5\varepsilon \kappa^2 \partial_s^2 \kappa - \varepsilon \kappa^5.$$

Proof. We have

$$\partial_t \kappa = \partial_t \langle \partial_s \tau, \nu \rangle = \langle \partial_t \partial_s \tau, \nu \rangle.$$

Therefore, using formula (3.5) we have

$$\begin{aligned} \partial_t \kappa &= \langle \partial_s \partial_t \partial_s \gamma, \nu \rangle - \kappa E^\varepsilon \langle \partial_s \tau, \nu \rangle \\ &= \langle \partial_s^2 \partial_t \gamma, \nu \rangle - \langle \partial_s [\kappa E^\varepsilon \partial_s \gamma], \nu \rangle - \kappa^2 E^\varepsilon. \end{aligned}$$

Using the evolution law (3.2) we get

$$\begin{aligned} \langle \partial_s^2 \partial_t \gamma, \nu \rangle &= -\langle \partial_s^2 (E^\varepsilon \nu), \nu \rangle = -\partial_s^2 E^\varepsilon + E^\varepsilon \langle \partial_s (\kappa \tau), \nu \rangle \\ &= -\partial_s^2 E^\varepsilon + \kappa^2 E^\varepsilon. \end{aligned}$$

In addition,

$$\langle \partial_s[\kappa E^\varepsilon \partial_s \gamma], \nu \rangle = \kappa E^\varepsilon \langle \partial_s \tau, \nu \rangle = \kappa^2 E^\varepsilon.$$

Hence $\partial_t \kappa = -\partial_s^2 E^\varepsilon - \kappa^2 E^\varepsilon$ and the last equality in (3.6) follows by expanding E^ε . q.e.d.

Remark 3.5. For $\varepsilon = 0$, formula (3.6) gives the well known evolution equation $\kappa_t = \partial_s^2 \kappa + \kappa^3$, valid for motion by curvature, see [17, Lemma 3.1.6].

We recall now the following interpolation inequalities for closed curves, see [5, p. 93].

Proposition 3.6. *Let γ be a regular closed curve in \mathbb{R}^2 with finite length L . Let u be a smooth function defined on γ , $m \geq 1$ and $p \in [2, +\infty]$. If $n \in \{0, \dots, m - 1\}$ we have the estimates*

$$(3.7) \quad \|\partial_s^n u\|_{L^p} \leq C_{n,m,p} \|\partial_s^m u\|_{L^2}^\sigma \|u\|_{L^2}^{1-\sigma} + \frac{B_{n,m,p}}{L^{m\sigma}} \|u\|_{L^2},$$

where

$$\sigma = \frac{n + 1/2 - 1/p}{m} \in [0, 1)$$

and the constants $C_{n,m,p}$ and $B_{n,m,p}$ are independent of γ .

Clearly inequalities (3.7) hold with uniform constants if applied to a family of curves having lengths uniformly bounded below by some positive value.

Remark 3.7. In the special case $p = +\infty$, we have $\sigma = \frac{n+1/2}{m}$, and

$$\|\partial_s^n u\|_{L^\infty} \leq C_{n,m} \|\partial_s^m u\|_{L^2}^\sigma \|u\|_{L^2}^{1-\sigma} + \frac{B_{n,m}}{L^{m\sigma}} \|u\|_{L^2}.$$

Remark 3.8. In the particular case $n = 0$, $m = 2$, $p = 6$ we get $\sigma = 1/6$ and

$$\|u\|_{L^6} \leq C \|\partial_s^2 u\|_{L^2}^{\frac{1}{6}} \|u\|_{L^2}^{\frac{5}{6}} + \frac{C}{L^{\frac{1}{3}}} \|u\|_{L^2}$$

for some $C > 0$; hence, by means of the Young inequality $|xy| \leq \frac{1}{a}|x|^a + \frac{1}{b}|y|^b$, $1/a + 1/b = 1$, choosing $a = b = 2$, $x = \sqrt{2} \|\partial_s^2 u\|_{L^2}$ and $y = \frac{C^6}{\sqrt{2}} \|u\|_{L^2}^5$, we obtain

$$(3.8) \quad \int_\gamma u^6 ds \leq \int_\gamma (\partial_s^2 u)^2 ds + C \left(\int_\gamma u^2 ds \right)^5 + \frac{C}{L^2} \left(\int_\gamma u^2 ds \right)^3.$$

In the particular case $n = 0$, $m = 1$, $p = 4$ we get $\sigma = 1/4$ and

$$\|u\|_{L^4} \leq C \|\partial_s u\|_{L^2}^{\frac{1}{4}} \|u\|_{L^2}^{\frac{3}{4}} + \frac{C}{L^{\frac{1}{4}}} \|u\|_{L^2};$$

hence, reasoning as before,

$$(3.9) \quad \int_\gamma u^4 ds \leq \int_\gamma (\partial_s u)^2 ds + C \left(\int_\gamma u^2 ds \right)^3 + \frac{C}{L} \left(\int_\gamma u^2 ds \right)^2.$$

We are now ready for the estimates. We recall that

$$(3.10) \quad \partial_t ds = \kappa E^\varepsilon ds = (-\kappa^2 + 2\varepsilon\kappa\partial_s^2\kappa + \varepsilon\kappa^4) ds.$$

Lemma 3.9. *We have*

$$(3.11) \quad \partial_t \int_\gamma \kappa^2 ds = \int_\gamma (-2(\partial_s\kappa)^2 + \kappa^4 - 4\varepsilon(\partial_s^2\kappa)^2 - \varepsilon\kappa^6 - 4\varepsilon\kappa^3\partial_s^2\kappa) ds.$$

Proof. From (3.10) and Lemma 3.4 we get

$$\begin{aligned} & \partial_t \int_\gamma \kappa^2 ds \\ &= 2 \int_\gamma \kappa \partial_t \kappa ds + \int_\gamma (-\kappa^4 + 2\varepsilon\kappa^3\partial_s^2\kappa + \varepsilon\kappa^6) ds \\ &= 2 \int_\gamma \kappa (\partial_s^2\kappa + \kappa^3 - 2\varepsilon\partial_s^4\kappa - 6\varepsilon\kappa(\partial_s\kappa)^2 - 5\varepsilon\kappa^2\partial_s^2\kappa - \varepsilon\kappa^5) ds \\ &\quad + \int_\gamma (-\kappa^4 + 2\varepsilon\kappa^3\partial_s^2\kappa + \varepsilon\kappa^6) ds \\ &= \int_\gamma (2\kappa\partial_s^2\kappa + \kappa^4 - 4\varepsilon\kappa\partial_s^4\kappa - 12\varepsilon\kappa^2(\partial_s\kappa)^2 - 8\varepsilon\kappa^3\partial_s^2\kappa - \varepsilon\kappa^6) ds. \end{aligned}$$

Therefore, integrating by parts, we obtain

$$\begin{aligned} & \partial_t \int_\gamma \kappa^2 ds \\ &= \int_\gamma (-2(\partial_s\kappa)^2 + \kappa^4 - 4\varepsilon(\partial_s^2\kappa)^2 - \varepsilon\kappa^6 - 12\varepsilon\kappa^2(\partial_s\kappa)^2 - 8\varepsilon\kappa^3\partial_s^2\kappa) ds \\ &= \int_\gamma (-2(\partial_s\kappa)^2 + \kappa^4 - 4\varepsilon(\partial_s^2\kappa)^2 - \varepsilon\kappa^6 - 4\varepsilon\kappa^3\partial_s^2\kappa) ds, \end{aligned}$$

where in the last equality we used the fact that $-3 \int_\gamma \kappa^2(\partial_s\kappa)^2 ds = \int_\gamma \kappa^3\partial_s^2\kappa ds$. q.e.d.

Proposition 3.10. *The following estimate holds:*

$$(3.12) \quad \partial_t \int_\gamma \kappa^2 ds \leq C \left(\int_\gamma \kappa^2 ds \right)^3 + C \left(\int_\gamma \kappa^2 ds \right)^5,$$

where C is a constant independent of ε .

Proof. Adding to the right hand side of equation (3.11) the positive quantity $2\varepsilon(\partial_s^2\kappa + \kappa^3)^2$ we get

$$\partial_t \int_\gamma \kappa^2 ds \leq \int_\gamma (-2(\partial_s\kappa)^2 + \kappa^4 - 2\varepsilon(\partial_s^2\kappa)^2 + \varepsilon\kappa^6) ds.$$

Now using inequalities (3.8) and (3.9) we obtain

$$\begin{aligned} \partial_t \int_{\gamma} \kappa^2 &\leq \int_{\gamma} (- (\partial_s \kappa)^2 - \varepsilon (\partial_s^2 \kappa)^2) ds + C\varepsilon \left(\int_{\gamma} \kappa^2 ds \right)^5 \\ &\quad + \frac{C\varepsilon}{L^2} \left(\int_{\gamma} \kappa^2 ds \right)^3 + C \left(\int_{\gamma} \kappa^2 ds \right)^3 + \frac{C}{L} \left(\int_{\gamma} \kappa^2 ds \right)^2 \\ &\leq C \left(\int_{\gamma} \kappa^2 ds \right)^5 + C \left(\int_{\gamma} \kappa^2 ds \right)^3 + \frac{C}{L} \left(\int_{\gamma} \kappa^2 ds \right)^2 \\ &\quad + \frac{C}{L^2} \left(\int_{\gamma} \kappa^2 ds \right)^3 \\ &\leq C \left(\int_{\gamma} \kappa^2 ds \right)^5 + C \left(\int_{\gamma} \kappa^2 ds \right)^3, \end{aligned}$$

where we supposed $\varepsilon < 1$ and in the last inequality we used the geometric estimate (3.4). q.e.d.

We deal now with the higher derivatives of the curvature.

Since here we are working in dimension and codimension one, for the rest of this section all polynomials in the curvature κ and its derivatives are completely contracted, that is they belong to the “family” $\mathfrak{q}^r(\partial_s^l \kappa)$ (see Definition 2.3); moreover, each of their monomials is of the form

$$\prod_{i=1}^N \partial_s^{j_i} \kappa \quad \text{with } 0 \leq j_i \leq l \text{ and } N \geq 1$$

with

$$r = \sum_{i=1}^N (j_i + 1),$$

as the $*$ -product in this case is simply the usual product.

Lemma 3.11. *For any $j \in \mathbb{N}$ the following formula holds:*

$$(3.13) \quad \partial_t \partial_s^j \kappa = \partial_s^{j+2} \kappa + \mathfrak{q}^{j+3}(\partial_s^j \kappa) - 2\varepsilon \partial_s^{j+4} \kappa - 5\varepsilon \kappa^2 \partial_s^{j+2} \kappa + \varepsilon \mathfrak{q}^{j+5}(\partial_s^{j+1} \kappa).$$

Proof. We argue by induction on j .

The case $j = 0$ in (3.13) is equation (3.6), where $\mathfrak{q}^5(\partial_s \kappa) = -6\kappa(\partial_s \kappa)^2 - \kappa^5$.

Suppose that (3.13) holds for $(j-1)$; using the commutation rule (3.5), we get

$$\begin{aligned} \partial_t \partial_s^j \kappa &= \partial_s \partial_t \partial_s^{j-1} \kappa + \kappa(\kappa - 2\varepsilon \partial_s^2 \kappa - \varepsilon \kappa^3) \partial_s^j \kappa \\ &= \partial_s [\partial_s^{j+1} \kappa + \mathfrak{q}^{j+2}(\partial_s^{j-1} \kappa) - 2\varepsilon \partial_s^{j+3} \kappa - 5\varepsilon \kappa^2 \partial_s^{j+1} \kappa + \varepsilon \mathfrak{q}^{j+4}(\partial_s^j \kappa)] \\ &\quad + \mathfrak{q}^{j+3}(\partial_s^j \kappa) + \varepsilon \mathfrak{q}^{j+5}(\partial_s^{j+1} \kappa), \end{aligned}$$

where we expressed $\mathfrak{q}^{j+3}(\partial_s^j \kappa) = \kappa^2 \partial_s^j \kappa$ and $\mathfrak{q}^{j+5}(\partial_s^{j+1} \kappa) = -(2\kappa \partial_s^2 \kappa + \kappa^4) \partial_s^j \kappa$. Hence, we deduce

$$\partial_t \partial_s^j \kappa = \partial_s^{j+2} \kappa + \mathfrak{q}^{j+3}(\partial_s^j \kappa) - 2\varepsilon \partial_s^{j+4} \kappa - 5\varepsilon \kappa^2 \partial_s^{j+2} \kappa + \varepsilon \mathfrak{q}^{j+5}(\partial_s^{j+1} \kappa),$$

which gives the inductive step. q.e.d.

Lemma 3.12. *For any $j \in \mathbb{N}$ we have*

$$\begin{aligned} \partial_t \int_{\gamma} |\partial_s^j \kappa|^2 ds &= -2 \int_{\gamma} |\partial_s^{j+1} \kappa|^2 ds - 4\varepsilon \int_{\gamma} |\partial_s^{j+2} \kappa|^2 ds \\ &\quad + \int_{\gamma} \mathfrak{q}^{2j+4}(\partial_s^j \kappa) ds + \varepsilon \int_{\gamma} \mathfrak{q}^{2j+6}(\partial_s^{j+1} \kappa) ds. \end{aligned}$$

Proof. Using (3.10), (3.13) and integrating by parts we deduce

$$\begin{aligned} (3.14) \quad \partial_t \int_{\gamma} |\partial_s^j \kappa|^2 ds &= 2 \int_{\gamma} \partial_s^j \kappa \partial_t \partial_s^j \kappa ds + \int_{\gamma} |\partial_s^j \kappa|^2 \kappa E^\varepsilon ds \\ &= 2 \int_{\gamma} \partial_s^j \kappa (\partial_s^{j+2} \kappa + \mathfrak{q}^{j+3}(\partial_s^j \kappa)) ds \\ &\quad + \varepsilon \int_{\gamma} 2\partial_s^j \kappa (-2\partial_s^{j+4} \kappa - 5\kappa^2 \partial_s^{j+2} \kappa + \mathfrak{q}^{j+5}(\partial_s^{j+1} \kappa)) ds \\ &\quad - \int_{\gamma} |\partial_s^j \kappa|^2 \kappa (\kappa - 2\varepsilon \partial_s^2 \kappa - \varepsilon \kappa^3) ds \\ &= -2 \int_{\gamma} (|\partial_s^{j+1} \kappa|^2 + \mathfrak{q}^{2j+4}(\partial_s^j \kappa)) ds \\ &\quad - 4\varepsilon \int_{\gamma} (|\partial_s^{j+2} \kappa|^2 + \mathfrak{q}^{2j+6}(\partial_s^{j+1} \kappa)) ds. \end{aligned}$$

q.e.d.

Proposition 3.13. *For any $j \in \mathbb{N}$ we have the ε -independent estimate, for $\varepsilon < 1$,*

$$(3.15) \quad \partial_t \int_{\gamma} |\partial_s^j \kappa|^2 ds \leq C \left(\int_{\gamma} \kappa^2 ds \right)^{2j+3} + C \left(\int_{\gamma} \kappa^2 ds \right)^{2j+5} + C$$

where the constant C depends only on $1/L$.

Proof. We estimate the term $\int_{\gamma} \mathfrak{q}^{2j+4}(\partial_s^j \kappa) ds$ as in [26, Sect. 7]. By definition, we have

$$\mathfrak{q}^{2j+4}(\partial_s^j \kappa) = \sum_m \prod_{l=1}^{N_m} \partial_s^{c_{ml}} \kappa$$

with all the c_{ml} less than or equal to j and

$$\sum_{l=1}^{N_m} (c_{ml} + 1) = 2j + 4$$

for every m . Hence,

$$|\mathfrak{q}^{2j+4}(\partial_s^j \kappa)| \leq \sum_m \prod_{l=1}^{N_m} |\partial_s^{c_{ml}} \kappa|$$

and setting

$$Q_m = \prod_{l=1}^{N_m} |\partial_s^{c_{ml}} \kappa|,$$

we clearly obtain

$$\int_{\gamma} |\mathfrak{q}^{2j+4}(\partial_s^j \kappa)| ds \leq \sum_m \int_{\gamma} Q_m ds.$$

We now estimate any term Q_m via interpolation inequalities. After collecting derivatives of the same order in Q_m we can write

$$(3.16) \quad Q_m = \prod_{i=0}^j |\partial_s^i \kappa|^{\alpha_{mi}} \quad \text{with} \quad \sum_{i=0}^j \alpha_{mi}(i+1) = 2j+4.$$

Then

$$\begin{aligned} \int_{\gamma} Q_m ds &= \int_{\gamma} \prod_{i=0}^j |\partial_s^i \kappa|^{\alpha_{mi}} ds \\ &\leq \prod_{i=0}^j \left(\int_{\gamma} |\partial_s^i \kappa|^{\alpha_{mi} \lambda_i} ds \right)^{\frac{1}{\lambda_i}} \\ &= \prod_{i=0}^j \|\partial_s^i \kappa\|_{L^{\alpha_{mi} \lambda_i}}^{\alpha_{mi} \lambda_i} \end{aligned}$$

where the values λ_i are chosen as follows: $\lambda_i = 0$ if $\alpha_{mi} = 0$ (in this case the corresponding term is not present in the product) and $\lambda_i = \frac{2j+4}{\alpha_{mi}(i+1)}$ if $\alpha_{mi} \neq 0$. Clearly, $\alpha_{mi} \lambda_i = \frac{2j+4}{i+1} \geq \frac{2j+4}{j+1} > 2$ and by the condition in (3.16), $\sum_{\substack{i=0 \\ \lambda_i \neq 0}}^j \frac{1}{\lambda_i} = \sum_{\substack{i=0 \\ \lambda_i \neq 0}}^j \frac{\alpha_{mi}(i+1)}{2j+4} = 1$.

As $\alpha_{mi} \lambda_i > 2$ these values are allowed as exponents p in inequality (3.7) and taking $m = j + 1$, $n = i$, $u = \kappa$, we get

$$\begin{aligned} \|\partial_s^i \kappa\|_{L^{\alpha_{mi} \lambda_i}} &\leq C \|\partial_s^{j+1} \kappa\|_{L^2}^{\sigma_{mi}} \|\kappa\|_{L^2}^{1-\sigma_{mi}} + \frac{C}{L^{(j+1)\sigma_{mi}}} \|\kappa\|_{L^2} \\ &\leq C \left(\|\partial_s^{j+1} \kappa\|_{L^2} + \|\kappa\|_{L^2} \right)^{\sigma_{mi}} \|\kappa\|_{L^2}^{1-\sigma_{mi}} \end{aligned}$$

with

$$\sigma_{mi} = \frac{i + 1/2 - 1/(\alpha_{mi} \lambda_i)}{j + 1},$$

and the constant C depends only on $1/L$.

Multiplying together all the estimates,

$$(3.17) \quad \begin{aligned} \int_{\gamma} Q_m ds &\leq C \prod_{i=0}^j (\|\partial_s^{j+1} \kappa\|_{L^2} + \|\kappa\|_{L^2})^{\alpha_{mi} \sigma_{mi}} \|\kappa\|_{L^2}^{\alpha_{mi}(1-\sigma_{mi})} \\ &\leq C (\|\partial_s^{j+1} \kappa\|_{L^2} + \|\kappa\|_{L^2})^{\sum_{i=0}^j \alpha_{mi} \sigma_{mi}} \|\kappa\|_{L^2}^{\sum_{i=0}^j \alpha_{mi}(1-\sigma_{mi})}. \end{aligned}$$

Then we compute

$$\begin{aligned} \sum_{i=0}^j \alpha_{mi} \sigma_{mi} &= \sum_{i=0}^j \frac{\alpha_{mi}(i+1/2) - 1/\lambda_i}{j+1} \\ &= \frac{\sum_{i=0}^j \alpha_{mi}(i+1/2) - 1}{j+1} \end{aligned}$$

and using again the rescaling condition in (3.16),

$$\sum_{i=0}^j \alpha_{mi} \sigma_{mi} = \frac{4j+6 - \sum_{i=0}^j \alpha_{mi}}{2(j+1)}.$$

Since

$$\begin{aligned} \sum_{i=0}^j \alpha_{mi} &\geq \sum_{i=0}^j \alpha_{mi} \frac{i+1}{j+1} \\ &= \frac{2j+4}{j+1} \end{aligned}$$

we get

$$\begin{aligned} \sum_{i=0}^j \alpha_{mi} \sigma_{mi} &\leq \frac{2j^2 + 4j + 1}{(j+1)^2} \\ &= 2 - \frac{1}{(j+1)^2} < 2. \end{aligned}$$

Hence, we can apply the Young inequality to the product in the last term of inequality (3.17), in order to get the exponent 2 on the first quantity, that is,

$$\begin{aligned} \int_{\gamma} Q_m ds &\leq \frac{\delta_m}{2} (\|\partial_s^{j+1} \kappa\|_{L^2} + \|\kappa\|_{L^2})^2 + C_m \|\kappa\|_{L^2}^{\beta} \\ &\leq \delta_m \int_{\gamma} |\partial_s^{j+1} \kappa|^2 ds + \delta_m \int_{\gamma} \kappa^2 ds + C_m \|\kappa\|_{L^2}^{\beta}, \end{aligned}$$

for arbitrarily small $\delta_m > 0$ and some constant $C_m > 0$. The exponent β is given by

$$\begin{aligned} \beta &= \sum_{i=0}^j \alpha_{mi}(1 - \sigma_{mi}) \frac{1}{1 - \frac{\sum_{i=0}^j \alpha_{mi} \sigma_{mi}}{2}} \\ &= \frac{2 \sum_{i=0}^j \alpha_{mi}(1 - \sigma_{mi})}{2 - \sum_{i=0}^j \alpha_{mi} \sigma_{mi}} \\ &= \frac{2 \sum_{i=0}^j \alpha_{mi} - \frac{4j+6 - \sum_{i=0}^j \alpha_{mi}}{j+1}}{2 - \frac{4j+6 - \sum_{i=0}^j \alpha_{mi}}{2(j+1)}} \\ &= 2 \frac{2(j+1) \sum_{i=0}^j \alpha_{mi} - 4j - 6 + \sum_{i=0}^j \alpha_{mi}}{4j + 4 - 4j - 6 + \sum_{i=0}^j \alpha_{mi}} \\ &= 2 \frac{(2j+3) \sum_{i=0}^j \alpha_{mi} - 2(2j+3)}{\sum_{i=0}^j \alpha_{mi} - 2} \\ &= 2(2j+3). \end{aligned}$$

Therefore, we conclude

$$\int_{\gamma} Q_m ds \leq \delta_m \int_{\gamma} |\partial_s^{j+1} \kappa|^2 ds + \delta_m \int_{\gamma} \kappa^2 ds + C_m \left(\int_{\gamma} \kappa^2 ds \right)^{2j+3}.$$

Repeating this argument for all the Q_m and choosing suitable δ_m whose sum over m is less than one, we conclude that there exists a constant C depending only on $1/L$ and $j \in \mathbb{N}$ such that

$$\int_{\gamma} \mathfrak{q}^{2j+4} (\partial_s^j \kappa) ds \leq \int_{\gamma} |\partial_s^{j+1} \kappa|^2 ds + C \left(\int_{\gamma} \kappa^2 ds \right)^{2j+3} + C.$$

Reasoning similarly for the term $\mathfrak{q}^{2j+6} (\partial_s^{j+1} \kappa)$, we obtain

$$\int_{\gamma} \mathfrak{q}^{2j+6} (\partial_s^j \kappa) ds \leq \int_{\gamma} |\partial_s^{j+2} \kappa|^2 ds + C \left(\int_{\gamma} \kappa^2 ds \right)^{2j+5} + C.$$

Hence, from (3.14) we get

$$\begin{aligned} \partial_t \int_{\gamma} |\partial_s^j \kappa|^2 ds &\leq - \int_{\gamma} |\partial_s^{j+1} \kappa|^2 ds - \varepsilon \int_{\gamma} |\partial_s^{j+2} \kappa|^2 ds \\ &\quad + C \left(\int_{\gamma} \kappa^2 ds \right)^{2j+3} + C\varepsilon \left(\int_{\gamma} \kappa^2 ds \right)^{2j+5} + C \\ &\leq C \left(\int_{\gamma} \kappa^2 ds \right)^{2j+3} + C \left(\int_{\gamma} \kappa^2 ds \right)^{2j+5} + C \end{aligned}$$

when $\varepsilon < 1$ and the constant C depends only on $1/L$. q.e.d.

By means of Propositions 3.10 and 3.13 we have then the following result.

Theorem 3.14. *For any $j \in \mathbb{N}$ there exists a smooth function $Z^j : \mathbb{R} \rightarrow (0, +\infty)$ such that*

$$\partial_t \int_{\gamma} |\partial_s^j \kappa|^2 ds \leq Z^j \left(\int_{\gamma} \kappa^2 ds \right)$$

for every $\varepsilon < 1$ and curve γ evolving by the gradient of the functional \mathcal{G}^ε .

Proof. The statement clearly follows by Propositions 3.10 and 3.13, since by Lemma 3.2 the quantity $1/L$ is controlled by $\int_{\gamma} \kappa^2 ds$.

The smoothness of the functions Z^j is obtained choosing possibly slightly larger constants in inequalities (3.15) and (3.12). q.e.d.

This proposition, like its analogue for the general case (Theorem 6.2), is the key tool in order to get ε -independent compactness estimates. Indeed, for example, one can see that, by an ODE's argument, since all the flows (letting $0 < \varepsilon < 1$ vary) start from a common initial smooth curve, fixing any $j \in \mathbb{N}$, there exists a common positive interval of time such that all the quantities $\|\partial_s^i \kappa\|_{L^2}$, for $i \in \{0, \dots, j\}$ are equibounded. This will allow us to get compactness and C^∞ convergence to the mean curvature flow as $\varepsilon \rightarrow 0$.

4. The General Case

If $k > [n/2] + 2$ it is shown in [14] that for every $\varepsilon > 0$ all the flows φ^ε , associated with the functionals

$$\mathcal{G}_k^\varepsilon(\varphi) = \int_M \left(1 + \varepsilon |A^k|^2 \right) d\mu,$$

and starting from a common initial n -dimensional smooth compact immersed submanifold of \mathbb{R}^{n+m} , are smooth for every positive time.

By means of Theorem 4.5 and Theorem 5.9 in [2] and the results of [14], the *first variation* of the functional $\mathcal{G}_k^\varepsilon$ is given by

$$E^\varepsilon = -H + 2\varepsilon k (-1)^{k-1} \overbrace{\left(\Delta^M \circ \Delta^M \circ \dots \circ \Delta^M \right)}^{(k-2)\text{-times}} H \Big|^\perp + \varepsilon \mathfrak{q}^{2k-3} (\nabla^{2k-5} B)^\perp$$

where $\mathfrak{q}^{2k-3} (\nabla^{2k-5} B)$ takes values in \mathbb{R}^{n+m} .

Here we denote with Δ^M the Laplacian of the smooth compact Riemannian manifold without boundary $M = (M, g)$, where g is the metric induced on M by the immersion.

Then we have a solution of the geometric evolution problem for any initial smooth immersion $\varphi_0 : M \rightarrow \mathbb{R}^{n+m}$, that is, a smooth function $\varphi^\varepsilon : M \times [0, +\infty) \rightarrow \mathbb{R}^{n+m}$ such that

- 1) the map $\varphi^\varepsilon(\cdot, t) : M \rightarrow \mathbb{R}^{n+m}$ is an immersion for every $t \in [0, +\infty)$;
- 2) $\varphi^\varepsilon(p, 0) = \varphi_0(p)$ for every $p \in M$;
- 3) the following parabolic system is satisfied:

$$\begin{aligned} \frac{\partial \varphi^\varepsilon}{\partial t} &= -E^\varepsilon \\ &= H + 2\varepsilon k(-1)^k \overbrace{(\Delta^M \circ \Delta^M \circ \dots \circ \Delta^M)}^{(k-2)\text{-times}} H \Big|^\perp + \varepsilon q^{2k-3} (\nabla^{2k-5} B)^\perp. \end{aligned}$$

5. Evolution of Geometric Quantities

We work out some evolution equations for the geometric quantities under the flow by the gradient of $\mathcal{G}_k^\varepsilon$.

In general, if a family of immersed manifolds $\varphi(\cdot, t) : M \rightarrow \mathbb{R}^{n+m}$ moves by $\partial_t \varphi = -E$, with the field E normal, we have

$$\frac{\partial g_{ij}}{\partial t} = 2 \langle B_{ij}, E \rangle, \quad \frac{\partial g^{ij}}{\partial t} = -2g^{is} \langle B_{sl}, E \rangle g^{lj}.$$

Now for the Christoffel symbols $\Gamma_{ij}^s = \left\langle \frac{\partial^2 \varphi}{\partial x_i \partial x_j}, \frac{\partial \varphi}{\partial x_l} \right\rangle g^{ls}$ we have

$$\begin{aligned} \frac{\partial \Gamma_{ij}^s}{\partial t} &= - \left\langle \frac{\partial^2 E}{\partial x_i \partial x_j}, \frac{\partial \varphi}{\partial x_l} \right\rangle g^{ls} - \left\langle \frac{\partial^2 \varphi}{\partial x_i \partial x_j}, \frac{\partial E}{\partial x_l} \right\rangle g^{ls} \\ &\quad + \left\langle \frac{\partial^2 \varphi}{\partial x_i \partial x_j}, \frac{\partial \varphi}{\partial x_l} \right\rangle \frac{\partial g^{ls}}{\partial t}. \end{aligned}$$

Then, supposing to work in normal coordinates, $\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = B_{ij}$ is a normal vector; hence

$$\begin{aligned} \frac{\partial \Gamma_{ij}^s}{\partial t} &= - \left\langle \frac{\partial^2 E}{\partial x_i \partial x_j}, \frac{\partial \varphi}{\partial x_l} \right\rangle g^{ls} - \left\langle B_{ij}, \frac{\partial E}{\partial x_l} \right\rangle g^{ls} \\ &= - \frac{\partial^2}{\partial x_i \partial x_j} \left\langle E, \frac{\partial \varphi}{\partial x_l} \right\rangle g^{ls} + \left\langle \frac{\partial E}{\partial x_i}, \frac{\partial^2 \varphi}{\partial x_j \partial x_l} \right\rangle g^{ls} \\ &\quad + \left\langle \frac{\partial E}{\partial x_j}, \frac{\partial^2 \varphi}{\partial x_i \partial x_l} \right\rangle g^{ls} + \left\langle E, \frac{\partial^3 \varphi}{\partial x_i \partial x_j \partial x_l} \right\rangle g^{ls} - \left\langle B_{ij}, \frac{\partial E}{\partial x_l} \right\rangle g^{ls} \\ &= \langle B_{jl}, \nabla_i E \rangle g^{ls} + \langle B_{il}, \nabla_j E \rangle g^{ls} + \langle E, \nabla_i B_{jl} \rangle g^{ls} - \langle B_{ij}, \nabla_l E \rangle g^{ls}. \end{aligned}$$

Remark 5.1. By this computation, since the Christoffel symbols are symmetric in the ij -indexes, the covariant 3-tensor $(\nabla B)^\perp$ is symmetric (as in the codimension one case).

Then, we compute the evolution of B,

$$\begin{aligned}
 (5.1) \quad \frac{\partial B_{ij}}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j} - \Gamma_{ij}^s \frac{\partial \varphi}{\partial x_s} \right) \\
 &= -\frac{\partial^2 E}{\partial x_i \partial x_j} - \frac{\partial \Gamma_{ij}^s}{\partial t} \frac{\partial \varphi}{\partial x_s} + \Gamma_{ij}^s \frac{\partial E}{\partial x_s} \\
 &= -\nabla_{ij}^2 E - \frac{\partial \Gamma_{ij}^s}{\partial t} \frac{\partial \varphi}{\partial x_s} \\
 &= -\nabla_{ij}^2 E + \mathfrak{f}_{ij}^s(\nabla B, \nabla E) \frac{\partial \varphi}{\partial x_s},
 \end{aligned}$$

where $\mathfrak{f}_{ij}^s(\nabla B, \nabla E)$ is the polynomial expression above in B, E and their derivatives.

Before proceeding we need the following technical lemma.

Lemma 5.2. *If X is a vector field on M with values in \mathbb{R}^{n+m} , we have*

$$\begin{aligned}
 (\nabla_i X)^\perp &= \nabla_i X^\perp + B_{ij} g^{js} \langle X, \partial \varphi / \partial x_s \rangle + \langle X, B_{ij} \rangle g^{js} \frac{\partial \varphi}{\partial x_s}, \\
 (\nabla_i X)^M &= \nabla_i X^M - B_{ij} g^{js} \langle X, \partial \varphi / \partial x_s \rangle - \langle X, B_{ij} \rangle g^{js} \frac{\partial \varphi}{\partial x_s}.
 \end{aligned}$$

More generally,

$$\begin{aligned}
 (\nabla_{i_1 \dots i_k} X)^\perp &= \nabla_{i_1 \dots i_k} X^\perp + \sum_{j=0}^{k-1} \mathfrak{p}_s^{k-j} (\nabla^{k-j-1} B) \langle \nabla^j X, \partial \varphi / \partial x_s \rangle \\
 &\quad + \sum_{j=0}^{k-1} \left\langle \nabla^j X, \mathfrak{p}_s^{k-j} (\nabla^{k-j-1} B) \right\rangle \frac{\partial \varphi}{\partial x_s}.
 \end{aligned}$$

Proof. We compute

$$\begin{aligned}
 (\nabla_i X)^\perp &= \nabla_i X^\perp + (\nabla_i X^M)^\perp - (\nabla_i X^M)^M \\
 &= \nabla_i X^\perp + B_{ij} g^{js} \langle X, \partial \varphi / \partial x_s \rangle - \left\langle \frac{\partial X^\perp}{\partial x_i}, \frac{\partial \varphi}{\partial x_j} \right\rangle g^{js} \frac{\partial \varphi}{\partial x_s} \\
 &= \nabla_i X^\perp + B_{ij} g^{js} \langle X, \partial \varphi / \partial x_s \rangle + \left\langle X^\perp, \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\rangle g^{js} \frac{\partial \varphi}{\partial x_s} \\
 &= \nabla_i X^\perp + B_{ij} g^{js} \langle X, \partial \varphi / \partial x_s \rangle + \langle X, B_{ij} \rangle g^{js} \frac{\partial \varphi}{\partial x_s}.
 \end{aligned}$$

The second formula is similar.

The third formula follows by induction, once one works in normal coordinates where $\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = B_{ij}$, which is a normal vector. q.e.d.

Remark 5.3. Roughly, this lemma says that the interchange of differentiation and projection operators introduces some extra terms in B,

X and their derivatives, and the order of differentiation of X is lower than the initial one.

This is useful when X is a function of B ; in particular, if X is the mean curvature vector H we have

$$\begin{aligned}
 (5.2) \quad (\nabla_{i_1 \dots i_k} H)^\perp &= \nabla_{i_1 \dots i_k} H^\perp + \sum_{j=0}^{k-1} \mathfrak{p}_s^{k-j} (\nabla^{k-j-1} B) \langle \nabla^j H, \partial\varphi/\partial x_s \rangle \\
 &\quad + \sum_{j=0}^{k-1} \left\langle \nabla^j H, \mathfrak{p}_s^{k-j} (\nabla^{k-j-1} B) \right\rangle \frac{\partial\varphi}{\partial x_s} \\
 &= \nabla_{i_1 \dots i_k} H + \mathfrak{p}^k (\nabla^{k-1} B) + \mathfrak{p}_s^k (\nabla^{k-1} B) \frac{\partial\varphi}{\partial x_s},
 \end{aligned}$$

as H is a normal vector.

Lemma 5.4. *Flowing by the gradient of the functional $\mathcal{G}_k^\varepsilon$, for any $s \in \mathbb{N}$ we have*

$$\begin{aligned}
 (5.3) \quad \frac{\partial}{\partial t} \int_M |\nabla^s B|^2 d\mu &= -2 \int_M |\nabla^{s+1} B|^2 d\mu + \int_M \mathfrak{q}^{2s+4} (\nabla^{s+1} B) d\mu \\
 &\quad - 4\varepsilon k \int_M |\nabla^{s+k-1} B|^2 d\mu + \varepsilon \int_M \mathfrak{q}^{2k+2s} (\nabla^{2k+s-3} B) d\mu
 \end{aligned}$$

and

$$\begin{aligned}
 (5.4) \quad \frac{\partial}{\partial t} \int_M |B|^{2s+2} d\mu &= \int_M \mathfrak{q}^{2s+4} (\nabla^2 B) d\mu + \varepsilon \int_M \mathfrak{q}^{2k+2s} (\nabla^{2k-3} B) d\mu.
 \end{aligned}$$

Moreover, every monomial of the terms \mathfrak{q} in the two formulas contains at least three factors, for the $*$ -product.

Proof. Substituting E^ε in place of E in (5.1) and expanding, after some computation using Lemma 5.2, we get

$$\begin{aligned}
 (5.5) \quad \frac{\partial B}{\partial t} &= 2\varepsilon k (-1)^k \nabla^2 \left(\overbrace{\Delta^M \circ \Delta^M \circ \dots \circ \Delta^M}^{(k-2)\text{-times}} H \right)^\perp + \nabla^2 H \\
 &\quad + \varepsilon \mathfrak{p}^{2k-1} (\nabla^{2k-3} B) + \varepsilon \mathfrak{p}_j^{2k-1} (\nabla^{2k-3} B) \frac{\partial\varphi}{\partial x_j} + \mathfrak{p}_j^3 (\nabla^2 B) \frac{\partial\varphi}{\partial x_j}.
 \end{aligned}$$

Applying now formula (5.2) to the first term on the right hand side of (5.5), we obtain

$$\begin{aligned} \frac{\partial B}{\partial t} &= 2\varepsilon k(-1)^k \nabla^2 \overbrace{\Delta^M \circ \Delta^M \circ \dots \circ \Delta^M}^{(k-2)\text{-times}} H + \nabla^2 H \\ &\quad + \varepsilon \mathbf{p}^{2k-1}(\nabla^{2k-3} B) + \varepsilon \mathbf{p}_j^{2k-1}(\nabla^{2k-3} B) \frac{\partial \varphi}{\partial x_j} + \mathbf{p}_j^3(\nabla^2 B) \frac{\partial \varphi}{\partial x_j}. \end{aligned}$$

We now observe that for any tensor T , we have

$$\begin{aligned} \frac{\partial}{\partial t} \nabla T &= \nabla \frac{\partial}{\partial t} T + T * B * \nabla E^\varepsilon + T * \nabla B * E^\varepsilon \\ &= \nabla \frac{\partial}{\partial t} T + T * \mathbf{p}^3(\nabla B) + \varepsilon T * \mathbf{p}^{2k-1}(\nabla^{2k-3} B). \end{aligned}$$

Then, starting from equation (5.5) and working by induction, again in normal coordinates, we get

$$\begin{aligned} \frac{\partial}{\partial t} \nabla^s B &= 2\varepsilon k(-1)^k \nabla^{s+2} \overbrace{\Delta^M \circ \Delta^M \circ \dots \circ \Delta^M}^{(k-2)\text{-times}} H + \nabla^{s+2} H \\ &\quad + \varepsilon \mathbf{p}^{2k+s-1}(\nabla^{2k+s-3} B) + \mathbf{p}^{s+3}(\nabla^{s+1} B) \\ &\quad + \varepsilon \mathbf{p}_j^{2k+s-1}(\nabla^{2k+s-3} B) \frac{\partial \varphi}{\partial x_j} + \mathbf{p}_j^{s+3}(\nabla^{s+2} B) \frac{\partial \varphi}{\partial x_j}. \end{aligned}$$

We notice that every monomial of the terms $\mathbf{p}^{2k+s-1}(\nabla^{2k+s-3} B)$ and $\mathbf{p}^{s+3}(\nabla^{s+1} B)$ contains at least *two* factors, since for both the difference between the *rescaling order* and the highest possible order of differentiation of B is two.

Hence,

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla^s B|^2 &= 4\varepsilon k(-1)^k \nabla_{i_1 \dots i_s l w}^{s+2} \overbrace{\Delta^M \circ \Delta^M \circ \dots \circ \Delta^M}^{(k-2)\text{-times}} H \\ &\quad \cdot \nabla_{j_1 \dots j_s}^s B_{pz} g^{i_1 j_1} \dots g^{i_s j_s} g^{lp} g^{wz} \\ &\quad + 2 \nabla_{i_1 \dots i_s l w}^{s+2} H \nabla_{j_1 \dots j_s}^s B_{pz} g^{i_1 j_1} \dots g^{i_s j_s} g^{lp} g^{wz} \\ &\quad + [\varepsilon \mathbf{p}^{2k+s-1}(\nabla^{2k+s-3} B) + \mathbf{p}^{s+3}(\nabla^{s+1} B)] * \nabla^s B \\ &= 4\varepsilon k(-1)^k \nabla_{i_1 \dots i_s l w}^{s+2} \overbrace{\Delta^M \circ \Delta^M \circ \dots \circ \Delta^M}^{(k-2)\text{-times}} H \\ &\quad \cdot \nabla_{j_1 \dots j_s}^s B_{pz} g^{i_1 j_1} \dots g^{i_s j_s} g^{lp} g^{wz} \\ &\quad + 2 \nabla_{i_1 \dots i_s l w}^{s+2} H \nabla_{j_1 \dots j_s}^s B_{pz} g^{i_1 j_1} \dots g^{i_s j_s} g^{lp} g^{wz} \\ &\quad + \varepsilon \mathbf{q}^{2k+2s}(\nabla^{2k+s-3} B) + \mathbf{q}^{2s+4}(\nabla^{s+1} B) \end{aligned}$$

and every monomial of the terms $\mathbf{q}^{2k+2s}(\nabla^{2k+s-3} B)$ and $\mathbf{q}^{2s+4}(\nabla^{s+1} B)$ contains at least *three* factors, by the previous remark.

Thus, we have that the time derivative of the quantity $\int_M |\nabla^s B|^2 d\mu$ is given by

$$\begin{aligned} & 4\varepsilon k(-1)^k \int_M \nabla_{i_1 \dots i_s l w}^{s+2} \overbrace{\Delta^M \circ \Delta^M \circ \dots \circ \Delta^M}^{(k-2)\text{-times}} H \\ & \quad \cdot \nabla_{j_1 \dots j_s}^s B_{pz} g^{i_1 j_1} \dots g^{i_s j_s} g^{lp} g^{wz} d\mu \\ & + 2 \int_M \nabla_{i_1 \dots i_s l w}^{s+2} H \nabla_{j_1 \dots j_s}^s B_{pz} g^{i_1 j_1} \dots g^{i_s j_s} g^{lp} g^{wz} d\mu \\ & + \varepsilon \int_M \mathfrak{q}^{2k+2s} (\nabla^{2k+s-3} B) d\mu + \int_M \mathfrak{q}^{2s+4} (\nabla^{s+1} B) d\mu, \end{aligned}$$

where we used that

$$\frac{\partial}{\partial t} d\mu = \langle H, E^\varepsilon \rangle d\mu;$$

hence its contribution can be absorbed in the last two terms (notice that, in doing this, the condition of at least three factors in the monomials of $\mathfrak{q}^{2k+2s} (\nabla^{2k+s-3} B)$ and $\mathfrak{q}^{2s+4} (\nabla^{s+1} B)$ is preserved).

To conclude, we need the following formula, which is obtained by direct computation:

$$(5.6) \quad \nabla_i B_{jl} = \nabla_j B_{il} + (B * B)_z \frac{\partial \varphi}{\partial x_z}.$$

Indeed, the 3-tensor $\nabla_i B_{jl}$ taking values in \mathbb{R}^{n+m} is not symmetric (see also Remark 5.1).

Reasoning then as in [26, Lemmas 7.2, 7.3, Prop. 7.4] (integrating by parts and interchanging derivatives), with the only care that instead of applying formula (2.4) in that paper, we use formula (5.6), we eventually obtain equation (5.3).

The final polynomials $\mathfrak{q}^{2k+2s} (\nabla^{2k+s-3} B)$ and $\mathfrak{q}^{2s+4} (\nabla^{s+1} B)$ still have the three factors property, as every interchange of covariant derivatives in the previous process always introduces an extra lower order term, with one more factor of kind $\nabla^l B$ (since the formula of interchange of covariant derivatives involves the Riemann tensor, that we express in terms of B), which is absorbed in $\mathfrak{q}^{2k+2s} (\nabla^{2k+s-3} B)$ and $\mathfrak{q}^{2s+4} (\nabla^{s+1} B)$.

The same happens when one uses formula (5.6).

Equation (5.4) follows analogously. q.e.d.

6. ε -Independent Estimates

For any integer $s > n/2$ and $\varepsilon > 0$ we set

$$(6.1) \quad Q_\varepsilon^s(t) = \int_M (1 + |\nabla^s B|^2 + |B|^{2s+2}) d\mu, \quad t \in [0, +\infty).$$

Letting $\varepsilon > 0$ vary, we want to study the evolution of Q_ε^s under the flows φ^ε associated with the functionals $\mathcal{G}_k^\varepsilon$ (we recall that $k > [n/2] + 2$).

By Lemma 5.4, we have

$$(6.2) \quad \frac{\partial Q_\varepsilon^s}{\partial t} = -2 \int_M |\nabla^{s+1} B|^2 d\mu - 4\varepsilon k \int_M |\nabla^{s+k-1} B|^2 d\mu \\ + \int_M \mathfrak{q}^{2s+4} (\nabla^{s+1} B) d\mu + \varepsilon \int_M \mathfrak{q}^{2k+2s} (\nabla^{2k+s-3} B) d\mu.$$

In order to deal with the polynomial terms we state in other words Proposition 6.5 in [26] (see all Section 6 in the same paper).

Proposition 6.1. *Choosing some $\delta > 0$ and setting $D = \text{Vol}(M) + \|\mathbf{H}\|_{L^{n+\delta}(\mu)}$, there exists a constant C depending only on $n, m, l, z, j, p, q, r, \delta$ and D , such that for every compact, n -dimensional manifold (M, g) , isometrically immersed in \mathbb{R}^{n+m} , and covariant tensor $T = T_{i_1 \dots i_l}$, the following inequality holds*

$$(6.3) \quad \|\nabla^j T\|_{L^p(\mu)} \leq C \|T\|_{W^{z,q}(\mu)}^\sigma \|T\|_{L^r(\mu)}^{1-\sigma},$$

for all $z \in \mathbb{N}$, $j \in \{0, \dots, z\}$, $p, q, r \in [1, +\infty)$ and $\sigma \in [j/z, 1]$ with the compatibility condition

$$\frac{1}{p} = \frac{j}{n} + \sigma \left(\frac{1}{q} - \frac{z}{n} \right) + \frac{1-\sigma}{r}.$$

If such a condition gives a negative value for p , the inequality holds in (6.3) for every $p \in [1, +\infty)$ on the left hand side.

This clearly implies, looking at the definition of the quantities Q_ε^s , that we can alternatively let the constant C in inequality (6.3) depend on $n, m, s, l, z, j, p, q, r$ and $Q_\varepsilon^{\lfloor n/2 \rfloor + 1}$.

Working now as in Section 7 of [26], with $s > n/2$ fixed, we can interpolate the polynomial terms as follows,

$$\int_M \mathfrak{q}^{2s+4} (\nabla^{s+1} B) d\mu \leq \int_M |\nabla^{s+1} B|^2 d\mu + C_1 (Q_\varepsilon^{\lfloor n/2 \rfloor + 1}) \\ \int_M \mathfrak{q}^{2k+2s} (\nabla^{2k+s-3} B) d\mu \leq 3k \int_M |\nabla^{s+k-1} B|^2 d\mu + C_2 (Q_\varepsilon^{\lfloor n/2 \rfloor + 1})$$

where $C_1(Q_\varepsilon^{\lfloor n/2 \rfloor + 1})$ and $C_2(Q_\varepsilon^{\lfloor n/2 \rfloor + 1})$ are some constants depending only on n, m, k, s and $Q_\varepsilon^{\lfloor n/2 \rfloor + 1}$.

We discuss briefly a key point of such estimates.

By Lemma 5.4, we know that every monomial of $\mathfrak{q}^{2s+4} (\nabla^{s+1} B)$ and $\mathfrak{q}^{2k+2s} (\nabla^{2k+s-3} B)$ contains at least three factors for the $*$ -product.

Then, if a monomial of $\mathfrak{q}^{2s+4} (\nabla^{s+1} B)$ contains a factor $\nabla^{s+1} B$, all the other factors $\nabla^l B$ (which are at least two) must have $0 \leq l < s+1$, because every other factor contributes with $l+1 \geq 1$ to the total sum $2s+4$, giving the rescaling order. Hence, since $\nabla^{s+1} B$ can eventually occur only one time, we can “eliminate” it by means of Young inequality and interpolate.

Regarding the term $\mathfrak{q}^{2k+2s}(\nabla^{2k+s-3}\mathbf{B})$, by the same argument, if some monomial contains at least one factor $\nabla^{s+k-1+j}\mathbf{B}$ for some integer $j \geq 0$, then all the other factors $\nabla^l\mathbf{B}$ (which are at least two) must have $0 \leq l < s + k - j - 1$. Then, integrating repeatedly by parts, we can “move” j derivatives from the factor $\nabla^{s+k-1+j}\mathbf{B}$ to the other factors, obtaining a polynomial $\mathfrak{q}^{2k+2s}(\nabla^{k+s-1}\mathbf{B})$ whose monomials contain at most only one factor $\nabla^{s+k-1}\mathbf{B}$. At this point, we conclude as for the other polynomials, using the Young inequality and interpolation.

Hence, for every $s > n/2$, by (6.2) we have the estimate

$$\frac{\partial Q_\varepsilon^s}{\partial t} \leq - \int_M |\nabla^{s+1}\mathbf{B}|^2 d\mu - \varepsilon k \int_M |\nabla^{s+k-1}\mathbf{B}|^2 d\mu + C(Q_\varepsilon^{[n/2]+1}),$$

where $C = C_1 + \varepsilon C_2$ and the constants C_1, C_2 depend on ε only through $Q_\varepsilon^{[n/2]+1}$.

Theorem 6.2. *For any integer $s > n/2$ there exists a smooth function $Z^s : \mathbb{R} \rightarrow (0, +\infty)$ such that*

$$(6.4) \quad \begin{aligned} \partial_t \int_M (1 + |\nabla^s\mathbf{B}|^2 + |\mathbf{B}|^{2s+2}) d\mu \\ \leq Z^s \left(\int_M (1 + |\nabla^{[n/2]+1}\mathbf{B}|^2 + |\mathbf{B}|^{2[n/2]+4}) d\mu \right) \end{aligned}$$

for every $\varepsilon \in (0, 1)$ and any smooth evolution by the gradient of the functional $\mathcal{G}_k^\varepsilon$.

Proof. The functions Z^s can be clearly chosen to be smooth, possibly slightly enlarging the constants in the last inequality above. q.e.d.

As a consequence we get the following proposition.

Proposition 6.3. *In the same hypotheses of Theorem 6.2, there exists a continuous nonincreasing function $\Theta : (0, +\infty) \rightarrow (0, +\infty)$, independent of $\varepsilon \in (0, 1)$, such that for every $T \geq 0$ and $t \in [T, T + \Theta(Q_\varepsilon^{[n/2]+1}(T))]$ we have $Q_\varepsilon^{[n/2]+1}(t) \leq 2Q_\varepsilon^{[n/2]+1}(T)$.*

Proof. The statement follows by a standard ODE’s argument applied to the differential inequality

$$\begin{aligned} \partial_t \int_M (1 + |\nabla^{[n/2]+1}\mathbf{B}|^2 + |\mathbf{B}|^{2[n/2]+4}) d\mu \\ \leq Z^{[n/2]+1} \left(\int_M (1 + |\nabla^{[n/2]+1}\mathbf{B}|^2 + |\mathbf{B}|^{2[n/2]+4}) d\mu \right) \end{aligned}$$

which is the first case of Theorem 6.2. q.e.d.

In other words, this proposition says that (for ε small enough) we have a uniform control $Q_\varepsilon^{[n/2]+1}(t) \leq C$ in some time interval $[T, T + \Theta]$ (hence also a control on the constants in Proposition 6.1 and on the right hand side of inequalities (6.4) for every $s > n/2$), with C and $\Theta > 0$

depending (smoothly) only on the value of $Q_\varepsilon^{[n/2]+1}$ at the starting time T .

7. Convergence to the Mean Curvature Flow

In this section we prove the convergence of solutions $\varphi^\varepsilon : M \times [0, +\infty) \rightarrow \mathbb{R}^{n+m}$ to (1.1) (all starting from a common immersion φ_0) to the mean curvature flow $\varphi : M \times [0, T_{\text{sing}}) \rightarrow \mathbb{R}^{n+m}$ before its first singularity time. Since we are considering $\varepsilon \rightarrow 0$, we can assume in all this section that $\varepsilon \in (0, 1)$.

We need the following result (which can be proved as in the codimension one case following [26, Proposition 6.3]).

Proposition 7.1. *If a compact, n -dimensional manifold (M, g) , isometrically immersed in \mathbb{R}^{n+m} , satisfies $\text{Vol}(M) + \|\mathbf{H}\|_{L^{n+\delta}(\mu)} \leq D$ for some $\delta > 0$ then for every covariant tensor $S = S_{i_1 \dots i_l}$ we have*

$$\max_M |S| \leq C (\|\nabla S\|_{L^p(\mu)} + \|S\|_{L^p(\mu)}) \quad \text{if } p > n,$$

where the constants depend only on n, m, l, p, δ and D .

Let $Q^s(t)$ denote, for each nonnegative time t before the first singularity, the right hand side of equation (6.1) for the mean curvature flow φ at time t .

Lemma 7.2. *Let ε belong to some interval $(0, \varepsilon_0)$ with $\varepsilon_0 < 1$.*

If the family of immersions $\varphi^\varepsilon(\cdot, T) : M \rightarrow \mathbb{R}^{n+m}$ are bounded in the C^∞ topology, for any $s \in \mathbb{N}$ all the quantities $|\nabla^s \mathbf{B}|$ are uniformly bounded by ε -independent constants $C_s < +\infty$ in the time interval $[T, T + \theta]$, where $\theta = \Theta(\sup_{\varepsilon \in (0, \varepsilon_0)} Q_\varepsilon^{[n/2]+1}(T)) > 0$ and Θ is the function in Proposition 6.3.

Proof. By the C^∞ boundedness of the family $\varphi^\varepsilon(\cdot, T) : M \rightarrow \mathbb{R}^{n+m}$, all the quantities $Q_\varepsilon^{[n/2]+1}(T)$ are equibounded, as $\varepsilon \in (0, \varepsilon_0)$. Since the function Θ is continuous and nonincreasing, setting

$$\theta = \Theta \left(\sup_{\varepsilon \in (0, \varepsilon_0)} Q_\varepsilon^{[n/2]+1}(T) \right) > 0,$$

by Proposition 6.3 there exists a constant $C > 0$ such that $Q_\varepsilon^{[n/2]+1}(t) \leq C$ for every $\varepsilon \in (0, \varepsilon_0)$ and $t \in [T, T + \theta]$.

Then, again by the boundedness of the family $\varphi^\varepsilon(\cdot, T)$ and Theorem 6.2, in the same time interval $[T, T + \theta]$ all the quantities

$$\int_M (1 + |\nabla^s \mathbf{B}|^2 + |\mathbf{B}|^{2s+2}) d\mu,$$

for every $s > n/2$, are bounded by ε -independent constants $C_s < +\infty$. Moreover, all the constants in the interpolation inequalities of Propositions 6.1 and 7.1 are also bounded.

Now, as a first step we see that, by means of Proposition 6.1, we get the following estimates, for every $p \in [2, +\infty)$ and $s > n/2$:

$$\int_M |\nabla^s B|^p d\mu \leq C_{s,p}$$

in the same time interval $[T, T + \theta]$. Here again the constants $C_{s,p} < +\infty$ are ε -independent.

Then, we conclude the proof by means of Proposition 7.1. q.e.d.

Lemma 7.3. *Assume that at time $t = T$ the family of maps $\varphi^\varepsilon(\cdot, T) : M \rightarrow \mathbb{R}^{n+m}$ converges as $\varepsilon \rightarrow 0$ in the C^∞ topology to the immersion $\varphi_T : M \rightarrow \mathbb{R}^{n+m}$. Then the maps φ^ε smoothly converge in the time interval $[T, T + \Theta(Q^{[n/2]+1}(T))]$ to the solution of the mean curvature flow starting from φ_T (here, $Q^{[n/2]+1}(T)$ is the quantity relative to the immersion φ_T).*

Proof. Choosing any $\varepsilon_0 < 1$, by the previous lemma, we have uniform bounds on B and its derivatives in the time interval $[T, T + \theta]$ with $\theta = \Theta(\sup_{\varepsilon \in (0, \varepsilon_0)} Q_\varepsilon^{[n/2]+1}(T))$. Then, there exists $C > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that

$$\left| \frac{\partial \varphi^\varepsilon(p, t)}{\partial t} \right| = |E^\varepsilon(p, t)| < C \quad \forall (p, t) \in M \times [T, T + \theta], \varepsilon \in (0, \varepsilon_0).$$

Now we consider the metric tensors $g_{ij}^\varepsilon(p, t) = \left\langle \frac{\partial \varphi^\varepsilon(p, t)}{\partial x_i}, \frac{\partial \varphi^\varepsilon(p, t)}{\partial x_j} \right\rangle$, and fix a vector $V = \{v^i\} \in T_p M$. Then we have

$$\begin{aligned} \left| \frac{\partial}{\partial t} |V|_{g^\varepsilon(p, t)}^2 \right| &= |\partial_t g_{ij}^\varepsilon(p, t) v^i v^j| = 2 |\langle E^\varepsilon, B_{ij} \rangle v^i v^j| \\ &\leq 2|E^\varepsilon| |B|_{g^\varepsilon(p, t)} |V|_{g^\varepsilon(p, t)}^2 \leq C |V|_{g^\varepsilon(p, t)}^2 \end{aligned}$$

where C does not depend on $\varepsilon \in (0, \varepsilon_0)$.

Then a simple ODE's argument shows that the metrics g_{ij}^ε are all equivalent; more precisely, there exists a positive constant C depending only on φ_T such that

$$(7.1) \quad \frac{\text{Id}}{C} \leq g_{ij}^\varepsilon(p, t) \leq C \text{Id},$$

as matrices.

Moreover, as functions, all the $g_{ii}^\varepsilon = \left| \frac{\partial \varphi^\varepsilon}{\partial x_i} \right|^2$ are equibounded above by a common constant.

Hence, by Ascoli–Arzelà’s Theorem, up to a subsequence, the immersions φ^ε uniformly converge, as $\varepsilon \rightarrow 0$, to some Lipschitz map $\widehat{\varphi} : M \times [T, T + \theta] \rightarrow \mathbb{R}^{n+m}$, which clearly satisfies $\widehat{\varphi}(p, T) = \varphi_T(p)$ for every $p \in M$.

Similarly, as the time derivative of the Christoffel symbols is given by

$$(7.2) \quad \frac{\partial}{\partial t} \Gamma_{ij}^l = \nabla E^\varepsilon * B + E^\varepsilon * \nabla B$$

(see the beginning of Section 5) and all the metrics are equivalent, it follows that all the Christoffel symbols are equibounded. This means that estimating the covariant derivatives is equivalent to estimating the standard derivatives in coordinates; hence, we immediately have $|\partial^s \nabla^l B| \leq C_{s,l}$ for every $s, l \in \mathbb{N}$.

Since

$$\frac{\partial}{\partial t} g_{ij}^\varepsilon = 2 \langle E^\varepsilon, B_{ij} \rangle$$

we get

$$\left| \nabla^s \frac{\partial}{\partial t} g_{ij} \right| \leq C_s,$$

and, by formula (7.2),

$$\left| \nabla^s \frac{\partial}{\partial t} \Gamma_{ij}^l \right| \leq C_s$$

for every $s \in \mathbb{N}$.

Hence, we get $\left| \partial^s \frac{\partial}{\partial t} \Gamma_{ij}^l \right| \leq C_s$ which implies, as the family of maps φ_T^ε is bounded in the C^∞ -topology, that $|\partial^s \Gamma_{ij}^l| \leq C_s$.

Since we already know that $|\varphi^\varepsilon|$ are equibounded, $|\partial \varphi^\varepsilon| \leq C$ and $\partial^2 \varphi^\varepsilon = \Gamma \partial \varphi^\varepsilon + B$, by the estimates $|\partial^s \nabla^l B| \leq C_{s,l}$ we can conclude that the derivatives $|\partial^s \varphi^\varepsilon|$ are all bounded by ε -independent constants C_s , for every $s \in \mathbb{N}$.

Finally, the uniform control on the time and mixed derivatives of φ^ε follows using the evolution equation.

Hence, the sub-convergence $\varphi^\varepsilon \rightarrow \widehat{\varphi}$, as $\varepsilon \rightarrow 0$, is in the C^∞ topology and $\widehat{\varphi}$ is smooth; moreover, the limit metric is positive definite by (7.1).

Passing to the limit in the evolution equation $\partial_t \varphi^\varepsilon = E^\varepsilon$, by the bounds on B and its derivatives, shows that $\widehat{\varphi} : M \times [T, T + \theta] \rightarrow \mathbb{R}^{n+m}$ is the flow by mean curvature of the starting smooth datum φ_T . Since this flow is unique, the entire sequence of maps φ^ε converges to $\widehat{\varphi}$ in the time interval $[T, T + \theta]$.

Choosing now any $\delta > 0$, we take $\varepsilon_0 > 0$ such that

$$\sup_{\varepsilon \in (0, \varepsilon_0)} Q_\varepsilon^{[n/2]+1}(T) - Q^{[n/2]+1}(T) < \delta;$$

this is clearly possible as $\varphi^\varepsilon(\cdot, T)$ converges smoothly to φ_T .

Since the function Θ is nonincreasing (see Lemma 7.2), then we conclude that for any $\delta > 0$ the sequence φ^ε converges to the mean curvature flow of φ_T in the time interval $[T, T + \Theta(Q^{[n/2]+1}(T) + \delta)]$.

Letting δ go to zero, as the function Θ is continuous, we get the thesis. q.e.d.

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. Let T_{\max} be the maximal time such that φ^ε converge to the solution of the mean curvature flow equation φ in $C^\infty(M \times [0, T_{\max}))$ starting at time $t = 0$ from the common immersion φ_0 . Observe that T_{\max} is positive by Lemma 7.3. We want to show that T_{\max} coincides with the first singularity time T_{sing} for φ .

Assume by contradiction that $T_{\max} < T_{\text{sing}}$. Then $\varphi(\cdot, t) \rightarrow \varphi(\cdot, T_{\max})$ in $C^\infty(M)$ as $t \rightarrow T_{\max}$. As the function Θ is continuous, there exists

$$\lim_{t \rightarrow T_{\max}} \Theta(Q^{[n/2]+1}(t)) = \Theta(Q^{[n/2]+1}(T_{\max})) = \tau > 0.$$

Choosing now a time $T \in [T_{\max} - \tau/4, T_{\max})$ such that $\Theta(Q^{[n/2]+1}(T)) > \tau/2$, and applying Lemma 7.3, we see that $\varphi^\varepsilon(\cdot, t)$ converges to the mean curvature flow also for t in the interval $[T, T + \tau/2]$. As $T + \tau/2 \geq T_{\max} - \tau/4 + \tau/2 > T_{\max}$, we have a contradiction. q.e.d.

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