Some Upper Bound Formulas For Ramsey Numbers And Their Applications

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Abstract

In this paper, we will discuss some upper bound formulas for Ramsey numbers. At first, we will derive a more unified form from two parameter formulas which was obtained by Yiru Huang etc. Then, similar to their methods, we introduce two new bound formulas for Ramsey numbers in this paper. At last, as the applications of our new bound formulas, some upper bounds for small Ramsey numbers will be obtained.

1 Introduction

1.1 Ramsey numbers

It is well known that of any six persons in the world, three of them must be either acquainted to each other or totally unfamiliar with. Somehow, there are lots of other similar interesting cases, like of any nine persons in the world, three of them should know each other well, while four of them are strangers; of fourteen people, three of them must know each other, while five are strangers. Now let us generalize these cases to find out the rules behind them.

Let $p \geq 2$, $q \geq 2$ be two given positive integers, define $R(p, q)$ as the minimum value of integer n which satisfies the following statement: if we randomly color each side of an n order's complete graph with red or blue, then there must be a wholly red complete graph of p order K_p or a totally blue complete graph of q order K_q . Such $R(p, q)$ is called the Ramsey number of K_p and K_q.

Here are some Ramsey numbers: $R(3, 3) = 6$, $R(3, 4) = 9$, $R(3, 5) = 14$. And a Ramsey number meets two qualities: 1. For any positive integer $p \geq 2$, $q \geq 2$, we have $R(p, q) = R(q, p)$. 2. For any positive integer $q \geq 2$, we also have $R(2, q) = q$.

So far, only a few Ramsey numbers have been discovered. Except for $R(2, q) = q$, we can easily find out some rather small Ramsey numbers, such as $R(3, 3)$, $R(3, 4)$, $R(3, 5)$, however the exploration of Ramsey numbers still has a long way to go. Let us take a look at $R(3,3) = 6$ first.

To prove that of any six persons in the world, three of them must be either acquainted to each other or totally unfamiliar with. The assignment equals to prove that if we color every side of a six order complete graph K_6 with red or blue, than there must be a single colored triangle. (In short, the assignment means to prove $R(3,3) \leq 6$.)

Demonstration: we choose a vertex v of K_6 randomly, and v has 5 attaching sides, which are either red or blue. According to the average number principle, there are at least 3 same colored sides. And we suppose vv_1, vv_2, vv_3 are red.

Consider the sides attached by v_1, v_2, v_3, if one of them are red, then a red triangle will appear in the graph. On the other hand, if v_1v_2, v_2v_3, v_3v_1 are all blue, then $v_1v_2v_3$ is a blue triangle. Thus proves that there must be single colored triangles in the graph. (In fact, we can prove the two-colored K_6 must have at least 2 single colored triangles.)

Since it is easy to find a graph of 5 order which contains neither K_3 nor E_3 (a graph has only 3 vertices, but no sides), we can say $R(3,3) > 5$. Thus $R(3,3) = 6$.

For some small Ramsey numbers $R(k,l)$, S.P. Radziszowski has compiled all the results we have known by now in his survey paper [7].

From S.P. Radziszowski's surgery paper, we only know 9 exact values for Ramsey numbers. For other Ramsey numbers, despite don't know their exact values, people are interested in estimating their bounds. This paper will only focus on the upper bounds for Ramsey numbers.

1.2 Classical upper bound formula

For the general upper bound formulas of Ramsey numbers, we have know the following classical results:

1. In 1935, *P.Erodös* and G. Szekeres got

$$R(k,l) \leq R(k-1,l) + R(k,l-1).$$

2. In 1968, Walker[8] obtained

$$R(k,k) \leq 4R(k,k-2) + 2.$$

3. In 1983, Griggs [1] obtained

$$R(3,n) \leq Cn^2/ln(n), for \quad n \geq 15$$

where C is some positive constant.

Besides the above formula, many upper bound formulas appear in recent decades, we recommend the survey paper [7] for a review.

1.3 Main results

In this paper, we will introduce the following two new upper bound formulas

Theorem 1.1. *Let $m \geq 4$, $n \geq 4$, $R(G_1^{m-2}, G_2) \leq a+1$, $R(G_1, G_2^{n-2}) \leq b+1$, $R(G_1^{m-1}, G_2) \leq c+1$ and $R(G_1, G_2^{n-1}) \leq d+1$, then we have the following upper bounds for $R(G_1, G_2)$:*

$$R(G_1, G_2) \leq 3c + b + 2$$

Furthermore, if $b \geq a$, then we have

$$R(G_1, G_2) \leq \frac{5}{2} + \frac{a}{2} + \frac{\sqrt{1 + 2a + a^2 + 12cd + 4bd - 4ad}}{2}$$

The notations $G_1, G_2, G_1^{m-2}, G_2^{n-2}$ will be explained in section 2. As the application, we get the upper bounds $R(5, 13) \leq 1139$, $R(5, 15) \leq 1878$, $R(6, 13) \leq 3705$, these are the same best upper bounds we have known by now showed in [7].

Besides, we also have $R(10, 13) \leq 145975$, $R(10, 14) \leq 233569$, and $R(10, 15) \leq 388645$. It is clear that these upper bounds are better than which derived from the formula

$$R(k, l) \leq R(k-1, l) + R(k, l-1)$$

Thus, they can be considered as the new upper bounds.

2 Upper bound formulas for Ramsey numbers

In this section, at first, we will introduce some parameter theorems obtained by Y. Huang etc. in recent years [5, 3, 6]. Then, we will derive two upper bound formulas by the similar methods.

We need some notations followed from [3], the Ramsey number $R(G_1, G_2)$ for two given graphs G_1, G_2 is the smallest positive integer $p+1$ such that for any graph G of order $p+1$ either G contains G_1 or G^c contains G_2, where G^c is the complement of G. A graph H of order p is called a $(G_1, G_2; p)$-Ramsey graph if H does not contain G_1 and H^c does not contain G_2. Thus $R(K_m, K_n)$ is $R(m, n)$ described in the introduction. When an edge e is removed from G, we denote the graph by $G - e$. Let d_i be the degree of vertex i in G of order p, and let $\overline{d}_i = p - 1 - d_i$, where $1 \leq i \leq p$. And let $f(K_r)$ ($g(K_r)$,resp.) denote the number of K_r in G(G^c,resp.). We always assume that $G_1 = K_m$ or $K_m - e$, $G_2 = K_n$ or $K_n - e$, $G_1^{m-i} = K_{m-i}$ or $K_{m-i} - e$, $G_2^{n-i} = K_{n-i}$ or $K_{n-i} - e$.

Moreover, we need two lemmas:

Lemma 2.1. *[2] For any graph G of order p, we denote i as its vertex, d_i as the degree of vertex i, $\overline{d}_i = p - 1 - d_i$, $1 \leq i \leq p$, Then*

$$f(K_3) + g(K_3) = \binom{p}{3} - \frac{1}{2}\sum_{i=1}^{n} d_i \overline{d}_i \tag{2.1}$$

Lemma 2.2. *[6] For any $(G_1, G_2; p)$-Ramsey graph G of order p, the following inequalities hold:*

$$(s+1) f(K_{s+1}) \leq f(K_s)[R(G_1^{m-s}, G_2) - 1] \tag{2.2}$$

$$(t+1) g(K_{t+1}) \leq g(K_t)[R(G_1, G_2^{n-t}) - 1] \tag{2.3}$$

By the above two lemmas, we have the following main theorem which appears in series papers of Y. Huang etc. [5, 3, 6].

Theorem 2.3. *[3] Let $m \geq 4$, $n \geq 4$, $R(G_1^{m-2}, G_2) \leq a+1$, $R(G_1, G_2^{n-2}) \leq b+1$, $R(G_1^{m-1}, G_2) \leq c+1$ and $R(G_1, G_2^{n-1}) \leq d+1$, then we have the following parameter formulas for $R(G_1, G_2)$:*

(i)Let $t \geq 0$, $A = 2c - 2 - \frac{1}{3}(4a+2b)$, $B = (a+b+2)^2 + \frac{1}{3}(b-a)^2$, and $F(t) = a+b+4-t+\sqrt{\frac{4}{3}t^2 + 2At + B}$, Then $R(G_1, G_2) \leq F(t)$. Moreover, when $4B - 3A^2 > 0$, we have

$$R(G_1, G_2) \leq \begin{cases} a+b+4+\dfrac{3A}{4}+\dfrac{1}{4}\sqrt{-3A^2+4B}, & t_0 = \frac{3}{4}(\sqrt{4B-3A^2}-A) > 0, \\ a+b+4+\sqrt{B}, & t_0 = \frac{3}{4}(\sqrt{4B-3A^2}-A) \leq 0. \end{cases}$$

(ii) Let two parameters $x \in (0,3)$, $y \in \mathbb{R}$. And let

$$f(x,y) = C + \sqrt{C^2 - D}, g(x,y) = C - \sqrt{C^2 - D},$$
$$C = \frac{3(y+a-b) - 2(1+a)x}{9-4x}, D = \frac{(3-x)(y+a-b)^2 + xy^2}{(3-x)(9-4x)}$$

Then

(a)$R(G_1, G_2) \geq 2 + f(x,y)$ or $R(G_1, G_2) \leq 2 + g(x,y)$ if $0 < x < \frac{9}{4}$;

(b)$R(G_1, G_2) \leq 2 + f(x,y)$, if $x \in (\frac{9}{4}, 3)$;

(c)$R(G_1, G_2) \leq a+b+4+\frac{2}{3}\sqrt{(a+2b+3)(2a+b+3)+(b-a)^2}$ if $x = \frac{9}{4}$.

Remark 2.4. In fact, theorem 2.3 is a unified form for the main results of paper [3]. The theorem 1 in [3] is the first part of theorem 2.3, and theorem 2 in [3] is a special case of second part of theorem 2.3.

Proof. We will follow the method introduced in paper [3]. Let $p = R(G_1, G_2) - 1$, and G is a $(G_1, G_2; p)$-Ramsey graph G. Let $s = 2$ in Lemma 2, we have

$$3f(K_3) \leq \frac{1}{2} a \sum_{i=1}^{p} d_i,$$

$$3g(K_3) \leq \frac{1}{2} b \sum_{i=1}^{p} \overline{d}_i$$

where we have used $f(K_2) \leq \frac{1}{2}\sum_{i=1}^{p} d_i$, $g(K_2) \leq \frac{1}{2}\sum_{i=1}^{p} \overline{d}_i$, and $R(G_1^{m-2}, G_2) \leq a+1$, $R(G_1, G_2^{n-2}) \leq b+1$.

By lemma 1, we have,

$$3\left(\binom{p}{3} - \frac{1}{2}\sum_{i=1}^{p} d_i \overline{d}_i\right) \leq \frac{1}{2}\left(a\sum_{i=1}^{p} d_i + b\sum_{i=1}^{p} \overline{d}_i\right)$$

After recollecting, by $d_i + \overline{d}_i = p - 1$, then

$$p(p-1)(p-2-a) \le \sum_{i=1}^{p}(p-1-d_i)(3d_i+b-a) \le \sum_{i=1}^{p}(-3\overline{d}_i^2 + (3p-3+b-a)\overline{d}_i) \tag{2.5}$$

The next thing we need to do is to get an upper bound of $-3\overline{d}_i^2 + (3p-3+b-a)\overline{d}_i$, then by (2.5), through solving the inequality of (2.5), we can obtain an upper of p.

At first, we note that when $0 \le \overline{d}_i \le p-1$,

$$-3\overline{d}_i^2 + (3p-3+b-a)\overline{d}_i \le \frac{(3p-3+b-a)^2}{12}.$$

Thus, by (2.5), we have

$$(p-1)(p-2-a) \le \frac{(3p-3+b-a)^2}{12} \tag{2.6}$$

Solving the inequality (2.6),

$$3+a+b-\sqrt{B} \le p \le 3+a+b+\sqrt{B}$$

where B is defined above. Hence

$$4+a+b-\sqrt{B} \le R(G_1,G_2) \le 4+a+b+\sqrt{B} \tag{2.7}$$

which is a bound of $R(G_1,G_2)$. However, unfortunately, $4+a+b-\sqrt{B} \le 2$ and $4+a+b+\sqrt{B} \ge 2a+2b+6$, so the bounds of $R(G_1,G_2)$ obtained from (2.7) is not better than the results we have known in [7].

Now, we will introduce the parameters to deal with right hand side of (2.5). In paper [3] has used the following two methods:

(i)let $t \ge 0$, then

$$\begin{aligned} p(p-1)(p-2-a) &\le \sum_{i=1}^{p}(-3\overline{d}_i^2 + (3p-3+b-a)\overline{d}_i) \\ &\le \sum_{i=1}^{p}(-3\overline{d}_i^2 + (3p-3+b-a+t)\overline{d}_i - t(p-1) + tc) \\ &\le p\left(\frac{1}{12}(3p-3+b-a+t)^2 - t(p-1) + tc\right) \end{aligned}$$

In the last "$\le$", we have used the fact [4] that

$$d_i \le c, \overline{d}_i \le d \tag{2.8}$$

Thus, we have

$$a+b+3-t-\sqrt{\frac{4}{3}t^2+2At+B} \le p \le a+b+3-t+\sqrt{\frac{4}{3}t^2+2At+B}$$

i.e.

$$a+b+4-t-\sqrt{\frac{4}{3}t^2+2At+B} \le R(G_1,G_2) \le a+b+4-t+\sqrt{\frac{4}{3}t^2+2At+B}$$

Let $G(t)=a+b+4-t-\sqrt{\frac{4}{3}t^2+2At+B}$ and $F(t)=a+b+4-t+\sqrt{\frac{4}{3}t^2+2At+B}$.

In order to get a better upper bound of $R(G_1,G_2)$, we need to estimate the minimal value of F(t) when $t \ge 0$.

Calculate the derivation of t,

$$\frac{d}{dt}F(t) = -1+\frac{24t+18A}{6\sqrt{12t^2+18At+9B}}$$

$$\frac{d^2}{dt^2}F(t) = \frac{-(24t+18A)^2+48(12t^2+18At+9B)}{12(12t^2+18At+9B)}$$

When, $4B-3A^2 \ge 0$, the may solution of $\frac{d}{dt}F(t)=0$ in $t \ge 0$ is $t_0=\frac{3}{4}(\sqrt{4B-3A^2}-A)$, and $\frac{d^2}{dt^2}F(t_0) = \frac{1}{3\sqrt{-3A^2+4B}} > 0$, hence t_0 is the minimal point of F(t). Thus, if $t_0 = \frac{3}{4}(\sqrt{4B-3A^2}-A) \ge 0$, we have

$$R(G_1,G_2) \le F(t_0) = a+b+4+\frac{3A}{4}+\frac{\sqrt{-3A^2+4B}}{4}$$

We obtain the part (i) of theorem 2.1.

(ii)In the second way, we introduce two parameters x, y,

$$\begin{aligned} p(p-1)(p-2-a) &\le \sum_{i=1}^{p}(-3\overline{d}_i^2+(3p-3+b-a)\overline{d}_i) \\ &\le \sum_{i=1}^{p}(-x\overline{d}_i^2+(3p-3+b-a-y)\overline{d}_i-(3-x)\overline{d}_i^2+y\overline{d}_i) \\ &\le \frac{p}{4x}(3p-3+b-a-y)^2+\frac{y^2p}{4(3-x)} \end{aligned}$$

Then, by an easy discussion, we have the part 2 of theorem 2.1. □

Motivated by the proof of theorem 2.3, now we will give two upper bound formulas. Our main results is

Theorem 2.5. *Let* $m \ge 4$, $n \ge 4$, $R(G_1^{m-2},G_2) \le a+1$, $R(G_1,G_2^{n-2}) \le b+1$, $R(G_1^{m-1},G_2) \le c+1$ *and* $R(G_1,G_2^{n-1}) \le d+1$, *then we have the following upper bounds for* $R(G_1,G_2)$*:*

$$R(G_1,G_2) \le 3c+b+2 \tag{2.9}$$

Furthermore, if $b \geq a$, then we have

$$R(G_1, G_2) \leq \frac{5}{2} + \frac{a}{2} + \frac{\sqrt{1 + 2a + a^2 + 12cd + 4bd - 4ad}}{2} \tag{2.10}$$

Proof. In the proof of above theorem 2.1, we have got

$$p(p-1)(p-2-a) \leq \sum_{i=1}^{p}(p-1-d_i)(3d_i + b - a)$$

by (2.8), we know $0 \leq d_i \leq c$, hence

$$p(p-1)(p-2-a) \leq p(p-1)(3c + b - a)$$

i.e.

$$p \leq 3c + b + 1 \tag{2.11}$$

by the definition p and $R(G_1, G_2)$, (2.11) is just (2.9) in theorem 2.3.

Furthermore, if $b \geq a$, then $(3d_i + b - a) \geq 0$, thus by (2.8)

$$\begin{aligned} p(p-1)(p-2-a) &\leq \sum_{i=1}^{p} \overline{d}_i(3d_i + b - a) \\ &\leq \sum_{i=1}^{p} d(3d_i + b - a) \\ &\leq \sum_{i=1}^{p} d(3c + b - a) \\ &= pd(3c + b - a) \end{aligned}$$

so, we have

$$p^2 - (a+3)p + a + ad + 2 - 3cd - bd \leq 0 \tag{2.12}$$

solving (2.12),

$$\frac{3}{2} + \frac{a}{2} - \frac{\sqrt{1 + 2a + a^2 + 12cd + 4bd - 4ad}}{2} \leq p \leq \frac{3}{2} + \frac{a}{2} + \frac{\sqrt{1 + 2a + a^2 + 12cd + 4bd - 4ad}}{2}$$

Therefore, we get the upper bound

$$R(G_1, G_2) \leq \frac{5}{2} + \frac{a}{2} + \frac{\sqrt{1 + 2a + a^2 + 12cd + 4bd - 4ad}}{2}$$

□

3 Some calculations

In this section, we will discuss the applications of our new upper bound formulas introduced in this paper. In fact, we have made a computer program to calculate the upper bounds from our new upper bound formulas combining the result of the latest version of Radziszowski's survey paper [7].

For the classical two color Ramsey numbers $R(k, l)$, we have the following results calculated from our upper bound formula (2.10) in theorem 2.5.

$R(5, 13) \leq 1139$, $R(5, 15) \leq 1878$, $R(6, 13) \leq 3705$, these are the same best upper bounds we have known by now showed in [7].

Furthermore, from our calculation, we also have that

$R(10, 13) \leq 145975$, $R(10, 14) \leq 233569$, and $R(10, 15) \leq 388645$. It is clear that these upper bounds are better than which derived from the classical upper bound formula

$$R(k, l) \leq R(k-1, l) + R(k, l-1)$$

Thus, they can be considered as the new upper bounds.

4 Further directions

In the last section, we will point out some further researches in the future.

(1). How to get better lower bound formulas by the methods showed in this paper?

In fact, in the proof of theorem 2.1, we have gotten $G(t) \leq R(G_1, G_2)$, thus to calculate the maximal value of $G(t)$, we can get some lower bound of Ramsey numbers.

(2). How to generalize the upper bound formula to multi-Ramsey numbers $R(G_1, .., G_n)$'s case?

It is interesting to generalize lemmas used in this paper to multi-color's case, and then to derive the same type upper bound formulas for multi-Ramsey numbers.

(3). How to generalize the bound formula to other types graphs Ramsey number?

In this paper, we only consider the particular graphs K_n and $K_n - e$. In [7], many small Ramsey numbers results for other types graphs are also listed. It is also interesting to generalize the method in this paper to these graphs.

References

[1] J. Griggs *An Upper Bound on the Ramsey Numbers R(3, k)*, J. Comb. Theory. A . 35(1983) 145-153.

[2] A. Goodman *On sets of acquaintances and strangers at any party*, Amer.MAth.Monthly 66(1959) 778-783.

[3] Y. Huang, Y. Wang, W. Sheng, J. Yang, K. Zhang and J Huang *New upper bound formula with parameters for Ramsey numbers*, Discrete.Math. 307(2007) 760-763.

[4] Y. Huang, J. Yang and Y. Wang *New upper and lower bounds for Ramsey numbers with two parameters*, OR Transactions, Vol.9 No.3(2005), 45-48.

[5] Y. Huang, J. Yang and K. Zhang *A note on Ramsey numbers with two parameters*, European Journal of Combinatorics.

[6] Y. Huang and K. Zhang *A new upper bound formula for two color classical Ramsey numbers*, JCMCC 28(1998), 347-350.

[7] S. Radziszowski *Small Ramsey numbers*, The Electronic Journal of Combinatorics 1. Latest version: Augest 4, 2009.

[8] K. Walker *Dichromatic graphs and Ramsey numbers*, Journal of Combinatorial Theory. 5(1968) 238-243.