J. DIFFERENTIAL GEOMETRY
78 (2008) 13-32

CALABI-YAU CONNECTIONS WITH TORSION ON TORIC BUNDLES

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Abstract

We find sufficient conditions for principal toric bundles over compact Kähler manifolds to admit Calabi-Yau connections with torsion, as well as conditions to admit strong Kähler connections with torsion. With the aid of a topological classification, we construct such geometry on $(k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$ for all $k \geq 1$.

1. Introduction

In this article, we investigate a construction of Hermitian connections with special holonomy on Hermitian non-Kählerian manifolds. On Hermitian manifolds, there is a one-parameter family of Hermitian connections canonically depending on the complex structure J and the Riemannian metric g [**22**]. Among them is the Chern connection on holomorphic tangent bundles. In this paper, we are interested in what physicists call the Kähler-with-torsion connection (a.k.a. KT connection) [**41**]. It is the unique Hermitian connection whose torsion tensor is totally skew-symmetric when 1-forms are identified to their dual vectors with respect to the Riemannian metric. If T is the torsion tensor of a KT connection, it is characterized by the identity [**22**]

$$g(T(A,B),C) = dF(JA,JB,JC)$$

where F is the Kähler form; $F(A,B) = g(JA,B)$, and A, B, C are any smooth vector fields.

As a Hermitian connection, the holonomy of a KT connection is contained in the unitary group $\mathrm{U}(n)$. If the holonomy of the KT connection is reduced to $\mathrm{SU}(n)$, the Hermitian structure is said to be Calabi-Yau with torsion (a.k.a. CYT).

Such geometry in physical context was considered first by A. Strominger [**41**] and C. Hull [**32**]. More recently CYT structures on non-Kähler manifolds attracted attention as models for string compactifications. Many examples were found [**5**], [**2**], [**14**], [**13**], [**24**], [**26**]. This led

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The second author was partially supported by NSF DMS-0333172. The third author was partially supported by NSF DMS-0204002.

Received 07/25/2005.

to a conjecture [**29**] that *any* compact complex manifold with vanishing first Chern class admits a Hermitian metric and connection with totally skew-symmetric torsion and (restricted) holonomy in $\mathrm{SU}(n)$. Counterexamples to this conjecture appear in [**17**]. There are also examples of CYT connections unstable under deformations. These two features of CYT connections are in sharp contrast to well known moduli theory of Calabi-Yau (Kähler) metric. In this paper, we shall see more examples relevant to the moduli problem for CYT connections.

Below we begin our general construction on toric bundles on Hermitian manifolds. Inspired by the recent results of [**26**], we will focus our attention to two-dimensional bundles of torus over compact complex surfaces. The main technical observation is the following.

Proposition. *Suppose that X is a compact Kähler manifold. Let the harmonic part of the Ricci form of its Kähler metric be ρ^{har}. Suppose that M is a principal toric bundle with curvature $(\omega_1, \dots, \omega_{2k})$ and that all curvature forms are harmonic type $(1,1)$-forms. Then M admits a KT connection with restricted holonomy in $\mathrm{SU}(n)$ if $\rho^{har} = \sum_{\ell=1}^{2k}(\Lambda\omega_\ell)\omega_\ell$, where Λ is a contraction with respect to the Kähler form on X.*

Combining the above technical observation with various algebraic geometrical and algebraic topological results, we find a large class of examples of compact simply-connected CYT manifolds. A slight modification of our construction produces strong KT (a.k.a. SKT) structures. SKT structures appear in physics literature and refer to Hermitian structures with $dd^cF = 0$ [**18**] [**31**]. Combining Theorem 13 on a construction of CYT structures and Theorem 15 on a construction of SKT structures, we establish the following observation.

Theorem. *For any positive integer $k \geq 1$, the manifold $(k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$ admits a CYT structure and a SKT structure.*

The CYT structure and SKT structure in the above theorem do not necessarily coincide. A CYT structure that also satisfies $dd^cF = 0$ is called strong CYT. It remains a challenge to see if these manifolds admit strong CYT structures. On the other hand, the existence of complex structures on these spaces is well known [**37**].

Our constructive approach to CYT structures and SKT structures is in contrast to the obstruction theories developed by other authors [**17**], [**33**]. It also enriches the set of examples found in [**29**].

Although this paper focuses on constraints on KT connections, much of its methods could be modified to construct canonical connections subjected to similar constraints. The departure point would be Proposition 5.

In the rest of this article, by a CYT connection we mean a KT connection having *restricted* holonomy in $\mathrm{SU}(n)$. Strictly speaking, we should have called it locally CYT. Obviously, on simply connected manifolds such as $(k-1)(S^2\times S^4)\#k(S^3\times S^3)$, the distinction between restricted holonomy and holonomy disappears.

Acknowledgement. We are very grateful to Prof. S.-T. Yau for his interest in our work. We are also in debt to the very knowledgable and thoughtful referee(s) for the extremely useful suggestions. Finally, we want to thank J. Edward for the help in editing.

2. Canonical connections on toric bundles

Let g be a Riemannian metric and J be an integrable complex structure such that they together form a Hermitian structure on a manifold M. Let F be the Kähler form; $F(A,B) := g(JA,B)$. Let d^c be the operator $(-1)^n JdJ$ on n-forms [**8**] and D be the Levi-Civita connection of the metric g. Then a family of canonical connections is given by

$$(1)\quad \begin{aligned} g(\nabla^t_A B, C) &= g(D_A B, C) \\ &\quad + \frac{t-1}{4}(d^cF)(A,B,C) + \frac{t+1}{4}(d^cF)(A,JB,JC), \end{aligned}$$

where A,B,C are any smooth vector fields and the real number t is a free parameter [**22**]. The connections ∇^t are called *canonical* connections. The connection ∇^1 is the Chern connection on the holomorphic tangent bundle. The connection ∇^{-1} is called the KT connection by physicists and the Bismut connection by some mathematicians. The mathematical features and background of these connections are articulated in [**22**]. When the Hermitian metric is a Kähler metric, the entire family of canonical connections collapses to a single connection, namely the Levi-Civita connection.

In this section, we construct these connections on toric bundles over Hermitian manifolds. We begin with a standard construction of complex structures.

Lemma 1. *Suppose that M is the total space of a principal toric bundle over a Hermitian manifold X with characteristic classes of type* $(1,1)$. *If the fiber is even-dimensional, then M admits an integrable complex structure so that the projection map from M to X is holomorphic.*

Proof. Choose a connection $(\theta_1,\theta_2,\ldots,\theta_{2k})$ on the principal bundle M. Let π denote the projection from M onto X. The curvature form of this connection is $(d\theta_1,d\theta_2,\ldots,d\theta_{2k})$. By assumption, for each j there exists (1,1)-form ω_j on X such that $d\theta_j=\pi^*\omega_j$. Define $\omega=(\ldots,\omega_j,\ldots)$.

To construct an almost complex structure J on M, we use the horizontal lift of the base complex structure on the horizontal space of the given connection. The vertical space consists of vectors tangent to an even-dimensional torus, and hence carries a complex structure. We choose J so that $J\theta_{2j-1} = \theta_{2j}$ for $1 \leq j \leq k$.

Since the fibers are complex submanifolds, if V and W are vertical (1,0) vector fields, then $[V, W]$ is a vertical (1,0) vector field. If A^h and B^h are horizontal lifts of (1,0) vector fields A and B on X, $[A^h, B^h] = \omega(A, B)$. This is equal to zero because ω is of type (1,1). Finally, if V is a vertical (1,0) vector field and A^h is the horizontal lift of a (1,0) vector field A on the base manifold X, then $[V, A^h] = 0$ because horizontal distributions are preserved by the action of the structure group of a principal bundle. It follows that the complex structure on M is integrable.

Since the projection from M onto X preserves type decomposition, it is holomorphic. q.e.d.

Suppose that g_X is a Hermitian metric on the base manifold X with Kähler form F_X. Consider a Hermitian metric g_M on M defined by

$$g_M := \pi^* g_X + \sum_{\ell=1}^{2k} (\theta_\ell \otimes \theta_\ell). \tag{2}$$

Since $J\theta_{2j-1} = \theta_{2j}$, the Kähler form of the metric g_M is

$$F_M = \pi^* F_X + \sum_{j=1}^{k} \theta_{2j-1} \wedge \theta_{2j}. \tag{3}$$

Here we use the convention that $\theta_1 \wedge \theta_2 = \theta_1 \otimes \theta_2 - \theta_2 \otimes \theta_1$.

Let Λ be the contraction of differential forms on the manifold X with respect to the Kähler form F_X. If $e_1, \ldots, e_{2n}$ is a local Hermitian frame on X such that $Je_{2a-1} = e_{2a}$ for $1 \leq a \leq n$, and ω is a type (1,1) form, then

$$\Lambda\omega = \sum_{a=1}^{n} \omega(e_{2a-1}, e_{2a}).$$

Lemma 2. *If δF_M and δF_X are the codifferentials of the Kähler forms on M and X respectively, then*

$$\delta F_M = \pi^* \delta F_X + \sum_{\ell=1}^{2k} \pi^*(\Lambda\omega_\ell)\theta_\ell. \tag{4}$$

Proof. Extend a local Hermitian frame $\{e_1, \ldots, e_{2n}\}$ on an open subset of X to a Hermitian frame $\{e_1, \ldots, e_{2n}, t_1, \ldots, t_{2k}\}$ on an open subset of M such that the vector fields $\{t_1, \ldots, t_{2k}\}$ are dual to the 1-forms $\{\theta_1, \ldots, \theta_{2k}\}$.

Recall that the co-differential of a tensor could be expressed in terms of the contraction of Levi-Civita connection D.

$$\begin{aligned}\delta F_M(V) = &-\sum_{a=1}^{n}\left(D_{e_{2a-1}}F_M\right)(e_{2a-1},V) - \sum_{a=1}^{n}\left(D_{e_{2a}}F_M\right)(e_{2a},V)\\ &-\sum_{j=1}^{k}\left(D_{t_{2j-1}}F_M\right)(t_{2j-1},V) - \sum_{j=1}^{k}\left(D_{t_{2j}}F_M\right)(t_{2j},V).\end{aligned}$$

It is a standard calculation to show that for any smooth vector fields A, B, and C on a Hermitian manifold,

$$-2(D_AF)(B,C) = dF(A,JB,JC) - dF(A,B,C).$$

It follows that

$$\begin{aligned}\delta F_M(V) &= \sum_{a=1}^{n} dF_M(e_{2a-1},e_{2a},JV) + \sum_{j=1}^{k} dF_M(t_{2j-1},t_{2j},JV)\\ &= \sum_{a=1}^{n} dF_M(JV,e_{2a-1},e_{2a}).\end{aligned}$$

Due to (3),

$$\begin{aligned}dF_M &= \pi^*dF_X + \sum_{j=1}^{k}(d\theta_{2j-1}\wedge\theta_{2j} - \theta_{2j-1}\wedge d\theta_{2j})\\ &= \pi^*dF_X + \sum_{j=1}^{k}(\pi^*\omega_{2j-1}\wedge\theta_{2j} - \theta_{2j-1}\wedge\pi^*\omega_{2j}).\end{aligned}\tag{5}$$

Therefore,

$$\begin{aligned}\delta F_M(V) &= \sum_{a=1}^{n}\pi^*dF_X(JV,e_{2a-1},e_{2a})\\ &\quad+\sum_{j=1}^{k}\sum_{a=1}^{n}\pi^*\omega_{2j-1}(e_{2a-1},e_{2a})\theta_{2j}(JV)\\ &\quad-\sum_{j=1}^{k}\sum_{a=1}^{n}\pi^*\omega_{2j}(e_{2a-1},e_{2a})\theta_{2j-1}(JV)\\ &= \pi^*\delta F_X(V) - \sum_{j=1}^{k}(\Lambda\omega_{2j-1})(J\theta_{2j})(V) + \sum_{j=1}^{k}(\Lambda\omega_{2j})(J\theta_{2j-1})(V)\\ &= \pi^*\delta F_X(V) + \sum_{j=1}^{k}(\Lambda\omega_{2j-1})\theta_{2j-1}(V) + \sum_{j=1}^{k}(\Lambda\omega_{2j})\theta_{2j}(V).\end{aligned}\tag{6}$$

So, the proof is now complete. q.e.d.

Hermitian manifolds with $\delta F = 0$ are called *balanced* [**36**].

Lemma 3. *Let ρ^1_X and ρ^1_M be the Ricci forms of the Chern connections on X and M respectively. Then $\rho^1_M = \pi^*\rho^1_X$.*

Proof. Let Θ_M and Θ_X be the holomorphic tangent bundles of X and M respectively. Since M is a holomorphic principal toric bundle over X, Θ_M fits into the following exact sequence of holomorphic vector bundles:

$$0 \to \underline{\mathbb{C}}^k \to \Theta_M \to \pi^*\Theta_X \to 0,$$

where $\underline{\mathbb{C}}^k$ is the rank-k trivial holomorphic vector bundle on M. It follows that the pull-back map induces a holomorphic isomorphism between the canonical bundles K_M and K_X over M and X respectively: $K_M = \pi^*K_X$. On the other hand, as a differentiable vector bundle Θ_M is isomorphic to the direct sum $\underline{\mathbb{C}}^k \oplus \pi^*\Theta_X$. Therefore, the induced Hermitian metric on K_M is isometric to the pull-back metric on K_X. Due to the uniqueness of Chern connection in terms of Hermitian structure and holomorphic structure, the induced Chern connection on K_M is the pull-back of the induced Chern connection on K_X. Therefore, the curvature on K_M is the pull-back of the curvature of K_X. The same can be said for the curvatures on the anti-canonical bundles. Since up to a universal constant, the Ricci form is equal to the curvature form of the Chern connection on anti-canonical bundle, the proposition follows. q.e.d.

On any manifold Y with a Hermitian metric g_Y, each canonical connection ∇^t on the holomorphic tangent bundle induces a connection on the anti-canonical bundle K_Y^{-1}. Let R^t be the curvature of ∇^t and ρ^t_Y be the Ricci form. Then $i\rho^t_Y$ is the curvature of the induced connection of ∇^t on K_Y^{-1}. By [**22**, (2.7.6)], for any smooth section s of the anti-canonical bundle K_Y^{-1} and for any real numbers t and u,

$$\nabla^t s - \nabla^u s = i\frac{t-u}{2}\delta F_Y \otimes s. \tag{7}$$

It follows that

$$\rho^t_Y - \rho^u_Y = \frac{t-u}{2} d\delta F_Y. \tag{8}$$

Proposition 4. *Let ρ^t_M and ρ^t_X be the Ricci forms of the canonical connections on M and X respectively; then*

$$\rho^t_M = \pi^*\rho^t_X + \frac{t-1}{2}\sum_{\ell=1}^{2k} d((\Lambda\omega_\ell)\theta_\ell). \tag{9}$$

Proof. Applying (8) on M and using Lemma 2, we find that

$$\rho_M^t - \rho_M^1 = \frac{t-1}{2} d\delta F_M = \frac{t-1}{2}\left(\pi^* d\delta F_X + \sum_{\ell=1}^{2k} d\left((\Lambda\omega_\ell)\theta_\ell\right)\right)$$

$$= \pi^*(\rho_X^t - \rho_X^1) + \frac{t-1}{2}\sum_{\ell=1}^{2k} d\left((\Lambda\omega_\ell)\theta_\ell\right).$$

Now the conclusion follows Lemma 3. q.e.d.

Proposition 5. *Suppose that the base manifold X is compact and the metric g_X is Kähler. Let ρ_X be the Ricci form of g_X. If each curvature form ω_ℓ is chosen to be harmonic, then*

$$\rho_M^t = \pi^*\left(\rho_X + \frac{t-1}{2}\sum_{\ell=1}^{2k}(\Lambda\omega_\ell)\omega_\ell\right). \tag{10}$$

Proof. Every curvature form ω_ℓ is closed. Up to an addition of an exact 2-form, we may assume that ω_ℓ is harmonic. It amounts to modifying the connection 1-form θ_ℓ by the pullback of a 1-form on X.

Since ω_ℓ is a harmonic (1,1)-form and the metric is Kähler, its trace is constant [**8**, 2.33]. Therefore,

$$d\pi^*((\Lambda\omega_\ell)\theta_\ell) = \pi^* d((\Lambda\omega_\ell)\theta_\ell) = \pi^*(\Lambda\omega_\ell)d\theta_\ell = \pi^*((\Lambda\omega_\ell)\omega_\ell).$$

As g_X is a Kähler metric, all Ricci forms ρ_X^t are equal to the Ricci form ρ_X of the Levi-Civita connection. The proposition follows Lemma 4. q.e.d.

3. CYT connections

When the holonomy of the KT connection is contained in the special unitary group, the KT connection is called a CYT connection. Locally, it is determined by the vanishing of its corresponding Ricci form. Since the KT connection is uniquely determined by the Hermitian structure, we address the Hermitian structure as a CYT structure when the KT connection is CYT. In this section, we focus on toric bundles over compact Kählerian bases with various geometrical or differential topological features.

The last proposition implies that the metric g_M is CYT if the base manifold X is Kähler and its Ricci curvature satisfies the following:

$$\rho_X = \sum_{\ell=1}^{2k}(\Lambda\omega_\ell)\omega_\ell. \tag{11}$$

However, it is not easy to find solutions to this equation, as it requires the Ricci form of a compact Kähler manifold to be harmonic. We could significantly relax the above condition by the following observation.

Lemma 6. *Suppose that the Ricci form of the KT connection of a Hermitian metric g_M is dd^c-exact on a manifold M of dimension greater than two. Then the metric g_M is conformally a CYT structure. In other words, there exists a conformal change of g_M such that the Ricci form of the induced KT connection vanishes.*

Proof. Let ϕ be an everywhere positive function on the manifold M. Let $\tilde{g}_M = \phi^2 g_M$ be a conformal change. The corresponding Kähler forms are related by $\tilde{F}_M = \phi^2 F_M$. The Ricci forms of the Chern connections are related by

$$\tilde{\rho}^1_M = \rho^1_M - m dd^c \log\phi, \tag{12}$$

where m is the complex dimension of M [**21**, Equation (21)]. The change of $d\delta F_M$ is

$$d\tilde{\delta}\tilde{F}_M = d\delta F_M - 2(m-1)dd^c\log\phi.$$

Given the universal relation among canonical connections (8), we derive the relation between the KT connections of conformally related metrics (the formula appears also in [**29**, Section 17, Lemma 1]):

$$\tilde{\rho}^{-1}_M = \rho^{-1}_M + (m-2)dd^c\log\phi.$$

If there exists a function Ψ such that $\rho^{-1}_M = dd^c\Psi$, then

$$\tilde{\rho}^{-1}_M = dd^c(\Psi + (m-2)\log\phi).$$

Given Ψ, one could solve the equation $\Psi + (m-2)\log\phi = 0$ for ϕ with $\phi(p) > 0$ for every point p on M. Therefore, $\tilde{\rho}^{-1}_M = 0$. q.e.d.

Proposition 7. *Suppose that X is a compact Kähler manifold with Ricci form ρ_X. The toric bundle M admits a CYT connection if ρ_X^{har}, the harmonic part of ρ_X, satisfies the following:*

$$\rho_X^{\mathrm{har}} = \sum_{\ell=1}^{2k}(\Lambda\omega_\ell)\omega_\ell. \tag{13}$$

Proof. By $\partial\overline{\partial}$-Lemma, there exists a function Φ such that

$$\rho_X = \rho_X^{\mathrm{har}} + dd^c\Phi.$$

The assumption on ρ_X^{har} and Proposition 5 together imply that

$$\rho^{-1}_M = \pi^*(\rho_X - \rho_X^{\mathrm{har}}) = \pi^* dd^c\Phi = dd^c\pi^*\Phi.$$

Due to the last lemma, the metric g_M is conformally equivalent to a CYT metric. q.e.d.

Note that Equation (13) has a topological interpretaion. Since the curvature form ω_ℓ is a harmonic (1,1)-form, $g(\omega_\ell, F_X) = \Lambda(\omega_\ell)$ is a constant. Therefore,

$$\int_X g_X(\omega_\ell, F_X) dvol_X = g_X(\omega_\ell, F_X) vol_X = \frac{g_X(\omega_\ell, F_X)}{n!}\int_X F_X^n.$$

On the other hand,

$$\int_X g_X(\omega_\ell, F_X) dvol_X = \int_X \omega_\ell \wedge *F_X = \frac{1}{(n-1)!}\int_x \omega_\ell \wedge F_X^{n-1}.$$

Therefore, Equation (13) is reformulated in terms of cohomology classes and their intersections as follows:

$$c_1(M) = n\sum_{\ell=1}^{2k} \frac{[\omega_\ell]\cup[F]^{n-1}}{[F]^n}[\omega_\ell].$$

When the complex dimension of the manifold is equal to 2, we have the next result.

Corollary 8. *Suppose in addition that the base manifold X is complex two-dimensional. Let Q be the intersection form of X. Then M admits a CYT connection if*

$$\rho_M^{har} = 2\frac{Q(F_X,\omega_1)}{Q(F_X,F_X)}\omega_1 + 2\frac{Q(F_X,\omega_2)}{Q(F_X,F_X)}\omega_2. \tag{14}$$

When the base manifold is Kähler Einstein, we could solve Equation (11) directly without going through a conformal change as in Proposition 7.

Proposition 9. *Suppose that X is compact real $2n$–dimensional Kähler Einstein manifold with positive scalar curvature. Let its scalar curvature be normalized to be $2n^2$. Suppose that M is an even-dimensional toric bundle with curvature $(\omega_1,\ldots,\omega_{2k})$ such that $\omega_1 = F_X$ and for all $2 \le \ell \le 2k$, ω_ℓ is primitive; then M admits a CYT structure.*

Proof. Since X is Kähler Einstein, $\rho_X^t = \frac{2n^2}{2n}F_X = nF_X$ for all t. When ω_ℓ is primitive, $\Lambda\omega_\ell = 0$. By Proposition 5

$$\rho_M^{-1} = \pi^*(nF_X - (\Lambda\omega_1)\omega_1) = n\pi^*(F_X - \omega_1) = 0. \tag{15}$$

q.e.d.

Corollary 10. *Let P be the principal $U(1)$-bundle of the maximum root of the anti-canonical bundle of a compact Kähler Einstein manifold with positive scalar curvature. Then $P \times S^1$ admits a CYT-structure.*

Proposition 11. *Let X be a compact Ricci-flat Kähler manifold. Suppose that M is an even-dimensional toric bundle with curvature $(\omega_1,\ldots,\omega_{2k})$ such that every ω_ℓ is primitive; then there is a Hermitian metric on M such that all canonical connections are Ricci flat. In particular, it admits a CYT structure.*

Proof. Since X is Ricci-flat Kähler, $\rho_X^t = 0$ for all t. When all ω_ℓ are primitive, $\Lambda\omega_\ell = 0$. By Proposition 5 or Equation (11),

$$\rho_M^t = \frac{t-1}{2}\pi^*\left(\sum_\ell (\Lambda\omega_\ell)\omega_\ell\right) = 0 \tag{16}$$

for all t. q.e.d.

The condition on ω_ℓ being primitive in the last proposition is necessary, as there exists an example of real 2-dimensional holomorphic principal toric bundle over a real 4-dimensional flat torus admitting no CYT connections [**17**, Theorem 4.2].

The last proposition is applicable to K3-surfaces with Calabi-Yau metrics. The abundance of primitive harmonic (1,1)-forms generates a large collection of CYT structures on toric bundles on K3-surfaces. Some explicit constructions can be found in [**26**]. We shall remark on the topology of these examples and their relation with Strominger's equations later in this article.

On the other hand, the last two propositions could not be extended to include Kähler Einstein manifolds with negative scalar curvature as base manifolds. For instance, if X is a compact complex surface of general type, the space of holomorphic sections of K_X^m for some positive integer m is at least two dimensional. Since $K_M = \pi^* K_X$, we have a contradiction to a vanishing theorem [**1**].

4. Examples of CYT structures

4.1. Product of Spheres $S^3 \times S^3$. The second integral cohomology of the product of two complex projective lines $X = \mathbf{CP}^1 \times \mathbf{CP}^1$ is generated by two effective divisor classes C and D with the properties that

$$Q(C, C) = Q(D, D) = 0, \quad Q(C, D) = 1.$$

They are the pullback of the hyperplane class from the respective factors onto the product space. The anti-canonical class is $-K_X = 2C + 2D$. The class $C + D$ is positive as its associated map embeds $\mathbf{CP}^1 \times \mathbf{CP}^1$ into the complex projective 3-space $\mathbf{CP}^3$. Therefore, $F_X := \frac{1}{2}(C + D)$ is a Kähler class. Let

$$\omega_1 = C, \quad \omega_2 = D.$$

Then $Q(F_X, F_X) = \frac{1}{2}$, and $Q(F_X, \omega_1) = Q(F_X, \omega_2) = \frac{1}{2}$. Therefore,

$$2\frac{Q(F_X, \omega_1)}{Q(F_X, F_X)}\omega_1 + 2\frac{Q(F_X, \omega_2)}{Q(F_X, F_X)}\omega_2 = 2C + 2D = -K_X.$$

By Proposition 7 there exists a CYT structure on the total space of the toric bundle with curvature (ω_1, ω_2). Since ω_1 and ω_2 are the curvature of the Hopf bundle on $\mathbf{CP}^1$, the total space of the toric bundle is simply $S^3 \times S^3$. The existence of CYT structure or a flat invariant Hermitian connection on this space is well known [**18**].

4.2. Toric bundles on blow-up of $\mathbf{CP}^2$ twice. Let X be the blow-up of $\mathbf{CP}^2$ at two distinct points. Let H be the hyperplane class of the complex projective plane and E_ℓ be the exceptional divisor of blowing-up the ℓ-th point on the complex projective plane. The anti-canonical class is $3H - E_1 - E_2$, and it is ample. The class $H - 2E_1 - E_2$ is primitive with respect to the anti-canonical class. By taking

$$F_X = \omega_1 = 3H - E_1 - E_2 \quad \text{and} \quad \omega_2 = H - 2E_1 - E_2, \tag{17}$$

we solve the geometric equation in Corollary 8. Therefore, there exists a CYT structure of the total space of the bundle whose curvature is (ω_1, ω_2).

4.3. Blow-up of $\mathbf{CP}^2$ three to eight times. Let X be the blow-up of $\mathbf{CP}^2$ at k distinct points at general position on the complex projective plane. Assume that $3 \le k \le 8$. It is well known that the anti-canonical class on X is positive and the manifold admits a Kähler Einstein metric with positive scalar curvature.

Let H be the hyperplane class of the complex projective plane and E_ℓ be the exceptional divisor of blowing up the ℓ-th point. The anti-canonical class is $-K_X = 3H - E_1 - \cdots - E_k$. Let

$$\begin{aligned} &\omega_0 = H, \quad \omega_1 = -K_X = 3H - E_1 - \cdots - E_k, \quad \omega_2 = E_1 - E_2, \\ &\omega_j = E_j, \text{ for all } 3 \le j \le k. \end{aligned} \tag{18}$$

Then $\{\omega_0, \ldots, \omega_k\}$ forms an integral basis for $H^2(X, \mathbb{Z})$. Let g_X be a Kähler Einstein metric on X whose Kähler class is equal to ω_1; then ω_2 is primitive with respect to g_X. By Proposition 9, the toric bundle with curvature (ω_1, ω_2) admits a CYT structure.

4.4. Blow-ups of $\mathbf{CP}^2$ many times. Next for $k \ge 9$, let X be the blow-up of $\mathbf{CP}^2$ at k distinct points on an irreducible smooth cubic curve. Let

$$\omega_1 = 4H - 2\sum_{\ell=1}^{4} E_\ell - \sum_{\ell=5}^{k} E_\ell, \quad \omega_2 = -H + \sum_{\ell=1}^{4} E_\ell. \tag{19}$$

Consider a real cohomology class $F_X = nH - \sum_{\ell=1}^{k} n_\ell E_\ell$ on X. We now seek n and n_ℓ for $1 \le \ell \le k$ such that

$$Q(F_X, F_X) = 4, \quad Q(\omega_1, F_X) = Q(\omega_2, F_X) = 2, \tag{20}$$

because the resulting cohomology class F_X will solve Equation (14) in Corollary 8. The above equations are equivalent to the following set of equations in n and n_ℓ.

$$n^2 - \sum_{\ell=1}^{k} n_\ell^2 = 4, \quad 4n - 2\sum_{\ell=1}^{4} n_\ell - \sum_{\ell=5}^{k} n_\ell = 2, \quad -n + \sum_{\ell=1}^{4} n_\ell = 2.$$

To illustrate the existence of a solution, we further assume that $n_1 = n_2 = n_3 = n_4$ and $n_5 = \cdots = n_k$. In terms of n, the above system becomes

$$n_5 = \cdots = n_k = \frac{2n-6}{k-4}, \quad n_1 = n_2 = n_3 = n_4 = \frac{1}{4}(n+2), \tag{21}$$

$$(3k-28)n^2 + (112-4k)n - (20k+64) = 0. \tag{22}$$

An elementary computation demonstrates that when $k \geq 9$ this system of equations has a solution such that $n > 3$. Note that all n_ℓ are strictly positive and $n > n_\ell + n_j$ for all ℓ and j.

Next, we need to demonstrate that F_X with the given n and n_ℓ above is also a Kähler class. Due to an improved Nakai-Moishezon criteria [**11**], a class F_X in $H^{(1,1)}(X)$ is Kähler if the following conditions are met:

1) $Q(F_X, F_X) > 0$.
2) $Q(F_X, D) > 0$ for any irreducible curve D with negative self-intersection.
3) $Q(F_X, C) > 0$ for an ample divisor C.

On X, the divisor class $aH - E_1 - \cdots - E_k$ is ample when a is sufficiently large. Therefore, the last condition is fulfilled because n is positive.

According to [**19**], on the blow-up of distinct points on a smooth cubic in $\mathbf{CP}^2$, the irreducible curves with negative self-intersections are E_ℓ, $H - E_\ell - E_j$ with $\ell \neq j$, and the proper transform of the cubic containing every point of blow-up when $k \geq 10$. The latter is linearly equivalent to $-K_X = 3H - \sum_{\ell=1}^{k} E_\ell$. Since $\omega_1 + \omega_2 = -K_X$ and F_X satisfy the conditions in (20), $Q(-K_X, F_X) > 0$. Therefore, F_X is in the Kähler cone as long as the intersection numbers of F_X with E_ℓ and with $H - E_j - E_\ell$ are positive. It is equivalent to the constraints $n > n_\ell + n_j$, and $n_\ell > 0$ for $1 \leq \ell, j \leq k$. Since solutions to Equation (22) fulfill these conditions, the corresponding F_X is a Kähler class.

4.5. CYT structures on $(k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$. We now examine the topology of the total spaces of the toric bundles found in the last three sections.

Proposition 12. *Suppose that X is a compact and simply connected manifold. Let M be the total space of a principal T^2-bundle over X with curvature forms (ω_1, ω_2). Suppose that the curvature forms are part of a set of generators $\{\omega_1, \ldots, \omega_b\}$ of $H^2(X, \mathbb{Z})$. If there exist α, β in $H^2(X, \mathbb{Z})$ fulfilling the following equations on X,*

$$\omega_1 \wedge \alpha = \pm \mathrm{vol}_X, \quad \omega_2 \wedge \alpha = 0, \quad \omega_2 \wedge \beta = \pm \mathrm{vol}_X, \quad \omega_1 \wedge \beta = 0, \tag{23}$$

then $b = b_2(X) - 2$, $H^2(M, \mathbb{Z}) = \mathbb{Z}^{b_2(X)-2}$ and the cohomology ring of M has no torsion.

Proof. There exist connection forms (θ_1, θ_2) on M with curvatures $(d\theta_1, d\theta_2) = \pi^*(\omega_1, \omega_2)$. Let T^2_x be the fiber of M over a point x on the base manifold. Then the restrictions $\theta_1|_{T^2_x}$ and $\theta_2|_{T^2_x}$ generate the

$\mathbb{Z}$-module $H^*(T^2_x, \mathbb{Z})$. The standard Leray's theorem implies that the E_2-terms of the Leray spectral sequence for the T^2-bundle M over X are given in the table below [**12**, 15.11].

$\mathbb{Z} = \langle \theta_1 \wedge \theta_2 \rangle$	0	$\mathbb{Z}^b$	0	$\mathbb{Z}$
$\mathbb{Z}^2 = \langle \theta_1, \theta_2 \rangle$	0	$\mathbb{Z}^2 \otimes \mathbb{Z}^b$	0	$\mathbb{Z}^2$
$\mathbb{Z}$	0	$\mathbb{Z}^b = \langle \omega_1, \omega_2, \dots, \omega_b \rangle$	0	$\mathbb{Z} = \langle \mathrm{vol}_X \rangle$

.

Next we calculate the E_3-terms. The map $d_2 : E_2^{0,1} \to E_2^{2,0}$ is given by $\theta_1, \theta_2 \mapsto \omega_1, \omega_2$. It is an injection and therefore $E_3^{0,1} = 0$. Since $d_2(E_2^{2,0}) = 0$, $E_3^{2,0} = \mathbb{Z}^{b-2} \cong \langle \omega_3, \omega_4, \dots, \omega_b \rangle$.

Similarly, the map $d_2 : E_2^{0,2} \to E_2^{2,1}$ is given by $\theta_1 \wedge \theta_2 \mapsto \omega_1 \wedge \theta_2 - \theta_1 \wedge \omega_2$, so it is injective. Therefore, $E_3^{0,2} = 0$. The map $d_2 : E_2^{2,1} \to E_2^{4,0}$ is given by $\theta_i \wedge \omega_j \mapsto \omega_i \wedge \omega_j$. It is surjective because $d_2(\theta_1 \wedge \alpha) = \pm \mathrm{vol}_X$. Therefore, $E_3^{2,1} = \mathbb{Z}^{2b-2}$.

Finally, the map $d_2 : E_2^{2,2} \to E_2^{4,1}$ is given by

$$\theta_1 \wedge \theta_2 \wedge \omega \mapsto \omega_1 \wedge \theta_2 \wedge \omega - \theta_1 \wedge \omega_2 \wedge \omega$$

for any $\omega \in H^2(X, \mathbb{Z})$. In particular,

$$d_2(\theta_1 \wedge \theta_2 \wedge \alpha) = \pm \mathrm{vol}_X \wedge \theta_2 \quad \text{and} \quad d_2(\theta_1 \wedge \theta_2 \wedge \beta) = \pm \mathrm{vol}_X \wedge \theta_1.$$

Therefore, the restriction d_2 on the $E_2^{2,2}$-term is surjective. It follows that $E_3^{2,2} = \mathbb{Z}^{b-2}$. With the other E_3 terms easily computed, the above computation yields the table of E_3-terms below.

0	0	$\mathbb{Z}^{b-2}$	0	$\mathbb{Z}$
0	0	$\mathbb{Z}^{2b-2}$	0	0
$\mathbb{Z}$	0	$\mathbb{Z}^{b-2}$	0	0

It follows that the spectral sequence degenerates at the E_3-level and

$$H^2(M, \mathbb{Z}) \cong E_3^{2,0} \oplus E_3^{1,1} \oplus E_3^{0,2} \cong E_3^{2,0} \cong \langle \omega_3, \omega_4, \dots, \omega_b \rangle.$$

q.e.d.

Theorem 13. *For every positive integer $k \geq 1$, the manifold $(k-1)(S^2 \times S^4) \# k(S^3 \times S^3)$ admits a CYT structure.*

Alternatively, any 6-dimensional compact simply-connected spin manifold with torsion free cohomology and free S^1-action admits a CYT structure.

Proof. We have seen a CYT structure on $S^3 \times S^3$ in a previous section.

For $k \geq 2$, let X_k be the blow-up of $\mathbf{CP}^2$ at k distinct points on an irreducible smooth cubic. Let M_k be the total space of the toric bundles over X_k obtained in Sections 4.2, 4.3, or 4.4. Since M_k admits a CYT structure, $c_1(M_k) = 0$. In particular, the second Stiefel-Whitney class vanishes and M_k is a spin manifold.

We now examine the cohomology of the space M_k through the last proposition. When $k = 2$, let ω_1 and ω_2 be given as in (17) and let

$$\alpha = H + E_1 - 3E_2, \quad \beta = E_1 - E_2.$$

When $3 \leq k \leq 8$, let ω_1 and ω_2 be given as in (18) and let

$$\alpha = E_k, \quad \beta = E_1 - E_k.$$

When $k \geq 9$, let ω_1 and ω_2 be given as in (19) and let

$$\alpha = E_k, \quad \beta = H - E_5 - E_6 - E_7 - E_8.$$

Then the set of data $\{\omega_1, \omega_2, \alpha, \beta\}$ on respective manifolds satisfies the hypothesis of Proposition 12. In particular, $b_2(M_k) = k - 1$, for each $k \geq 2$.

Since M_k is a toric bundle, it admits a free S^1-action. By [**25**], a compact smooth simply-connected spin 6-manifold with torsion-free cohomology, $b_2(M_k) = k - 1$, and free S^1-action is diffeomorphic to $(k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$. Therefore, we can complete the proof of this theorem if we demonstrate that the space M_k is simply connected.

When $3 \leq k \leq 8$, $Q(\omega_1, \alpha) = 1$ and $Q(\omega_2, \alpha) = 0$, and the restriction of the bundle M_k onto the 2-sphere representing the homotopy class of α is the projection from $S^3 \times S^1$ onto S^2 via the Hopf fibration $S^3 \to S^2$. In particular, the map $\pi_2(M_k) \to \pi_1(T^2)$ in the homotopy sequence of the fibration from M_k onto X_k sends α to a generator of $\pi_1(T^2)$ [**40**]. Similarly, the Poincaré dual of β is represented topologically by a smooth 2-sphere. As $Q(\omega_1, \beta) = 0$ and $Q(\omega_2, \beta) = -1$, the map $\pi_2(M_k) \to \pi_1(T^2)$ sends the Poincaré dual of β to a different generator of $\pi_1(T^2)$. Therefore, the map $\pi_2(M_k) \to \pi_1(T^2)$ is surjective. Since X_k is simply connected, by the homotopy sequence of the fibration $M_k \to X_k$, M_k is simply connected.

When $k \geq 9$, a similar analysis shows that the map $\pi_2(M_k) \to \pi_1(T^2)$ sends the Poincaré dual of α and β onto the generators of $\pi_1(T^2)$. Hence, M_k is simply connected.

When $k = 2$, the Poincaré dual of β is an embedded 2-sphere. Since $Q(\omega_1, \beta) = 0$ and $Q(\omega_2, \beta) = 1$, the restriction of the bundle $M_2 \to X_2$ onto the Poincaré dual of α is the Hopf fibration $S^1 \times S^3 \to S^2$. The map $\pi_2(M_2) \to \pi_1(T^2)$ sends β to a generator in $\pi_1(T^2)$. The Poincaré dual of $\gamma = H - E_1 - E_2$ is also an embedded sphere. Since $Q(\omega_1, \gamma) = 1$, under the map $\pi_2(M_2) \to \pi_1(T^2)$ the images of γ and β form a set of generators for $\pi_1(T^2)$. q.e.d.

5. SKT connections

A KT connection is strong (a.k.a. SKT) if its torsion is a *closed* three-form. It is equivalent to require the Kähler form to be dd^c-closed. Such structures recently appeared in the theory of generalized Kähler geometry [**28**] [**30**]. In real six dimension, a Hermitian metric with strong KT connection is an astheno-Kähler metric [**34**]. Results on SKT structures on nilmanifolds could be found in [**18**].

To construct SKT connections, we return to the general set-up leading to Lemma 2. Since the projection map π is holomorphic and the curvature forms $(\omega_{2j-1}, \omega_{2j})$ are type (1,1), given Equation (5) we have

$$\begin{aligned} d^cF_M &= JdF_M \\ &= J\pi^*dF_X + \sum_{j=1}^{k}(\pi^*\omega_{2j-1} \wedge J\theta_{2j} - J\theta_{2j-1} \wedge \pi^*\omega_{2j}) \\ &= \pi^*d^cF_X - \sum_{j=1}^{k}(\pi^*\omega_{2j-1} \wedge \theta_{2j-1} + \theta_{2j} \wedge \pi^*\omega_{2j}). \end{aligned}$$

Therefore, $dd^cF_M = \pi^*dd^cF_X - \sum_{j=1}^{k} \pi^*(\omega_{2j-1} \wedge \omega_{2j-1} + \omega_{2j} \wedge \omega_{2j})$.

Proposition 14. *Suppose that a Hermitian structure on a toric bundle M over a Hermitian manifold X is given as in* Equation (2)*. Its KT connection is strong if and only if*

$$\sum_{j=1}^{k}(\omega_{2j-1} \wedge \omega_{2j-1} + \omega_{2j} \wedge \omega_{2j}) = dd^cF_X. \tag{24}$$

In particular, suppose that X is a compact complex Kähler surface, Q is the intersection form on X, and M is a real two-dimensional toric bundle over X. If the KT connection on M is strong, then

$$Q(\omega_1, \omega_1) + Q(\omega_2, \omega_2) = 0. \tag{25}$$

Note that unlike the trace, the square of a harmonic form is not harmonic so (25) is not equivalent to (24). By Hodge-Riemann bilinear relations, the intersection form on primitive type (1,1) classes on compact complex surfaces is negative definite [**27**]. There is little chance of using our construction here to produce strong CYT structures on 2-toric bundles over K3-surfaces. However, on $S^2 \times S^2$, when $\omega_1 = C$ and $\omega_2 = D$ as given in Section 4.1, they solve the equation (24). Therefore, the CYT-structure on $S^3 \times S^3$ is strong, which is a well known fact.

It is known that on most rational surfaces a product of harmonic forms is not harmonic and every harmonic anti-self-dual 2-form has at least one zero [**35**]. So we must use a non-Kähler metric on X. With this observation in mind we are ready to prove the following:

Theorem 15. *For every positive integer $k \geq 1$, the manifold $(k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$ admits a strong KT structure.*

Alternatively, any 6-dimensional compact simply-connected spin manifold with torsion free cohomology and free S^1-action admits a SKT structure.

Proof. Consider a blow-up of $\mathbf{CP}^2$ at k ($k \geq 2$) points on a smooth irreducible cubic. Choose two arbitrary closed forms ω_1 and ω_2 satisfying (25). By the dd^c-lemma, $\omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 = dd^c\alpha$ for some real (1,1)-form α. We can choose a dd^c-closed (1,1)-form β (e.g., any Kähler form multiplied by an appropriate constant) such that

$$\min_{p\in X}(\min_{||Y||=1} \beta_p(Y, JY)) > -\min_{p\in X}(\min_{||Y||=1} \alpha_p(Y, JY)).$$

Then the form $\alpha + \beta$ is positive definite everywhere and defines a Hermitian metric on X, which will produce SKT metric on M. q.e.d.

6. Remarks

6.1. Non-uniqueness. The construction on bundles over the blow-ups of $\mathbf{CP}^2$ in Section 4.4 could be used to produce apparently different CYT structures on $(k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$ using different Kähler classes on the same base manifold. For example, when $k \geq 11$, we may choose

$$\omega_1 = 4H - 2(E_1 + E_2) - \sum_{\ell=3}^{k} E_\ell, \quad \omega_2 = -H + E_1 + E_2,$$

and then solve the equations $Q(\omega_1, F_X) = Q(\omega_2, F_X) = 2$, $Q(F_X, F_X) = 4$ for n, a, b in $F_X = nH - a(E_1 + E_2) - b(E_3 + \cdots + E_k)$.

It is also possible to use topologically different base manifolds and toric bundles to produce CYT structures on the same real six-dimensional manifold. For instance, let the base manifold X be a Kummer surface. It admits sixteen smooth rational curves C_i with $Q(C_i, C_j) = -2\delta_{ij}$. Due to Piateckii-Shapiro and Shafarevich's description of the cohomology ring of X [**38**, pp. 568–571], if we choose $\omega_1 = C_1 \pm C_2$ and $\omega_2 = C_3 \pm C_4$ (with arbitrary signs), there are elements α and β in $H^2(X, \mathbb{Z})$ satisfying all conditions in Proposition 12. This proposition enables us to identify the total space M of the toric bundle with (ω_1, ω_2) as curvature forms of $20(S^2 \times S^4)\#21(S^3 \times S^3)$. Since the canonical bundle of M is the pullback of the one on M, it is holomorphically trivial.

Moreover the Kähler cone of X in $H^2(X, \mathbb{R})$ is one of the chambers with walls $\Pi_i = \{E \in H^2(X, \mathbb{Z}) : Q(C_i, E) = 0\}$. Then we can choose a Ricci-flat Kähler metric F_X on X such that $Q(C_i, F_X) = \pm 1$ for $i = 1, 2, 3, 4$. For this choice we have $Q(F_X, \omega_j) = 0$ for $j = 1, 2$ after fixing the signs in the definition of ω_i. As a consequence we obtain a balanced CYT structure on $20(S^2 \times S^4)\#21(S^3 \times S^3)$ which implicitly appears in [**26**]. It is also half-flat in the terminology of [**15**]. The structure is not strong, which is in accordance to the result in [**33**] that a strong and balanced CYT structure on compact manifold is Kähler.

6.2. Relation with Strominger's equations. Our construction on the connected sums of $S^2 \times S^4$ with $S^3 \times S^3$ is related to a set of Strominger's equations in string theory. In [**41**], Strominger analyzes heterotic superstring background with spacetime supersymmetry. His model can be translated to our situation in the following terms: First we need a conformally balanced CYT manifold with holomorphic (3,0)-form of constant norm. The manifold is endowed with an auxiliary semistable bundle with Hermitian-Einstein connection A with curvature F_A. The last and most restrictive equation in [**41**] is

$$dH = \alpha'(\mathrm{Tr}R \wedge R - \mathrm{tr}F_A \wedge F_A). \tag{26}$$

Here "Tr" and "tr" are the traces in the tangent bundle and the auxiliary bundle respectively, R is the curvature of any metric connection ∇, and α' is a positive constant. Solutions to the Strominger's equations with the choice of R being the curvature of the Chern connection ∇^1 have recently been found by Fu and Yau [**20**]. The Hermitian metric has Kähler form as in (2) with F_X being conformally Kähler. With this data, they solve the system proposed by Strominger for an unknown conformal factor on a K3-surface as base space. Since the anomalies can be cancelled for *any* choice of metric connection, it is important progress towards a realistic string theory [**3**]. However, the requirement that the connection ∇ preserves both worldsheet conformal invariance and spacetime supersymmetry leads the connection ∇ to be equal to $D - \frac{1}{2}H$, where H is the torsion of the KT (Bismut) connection, i.e., $\nabla^{-1} = D + \frac{1}{2}H$ [**32**]. Therefore, the term R in Equation (26) is the curvature of the connection $D - \frac{1}{2}H$. It is an open question whether such a connection exists on compact manifolds.

6.3. Orbifolds. In this article, we focus on toric bundles over smooth complex manifolds. However, most of the local geometric considerations could be extended to toric bundles as V-bundles over orbifolds. For example, suppose that the base space X is a Kähler Einstein orbifold with positive scalar curvature [**16**]. Let P be the principal $U(1)$-bundle of the maximal root of the anti-canonical bundle. Our construction shows that $S^1 \times P$ carries a CYT structure. We refer the readers to [**9**] for an analysis of the geometric and topological consideration of P.

6.4. Other geometric structures. The spaces $M = (k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$ are S^1-bundles over the Sasakian 5-manifolds $(k-1)(S^2 \times S^3)$. Since $(k-1)(S^2 \times S^3)$ admits a Ricci-positive metric for any $k \geq 2$ [**39**] [**10**], a result of Bérard-Bergery [**7**] implies that the manifolds M admit Riemannian metrics with positive Ricci curvature.

It would be interesting to see to what extent the CYT metric, the SKT metric and the positive Ricci curvature metrics on M are related.

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