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THE LENGTH OF A SHORTEST GEODESIC LOOP AT A POINT

Regina Rotman

Abstract

In this paper we prove that given a point $p \in M^n$, where M^n is a closed Riemannian manifold of dimension n, the length of a shortest geodesic loop $l_p(M^n)$ at this point is bounded above by $2nd$, where d is the diameter of M^n. Moreover, we show that on a closed simply connected Riemannian manifold M^n with a non-trivial second homotopy group there either exist at least three geodesic loops of length less than or equal to $2d$ at each point of M^n, or the length of a shortest closed geodesic on M^n is bounded from above by 4d.

Introduction and main results

Let M^n be a closed Riemannian manifold of dimension n. In 1983, M. Gromov asked whether one can bound above the length of a shortest closed geodesic $l(M^n)$ on M^n by $c(n)\mathrm{vol}\,(M^n)^{\frac{1}{n}}$, where $\mathrm{vol}\,(M^n)$ is the volume of M^n and $c(n)$ is a constant that depends on the dimension of M^n only. A similar question can be asked about the relationship between $l(M^n)$ and the diameter d of a manifold. The fact that on each manifold there exists a closed geodesic was shown by L. Lusternik and A. Fet. A similar argument shows that there exists a geodesic loop at each point of a closed Riemannian manifold. So, one can also ask if there exists a constant $k(n)$ such that for each point $p \in M^n$, the length of a shortest geodesic loop $l_p(M^n)$ at this point is bounded above by $k(n)d$. Note that, although it is quite easy to see that $l_p(M^n) \leq 2d$ in the case of a closed Riemannian manifold that is not simply connected, this is not true in general, as it was recently shown by F. Balacheff, C.B. Croke, and M. Katz in [**BlCK**].

Note also, that for no constant $C(n)$ we can bound above $l_p(M^n)$ by $C(n)\mathrm{vol}\,(M^n)^{\frac{1}{n}}$ for every $p \in M^n$. For example, consider a prolate ellipsoid E that is an ellipsoid generated by an ellipse rotated around its major axis. Let us denote its polar radius by R. Let $p \in E$ be the north pole of E. Then all geodesics and, thus, geodesic loops passing through

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p are ellipses (see Fig. 1). Therefore, the ratio $\frac{l_p(E)}{\sqrt{A(E)}}$ will approach infinity as R goes to infinity and the smaller semiaxis is fixed.

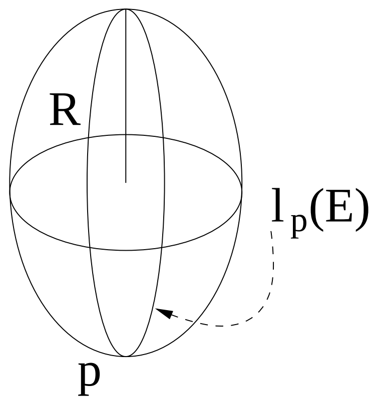

Figure 1. Prolate Ellipsoid.

Here is the main result of our paper.

Theorem 0.1. *Let M^n be a closed Riemannian manifold of dimension n. Let q denote the smallest integer for which $\pi_q(M^n) \neq \{0\}$. Then for each $p \in M^n$ there exists a geodesic loop based at p of length $\leq 2qd$, where d is the diameter of M^n. In particular, the length of a shortest geodesic loop based at p is $\leq 2nd$.*

A related problem is the problem of estimating the length of a shortest geodesic loop, $\alpha(M^n)$ on a closed Riemannian manifold M^n without fixing a basepoint. The first such curvature-free estimates were obtained in 2004 and are due to S. Sabourau, who established that $\alpha(M^n)$ is bounded above by $c(n)\mathrm{vol}\,(M^n)^{\frac{1}{n}}$ for some constant $c(n)$ that was not explicitely calculated in his paper [**S2**]. He also demonstrated that $\alpha(M^n) \leq \frac{(8\cdot 3^n-2)d}{3}$. Our estimate for the length of a geodesic loop in terms of the diameter is, however, qualitatively different from that of Sabourau, since we estimate the length of a geodesic loop at each point of a manifold, whereas Sabourau shows that there exists a point, at which the length of a geodesic loop can be estimated. In this paper we obtain an estimate for $\alpha(M^n)$ in terms of the Filling Radius of M^n. The following definition is due to M. Gromov (see [**G**]).

Definition 0.2 (Filling Radius)**.** Let M be a Riemannian manifold topologically imbedded into an arbitrary metric space X. Then the filling radius FillRad$\,(M \subset X)$ is the infimum of $\varepsilon > 0$, such that M bounds in the ε-neighborhood $N_\varepsilon(M)$, that is $i_*(H_n(M^n)) = \{0\}$, where i_* is induced by inclusion $i : M^n \longrightarrow N_\varepsilon(M^n)$ and where $H_n(M^n)$ is taken with coefficients in $\mathbf{Z}$, when M^n is orientable, and with coefficients in $\mathbf{Z}_2$, when M^n is nonorientable. Filling Radius of an abstract

Riemannian manifold is $\mathrm{FillRad}\,(M \subset X)$, where $X = L^\infty(M)$, i.e., the Banach space of bounded Borel functions f on M, and the embedding of M into X is a map that to each point p of M assigns a distance function $p \longrightarrow f_p = d(p,q)$ (see [**G**]). Equivalently, $\mathrm{FillRad}\, M$ can be defined as the infimum of $\mathrm{FillRad}\,(M \subset X)$ over all metric spaces X and isometric embeddings of M into X.

Theorem 0.3. *Let M^n be a closed Riemannian manifold. Then the length of a shortest geodesic loop, $\alpha(M^n)$ on M^n, is bounded above by $c(n)\mathrm{FillRad}\, M^n$, where $c(n) = 3 \cdot 4^{n-1}$.*

The volume inequality then follows from the previous theorem and from the volume upper bound for the filling radius due to Gromov.

Corollary 0.4. *Let M^n be a closed Riemannian manifold M^n. Then the length of a shortest geodesic loop $\alpha(M^n) \leq 3 \cdot 4^{n-1}(n+1)n^n(n+1)!^{\frac{1}{2}}\mathrm{vol}\,(M^n)^{\frac{1}{n}}$, where $\mathrm{vol}\,(M^n)$ is the volume of M^n.*

Proof. This corollary immediately follows from the above theorem and from Gromov's estimate for the filling radius of M^n in terms of the volume of M^n, namely $\mathrm{FillRad}\, M^n \leq (n+1)n^n(n+1)!^{\frac{1}{2}}\mathrm{vol}\,(M^n)^{\frac{1}{n}}$ (see [**G**]). □

This corollary provides an explicit value for the constant in the inequality $\alpha(M^n) \leq \mathrm{const}\,(n)\mathrm{vol}\,(M^n)^{\frac{1}{n}}$. We believe that this value is better than the one that can be obtained after some computations using the methods of [**S2**]. Our proof is also simpler than the proof in [**S2**], as it does not involve the results from [**GrP**].

At present, there do not exist similar curvature-free upper bounds for the length of a shortest closed geodesic $l(M^n)$ in the general case of a closed Riemannian manifold M^n, though such bounds do exist for stationary 1-cycles, ([**NR2**]) and minimal surfaces, ([**NR3**]), as well as for some topological types of Riemannian manifolds, namely, 2-dimensional sphere, ([**C**, **M**, **S1**, **NR1**, **R1**, **R2**]), and 1-essential manifolds, ([**G**]). (Gromov's estimate generalizes results of many people, who worked on estimating systoles in case of surfaces, namely, C. Loewner, P. Pu, R. Accola, C. Blatter, C. Bavard, T. Sakai, Ju. Burago and V. Zalgaller, J. Hebda and others (see [**BZ**], [**CK**])). Thus one of the central problems in this subject remains to find upper bounds of similar nature for $l(M^n)$. With this goal in mind, we will prove the following theorem.

Theorem 0.5. *Let M^n be a simply connected closed Riemannian manifold with $\pi_2(M^n) \neq \{0\}$. Then either the length of a shortest closed geodesic is bounded above by $4d$, or at each point of M^n there exist three distinct geodesic loops based at that point of length bounded above by $2d$.*

One can view this theorem in the following way: unless there are three geodesic loops of length $\leq 2d$ based at each point of M^n, which seems to be unlikely for many Riemannian manifolds, there exists a closed geodesic of length $\leq 4d$. This theorem strengthens a previous result by F. Balacheff, (see [**Bl**]).

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1. The proof of Theorem 0.1: The basic ideas

We will begin with the two simple lemmas. The following is a basic Morse-theoretic type lemma.

Lemma 1.1. *Let M^n be a complete Riemannian manifold. Let $p \in M^n$. Suppose that the length of a shortest geodesic loop $l_p(M^n)$ based at p is greater than L. Then given any piecewise differentiable closed curve $\gamma : [0,1] \longrightarrow M^n$ of length $\leq L$ such that $\gamma(0) = \gamma(1) = p$, there exists a length decreasing path homotopy that connects this curve with p. Moreover, this homotopy depends continuously on a curve γ. In other words, the space of loops of length $\leq L$ based at p is contractible.*

Proof. A standard way to prove the assertion of the lemma is via the Birkhoff curve shortening process (BCSP), which is an explicit length nonincreasing deformation of the space of loops based at p of length $\leq L$, to the space of constant loops. One can find a detailed description of the BCSP for closed curves in [**C**]. The BCSP for loops is completely analogous. One just keeps the end points fixed. $\square$

Lemma 1.2. *Let M^n be a Riemannian manifold. Let $p, q \in M^n$. Let $\gamma_1(t), \gamma_2(t)$ be two curves connecting the point p to the point q of lengths l_1, l_2 respectively. Consider the curve $\gamma_2 * -\gamma_1$, that is a product of γ_2 and $-\gamma_1$. This curve is a loop based at p. If this loop is contractible to p by a path homotopy along the curves of length $\leq l_1 + l_2$, then there is a path homotopy $h_\tau(t), \tau \in [0,1]$, such that $h_0(t) = \gamma_1(t), h_1(t) = \gamma_2(t)$ and the length of curves during this homotopy is bounded above by $2l_1 + l_2$. (Note that by a path homotopy we mean a homotopy that fixes the end points of a curve.) Moreover, when M^n has no geodesic loops of length $\leq l_1 + l_2$, this path homotopy can be made to continuously depend on a digon formed by γ_1 and γ_2. Alternatively, we can find a path homotopy with the same properties in which the length of curves is bounded by $l_1 + 2l_2$.*

Proof. Let $\tilde{h}_\tau(t)$ be a homotopy that connects $\gamma_2 * -\gamma_1$ with a point p (see Fig. 2 (a) and (b)). Then let us consider the following homotopy $\gamma_1 \sim \tilde{h}_{1-\tau} * \gamma_1 \sim \gamma_2 * -\gamma_1 * \gamma_1 \sim \gamma_2$ (see Fig. 2 (a)-(g)). The length of curves during this homotopy is $\leq 2l_1 + l_2$.

Note that, assuming there are no geodesic loops of length $\leq l_1 + l_2$, one can contract $\gamma_2 * -\gamma_1$ via the BCSP, which will then continuously depend on the initial curve (see Lemma 1.1). Thus, the path homotopy between $\gamma_1(t)$ and $\gamma_2(t)$ will also continuously depend on the initial digon.

The last assertion follows from the fact that we can reverse the role of l_1 and l_2 and construct a path homotopy between l_2 and l_1 passing through curves of length $l_1 + 2l_2$. Then we reverse the direction of this path homotopy, obtaining a path homotopy from l_1 to l_2 with the required properties.

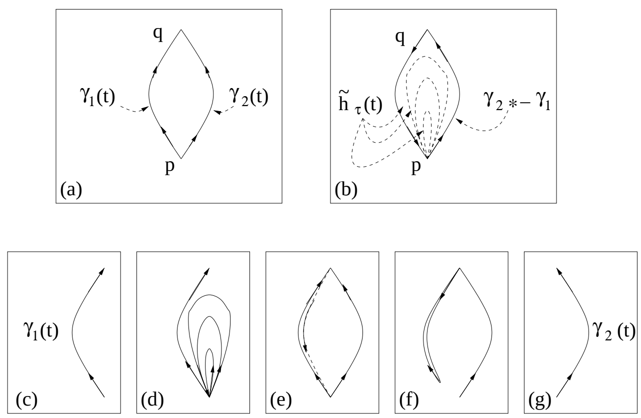

Figure 2. Illustration of the proof of Lemma 1.2.

□

(A similar argument is used by C.B. Croke to prove Lemma 3.1 in [**C**].)

Let us first provide a short explanation of the proof of Theorem 0.1. The proof of Theorem 0.3 will be similar.

Note that in the case of a manifold with $\pi_1(M^n) \neq \{0\}$ (which is the case of Theorem 0.1 in which $q = 1$), there are many easy and well-known proofs. For instance, given a closed Riemannian manifold M^n with a nontrivial fundamental group and a point $p \in M^n$, one can consider its universal covering space E together with the covering metric. Consider two distinct points $p_1, p_2 \in \pi^{-1}(p)$ from the adjacent fundamental domains, where $\pi : E \longrightarrow M^n$ is a covering map. Join two points by a minimal geodesic in the covering space of length $\leq 2d$, where d is the diameter of M^n. This geodesic segment projects to a geodesic loop based at p of length $\leq 2d$. Another proof, as suggested by the referee, goes as follows: consider a shortest noncontractible geodesic loop based at a fixed point $p \in M^n$. Suppose that its length $L > 2d$, and connect the midpoint of the loop with p by a shortest geodesic segment of length $\leq d$. We thus obtain two loops based at p, each of length $< L$, at least one of them noncontractible, and reach a contradiction with the assumption that the original loop was shortest.

We will, however, present a different, thought slightly more complicated, proof of the lemma, which is more indicative of what will happen in higher dimensions.

Lemma 1.3. *Let M^n be a closed Riemannian manifold with a non-trivial fundamental group. Then at each point $p \in M^n$ there exists a geodesic loop of length $\leq 2d$.*

Proof. Consider any non-contractible map $f : S^1 \longrightarrow M^n$. Suppose S^1 is partitioned (triangulated) into very small segments, so that the diameter of each edge in the induced triangulation on $f(S^1)$ is smaller than some $\delta > 0$. Let D^2 be the standard disc that is triangulated as a cone over S^1. Assume that for some $p \in M^n$ the length of a shortest geodesic loop $l_p > 2d + \delta$. We will show that in this case we can extend $f : S^1 \longrightarrow M^n$ to D^2, thus reaching a contradiction with the fact that this map is non-contractible. The extension procedure will be inductive to skeleta of D^2. The 0-skeleton of D^2 consists of one additional simplex, namely, the center of the disc that we will denote by $\tilde{p}$. We will let $f(\tilde{p}) = p$. Next to extend to the 1-skeleton, consider an arbitrary edge of the form $[\tilde{p}, \tilde{v}_i]$, where $\tilde{v}_i$ is a vertex of the triangulation of S^1. We will assign to this edge a minimal geodesic segment $[p, v_i]$ connecting the point p with $v_i = f(\tilde{v}_i)$. Next to extend to the 2-skeleton, consider a 2-simplex $[\tilde{p}, \tilde{v}_i, \tilde{v}_{i+1}]$. The boundary of this simplex is mapped to a closed curve of length $\leq 2d + \delta$, consisting of two minimizing geodesic segments and an edge $[v_i, v_j]$ of length $\leq \delta$. This curve passes through p. Let us apply the BCSP with a fixed p. Since we have assumed that there are no geodesic loops of length $\leq 2d + \delta$ based at p, this curve is contractible to p. Thus, we can assign to the above 2-simplex, surface generated by the homotopy contracting this curve to p. Therefore, we have succeeded at extending $f : S^1 \longrightarrow M^n$ to D^2, which contradicts our assumption about non-contractibility of f.

This shows that there must be a geodesic loop of length $\leq 2d + \delta$ based at p. We conclude by letting δ approach 0, which shows that there must be a geodesic loop of length $\leq 2d$. □

Now we are going to explain how to prove Theorem 0.1 in the case when $\pi_1(M^n) = \{0\}$, but $\pi_2(M^n) \neq \{0\}$. This is the case of Theorem 0.1, in which $q = 2$. This will only be done to better illustrate the ideas of the proof in the general case.

Proposition 1.4. *Let M^n be a closed Riemannian manifold with the trivial fundamental group, but with $\pi_2(M^n) \neq \{0\}$. Then at each point $p \in M^n$ there exists at least one (non-constant) geodesic loop of length $\leq 4d$.*

Proof. Let $f : S^2 \longrightarrow M^n$ be a non-contractible map from the standard 2-dimensional sphere to M^n. Suppose S^2 is endowed with a fine triangulation in such a way that the diameter of any simplex in the induced triangulation of $f(S^2)$ is smaller than some $\delta > 0$. Furthermore, suppose that D^3 is a disc that is triangulated as the cone over S^2. Assume that $l_p(M^n) > 4d$ for some $p \in M^n$. We will extend the map $f : S^2 \longrightarrow M^n$ to D^3, thus reaching a contradiction. The procedure will be inductive to skeleta of D^3. We will begin by extending f to the 0-skeleton of D^3 that consists of a single additional point $\tilde{p}$ at the center of the disc. We will let the image of $\tilde{p}$ be the given point $p \in M^n$. Next, let us extend to the 1-skeleton as follows: we will assign to an edge $[\tilde{p}, \tilde{v}_i]$ that connects the center of the disc with a vertex $\tilde{v}_i$ a minimal geodesic segment $[p, v_i]$ connecting the point p with the vertex $v_i = f(\tilde{v}_i)$. Next we extend to the 2-skeleton. Consider an arbitrary 2-simplex $\tilde{\sigma}_i = [\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_2}]$. Its boundary $\partial\tilde{\sigma}_i^2 = [\tilde{p}, \tilde{v}_{i_1}] - [\tilde{p}, \tilde{v}_{i_2}] + [\tilde{v}_{i_1}, \tilde{v}_{i_2}]$ is mapped to a closed curve of length $\leq 2d + \delta$. Assuming that the length of a shortest geodesic loop based at p is greater than $2d + \delta$, this curve can be contracted to a point by the BCSP that fixes p, i.e., all the curves in the homotopy will start and end at p. Recall that the BCSP denotes the Birkhoff Curve Shortening Process (see the proof of Lemma 1.1). We will map the 2-simplex to the surface generated by this homotopy, which will be denoted as σ_i^2.

Note that we should not be able to extend the map f any further. That means that there exists a 3-simplex $\tilde{\sigma}_i^3$ such that $f : \partial\tilde{\sigma}_i^3 \longrightarrow M^n$ is not contractible. On the other hand, let $\tilde{\sigma}_i^3 = [\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_2}, \tilde{v}_{i_3}]$. Then $\partial\tilde{\sigma}_i^3 = \Sigma_{j=0}^3(-1)^j[\tilde{v}_{i_0}, \ldots \hat{\tilde{v}}_{i_j}, \ldots, \tilde{v}_{i_3}]$, where $\tilde{v}_{i_0} = \tilde{p}$. Let us denote $[\tilde{p}, \tilde{v}_{i_j}]$ by $\tilde{e}_j$ and $[p, v_{i_j}]$ by e_j. Since $[v_{i_1}, v_{i_2}, v_{i_3}]$ can be made arbitrarily small, we will treat it here as a point q for the sake of simplicity of the exposition (see the Remark and Lemma 1.5 that follows the proof). Note also that assuming that there are no geodesic loops based at p of length $\leq 4d$ we can contract $f : \partial\tilde{\sigma}_i^3 \longrightarrow M^n$ to the point p as follows:

1. By Lemma 1.2 there is a path homotopy between e_1 and e_2 that passes through curves $e_{\tau_{12}}$, where $1 \leq \tau_{12} \leq 2$ of length $\leq 3d$. This homotopy continuously depends on e_1 and e_2. We claim that it can be used to construct a homotopy between the above sphere and a point.
2. We will define $S^2_{\tau_{12}}$ as follows: consider the two points p and q joined by two geodesic segments e_2, e_3 and the curve $e_{\tau_{12}}$, (see fig. 3 (a)). Assuming that there are no geodesic loops based at p of length $\leq 4d$, both curves $e_2 * -e_{\tau_{12}}$ and $e_{\tau_{12}} * -e_3$ are contractible

to the point p without their lengths increasing by the BCSP. Let us call the discs obtained during this homotopy $(D_2^2)_{\tau_{12}}$ (see Fig. 3 (b)), and $(D_3^2)_{\tau_{12}}$ respectively. They change continuously with τ_{12}. Then $S^2_{\tau_{12}}$ is obtained by the obvious gluing of the three discs: $\sigma^2_{i_0,i_2,i_3}$, $(D_2^2)_{\tau_{12}}$ and $(D_3^2)_{\tau_{12}}$ along their boundaries. There $\sigma^2_{i_0,i_2,i_3}$ denotes the two-dimensional simplex obtained by filling the digon formed by e_2 and e_3 by a 2-disc during the previous step of the induction. Note that when $\tau_{12}=1$, $S^2_{\tau_{12}}$ is the original sphere and when $\tau_{12}=2$ it is a sphere constructed as follows: we begin with two points p and q, join them with three segments, two of which coincide: e_2, e_2, e_3. Next obtain three discs. One of them is degenerate and constructed by contracting a curve $e_2 * -e_2$ along itself, and the other two coincide, but have opposite orientation: both of the discs are obtained by contracting the curve e_2*-e_3, but the curves in each disc are taken with the opposite orientations (see Fig. 3 (c)). So the sphere that we obtain consists of two identical discs with opposite orientation glued along their boundary, and it is contractible along itself. Thus, we obtain a homotopy between the above sphere and a point and, therefore, reach a contradiction.

□

Here it is also important to note that the main idea behind the above proof is the following: Given two points and three geodesic segments connecting them and faces between each pair of geodesics that together form a 2-sphere, we can first construct a 3-disc that “fills” this 2-sphere, assuming that there are no geodesic loops of length $\leq 4d$. In fact, the faces were first constructed from each of the three given pairs using the assumption that there are no geodesic loops of length $\leq 2d$.

A similar operation can be performed for a pair of points and arbitrary three paths that connect those points, assuming there are no “short” geodesic loops. Moreover, these fillings continuously depend on the initial triple of paths. A multidimensional generalization of this idea will be used later in the proof.

Remark. Let M^n be a closed Riemannian manifold. Let us consider a sphere in the manifold M^n obtained by taking a small 2-simplex $[v_{i_1}, v_{i_2}, v_{i_3}]$ and a point p, connecting p with each v_{i_j} by a minimal geodesic segment $e_j, j=1,2,3$, and finally, by contracting each of the closed curves $e_j + [v_{i_j}, v_{i_{j \bmod 3+1}}] - e_{j \bmod 3+1}$, where $j=1,2,3$ to the point p as loops (see Fig. 4). Denote this sphere by S_0. Next, let us define a sphere S_1. Take a point $q \in [v_{i_1}, v_{i_2}, v_{i_3}]$, and connect it with each v_{i_j} in $[v_{i_1}, v_{i_2}, v_{i_3}]$ by a short segment σ_j (see Fig. 4 (b)), $j=1,2,3$ of length $\leq \delta$. Then instead of curves e_j we can consider the new curves $e_j^* = e_j + \sigma_j$ of length $\leq d+\delta$ (see Fig. 4 (c)). Note that each of the digons of the form $e^*_{j \bmod 3+1} * -e^*_j$ is contractible to p as loops without

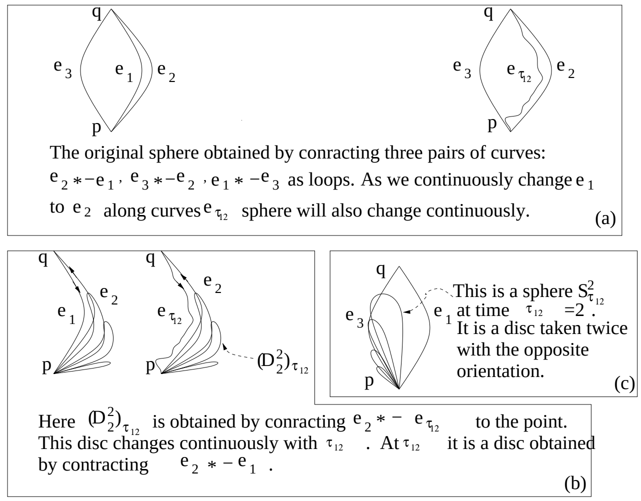

Figure 3. Construction of $S^2_{\tau_{12}}$

the length increase (see Fig. 4 (d)). These three homotopies give rise to three discs. Gluing them together results in a sphere that we will denote by S_1.

Lemma 1.5. *For small enough δ the spheres S_0 and S_1 are homotopic.*

Proof. For small δ spheres S_0 and S_1 are homotopic via the intermediate spheres S_t that are depicted on fig. 5. □

Therefore, S_1 is not contractible whenever S_0 is not contractible. Note also, that one can apply Lemma 1.2 to show that e_j^* is path homotopic to $e^*_{j \bmod 3+1}$ and the length of curves in this path homotopy is bounded by $3d + 3\delta$. We can eventually let δ go to 0. Thus, we can contract S_1 instead of S_0 in the proof of Proposition 1.4. So, for all practical purposes, $[v_{i_1}, v_{i_2}, v_{i_3}]$ can be treated as a point q.

Now, let us present the proof of the main theorem.

2. The proof of Theorem 0.1

Before giving the proof of Theorem 0.1 in the general case, let us describe the main ideas behind it. Let M^n be a closed Riemannian manifold, and suppose that $q > 0$ is the smallest natural number such that $\pi_q(M^n)$ is not trivial. We consider a non-contractible sphere $f : S^q \longrightarrow M^n$ and show that, assuming there are no "short" geodesic loops, it can be filled by a disc. To construct this disc we use the following bootstrap procedure of constructing spheres and discs of progressively growing dimensions. One begins with two points p and q joined by k segments. Now, to construct a sphere of dimension $s < k$, one selects

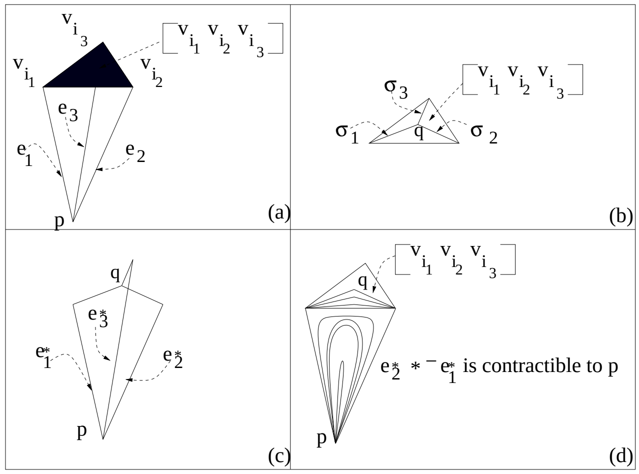

Figure 4. Small 2-simplex can be ignored.

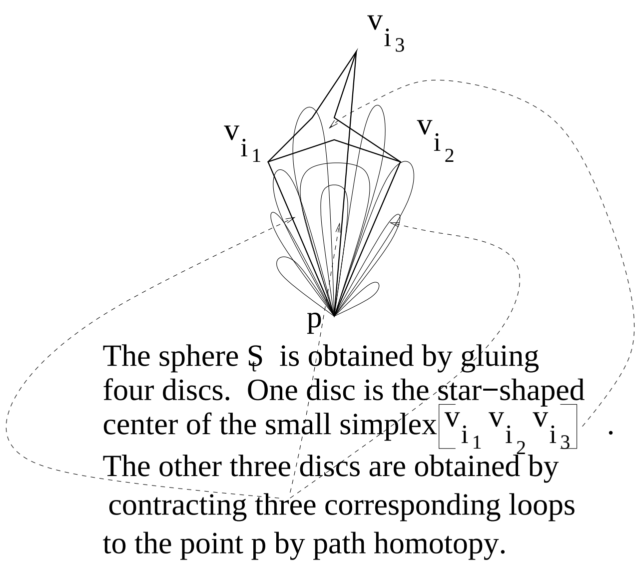

Figure 5. Spheres S_t.

$s+1$ segments. The sphere is constructed by a natural gluing of $s+1$ s-discs. These discs are glued as the simplices in the boundary of $s+1$-dimensional simplex, where one of the simplices degenerates to a point.

Each such disc corresponds to s segments that are selected out of the given $s+1$ segments, and is generated by a one-parameter family of $(s-1)$-dimensional spheres that starts with a sphere that is constructed from those s-segments on the previous step of induction and ends with a point.

Definition 2.1. An m-cage Cg^m is a graph on a Riemannian manifold M^n that consists of two vertices $p, q \in M^n$ and m piecewise smoothly immersed edges $e_1, \ldots, e_m$ that connect these vertices. We will denote the space of all m cages with vertices p, q by $C_{p,q,m}$.

Thus, the idea behind the proof of Theorem 0.1 is that in the absence of short geodesic loops, short m-cages can be "filled" first by spheres and then by discs.

Definition 2.2. Let $C^{l_1,l_*}_{p,q,m} \subset C_{p,q,m}$ be a set of m-cages, such that the length of e_1 is bounded by l_1 and the length of every other edge $e_i, i > 1$ is bounded above by l_*. An N-filling of cages (from $C^{l_1,l*}_{p,q,m}, m \leq N$) is a set of continuous maps $\phi_m : C^{l_1,l_*}_{p,q,m} \longrightarrow C(\sigma^m, M^n)$ for all $m = 1, 2, 3, \ldots, N$, where $C(\sigma^m, M^n)$ denotes the space of singular m-simplices in M^n satisfying the following properties:

(1) $\phi_m(Cg^m)$ maps the $(m-1)$-dimensional face $[v_1, \ldots, v_m]$ of the standard simplex $[v_0, \ldots, v_m]$ into q.
(2) $\phi_m(Cg^m)$ maps the edges $[v_0, v_i]$ to edges e_i of Cg^m.
(3) The restriction of ϕ_m to $[v_0, \ldots, \hat{v}_i, \ldots, v_m]$ coincides with $\phi_{m-1}(Cg^{m-1}_i)$, where Cg^{m-1}_i denotes the cage obtained from Cg^m by removing e_i.

The informal meaning of this definition is the following. An N-filling is a way to "fill" every "short" i-cage by an i-disc for every i. (The edges of cages then become meridians in the boundary sphere of this i-disc.) Moreover, the i-disc should depend continuously on the i-cage. In addition, one should have the coherence condition (Definition 2.2, part (3)) that asserts that the appropriate restrictions of a "filling" of an i-cage should provide fillings of the $(i-1)$-cages obtained from the i-cage by deleting one of its edges. By a "short" i-cage we mean a cage such that the length of its first edge is $\leq l_1$ and the lengths of remaining edges are $\leq l_*$. The following lemma asserts the existence of such an N-filling provided that there are no "short" geodesic loops. This lemma plays a crucial part in the proof of Theorem 0.1, as the existence of such an N-filling almost immediately leads to a contradiction with the assumption that $\pi_q(M^n) \neq \{0\}$.

Lemma 2.3. *Let N be a positive integer. Fix a point $p \in M^n$ and let q be any point in M^n. Suppose there are no geodesic loops in M^n of length $\leq l_1 + (2N-3)l_*$ based at p. Then there exists an N-filling of m-cages in $C^{l_1,l_*}_{p,q,m}$, $m = 1, 2, 3, \ldots, N$.*

Proof. The proof is by induction with respect to N. In the case of $N = 1, \phi_1(Cg^1) = Cg^1$. Suppose, we have constructed a filling of $(N-1)$-cages in $C^{l_1,l_*}_{p,q,N-1}$. We will next construct an N filling. In order to do that, consider a cage Cg^N that is obtained by connecting points

p and q by segments $e_1, \dots, e_N$ each of length $\leq l_*$. By Lemma 1.2 there exists a path homotopy between e_1 and e_2 that passes through curves $e_{\tau_{12}}$ of length at most $l_1 + 2l_*$. At each time $1 \leq \tau_{12} \leq 2$ consider a cage $Cg^N_{\tau_{12}}$ that consists of vertices p and q joined by N segments $e_{\tau_{12}}, e_2, \dots, e_N$. For each $(N-1)$-tuple selected out of this N tuple, there exists an $(N-1)$-filling of this $(N-1)$-tuple because the length of $e_{\tau_{12}}$ plus $(2(N-1)-3)l_*$ is less than or equal to $l_1 + 2l_* + (2(N-1)-3))l_* = l_1 + (2N-3)l_*$, which, by our hypothesis, is the lower bound for the length of a shortest geodesic loop at p required to apply the induction assumption. Thus, by induction hypothesis, we have an $(N-1)$-disc corresponding to each $(N-1)$-tuple. Glue these discs together as in the boundary of an N-simplex and obtain an $(N-1)$-dimensional sphere $S^{N-1}_{\tau_{12}}$. At the time $\tau_{12} = 2$, $S^{N-1}_{\tau_{12}}$ corresponds to two identical, but oppositely oriented discs, by which we mean that the curves that generate these discs are oppositely oriented. Thus, this sphere can be deformed to a point over itself. The resulting 1-parameter family of spheres generates an N-disc, that corresponds to an N-filling of $e_1, \dots, e_N$. As the BCSP is continuous with respect to a loop that is being contracted, and as the $(N-1)$-filling that has already been constructed in the previous step of the induction is also continuous (continuity being a part of its definition), the N-filling also continuously depends on $e_1, \dots, e_N$. $\square$

Remark. Note that this construction of N-fillings is recursive. That is, in order to construct the N-filling, we use the construction of $(N-1)$-fillings.

Proof of Theorem 0.1. Assume that the length of a shortest geodesic loop based at p is $\geq 2qd$. Then, according to Lemma 2.3, there exists a filling of all m-cages $e_1, \dots, e_m$, where $m \leq q$, the length of e_1 is $\leq 3d$ and the length of e_i is $\leq d$ for $i > 1$.

Let $f : S^q \longrightarrow M^n$ be a non-contractible map. Assume S^q is triangulated into fine simplices, and that $f(S^q)$ has induced triangulation such that diameter of any simplex in this triangulation is smaller than δ. Number all vertices of this triangulation by integers beginning with 1. Let D^{q+1} be triangulated as a cone over S^q. Assuming that the length of a shortest geodesic loop based at $p \in M^n$ is greater than $2qd$, we will extend our map to D^{q+1}, thus reaching a contradiction. To construct this extension we use induction on skeleta of D^{q+1}. To perform the first step of the induction procedure, we will map the center of the disc $\tilde{p}$ to the point p above. Then we map new 1-simplices $[\tilde{p}, \tilde{v}_{i_0}]$ into arbitrary minimizing geodesics connecting p with $v_{i_0} = f(\tilde{v}_{i_0})$. On each of the remaining steps we proceed as follows: we need to fill a sphere $S_0 = f : \partial[\tilde{p}, \tilde{v}_{i_1}, \dots, \tilde{v}_{i_m}] \longrightarrow M^n$, $i_1 < i_2, \dots, < i_n$, that is constructed in the previous steps. Take a point $v^* \in [v_{i_1}, \dots, v_{i_m}] = f([\tilde{v}_{i_1}, \dots, \tilde{v}_{i_m}])$.

Extend paths e_j connecting p with v_{i_j} by adding small paths σ_j connecting v_{i_j} with v^*. Denote the result by e_j^*. Construct the sphere S_1 by filling m $m-1$ cages $e_1^*, \ldots, \hat{e}_{i_j}^*, \ldots, e_m^*$ as in Lemma 2.3. The spheres S_0 and S_1 are homotopic (see the Remark at the end of the previous section). The homotopy between S_0 and S_1 is based on shrinking small simplex $[v_{i_1}, \ldots, v_{i_m}]$ over itself to v^* (see Fig. 5). Finally, we contract S_1 using the m-filling constructed in Lemma 2.3. (S_1 will be the boundary of the m-disc that "fills" $e_1^*, \ldots, e_m^*$ because of condition (3) in Definition 2.2, which guarantees that the fillings agree as the dimension increases.) □

For the sake of exposition, we are going to give a more detailed explanation of how the extension process in the above proof of Theorem 0.1 via Lemma 2.3 really works. This explanation can be regarded as a less formal but more transparent version of the above proof. In Section 1 we have already discussed the extension to the 3-skeleton. Let us now describe how to extend to the 4-skeleton:

Assume that we have already extended to the 1, 2, and 3-skeleta, as described in Section 1. Let us denote the image of a 3-simplex $\tilde{\sigma}^3_{i_0,\ldots,i_3} = [\tilde{v}_{i_0}, \tilde{v}_{i_1}, \ldots, \tilde{v}_{i_3}]$, where $\tilde{v}_{i_0} = \tilde{p}$ by $\sigma^3_{i_0,\ldots,i_3}$.

Now suppose we want to extend our map to the 4-skeleton. Let us consider an arbitrary simplex $\tilde{\sigma}_{i_0,i_1,i_2,i_3,i_4} = [\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_2}, \tilde{v}_{i_3}, \tilde{v}_{i_4}]$. Its boundary is mapped to the following 3-sphere $\Sigma_{j=0}^4 (-1)^j \sigma_{i_0,..,\hat{i}_j,\ldots,i_4}$. Now let us construct the following homotopy contracting this sphere to a point. Again, without loss of generality, assume that the simplex $[v_{i_1}, \ldots, v_{i_4}]$ is so small that it can, for our purposes, be treated as a point, which we will denote by q. Each of the four edges $[p, v_{i_j}]$ will be denoted by e_j. So we want to demonstrate that a 4-cage Cg^4 that consists of points p, q and four edges e_1, e_2, e_3, e_4 can be coherently filled by a 4-disc. We know that e_1 is homotopic to e_2 by a path homotopy along the curves $e_{\tau_{12}}$, $1 \leq \tau_{12} \leq 2$ of length $\leq 3d$ (see Lemma 1.2). Let us "move" e_1 to e_2 and construct a homotopy of the 3-sphere that will "follow" this move. That is, for each τ_{12}, we want to construct a sphere $S^3_{\tau_{12}}$ that continuously depends on τ_{12}. This sphere will be made of four discs glued together. These discs are glued as four simplices in the boundary of the 4-simplex, where the fifth simplex degenerates to a point.

Disc $(D^3_1)_{\tau_{12}}$ will stay constantly equal to $\sigma^3_{i_2,i_3,i_4}$.

$(D^3_2)_{\tau_{12}}$ is constructed as follows: take two points p, q connected by three segments: $e_{\tau_{12}}, e_3, e_4$ (see Fig. 6 (a)). We know that in this situation, we can construct a sphere $S^2_{\tau_{12}}$ and also continuously deform it to a point as follows:

1. We construct $S^2_{\tau_{12}}$ by taking three loops $e_3 * -e_{\tau_{12}}$, $e_4 * -e_3$, $e_{\tau_{12}} * -e_4$ and contracting them to p by a length decreasing path

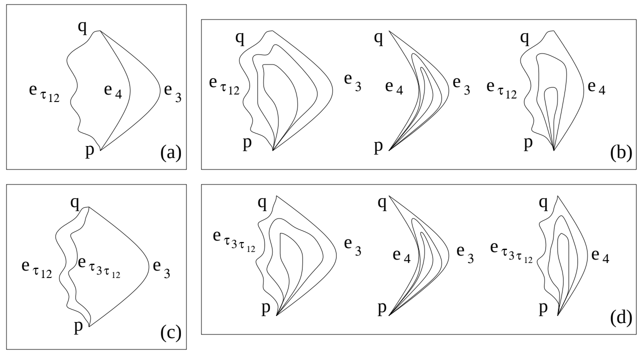

Figure 6. Constructing $(D^3_2)_{\tau_{12}}$.

homotopy (see Fig. 6 (b)). Here we use the assumption that the length of a shortest geodesic loop at p is greater than $2qd$, and, thus, greater than $4d$. So each of the loops is contractible to p without the length increase.

2. Now, by Lemma 1.2 there exists a path homotopy that connects $e_{\tau_{12}}$ with e_3 along the curves $e_{\tau_{3\tau_{12}}}$, $1 \leq \tau_{3\tau_{12}} \leq 2$ of length $\leq 5d$. This is due to the fact that the loop $e_{\tau_{12}} * -e_3$ is contractible to p without the length increase (see Fig. 6 (c)).
3. As $e_{\tau_{12}}$ moves to e_3, we use the fact that the length of a geodesic loop is also greater than $6d$ to construct a family of 2-dimensional spheres $S^2_{\tau_{3\tau_{12}}}$ that continuously depends on $\tau_{3\tau_{12}}$ and that coincides with $S^2_{\tau_{12}}$, when $\tau_{3\tau_{12}} = 1$. That is we repeat Step 1, but with $e_{\tau_{3\tau_{12}}}$ replacing $e_{\tau_{12}}$. Note also that when $\tau_{3\tau_{12}} = 2$, we obtain a degenerate sphere, consisting of a 2-disc taken twice with the opposite orientation, that can be contracted to a point. This family of spheres corresponds to a 3-disc $(D^3_2)_{\tau_{12}}$. Note that at $\tau_{12} = 1$ it is $\sigma^3_{i_1,i_3,i_4}$ and at $\tau_{12} = 2$ it is $-\sigma^2_{i_2,i_3,i_4}$.
4. The other two discs $(D^3_3)_{\tau_{12}}$ and $(D^3_4)_{\tau_{12}}$ are obtained in a similar way.
5. The sphere $S^3_{\tau_{12}}$ is obtained by the obvious gluing. Furthermore, S^3_1 is the original sphere and S^3_2 is a sphere that is obtained by gluing $\sigma^3_{i_2,i_3,i_4}$ and $-\sigma^3_{i_2,i_3,i_4}$, and so it is contractible to a point. We will map $\tilde{\sigma}^4_{i_0,\ldots,i_4}$ to the disc generated by this family of 3-spheres. Let us denote this disc by $\sigma^4_{i_0,\ldots,i_4}$.

Now suppose we have extended the map f to the k-skeleton of D^{q+1} in a similar fashion and now we want to extend it to the $(k+1)$-skeleton. Consider an arbitrary $(k+1)$-simplex $\tilde{\sigma}^{k+1}_{i_0,\ldots,i_{k+1}}$. Its boundary is mapped to $\Sigma_{j=0}^{k+1}(-1)^j\sigma^k_{i_0,\ldots,\hat{i}_j,\ldots,i_{k+1}}$. As before, we observe that $\sigma_{i_1,\ldots,i_{k+1}}$ can be treated as a point denoted by q, and we denote edges $[p, v_{i_j}]$ as e_j.

We know that there is a path homotopy between e_1 and e_2 that passes through the curves $e_{\tau_{12}}$ of length $\leq 3d$. We can extend this homotopy to the homotopy between the underlying map of the boundary of a simplex $\tilde{\sigma}^{k+1}_{i_0,\dots,i_{k+1}}$ and a k-sphere that will then be contracted to a point. This homotopy can be explained as follows. Its image is a $q+1$-dimensional disc $\sigma^{k+1}_{i_0,\dots,i_{k+1}}$ that is generated by the family of spheres $S^k_{\tau_{12}}$, $1 \leq \tau_{12} \leq 2$, such that $S^k_1 = \partial\sigma^k_{i_0,\dots,i_{k+1}} = f(\partial\tilde{\sigma}^k_{i_0,\dots,i_{k+1}})$ and S^k_2 is a sphere that consists of two copies of a k-disc with opposite orientations glued along their common boundary and is contractible along itself. This family of spheres $S^k_{\tau_{12}}$ is constructed by taking two points p, q, joining them by $e_{\tau_{12}}, e_2, \dots, e_{k+1}$, and repeating the whole process of constructing the k-sphere based on two vertices and $k+1$ curves connecting them, but with $e_{\tau_{12}}$ replacing e_1. (We learned to construct such k-spheres on the previous step of induction.) As the length of $e_{\tau_{12}}$ can exceed the length of e_1 by $2d$, so the length of curves in all of the homotopies can increase by $2d$ as well. At this step we use the assumption that $l_p(M^n) > 2qd > 2kd$. Recall that the family of spheres $S^k_{\tau_{12}}$ is constructed by gluing of k discs. The disc $(D^k_1)_{\tau_{12}}$ will be constantly equal to $\sigma^k_{i_2,\dots,i_{k+1}}$. And, of course, $(D^k_2)_{\tau_{12}} = D^k_2(\tau_{12})$ is constructed using the previous step of an inductive construction: we begin with the two points p, q joined by k segments: $e_{\tau_{12}}, e_3, \dots, e_{k+1}$. The disc is constructed by constructing a family of spheres $S^{k-1}_{\tau_3\tau_{12}}$ that start with a sphere $S^{k-1}_{\tau_{12}}$ and with a sphere that is easily contractible to a point, which we already learned to do at the previous stage, etc. Other discs are constructed in a similar fashion.

Thus, we can continue until we extend to the $(q+1)$-skeleton of D^{q+1}, reaching a contradiction.

3. Proof of Theorem 0.3

To prove Theorem 0.3 we will first need to define the notion of N-filling of complete graphs in Riemannian manifolds that is similar to the notion of N-filling of cages introduced in the previous section.

Definition 3.1. For a fixed value of l consider the spaces $K_{m,l}$ of piecewise smooth immersions of the 1-skeleton of the standard simplex σ^{m+1} (or, equivalently, of the complete graph with $(m+2)$ vertices) into M^n, such that each edge is mapped into a curve of length $\leq l$. An *N-filling of $K_{m,l}$* is, by definition, a collection of continuous maps $\phi_m : K_{m,l} \longrightarrow C(\sigma^{m+1}, M^n)$, $m = 1, 2, \dots, N$, satisfying the following properties:

1) For every $k \in K_{m,l}$ the restriction of $\phi_m(k)$ to the 1-skeleton of σ^m coincides with k. (This property means that $\phi_m(k)$ fills k.)
2) For every $k \in K_{m,l}$, $(m \leq N)$, the restriction of $\phi_m(k)$ to a m-dimensional face of σ^{m+1} coincides with ϕ_{m-1} evaluated on the

element of $K_{m-1,l}$ obtained from k by restricting k to the set of all 1-dimensional simplices in the 1-skeleton of this face of σ^m.

Note that this definition, in particular, means that each n-filling agrees with all of its subfillings and depends continuously on its 1-skeleton.

Lemma 3.2. *Assume that the length of the shortest geodesic loop on M^n is greater than $\bar{c}(n)l$, where $\bar{c}(n) = 3 \cdot 4^{n-1}$. Then there exists an n-filling of $K_{m,l}$, $m = 1, \ldots, n$.*

Proof. We are going to prove the existence of i-fillng of $K_{m,l}$, $(m \leq i)$, for every $i \leq n$. The proof will be by inducion with respect to i. On every step we already have the restriction of ϕ_j that we want to construct to $\partial\sigma^{j+1}$. This restriction is uniquely determined by condition 2 of the definition of N-fillings and, if $i > 1$, by the previous steps of induction. In other words, for every $k \in K_{j,l}$ we already have a filling of k by a j-sphere. It remains only to contract this sphere. The idea is to find a path k_t connecting k with a complete graph with $(j+2)$ vertices immersed in M^n such that all its edges are mapped to some paths in a tree. This path k_t should continuously depend on k. Filling all k_ts by spheres S^j_t we obtain a homotopy between the sphere $\phi_j(\partial\sigma^{j+1})$ and the degenerate sphere S^j_1 that lives in a tree and is, therefore, contractible.

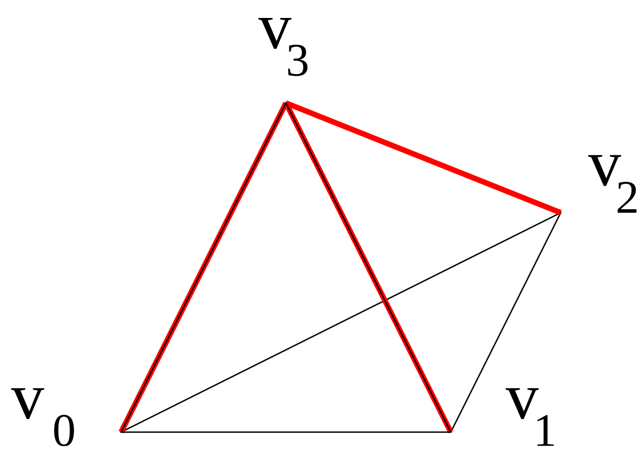

Figure 7. Collapsing triangles

To describe the homotopy k_t we introduce the notion of a *collapsing of a triangle*. Let k_a, k_b, k_c be any three edges of k. As there are no geodesic loops of length $\leq 3l$ in M^n, we can apply Lemma 1.2 to construct a path homotopy between k_a and $k_b * k_c$. This homotopy passes through paths of length $\leq 2\,\text{length}\,(k_a) + \text{length}\,(k_b) + \text{length}\,(k_c) \leq 4l$. This homotopy induces a homotopy of triangles $(k_a)_t, k_b, k_c, t \in [0,1]$ that we call a collapsing of the triangle k_a, k_b, k_c. At the end of this homotopy k_a is being replaced by another edge that goes along $k_b * k_c$, and the considered triangle becomes thin.

After collapsing finitely many triangles, we can obtain an element of $K_{j,l}$, where all edges run along the tree-shaped union k_1 of edges of k adjacent to one vertex of k, let's say the vertex with the highest number (see Fig. 7, which illustrates that the edge $[v_0, v_1]$ is being collapsed to the path $[v_0, v_3, v_1]$, the edge $[v_1, v_2]$ is being collapsed to $[v_1, v_3, v_2]$, and the edge $[v_0, v_2]$ is being collapsed to $[v_0, v_3, v_2]$).

Now we can continue the homotopy of complete graphs by contracting all edges of k_1 to a point (to v_3 on Fig. 7) via the considered tree by a length non-increasing homotopy. We can fill these complete graphs by j-spheres using the induction assumption.

However, note that in order to construct the sphere S^j_t we will need to apply the induction assumption to $K_{m,4l}$, as the length of edges obtained during the collapsing of triangles is bounded above by $4l$. Therefore, $\bar{c}(j+1) \leq 4\bar{c}(j)$. Note that Lemma 1.2 immediately implies that $\bar{c}_1(1) = 3$. Indeed, if $j = 1$, we need to contract a triangle of perimeter $\leq 3l$. Thus, we can take $\bar{c}(n) = 3 \cdot 4^{n-1}$. $\square$

Proof of Theorem 0.3. Let $X = L^\infty(M^n)$. The fact that $\mathrm{FillRad}\,(M^n)$ is the filling radius of M^n means, by definition, that for every $\varepsilon > 0$ there exists a singular chain c in the $(\mathrm{FillRad}\,(M^n) + \frac{\varepsilon}{2})$-neighborhood of M^n in X, such that $[M^n] = \partial c$. Using a simplicial approximation we can replace c by a polyhedron W^{n+1} in the $(\mathrm{FillRad}\,(M^n)+\varepsilon)$-neighborhood of M^n. So, M^n bounds a polyhedron W^{n+1} in its $(\mathrm{FillRad}\,(M^n) + \varepsilon)$-neighborhood. (This simple fact had been first observed and used by Gromov, see Statement 1.2.C on page 10 in [**G**].) Also, it is important to remember that the (Kuratowski) embedding of M^n in X is isometric.

Assume that the length of a shortest geodesic loop in M^n, $\alpha(M^n)$ satisfies the following inequality:

$$(*) \qquad \alpha(M^n) > 2\bar{c}(n)\mathrm{vol}\,(M^n)^{\frac{1}{n}} = 6 \cdot 4^{n-1}\ \mathrm{FillRad}\,(M^n).$$

We are going to bring this assumption to a contradiction by extending the identity map of $M^n = \partial W^{n+1}$ into itself to a map τ of $W^{n+1} \longrightarrow M^n$, which would imply $\mathrm{FillRad}\,(M^n) = 0$. (To see that this extension is impossible, consider the induced homomorphisms on the nth homology groups of M^n and W^{n+1} with coefficients in $\mathbf{Z}$ if M^n is orientable, and with coefficients in $\mathbf{Z_2}$ if M^n is not orientable, and observe that the homomorphism induced by the inclusion of M^n into W^{n+1} is trivial. Yet the composition of the homomorphisms induced by τ and the inclusion is the identity homomorphism. This is clearly impossible.)

To construct τ we first consider a very fine triangulation of W^{n+1}, such that the diameter of any simplex of W^{n+1} does not exceed ε. Each of the 0-simplices of W^{n+1} will be mapped to one of the closest points in M^n (in the metric of the ambient space X). Each of the 1-simplices of $W^{n+1} \setminus M^n$ will be mapped into one of the minimizing geodesics in M^n between the images of the endpoint of this simplex. Let v_{i_1}, v_{i_2} denote the respective images of two vertices $\tilde{v}_{i_1}, \tilde{v}_{i_2}$ of W^{n+1}. Then $\operatorname{dist}(v_{i_1}, v_{i_2}) \leq \operatorname{dist}(v_{i_1}, \tilde{v}_{i_1}) + \operatorname{dist}(\tilde{v}_{i_1}, v_{i_2}) \leq \operatorname{dist}(v_{i_1}, \tilde{v}_{i_1}) + \operatorname{dist}(\tilde{v}_{i_1}, \tilde{v}_{i_2})$ $+\operatorname{dist}(\tilde{v}_{i_2}, v_{i_2}) \leq 2\text{FillRad} + \tilde{\varepsilon}$, where $\tilde{\varepsilon} = 3\varepsilon$. Thus, the length of the image of each 1-simplex of $W^{n+1} \setminus M^n$ does not exceed $2\text{FillRad}(M^n) + \tilde{\varepsilon}$, where $\tilde{\varepsilon}$ can be made arbitrarily small by selecting a sufficiently small ε and by refining the chosen triangulation of W^{n+1}.

Now the extension of τ to any closed $(n+1)$-dimensional simplex of $W^{n+1} \setminus M^n$ is equivalent to filling the image of its 1-skeleton. Enumerate all the vertices of the chosen triangulation of W^{n+1} by increasing successive integers. We will apply Lemma 3.2 to already constructed images of 1-skeleta of all $(n+1)$-dimensional simplices of W^{n+1}. In order to do that we need to number vertices of every $(n+1)$-dimensional simplex of W^{n+1} by numbers $0, 1, \ldots, n+2$. We use the already chosen numbering of all of the vertices of W^{n+1} and organize the vertices of every $(n+1)$-simplex in the increasing order. This would guarantee that, when we apply Lemma 3.2, every k-simplex will be filled in the same way when we consider it as a k-face of any $(n+1)$-dimensional simplex of W^{n+1} adjacent to it.

Using the assumption in $(*)$ we can apply Lemma 3.2 to all $(n+1)$-dimensional simplices of W^{n+1} providing that $\tilde{\varepsilon} < \frac{\alpha(M^n)}{3 \cdot 4^{n-1}} - 2\,\text{FillRad}(M^n)$. Its application completes the proof of the theorem. □

4. Proof of Theorem 0.5

Now we are going to prove Theorem 0.5. In this theorem we show that either there exists a "short" geodesic on a closed Riemannian manifold with a non-trivial second homology group, or at each point of this manifold there exist at least three distinct "short" geodesic loops, i.e., of length $\leq 2d$.

Also note that F. Balacheff had shown that on a closed Riemannian manifold with a non-trivial second homology group, there either exists a "short" closed geodesic or at least one "short" geodesic loop at each point of a manifold, (see [**Bl**]).

In 1951, A. Fet and L. Lusternik showed that on any closed Riemannian manifold M^n there exists at least one periodic geodesic. Their proof uses Morse theory on the space ΛM^n of all piecewise differentiable closed curves on M^n, by taking the smallest integer q, such that $\pi_q(M^n) \neq \{0\}$, which implies that $\pi_{q-1}(M^n) = \{0\}$, whereas $\pi_{q-1}(\Lambda M^n) \neq \{0\}$. One then demonstrates that, unless there is a periodic geodesic, any map $f: S^{q-1} \longrightarrow \Lambda M^n$ can be deformed to M^n along the integral curves of the energy functional. The idea of the proof of Theorem 0.5 is an effectivization of the proof of Fet and Lusternik's existence theorem. That is, assuming there is a point $p \in M^n$ at which there do not exist at least three geodesic loops of length $\leq 2d$, we will construct a non-contractible map $f : S^1 \longrightarrow \Lambda M^n$, where $\max_t \text{length}\,(f(t)) \leq 4d$. This will show that there exists a periodic geodesic on M^n of length $\leq 4d$. The map $f : S^1 \longrightarrow \Lambda M^n$ is constructed by providing appropriate sweep out of a non-contractible sphere $g : S^2 \longrightarrow M^n$ by circles. Though there are many ways of constructing sweep-outs of a 2-sphere, resulting in loops in the space ΛM^n of (piecewise differentiable) curves, the main idea behind the following proof is to construct sweep-outs with the controlled length upper bounds, assuming that at some point $p \in M^n$ there are not enough short geodesic loops.

Proof of Theorem 0.5. Once again, let us begin with a non-contractible map $f : S^2 \longrightarrow M^n$, where S^2 is the standard 2-sphere endowed with a fine triangulation. Let $p \in M^n$. We will try to extend this map to D^3 triangulated as a cone over S^2, which, of course, is impossible. The procedure will be inductive to the skeleta of D^3. We will begin as usual, by extending to the 0-skeleton. This is done by assigning to the center of the disc, $\tilde{p}$, the given point p. Next we extend to the 1-skeleton, by assigning to an edge $[\tilde{p}, \tilde{v}_i]$ a minimal geodesic segment $[p, v_i]$ of length smaller than d. Next we extend to the 2-skeleton. Consider an arbitrary 2-simplex $\tilde{\sigma}_{i_0,i_1,i_2}$. Its boundary is mapped to a closed curve of length $\leq 2d + \delta$. This curve is either contractible to p by a path homotopy, or there exists a geodesic loop based at p of length $\leq 2d$. In such a case we will release the point and will let the curve contract to a point by a regular (not path) homotopy (see Fig. 8). In either of these cases, the image of this simplex will be a disc generated by the homotopy connecting the curve with a point. We will denote it as σ_{i_0,i_1,i_2}.

It is impossible to extend $f : S^2 \longrightarrow M^n$ to the 3-skeleton of D^3. Therefore, there exists a 3-simplex $\tilde{\sigma}_{i_0,i_1,i_2,i_3}$ such that the map $f : \partial\tilde{\sigma}_{i_0,i_1,i_2,i_3} \longrightarrow M^n$ is a non-contractible sphere. Let us consider this sphere. It consists of three "big" discs: $-\tilde{\sigma}_{i_0,i_2,i_3}$, $\tilde{\sigma}_{i_0,i_1,i_3}$, $-\tilde{\sigma}_{i_0,i_1,i_2}$, and a "small" one σ_{i_1,i_2,i_3}. The "small" one is so small that it can be regarded as a point q for all practical purposes, (see Lemma 1.5). The rest of the discs were obtained by contracting their corresponding boundaries to a point. Moreover, those three discs were either generated by a path homotopy that connects the boundary to a point, or by a homotopy that was a path homotopy until we encountered a critical geodesic loop, and which then became a regular homotopy (see Fig. 8).

Let us consider the following three cases.

(1) The boundary of each face gets "stuck" on a distinct geodesic loop based at p of length $\leq 2d + \delta$.

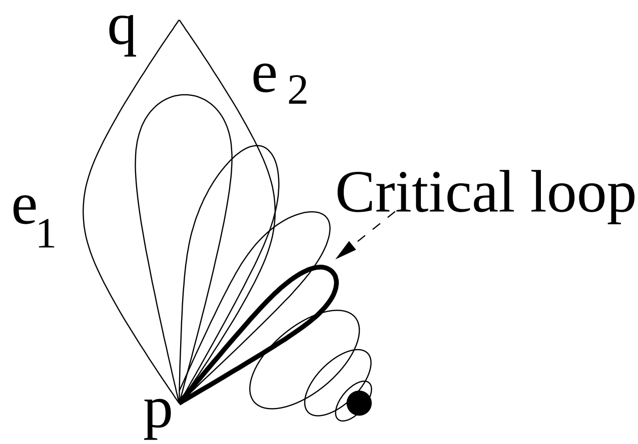

Figure 8. Extending to the 2-skeleton.

(2) The boundary of one of the simplices is contractible to p via path homotopy.
(3) None of the boundaries are contractible to p via length-decreasing path homotopy, but at least two of the geodesic loops that obstruct this coincide.

In the first case, we are done. We have three distinct loops based at p of length $\leq 2d + \delta$. We just need to let δ go to 0.

In the second case, without loss of generality, assume that $e_2 * -e_1$ is contractible to a point p with a length-decreasing path homotopy. Then, by Lemma 1.2, we know that e_1 is path homotopic to e_2 through curves $e_{\tau_{12}}$ of length less than or equal to $3d$. Assume $e_1 * -e_3$ is contractible to a point q_1 along the curves $\gamma(\tau)$ and that $e_2 * -e_3$ is contractible to a point q_2 along the curves $\alpha(\tau)$ of length $\leq 2d$ (see Fig. 9 (a)).

Therefore, we can construct the following homotopy in the space ΛM^n of closed curves. Here is a loop in ΛM^n. $q_1 \sim \gamma(1-\tau) \sim e_1 * -e_3 \sim e_{\tau_{12}} * -e_3 \sim e_2 * -e_3 \sim \alpha(\tau) \sim q_2 \sim q_1$ (see Fig. 9 (b)-(d)). This loop corresponds to the non-contractible sphere $f : \partial\tilde{\sigma}^3_{i_0,\ldots,i_3} \longrightarrow M^n$, as it was obtained from the above sphere by a sweep-out. Therefore, it is a non-contractible loop that passes through curves of length $\leq 4d$. Therefore, there exists a closed geodesic of length $\leq 4d$.

Finally, in the third case, we will construct a non-contractible loop in the space ΛM^n as follows.

Let us assume that $e_1 * -e_2$ and $e_2 * -e_3$ get "stuck" on the same loop α_1 (see Fig. 10 (a)), and $e_1 * -e_3$ gets "stuck" on the loop α_2, which might or might not coincide with α_1. Those loops are then contractible to points $\tilde{q}_1$ and $\tilde{q}_2$ respectively (see Fig. 10 (a) and (b)). Denote the curves in the homotopy that connects α_1 with q_1 by α_τ, $1 \leq \tau \leq 2$. Further, denote the curves in the homotopy that connects $e_3 * -e_1$ and

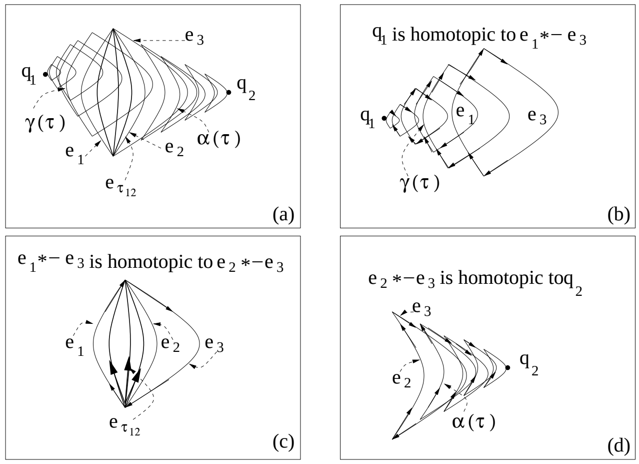

Figure 9. Loop in the space ΛM.

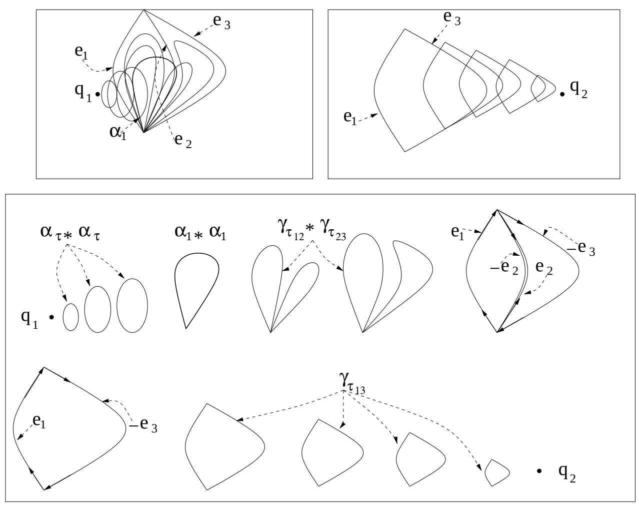

Figure 10. Loop in the space ΛM.

q_2 by $\gamma_{\tau_{13}}$, $1 \leq \tau_{13} \leq 2$. Finally, denote the curves in the homotopy that connects $e_1 * -e_2$ and α_1 by $\gamma_{\tau_{12}}$ and the curves in the homotopy that connects $e_2 * -e_3$ and α_1 by $\gamma_{\tau_{23}}$, $1 \leq \tau_{12}, \tau_{23} \leq 2$.

We will now describe a non-contractible loop in the space ΛM^n (see Fig. 10 (c)). It will be a sweep-out of the non-contractible sphere $f : \partial\tilde{\sigma}^3_{i_0,\dots,i_3} \longrightarrow M^n$ by short loops.

$q_1 \sim \alpha_\tau * \alpha_\tau \sim \alpha_1 * \alpha_1$ (that is we go around $\alpha_{1\tau}$ twice). Here we use the "reverse" of the BCSP without a fixed point. The length of curves in this homotopy is bounded by $4d$, since the length of $\alpha_{1\tau} \leq 2d$.

$\alpha_1 * \alpha_1 \sim e_{\tau_{12}} * e_{\tau_{23}} \sim e_1 * -e_2 * e_2 * -e_3$. Here we use the "reverse" of the BCSP applied to loops based at q_1. The length of curves during this homotopy is bounded by $4d$ as well.

$e_1 * -e_2 * e_2 * -e_3 \sim e_1 * -e_3$. This is a homotopy of shortening $-e_2 * e_2$. The length of the original curve was $\leq 4d$ and the curve becomes shorter during the homotopy, so that the length of the final curve is $\leq 2d$.

$e_1 * -e_3 \sim \gamma_{\tau_{13}} \sim q_2 \sim q_1$. This is simply the BCSP applied to $e_1 * -e_3$, so the length of curves in this homotopy is bounded by $2d$.

Note that we have constructed the loop in the space ΛM^n that starts and ends with a constant curve q_1. $\square$

References

[Bl] F. Balacheff, *Sur des problemes de la géométrie systolique*, Séminaire de Théorie Spectrale et Géométrie, Vol. 22, Année 2003-2004, 71–82, Sémin. Théor. Spectr. Géom., **22**, Univ. Grenoble I, Saint-Martin-d'Hères, 2004, MR 2136136.

[BlCK] F. Balacheff, C.B. Croke, & M. Katz, *A Zoll counterexample to a geodesic length conjecture*, preprint, to appear in GAFA.

[BZ] Ju. Burago & V. Zalgaller, *Geometric Inequalities*, Berlin, Springer-Verlag, 1988, MR 0936419, Zbl 0633.53002.

[C] C.B. Croke, *Area and the length of the shortest closed geodesic*, J. Differential Geom. **27** (1988) 1–21, MR 0918453, Zbl 0642.53045.

[CK] C.B. Croke & M. Katz, *Universal volume bounds in Riemannian manifolds*, Surveys in differential geometry, Vol. VIII (Boston, MA 2002), 109–137, Int. Press., Somerville, MA, 2003, MR 2039987, Zbl 1051.53026.

[GrP] R.E. Greene & P. Petersen, *Little topology, big volume*, Duke Math. J. **67**(**2**) (1992) 273–290, MR 1177307, Zbl 0772.53033.

[G] M. Gromov, *Filling Riemannian manifolds*, J. Differential Geom. **18** (1083) 1–147, MR 0697984, Zbl 0515.53037.

[M] M. Maeda, *The length of a closed geodesic on a compact surface*, Kyushu J. Math. **48**(**1**) (1994) 9–18, MR 1269063, Zbl 0818.53064.

[NR1] A. Nabutovsky & R. Rotman, *The length of the shortest closed geodesic on a 2-dimensional sphere*, IMRN 2002, **23**, 1211–1222, MR 1903953, Zbl 1003.53030.

[NR2] ———, *Volume, diameter and the minimal mass of a stationary 1-cycle*, Geom. Funct. Anal. **14**(**4**) (2004) 748–790, MR 2084979, Zbl 1073.53057.

[NR3] ———, *Curvature-free upper bounds for the smallest area of a minimal surface*, Geom. Funct. Anal. **16**(**2**) (2006) 453–475, MR 2231470, Zbl 1110.53047.

[R1] R. Rotman, *The length of a shortest closed geodesic on a two-dimensional sphere and coverings by metric balls*, Geom. Dedicata **110** (2005) 143–157, MR 2136024, Zbl 1080.53031.

[R2] ———, *The length of a shortest closed geodesic and the area of a 2-dimensional sphere*, Proc. Amer. Math. Soc. **134**(**10**) (2006) 3041–3047, MR 2231630, Zbl 1098.53035.

[S1] S. Sabourau, *Filling radius and short closed geodesics of the* 2*-sphere*, Bull. Soc. Math. France **132**(**1**) (2004) 105–136, MR 2075918, Zbl 1064.53020.

[S2] ———, *Global and local volume bounds and the shortest geodesic loops*, Commun. Anal. Geom. **12** (2004) 1039–1053, MR 2103310, Zbl 1075.53030.

Department of Mathematics
University of Toronto
Toronto, Ontario M5S 2E4
Canada

E-mail address: rina@math.toronto.edu

Department of Mathematics
Penn State University
University Park, PA 16802

E-mail address: rotman@math.psu.edu