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ON THE GENUS OF TRIPLY PERIODIC MINIMAL SURFACES

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Abstract

We prove the existence of embedded minimal surfaces of arbitrary genus $g \geq 3$ in any flat 3-torus. In fact, we construct a sequence of such surfaces converging to a planar foliation of the 3-torus. In particular, the area of the surface can be chosen arbitrarily large.

1. Introduction

Triply periodic minimal surfaces in euclidean space are invariant by three independant translations. In the nineteenth century, five embedded triply periodic minimal surfaces were known to H.A. Schwarz and his school. In 1970, Alan Schoen described 12 further families of examples. The arguments were completed by H. Karcher, who also proved the existence of many further examples.

If M is a triply periodic minimal surface and Λ is the lattice generated by its three periods, then M projects to a minimal surface in the flat 3-torus $\mathbb{R}^3/\Lambda$. Conversely, a (non-flat) minimal surface in $\mathbb{R}^3/\Lambda$ lifts to a triply periodic minimal surface in $\mathbb{R}^3$.

A natural question is whether there exist non-flat minimal surfaces in any flat 3-torus. The examples constructed by H.A. Schwarz, A. Schoen and H. Karcher are very symmetric by construction, so they only construct examples in very particular 3-tori. In 1990, W. Meeks made the following conjecture ([**6**], Conjecture 3.2):

Conjecture 1 (Meeks)**.** For any flat 3-torus $\mathbb{R}^3/\Lambda$ and any integer $g \geq 3$ there exist an embedded, orientable minimal surface of genus g in $\mathbb{R}^3/\Lambda$.

A related question asked by H. Karcher is the following ([**4**], question 4):

Can triply periodic embedded minimal surfaces of arbitrary large genus exist in a nontrivial way – or does the genus stay bounded if one divides out by all translational symmetries?

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In this paper we prove that the conjecture is true and answer this question:

Theorem 1. *For any flat 3-torus $\mathbb{R}^3/\Lambda$, and for any integer $g \geq 3$, there exists a sequence of orientable, compact, embedded minimal surfaces $(M_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^3/\Lambda$ which have genus g. The area of M_n goes to infinity when $n \to \infty$. Moreover, if $g \geq 4$, the only translation of $\mathbb{R}^3/\Lambda$ which leaves M_n invariant is the identity (so M_n lifts to a triply periodic minimal surface in $\mathbb{R}^3$, whose lattice of periods is precisely Λ).*

Here is what is known about this conjecture. It is known that a minimal surface of genus one in a flat 3-torus must be flat (a plane) and minimal surfaces of genus two in flat 3-tori do not exist ([**6**], Corollary 3.1). W. Meeks has proved ([**6**], Corollary 10.1) that the conjecture is true in the genus 3 case by using a min-max argument. Then, by taking covers, he concluded that the conjecture holds for arbitrary *odd* genus $g \geq 3$. In other words, if Λ' is a sub-lattice of Λ, a genus 3 minimal surface in $\mathbb{R}^3/\Lambda$ lifts to an odd genus minimal surface in $\mathbb{R}^3/\Lambda'$. The case of even genus remained open. Also, these examples do not answer the question of H. Karcher.

Let us now explain the idea of the construction, which was suggested to the author by Antonio Ros. Consider a plane in space and its projection in the flat 3-torus $\mathbb{R}^3/\Lambda$. Depending on the plane, the area of the projection may be finite or infinite. Choose the plane so that the area is very large. In the quotient, the plane will wrap around many times, and what we see locally is many parallel sheets very close to each other. Take another plane parallel to the first one. Our main result, Theorem 4, allows us to open small catenoidal necks between the two planes (on both sides), producing an embedded minimal surface in $\mathbb{R}^3/\Lambda$ (see Figure 1). Its genus is equal to the number of necks plus one.

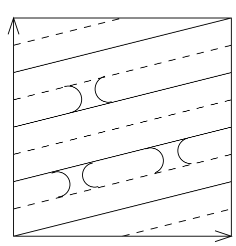

Figure 1. A minimal surface of genus 4 in the cubic 3-torus. One of the two planes is represented with dots for clarity.

It turns out that the placement of the necks is not arbitrary: they must satisfy a *balancing condition* which we will explain. One of our tasks in this paper is to give examples of balanced configurations of necks with an arbitrary number of points.

The proof of the main theorem is in the spirit of recent *gluing* constructions [**8**], [**9**]. We use the Weierstrass Representation of minimal surfaces in its simplest form: consider a Riemann surface Σ and three holomorphic 1-forms ϕ_1, ϕ_2, ϕ_3 on Σ. Then

$$X(z) = \mathrm{Re} \int_{z_0}^{z} (\phi_1, \phi_2, \phi_3) \mod \Lambda$$

defines a conformal minimal immersion $X : \Sigma \to \mathbb{R}^3/\Lambda$ provided the following conditions are satisfied:

$$\forall \gamma \in H_1(\Sigma), \quad \mathrm{Re} \int_{\gamma} (\phi_1, \phi_2, \phi_3) \in \Lambda$$

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$$

$$|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 > 0.$$

The first equation says that X is well defined and the second that it is conformal, hence minimal since it is harmonic. The third one says that X is a regular immersion.

We define Σ by *opening the nodes* of a (singular) Riemann surface with nodes. We define ϕ_1, ϕ_2, ϕ_3 by prescribing their periods. We solve the above equations using the implicit function theorem.

The main difference with previous constructions is that in [**8**] we used the Weierstrass representation in its classical form

$$X(z) = \mathrm{Re} \int_{z_0}^{z} \left(\frac{1}{2}(g^{-1} - g)dh, \frac{\mathrm{i}}{2}(g^{-1} + g)dh, dh \right),$$

where dh is a holomorphic 1-form and g is a meromorphic function on Σ (the Gauss map). Now on a high genus Riemann surface it is more natural to define holomorphic 1-forms (by prescribing periods) than meromorphic functions (where we have to face Abel's theorem). In fact, in [**8**] the Riemann surface Σ and the Gauss map g were defined at the same time, but that made the construction a little bit artificial. Also, the construction had some technical complications in the case where the Gauss map had multiple zeros or poles. It seems more natural to use ϕ_1, ϕ_2 and ϕ_3 and avoid meromorphic functions completely.

The paper is organized as follows. In Section 2 we explain the balancing condition and state our main theorem. In Section 3 we prove the conjecture of Meeks. In Section 4 we give examples of balanced configurations. In particular we prove the existence of triply periodic minimal surfaces with no symmetries beside translations. In Section 5 we prove our main theorem.

Acknowledgements. This project was initiated when I visited Granada in 2003. It's a pleasure to thank Antonio Ros for the invitation and for suggesting that the conjecture of Meeks could be solved by desingularising a planar foliation of the 3-torus. I would also like to thank Laurent Mazet for carefuly reading the first draft of this paper, finding several mistakes and suggesting many improvements.

2. Main result

2.1. Simply periodic surfaces. Our main theorem in this paper is an adaptation, to the triply periodic case, of a construction of simply periodic minimal surfaces with ends asymptotic to horizontal planes [**8**]. This construction may be described as follows.

Consider an infinite number of copies of the plane $\mathbb{R}^2$, labeled P_k, $k \in \mathbb{Z}$. In each plane P_k, choose a finite number n_k of points $p_{k,i}$, $1 \leq i \leq n_k$. Identify, for each (k,i), the point $p_{k,i}$ in P_k with the same point in P_{k+1}. This creates an abstract singular 2-manifold with double points (or nodes) which we call M_0. Consider the map $X_0 : M_0 \to \mathbb{R}^3$ which sends each point $(x,y) \in P_k$ to the point $(x,y,0)$. We may think of X_0 as a minimal, isometric immersion of the singular 2-manifold M_0 into $\mathbb{R}^3$. We want to *desingularise* this singular minimal surface, namely: perturb M_0 into a regular 2-manifold M_ε and X_0 into a minimal immersion $X_\varepsilon : M_\varepsilon \to \mathbb{R}^3$. Intuitively, we do this by slightly pushing the planes away from each other and replacing each double point by a small catenoid.

We assume the following periodicity: there exists an even integer $N \geq 2$ and a vector $T \in \mathbb{R}^2$ such that for all (k,i), $p_{k+N,i} = p_{k,i} + T$. We construct a family of minimal surfaces M_ε which are periodic with period (T, ε), and have N ends asymptotic to horizontal planes in the quotient. (Here N must be even so that the quotient is orientable.) We call the collection $\{p_{k,i}\}$ a *simply periodic configuration*. We call $p_{k,i}$ a point at *level* k of the configuration. Note however that all points $p_{k,i}$ are in $\mathbb{R}^2$. It will be convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$ and see each point $p_{k,i}$ as a complex number.

For the construction to work, the configuration must satisfy a balancing condition which we now explain. Let $c_k = 1/n_k$. Let $\widetilde{\omega}_k$ be the unique meromorphic 1-form on $\mathbb{C} \cup \{\infty\}$ with n_{k-1} simple poles at $p_{k-1,i}$ with residue c_{k-1} and n_k simple poles at $p_{k,i}$ with residue $-c_k$. (We assume that for each k, the points $p_{k,i}$, $p_{k-1,j}$ are distinct). Let

$$\widetilde{F}_{k,i} = \frac{1}{2}\mathrm{Res}_{p_{k,i}}\left(\frac{(\widetilde{\omega}_k)^2}{dz} + \frac{(\widetilde{\omega}_{k+1})^2}{dz}\right).$$

Explicitly,

$$\widetilde{\omega}_k = \sum_{i=1}^{n_{k-1}} \frac{c_{k-1}}{z - p_{k-1,i}} dz - \sum_{i=1}^{n_k} \frac{c_k}{z - p_{k,i}} dz,$$

which gives after a straightforward computation

$$\widetilde{F}_{k,i} = 2 \sum_{\substack{j=1 \\ j \neq i}}^{n_k} \frac{(c_k)^2}{p_{k,i} - p_{k,j}} - \sum_{j=1}^{n_{k+1}} \frac{c_k c_{k+1}}{p_{k,i} - p_{k+1,j}} - \sum_{j=1}^{n_{k-1}} \frac{c_k c_{k-1}}{p_{k,i} - p_{k-1,j}}.$$

The analogy with 2-dimensional electrostatic forces suggests that we call $\widetilde{F}_{k,i}$ a force. Each point $p_{k,i}$ interacts with the points $p_{k,j}$ at the same level and with points $p_{k\pm1,j}$ at the levels below and above it.

Definition 1. A simply periodic configuration is balanced if all forces $\widetilde{F}_{k,i}$ are zero. It is non-degenerate if the differential of the map $\mathbf{p} \to \widetilde{\mathbf{F}}$ has real co-rank 2, where $\mathbf{p}$ and $\widetilde{\mathbf{F}}$ stand for the collection of $p_{k,i}$ and $\widetilde{F}_{k,i}$ for $1 \leq k \leq N$ and $1 \leq i \leq n_k$.

The forces are clearly invariant by translation of all points, so vectors of the form $(v, v, \dots, v)$, where $v \in \mathbb{R}^2$, are in the kernel of the differential. The non-degeneracy condition asks that these are the only ones. The following theorem is proven in [**8**]:

Theorem 2. *Given a balanced, non-degenerate, simply periodic configuration, there exists a smooth family of embedded simply periodic minimal surfaces M_ε, for $\varepsilon > 0$ small enough, which have period (T, ε). The quotient has genus $n-1$, where $n = n_1 + \cdots + n_N$ is the total number of necks, and has N horizontal planar ends. Moreover, M_ε converges to the singular minimal surface M_0 when $\varepsilon \to 0$.*

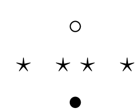

Figure 2. A simply periodic configuration with $N = 2$, $n_1 = 4$ and $n_2 = 1$. Points at level 0, 1 and 2 are represented respectively by a black dot, a star, and a white dot. The configuration extends periodically.

The last statement of the theorem should be understood as follows: if we see M_ε as an abstract 2-manifold isometrically embedded in $\mathbb{R}^3$ by the canonical injection $X_\varepsilon : M_\varepsilon \to \mathbb{R}^3$, then the abstract manifold M_ε converges to M_0 and X_ε converges to X_0 on compact subsets of M_0 minus the double points.

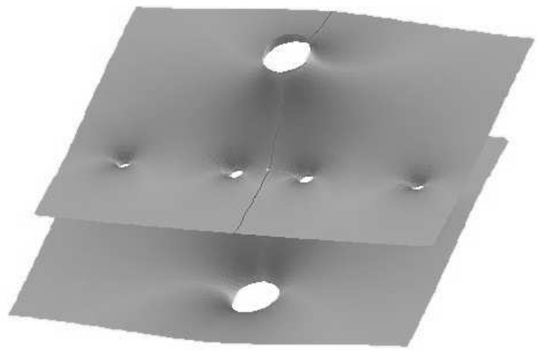

Figure 3. An element in the corresponding family of minimal surfaces. Computer image made by the author using J. Hoffman MESH software.

Remark 1. In the sense of laminations, M_ε converges to the lamination of $\mathbb{R}^3$ by horizontal planes, with singular set $\{(p_{k,i}, 0)\}$. Note however that this limit object does not contain enough information to reconstruct M_ε. This is why we prefer to see the limit object M_0 as a minimal immersion of a singular 2-manifold.

Section 2.1 of [**8**] provides us with an explicit family of balanced configurations, depending on an integer parameter $m \geq 1$.

Theorem 3. *The following simply periodic configuration is balanced and non-degenerate:* $N = 2$, $n_1 = m$, $n_2 = 1$, $p_{1,j} = \operatorname{cotan}\left(\dfrac{j\pi}{m+1}\right)$, $p_{2,1} = \mathrm{i}$ *and* $T = 2\mathrm{i}$ *(so* $p_{0,1} = -\mathrm{i}$*).*

For example, in the case $m = 1$, we have $p_{k,1} = (k-1)\mathrm{i}$. The associated minimal surfaces M_ε are the Riemann examples.

The case $m = 4$ is represented on Figure 2. In the particular case of the configuration given by Theorem 3, the corresponding minimal surfaces are hyper-elliptic, so it is not too hard to write explicitly their Weierstrass representation and produce computer images, see Figure 3.

2.2. Triply periodic surfaces. A natural and interesting question is whether we can carry a similar construction when there is an infinite number of points at each level, namely $n_k = \infty$ for all k. It is not clear in general what the balancing condition should be, because if there is an infinite number of points, each force might be a diverging series.

In this paper we answer this question in the case where the points at each level are arranged in a doubly periodic way, namely they are invariant by two independant translations with vectors T_1 and T_2. We assume that modulo T_1 and T_2, there is a finite number n_k of points at level k and we call them $p_{k,i}$, $1 \leq i \leq n_k$. We see each point $p_{k,i}$ as an element of the torus $\mathbb{T} = \mathbb{R}^2/\Gamma$, where Γ is the 2-dimensional lattice generated by T_1 and T_2. We assume the following periodicity: there exists again an even integer $N \geq 2$ and a vector $T_3 \in \mathbb{R}^2$ (which replaces the vector T in the simply periodic case) so that $p_{k+N,i} = p_{k,i} + T_3$. We call the collection $\{p_{k,i}\}$ a *triply periodic configuration.* We want to construct a family of triply periodic minimal surfaces with periods $(T_1, 0)$, $(T_2, 0)$ and (T_3, ε).

Forces are defined as in the simply periodic case. Again let $c_k = 1/n_k$. Consider $\mathbb{T}$ as the genus one compact Riemann surface $\mathbb{C}/\Gamma$. Let ω_k be the meromorphic 1-form on $\mathbb{T}$ with n_{k-1} simple poles at $p_{k-1,i}$ with residue c_{k-1}, n_k simple poles at $p_{k,i}$ with residue $-c_k$, and pure imaginary periods. In other words,

$$\operatorname{Re}\int_0^{T_1} \omega_k = \operatorname{Re}\int_0^{T_2} \omega_k = 0.$$

The meromorphic 1-form ω_k exists because the sum of the residues is zero, and the period condition makes it unique. Define forces as in the simply periodic case by

$$F_{k,i} = \frac{1}{2}\operatorname{Res}_{p_{k,i}}\left(\frac{(\omega_k)^2}{dz} + \frac{(\omega_{k+1})^2}{dz}\right).$$

The main difference with the simply periodic case is that forces cannot be computed as explicitly. They may be written in various ways in term of elliptic functions. We will do this in Section 4.

Definition 2. A triply periodic configuration is balanced if all forces $F_{k,i}$ are zero. It is non-degenerate if the differential of the map $\mathbf{p} \to \mathbf{F}$ has real co-rank 2, where $\mathbf{p}$ and $\mathbf{F}$ are as in Definition 1.

The main result of the paper is the following:

Theorem 4. *Given a balanced, non-degenerate triply periodic configuration, there exists a smooth family of embedded triply periodic minimal surfaces M_ε, for $\varepsilon > 0$ small enough, which have period $(T_1, 0)$, $(T_2, 0)$ and (T_3, ε). The quotient has genus $n + 1$, where $n = n_1 + \cdots + n_N$. Its area is close to N times the area of $\mathbb{T}$. Moreover, M_ε converges to M_0 when $\varepsilon \to 0$, where M_0 is a singular minimal surface defined as in the beginning of section* 2.1.

In fact, we construct a family of surfaces depending smoothly on ε, T_1, T_2 and T_3 (in a neighborhood of the given values). This gives

Corollary 1. *Under the same hypothesis, there exists $\eta > 0$ such that the following is true: if V_1, V_2 and V_3 are three independant vectors in space such that $|V_i - (T_i, 0)| < \eta$ for $i = 1, 2, 3$, then there exists a triply periodic minimal surface with periods V_1, V_2, V_3, which satisfies the same conclusion as the theorem.*

Hence up to isometries, we construct a 6-dimensional family of surfaces. This is also the dimension of the space of flat 3-tori.

Proof. Simply consider the rotation with the smallest angle which sends V_1 and V_2 to the horizontal plane. q.e.d.

3. Proof of the conjecture of Meeks

In Section 4 we will prove

Theorem 5. *For any $g \geq 3$, there exists T_1, T_2, T_3, and a triply periodic, balanced, non-degenerate configuration with periods T_1, T_2, T_3 which has $g-1$ points, so the associated minimal surfaces have genus g.*

By Corollary 1, if V_1, V_2 and V_3 are three independant vectors in space close to the horizontal vectors $(T_1, 0)$, $(T_2, 0)$, $(T_3, 0)$, there exists an embedded minimal surfaces of genus g with periods V_1, V_2, V_3. It turns out that this already gives minimal surfaces of genus g in arbitrary flat 3-tori, up to scale ! To understand why, let us first consider an example.

Assume that $T_1 = (1, 0)$, $T_2 = (0, 1)$ and $T_3 = (0, 0)$. How can we recover minimal surfaces of genus g in the cubic torus, namely with periods $V_1 = (1, 0, 0)$, $V_2 = (0, 1, 0)$, $V_3 = (0, 0, 1)$?

Consider the following sequence of matrices

$$A_n = \begin{bmatrix} n & 0 & 1 \\ 1 & n & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since this matrix has determinant 1, its columns form a basis of the cubic lattice. On the other hand, when $n \to \infty$, $n^{-1}A_n$ converges to the matrix with columns $(T_1, 0)$, $(T_2, 0)$ and $(T_3, 0)$. By Corollary 1, for n large enough, there exists a minimal surface of genus g and periods the columns of $n^{-1}A_n$. Scaling by n gives a sequence of minimal surfaces M_n in the cubic torus.

In the general case, we need a result which we will prove in the next section. Let $SL(3, \mathbb{Z})$ be the group of 3×3 matrices with integer coefficients and determinant one.

Proposition 1. *Consider a matrix $M \in \mathcal{M}_3(\mathbb{R})$ such that* $\det(M) = 0$*. Then there exists a sequence of matrices $A_n \in SL(3, \mathbb{Z})$ and a sequence of reals λ_n such that $\lambda_n A_n \to M$ (so $\lambda_n \to 0$).*

Using this proposition, we prove Theorem 1. Let Λ be an arbitrary lattice in $\mathbb{R}^3$. Let V_1, V_2, V_3 be a basis of Λ. We write $[V_1\ V_2\ V_3]$ for the matrix whose columns are V_1, V_2, V_3. Recall that $SL(3,\mathbb{Z})$ acts on bases of Λ as follows: if $A \in SL(3,\mathbb{Z})$, define V_1', V_2', V_3' by

$$[V_1'\ V_2'\ V_3'] = [V_1\ V_2\ V_3]A.$$

In other words, $V_j' = \sum_i a_{ij} V_i$. Then V_1', V_2', V_3' is again a basis of Λ. Indeed, because A has integer coefficients, the lattice generated by V_1', V_2' and V_3' is included in Λ. The converse is true because A is invertible. Let $M \in \mathcal{M}_3(\mathbb{R})$ be the rank 2 matrix defined by

$$\begin{bmatrix} T_1 & T_2 & T_3 \\ 0 & 0 & 0 \end{bmatrix} = [V_1\ V_2\ V_3]M.$$

By Proposition 1 there exists a sequence $A_n \in SL(3,\mathbb{Z})$ and $\lambda_n \in \mathbb{R}$ such that $\lambda_n \to 0$ and $\lambda_n A_n \to M$. Define V_1^n, V_2^n, V_3^n by

$$[V_1^n, V_2^n, V_3^n] = \lambda_n [V_1, V_2, V_3] A_n.$$

Then V_1^n, V_2^n, V_3^n is a basis of the lattice $\lambda_n \Lambda$ and $V_i^n \to (T_i, 0)$ when $n \to \infty$, for $i = 1, 2, 3$. By Corollary 1, for n large enough, there exists an embedded triply periodic minimal surface of genus g with periods V_1^n, V_2^n, V_3^n. Scaling by $1/\lambda_n$, we have a sequence of embedded minimal surfaces of genus g in $\mathbb{R}^3/\Lambda$ whose area goes to ∞. This proves Theorem 1. q.e.d.

3.1. On singular matrices. We state a general result which we only use in the case of dimension $d = 3$. We use the following notations. Let $\mathcal{M}_d(A)$ be the set of $d \times d$ matrices with coefficients in A, where A is $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. Let $\mathcal{N}_d(A)$ be the set of matrices in $\mathcal{M}_d(A)$ whose determinant is zero. Let $\mathcal{A}_d$ be the set of matrices $M \in \mathcal{M}_d(\mathbb{R})$ such that there exists a sequence of matrices $A_n \in SL(d,\mathbb{Z})$ and a sequence of reals λ_n such that $\lambda_n \to 0$ and $\lambda_n A_n \to M$.

Proposition 2.

$$\mathcal{A}_d = \mathcal{N}_d(\mathbb{R}).$$

Proof. The inclusion $\subset$ is clear. For the reverse inclusion, first observe the following facts.

(i) If $M \in \mathcal{A}_d$ and $P \in SL(d,\mathbb{Z})$ then $PM \in \mathcal{A}_d$ and $MP \in \mathcal{A}_d$.
(ii) If $M \in \mathcal{A}_d$ and $\lambda \in \mathbb{R}$ then $\lambda M \in \mathcal{A}_d$.
(iii) $\mathcal{A}_d$ is closed.
(iv) $\mathcal{N}_d(\mathbb{Q})$ is dense in $\mathcal{N}_d(\mathbb{R})$.

By points (iii) and (iv), it suffices to prove that $\mathcal{N}_d(\mathbb{Q}) \subset \mathcal{A}_d$. By point (ii) it suffices to prove that $\mathcal{N}_d(\mathbb{Z}) \subset \mathcal{A}_d$.

Let $M \in \mathcal{M}_d(\mathbb{Z})$. By standard theory of matrices with integer coefficients, M may be put into reduced form, namely

$$M = P \begin{bmatrix} a_1 & & \\ 0 & \ddots & 0 \\ & & a_d \end{bmatrix} Q$$

where $P, Q \in SL(d, \mathbb{Z})$ and $a_1, \dots, a_d \in \mathbb{Z}$. Moreover, if r is the rank of M, then $a_i = 0$ for $i > r$. Now assume that $\det M = 0$ so $a_d = 0$. By point (i) we only need to prove that the above diagonal matrix is in $\mathcal{A}_d$. When $d = 3$, which is the case we are interested in, simply write

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \lim_{n\to\infty} \frac{1}{n} \begin{bmatrix} a_1 n & 1 & 0 \\ 0 & a_2 n & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The matrix on the right has determinant 1. In the general case, write

$$\begin{bmatrix} a_1 & & & \\ & \ddots & 0 & \\ & 0 & a_{d-1} & \\ & & & 0 \end{bmatrix} = \lim_{n\to\infty} \frac{1}{n} \begin{bmatrix} a_1 n & 1 & & 0 \\ 0 & \ddots & \ddots & \\ & & \ddots \; a_{d-1} n & 1 \\ (-1)^{d+1} & & 0 & 0 \end{bmatrix}.$$

q.e.d.

4. Examples

4.1. Preliminary observations. First observe that

$$F_{k,i} = \frac{1}{2} \mathrm{Res}_{p_{k,i}} \sum_{\ell=1}^{N} \frac{(\omega_\ell)^2}{dz}$$

because ω_ℓ has no pole at $p_{k,i}$ if $\ell \neq k, k+1$. Hence by the residue theorem, the forces sum up to zero:

$$\sum_{k=1}^{N} \sum_{i=1}^{n_k} F_{k,i} = 0.$$

Also, we have

$$F_{k,i} = \frac{1}{4} \mathrm{Res}_{p_{k,i}} \frac{(\omega_k - \omega_{k+1})^2}{dz}.$$

To see this, observe that

$$(\omega_k + \omega_{k+1})^2 + (\omega_k - \omega_{k+1})^2 = 2((\omega_k)^2 + (\omega_{k+1})^2)$$

and $\omega_k + \omega_{k+1}$ has no pole at $p_{k,i}$ as the residues cancel.

4.2. Simplest configurations.

Proposition 3. *Take $N = 2$, $n_1 = n_2 = 1$. Consider a 2-dimensional lattice $\Gamma = \mathbb{Z}T_1 + \mathbb{Z}T_2$ and let $\mathbb{T} = \mathbb{C}/\Gamma$. Consider some $a \in \mathbb{T}$ such that 0, a and $-a$ are distinct points in $\mathbb{T}$. The following triply periodic configuration is balanced: $p_{1,1} = 0$, $p_{2,1} = a$, $T_3 = 2a$ (so $p_{0,1} = -a$). The corresponding minimal surfaces have genus 3.*

We will study non-degeneracy of this configuration in section 4.3.3. Observe that if we replace a by $a+T/2$ where $T \in \{T_1, T_2, T_1+T_2\}$, we get three other balanced configurations which have the same period T_3 modulo Γ.

Proof. The configuration is balanced by symmetry. Indeed, the meromorphic 1-form $\omega_1 - \omega_2$ has three simple poles at 0, $\pm a$, with respective residues -2 and 1. Let $\sigma(z) = -z$. Then $\sigma^*(\omega_1 - \omega_2)$ has the same poles as $\omega_1 - \omega_2$ with the same residues, so their difference is a holomorphic 1-form on $\mathbb{C}/\Gamma$ with imaginary periods, so they are equal. In other words, the function $(\omega_1 - \omega_2)/dz$ is an odd function. Hence its square is an even function, so it has no residue at zero, and so $F_{1,1} = 0$. Since the forces sum up to zero, $F_{2,1} = 0$, so the configuration is balanced.

q.e.d.

4.3. Balanced configurations using Weierstrass ζ function. In this section we compute explicitly the forces in term of Weierstrass ζ function. This has several applications: first it may be used to compute the forces numerically and find numerical examples of balanced configurations. It will also give us an electrostatic interpretation of the forces. Finally, we use it to prove the existence of balanced, non-degenerate triply periodic configurations, as perturbations of simply periodic configurations.

4.3.1. Forces in term of Weierstrass ζ function. Given a lattice $\Gamma = \mathbb{Z}T_1 + \mathbb{Z}T_2$ in the complex plane, the Weierstrass ζ and $\wp$ functions are defined by

$$\zeta(z) = \frac{1}{z} + \sum_{\substack{w\in\Gamma \\ w\neq 0}} \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2} \right),$$

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{w\in\Gamma \\ w\neq 0}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

We refer the reader to Ahlfors [**1**] for standard properties of these functions. The function ζ is odd, its derivative is $-\wp$ and it is quasi-periodic, namely for $i = 1, 2$

$$\zeta(z+T_i) = \zeta(z) + \eta_i \quad \text{with } \eta_i = 2\zeta(T_i/2).$$

It has a simple pole at each point of the lattice Γ with residue 1. Hence we may write

$$\omega_k = \left(\sum_{j=1}^{n_{k-1}} c_{k-1}\zeta(z - p_{k-1,j}) - \sum_{j=1}^{n_k} c_k \zeta(z - p_{k,j}) + \lambda_k \right) dz,$$

where λ_k is some constant. Indeed, the right term is well defined on $\mathbb{T} = \mathbb{C}/\Gamma$ because the sum of the residues is zero, so the quasi-periodicities of ζ cancel, and two meromorphic 1-forms which have the same principal part differ by a holomorphic 1-form, so a constant times dz. The constant λ_k is determined by the condition that the periods of ω_k are imaginary. A straightforward computation gives

$$F_{k,i} = \sum_{\substack{j=1 \\ j\neq i}}^{n_k} 2(c_k)^2 \zeta(p_{k,i} - p_{k,j})$$
$$- \sum_{\pm} \sum_{j=1}^{n_{k\pm 1}} c_k c_{k\pm 1} \zeta(p_{k,i} - p_{k\pm 1,j}) + (\lambda_{k+1} - \lambda_k) c_k.$$

Here the sum on $\pm$ means that there are two terms, one for the $+$ sign and one for the $-$ sign. Let us now compute λ_k. First observe that in our formula for ω_k, we have chosen a representative in $\mathbb{C}$ for each point $p_{k,i}$. The constant λ_k depends on the choice of representatives. Let γ_1 and γ_2 be the standard generators of the homology of $\mathbb{T}$ (in other words, the homology class of the path from 0 to T_1 and 0 to T_2). Given two points a, b in $\mathbb{C}$, I claim that for $i = 1, 2$,

$$\int_{\gamma_i} \zeta(z-a)dz - \zeta(z-b)dz \equiv -\eta_i(a-b)\ [2\pi\mathrm{i}\,].$$

Indeed, the integrant is a well defined meromorphic 1-form on $\mathbb{T}$ with two simple poles of residue ± 1, so its periods are well defined mod $2\pi\mathrm{i}$. The formula clearly holds when $a = b$, and using the quasi-periodicity of the function ζ, both sides have the same derivative with respect to a, so the claim is true. Using this formula with $a = p_{k-1,j}$ or $a = p_{k,j}$, and $b = z_0$ where z_0 is some fixed point, we obtain for $i = 1, 2$ (the z_0 terms cancel because the sum of the residues is zero)

$$\operatorname{Re} \int_{\gamma_i} \omega_k = \operatorname{Re}\left(-\eta_i \left(\sum_{j=1}^{n_{k-1}} c_{k-1} p_{k-1,j} - \sum_{j=1}^{n_k} c_k p_{k,j} \right) + \lambda_k T_i \right).$$

Let us define the center of mass μ_k of the points at level k by

$$\mu_k = \sum_{j=1}^{n_k} c_k p_{k,j}.$$

Let $(x_k, y_k) \in \mathbb{R}^2$ be the coordinates of μ_k in the basis T_1, T_2. Recall that η_1 and η_2 satisfy the Legendre relation ([**1**], p. 274)

$$\eta_1 T_2 - \eta_2 T_1 = 2\pi\mathrm{i}\,.$$

This gives for $i = 1, 2$

$$\operatorname{Re} \int_{\gamma_i} \omega_k = \operatorname{Re}\left((-\eta_1(x_{k-1} - x_k) - \eta_2(y_{k-1} - y_k) + \lambda_k) T_i \right).$$

Hence

$$\lambda_k = \eta_1(x_{k-1} - x_k) + \eta_2(y_{k-1} - y_k).$$

4.3.2. Electrostatic interpretation of the forces. Assume that $2\mu_k = \mu_{k-1} + \mu_{k+1}$ for all k. This means that $\mu_{k+1} - \mu_k$ is constant, so the centers of mass μ_k are regularly spaced. Then $\lambda_{k+1} = \lambda_k$. Using the definition of the ζ function as a series, and after a tedious computation, we obtain the following formula for the force:

$$\begin{aligned} F_{k,i} = & \sum_{\substack{j=1\\ j\neq i}}^{n_k} \frac{2(c_k)^2}{p_{k,i} - p_{k,j}} - \sum_{\pm} \sum_{j=1}^{n_{k\pm1}} \frac{c_k c_{k\pm1}}{p_{k,i} - p_{k\pm1,j}} \\ & + \sum_{\substack{w\in\Gamma\\ w\neq 0}} \left(\sum_{j=1}^{n_k} \frac{2(c_k)^2}{p_{k,i} - p_{k,j} - w} - \sum_{\pm} \sum_{j=1}^{n_{k\pm1}} \frac{c_k c_{k\pm1}}{p_{k,i} - p_{k\pm1,j} - w} \right). \end{aligned}$$

This formula may be interpreted as follows: this is the same formula as the force $\widetilde{F}_{k,i}$ in the simply periodic case, except that there is an infinite number of terms: $p_{k,i}$ interacts with all the other points $p_{\ell,j} + w$, except itself, namely for $(\ell, j, w) = (k, i, 0)$. This provides an electrostatic interpretation of the force.

Remark 2. It is not hard to see that the sum in the last formula converges, as written, if and only if the configuration satisfies $2\mu_k = \mu_{k-1} + \mu_{k+1}$. This makes our hypothesis natural if we are to give an electrostatic interpretation of the forces.

4.3.3. The genus 3 case. We now return to the case $N = 2$, $n_1 = n_2 = 1$ which gives genus 3 surfaces. We may assume by translating the configuration that $p_{0,1} + p_{2,1} = 0$. Writing $p_{1,1} = xT_1 + yT_2$ we obtain

$$F_{1,1} = -\zeta(xT_1 + yT_2 - T_3/2) - \zeta(xT_1 + yT_2 + T_3/2) + 2x\eta_1 + 2y\eta_2.$$

When $x = y = 0$, which is the configuration we considered in section 4.2, we have $F_{1,1} = 0$ so the configuration is balanced, and using that the derivative of ζ is $-\wp$,

$$\begin{aligned} \frac{\partial F_{1,1}}{\partial x} &= 2\wp(T_3/2)T_1 + 2\eta_1, \\ \frac{\partial F_{1,1}}{\partial y} &= 2\wp(T_3/2)T_2 + 2\eta_2. \end{aligned}$$

Given T_1 and T_2, for generic values of T_3, the differential of $F_{1,1}$ with respect to (x, y) has real rank 2 so the configuration is non-degenerate. But there is a real one dimensional family of values of T_3 for which the differential has rank one so the configuration is degenerate.

Remark 3. Using the above formula, we find numerically that there are other balanced configurations which are not as symmetric as the one we discussed in Section 4.2. This confirms the already suspected fact that the space of genus 3 minimal surfaces in a 3-torus is quite intricate. We will not discuss the subject any further, as we are mainly interested in higher genus surfaces.

4.3.4. Triply periodic perturbation of a simply periodic configuration. In this section we start with a balanced, non-degenerate simply periodic configuration $\{p_{k,i}\}$ with period T (which cannot be zero, see Proposition 2.4 in [**8**]). Consider two independant vectors T_1 and T_2, let Γ be the lattice generated by T_1, T_2, and let $T_3 = T$. Given a real $t \neq 0$, we may consider the triply periodic configuration with periods T_1/t, T_2/t and T_3, whose points are $p_{k,i} \bmod \Gamma/t$. This configuration is of course not balanced anymore, so our goal is to perturb it into a balanced triply periodic configuration for t small enough. We write $F_{k,i}(t)$ for the forces of the configuration $p_{k,i} \bmod \Gamma/t$ and $\zeta(z, \Gamma/t)$ for the ζ function associated to the lattice Γ/t. Using the definition of the function ζ we have

$$\zeta(z, \Gamma/t) = t\zeta(tz, \Gamma)$$
$$\lim_{t \to 0} \zeta(z, \Gamma/t) = \frac{1}{z}$$

and the limit is uniform with respect to z in compact subsets of $\mathbb{C}$. Also $\eta_i(\Gamma/t) = t\eta_i(\Gamma)$, which gives $\lambda_k(t) = \mathcal{O}(t^2)$. It follows from our formula in Section 4.3.1 that

$$\lim_{t \to 0} F_{k,i}(t) = \widetilde{F}_{k,i}$$

where $\widetilde{F}_{k,i}$ is the force associated to the simply periodic configuration $p_{k,i}$. Moreover, $F_{k,i}(t)$ extends real analytically to $t = 0$. The intuitive idea behind this formula, from the point of view of our electrostatic interpretation in Section 4.3.2, is that when $t \to 0$, T_1/t and T_2/t go to infinity, so $p_{k,i}$ does not interact anymore with the points $p_{\ell,j} + w/t$, $w \neq 0$.

Theorem 6. *Let $\{p^0_{k,i}\}$ be a balanced, non-degenerate, simply periodic configuration with period T_3. Fix some lattice $\Gamma = \mathbb{Z}T_1 + \mathbb{Z}T_2$. Then for t close to 0, there exist analytic functions $p_{k,i}(t)$, such that $p_{k,i}(0) = p^0_{k,i}$ and for $t \neq 0$, the triply periodic configuration $\{p_{k,i}(t) \bmod \Gamma/t\}$ is balanced and non-degenerate.*

Theorem 5 follows as a corollary of this theorem and Theorem 3.

Proof. This is a quite straightforward application of the implicit function theorem. We fix $p_{1,1} = p^0_{1,1}$: this normalizes the translation invariance of the balancing condition. Let $\mathbf{p}$ be the collection of all variables $p_{k,i}$, $1 \leq k \leq N$, $1 \leq i \leq n_k$ except $p_{1,1}$, and let $\mathbf{F}(t, \mathbf{p})$ be the collection of the forces $F_{k,i}(t)$ for $1 \leq k \leq N$, $1 \leq i \leq n_k$ except $F_{1,1}(t)$. Then $\mathbf{F}$ is a real analytic map, $\mathbf{F}(0, \mathbf{p}^0) = 0$ and the partial differential of $\mathbf{F}$ with respect to $\mathbf{p}$ at $(0, \mathbf{p}^0)$ is an isomorphism. By the implicit function theorem, for t close to 0, there exists $\mathbf{p}(t)$ such that $\mathbf{F}(t, \mathbf{p}(t)) = 0$. If

$t \neq 0$, the associated triply periodic configuration $p_{k,i}(t) \bmod \Gamma/t$ has all its forces zero except maybe $F_{1,1}(t)$. Since the sum of the forces is zero, $F_{1,1}(t)$ is also zero. Hence the configuration $p_{k,i}(t) \bmod \Gamma/t$ is balanced. It is also non-degenerate for t small enough by continuity.

q.e.d.

Remark 4. In the simply periodic case, the force $\widetilde{F}_{k,i}$ is a complex analytic function of the variables $p_{\ell,j}$, so non-degeneracy is equivalent to the fact that the complex matrix of partial derivatives $\partial\widetilde{F}_{k,i}/\partial p_{\ell,j}$ has complex co-rank 1, which can be checked by computing a determinant.

In the triply periodic case, the force $F_{k,i}$ is only real analytic because of the period normalization. For this reason, non-degeneracy is much harder to deal with in this case. This makes the above theorem particularly valuable since non-degeneracy comes for free.

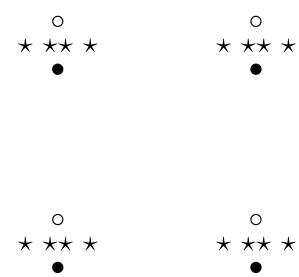

Figure 4. A numerically computed triply periodic configuration with $N = 2$, $n_1 = 4$ and $n_2 = 1$, obtained by perturbation of the simply periodic configuration of Figure 2.

4.4. Surfaces with no symmetries. In the introduction of [**3**], D. Hoffman asked the following question (question 2):

Does there exist a triply periodic embedded minimal surface with no symmetries other than translations?

We give a positive answer to this question. First observe that for generic lattices Λ, the only symmetries of the torus $\mathbb{R}^3/\Lambda$ are translations: $x \mapsto x + v$ and symmetries about a point: $x \mapsto -x + v$. This means that in a generic flat 3-torus, we do not have to worry about planar symmetries since they are just impossible. Now consider the simply periodic configuration represented in Figure 5. This configuration is obtained by combining two configurations of Theorem 3, one with $m = 2$ and one with $m = 3$ scaled by $8/9$ (the scale is chosen so that the forces at the points with $n_k = 1$ cancel). It is proven in [**8**], Proposition 2.3, that configurations obtained this way are balanced and non-degenerate.

The only symmetry of this configuration is the symmetry about the Ox_2 axis. Using Theorem 5 to perturb this configuration into a triply periodic configuration, and arguing as in Section 3, we have proven the existence, in any 3-torus $\mathbb{R}^3/\Lambda$, of a minimal surface whose only possible symmetry, if any, is a planar symmetry. Hence in a generic 3-torus, the surface will have no symmetry.

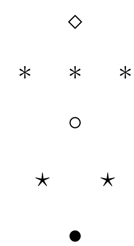

Figure 5. A simply periodic configuration with $N = 4$, $n_1 = 2$, $n_2 = 1$, $n_3 = 3$ and $n_4 = 1$.

4.5. Balanced configurations using Weierstrass $\wp$ function. In this section we briefly explain how balanced configurations may be computed by solving purely algebraic (i.e., polynomial) equations. However, it does not seem possible to study non-degeneracy along these lines, so we won't go into very much detail.

Consider, for example, the case where $N = 2$, $n_1 = 2m$ is even and $n_2 = 1$ (which gives genus $2m + 2$). We assume that the configuration is symmetric about the origin, namely $p_{1,i+m} = -p_{1,i}$ for $1 \leq i \leq m$, $p_{2,1} = T_3/2$ so $p_{0,1} = -T_3/2$. Then we can compute the forces in term of the Weierstrass $\wp$ function as follows: to simplify the notations let us write $p_i = p_{1,i}$. Then

$$\omega_1 - \omega_2 = \frac{-2}{n_1} \sum_{i=1}^{m} \frac{\wp' dz}{\wp - \wp(p_i)} + \frac{\wp' dz}{\wp - \wp(T_3/2)}.$$

Indeed, the right side has the right poles and residues and has imaginary periods. An easy computation gives

$$\mathrm{Res}_{p_i} \left(\frac{\wp'}{\wp - \wp(p_i)} \right)^2 = \frac{\wp''(p_i)}{\wp'(p_i)}.$$

Recalling that $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, where g_2 and g_3 are the modular invariants of $\mathbb{C}/\Gamma$ ([**1**] p. 276), we obtain after elementary computations

$$F_{1,i} = \frac{\wp'(p_i)}{n_1^2} \Bigg(\frac{6\wp^2(p_i) - g_2/2}{4\wp^3(p_i) - g_2\wp(p_i) - g_3} + \sum_{j \neq i} \frac{2}{\wp(p_i) - \wp(p_j)} - \frac{n_1}{\wp(p_i) - \wp(T_3/2)} \Bigg).$$

By symmetry we have $F_{1,i+m} = -F_{1,i}$ and $F_{2,1} = 0$, so we have to solve the m equations $F_{1,1} = 0, \dots, F_{1,m} = 0$. These are purely algebraic equations in the unknowns $X_i = \wp(p_i)$. For example, in the case $m = 1$, $T_3 = 0$, $g_2 = 4$ and $g_3 = 0$ (this is a square torus), we obtain the equation $\wp^2(p_1) = 1/3$. For modest values of m, the above system is easy to solve using, for example, Maple. This is how the configuration in Figure 4 was computed.

5. Proof of Theorem 4

5.1. The initial Riemann surface with nodes Σ_0. For each $k = 1, \dots, N$, consider some complex parameter τ_k such that $\mathrm{Im}(\tau_k) \neq 0$. Let $\mathbb{T}_k$ be the genus one compact Riemann surface $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau_k)$.

For each $k = 1, \dots, N$ and each $i = 1, \dots, n_k$, consider a pair of points $(a_{k,i}, b_{k,i})$ such that $(a_{k,i}, b_{k,i}) \in \mathbb{T}_k \times \mathbb{T}_{k+1}$ if $k < N$, and $(a_{N,i}, b_{N,i}) \in \mathbb{T}_N \times \mathbb{T}_1$. (Here we assume that in each $\mathbb{T}_k$, the points $a_{k,i}$, $b_{k-1,j}$ are distinct.)

Consider the disjoint union $\mathbb{T}_1 \cup \dots \cup \mathbb{T}_N$. Identify, for each pair of points, $a_{k,i}$ with $b_{k,i}$ to create a node (or double point). This creates a singular Riemann surface with nodes which we call Σ_0. It depends on the complex parameters τ_k, $a_{k,i}$, $b_{k,i}$.

5.2. Opening nodes. Following Fay [**2**] and Masur [**5**] we define a (regular) Riemann surface Σ by "opening nodes".

Consider the local complex coordinates $v_{k,i} = z - a_{k,i}$ in a neighborhood of $a_{k,i}$ in $\mathbb{T}_k$, and $w_{k,i} = z - b_{k,i}$ in a neighborhood of $b_{k,i}$ in $\mathbb{T}_{k+1}$. Here, and in what follows, $\mathbb{T}_{N+1}$ should always be understood as $\mathbb{T}_1$.

Consider, for each $k = 1, \dots, N$ and each $i = 1, \dots, n_k$, some complex parameter $t_{k,i}$ close to 0. Let $t \in \mathbb{C}^n$ be the collection of these parameters. Remove the disks $|v_{k,i}| < \sqrt{|t_{k,i}|}$ and $|w_{k,i}| < \sqrt{|t_{k,i}|}$. Identify pairs of points on the boundary of these disks with the rule

$$v_{k,i} w_{k,i} = t_{k,i}.$$

This identifies the two circles, with a Dehn twist of angle equal to the argument of $t_{k,i}$, and defines a (possibly noded) Riemann surface Σ_t which depends holomorphically on t (as well as on the other parameters τ_k, $a_{k,i}$, $b_{k,i}$).

When all the $t_{k,i}$ are nonzero, Σ_t is a genuine compact Riemann surface (without nodes). From the topological point of view, its genus g is easily seen to be equal to $n+1$, where n is the total number of nodes.

When $t_{k,i} = 0$, this construction identifies the point $a_{k,i}$ with $b_{k,i}$ and Σ_t has a node. In particular when $t = 0$, Σ_t is the noded Riemann surface we started from.

5.3. Regular differentials. In this section, we explain all the tools we need to solve our problem. Following Masur [**5**], we extend the notions

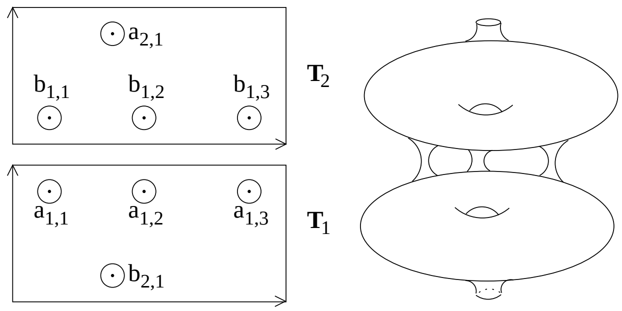

Figure 6. Left: definition of Σ_t in the case $N = 2$, $n_1 = 3$, $n_2 = 1$. Right: topological model of Σ_t. The top circle is identified with the bottom one.

of the holomorphic 1-form and the holomorphic quadratic differential to the case of Riemann surfaces with nodes. Consider a Riemann surface with nodes Σ where we see each node as a pair of points (a, b) which have been identified.

Definition 3 ([**5**])**.** A regular q-differential ω on Σ is a form of type $(q, 0)$ which is holomorphic outside the nodes and which, for each pair of points (a, b) which are identified to create a node, has two poles of order $\leq q$ at a and b. Moreover, the residues of ω at a and b must be opposite if q is odd and equal if q is even.

Here the residue of ω at some point p is the coefficient of dz^q/z^q in the expression of ω in term of a local coordinate z such that $z(p) = 0$.

When Σ has no nodes, a regular q-differential is just a holomorphic q-differential. We are only interested in the cases $q = 1$ and $q = 2$, which correspond respectively to holomorphic 1-forms and holomorphic quadratic differentials.

We write $\Omega^q(\Sigma)$ for the space of regular q-differentials on Σ. The space $\Omega^1(\Sigma)$ has dimension g and $\Omega^2(\Sigma)$ has dimension $3g - 3$, where g is the genus of Σ (provided $g \geq 2$).

Moreover, these spaces depend holomorphically on parameters, in the following sense: Proposition 4.1 of Masur [**5**] says that there exists a basis $\omega_{1,t}, \ldots, \omega_{g,t}$ of $\Omega^1(\Sigma_t)$, which "depends holomorphically" on t in a neighborhood of 0. This is fundamental for the construction we have in mind, since we will apply the implicit function theorem at $t = 0$.

For our purpose, it will be enough to know that for any $\delta > 0$, the restriction of $\omega_{j,t}$ to the domain Ω_δ defined by $\forall k$, $\forall i$, $|v_{k,i}| > \delta$, $|w_{k,i}| > \delta$, depends holomorphically on (z, t), $z \in \Omega_\delta$ and t in a neighborhood of 0. This makes sense because the domain Ω_δ is independent of t provided $t < \delta^2$. There is a deeper way of expressing that ω_t depends holomorphically on t [**5**], but we will not need it.

The isomorphism between $\Omega^1(\Sigma_t)$ and $\mathbb{C}^g$ can be made explicit by computing periods as follows. Let A_k, B_k be the standard generators of the homology of the torus $\mathbb{T}_k$, namely A_k is the straight path from 0 to 1 and B_k from 0 to τ_k. (Here we assume that A_k and B_k stay away from the nodes, taking different representatives of their homology class if necessary.) We write $C(a_{k,i})$ for the circle $|z - a_{k,i}| = \epsilon$ in $\mathbb{T}_k$, where ϵ is some fixed small positive number, and $C(b_{k,i})$ for the circle $|z - b_{k,i}| = \epsilon$ in $\mathbb{T}_{k+1}$. The circle $C(a_{k,i})$ is homologous in Σ_t to the circle $-C(b_{k,i})$.

Again, when $k = N$, $\mathbb{T}_{N+1}$ should be understood as $\mathbb{T}_1$. More generally, we adopt the following cyclic convention: $\mathbb{T}_{k+N} = \mathbb{T}_k$, and in the same way, $a_{k+N,i} = a_{k,i}$, $b_{k+N,i} = b_{k,i}$.

Lemma 1. *The map* $\omega \to \Big(\underbrace{\int_{A_k} \omega}_{1\leq k\leq N},\ \underbrace{\int_{C(a_{k,i})} \omega}_{\substack{1\leq k\leq N\\ 2-\delta_{k,1}\leq i\leq n_k}} \Big)$ *is an isomorphism from* $\Omega^1(\Sigma_t)$ *to* $\mathbb{C}^g$.

In other words, we may define a regular 1-differential on Σ_t by prescribing its periods on all indicated cycles. The condition $2 - \delta_{k,1} \leq i$ reads as $i \geq 2$ if $k \neq 1$ and $i \geq 1$ if $k = 1$. The total number of cycles is equal to the genus of Σ_t.

Proof. One way to prove the lemma is to observe that the indicated set of cycles is the set of A-curves of a canonical homology basis. Here is another argument, which illustrates the use of Riemann surfaces with nodes, and will also work for the next lemma. We check that the statement is true in the case of the noded Riemann surface Σ_0. By continuity, the statement will be true provided t is small enough, which is a weaker statement but enough for our needs.

We prove that the map is injective. Let ω be a regular 1-form on Σ_0. In $\mathbb{T}_k$, ω has simple poles at $a_{k,i}$ and $b_{k-1,i}$, and the residue of ω at $b_{k-1,i}$ is minus its residue at $a_{k-1,i}$. By the residue theorem in $\mathbb{T}_k$

$$\sum_{i=1}^{n_k} \mathrm{Res}_{a_{k,i}}\omega = -\sum_{i=1}^{n_{k-1}} \mathrm{Res}_{b_{k-1,i}}\omega = \sum_{i=1}^{n_{k-1}} \mathrm{Res}_{a_{k-1,i}}\omega,$$

so this sum is independant of k. When ω is in the kernel of the map of the lemma, this implies that all residues of ω are zero, so ω is in fact holomorphic on each $\mathbb{T}_k$, and has vanishing A_k-period, so $\omega = 0$. q.e.d.

Concerning regular quadratic differentials, Proposition 5.1 of Masur [**5**] says that there exists a basis $\psi_{1,t}, \ldots, \psi_{3g-3,t}$ of $\Omega^2(\Sigma_t)$, which depends holomorphically on t.

Lemma 2. *For t close to 0, the map*

$$\psi \to L(\psi) = \Big(\underbrace{\int_{A_k} \frac{\psi}{dz}}_{1\le k\le N}, \underbrace{\int_{C(a_{k,i})} \frac{(z-a_{k,i})\psi}{dz}}_{\substack{1\le k\le N\\ 1\le i\le n_k}}, \underbrace{\int_{C(a_{k,i})} \frac{\psi}{dz}}_{\substack{1\le k\le N\\ 2-\delta_{k,1}\le i\le n_k}}, \underbrace{\int_{C(b_{k,i})} \frac{\psi}{dz}}_{\substack{1\le k\le N\\ 1+\delta_{k,N}\le i\le n_k}} \Big)$$

is an isomorphism from $\Omega^2(\Sigma_t)$ to $\mathbb{C}^{3g-3}$.

In this lemma, dz stands for the standard holomorphic 1-form on each $\mathbb{T}_k$, so dz is not globally defined on Σ_t, which is why integrating ψ/dz on $C(a_{k,i})$ and $C(b_{k,i})$ gives independant results.

Proof. As in the previous lemma, it suffices to prove that the map is injective when $t = 0$. Consider an element ψ in the kernel. Recall that a regular quadratic differential on the noded Riemann surface Σ_0 has at most double poles at the nodes. The second summand of the map tells us that ψ has at most simple poles at all $a_{k,i}$, hence at all $b_{k,i}$ (recall the definition of a regular quadratic differential). The third and fourth summands tell us that the only points where ψ might have poles are $a_{k,1}$ for $2 \le k \le N$ and $b_{N,1}$. This means that ψ has at most one simple pole in each $\mathbb{T}_k$. Applying the residue theorem to the holomorphic 1-form ψ/dz in $\mathbb{T}_k$, we conclude that ψ has no poles at all so ψ is holomorphic in each $\mathbb{T}_k$. The first summand gives $\psi = 0$. q.e.d.

Let us now return to regular 1-forms. Let ω_t be a regular 1-form on Σ_t defined by prescribing periods (independant of t) on cycles as in Lemma 1. We need to compute the derivative of ω_t with respect to the parameters $t_{k,i}$ at $t = 0$.

Lemma 3. *The derivative $\partial\omega_t/\partial t_{k,i}$ at $t = 0$ is a meromorphic 1-form on Σ_0 which has double poles at $a_{k,i}$ and $b_{k,i}$, is otherwise holomorphic, and has vanishing periods on all cycles of Lemma 1. The principal part at the poles are*

$$\frac{dz}{(z-a_{k,i})^2}\frac{-1}{2\pi \mathrm{i}}\int_{C(b_{k,i})}\frac{\omega_0}{z-b_{k,i}} \qquad \text{at } a_{k,i}$$

$$\frac{dz}{(z-b_{k,i})^2}\frac{-1}{2\pi \mathrm{i}}\int_{C(a_{k,i})}\frac{\omega_0}{z-a_{k,i}} \qquad \text{at } b_{k,i}.$$

Note that these conditions (principal parts + periods) determine a unique meromorphic 1-form.

Proof. First recall that when we say that ω_t depends holomorphically on t, we only mean that its restriction to Ω_δ depends holomorphically on (z,t). So the statement of the lemma is that the restriction to Ω_δ of the derivative of ω_t agrees with the indicated meromorphic 1-form.

To simplify the notations, write $t = t_{k,i}$, $v = v_{k,i}$ and $w = w_{k,i}$. Let us write the Laurent series of ω_t in the annular region $\delta < |v| < \epsilon$

$$\omega_t = \sum_{n \in \mathbb{Z}} c_n(t) v^n dv$$

where

$$c_n(t) = \frac{1}{2\pi \mathrm{i}} \int_{|v|=\epsilon} \frac{\omega_t}{v^{n+1}}.$$

Hence

$$\frac{\partial \omega_t}{\partial t} = \sum_{n \in \mathbb{Z}} \frac{\partial c_n(t)}{\partial t} v^n dv.$$

The above series converges uniformly in the region $\delta < |v| < \epsilon$, so the series for $n \geq 0$ converges uniformly in the disk $|v| < \epsilon$ and defines a holomorphic function. To compute the derivative of the coefficient c_n for $n < 0$, first observe that c_{-1} is constant so its derivative is zero, and for $n \leq -2$ and $t \neq 0$

$$c_n(t) = \frac{-1}{2\pi \mathrm{i}} \int_{|w|=\epsilon} \omega_t \frac{w^{n+1}}{t^{n+1}}$$

$$\frac{\partial c_n(t)}{\partial t} = \frac{n+1}{2\pi \mathrm{i}} \int_{|w|=\epsilon} \omega_t \frac{w^{n+1}}{t^{n+2}} - \frac{1}{2\pi \mathrm{i}} \int_{|w|=\epsilon} \left(\frac{\partial \omega_t}{\partial t} \right) \frac{w^{n+1}}{t^{n+1}}.$$

Now ω_t and its derivative are both uniformly bounded on the circle $|w| = \epsilon$, so if we let $t \to 0$, we see that for $n \leq -3$, the derivative of $c_n(t)$ at $t = 0$ is zero, and for $n = -2$,

$$\frac{\partial c_{-2}}{\partial t}(0) = \frac{-1}{2\pi \mathrm{i}} \int_{|w|=\epsilon} \omega_0 w^{-1}.$$

This proves that the derivative of ω_t with respect to $t_{k,i}$ has a double pole at $a_{k,i}$ with the indicated principal part. Entirely similar computations give that the derivative of ω_t has a double pole at $b_{k,i}$ and no poles at the other nodes. q.e.d.

We need one more result to compute the integral of ω_t on a path from the point $a_{k,i} + \epsilon$ to the point $b_{k,i} + \epsilon$, which goes through the neck.

Lemma 4. *The difference*

$$\int_{a_{k,i}+\epsilon}^{b_{k,i}+\epsilon} \omega_t - (\log t_{k,i}) \, \frac{1}{2\pi \mathrm{i}} \int_{C(a_{k,i})} \omega_t$$

is a well defined analytic function of $t_{k,i}$ which extends analytically to $t_{k,i} = 0$.

This lemma is proven in [**9**], Section 3.6, Lemma 1, using a Laurent series to estimate ω_t on the neck as in the proof of the previous lemma. Observe that the logarithm is not well defined because $t_{k,i}$ is a complex number, nor is the left integral because there is no canonical way to choose the integration path, but the multi-valuations cancel.

5.4. The Weierstrass data.

5.4.1. Homology basis. Let $A_{k,i}$ be the homology class of the circle $C(a_{k,i})$ (positively oriented) in $\mathbb{T}_k$. For $1 \leq k \leq N$ and $2 \leq i \leq n_k$, let $B_{k,i}$ be a cycle as on Figure 7, which intersects only $A_{k,1}$ and $A_{k,i}$, with respective intersection numbers -1 and $+1$. Let B be a cycle as on Figure 7, which intersects all circles $A_{k,1}$ with intersection numbers $+1$. Then the following set of cycles forms a homology basis of Σ_t : $A_{1,1}$, B, $A_{k,i}, B_{k,i}$ for $1 \leq k \leq N$, $2 \leq i \leq n_k$, and A_k, B_k for $1 \leq k \leq N$. (Observe that if we replace $B_{1,i}$ by $B_{1,i} + B$, then we have a canonical homology basis; namely, each A-cycle intersects precisely one B-cycle with intersection number 1, and all other intersection numbers are zero.)

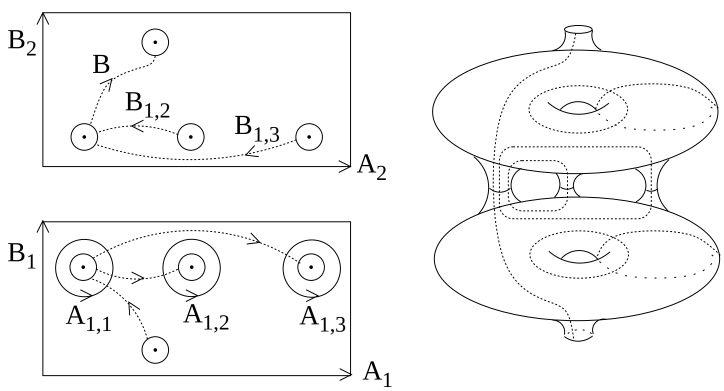

Figure 7. Homology basis (genus 5).

5.4.2. The period problem. Without loss of generality we may assume that $T_1 = (1, 0)$. We need to define three regular 1-forms ϕ_1, ϕ_2, ϕ_3 on Σ_t such that $\mathrm{Re} \int (\phi_1, \phi_2, \phi_3)$ on each cycle of the homology basis belongs to the lattice Λ generated by $(T_1, 0)$, $(T_2, 0)$ and (T_3, ε). From the geometric picture of the surface we want to construct, we know in fact exactly what each period should be, namely

$$\mathrm{Re} \int_{A_{k,i}} (\phi_1, \phi_2, \phi_3) = 0$$

$$\mathrm{Re} \int_{B_{k,i}} (\phi_1, \phi_2, \phi_3) = 0$$

$$\mathrm{Re} \int_{A_k} (\phi_1, \phi_2, \phi_3) = (T_1, 0) = (1, 0, 0)$$

$$\mathrm{Re} \int_{B_k} (\phi_1, \phi_2, \phi_3) = (T_2, 0)$$

$$\mathrm{Re}\int_B(\phi_1,\phi_2,\phi_3)=(T_3,\varepsilon).$$

The period problem for the A-curves will be automatically solved by definition of ϕ_1, ϕ_2, ϕ_3. The period problem for the B-curves is a system of equations we will have to solve. It is not necessary to solve the last equation $\mathrm{Re}\int_B\phi_3=\varepsilon$; we will use it to define ε.

5.4.3. Definition of ϕ_1, ϕ_2, ϕ_3. Using Lemma 1, we define three regular 1-forms ϕ_1, ϕ_2 and $\widetilde{\phi}_3$ by prescribing the following periods:

$$\int_{C(a_{k,i})}(\phi_1,\phi_2,\widetilde{\phi}_3)=2\pi\mathrm{i}\,(\alpha_{k,i;1},\ \alpha_{k,i;2},\ \alpha_{k,i;3})$$

$$(1\le k\le N,\quad 2-\delta_{k,1}\le i\le n_k),$$

$$\int_{A_k}(\phi_1,\phi_2,\widetilde{\phi}_3)=(1+\mathrm{i}\,\alpha_{k;1},\ \mathrm{i}\,\alpha_{k;2},\mathrm{i}\ \alpha_{k;3})\quad(1\le k\le N).$$

All the α numbers in the right hand side are real parameters. The reason for the $2\pi\mathrm{i}$ factor is that we will interpret the parameters $\alpha_{k,i;\ell}$ as residues. We define $\phi_3=x\widetilde{\phi}_3$, where x is a real number in a neighborhood of 0. When $x=0$, $\phi_3=0$ so the surface will be locally a flat horizontal plane: this is the limit case. When $x\neq 0$, we want to adjust the other parameters, so that the period problem is solved and $\phi_1^2+\phi_2^2+\phi_3^2=0$. We will do this using the implicit function theorem at $x=0$. We will prove regularity and embeddedness in Section 5.8.

Note that we do not prescribe the periods of ϕ_1, ϕ_2, $\widetilde{\phi}_3$ on the circles $C(a_{2,1}),\dots,C(a_{N,1})$. It will be convenient to define

$$(\alpha_{k,1;1}\ ,\ \alpha_{k,1;2}\ ,\ \alpha_{k,1;3})=\frac{1}{2\pi\mathrm{i}}\int_{C(a_{k,1})}(\phi_1,\phi_2,\widetilde{\phi}_3)\qquad(2\le k\le N).$$

These can be expressed as functions of the other parameters:

$$\alpha_{k,1;\ell}=\sum_{i=1}^{n_1}\alpha_{1,i;\ell}-\sum_{i=2}^{n_k}\alpha_{k,i;\ell}. \tag{1}$$

To see this, let ω be a holomorphic 1-form on Σ_t. Using Cauchy theorem in the domain of $\mathbb{T}_k$ bounded by the circles $C(a_{k,i})$ and $C(b_{k-1,i})$, we have

(2)
$$\sum_{i=1}^{n_k}\int_{C(a_{k,i})}\omega=-\sum_{i=1}^{n_{k-1}}\int_{C(b_{k-1,i})}\omega=\sum_{i=1}^{n_{k-1}}\int_{C(a_{k-1,i})}\omega=\sum_{i=1}^{n_1}\int_{C(a_{1,i})}\omega.$$

5.4.4. Parameters count. Let us count the number of parameters we have. Let $n=n_1+\cdots+n_N$ be the number of necks. We have the conformal parameters $a_{k,i}$, $b_{k,i}$, $t_{k,i}$ and τ_k, for a total of $3n+N$ complex parameters. We have the period parameters $\alpha_{k,i;\ell}$, $\alpha_{k;\ell}$, for a total of $3(n-N+1)+3N=3(n+1)$ real parameters. Finally we have the real parameter x. If we see each complex parameter as two real parameters, this gives us a total of $9n + 2N + 4$ real parameters.

Let us now count the number of equations we have to solve. We have the conformality equation $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$, which amounts to $3g - 3 = 3n$ complex equations. We have the period conditions on the cycles $B_{k,i}$, B_k and B, for a total of $3(n - N) + 3N + 2 = 3n + 2$ real equations. This gives us a total of $9n + 2$ real equations.

So we see that we have $2N + 2$ too many parameters, a lot more than expected. This is because the parameters are not independent and we can normalize the value of some of them. Indeed, if we multiply all parameters $\alpha_{k,i;3}$ and $\alpha_{k;3}$ by some λ and divide x by λ, we do not change the Weierstrass data, so we may impose one condition on these parameters. Also, if we translate all points $a_{k,i}$ and $b_{k-1,j}$ in $\mathbb{T}_k$ by some fixed amount, we get an isomorphic Riemann surface Σ_t, so we may fix the value of one point in each $\mathbb{T}_k$. So we have one real and N complex normalizations, for a total of $2N + 1$ real normalizations.

Taking this into account, we have one more parameter than equations, so we expect to construct a one parameter family of minimal surfaces.

5.5. The conformality equations. Let $\mathcal{Q} = \phi_1^2 + \phi_2^2 + \phi_3^2$. This is a regular quadratic differential on Σ_t. To ensure that $\mathcal{Q} = 0$ we will solve the system $L(\mathcal{Q}) = 0$, where L is the linear operator defined in Lemma 2. We start with the first three equations. We will solve the last one in Section 5.7.

Proposition 4. *For x close to 0, there exist (locally unique) values of the parameters $\alpha_{k;1}$, $\alpha_{k;2}$, $\alpha_{k,i;1}$, $\alpha_{k,i;2}$, $t_{k,i}$ (for all possible values of the indices k, i) such that the following equations are satisfied:*

$$\int_{A_k} \frac{\mathcal{Q}}{dz} = 0 \qquad (1 \leq k \leq N)$$

$$\int_{C(a_{k,i})} \frac{(z - a_{k,i})\mathcal{Q}}{dz} = 0 \qquad (1 \leq k \leq N,\ 1 \leq i \leq n_k)$$

$$\int_{C(a_{k,i})} \frac{\mathcal{Q}}{dz} = 0 \qquad (1 \leq k \leq N,\ 2 - \delta_{k,1} \leq i \leq n_k).$$

These values depend real analytically on $X = x^2$ (and on all other parameters $a_{k,i}$, $b_{k,i}$, τ_k, $\alpha_{k,i;3}$ and $\alpha_{k;3}$ as well). Moreover, at the point $x = 0$, we have

$$t_{k,i} = 0,\ \frac{\partial t_{k,i}}{\partial X} = \frac{(\alpha_{k,i;3})^2}{4}$$

$$\alpha_{k;1} = 0,\ \alpha_{k;2} = (-1)^k,\ \alpha_{k,i;1} = \alpha_{k,i;2} = 0$$

and for $i \geq 2 - \delta_{k,1}$,

$$\frac{\partial}{\partial X}(\alpha_{k,i;1} + (-1)^k \mathrm{i}\, \alpha_{k,i;2}) = \frac{-1}{2} \mathrm{Res}_{a_{k,i}} \frac{(\widetilde{\phi_3})^2}{dz}.$$

Note that when the parameters have these values, $x \neq 0$ implies $t_{k,i}(x) \neq 0$ (for x small enough, provided $\alpha_{k,i;3} \neq 0$) so $\Sigma_{t(x)}$ is a Riemann surface without nodes.

Proof. We use the implicit function theorem. First assume that $x = 0$, $t_{k,i} = 0$, $\alpha_{k,i;1} = \alpha_{k,i;2} = 0$ (all indices k, i). Then $\phi_3 = 0$ and ϕ_1, ϕ_2 are holomorphic 1-forms on $\mathbb{T}_k$, so they are entirely determined by their period on A_k, namely

$$\phi_1 = (1 + \mathrm{i}\,\alpha_{k;1})dz\ ,\ \phi_2 = \mathrm{i}\,\alpha_{k;2}dz$$

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = (1 + 2\mathrm{i}\,\alpha_{k;1} - \alpha_{k;1}^2 - \alpha_{k;2}^2)dz^2.$$

Hence, we need $\alpha_{k;1} = 0$ and $\alpha_{k;2} = \pm 1$. The choice ± 1 is determined by the orientation of the surface: $\phi_1 = dz$, $\phi_2 = \mathrm{i}\,dz$ gives the Weierstrass representation of a horizontal plane with normal $(0, 0, -1)$ while $\phi_1 = dz$, $\phi_2 = -\mathrm{i}\,dz$ gives a horizontal plane with the opposite normal. Hence from the geometry of the surface we want to construct, the sign should alternate between consecutive $\mathbb{T}_k$. We choose $\alpha_{k;2} = (-1)^k$, which means that in $\mathbb{T}_k$, the normal will point down for k even and up for k odd. Then $\phi_1 = dz$, $\phi_2 = (-1)^k\mathrm{i}\,dz$ in $\mathbb{T}_k$ and $\mathcal{Q} = 0$ so all equations are satisfied. We now compute the derivatives with respect to all parameters, except x, at this point. Let p be any of the parameters and γ be any cycle; then

$$\begin{aligned}\frac{\partial}{\partial p}\int_\gamma \frac{\mathcal{Q}}{dz} &= \int_\gamma \frac{2\phi_1}{dz}\frac{\partial \phi_1}{\partial p} + \frac{2\phi_2}{dz}\frac{\partial \phi_2}{\partial p}\\ &= \int_\gamma 2\frac{\partial \phi_1}{\partial p} + 2(-1)^k\mathrm{i}\,\frac{\partial \phi_2}{\partial p}\\ &= 2\frac{\partial}{\partial p}\int_\gamma \phi_1 + (-1)^k\mathrm{i}\,\phi_2.\end{aligned}$$

This gives (recall the definition of these forms)

$$\frac{\partial}{\partial \alpha_{k,i;1}}\int_{C(a_{k,i})}\frac{\mathcal{Q}}{dz} = 4\pi\mathrm{i}\,, \quad \frac{\partial}{\partial \alpha_{k,i;2}}\int_{C(a_{k,i})}\frac{\mathcal{Q}}{dz} = -4\pi(-1)^k$$

$$\frac{\partial}{\partial \alpha_{k;1}}\int_{A_k}\frac{\mathcal{Q}}{dz} = 2\mathrm{i}\,, \quad \frac{\partial}{\partial \alpha_{k;2}}\int_{A_k}\frac{\mathcal{Q}}{dz} = -2(-1)^k$$

and all other partial derivatives of these functions are zero. By Lemma 3, the derivatives of ϕ_1 and ϕ_2 with respect to the parameter $t_{k,i}$ have a double pole at $a_{k,i}$. Since $\phi_1 = dz$ and $\phi_2 = (-1)^{k+1}\mathrm{i}\,dz$ in $\mathbb{T}_{k+1}$, their principal parts at $a_{k,i}$ are respectively $-dz/(z-a_{k,i})^2$ and $(-1)^k\mathrm{i}\,dz/(z-a_{k,i})^2$. This gives

$$\frac{\partial}{\partial t_{k,i}}\int_{C(a_{k,i})}\frac{(z-a_{k,i})\mathcal{Q}}{dz} = \int_{C(a_{k,i})}(z-a_{k,i})\left(\frac{2\phi_1}{dz}\frac{\partial \phi_1}{\partial t_{k,i}} + \frac{2\phi_2}{dz}\frac{\partial \phi_2}{\partial t_{k,i}}\right) = -8\pi\mathrm{i}\,.$$

The derivative of this function with respect to any of the other parameters is zero (because the derivatives of ϕ_1 and ϕ_2 have at most simple poles).

The first statement of the proposition now follows from the implicit function theorem. Note that the map is only real analytic because all the α parameters are real. Since $(\phi_3)^2 = X(\widetilde{\phi}_3)^2$, with $X = x^2$, we have

$$\frac{\partial}{\partial X}\int_{C(a_{k,i})}\frac{\mathcal{Q}}{dz} = \int_{C(a_{k,i})}\frac{(\widetilde{\phi}_3)^2}{dz} = 2\pi\mathrm{i}\,\mathrm{Res}_{a_{k,i}}\frac{(\widetilde{\phi}_3)^2}{dz}$$

$$\frac{\partial}{\partial X}\int_{C(a_{k,i})}\frac{(z-a_{k,i})\mathcal{Q}}{dz} = \int_{C(a_{k,i})}\frac{(z-a_{k,i})(\widetilde{\phi}_3)^2}{dz} = 2\pi\mathrm{i}\,(\alpha_{k,i;3})^2.$$

This gives the indicated derivatives of $\alpha_{k,i;1}$, $\alpha_{k,i;2}$ and $t_{k,i}$ with respect to X. q.e.d.

5.6. The period problem.

5.6.1. Solution of the Period problem for ϕ_3.

Proposition 5. *Assume the parameters $t_{k,i}$ are given as functions of x by Proposition 4. Then for x close to 0, there exist (locally unique) values of the parameters $\alpha_{k;3}$, $\alpha_{k,i;3}$ (all indices) such that the following equations are satisfied:*

$$\mathrm{Re}\int_{B_k}\widetilde{\phi}_3 = 0 \qquad (1 \le k \le N)$$

$$\mathrm{Re}\int_{B_{k,i}}\widetilde{\phi}_3 = 0 \qquad (1 \le k \le N,\ 2 \le i \le n_k)$$

$$\sum_{i=1}^{n_1}\alpha_{1,i;3} = -1.$$

Moreover, when $x = 0$, we have $\alpha_{k,i;3} = -c_k$ where $c_k = 1/n_k$. Finally, for $x \neq 0$ close to 0, we have

$$\mathrm{Re}\int_B\widetilde{\phi}_3 \sim -\log(x^2)(c_1 + \cdots + c_N). \tag{3}$$

The third equation is a normalization, see Section 5.4.4. The reason for the minus sign is that given the geometric picture of the surface we want to construct and the orientation of $A_{k,i}$, we expect that the flux of each cycle $A_{k,i}$ points downward, so the imaginary part of the period of ϕ_3 is negative.

Proof. First observe that $\widetilde{\phi}_3$ depends linearly on the parameters $\alpha_{k,i;3}$ and $\alpha_{k;3}$, so we have to solve linear equations: no need to use the implicit function theorem here! The period of $\widetilde{\phi}_3$ on the cycle B_k depends analytically on x because we can choose a representative of B_k which stays away from the nodes. We represent $B_{k,i}$ by the composition of the following paths:

1) a path from $a_{k,1}+\epsilon$ to $a_{k,i}+\epsilon$ in $\mathbb{T}_k$, (depending continuously on parameters and avoiding nodes),
2) a path from $v_{k,i}=\epsilon$ to $w_{k,i}=\epsilon$ (going through the neck),
3) a path from $b_{k,i}+\epsilon$ to $b_{k,1}+\epsilon$ in $\mathbb{T}_{k+1}$,
4) a path from $w_{k,1}=\epsilon$ to $v_{k,1}=\epsilon$.

The integral of $\widetilde{\phi}_3$ on the first and third paths depends analytically on x. We use Lemma 4 to estimate the integral on the second and fourth paths. This gives

$$\operatorname{Re}\int_{B_{k,i}}\widetilde{\phi_3}=\operatorname{Re}(\alpha_{k,i;3}\log t_{k,i}-\alpha_{k,1;3}\log t_{k,1})+\text{analytic}$$

and in the same way,

$$\operatorname{Re}\int_{B}\widetilde{\phi_3}=\sum_{k=1}^{N}\operatorname{Re}(\alpha_{k,1;3}\log t_{k,i})+\text{analytic.}$$

By Proposition 4, $t_{k,i}\simeq\frac{1}{4}(x\alpha_{k,i;3})^2$. We assume that all the numbers $\alpha_{k,i;3}$ are non zero. This gives

$$\operatorname{Re}\int_{B_{k,i}}\widetilde{\phi_3}=(\alpha_{k,i;3}-\alpha_{k,1;3})\log(x^2)+\text{analytic.}$$

Hence the function $(\log(x^2))^{-1}\operatorname{Re}\int_{B_{k,i}}\widetilde{\phi}_3$ extends continuously to $x=0$ with value $\alpha_{k,i;3}-\alpha_{k,1;3}$. Proposition 5 boils down to the following statement:

Claim 1. When $x=0$, the following linear map is an isomorphism

$$\Big(\underbrace{\alpha_{k,i;3}}_{\substack{1\le k\le N\\ 2-\delta_{k,1}\le i\le n_k}},\ \underbrace{\alpha_{k;3}}_{1\le k\le N}\Big)\mapsto\Big(\underbrace{\alpha_{k,i;3}-\alpha_{k,1;3}}_{\substack{1\le k\le N\\ 2\le i\le n_k}},\ \underbrace{\operatorname{Re}\int_{B_k}\widetilde{\phi}_3}_{1\le k\le N},\ \sum_{i=1}^{n_1}\alpha_{1,i;3}\Big).$$

Proof. The domain and target spaces have the same dimension. Using equation (1), it is straightforward to check that the kernel is zero. q.e.d.

Remark 5. The function $1/\log(x^2)$ does not extend to a smooth function at 0, so the solutions to our equations are not smooth functions of x. To deal with this problem, we write $x=\exp(-1/\xi^2)$, which extends to a smooth function at $\xi=0$. Then $1/\log(x^2)=-\frac{1}{2}\xi^2$, so our equations and their solutions depend smoothly on the auxiliary variable ξ in a neighborhood of 0. Note, however, that we (sadly) leave the realm of analytic functions, but there seems to be no way to avoid this!

5.6.2. Solution of the period problem for ϕ_1 and ϕ_2. For $\ell=1,2,3$, write $T_\ell=(T_{\ell;1},T_{\ell,2},0)$.

Proposition 6. *Assume the parameters $t_{k,i}$, $\alpha_{k;\ell}$ and $\alpha_{k,i;\ell}$ ($\ell = 1,2,3$) are given as functions of x by Propositions 4 and 5. For x close to 0, there exist (locally unique) values of the parameters τ_k and $b_{k,i}$ (all indices) such that the following equations are satisfied:*

$$\mathrm{Re}\int_{B_k}\phi_\ell = T_{2;\ell} \qquad (1\le k\le N,\ \ell=1,2)$$

$$\mathrm{Re}\int_{B_{k,i}}\phi_\ell = 0 \qquad (1\le k\le N,\ 2\le i\le n_k,\ \ell=1,2)$$

$$\mathrm{Re}\int_{B}\phi_\ell = T_{3;\ell} \qquad (\ell=1,2).$$

Moreover, when $x=0$, we have $\tau_k = T_2$ if k is odd and $\tau_k=\overline{T_2}$ if k is even, $b_{k,i}=\overline{a_{k,i}}$ if $1\le k\le N-1$, and $b_{N,i}=\overline{a_{N,i}}-T_3$.

Proof. We first solve the equations when $x=0$ and then conclude using the implicit function theorem. When $x=0$, we have $\phi_1 = dz$ and $\phi_2 = (-1)^k \mathrm{i}\,dz$ in $\mathbb{T}_k$. Hence

$$T_{2;1} = \mathrm{Re}\int_{B_k}\phi_1 = \mathrm{Re}\int_0^{\tau_k} dz = \mathrm{Re}(\tau_k)$$

$$T_{2;2} = \mathrm{Re}\int_{B_k}\phi_2 = \mathrm{Re}\int_0^{\tau_k}(-1)^k\mathrm{i}\,dz = (-1)^{k+1}\mathrm{Im}(\tau_k).$$

This determines τ_k. Then

$$\begin{aligned}0 = \mathrm{Re}\int_{B_{k,i}}\phi_1 &= \mathrm{Re}\left(\int_{a_{k,1}}^{a_{k,i}} dz + \int_{b_{k,i}}^{b_{k,1}} dz\right)\\ &= \mathrm{Re}(a_{k,i}-a_{k,1}+b_{k,1}-b_{k,i}),\\ 0 = \mathrm{Re}\int_{B_{k,i}}\phi_2 &= \mathrm{Re}\left(\int_{a_{k,1}}^{a_{k,i}}(-1)^k\mathrm{i}\,dz + \int_{b_{k,i}}^{b_{k,1}}(-1)^{k+1}\mathrm{i}\,dz\right)\\ &= (-1)^{k+1}\mathrm{Im}(a_{k,i}-a_{k,1}-b_{k,1}+b_{k,i}).\end{aligned}$$

This gives

$$a_{k,i}-a_{k,1} = \overline{b_{k,i}}-\overline{b_{k,1}} \qquad (1\le k\le N,\ 2\le i\le n_k). \tag{4}$$

As explained in Section 5.4.4, we may normalize translation by fixing one point in each $\mathbb{T}_k$. For $k=1,\dots,N-1$, we fix the value of $b_{k,1}$ in $\mathbb{T}_{k+1}$ by asking that $b_{k,1}=\overline{a_{k,1}}$. This normalizes translation in $\mathbb{T}_2,\dots,T_N$. (We will normalize translation in $\mathbb{T}_1$ later). Equation (4) gives $b_{k,i}=\overline{a_{k,i}}$ for $1\le k\le N-1$. Computations entirely similar to the previous ones give

$$T_{3;1} = \mathrm{Re}\int_B\phi_1 = \mathrm{Re}\left(\int_{b_{N,1}}^{a_{1,1}} dz + \sum_{k=1}^{N-1}\int_{b_{k,1}}^{a^{k+1,1}} dz\right) = \mathrm{Re}(a_{N,1}-b_{N,1})$$

$$T_{3;2} = \mathrm{Re}\int_B\phi_2 = \mathrm{Im}(-a_{N,1}-b_{N,1}).$$

This gives $b_{N,1} = \overline{a_{N,1}} - T_3$. Equation (4) gives $b_{N,i} = \overline{a_{N,i}} - T_3$.

To apply the implicit function theorem, we need to study the smoothness of the above functions. The integrals on the cycles B_k are clearly analytic functions of x. By Proposition 4, the periods of ϕ_1 and ϕ_2 on the circles $C(a_{k,i})$, namely the parameters $\alpha_{k,i;1}$ and $\alpha_{k,i;2}$, are of order x^2. By Lemma 4, their periods on the cycles $B_{k,i}$ are analytic functions plus terms of order $x^2 \log t_{k,i} = \mathcal{O}(x^2 \log(x^2))$. This extends to a smooth function of the auxiliary parameter ξ at $\xi = 0$, see Remark 5. Hence the periods are smooth functions of ξ and the other parameters. The partial differential with respect to the parameters $b_{k,i}$ and τ_k is clearly an isomorphism. The proposition then follows from the implicit function theorem at $\xi = 0$. q.e.d.

5.7. The balancing condition.

It remains to solve the last equation in Lemma 2.

Proposition 7. *Assume the parameters $t_{k,i}$, $\alpha_{k;1}$, $\alpha_{k;2}$, $\alpha_{k,i;1}$ and $\alpha_{k,i;2}$ are given as analytic functions of $X = x^2$ by Proposition 4. Let*

$$f_{k,i} = \frac{1}{2\pi \mathrm{i}} \int_{C(b_{k,i})} \frac{\mathcal{Q}}{dz} \qquad (1 \le k \le N\ ,\ 1 + \delta_{k,N} \le i \le n_k).$$

Then $\widetilde{f}_{k,i} = x^{-2} f_{k,i}$ extends analytically to $x = 0$ and its value at $x = 0$ is

$$\widetilde{f}_{k,i}(0) = R_{k,i} \qquad \textit{if } 1 \le k \le N,\ 2 - \delta_{k,1} \le i \le n_k$$

$$\widetilde{f}_{k,1}(0) = R_{k,1} + \sum_{\ell=1}^{k-1} \sum_{i=1}^{n_\ell} \varphi^{k-\ell}(R_{\ell,i}) \qquad \textit{if } 2 \le k \le N-1$$

where $\varphi(z) = \overline{z}$ denotes conjugation in $\mathbb{C}$ and

$$R_{k,i} = \varphi\left(\mathrm{Res}_{a_{k,i}} \frac{(\widetilde{\phi}_3)^2}{dz}\right) + \mathrm{Res}_{b_{k,i}} \frac{(\widetilde{\phi}_3)^2}{dz}.$$

Proof. If $f(z)$ is an analytical function of z such that $f(0) = 0$, then $f(z)/z$ extends analytically to $z = 0$ with value $f'(0)$. So all we have to do is to compute the derivative of $f_{k,i}$ with respect to X at $X = 0$. We have

$$\frac{\partial f_{k,i}}{\partial \alpha_{k,i;1}} = -2, \quad \frac{\partial f_{k,i}}{\partial \alpha_{k,i;2}} = 2(-1)^k \mathrm{i}\,,$$

and the derivatives of $f_{k,i}$ with respect to the other parameters (except X) are zero. Hence, by the chain rule,

$$\frac{\partial f_{k,i}}{\partial X} = -2\frac{\partial \alpha_{k,i;1}}{\partial X} + 2(-1)^k \mathrm{i}\, \frac{\partial \alpha_{k,i;2}}{\partial X} + \mathrm{Res}_{b_{k,i}} \frac{(\widetilde{\phi}_3)^2}{dz}.$$

When $2-\delta_{k,1} \leq i \leq n_k$, using the last formula in Proposition 4, this gives $\partial f_{k,i}/\partial X = R_{k,i}$. To compute the derivative of $f_{k,1}$ for $2 \leq k \leq N-1$, we use equation (1) to obtain (the α terms cancel)

$$\begin{aligned}
&\sum_{i=1}^{n_k} \frac{\partial f_{k,i}}{\partial X} - \sum_{i=1}^{n_{k-1}} \varphi\left(\frac{\partial f_{k-1,i}}{\partial X}\right) \\
&= \sum_{i=1}^{n_k} \mathrm{Res}_{b_{k,i}} \frac{(\widetilde{\phi}_3)^2}{dz} - \sum_{i=1}^{n_{k-1}} \varphi\left(\mathrm{Res}_{b_{k-1,i}} \frac{(\widetilde{\phi}_3)^2}{dz}\right) \\
&= \sum_{i=1}^{n_k} \mathrm{Res}_{b_{k,i}} \frac{(\widetilde{\phi}_3)^2}{dz} + \sum_{i=1}^{n_k} \varphi\left(\mathrm{Res}_{a_{k,i}} \frac{(\widetilde{\phi}_3)^2}{dz}\right) \\
&= \sum_{i=1}^{n_k} R_{k,i}.
\end{aligned}$$

In the second line, we have used the residue theorem in $\mathbb{T}_k$. The result follows by induction on k. q.e.d.

Assume the parameters $t_{k,i}$, $\alpha_{k;\ell}$, $\alpha_{k,i;\ell}$ ($\ell = 1, 2, 3$) and $b_{k,i}$ are given as functions of x and the remaining parameters (namely $a_{k,i}$) by Proposition 4, 5 and 6.

Proposition 8. *For x small enough, there exist (locally unique) values of the parameters $a_{k,i}$ such that $\widetilde{f}_{k,i} = 0$ for $1 \leq k \leq N$, $1 + \delta_{k,N} \leq i \leq n_k$.*

This determines the value of all parameters as functions of x so that the period problem is solved and $\phi_1^2+\phi_2^2+\phi_3^2 = 0$. The solution depends smoothly on the auxiliary parameter ξ (see Remark 5). We also still have T_1, T_2 and T_3 as free parameters. The solution depends smoothly on these parameters.

Proof. First assume that $x = 0$. Given the value of τ_k found in Proposition 6, we have $\mathbb{T}_k = \varphi^{k+1}(\mathbb{T})$ where $\mathbb{T} = \mathbb{C}/\Gamma$ and $\varphi(z) = \overline{z}$. Let

$$a_{k,i} = \varphi^{k+1}(p_{k,i})$$

where $p_{k,i}$ is the given configuration. Then from Proposition 6, $b_{k,i} = \varphi^k(p_{k,i})$ for $1 \leq k \leq N-1$ and $b_{N,i} = p_{N,i} - T_3 = p_{0,i}$. It follows that

$$\widetilde{\phi}_3 = \begin{cases} \omega_k & \text{in } \mathbb{T}_k \text{ if } k \text{ odd}, \\ \overline{\varphi^*\omega_k} & \text{in } \mathbb{T}_k \text{ if } k \text{ even}, \end{cases}$$

where ω_k is the meromorphic 1-form on $\mathbb{T}$ defined in Section 2.2. (Indeed, these forms are holomorphic and have the same poles and residues, and both have imaginary periods.) This implies that

$$R_{k,i} = 2\varphi^k(F_{k,i})$$

where $F_{k,i}$ is the force defined in the Section 2.2. Hence, since the configuration is balanced, we have $\widetilde{f}_{k,i} = 0$. We may normalize translation in $\mathbb{T}_1$ by fixing the position of $p_{1,1}$. Then, since the configuration is non-degenerate, it follows from elementary linear algebra that the partial differential of $(\widetilde{f}_{k,i} : \ 1 \leq k \leq N, \ 1+\delta_{k,N} \leq i \leq n_k)$ with respect to $(p_{\ell,j} : \ 1 \leq k \leq N, \ 1+\delta_{k,1} \leq i \leq n_k)$ at $x = 0$ is an ($\mathbb{R}$ linear) isomorphism. The conclusion follows by the implicit function theorem again. q.e.d.

5.8. Embeddedness. At this point, we have constructed a one parameter family of conformal, minimal immersions from $\Sigma_{t(x)}$ to $\mathbb{R}^3/\Lambda_x$, x in a neighborhood of 0, where Λ_x is the lattice generated by the periods $(T_1, 0)$, $(T_2, 0)$, $(T_3, \varepsilon(x))$ and by equation (3) in Proposition 5,

$$\varepsilon(x) \simeq -x^2 \log(x^2)(c_1 + \cdots + c_N).$$

Here we assume that $x > 0$, as changing x into $-x$ only changes the sign of the third coordinate. We need to prove that the immersion is regular (free of branch points) and an embedding. Given a small $\delta > 0$, let $\Omega_{k,\delta}$ be the set of points in $\mathbb{T}_k$ which are at distance greater than δ from the points $a_{k,i}$ and $b_{k-1,i}$ (all indices i), and $\Omega_\delta = \Omega_{1,\delta} \cup \cdots \cup \Omega_{N,\delta}$.

The immersion is regular if $|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 > 0$. It is straightforward to check that this is true on Ω_δ because we know explicitly the limits of these forms on this domain. The problem is to prove that this holds on the necks. We prove that ϕ_3 has no zeroes on the necks. The zeros of ϕ_3 are the same as the zeros of $\widetilde{\phi}_3$. When $x = 0$, $\widetilde{\phi}_3$ has $n_k + n_{k-1}$ poles in $\mathbb{T}_k$, so it has $n_k + n_{k-1}$ zeros, which lie in $\Omega_{k,\delta}$ provided δ is small enough. By continuity, for x close to 0, $\widetilde{\phi}_3$ has $n_k + n_{k-1}$ zeros in $\Omega_{k,\delta}$. This gives a total of $2n$ zeros in Ω_δ, where $n = n_1 + \cdots + n_N$. But if $x \neq 0$, $\widetilde{\phi}_3$ is a holomorphic 1-form on a compact Riemann surface of genus $g = n + 1$, so it has $2g - 2 = 2n$ zeros. This means that $\widetilde{\phi}_3$ has no further zeros and proves that the surface is regular.

We now prove embeddedness. Let $X_x = (X_x^1, X_x^2, X_x^3)$ be the immersion given by the Weierstrass representation we have constructed, and let $\widetilde{X}_x = (X_x^1, X_x^2, x^{-1} X_x^3)$. We will prove that the image of $\widetilde{X}_x$ is embedded, and compute explicitly its limit (after suitable vertical translation) when $x \to 0$.

First observe that on each domain $\Omega_{k,\delta}$, the Gauss map converges to $(0, 0, -1)$ if k is even and $(0, 0, 1)$ if k is odd, hence its image is locally a graph. Then for $z \in \Omega_{k,\delta}$ we have, up to translation,

$$\lim_{x \to 0} X_x^1(z) + \mathrm{i}\, X_x^2(z) = \operatorname{Re} \int^z dz + \mathrm{i} \operatorname{Re} \int^z (-1)^k \mathrm{i}\, dz = \varphi^{k+1}(z)$$

where as before $\varphi(z) = \overline{z}$, and

$$\lim_{x \to 0} \widetilde{X}^3_x(z) = \mathrm{Re} \int^z (\varphi^{k+1})^* \omega_k = f_k(\varphi^{k+1}(z))$$

where $f_k(z) = \mathrm{Re} \int^z \omega_k$. This is a well defined function of $z \in \mathbb{T}$ because the residues of ω_k are real. It has logarithmic singularities at the poles of ω_k. Hence the image $\widetilde{X}_x(\Omega_\delta)$ converges (up to translation) when $x \to 0$ to the graph of f_k on $\mathbb{T}$ minus disks of radius δ around the singularities. So it is included in a slab whose width is bounded by some constant $C(\delta)$. By our computation in Section 5.6.1, the distance between consecutive slabs has order $\mathcal{O}(\log(x^2))$, so these slabs are disjoint for x small enough.

It remains to understand the behavior of $\widetilde{X}_x$ on the necks. There exists $c > 0$ such that for each k, the horizontal sections $x_3 = c$ (resp. $x_3 = -c$) of the graph of f_k consist of n_k (resp. n_{k-1}) disjoint convex curves. Hence, for x small enough, we may find numbers δ_k^+ and δ_k^- (depending on x), with $\delta_k^- < \delta_k^+ < \delta_{k+1}^-$, such that the intersection of the surface with the slab $\delta_k^- < x_3 < \delta_k^+$ is bounded by convex horizontal curves, and is a graph, and the intersection with the slab $\delta_k^+ < x_3 < \delta_{k+1}^-$ consists of n_k annuli, each bounded by two horizontal convex curves. By a theorem of Schiffman [**7**], a minimal annulus bounded by two horizontal convex curves is fibered by horizontal convex curves. It follows that the surface is embedded. This concludes the proof of Theorem 4.

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