Geometric Models: the Wonderful Tridimensional Kaleidoscope

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Abstract

The traditional kaleidoscope uses the prism as a basic unit. Through the reflections of the mirrors on the inner surfaces of a prism, we can see many symmetric axles and patterns. However, traditional kaleidoscope only presents two-dimensional images.

The three-dimensional kaleidoscope we designed uses the pyramid as a basic unit. We can see three-dimensional images via mirror faces on the inner surfaces of the pyramid. Our examination shows that not all pyramids are able to show three-dimensional images but only pyramids with specific angles do.

The kaleidoscope we designed differs from the traditional kaleidoscope in the following ways:

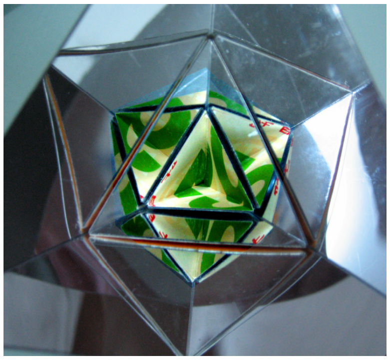
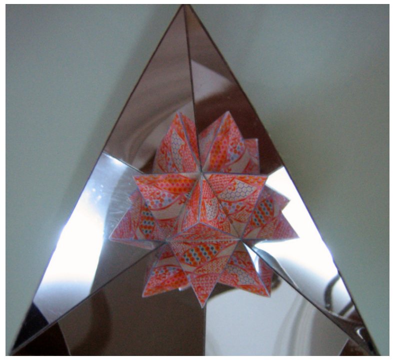
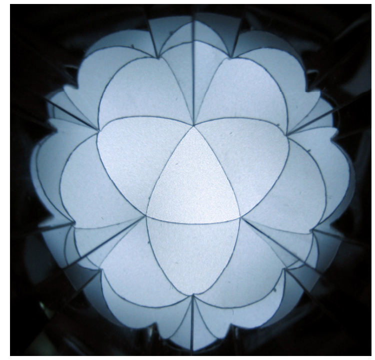
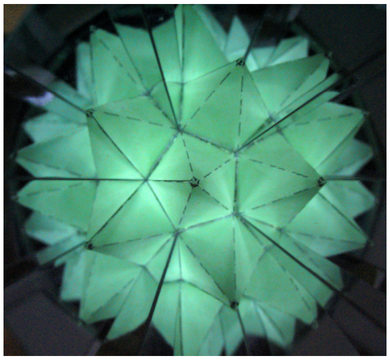
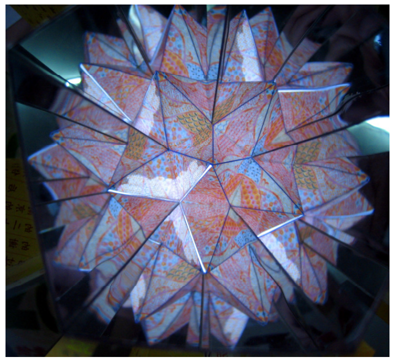
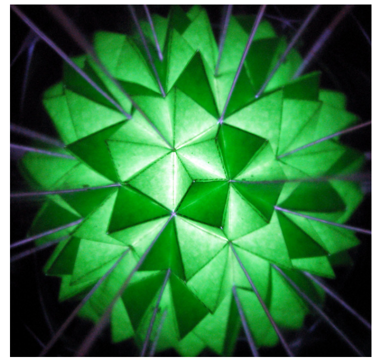
①Our three-dimensional kaleidoscope not only shows three-dimensional space, but also shows it through two-dimensional planes.

②Our three-dimensional kaleidoscope is based on three kinds of pyramid: three-sided pyramid (tetrahedron), four-sided pyramid (square pyramid) and five-sided pyramid (pentagonal pyramid). Each kind of pyramid has its own charm in showing its reflected images.

③Each kind of pyramid is separated from its entire geometry. However, it can present the whole view of the geometry after mirrored.

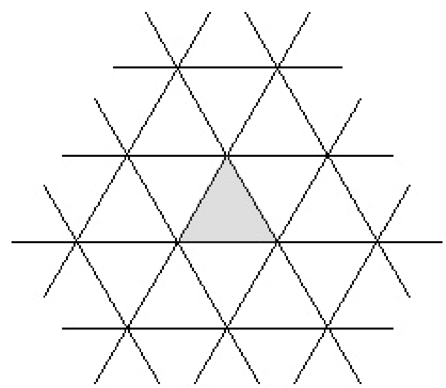
Our exploration shows that two dimensional and three dimensional worlds can change into one another under specific conditions. Perhaps before long we can discover more dimensions and explain the relationships between each kind of dimension.

This treatise mainly involves the principles of Euclidean Geometry and Projective Geometry.

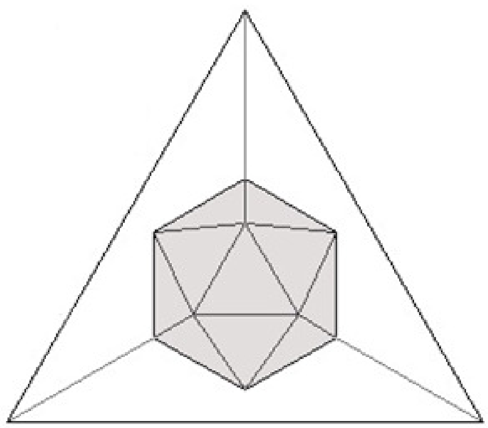
1. Foreword

Since the invention of the traditional kaleidoscope by English physicist David Brewster in 1816, its fascinating, omniform view has enriched the childhood life of generations of people including my parents. We can see a lot of symmetric axles from a simply designed kaleidoscope, and the mirrored patterns are governed by specific geometric principles. This has led us to our experimental effort to create a tridimensional kaleidoscope. Since a two-dimensional world can be so colorful, why not tridimensionize it and build a tridimensional kaleidoscope to further enhance the view?

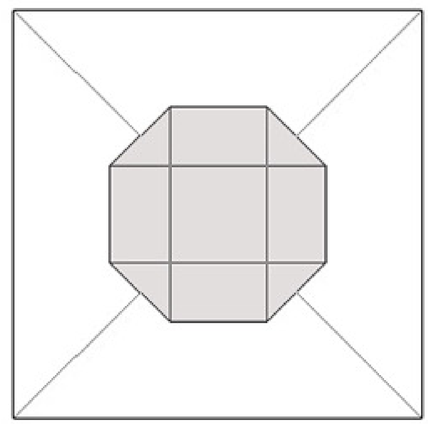

view in a traditional kaleidiscope

There are many branches of geometry in mathematics, and our treatise mainly applies the knowledge of Projective geometry and Euclidean Geometry to the study of pyramids enclosed by two dimensional shapes and the views formed by the reflected patterns.

The traditional kaleidoscope presents two dimensional images. Since its major component is a deltoid prism enclosed by rectangular mirrors, can we simply change the prism into a pyramid to produce tridimensional images? The answer is no. This is because a three-sided pyramid enclosed by three rectangular triangles only presents a vertical view of an icosahedron. Besides, a square pyramid enclosed by four isosceles triangles with top angles of 45°shows a vertical view of a solid figure of 26 faces. (Refer to figures below)



3-sided pyramid enclosed by regular triangles



square pyramid enclosed by isosceles right triangles

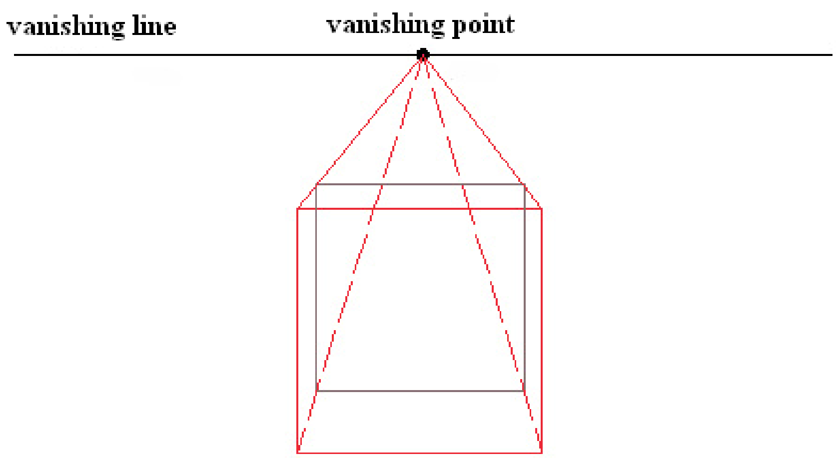
In other words, the results are not tridimensional geometric models. Only through pyramids (including three-sided, four-sided and five-sided pyramids) enclosed by triangles with specific top angles can we see perspective geometric models that are truly tridimensional.

Our study of tridimensional kaleidoscope starts from pyramids. Applying principles of Projective Geometry, we explore the relationship between solid geometry and perspective.

Euclidean Geometry, starting from two dimensional shapes, attains relationships such as location, magnitude and ratio among points, lines, faces and angles through logical inference. It also extends from the demonstration of equality and similarity of two dimensional shapes to that of the spacial relationships such as the magnitude of solid geometries and elements like dual.

Now let's start from three dimensional spaces and research the tridimensional geometric projections on two dimensional planes in light of principles of Projective Geometry, and then represent the tridimensional geometries on two dimensional planes. In Projective Geometry, two parallel lines meet at one point that is infinitely far from us. The point is called the infinite point. The locus of this infinite point is an infinitely far line. In the study of perspective, this point is called the vanishing point, while the line is called the vanishing line. Lines passing through a vanishing point are parallel to one another.

For instance, when you draw a cube on a two dimensional plane, you see that parallel lines focus on the vanishing point. The cube drawn can be regarded as a square frustum laid on its side. To compare the characteristics of a square pyramid with those of the original cube, and then find the variables and invariables of the two--that is one of the fundamental purposes of Projective Geometric research. (Figure below is the perspective of a cube)



It was this square frustum that enlightened us, so to speak, and led us to begin with pyramids, not prisms, in our attempt to search for the key to the tridimensional kaleidoscope.

2. Mathematical model establishment

Through the deep research on specular reflection and the five kinds of Plato solids, we finished making tridimensional kaleidoscopes, or basic units with mirror faces inside, that presented Plato solids with perspective by a series of calculation and repeated practices.

We found that the patterns in a tridimensional kaleidoscope are based on the number and the top angles of isosceles triangular mirror faces that enclose a basic unit. We define these top angles as A-C angles for more convenient demonstration. They are the breakthrough points for us to calculate the requisites for the formation of solids using mathematical knowledge such as the Pythagorean theorem and cosine theorem. Finally we saw five kinds of solid geometries through tridimensional kaleidoscopes.

Basing on five fundamental geometric models, we extended our treatise, hoping to find more kinds of solids that could be reflected in mirrors so that the world in tridimensional kaleidoscope would be more copious. This idea led us to reflect solid figures of 30, 60 and 90 faces through calculation.

From the research above, we arrived at a conclusion: any tridimensional kaleidoscope mentioned above can be enclosed by triangles (including isosceles triangles and scalene triangles) with specific value of top angles. Besides, we conclude the requisites for the formation of tridimensional kaleidoscopes:

① All the apexes of the solid reflected should be on a same circumball.

② The apex of a basic unit overlaps the center of the circumball.

③ The solid reflected must be enclosed by only one kind of polygons.

Since the way in which different tridimensional kaleidoscopes are calculated varies, we cannot temporarily find a theoretical equation to be used in calculation for all kinds of tridimensional kaleidoscopes. However, after calculating dihedral angles and A-C angles of the five kinds of Plato solids, we found equal and complementary relationships among them. This opened a window for us to study on the relationships among different solids in the future.

2. 1 From traditional to tridimensional

If you have ever taken a traditional kaleidoscope apart, you'll know that it consists of three mirrors vertical to the base, standing at an angle of 60°to each other. Each mirror has two images in the other two mirrors, and these two images are also able to reflect patterns. In this way, we'll get an infinite number of rectangular mirror faces, with an infinite number of symmetric patterns that are reflectd in an orderly manner in the mirrors.

If we tilt one end of the mirrors backwards, making the three-sided prism more like a pyramid, the view presented will not be a plane any more, but will curve like a ball. The symmetric patterns become out of shape. Angles (dihedral angles) emerge between two images. Now the view we see is closer to being tridimensional.

In Euclidean Geometry, there are five kinds of Plato Solids: tetrahedron, hexahedron, octahedron, dodecahedron and icosahedron. Basic faces of tetrahedron, octahedron and icosahedron are all regular triangles.

When we used three regular triangles to make a baseless three-sided pyramid, and then added mirrors inside, we saw a polyhedron that we thought to be a regular icosahedron. However, this supposition also raised the following questions:

1. As mentioned above, three kinds of Plato Solids (tetrahedron, octahedron and icosahedron) take regular triangles as basic faces, so why did we only see the icosahedron? What should we do so that a tetrahedron and an octahedron can be presented?

2. When we used 20 three-sided pyramids enclosed by regular triangles as basic units to make an icosahedron, we found there was some unfilled space left. That means 20 basic units like these can not be pieced together to form an icosahedron.

To look for an answer, we went back to "The Elements." This time, we tried to find the requisites for each type of polyhedron to be presented in mirrors through calculation, not congecture.

2.2 Requisites Calculation

Through the experiment of tilting deltoid prismatic mirrors, we found that the degree of tilt exerts an effect on the form of polyhedron. It is the top angle of triangular mirrors that determines the degree of tilt. Generally speaking, when the number of mirrors that enclose a pyramid is fixed, the bigger the top angle, the greater the inclination will be.

Thus, we need to calculate the exact value of this top angle in order to find the corresponding solid that can be tridimensionally presented.

If we put a Plato solid into a ball, making this solid inscribe to the ball, then the center of this solid will overlap the center of its circumball. Divide this solid through the center. For instance, a tetrahedron can be divided into 4 three-sided pyramids. In this way we get several basic units that can be pieced together to form this solid. Every basic unit is enclosed by three isosceles triangles. Different basic units will be obtained when we divide different solids, and the kind of isosceles triangles we get will vary, depending on the solid. So we have simplified the problem to find out this isosceles triangle, which will in turn give us the basic units and the solids.

There is a fairly easy way to calculate this isosceles triangle—by first finding out its top angle. Let's name it A-C angle. A stands for the arris of a polyhedron while C stands for the center of the circumball.

The degree of A-C angle can determine the formation of solids we see in mirrors.

When we calculate A-C angles of different solids, it's important to find the center of circumballs first to find the location of A-C angles. Link the center to two adjacent apexes of a solid. The angle between these two segments is A-C angle. Here we list the calculation of A-C angles for all five kinds of Plato solids.

Regular Octahedron:

An octahedron is enclosed by eight regular triangles. We can divide it into 8 three-sided pyramidal basic units as described above. Since an octahedron is up-down symmetric, we only need to use the upper part.

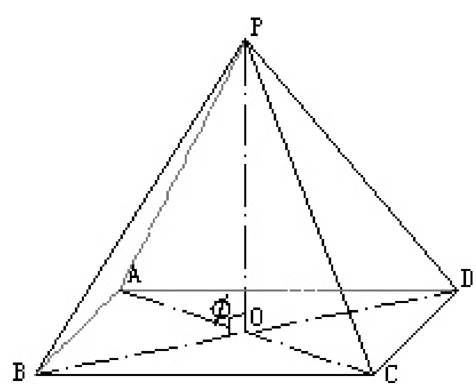
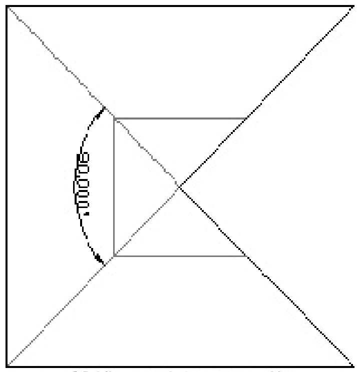


Figure at right shows the upper part of an octahedron. In the square pyramid P-ABCD, P is the apex and ABCD is the base.

Assume that AC and BD intersect at O. It's not hard to see that O is the center of the circumball of octahedron. Link OP, then $\angle$POA, $\angle$POB, $\angle$POC and $\angle POD \triangleq \phi$ are all A-C angles of the octahedron. Since



surface development(octahedron)

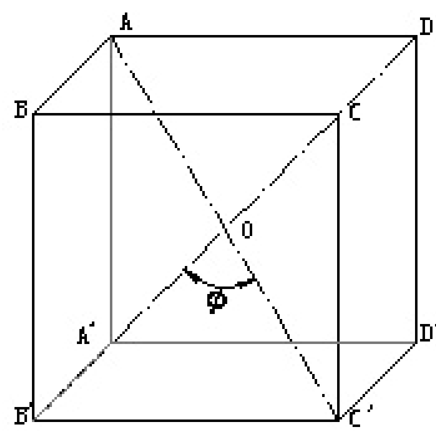
OP is perpendicular to the ABCD plane, the A-C angle is 90°.

That means the pyramid enclosed by three isosceles right triangular mirror faces can present a regular octahedron. In addition, this gives us the surface development of this pyramid. As we can see, the top angle of each triangle, and also the A-C angle of octahedron, is 90°.

Regular Hexahedron:

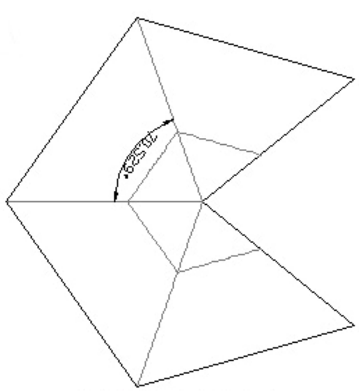
The calculation for hexahedron differs from that for octahedron because the cosine theorem is used this time.

Figure on the right shows a cube whose apexes are A, B, C, D, A', B', C' and D'.O is the intersection of segments AC' and B'D. Besides, it is the center of the circumball of this cube.



$\angle B'OC' \triangleq \phi$ is the A-C angle.

If the side length of the cube is x, then $A'C' = \sqrt{2}\,x$, $AC' = \sqrt{3}\,x$



surface development(hexagon)

$$OC' = \frac{1}{2}AC' = \frac{\sqrt{3}}{2}x$$

Thus, in △B'OC', using cosine theorem and we get

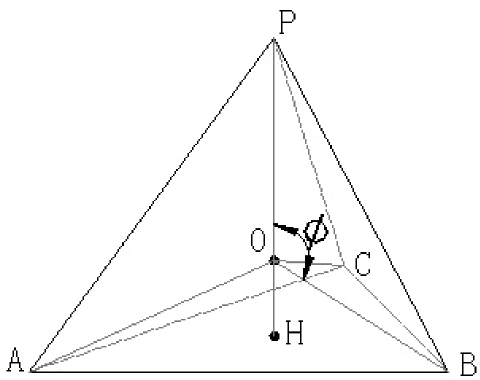
$$\text{Cos}\,\varnothing = \frac{(\frac{\sqrt{3}}{2}x)^2 \times 2 - x^2}{2 \times (\frac{\sqrt{3}}{2}x)^2} = \frac{1}{3}$$

so $\varnothing = \arccos\frac{1}{3} = 70.529°$

Therefore, in order to present a cube, we need four isosceles triangular mirror faces with the top angle of 70.529°.

Regular Tetrahedron:

In the figure at left, P-ABC is a tetrahedron. Assume PH is perpendicular to the ABC plane at the point H. Find a point O on PH to make OP=OA=OB=OC. Link AH. $\angle POB \triangleq \phi$ is the A-C angle.



Assume AP=x, OP=y.

Then PH= $\sqrt{AP^2 - AH^2} = \sqrt{x^2 - \frac{1}{3}x^2} = \frac{\sqrt{6}}{3}x$

Since PH=OP+OH,

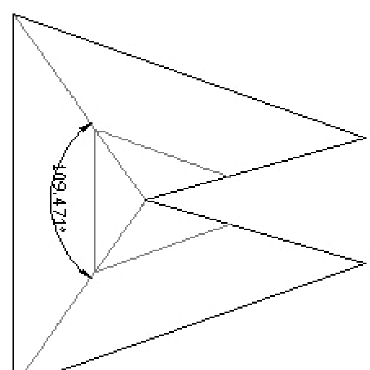
and OH= $\sqrt{OA^2 - AH^2} = \sqrt{y^2 - \frac{1}{3}x^2}$

so $\frac{\sqrt{6}}{3}x = y + \sqrt{y^2 - \frac{1}{3}x^2}$, and $x = \frac{2\sqrt{6}}{3}y$

In the triangle AOP, use the cosine theorem to get $Cos\phi = -\frac{1}{3}$, so $\phi \approx 109.471°$

Thus, in order to see tetrahedron in pyramidal mirrors, we need three isosceles triangular mirror faces whose top angles are 109.471°.

In this development, the top angle of each triangle, or the A-C angle of tetrahedron, is 109.471°.

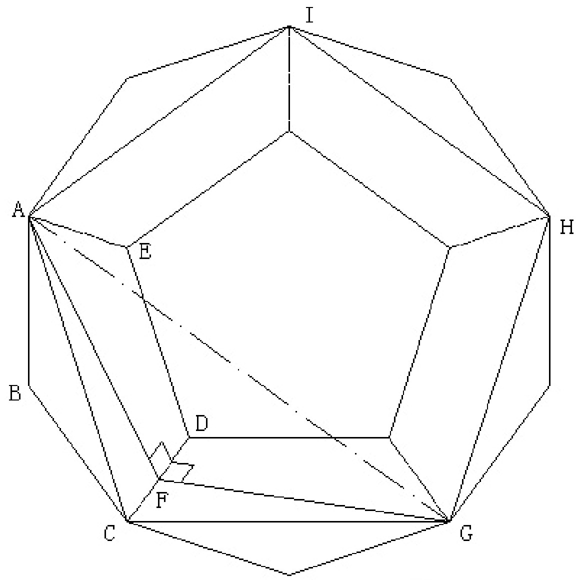


surface development(tetrahedron)

Regular Dodecahedron

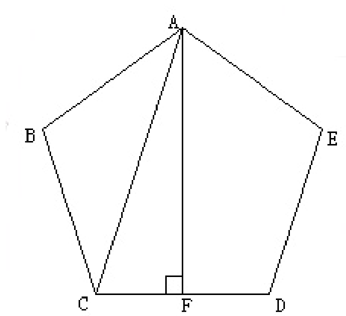
The calculation for dodecahedron is more complex. We must first calculate its dihedral angle in order to calculate A-C angle. Step 1 is the calculation for dihedral angle, and step 2 is the calculation for A-C angle.

Step 1: The figure below is a vertical view of regular dodecahedron with a top pentagon and five slant pentagons.



Choose one of the slant pentagons and mark the five apexes with A, B, C, D, and E. Both AF and FG are perpendicular to CD at point F. So ∠AFG≜θ is the dihedral angle. If we know the side lengths of △AFG, the value of θ can be calculated.

This figure shows the ABCDE pentagon in the previous figure. Link AC.



Assume the side length of each pentagon is x, and AC is y. y=2xSin54°.

Since ∠ABC=108°,

Then ∠BCA=36°, ∠ACD=108°-36°=72°

Obviously, AF=ySin72°=2xSin54°Sin72°

Look back to the vertical view of dodecahedron. Link CG, GH, HI and IA to get a new pentagon ACGHI. The ACGHI plane is parallel to the plane pantagon at top is in. Link AG, then AG=2ySin54°= $2xSin^2 54°$

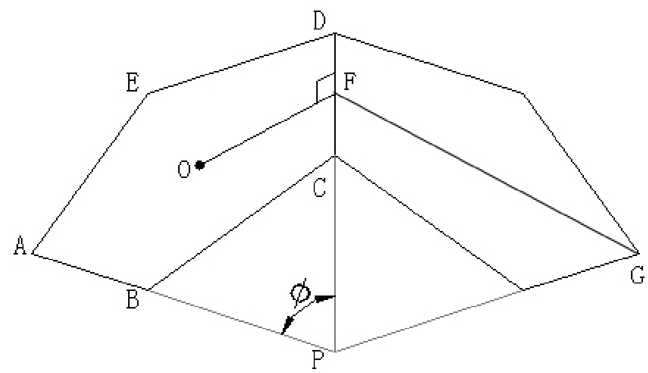
Thus, in △AFG, AF=2xSin54°Sin72°and AG= $2xSin^2 54°$. Use the cosine theorem and get

$$\text{Cos}\,\theta = \frac{2\cdot AF^2 - AG^2}{2AF^2} = \frac{2\cdot(2xSin54°Sin72°)^2 - (4xSin^2 54°)^2}{2\times(2xSin54°Sin72°)^2} \approx -0.45$$

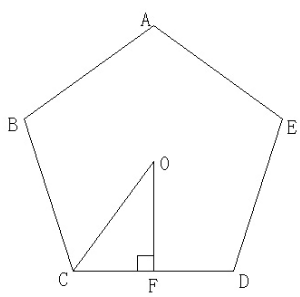
So θ =arcos-0.45≈116.565°

Step 2:

In the figure at right there are two basic units of dodecahedron in the first figure. Point P is the center of its circumball. ∠BPC≙ ϕ is the A-C angle we want to know. Assume O to be the center of pentagon ABCDE. Since ∠OFG is the dihedral angle of dodecahedron,



then ∠OFP= $\frac{1}{2}\angle OFG = \frac{\theta}{2}$ =58.283°(FP should be perpendicular to CD. We didn't mark it because of the limitation of view point.)



The pentagon ABCDE is shown in the figure on the left.

Link OC, because ∠OCD=54°,

so OF= $\frac{x}{2}\tan 54°$

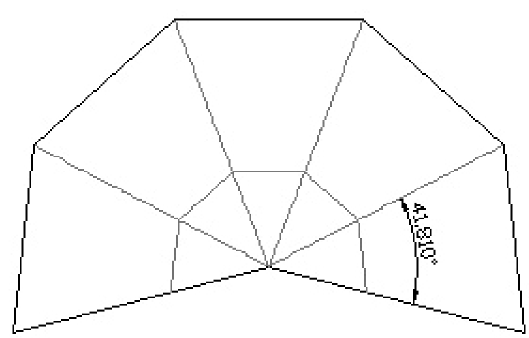
In △CPF, FP=CFcot $\frac{1}{2}$ ∠BPC = $\frac{x}{2}Cot\frac{1}{2}$ ∠BPC= $\frac{x}{2}Cot\frac{\phi}{2}$

Since OF= $\frac{x}{2}\tan 54°$, and FP= $\frac{x}{2}Cot\frac{\phi}{2}$

then OF= $\frac{x}{2}\tan 54°\,\text{Cos}\frac{\theta}{2}$

$$= \frac{OF}{FP} = \frac{\frac{x}{2}\tan 54°}{\frac{x}{2}Cot\frac{\phi}{2}} = \tan 54°\tan\frac{\phi}{2}$$

Thus, $\tan\frac{\phi}{2} = \frac{Cos\frac{\theta}{2}}{\tan 54°} \approx 0.382$



surface development(dodecahedron)

$\phi \approx 41.810°$

Therefore, in order to present a dodecahedron, we need five isosceles triangular mirror faces with the top angles of 41.810°.

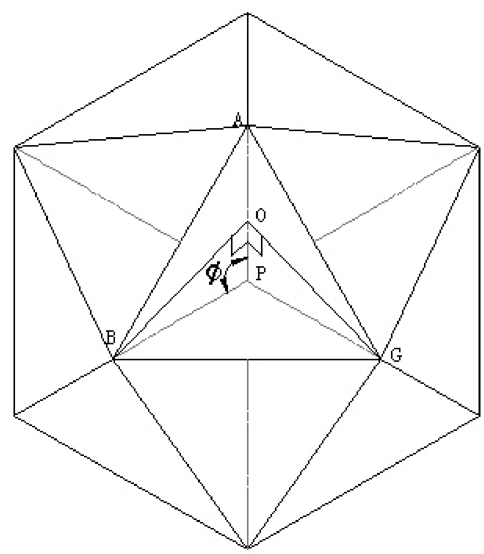
In the surface development, the top angle of each triangle, also the A-C angle of dodecahedron, is 41.810°.

Regular Icosahedron

Two methods can be used to calculate A-C angle of an icosahedron.

1st method:

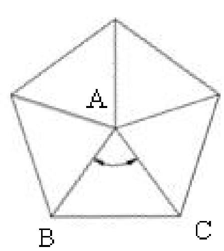
The figure on the right shows a concave icosahedron made of 20 basic units. △ABC is one of the faces of an icosahedron. P is the center of the circumball of icosahedron. O is a point on segment AP. BO and OC are perpendicular to AP at O. $\angle BPA \triangleq \phi$ is the A-C angle we want to know.



Since BO⊥AP, OC⊥AP,

Then AP is perpendicular to the BOC plane

The figure on the lower left shows part of the vertical view of this icosahedron with the top center point A. Because AP is perpendicular to the BOC plane, △ABC is the vertical view of △BOC. Therefore,



∠BOC=∠BAC=72°

Look back to the first figure. We assume the side length of regular icosahedron to be x and OB to be y.

In the △BOC, according to the cosine theorem,

$$Cos72° = \frac{2y^2 - x^2}{2y^2}$$, so y=0.851x

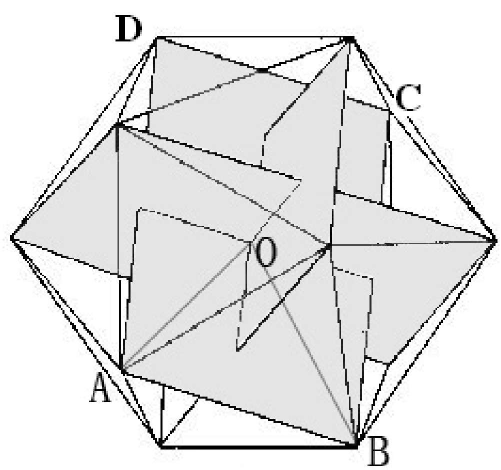
Because Sin∠BAO=$\frac{OB}{AB}=\frac{y}{x}$=0.851

So ∠BAO≈58.281°

In the △ABP, $\phi = 180° - 2\angle BAO \approx 63.435°$

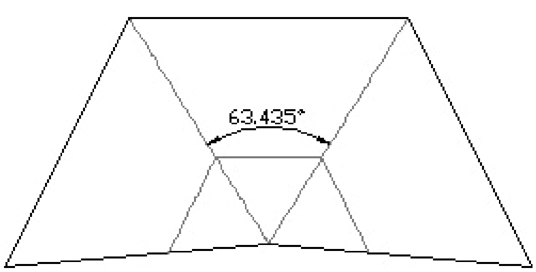
2nd method:

As we know, if we connect the two pairs of opposite sides of a regular icosahedron, we'll get a rectangular. The ratio of its length and width is the golden proportion. Choose one of the three rectangles ABCD, and let its two diagonals intersect at O, then O is the center of the circumball, and ∠AOB is the A-C angle of this icosahedron. Since we know the ratio of the width and length is 0.618, and we assume AD=x,



Then AO=$\frac{1}{2}\sqrt{x^2+(0.618x)^2} \approx 0.809x$

Using the cosine theorem in △AOB, it's not hard to conclude that ∠AOB=63.435°.The result is same to what we've got in the first method.



surface development(icosahedron)

In this surface development, the top angle of each triangles, also the A-C angle of an icosahedron, is 63.435°.

2.3 Conclusions and Implications

Based upon the above discussion, we can draw the following conclusions regarding the nature and characteristics of the tridimensional kaleidoscope:

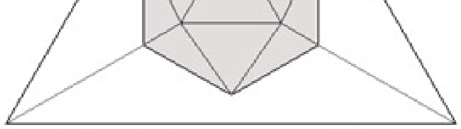
①The geometric images presented in a tridimensional kaleidoscope are trimensional.

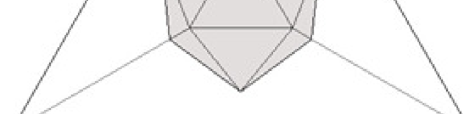
②The two dimensional shapes enclosed by mirror faces (eg. a triangle enclosed by mirror faces in a tetrahedron and a square enclosed by mirror faces in a square pyramid) can not only be triangles, but also squares and pentagons. However, in a traditional kaleidoscope, we won't achieve ideal reflectional results if we replace the 3-sided prism with 4-sided and 5-sided prisms. Therefore, by comparison, the forms of tridimensional kaleidoscope are more diverse and copious.

③The tridimensional kaleidoscope takes a pyramid separated from the solid as a basic unit. It is only part of the solid. After reflection, however, this part can represent the entire solid.

Perhaps after seeing all the calculations and discusion, you'll find they are not so complicated after all. What I have used is indeed the mathematical knowledge I've learnt in senior high school. The mathematical principles for designing a tridimensional kaleidoscope are not difficult. Nevertheless the tridimensional kaleidoscope deserves our research.

As we have mentioned, before we calculated all the A-C angles we needed to know, we had seen a polyhedron from a 3-sided pyramid enclosed by three regular triangular mirror faces, and we thought it might be an icosahedron. However, after we finished calculating, we found that the degree of A-C angle was 3.435°more than 60°, the degree of the top angle of a regular triangle. So we used three isosceles triangular mirror faces with top angles of 63.435°to make a 3-sided pyramid, and found that the polyhedron we saw was similar to that we saw in a regular 3-sided pyramid. But the amazing thing was that it wasn't a vertical view of an icosahedron, but an icosahedron in perspective. (Shown in figures below)


icosahedron in the regular pyramidal mirrors


icosahedron in mirror faces with top angles of 63.435°

We've already known that 20 basic units enclosed by three regular triangles each can not construct a regular icosahedron. But if we change the top angle into 63.435°,the basic units can form a complete icosahedron. So we made the other basic units according to our results of calculation, and found that the polyhedrons mirrored were also perspective in nature. That is to say, if the results of calculation are precise, polyhedrons we see in mirrors will be tridimensional.

Perspective paintings appeared during Renaissance. What the perspective paintings, and the theories proposed by French mathematicians Desargues, Pascal and Poncelet, have in common, is that they all intended to represent tridimensional objects on a two dimensional plane just like taking photos. However, pyramids made of mirror faces are doing just the opposite. They produce images in mirrors and make us see tridimensional space that is originally non-existent through two dimensional planes. The "photo" of a triangle can be a tetrahedron, an octahedron or an icosahedron. At the same time, we can see "photos" like hexahedron and dodecahedron through a square and a pentagon respectively.

For these reasons, our research is especially meaningful. We succeeded in seeing a solid through a plane and an entirety through a part via mirror faces. We believe it is not the only way to show three dimensions through two dimensions. But it is a new vision that helps us to deepen our understanding of the world we are familiar with; and it encourages us to open our mind, to be courageous in our search for truth, and to find new horizons in our future study. We are living in a multi-dimensional universe. Since the two-dimensional world and tridimensional world are mutually convertible, maybe in the future, we can discover more dimensions, explain the relationships between the different dimensions, and greatly enrich our knowledge of the objective world.

3 Studies on other data & Conjectures on relationships among different polyhedrons

At the same time we calculated A-C angles of different Plato Solids, we also calculated dihedral angles. We tabled these data below.

Polyhedron	Dihedral angle	A-C angle
tetrahedron	70.529°	109. 471°
hexahedron	90°	70.529°
octahedron	109.471°	90°
dodecahedron	116.565°	41.810°
icosahedron	138.190°	63.435°

※ Note: ⟶ means two angles are equal
—— means two angles are complementary

We find that their dihedral angles and A-C angles are relative to each other.

Dihedral angle of tetrahedron = A -C angle of hexahedron——→dihedral angle of hexahedron = A-C angle of octahedron——→dihedral angle of octahedron = A-C angle of tetrahedron;

Besides,

Dihedral angle of dodecahedron + A-C angle of icosahedron = 180°

A-C angle of dodecahedron + dihedral angle of icosahedron = 180°

The reasons for these rules are unknown at this point. However, there should be relationships among different solid geometric objects. Plato solids of Euclidean Geometry had demonstrated the dual relationships among hexahedron, octahedron, dodecahedron and icosahedron. Maybe the equal or complementary relationship among angles described in the table above is a kind of demonstration of duality principle in Projective Geometry.

4 References

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Appendix: Photos of Tridimensional Kaleidoscopes