Research on the Combinatorial Transform Mathematics Problem "Frog Leap"

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Abstract

In this paper we extend and deepen a shortlist for the 37th International Mathematical Olympiad (IMO) [1] and propose the Frog Leap Commute Theorem and the Queue Polynomial. We explore the problem from the following aspects:

(1) Make use of semi-invariants and propose the Frog Leap Commute Theorem.
(2) Make extensions regarding frogs leaping to opposite directions on a straight line.
(3) Research frogs leaping to the same direction on a straight line and solve the minimum number of frogs satisfying an infinite leap.
(4) Extend the problem to leaps on a plane or in space.
(5) Research and extend problems regarding frogs leaping on a circle.
(6) Estimate the function $c(n)$ and calculate the order of the function.

Key words

Frog Leap　Leap Commute Theorem　Queue Polynomial　Positive State

1. Original Problem

The original problem in *The 37th IMO Shortlist* is as follows:

A finite number of beans are placed on an infinite row of squares. A sequence of moves is performed as follows: at each stage a square containing more than one bean is chosen. Two beans are taken from this square; one of them is placed on the square immediately to the left, and the other is placed on the square immediately to the right of the chosen square. The sequence terminates if at some point there is at most one bean on each square. Given some initial configuration, it shows that any legal sequence of moves will terminate after the same number of steps and with the same final configuration [2].

For the convenience of description and further extension, we change the original problem above into the following one: A finite number of frogs are on a straight line. If two frogs are at the same point, one of them will leap one unit to the left and the other one unit to the right. Will the frogs leap infinitely or terminate after some sequence of steps?

Suppose there are n frogs and their coordinates are $x_1, x_2, ..., x_n (x_i \in Z)$ respectively. When two frogs on coordinate x leap, one coordinate changes into $x+1$, while the other into $x-1$.

We consider a semi-invariant (a variable that monotonically increases or decreases during the whole leaping progress) $S = x_1^2 + x_2^2 + ... + x_n^2$. Since $(x+1)^2 + (x-1)^2 - 2x^2 = 2$, S increases by 2 for each leap. If the frogs can leap infinitely, S increases continuously.

Consider two adjacent points. If there were frogs on them, there will always be frogs on them, regardless to the leaps. As a result, if the frogs keep leaping in one direction, they will leave at least one frog on every two adjacent points. Thus x_i is bounded, and S is bounded. The frogs must stop after a finite number of leaps.

For an initial state P, there are many leaping possibilities, each of which ends after a finite number of leaps. We find that any legal sequence of moves will terminate after the same number of steps and with the same final configuration. This is true because leaps can be commuted.

Theorem 1: Leap Commute Theory

Under the condition that each leap can be executed, for two adjacent leaps A and B, leaping at A and then B or leaping at B and then A will result in the same final state.

Proof: Leaping at point A results in the decrease of frog number at A by 2, and the increase of frog number at A+1 and A-1 by 1; leaping at point B results in the decrease of frog number at B by 2, and the increase of frog number at B+1 and B-1 by 1. Therefore, both leaping first at points A then B and leaping first at B then A results in the decrease of frog number at A and B by 2 each, and the increase of frog numbers at $A\pm1$ and $B\pm1$ by 1 each.

Now that two adjacent leaps can be exchanged, we can prove that any number of successive leaps can be exchanged.

Corollary: For an original state P, a series of leaps on the points $x_1, x_2, ..., x_n$ generate the final state Q. If we exchange the sequence of points from $x_1, x_2, ..., x_n$ to $y_1, y_2 ..., y_n$, we get a series of leaps on the points $y_1, y_2 ..., y_n$. Under the condition that each leap can be executed, the new leaping sequence will also result in the final state of Q.

Proof: Consider the number of frogs on a certain point A. If a leap occurs at point A, the number of frogs on that point decreases by 2. If a leap occurs at point A-1 or A+1, the number of frogs on point A increases by 1. If a leap occurs at any other points, the number of frogs on point A remains the same. Thus the number of frogs on a certain point is determined by a series of "-2"s and "+1"s. Because these "-" and "+" operations can be commuted, we can say that the final number of frogs on point A is certain, regardless of any leaping sequence.

For the same original state P_0, consider two kinds of leap X and Y:

$$X: T_1 \to T_2 \to ... \to T_m$$
$$Y: J_1 \to J_2 \to ... \to J_n$$

T_i represents the leap at x_i, J_i represents the leap at y_i

The first leap in X is T_1 which means the two frogs at x_1 leap to the opposite direction.

Within Y, the frogs at x_1 must leap, or else the leaps will not terminate. So there must be one of $y_1, y_2, ..., y_n$ that is equal to x_1, suppose that the one whose subscript is minimal as y_{k_1}. Thus $y_{k_1} = x_1$, $J_{k_1} = T_1$.

We can now adjust Y by moving the leap J_{k_1} at $y_{k_1} = x_1$ to the beginning of the leaping sequence. In other words, the first part of Y that $J_1 \to J_2 \to ... \to J_{k_1-1} \to J_{k_1}$ turns into $J_{k_1} \to J_1 \to J_2 \to ... \to J_{k_1-1}$. Because J_{k_1} is the first leap at point x_1, the number of frogs at other points will not decrease, and the circumstances where frogs that can leap before cannot leap later due to the lack of frogs will not occur. According to the deduction above, we can conclude that the two different leaping sequences will result in the same final state. So if we name the new commuted leaping sequence $J_{k_1} \to J_1 \to J_2 \to ... \to J_{k_1-1} \to J_{k_1+1} \to ... \to J_n$ by Y', we can see that Y' and Y share the same final state.

Now Y' and X has the same first leap. We can commute T_2, the second leap of X, in the same manner. Since there must be a same leap with T_2 in the leaps of Y', we can find the one with the minimal subscript, J_{k_2}, and place it behind J_{k_1} to obtain the new leap sequence $Y'': J_{k_1} \to J_{k_2} \to J_1 \to J_2 \to ....$ We can repeat this commuting progress until the leaping sequence becomes the same as X. As a result, sequence Y is only the rearrangement of sequence X, so X and Y must have the same final state and leaping steps. Thus we have proved the **Corollary**. In conclusion, for an initial state, any legal sequence of leaps will terminate after the same number of steps and with the same final state.

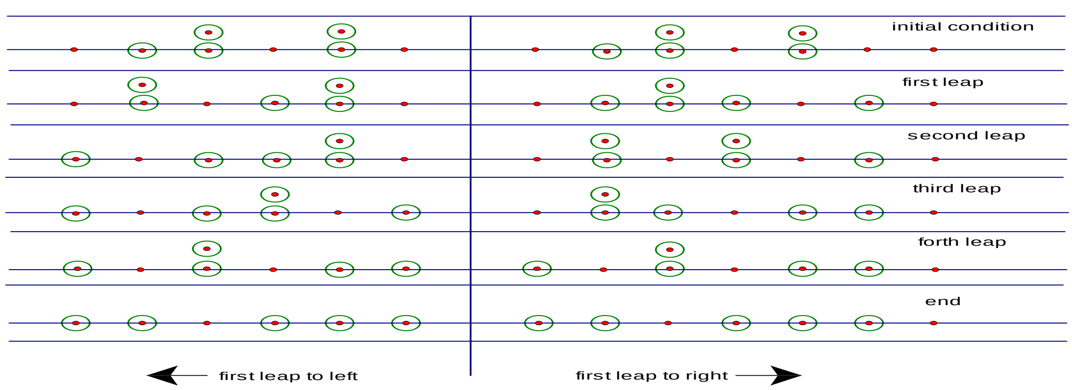

Figure 1: The Same Initial Condition Resulting into the Same End by Different Modes of Leaps

2. Two Extensions on Leaping to Opposite Directions on a Straight Line

We promote the following two extensions in this section:

1. The length of leaps to opposite sides are extended to "a units left, b units right."

Now that the two frogs on point x leap to $x+b$ and $x-a$ respectively, the same question arises: will the frogs leap infinitely?

The answer is also no. Suppose the coordinates for the n frogs are $x_1, x_2, ..., x_n (x_i \in Z)$ respectively. We consider another semi-variable: $S = x_1 + x_2 + ... + x_n$. As $(x-a)+(x+b)-2x=(b-a)$, supposing that $a<b$, the formula above will increase by $(b-a)$ for each leap executed. If the frogs leap infinitely, S increases unlimitedly. Consider $(a+b)$ consecutive points, if there are frogs on the points, there will always be frogs on them. This shows that for any leaping sequence, a frog cannot be more than $n(a+b)$ units away from its initial position, which means that x_i is bounded, and S is also bounded. Therefore, the frogs cannot leap infinitely.

2. The number of frogs in one leap extends from "two frogs" to "k frogs."

Assume that there are k frogs on coordinate x. After each leap the coordinates of the k frogs change to $x+a_1, x+a_2, ..., x+a_k$ (a_i may be positive or negative, regarding leaping left or right respectively). Suppose $a_1 \le a_2 \le ... \le a_k$. Since the frogs leap to opposite directions, $a_1 \le 0, a_k \ge 0$. Consider a set of consecutive points with the number of $a_k - a_1 = a_k + |a_1|$, we can prove, similar to extension 1 above, that if there are frogs on these points, there will always be frogs on them, so x_i is bounded. Consider a variable $S = x_1 + x_2 + ... + x_n$. The change of S after each leap is:

$$\Delta S = (x+a_1)+(x+a_2)+...+(x+a_k)-kx = a_1 + a_2 + ... + a_k$$

If $a_1 + a_2 + ... + a_k \ne 0$, S will continue to increase or decrease, but S is bounded. We have found a contradiction! Therefore the frogs cannot leap infinitely.

If $a_1 + a_2 + ... + a_k = 0$, consider a new variable $K = x_1^2 + x_2^2 + ... x_n^2$, the change of K with each leap is:

$\Delta K = (x+a_1)^2 + (x+a_2)^2 + ... + (x+a_k)^2 - kx^2 = a_1{}^2 + a_2{}^2 + ... + a_k{}^2 > 0$, which means K keeps increasing, but K is bounded. Contradiction! Thus the frogs cannot leap infinitely.

To sum up, as long as the frogs leap to opposite sides on a straight line, they will end up in the same final state after a finite amount of leaps.

3. Leaping in the Same Direction on a Straight Line

Now we have proved that the frogs cannot leap infinitely when they leap in two directions. From direct observation, we can see that the frogs become more dispersed after every leap, thus the leaping process must stop after a finite amount of steps. If the frogs leap in the same direction, well they manage to leap infinitely?

For example, let us assume that two frogs on the same point will leap one and two units to the right respectively. Suppose there are three frogs initially, two of which are located on $x=1$ while the third one is located on $x=2$. Note this state as $(1,1,2)$. After the two frogs on $x=1$ leap one and two units to the right respectively, the coordinates of the three frogs change into $(2,2,3)$, which is equivalent to shifting the three frogs to the right by one unit. Then the frogs on $x=2$ continue to leap to $x=3$ and $x=4$, rendering the coordinates of the three frogs into $(3,3,4)$, which is equivalent to shifting the three frogs by another one unit to the right. In this way, the frogs can leap on forever.

Now that the frogs can leap infinitely, our task is to find the least amount of frogs that can satisfy an infinite leap. Suppose the two frogs on a point leap p and q steps to the right, $p<q$ and they are relatively prime (if the length of leaps is amplified or reduced by the same factor, the circumstances are equivalent). We can also suppose the least amount of frogs is n.

Definition 1:

Positive State: a state where there exists at least one infinite leaping sequence.

Negative State: a state where there is no infinite leaping sequence.

Dead State: a state where no more leaps can be executed.

Theorem 2: Any leaping sequence following a Positive State is an infinite leaping sequence.

Proof: If a Positive State changes into a Dead State after a series of leaps, we can consider the first Negative State that appeared. Assume a Positive State P_1 changes into a Negative

State P_2 after T_0, a leap on point x_0. Then assume one of the infinite leaping sequences of the Positive State P_1 is $J_1 \to J_2 \to \dots \to J_n \to \dots$. Now let's consider whether a leap on x_0 exists in $J_1, J_2 \dots$.

If a leap on x_0 exists, we can assume the first one to be J_k. We can move J_k to the beginning of the sequence because leaping J_k first will not reduce the number of frogs on points other than x_0 and hamper the leaps on those points. Thus the leaping sequence of $J_k \to J_1 \to J_2 \to \dots \to J_{k-1} \to J_{k+1} \dots$ that started from the Positive State P_1 is an infinite leaping sequence, and the leaping sequence of $J_1 \to J_2 \to \dots \to J_{k-1} \to J_{k+1} \dots$ that started from the Negative State P_2 is also an infinite leaping sequence. This contradicts with P_2 being a Negative State.

If a leap on x_0 does not exist, there will always be at least two frogs on x_0, and the leaping process will never terminate. This also contradicts with P_2 being a Negative State. The Theorem has been proved.

Theorem 2 states that for an initial state, if there is an infinite leaping sequence, every leaping sequence will be infinite. Therefore, for an initial Positive State, we can arrange the leaping sequence on our own to shorten the length of the frogs' queue. We can accomplish this by leaping the frogs on the very left every time. Since n is the minimum number, $n-1$ frogs cannot leap infinitely, so no frogs can leap for a finite number of steps and then stop dead. This means that there must be an even number of frogs on the point in the very left, or else there will be one frog that remains unable to leap in the end.

In this way, the length of the frogs' queue continues to shorten until it reaches q. Since there is only a finite amount of arrays within a queue of the length q, we can find a state P and a state Q that share the same array of frogs, which means that one state is only the shift of another.

Now we can use a polynomial to represent the state of the n frogs.

Definition 2: Queue Polynomial

Suppose the coordinates of the n frogs are $a_1, a_2, \dots, a_n$ (some of which may be the

same), define $f(x)=x^{a_1}+x^{a_2}+...+x^{a_n}$ as the Queue Polynomial. In this way, the coefficient of x^{a_i} represents the number of frogs on point a_i, and $f(1)$, which is the sum of all coefficients, represents the total number of frogs n.

For two states P and Q that share the same array of frogs, suppose that P will be equivalent to Q after shifting a length of L. Thus the Queue Polynomial for the two states must satisfy: $g(x)=x^L\cdot f(x)$

Meanwhile, two frogs at point $x=a$ will leap to points $x=a+p$ and $x=a+q$. Thus the variation of the Queue Polynomial for every leap is $\Delta f=x^{a+p}+x^{a+q}-2x^a=x^a(x^p+x^q-2)$. Let $h(x)=x^p+x^q-2$, and for every leap, Δf is multiply of $h(x)$. Thus from state P to state Q, the variation of the Queue Polynomial is also a multiple of $h(x)$: $h(x)\left|g(x)-f(x)=(x^L-1)\cdot f(x)\right.$.

In order to find the greatest common factor of $h(x)$ and (x^L-1), consider their common roots. The roots of $x^L-1=0$ are unit roots on a unit circle. Then consider the roots of $h(x)=x^p+x^q-2$ on a unit circle.

$|x|=1$, $h(x)=x^p+x^q-2=0$, $x^p+x^q=2$.

Since $|x|=1$, $|x^p|=1,|x^q|=1$, $|x^p+x^q|\le|x^p|+|x^q|=1+1=2$.

This equation holds if and only if $x^p=x^q=1$. Since p and q are relatively prime, according to the Bezout Theorem, we know that $x=1$.

In this way, $(h(x),x^L-1)=(x-1)$, thus $\frac{h(x)}{x-1}\Big|f(x)$,

$\frac{h(x)}{x-1}=(1+x+...+x^{p-1})+(1+x+...+x^{q-1})\Big|f(x)$.

When $x=1$, $f(1)=1^{a_1}+1^{a_2}+...+1^{a_n}=n$.

Thus $(p+q)|n$, the minimum n should be $p+q$.

In fact, n can equal to $p+q$:

Suppose there are two frogs on points $1 \sim p$, and one frog on points $(p+1) \sim q$. During the leap, two frogs on point 1 leap to point $p+1$ and point $q+1$. Now there are two frogs on points $2 \sim p+1$ each, and one frog on points $p+2 \sim q+1$ each. We have shifted the queue of the frogs to the right by one unit, and by repeating this process, the frogs can leap infinitely.

The conclusion above is also applicable when k frogs leap together：

Suppose the k frogs leap $a_1 \sim a_k$ units to the right respectively. Suppose $a_1 \sim a_k$ is a group of positive integers that are relatively prime, with $a_1 < a_2 < ... < a_k$. Then the minimum value of n is $n = a_1 + a_2 + ... + a_k$. The array of the frogs is as follows: k frogs on point $1 \sim a_1$ each, $k-1$ frogs on point $(a_1+1) \sim a_2$ each… one frog on point $(a_{k-1}+1) \sim a_k$ each. Every leap is equivalent to shifting the frog array one unit rightwards, so the frogs can leap forever now. Figure 2 below shows the situation when $k = 3, a_1 = 1, a_2 = 2, a_3 = 3$.

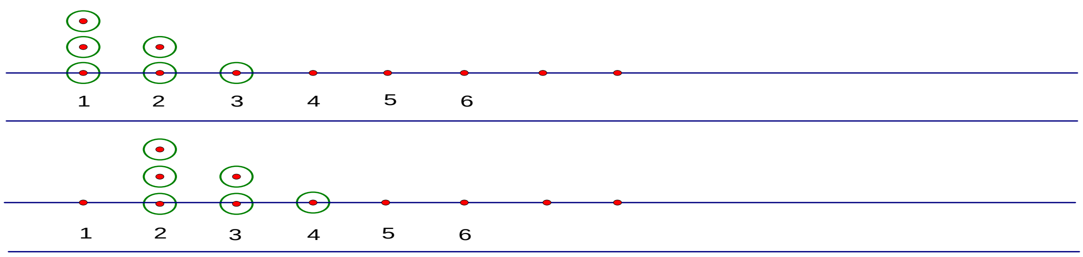

Figure 2： Infinite Leaps on a Straight Line

4. Leap on a plane and in space R^n

If two frogs on the same point leap in two different directions on a plane, we can project the two frogs on the exterior angle bisector of the two leaping directions. Consequently, the leaps of frogs are equivalent to the leaps of frog shadows. We notice that the leaps of the shadows are equivalent to the leaps to opposite sides on a straight line and therefore they cannot leap infinitely. Thus the frogs cannot leap infinitely either.

For an Euclidean space R^n, we define the leaps of k frogs as k vectors, $b_1 \sim b_k$. If the vectors $b_1 \sim b_k$ are not on the same line, we can find a line L, such as the exterior angle bisector of any two vectors that are not on the same line, so that the projections of vectors on L lie in two opposite directions. Thus the leaps of frogs are equivalent to the leaps of frog shadows, and since the shadows cannot leap infinitely, the frogs cannot leap infinitely either.

5. Leap on a Circle

Now that we have solved the leaping problems on lines, we turn our focus to leaps on circles.

Suppose the perimeter of the circle is n (the point on the circle is designated as 1,2,…,n), two frogs on the same point leap to the two adjacent points respectively. Since there is only a limited number of points, the frogs are sure to leap on forever if there is enough of them. So our question remains how many frogs needed at least to leap infinitely.

The answer is n. As leaping one step to the left is equivalent to leaping $n-1$ steps to the right on a circle, our problem is similar to leaping 1 unit and $n-1$ units in the same direction on a straight line. From the conclusions earlier in the paper, we can say that it is possible for $1+(n-1)=n$ frogs to leap infinitely. But is n the minimum number?

Since there is only a limited amount of points on a circle, according to the Drawer Principle, there must be at least one state that will appear twice after a number of leaps. Assume it took m steps for the state P_0 to appear the second time, a cycle of states will emerge: $P_0 \to P_1 \to P_2 \to ... \to P_m = P_0$.

For any state in the cycle (P_i), there should be at least one frog on any two adjacent points on the circle. Else, we can assume that no frogs are on point 1 or 2, which means that, for any state, there will never be frogs on point 1 or 2. Since neither point1 nor point 2 will ever have frogs, we can “cut” the circle into a straight line from between point 1 and 2. Thus the frogs cannot leap infinitely.

Then we can prove that there will be at least two frogs on three adjacent points. Suppose that only one frog is on point 1, 2, or 3. According to the paragraph above, that frog can only stay on point 2. Thus the number of frogs on point 1, 2 and 3 is $(0,1,0)$. Consider the leap that resulted in the state $(0,1,0)$ and define it as T. Right now there are no frogs are point 1 and 3, so leap T cannot happen at point 2 or 4. Neither can it happen at point further than 2 and 4. Thus T must have happened on point 1 or 3. Assume that T happened on point 1, then the number of frogs on point 1, 2 and 3 changed from $(2,0,0)$ to $(0,1,0)$. Under this condition, there are no frogs on the two consecutive points 2 and 3. This contradicts with the conclusion in the paragraph above. Thus there will be at least two frogs on three adjacent points.

Likewise, we can prove that there will be at least k-1 frogs on k adjacent points. For the initial state P, there are two frogs on one point to start the leap, and one frog on the rest n-1 points. These n frogs on the circle with n points can leap infinitely.

6. Extension of Leap on A Circle

We have figured out the circumstance of leap one step to the left and right in circle, then how about leaping “left a right b”? The same question is at least how many frogs are needed to leap infinitely. Note that the least number of frogs on the circle with perimeter n is $f(a,b,n)$, it’s difficult to deduce the accurate result of the least number, we could just make estimation.

If we magnify a, b, and n to the same multiple and the method of frog leap is the same, the number of frogs for infinite leap is the same as well ($f(ka,kb,kn) = f(a,b,n)$). Therefore, if a, b and n aren’t relatively prime, we can divide them with greatest common divisor and the result is the same.

Supposing $(a,b,n)=1$, we need to make out whether a and b are relatively prime, if not, we should turn it relatively prime and treat coordinates of n points as remainders of model n. In the light of model n, the coordinate of leaping a steps to the left is –a, and –b with leaping b steps to the right. Then multiply all the remainders of model n with k which is relatively prime with n, we can make one-to-one correspondence, so the result is that –a turns to –ka, +b turns to +kb, and “a steps to the left with b steps to the right” turns to “ka steps to the left with kb steps to the right”. In a circle with perimeter of n, the two methods of leap is corresponding in which the least number figured out is the same. When (k, n) =1, $f(ka,kb,n) = f(a,b,n)$. If the greatest common factor d of a and b is d, on assumption of $(d,n)=1$,

$f(a,b,n) = f(d\times\frac{a}{d}, d\times\frac{b}{d}, n) = f(\frac{a}{d},\frac{b}{d},n)$, then the greatest common factor d will be removed

If a and b are relatively prime, and $a \le b$, to figure out $f(a,b,n)$, we should make estimation of how many at least are needed. If there are several frogs can leap infinitely and they have limited states, there must be a state of circulation ($P_0 \to P_1 \to P_2 \to ... \to P_m = P_0$). It’s similar with leaping in a circle to leap “left a right b” on a straight line. If there is at least one frog at the consecutive a+b points, then at least one should be stayed. Due to the state of circulation, if no frogs on a+b point, then there won’t be forever. The circle is cut here, and there is no way to leap definitely. Therefore, at least one frog is at any point on the consecutive a+b points.

Then we will make use of mathematical induction to prove there are at least k frogs on the consecutive $a+kb$ points, or else, if k-1, according to the assumption, these frogs must be grouped on the segment $a+(k-2)b$, which is in the middle of $b+(a+(k-2)b)+b$. Because there should be frogs in section b which is in the terminal (the circle will be cut and it’s impossible for indefinite leap), at least k frogs should be in $a+kb$. Considering the last leap, there are k frogs before leap and k-1 after it, but where is the exact leaping point? It couldn’t occur out of $a+kb$, otherwise it will result in more frogs. Meanwhile, it couldn’t occur in segment $a+(k-2)b$, which will make two less frogs in the middle but one more both in the terminal and the inner switch won't change the total number. If we suppose it occur in the line b in the left terminal, two frogs must be in b before the leap and k-2 in the middle, with zero in the right b section. Therefore it's contradictory with assumption that k-2 frogs are in the middle $(a+(k-2)b)+b=a+(k-1)b$.

We do a division with the remainder $n \div b = q......r$, so $n=bq+r$. Take a point with at least two frogs, remaining $n-1$ points. Because of $n-1=bq+r-1 \geq bq-1 \geq b(q-1)+a$ and there being at least $q-1$ frogs on $n-1$ points, there are at least $q+1$ frogs together. So $f(a,b,n) \geq q+1 = \left[\frac{n}{b}\right]+1$.

We know if two frogs leap to the same side A and B, $A+B$ frogs can infinitely leap. In The line is like this, but the circle may be less, but $A+B$ is enough. We take consideration as this thinking. Because k and n are relatively prime, $f(ka,kb,n)=f(a,b,n)$, if left a and right b multiplied by k, it may make both left leap ka and right leap kb mean right moving a short distance(may be leap around the circle some times). Left leap ka means right leap $-ka \equiv A(\operatorname{mod} n)$; right leap kb means $db \equiv B(\operatorname{mod} n)$, $0<A,B<n$. In this condition, we know $A+B$ frogs are enough, $f(ka,kb,n) \leq A+B$. Now we should find a suitable k which make $A+B$ not large.

Simply estimate, $A \equiv -ka(\operatorname{mod} n)$, $B \equiv kb(\operatorname{mod} n)$, so

$$bA+aB \equiv b(-ka)+a(kb)=0(\operatorname{mod} n)$$

$bA+aB$ is a multiple of n, the smallest is n, $bA+aB \geq n$, so $b(A+B)>bA+aB>n$. $A+B>\frac{n}{b}$, so the $A+B$ at least is $\frac{n}{b}$. But this is not our purpose. We want a suitable k make $A+B$ almost be $\frac{n}{b}$.

At first, let's identify some inequalities, $A+B$ is almost equal to $\frac{n}{b}$, so $b(A+B)$ is almost equal to n. And $bA+aB$ >=n, which is multiple of n. Consequently, $b(A+B)-(bA+aB)=(b-a)B$ approximates to 0. That equation means B is so little that it is merely little greater than zero. Then $bA+aB=n$, so A is little less than $\left[\frac{n}{b}\right]$.

Because on the assumption that $kb = cn + B$, $k = \frac{cn+B}{b}$, an equation can be drawn:

$$A \equiv -ak = \frac{-a(cn+B)}{b} = \frac{-ac}{b}n - \frac{aB}{b} \pmod{n}$$

A is little less than $\left[\frac{n}{b}\right]$, so the decimal part of $\frac{-ac}{b}$ is $\frac{1}{b}$.In this case, c, which is limited by equation that $ac+1 \equiv 0 \pmod{b}$,exists and consist with an inequality $0 < c < b$ due to $(a,b)=1$.

However k is integer little greater than $\frac{c}{b}n$ and meets the equation $(k,n)=1$, so we find out the suitable k from $\left[\frac{c}{b}n\right]+1$. Assuming $k = \left[\frac{c}{b}n\right]+L$,which is the minimum relatively prime with n, then $B = b\cdot(\left[\frac{c}{b}n\right]+L)$ the balance of n is less than $b \cdot L$.

Additionally, $b(A+B)-n = b(A+B)-(bA+aB) = (b-a)\cdot B \le (b-a)\cdot bL$,

So $A+B \le \left[\frac{n}{b}\right]+(b-a)\cdot L$. If L is small, it is possible for $A+B$ to be equal to $\left[\frac{n}{b}\right]$. But what is L equal to? The answer is L must be infinitesimal.

Definition 3: $c(n)$ The maximum of consecutive numbers within $1 \sim n$ that are not relatively prime with n.

Since the numbers in $\left[\frac{n}{b}\right]+1 \sim \left[\frac{n}{b}\right]+(L-1)$ are relatively prime with n, we can conclude that $L-1 \le c(n)$, namely $L \le c(n)+1$.

$$\left[\frac{n}{b}\right]+1 \le f(a,b,n) \le A+B \le \left[\frac{n}{b}\right]+(b-a)\cdot L \le \left[\frac{n}{b}\right]+(b-a)\cdot(c(n)+1)$$

As n increases, $c(n)$ becomes infinitesimal compared to n, namely $\lim_{n\to\infty}\frac{c(n)}{n}=0$. In fact, we can prove that $c(n)$ is smaller than any mathematical power of n.

Theorem 3: $\forall \varepsilon > 0, \exists C,$ **so that** $c(n) \le C_\varepsilon \cdot n^\varepsilon$

Proof: Suppose $n = p_1^{a_1} p_2^{a_2} ......p_k^{a_k}$

The amount of numbers in $c(n)$ that are the multiple of p_1: $[\frac{c(n)}{p_1}]$ or $[\frac{c(n)}{p_1}]+1...$

The amount of numbers in $c(n)$ that are the multiple of p_1 and p_2: $[\frac{c(n)}{p_1 \cdot p_2}]$ or $[\frac{c(n)}{p_1 \cdot p_2}]+1...$

......

According to the Inclusion-Exclusion Principal:

$$c(n) \approx \frac{c(n)}{p_1}+\frac{c(n)}{p_2}+...+\frac{c(n)}{p_k}-\frac{c(n)}{p_1\cdot p_2}-...+\frac{c(n)\cdot(-1)^{m-1}}{p_{i_1}\cdot p_{i_2}\cdot ...\cdot p_{i_m}}+...$$

The error bond:

$$c(n)-(\frac{c(n)}{p_1}+\frac{c(n)}{p_2}+...+\frac{c(n)}{p_k}-\frac{c(n)}{p_1\cdot p_2}-...+\frac{c(n)\cdot(-1)^{m-1}}{p_{i_1}\cdot p_{i_2}\cdot ...\cdot p_{i_m}}+...) \le 2^k$$

$$c(n)\times(1-\frac{1}{p_1})\times(1-\frac{1}{p_2})\times...\times(1-\frac{1}{p_k}) \le 2^k$$

$$c(n) \le 2^k \times \frac{p_1}{p_1 - 1} \times \frac{p_2}{p_2 - 1} \times \ldots \times \frac{p_k}{p_k - 1} \qquad (1)$$

In order to prove $c(n) \le C_\varepsilon \cdot n^\varepsilon$, we only have to prove $\frac{c(n)}{n^\varepsilon} \le C_\varepsilon$

From (1) we have: $$\frac{c(n)}{n^\varepsilon} \le \frac{2p_1^{1-\varepsilon}}{p_1 - 1} \times \frac{2p_2^{1-\varepsilon}}{p_2 - 1} \times \ldots \times \frac{2p_k^{1-\varepsilon}}{p_k - 1} \qquad (2)$$

When $p \to \infty$, $\frac{2p^{1-\varepsilon}}{p-1} \to 0$.

This means that there are only a finite amount of prime numbers p that satisfy $\frac{2p^{1-\varepsilon}}{p-1} > 1$. We multiply these $\frac{2p^{1-\varepsilon}}{p-1}$ to get C_ε. Thus $\frac{2p_1^{1-\varepsilon}}{p_1 - 1} \times \frac{2p_2^{1-\varepsilon}}{p_2 - 1} \times \ldots \times \frac{2p_k^{1-\varepsilon}}{p_k - 1} \le C_\varepsilon$.

Thus $\frac{c(n)}{n^\varepsilon} \le C_\varepsilon$, which means that $c(n) \le C_\varepsilon \cdot n^\varepsilon$.

6. Conclusion and expectation

The essence of leaping to opposite side is that the frogs become more dispersed after every leap, thus they cannot leap infinitely. Leaping to the same side does not necessarily make the frogs dispersed and creates a cycle for the leaping states, which enables the frogs to leap infinitely. We found that $n = p + q$ frogs are enough to satisfy an infinite leap. We also proved that on a circle with n points, n is the minimum number of frogs that satisfy an infinite leap. For leaping "a units left, b units right" on a circle, we estimated that the order for the minimum number of frogs satisfying an infinite leap is $\frac{n}{b}$.

"Frog Leap" is a very complicated problem and there are still many problems yet to be solved. For instance, for an initial state where n frogs are on the same point and every two frogs leap one unit eastward and northward respectively, how many steps can the frogs leap before they stop? Also, we have proved that $c(n)$ is smaller than any mathematical power of n, but what is the exact value of $c(n)$? What is $c(n)$'s relationship with $\ln n$? These are just some of the problems that we can explore in the future.

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Bibliorgraphy

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Appendix (the proof of the IMO shortlist)

Let the squares be indexed serially by the integers: . . . , −1, 0, 1, 2, ...When a bean is moved from i to i + 1 or from i + 1 to i for the first time, we may assign the index i to it. Thereafter, whenever some bean is moved in the opposite direction, we shall assume that it is exactly the one marked by i, and so on. Thus, each pair of neighboring squares has a bean stuck between it, and since the number of beans is finite, there are only finitely pairs of neighboring squares, and thus finitely many squares on which moves are made. Thus we may assume w.l.o.g. that all moves occur between 0 and $l \in N$ and that all beans exist at all times within [0, l].

Defining b_i to be the number of beans in the ith cell

($i \in Z$) and b_i the total number of beans, we define the semi-invariant $S = \sum_{i \in Z} i^2 b_i S < b \cdot l^2 e_i$ Since all moves occur above 0, the semi-invariant S increases by 2 with each move, and since we always have $S < b \cdot l^2 \{b_i \geq 0\}$, it follows that the number of moves must be finite.

We now prove the uniqueness of the final configuration and the number of moves for some initial configuration $\{b_i\} x_i \geq 0$. Let $x_i \geq 0$ be the number of moves made in the ith cell ($i \in Z$) during the game. Since the game is finite, only finitely many of x_i's are nonzero. Also, the number of beans in cell i, denoted as e_i, at the end is

$$(\forall i \in Z) e_i = b_i + x_{i-1} + x_{i+1} - 2x_i \in \{0,1\} \qquad (1)$$

Thus it is enough to show that given $b_i \geq 0$, the sequence $\{x_i\}_{i \in Z}$ of nonnegative integers satisfying (1) is unique.

Suppose the assertion is false, I.e., that there exists at least one sequence $b_i \geq 0$ for which there exist distinct sequences $\{x_i\}$ and $\{x_i^{'}\}$ satisfying (1). We may choose such a $\{b_i\}$ for which min $\left\{\sum_{i \in Z} x_i, \sum_{i \in Z} x_i^{'}\right\}$ is minimal (since $\sum_{i \in Z} x_i$ is always finite). We choose any index j such that $b_j > 1$. Such an index j exists, since otherwise the game is over. Then one must make at least one move in the j th cell, which implies that $x_j, x_j^{'} \geq 1$. However, then the sequences $\{x_i\}$ and $\{x_i^{'}\}$ with x_j and $x_j^{'}$ decreased by 1 also satisfy (1) for a sequence $\{b_i\}$ where $b_{j-1}+1$, $b_j - 2$, b_{j+1} is replaced with $b_{j-1}+1$, $b_j - 2$, $b_{j+1}+1$ .This contradicts the assumption of minimal min$\left\{\sum_{i \in Z} x_i, \sum_{i \in Z} x_i^{'}\right\}$ for the initial $\{b_i\}$.