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A GENERALIZATION OF HAMILTON'S DIFFERENTIAL HARNACK INEQUALITY FOR THE RICCI FLOW

Simon Brendle

Abstract

In [**10**], R. Hamilton established a differential Harnack inequality for solutions to the Ricci flow with nonnegative curvature operator. We show that this inequality holds under the weaker condition that $M \times \mathbb{R}^2$ has nonnegative isotropic curvature.

1. Introduction

In [**10**], R. Hamilton established a differential Harnack inequality for solutions to the Ricci flow with nonnegative curvature operator (see [**9**] for an earlier result in the two-dimensional case). This inequality has since become one of the fundamental tools in the study of Ricci flow. We point out that H.D. Cao [**4**] has proved a differential Harnack inequality for solutions to the Kähler-Ricci flow with nonnegative holomorphic bisectional curvature.

In this paper, we prove a generalization of Hamilton's Harnack inequality replacing the assumption of nonnegative curvature operator by a weaker curvature condition. Throughout this paper, we assume that $(M, g(t))$, $t \in (0, T)$ is a family of complete Riemannian manifolds evolving under Ricci flow. Following R. Hamilton [**10**], we define

$$P_{ijk} = D_i \text{Ric}_{jk} - D_j \text{Ric}_{ik}$$

and

$$M_{ij} = \Delta \text{Ric}_{ij} - \frac{1}{2}\, D_i D_j \text{scal} + 2\, R_{ikjl}\, \text{Ric}^{kl} - \text{Ric}_i^k\, \text{Ric}_{jk} + \frac{1}{2t}\, \text{Ric}_{ij}.$$

Here, Ric and scal denote the Ricci and scalar curvature of $(M, g(t))$, respectively.

Theorem 1. *Suppose that $(M, g(t)) \times \mathbb{R}^2$ has nonnegative isotropic curvature for all $t \in (0, T)$. Moreover, we assume that*

$$\sup_{(x,t) \in M \times (\alpha, T)} \text{scal}(x, t) < \infty$$

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for all $\alpha \in (0,T)$. *Then,*

$$M(w,w) + 2\,P(v,w,w) + R(v,w,v,w) \geq 0$$

for all points $(x,t) \in M \times (0,T)$ *and all vectors* $v, w \in T_xM$.

As a consequence, we obtain a generalization of Hamilton's trace Harnack inequality (cf. [**10**]):

Corollary 2. *Suppose that* $(M, g(t)) \times \mathbb{R}^2$ *has nonnegative isotropic curvature for all* $t \in (0,T)$. *Moreover, we assume that*

$$\sup_{(x,t) \in M \times (\alpha,T)} \text{scal}(x,t) < \infty$$

for all $\alpha \in (0,T)$. *Then we have*

$$\frac{\partial}{\partial t}\text{scal} + \frac{1}{t}\,\text{scal} + 2\,\partial_i \text{scal}\, v^i + 2\,\text{Ric}(v,v) \geq 0$$

for all points $(x,t) \in M \times (0,T)$ *and all vectors* $v \in T_xM$.

The condition that $M \times \mathbb{R}^2$ has nonnegative isotropic curvature is preserved by the Ricci flow, and plays a key role in the proof of the Differentiable Sphere Theorem [**2**]. We point out that the following statements are equivalent:

(i) The product $M \times \mathbb{R}^2$ has nonnegative isotropic curvature.

(ii) For all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset T_xM$ and all $\lambda, \mu \in [-1,1]$, we have

$$\begin{aligned} &R(e_1,e_3,e_1,e_3) + \lambda^2\,R(e_1,e_4,e_1,e_4) \\ &+ \mu^2\,R(e_2,e_3,e_2,e_3) + \lambda^2\mu^2\,R(e_2,e_4,e_2,e_4) \\ &- 2\lambda\mu\,R(e_1,e_2,e_3,e_4) \geq 0. \end{aligned}$$

(iii) For all vectors $v_1, v_2, v_3, v_4 \in T_xM$, we have

$$\begin{aligned} &R(v_1,v_3,v_1,v_3) + R(v_1,v_4,v_1,v_4) \\ &+ R(v_2,v_3,v_2,v_3) + R(v_2,v_4,v_2,v_4) \\ &- 2\,R(v_1,v_2,v_3,v_4) \geq 0. \end{aligned}$$

The implication (i) $\Longrightarrow$ (ii) was established in [**2**]. Moreover, a careful examination of the proof of Proposition 21 in [**2**] shows that (ii) implies (iii). Finally, the implication (iii) $\Longrightarrow$ (i) is trivial.

2. The space-time curvature tensor and its evolution under Ricci flow

We first review the evolution equations for the various quantities that appear in the Harnack inequaltiy. The evolution equation of the

curvature tensor is given by

$$\begin{aligned}&\frac{\partial}{\partial t}R_{ijkl} - \Delta R_{ijkl}\\ &\quad + \mathrm{Ric}_i^m\, R_{mjkl} + \mathrm{Ric}_j^m\, R_{imkl} + \mathrm{Ric}_k^m\, R_{ijml} + \mathrm{Ric}_l^m\, R_{ijkm}\\ &= g^{pq}\, g^{rs}\, R_{ijpr}\, R_{klqs} + 2\, g^{pq}\, g^{rs}\, R_{ipkr}\, R_{iqls} - 2\, g^{pq}\, g^{rs}\, R_{iplr}\, R_{jpks}\end{aligned}$$

(cf. [**7**],[**8**]). Moreover, Hamilton proved that

$$\begin{aligned}&\frac{\partial}{\partial t}P_{ijk} - \Delta P_{ijk} + \mathrm{Ric}_i^m\, P_{mjk} + \mathrm{Ric}_j^m\, P_{imk} + \mathrm{Ric}_k^m\, P_{ijm}\\ &= 2\, g^{pq}\, g^{rs}\, R_{ipjr}\, P_{qsk} + 2\, g^{pq}\, g^{rs}\, R_{ipkr}\, P_{qjs} + 2\, g^{pq}\, g^{rs}\, R_{jpkr}\, P_{iqs}\\ &\quad - 2\,\mathrm{Ric}_p^m\, D^p R_{ijkm}\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial}{\partial t}M_{ij} - \Delta M_{ij} + \mathrm{Ric}_i^m\, M_{mj} + \mathrm{Ric}_j^m\, M_{im}\\ &= 2\, g^{pq}\, g^{rs}\, R_{ipjr}\, M_{qs} + 2\,\mathrm{Ric}_p^m\, (D^p P_{mij} + D^p P_{mji})\\ &\quad + 2g^{pq}g^{rs}P_{ipr}P_{jqs} - 4g^{pq}g^{rs}P_{ipr}P_{jsq} + 2\,\mathrm{Ric}^{lp}\,\mathrm{Ric}_p^m R_{iljm} - \frac{1}{2t^2}\mathrm{Ric}_{ij}\end{aligned}$$

(see [**10**], Lemma 4.3 and Lemma 4.4).

Chow and Chu [**5**] observed that the quantities M_{ij} and P_{ijk} can be viewed as components of a space-time curvature tensor (see also [**6**]). In the remainder of this section, we describe the definition of the space-time curvature tensor and its evolution under Ricci flow. Following [**6**], we define a connection $\tilde{D}$ on the product $M \times (0,T)$ by

$$\begin{aligned}\tilde{D}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} &= \Gamma_{ij}^k\, \frac{\partial}{\partial x^k}\\ \tilde{D}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial t} &= -\Big(\mathrm{Ric}_i^j + \frac{1}{2t}\,\delta_i^j\Big)\, \frac{\partial}{\partial x^j}\\ \tilde{D}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial x^i} &= -\Big(\mathrm{Ric}_i^j + \frac{1}{2t}\,\delta_i^j\Big)\, \frac{\partial}{\partial x^j}\\ \tilde{D}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} &= -\frac{1}{2}\,\partial^i \mathrm{scal}\, \frac{\partial}{\partial x^i} - \frac{3}{2t}\,\frac{\partial}{\partial t}.\end{aligned}$$

Here, Γ_{ij}^k denote the Christoffel symbols associated with the metric $g(t)$. We next define a $(0,4)$-tensor S by

$$\begin{aligned}S &= R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l\\ &\quad + P_{ijk}\, dx^i \otimes dx^j \otimes dt \otimes dx^k - P_{ijk}\, dx^i \otimes dx^j \otimes dx^k \otimes dt\\ &\quad + P_{ijk}\, dt \otimes dx^k \otimes dx^i \otimes dx^j - P_{ijk}\, dx^k \otimes dt \otimes dx^i \otimes dx^j\\ &\quad + M_{ij}\, dx^i \otimes dt \otimes dx^j \otimes dt - M_{ij}\, dx^i \otimes dt \otimes dt \otimes dx^j\\ &\quad - M_{ij}\, dt \otimes dx^i \otimes dx^j \otimes dt + M_{ij}\, dt \otimes dx^i \otimes dt \otimes dx^j.\end{aligned}$$

The tensor S is an algebraic curvature tensor in the sense that

$$S(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) = -S(\tilde{v}_2, \tilde{v}_1, \tilde{v}_3, \tilde{v}_4) = S(\tilde{v}_3, \tilde{v}_4, \tilde{v}_1, \tilde{v}_2)$$

and

$$S(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) + S(\tilde{v}_2, \tilde{v}_3, \tilde{v}_1, \tilde{v}_4) + S(\tilde{v}_3, \tilde{v}_1, \tilde{v}_2, \tilde{v}_4) = 0$$

for all vectors $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4 \in T_{(x,t)}(M \times (0,T))$.

Given any algebraic curvature tensor S, we define

$$\begin{aligned}
&\tilde{Q}(S)(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) \\
&= \sum_{p,q,r,s=1}^n g^{pq}\, g^{rs}\, S\Big(\tilde{v}_1, \tilde{v}_2, \frac{\partial}{\partial x^p}, \frac{\partial}{\partial x^r}\Big)\, S\Big(\tilde{v}_3, \tilde{v}_4, \frac{\partial}{\partial x^q}, \frac{\partial}{\partial x^s}\Big) \\
&\quad + 2 \sum_{p,q,r,s=1}^n g^{pq}\, g^{rs}\, S\Big(\tilde{v}_1, \frac{\partial}{\partial x^p}, \tilde{v}_3, \frac{\partial}{\partial x^r}\Big)\, S\Big(\tilde{v}_2, \frac{\partial}{\partial x^q}, \tilde{v}_4, \frac{\partial}{\partial x^s}\Big) \\
&\quad - 2 \sum_{p,q,r,s=1}^n g^{pq}\, g^{rs}\, S\Big(\tilde{v}_1, \frac{\partial}{\partial x^p}, \tilde{v}_4, \frac{\partial}{\partial x^r}\Big)\, S\Big(\tilde{v}_2, \frac{\partial}{\partial x^q}, \tilde{v}_3, \frac{\partial}{\partial x^s}\Big)
\end{aligned}$$

for all vectors $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4 \in T_{(x,t)}(M \times (0,T))$. It is straightforward to verify that

$$\tilde{Q}(S)(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) = -\tilde{Q}(S)(\tilde{v}_2, \tilde{v}_1, \tilde{v}_3, \tilde{v}_4) = \tilde{Q}(S)(\tilde{v}_3, \tilde{v}_4, \tilde{v}_1, \tilde{v}_2)$$

and

$$\tilde{Q}(S)(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) + \tilde{Q}(S)(\tilde{v}_2, \tilde{v}_3, \tilde{v}_1, \tilde{v}_4) + \tilde{Q}(S)(\tilde{v}_3, \tilde{v}_1, \tilde{v}_2, \tilde{v}_4) = 0$$

for all vectors $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4 \in T_{(x,t)}(M \times (0,T))$. Therefore, $\tilde{Q}(S)$ is again an algebraic curvature tensor.

Proposition 3. *The tensor S satisfies the evolution equation*

$$\tilde{D}_{\frac{\partial}{\partial t}} S = \tilde{\Delta} S + \frac{2}{t}\, S + \tilde{Q}(S).$$

Here,

$$\tilde{\Delta} S = \sum_{p,q=1}^n g^{pq}\, \tilde{D}^2_{\frac{\partial}{\partial x^p}, \frac{\partial}{\partial x^q}} S$$

denotes the Laplacian of S with respect to the connection $\tilde{D}$.

Proof. For abbreviation, let $W = \tilde{D}_{\frac{\partial}{\partial t}} S - \tilde{\Delta} S - \frac{2}{t}\, S$. Clearly, W is an algebraic curvature tensor. We claim that $W = \tilde{Q}(S)$. Note that

$$\begin{aligned}
&(\tilde{D}_{\frac{\partial}{\partial t}} S)\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big) \\
&= \frac{\partial}{\partial t} R_{ijkl} + \frac{2}{t}\, R_{ijkl} \\
&\quad + \text{Ric}_i^m\, R_{mjkl} + \text{Ric}_j^m\, R_{imkl} + \text{Ric}_k^m\, R_{ijml} + \text{Ric}_l^m\, R_{ijkm}
\end{aligned}$$

and

$$(\tilde{\Delta}S)\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big) = \Delta R_{ijkl}.$$

This implies

$$\begin{aligned}
&W\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big) \\
&= \frac{\partial}{\partial t} R_{ijkl} - \Delta R_{ijkl} \\
&\quad + \mathrm{Ric}_i^m \, R_{mjkl} + \mathrm{Ric}_j^m \, R_{imkl} + \mathrm{Ric}_k^m \, R_{ijml} + \mathrm{Ric}_l^m \, R_{ijkm} \\
&= g^{pq} \, g^{rs} \, R_{ijpr} \, R_{klqs} + 2 \, g^{pq} \, g^{rs} \, R_{ipkr} \, R_{iqls} - 2 \, g^{pq} \, g^{rs} \, R_{iplr} \, R_{jpks} \\
&= \tilde{Q}(S)\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&(\tilde{D}_{\frac{\partial}{\partial t}} S)\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial t}, \frac{\partial}{\partial x^k}\Big) \\
&= \frac{\partial}{\partial t} P_{ijk} + \frac{1}{2} \, \partial^m \mathrm{scal} \, R_{ijmk} + \frac{3}{t} \, P_{ijk} \\
&\quad + \mathrm{Ric}_i^m \, P_{mjk} + \mathrm{Ric}_j^m \, P_{imk} + \mathrm{Ric}_k^m \, P_{ijm}
\end{aligned}$$

and

$$\begin{aligned}
&(\tilde{\Delta}S)\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial t}, \frac{\partial}{\partial x^k}\Big) \\
&= \Delta P_{ijk} + D^p \mathrm{Ric}_p^m \, R_{ijmk} + 2 \, \mathrm{Ric}_p^m \, D^p R_{ijmk} + \frac{1}{t} \, D^m R_{ijmk} \\
&= \Delta P_{ijk} + \frac{1}{2} \, \partial^m \mathrm{scal} \, R_{ijmk} + 2 \, \mathrm{Ric}_p^m \, D^p R_{ijmk} + \frac{1}{t} \, P_{ijk}.
\end{aligned}$$

Using the evolution equation for the tensor P_{ijk}, we obtain

$$\begin{aligned}
&W\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial t}, \frac{\partial}{\partial x^k}\Big) \\
&= \frac{\partial}{\partial t} P_{ijk} - \Delta P_{ijk} - 2 \, \mathrm{Ric}_p^m \, D^p R_{ijmk} \\
&\quad + \mathrm{Ric}_i^m \, P_{mjk} + \mathrm{Ric}_j^m \, P_{imk} + \mathrm{Ric}_k^m \, P_{ijm} \\
&= 2 \, g^{pq} \, g^{rs} \, R_{ipjr} \, P_{qsk} + 2 \, g^{pq} \, g^{rs} \, R_{ipkr} \, P_{qjs} + 2 \, g^{pq} \, g^{rs} \, R_{jpkr} \, P_{iqs} \\
&= g^{pq} \, g^{rs} \, R_{ijpr} \, P_{qsk} + 2 \, g^{pq} \, g^{rs} \, R_{ipkr} \, P_{qjs} + 2 \, g^{pq} \, g^{rs} \, R_{jpkr} \, P_{iqs} \\
&= \tilde{Q}(S)\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial t}, \frac{\partial}{\partial x^k}\Big).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&(\tilde{D}_{\frac{\partial}{\partial t}} S)\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial t}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial t}\Big) \\
&= \frac{\partial}{\partial t} M_{ij} - \frac{1}{2} \, \partial^m \mathrm{scal} \, (P_{imj} + P_{jmi}) + \frac{4}{t} \, M_{ij} + \mathrm{Ric}_i^m \, M_{mj} + \mathrm{Ric}_j^m \, M_{im}
\end{aligned}$$

and

$$\begin{aligned}
&(\tilde{\Delta} S)\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial t}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial t}\Big) \\
&= \Delta M_{ij} - 2\left(\text{Ric}_p^m + \frac{1}{2t}\,\delta_p^m\right)(D^p P_{imj} + D^p P_{jmi}) \\
&\quad - D^p \text{Ric}_p^m\,(P_{imj} + P_{jmi}) + 2\left(\text{Ric}^{lp} + \frac{1}{2t}\,g^{lp}\right)\left(\text{Ric}_p^m + \frac{1}{2t}\,\delta_p^m\right) R_{iljm} \\
&= \Delta M_{ij} - 2\,\text{Ric}_p^m\,(D^p P_{imj} + D^p P_{jmi}) \\
&\quad - D^p \text{Ric}_p^m\,(P_{imj} + P_{jmi}) + 2\,\text{Ric}^{lp}\,\text{Ric}_p^m\,R_{iljm} \\
&\quad - \frac{1}{t}\,(D^m P_{imj} + D^m P_{jmi}) + \frac{2}{t}\,\text{Ric}^{lm}\,R_{iljm} + \frac{1}{2t^2}\,\text{Ric}_{ij} \\
&= \Delta M_{ij} - 2\,\text{Ric}_p^m\,(D^p P_{imj} + D^p P_{jmi}) \\
&\quad - \frac{1}{2}\,\partial^m \text{scal}\,(P_{imj} + P_{jmi}) + 2\,\text{Ric}^{lp}\,\text{Ric}_p^m\,R_{iljm} \\
&\quad + \frac{2}{t}\,M_{ij} - \frac{1}{2t^2}\,\text{Ric}_{ij}.
\end{aligned}$$

In the last step, we have used the formula

$$M_{ij} = -D^m P_{imj} + \text{Ric}^{lm}\,R_{iljm} + \frac{1}{2t}\,\text{Ric}_{ij}$$

(see [**10**], p. 235). Using the evolution equation for M_{ij}, we obtain

$$\begin{aligned}
&W\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial t}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial t}\Big) \\
&= \frac{\partial}{\partial t} M_{ij} - \Delta M_{ij} + 2\,\text{Ric}_p^m\,(D^p P_{imj} + D^p P_{jmi}) \\
&\quad + \text{Ric}_i^m\,M_{mj} + \text{Ric}_j^m\,M_{im} - 2\,\text{Ric}^{lp}\,\text{Ric}_p^m\,R_{iljm} + \frac{1}{2t^2}\,\text{Ric}_{ij} \\
&= 2\,g^{pq}\,g^{rs}\,R_{ipjr}\,M_{qs} + 2\,g^{pq}\,g^{rs}\,P_{ipr}\,P_{jqs} - 4\,g^{pq}\,g^{rs}\,P_{ipr}\,P_{jsq} \\
&= 2\,g^{pq}\,g^{rs}\,R_{ipjr}\,M_{qs} + 2\,g^{pq}\,g^{rs}\,P_{ipr}\,(P_{jqs} - P_{jsq}) - 2\,g^{pq}\,g^{rs}\,P_{ipr}\,P_{jsq} \\
&= 2\,g^{pq}\,g^{rs}\,R_{ipjr}\,M_{qs} - 2\,g^{pq}\,g^{rs}\,P_{ipr}\,P_{qsj} - 2\,g^{pq}\,g^{rs}\,P_{ipr}\,P_{jsq} \\
&= 2\,g^{pq}\,g^{rs}\,R_{ipjr}\,M_{qs} + g^{pq}\,g^{rs}\,P_{pri}\,P_{qsj} - 2\,g^{pq}\,g^{rs}\,P_{ipr}\,P_{jsq} \\
&= \tilde{Q}(S)\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial t}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial t}\Big).
\end{aligned}$$

Putting these facts together, we conclude that $W = \tilde{Q}(S)$. This completes the proof.

3. An invariant cone for the ODE $\frac{d}{dt}S = \tilde{Q}(S)$

We now consider the space of algebraic curvature tensors on $\mathbb{R}^n \times \mathbb{R}$. There is a natural mapping $\tilde{Q}$ which maps the space of algebraic

curvature tensors on $\mathbb{R}^n \times \mathbb{R}$ into itself. For each algebraic curvature tensor S on $\mathbb{R}^n \times \mathbb{R}$, the tensor $\tilde{Q}(S)$ is defined by

$$\begin{aligned}\tilde{Q}(S)(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) &= \sum_{p,q=1}^n S(\tilde{v}_1, \tilde{v}_2, e_p, e_q)\, S(\tilde{v}_3, \tilde{v}_4, e_p, e_q) \\ &\quad + 2 \sum_{p,q=1}^n S(\tilde{v}_1, e_p, \tilde{v}_3, e_q)\, S(\tilde{v}_2, e_p, \tilde{v}_4, e_q) \\ &\quad - 2 \sum_{p,q=1}^n S(\tilde{v}_1, e_p, \tilde{v}_4, e_q)\, S(\tilde{v}_2, e_p, \tilde{v}_3, e_q),\end{aligned}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$ (viewed as a subspace of $\mathbb{R}^n \times \mathbb{R}$).

Let K be the set of all algebraic curvature tensors on $\mathbb{R}^n \times \mathbb{R}$ such that

$$\begin{aligned}&S(\tilde{v}_1, \tilde{v}_3, \tilde{v}_1, \tilde{v}_3) + S(\tilde{v}_1, \tilde{v}_4, \tilde{v}_1, \tilde{v}_4) \\ &+ S(\tilde{v}_2, \tilde{v}_3, \tilde{v}_2, \tilde{v}_3) + S(\tilde{v}_2, \tilde{v}_4, \tilde{v}_2, \tilde{v}_4) - 2\, S(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) \geq 0\end{aligned}$$

for all vectors $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4 \in \mathbb{R}^n \times \mathbb{R}$. Clearly, K is a closed convex cone. Moreover, K is invariant under the natural action of $GL(n+1)$.

We claim that K is invariant under the ODE $\frac{d}{dt} S = \tilde{Q}(S)$. The proof relies on the following result:

Proposition 4. *Let S be an algebraic curvature tensor on $\mathbb{R}^n \times \mathbb{R}$, which lies in the cone K. Moreover, suppose that $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4$ are vectors in $\mathbb{R}^n \times \mathbb{R}$ satisfying*

$$\begin{aligned}&S(\tilde{v}_1, \tilde{v}_3, \tilde{v}_1, \tilde{v}_3) + S(\tilde{v}_1, \tilde{v}_4, \tilde{v}_1, \tilde{v}_4) \\ &+ S(\tilde{v}_2, \tilde{v}_3, \tilde{v}_2, \tilde{v}_3) + S(\tilde{v}_2, \tilde{v}_4, \tilde{v}_2, \tilde{v}_4) - 2\, S(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) = 0.\end{aligned}$$

Then, the expression

$$\begin{aligned}&S(\tilde{w}_1, \tilde{v}_3, \tilde{w}_1, \tilde{v}_3) + S(\tilde{w}_1, \tilde{v}_4, \tilde{w}_1, \tilde{v}_4) \\ &+ S(\tilde{w}_2, \tilde{v}_3, \tilde{w}_2, \tilde{v}_3) + S(\tilde{w}_2, \tilde{v}_4, \tilde{w}_2, \tilde{v}_4) \\ &+ S(\tilde{v}_1, \tilde{w}_3, \tilde{v}_1, \tilde{w}_3) + S(\tilde{v}_2, \tilde{w}_3, \tilde{v}_2, \tilde{w}_3) \\ &+ S(\tilde{v}_1, \tilde{w}_4, \tilde{v}_1, \tilde{w}_4) + S(\tilde{v}_2, \tilde{w}_4, \tilde{v}_2, \tilde{w}_4) \\ &- 2\, \big[S(\tilde{v}_3, \tilde{w}_1, \tilde{v}_1, \tilde{w}_3) + S(\tilde{v}_4, \tilde{w}_1, \tilde{v}_2, \tilde{w}_3)\big] \\ &- 2\, \big[S(\tilde{v}_4, \tilde{w}_1, \tilde{v}_1, \tilde{w}_4) - S(\tilde{v}_3, \tilde{w}_1, \tilde{v}_2, \tilde{w}_4)\big] \\ &+ 2\, \big[S(\tilde{v}_4, \tilde{w}_2, \tilde{v}_1, \tilde{w}_3) - S(\tilde{v}_3, \tilde{w}_2, \tilde{v}_2, \tilde{w}_3)\big] \\ &- 2\, \big[S(\tilde{v}_3, \tilde{w}_2, \tilde{v}_1, \tilde{w}_4) + S(\tilde{v}_4, \tilde{w}_2, \tilde{v}_2, \tilde{w}_4)\big] \\ &- 2\, S(\tilde{w}_1, \tilde{w}_2, \tilde{v}_3, \tilde{v}_4) - 2\, S(\tilde{v}_1, \tilde{v}_2, \tilde{w}_3, \tilde{w}_4)\end{aligned}$$

is nonnegative for all vectors $\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4 \in \mathbb{R}^n \times \mathbb{R}$.

Proof. The proof is similar to the proof of Proposition 8 in [**2**]. Since $S \in K$, we have

$$\begin{aligned}
0 \leq\ & S(\tilde{v}_1 + s\tilde{w}_1, \tilde{v}_3 + s\tilde{w}_3, \tilde{v}_1 + s\tilde{w}_1, \tilde{v}_3 + s\tilde{w}_3) \\
& + S(\tilde{v}_1 + s\tilde{w}_1, \tilde{v}_4 + s\tilde{w}_4, \tilde{v}_1 + s\tilde{w}_1, \tilde{v}_4 + s\tilde{w}_4) \\
& + S(\tilde{v}_2 + s\tilde{w}_2, \tilde{v}_3 + s\tilde{w}_3, \tilde{v}_2 + s\tilde{w}_2, \tilde{v}_3 + s\tilde{w}_3) \\
& + S(\tilde{v}_2 + s\tilde{w}_2, \tilde{v}_4 + s\tilde{w}_4, \tilde{v}_2 + s\tilde{w}_2, \tilde{v}_4 + s\tilde{w}_4) \\
& - 2\,S(\tilde{v}_1 + s\tilde{w}_1, \tilde{v}_2 + s\tilde{w}_2, \tilde{v}_3 + s\tilde{w}_3, \tilde{v}_4 + s\tilde{w}_4)
\end{aligned}$$

for all $s \in \mathbb{R}$. Taking the second derivative at $s = 0$, we obtain

$$\begin{aligned}
(1)\quad 0 \leq\ & S(\tilde{w}_1, \tilde{v}_3, \tilde{w}_1, \tilde{v}_3) + S(\tilde{w}_1, \tilde{v}_4, \tilde{w}_1, \tilde{v}_4) \\
& + S(\tilde{w}_2, \tilde{v}_3, \tilde{w}_2, \tilde{v}_3) + S(\tilde{w}_2, \tilde{v}_4, \tilde{w}_2, \tilde{v}_4) \\
& + S(\tilde{v}_1, \tilde{w}_3, \tilde{v}_1, \tilde{w}_3) + S(\tilde{v}_2, \tilde{w}_3, \tilde{v}_2, \tilde{w}_3) \\
& + S(\tilde{v}_1, \tilde{w}_4, \tilde{v}_1, \tilde{w}_4) + S(\tilde{v}_2, \tilde{w}_4, \tilde{v}_2, \tilde{w}_4) \\
& + 2\,S(\tilde{v}_1, \tilde{v}_3, \tilde{w}_1, \tilde{w}_3) + 2\,S(\tilde{v}_1, \tilde{w}_3, \tilde{w}_1, \tilde{v}_3) - 2\,S(\tilde{w}_1, \tilde{v}_2, \tilde{w}_3, \tilde{v}_4) \\
& + 2\,S(\tilde{v}_1, \tilde{v}_4, \tilde{w}_1, \tilde{w}_4) + 2\,S(\tilde{v}_1, \tilde{w}_4, \tilde{w}_1, \tilde{v}_4) - 2\,S(\tilde{w}_1, \tilde{v}_2, \tilde{v}_3, \tilde{w}_4) \\
& + 2\,S(\tilde{v}_2, \tilde{v}_3, \tilde{w}_2, \tilde{w}_3) + 2\,S(\tilde{v}_2, \tilde{w}_3, \tilde{w}_2, \tilde{v}_3) - 2\,S(\tilde{v}_1, \tilde{w}_2, \tilde{w}_3, \tilde{v}_4) \\
& + 2\,S(\tilde{v}_2, \tilde{v}_4, \tilde{w}_2, \tilde{w}_4) + 2\,S(\tilde{v}_2, \tilde{w}_4, \tilde{w}_2, \tilde{v}_4) - 2\,S(\tilde{v}_1, \tilde{w}_2, \tilde{v}_3, \tilde{w}_4) \\
& - 2\,S(\tilde{w}_1, \tilde{w}_2, \tilde{v}_3, \tilde{v}_4) - 2\,S(\tilde{v}_1, \tilde{v}_2, \tilde{w}_3, \tilde{w}_4).
\end{aligned}$$

Replacing $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4\}$ by $\{\tilde{v}_2, -\tilde{v}_1, \tilde{v}_4, -\tilde{v}_3\}$ yields

$$\begin{aligned}
(2)\quad 0 \leq\ & S(\tilde{w}_1, \tilde{v}_4, \tilde{w}_1, \tilde{v}_4) + S(\tilde{w}_1, \tilde{v}_3, \tilde{w}_1, \tilde{v}_3) \\
& + S(\tilde{w}_2, \tilde{v}_4, \tilde{w}_2, \tilde{v}_4) + S(\tilde{w}_2, \tilde{v}_3, \tilde{w}_2, \tilde{v}_3) \\
& + S(\tilde{v}_2, \tilde{w}_3, \tilde{v}_2, \tilde{w}_3) + S(\tilde{v}_1, \tilde{w}_3, \tilde{v}_1, \tilde{w}_3) \\
& + S(\tilde{v}_2, \tilde{w}_4, \tilde{v}_2, \tilde{w}_4) + S(\tilde{v}_1, \tilde{w}_4, \tilde{v}_1, \tilde{w}_4) \\
& + 2\,S(\tilde{v}_2, \tilde{v}_4, \tilde{w}_1, \tilde{w}_3) + 2\,S(\tilde{v}_2, \tilde{w}_3, \tilde{w}_1, \tilde{v}_4) - 2\,S(\tilde{w}_1, \tilde{v}_1, \tilde{w}_3, \tilde{v}_3) \\
& - 2\,S(\tilde{v}_2, \tilde{v}_3, \tilde{w}_1, \tilde{w}_4) - 2\,S(\tilde{v}_2, \tilde{w}_4, \tilde{w}_1, \tilde{v}_3) + 2\,S(\tilde{w}_1, \tilde{v}_1, \tilde{v}_4, \tilde{w}_4) \\
& - 2\,S(\tilde{v}_1, \tilde{v}_4, \tilde{w}_2, \tilde{w}_3) - 2\,S(\tilde{v}_1, \tilde{w}_3, \tilde{w}_2, \tilde{v}_4) + 2\,S(\tilde{v}_2, \tilde{w}_2, \tilde{w}_3, \tilde{v}_3) \\
& + 2\,S(\tilde{v}_1, \tilde{v}_3, \tilde{w}_2, \tilde{w}_4) + 2\,S(\tilde{v}_1, \tilde{w}_4, \tilde{w}_2, \tilde{v}_3) - 2\,S(\tilde{v}_2, \tilde{w}_2, \tilde{v}_4, \tilde{w}_4) \\
& + 2\,S(\tilde{w}_1, \tilde{w}_2, \tilde{v}_4, \tilde{v}_3) + 2\,S(\tilde{v}_2, \tilde{v}_1, \tilde{w}_3, \tilde{w}_4).
\end{aligned}$$

In the next step, we take the arithmetic mean of (1) and (2). This yields

$$
\begin{aligned}
(3)\quad 0 \leq\ & S(\tilde{w}_1, \tilde{v}_3, \tilde{w}_1, \tilde{v}_3) + S(\tilde{w}_1, \tilde{v}_4, \tilde{w}_1, \tilde{v}_4) \\
& + S(\tilde{w}_2, \tilde{v}_3, \tilde{w}_2, \tilde{v}_3) + S(\tilde{w}_2, \tilde{v}_4, \tilde{w}_2, \tilde{v}_4) \\
& + S(\tilde{v}_1, \tilde{w}_3, \tilde{v}_1, \tilde{w}_3) + S(\tilde{v}_2, \tilde{w}_3, \tilde{v}_2, \tilde{w}_3) \\
& + S(\tilde{v}_1, \tilde{w}_4, \tilde{v}_1, \tilde{w}_4) + S(\tilde{v}_2, \tilde{w}_4, \tilde{v}_2, \tilde{w}_4) \\
& + \Big[S(\tilde{v}_1, \tilde{v}_3, \tilde{w}_1, \tilde{w}_3) + S(\tilde{v}_1, \tilde{w}_3, \tilde{w}_1, \tilde{v}_3) - S(\tilde{w}_1, \tilde{v}_2, \tilde{w}_3, \tilde{v}_4) \\
& \quad + S(\tilde{v}_2, \tilde{v}_4, \tilde{w}_1, \tilde{w}_3) + S(\tilde{v}_2, \tilde{w}_3, \tilde{w}_1, \tilde{v}_4) - S(\tilde{w}_1, \tilde{v}_1, \tilde{w}_3, \tilde{v}_3) \Big] \\
& + \Big[S(\tilde{v}_1, \tilde{v}_4, \tilde{w}_1, \tilde{w}_4) + S(\tilde{v}_1, \tilde{w}_4, \tilde{w}_1, \tilde{v}_4) - S(\tilde{w}_1, \tilde{v}_2, \tilde{v}_3, \tilde{w}_4) \\
& \quad - S(\tilde{v}_2, \tilde{v}_3, \tilde{w}_1, \tilde{w}_4) - S(\tilde{v}_2, \tilde{w}_4, \tilde{w}_1, \tilde{v}_3) + S(\tilde{w}_1, \tilde{v}_1, \tilde{v}_4, \tilde{w}_4) \Big] \\
& + \Big[S(\tilde{v}_2, \tilde{v}_3, \tilde{w}_2, \tilde{w}_3) + S(\tilde{v}_2, \tilde{w}_3, \tilde{w}_2, \tilde{v}_3) - S(\tilde{v}_1, \tilde{w}_2, \tilde{w}_3, \tilde{v}_4) \\
& \quad - S(\tilde{v}_1, \tilde{v}_4, \tilde{w}_2, \tilde{w}_3) - S(\tilde{v}_1, \tilde{w}_3, \tilde{w}_2, \tilde{v}_4) + S(\tilde{v}_2, \tilde{w}_2, \tilde{w}_3, \tilde{v}_3) \Big] \\
& + \Big[S(\tilde{v}_2, \tilde{v}_4, \tilde{w}_2, \tilde{w}_4) + S(\tilde{v}_2, \tilde{w}_4, \tilde{w}_2, \tilde{v}_4) - S(\tilde{v}_1, \tilde{w}_2, \tilde{v}_3, \tilde{w}_4) \\
& \quad + S(\tilde{v}_1, \tilde{v}_3, \tilde{w}_2, \tilde{w}_4) + S(\tilde{v}_1, \tilde{w}_4, \tilde{w}_2, \tilde{v}_3) - S(\tilde{v}_2, \tilde{w}_2, \tilde{v}_4, \tilde{w}_4) \Big] \\
& - 2\, S(\tilde{w}_1, \tilde{w}_2, \tilde{v}_3, \tilde{v}_4) - 2\, S(\tilde{v}_1, \tilde{v}_2, \tilde{w}_3, \tilde{w}_4).
\end{aligned}
$$

Since S satisfies the first Bianchi identity, the assertion follows.

Proposition 5. *Let S be an algebraic curvature tensor on $\mathbb{R}^n \times \mathbb{R}$, which lies in the cone K. Moreover, suppose that $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4$ are vectors in $\mathbb{R}^n \times \mathbb{R}$ satisfying*

$$
\begin{aligned}
& S(\tilde{v}_1, \tilde{v}_3, \tilde{v}_1, \tilde{v}_3) + S(\tilde{v}_1, \tilde{v}_4, \tilde{v}_1, \tilde{v}_4) \\
& + S(\tilde{v}_2, \tilde{v}_3, \tilde{v}_2, \tilde{v}_3) + S(\tilde{v}_2, \tilde{v}_4, \tilde{v}_2, \tilde{v}_4) - 2\, S(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) = 0.
\end{aligned}
$$

Then,

$$
\begin{aligned}
& \tilde{Q}(S)(\tilde{v}_1, \tilde{v}_3, \tilde{v}_1, \tilde{v}_3) + \tilde{Q}(S)(\tilde{v}_1, \tilde{v}_4, \tilde{v}_1, \tilde{v}_4) \\
& + \tilde{Q}(S)(\tilde{v}_2, \tilde{v}_3, \tilde{v}_2, \tilde{v}_3) + \tilde{Q}(S)(\tilde{v}_2, \tilde{v}_4, \tilde{v}_2, \tilde{v}_4) - 2\, \tilde{Q}(S)(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) \geq 0.
\end{aligned}
$$

Proof. Consider the following $n \times n$ matrices:

$$
\begin{aligned}
a_{pq} &= S(\tilde{v}_1, e_p, \tilde{v}_1, e_q) + S(\tilde{v}_2, e_p, \tilde{v}_2, e_q), \\
b_{pq} &= S(\tilde{v}_3, e_p, \tilde{v}_3, e_q) + S(\tilde{v}_4, e_p, \tilde{v}_4, e_q), \\
c_{pq} &= S(\tilde{v}_3, e_p, \tilde{v}_1, e_q) + S(\tilde{v}_4, e_p, \tilde{v}_2, e_q), \\
d_{pq} &= S(\tilde{v}_4, e_p, \tilde{v}_1, e_q) - S(\tilde{v}_3, e_p, \tilde{v}_2, e_q), \\
e_{pq} &= S(\tilde{v}_1, \tilde{v}_2, e_p, e_q), \\
f_{pq} &= S(\tilde{v}_3, \tilde{v}_4, e_p, e_q)
\end{aligned}
$$

$(1 \leq p, q \leq n)$. It follows from Proposition 4 that the matrix

$$\begin{bmatrix} B & -F & -C & -D \\ F & B & D & -C \\ -C^T & D^T & A & -E \\ -D^T & -C^T & E & A \end{bmatrix}$$

is positive semi-definite. This implies

$$\operatorname{tr}(AB) + \operatorname{tr}(EF) - \operatorname{tr}(C^2) - \operatorname{tr}(D^2) \geq 0,$$

and hence

$$\sum_{p,q=1}^n a_{pq}\, b_{pq} - \sum_{p,q=1}^n e_{pq}\, f_{pq} - \sum_{p,q=1}^n c_{pq}\, c_{qp} - \sum_{p,q=1}^n d_{pq}\, d_{qp} \geq 0 \tag{4}$$

(cf. [**2**], Proposition 9). On the other hand, we have

$$\begin{aligned} \tilde{Q}(S)(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) &= \sum_{p,q=1}^n S(\tilde{v}_1, \tilde{v}_2, e_p, e_q)\, S(\tilde{v}_3, \tilde{v}_4, e_p, e_q) \\ &\quad + \sum_{p,q=1}^n S(\tilde{v}_1, \tilde{v}_3, e_p, e_q)\, S(\tilde{v}_2, \tilde{v}_4, e_p, e_q) \\ &\quad - \sum_{p,q=1}^n S(\tilde{v}_1, \tilde{v}_4, e_p, e_q)\, S(\tilde{v}_2, \tilde{v}_3, e_p, e_q) \\ &\quad + 2 \sum_{p,q=1}^n S(\tilde{v}_1, e_p, \tilde{v}_3, e_q)\, S(\tilde{v}_4, e_p, \tilde{v}_2, e_q) \\ &\quad - 2 \sum_{p,q=1}^n S(\tilde{v}_1, e_p, \tilde{v}_4, e_q)\, S(\tilde{v}_3, e_p, \tilde{v}_2, e_q) \end{aligned}$$

since S satisfies the first Bianchi identity. This implies

$$\begin{aligned} &\tilde{Q}(S)(\tilde{v}_1, \tilde{v}_3, \tilde{v}_1, \tilde{v}_3) + \tilde{Q}(S)(\tilde{v}_1, \tilde{v}_4, \tilde{v}_1, \tilde{v}_4) \\ &\quad + \tilde{Q}(S)(\tilde{v}_2, \tilde{v}_3, \tilde{v}_2, \tilde{v}_3) + \tilde{Q}(S)(\tilde{v}_2, \tilde{v}_4, \tilde{v}_2, \tilde{v}_4) - 2\,\tilde{Q}(S)(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) \\ &= \sum_{p,q=1}^n [S(\tilde{v}_1, \tilde{v}_3, e_p, e_q) - S(\tilde{v}_2, \tilde{v}_4, e_p, e_q)]^2 \\ &\quad + \sum_{p,q=1}^n [S(\tilde{v}_1, \tilde{v}_4, e_p, e_q) + S(\tilde{v}_2, \tilde{v}_3, e_p, e_q)]^2 \\ &\quad + 2 \sum_{p,q=1}^n a_{pq}\, b_{pq} - 2 \sum_{p,q=1}^n e_{pq}\, f_{pq} - 2 \sum_{p,q=1}^n c_{pq}\, c_{qp} - 2 \sum_{p,q=1}^n d_{pq}\, d_{qp}. \end{aligned} \tag{5}$$

The assertion follows immediately from (4) and (5).

4. Proof of Theorem 1

We define a Riemannian metric h on $M \times (0,T)$ by

$$h = \sum_{i,j=1}^{n} g_{ij}\,dx^i \otimes dx^j + \frac{1}{t^2}\,dt \otimes dt.$$

Lemma 6. *Suppose that*

$$\sup_{(x,t)\in M\times(0,T)} |\mathrm{Rm}| < \infty.$$

Then there exists a uniform constant C such that

$$\Big|\tilde{D}_{\frac{\partial}{\partial t}} h - \tilde{\Delta} h - \frac{1}{t}\,h\Big|_h \leq C$$

and

$$|\tilde{D}_v h|_h \leq C\,|v|$$

for all points $(x,t) \in M \times (0,T)$ and all vectors $v \in T_xM$.

Proof. By definition of $\tilde{D}$, we have

$$\begin{aligned}
\tilde{D}_{\frac{\partial}{\partial x^i}} dx^j &= -\Gamma_{ik}^j\,dx^k + \Big(\mathrm{Ric}_i^j + \frac{1}{2t}\,\delta_i^j\Big)\,dt \\
\tilde{D}_{\frac{\partial}{\partial x^i}} dt &= 0 \\
\tilde{D}_{\frac{\partial}{\partial t}} dx^j &= \Big(\mathrm{Ric}_i^j + \frac{1}{2t}\,\delta_i^j\Big)\,dx^i + \frac{1}{2}\,\partial^j \mathrm{scal}\,dt \\
\tilde{D}_{\frac{\partial}{\partial t}} dt &= \frac{3}{2t}\,dt.
\end{aligned}$$

This implies

$$\tilde{D}_{\frac{\partial}{\partial x^k}} h = \Big(\mathrm{Ric}_{ik} + \frac{1}{2t}\,g_{ik}\Big)\,(dx^i \otimes dt + dt \otimes dx^i).$$

Moreover, we have

$$\begin{aligned}
\tilde{D}_{\frac{\partial}{\partial t}} h &= \frac{1}{2}\,\partial_i \mathrm{scal}\,(dx^i \otimes dt + dt \otimes dx^i) \\
&\quad + \frac{1}{t}\,g_{ij}\,dx^i \otimes dx^j + \frac{1}{t^3}\,dt \otimes dt
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\Delta} h &= \frac{1}{2}\,\partial_i \mathrm{scal}\,(dx^i \otimes dt + dt \otimes dx^i) \\
&\quad + 2\,\Big(\mathrm{Ric}_k^i + \frac{1}{2t}\,\delta_k^i\Big)\,\Big(\mathrm{Ric}_i^k + \frac{1}{2t}\,\delta_i^k\Big)\,dt \otimes dt.
\end{aligned}$$

Putting these facts together, we obtain

$$\tilde{D}_{\frac{\partial}{\partial t}} h - \tilde{\Delta} h - \frac{1}{t}\,h = -2\,\Big(\mathrm{Ric}_k^i + \frac{1}{2t}\,\delta_k^i\Big)\,\Big(\mathrm{Ric}_i^k + \frac{1}{2t}\,\delta_i^k\Big)\,dt \otimes dt.$$

Thus, we conclude that

$$\Big|\tilde{D}_{\frac{\partial}{\partial t}} h - \tilde{\Delta} h - \frac{1}{t}\, h\Big|_h = 2t^2\, |\text{Ric}|^2 + 2t\, \text{scal} + \frac{n}{2}$$

and

$$|\tilde{D}_v h|_h^2 = 2t^2\, \text{Ric}^2(v,v) + 2t\, \text{Ric}(v,v) + \frac{1}{2}\, g(v,v)$$

for all points $(x,t) \in M \times (0,T)$ and all vectors $v \in T_xM$.

Lemma 7. *Suppose that $(M, g(t)) \times \mathbb{R}^2$ has nonnegative isotropic curvature for all $t \in (0,T)$. Moreover, we assume that*

$$\sup_{(x,t) \in M \times (0,T)} |D^m \text{Rm}| < \infty$$

for $m = 0, 1, 2, \ldots$. Then there exists a uniform constant C such that

$$S + \frac{1}{4}\, C\, t\, h \owedge h \in K$$

for all points $(x,t) \in M \times (0,T)$. Here, $\owedge$ denotes the Kulkarni-Nomizu product.

Proof. There exists a uniform constant C such that

$$\big|S - R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l\big|_h \leq C\, t$$

for all $(x,t) \in M \times (0,T)$. This implies

$$S - R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l + \frac{1}{4}\, C\, t\, h \owedge h \in K$$

for all $(x,t) \in M \times (0,T)$. Moreover, since $(M, g(t)) \times \mathbb{R}^2$ has nonnegative isotropic curvature, we have

$$R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l \in K$$

for all points $(x,t) \in M \times (0,T)$. Putting these facts together, the assertion follows.

Proposition 8. *Suppose that $(M, g(t)) \times \mathbb{R}^2$ has nonnegative isotropic curvature for all $t \in (0,T)$. Moreover, we assume that*

$$\sup_{(x,t) \in M \times (0,T)} |D^m \text{Rm}| < \infty$$

for $m = 0, 1, 2, \ldots$. Then, $S_{(x,t)} \in K$ for all $(x,t) \in M \times (0,T)$.

Proof. By Lemma 5.1 in [**10**], we can find a smooth function $\varphi : M \to \mathbb{R}$ with the following properties:

(i) $\varphi(x) \to \infty$ as $x \to \infty$,
(ii) $\varphi(x) \geq 1$ for all $x \in M$,
(iii) $\sup_{(x,t) \in M \times (0,T)} |\nabla \varphi(x)|_{g(t)} < \infty$,
(iv) $\sup_{(x,t) \in M \times (0,T)} |\Delta_{g(t)} \varphi(x)| < \infty$.

Let ε be an arbitrary positive real number. We define a $(0,4)$-tensor $\hat{S}$ by

$$\hat{S} = S + \frac{1}{4}\,\varepsilon\, e^{\lambda t}\,\varphi(x)\, h \owedge h,$$

where λ is a positive constant that will be specified later. Clearly, $\hat{S}$ is an algebraic curvature tensor. By Lemma 7, there exists a uniform constant C_1 such that

$$S + \frac{1}{4}\, C_1\, t\, h \owedge h \in K$$

for all points $(x,t) \in M \times (0,T)$. Hence, if $\varepsilon\, e^{\lambda t}\, \varphi(x) > C_1\, t$, then $\hat{S}_{(x,t)}$ lies in the interior of the cone K.

We claim that $\hat{S}_{(x,t)} \in K$ for all $(x,t) \in M \times (0,T)$. Suppose this is false. Then there exists a point $(x_0,t_0) \in M \times (0,T)$ such that $\hat{S}_{(x_0,t_0)} \in \partial K$ and $\hat{S}_{(x,t)} \in K$ for all $(x,t) \in M \times (0,t_0]$. Since $\hat{S}_{(x_0,t_0)} \in \partial K$, we can find vectors $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4 \in T_{(x_0,t_0)}(M \times (0,T))$ such that

$$|\tilde{v}_1 \wedge \tilde{v}_3 + \tilde{v}_4 \wedge \tilde{v}_2|_h^2 + |\tilde{v}_1 \wedge \tilde{v}_4 + \tilde{v}_2 \wedge \tilde{v}_3|_h^2 > 0$$

and

$$\begin{aligned}&\hat{S}(\tilde{v}_1, \tilde{v}_3, \tilde{v}_1, \tilde{v}_3) + \hat{S}(\tilde{v}_1, \tilde{v}_4, \tilde{v}_1, \tilde{v}_4)\\ &+ \hat{S}(\tilde{v}_2, \tilde{v}_3, \tilde{v}_2, \tilde{v}_3) + \hat{S}(\tilde{v}_2, \tilde{v}_4, \tilde{v}_2, \tilde{v}_4) - 2\,\hat{S}(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) = 0\end{aligned}$$

at (x_0,t_0). It follows from Proposition 5 that

(6)
$$\begin{aligned}&\tilde{Q}(\hat{S})(\tilde{v}_1, \tilde{v}_3, \tilde{v}_1, \tilde{v}_3) - \tilde{Q}(\hat{S})(\tilde{v}_1, \tilde{v}_4, \tilde{v}_1, \tilde{v}_4)\\ &+ \tilde{Q}(\hat{S})(\tilde{v}_2, \tilde{v}_3, \tilde{v}_2, \tilde{v}_3) - \tilde{Q}(\hat{S})(\tilde{v}_2, \tilde{v}_4, \tilde{v}_2, \tilde{v}_4) + 2\,\tilde{Q}(\hat{S})(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) \geq 0\end{aligned}$$

at (x_0,t_0). We may extend $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4$ to vector fields on $M \times (0,T)$ such that

$$\begin{aligned}&\tilde{D}_{\frac{\partial}{\partial x^i}}\,\tilde{v}_1 = 0 && \tilde{D}_{\frac{\partial}{\partial t}}\,\tilde{v}_1 - \tilde{\Delta}\tilde{v}_1 + \frac{1}{2t}\,\tilde{v}_1 = 0\\ &\tilde{D}_{\frac{\partial}{\partial x^i}}\,\tilde{v}_2 = 0 && \tilde{D}_{\frac{\partial}{\partial t}}\,\tilde{v}_2 - \tilde{\Delta}\tilde{v}_2 + \frac{1}{2t}\,\tilde{v}_2 = 0\\ &\tilde{D}_{\frac{\partial}{\partial x^i}}\,\tilde{v}_3 = 0 && \tilde{D}_{\frac{\partial}{\partial t}}\,\tilde{v}_3 - \tilde{\Delta}\tilde{v}_3 + \frac{1}{2t}\,\tilde{v}_3 = 0\\ &\tilde{D}_{\frac{\partial}{\partial x^i}}\,\tilde{v}_4 = 0 && \tilde{D}_{\frac{\partial}{\partial t}}\,\tilde{v}_4 - \tilde{\Delta}\tilde{v}_4 + \frac{1}{2t}\,\tilde{v}_4 = 0\end{aligned}$$

at (x_0,t_0). We now define a function $f : M \times (0,T) \to \mathbb{R}$ by

$$\begin{aligned}f = &\hat{S}(\tilde{v}_1, \tilde{v}_3, \tilde{v}_1, \tilde{v}_3) + \hat{S}(\tilde{v}_1, \tilde{v}_4, \tilde{v}_1, \tilde{v}_4)\\ &+ \hat{S}(\tilde{v}_2, \tilde{v}_3, \tilde{v}_2, \tilde{v}_3) + \hat{S}(\tilde{v}_2, \tilde{v}_4, \tilde{v}_2, \tilde{v}_4) - 2\,\hat{S}(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4).\end{aligned}$$

Clearly, $f(x_0, t_0) = 0$ and $f(x, t) \geq 0$ for all $(x, t) \in M \times (0, t_0]$. This implies

$$\frac{\partial}{\partial t} f - \Delta f \leq 0$$

at (x_0, t_0). Hence, if we put

$$Z = \tilde{D}_{\frac{\partial}{\partial t}} \hat{S} - \tilde{\Delta} \hat{S} - \frac{2}{t} \hat{S},$$

then we obtain

$$\begin{aligned} (7) \qquad & Z(\tilde{v}_1, \tilde{v}_3, \tilde{v}_1, \tilde{v}_3) + Z(\tilde{v}_1, \tilde{v}_4, \tilde{v}_1, \tilde{v}_4) \\ & + Z(\tilde{v}_2, \tilde{v}_3, \tilde{v}_2, \tilde{v}_3) + Z(\tilde{v}_2, \tilde{v}_4, \tilde{v}_2, \tilde{v}_4) - 2\, Z(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) \\ & = \frac{\partial}{\partial t} f - \Delta f \leq 0 \end{aligned}$$

at (x_0, t_0). On the other hand, it follows from Proposition 3 that

$$\begin{aligned} Z &= \tilde{D}_{\frac{\partial}{\partial t}} \hat{S} - \tilde{\Delta} \hat{S} - \frac{2}{t} \hat{S} \\ &= \tilde{Q}(S) + \frac{1}{4} \lambda\, \varepsilon\, e^{\lambda t}\, \varphi(x)\, h \owedge h - \frac{1}{4} \varepsilon\, e^{\lambda t}\, \Delta\varphi(x)\, h \owedge h \\ &\quad - \varepsilon\, e^{\lambda t} \sum_{j=1}^n \langle \nabla \varphi(x), e_j \rangle\, h \owedge \tilde{D}_{e_j} h \\ &\quad - \frac{1}{2} \varepsilon\, e^{\lambda t}\, \varphi(x) \sum_{j=1}^n \tilde{D}_{e_j} h \owedge \tilde{D}_{e_j} h \\ &\quad + \frac{1}{2} \varepsilon\, e^{\lambda t}\, \varphi(x)\, h \owedge \Big(\tilde{D}_{\frac{\partial}{\partial t}} h - \tilde{\Delta} h - \frac{1}{t} h \Big) \end{aligned}$$

for all $(x, t) \in M \times (0, T)$. In view of Lemma 6, there exists a uniform constant C_2 such that

$$\Big| Z - \tilde{Q}(S) - \frac{1}{4} \lambda\, \varepsilon\, e^{\lambda t}\, \varphi(x)\, h \owedge h \Big|_h \leq C_2\, \varepsilon\, e^{\lambda t}\, (\varphi(x) + |\nabla \varphi(x)| + |\Delta \varphi(x)|)$$

for all $(x, t) \in M \times (0, T)$. Since $\nabla \varphi(x)$ and $\Delta \varphi(x)$ are uniformly bounded, it follows that

$$\Big| Z - \tilde{Q}(S) - \frac{1}{4} \lambda\, \varepsilon\, e^{\lambda t}\, \varphi(x)\, h \owedge h \Big|_h \leq C_3\, \varepsilon\, e^{\lambda t}\, \varphi(x)$$

for all $(x, t) \in M \times (0, T)$.

We next observe that $\varepsilon\, e^{\lambda t_0}\, \varphi(x_0) \leq C_1\, t_0$. (Indeed, if $\varepsilon\, e^{\lambda t_0}\, \varphi(x_0) > C_1\, t_0$, then $\hat{S}_{(x_0, t_0)}$ would lie in the interior of the cone K, contrary to our choice of (x_0, t_0).) Hence, there exists a uniform constant C_4 such that

$$|S|_h + |\hat{S} - S|_h \leq C_4$$

at (x_0, t_0). This implies

$$\begin{aligned}|\tilde{Q}(\hat{S}) - \tilde{Q}(S)|_h &\leq C_5 \left(|S|_h \, |\hat{S} - S|_h + |\hat{S} - S|_h^2\right) \\ &\leq C_5 \, C_4 \, |\hat{S} - S|_h \\ &\leq C_6 \, \varepsilon \, e^{\lambda t} \, \varphi(x)\end{aligned}$$

at (x_0, t_0). Putting these facts together, we obtain

$$\Big|Z - \tilde{Q}(\hat{S}) - \frac{1}{4} \, \lambda \, \varepsilon \, e^{\lambda t} \, \varphi(x) \, h \owedge h\Big|_h \leq C_7 \, \varepsilon \, e^{\lambda t} \, \varphi(x)$$

at (x_0, t_0). This implies

$$Z - \tilde{Q}(\hat{S}) - \frac{1}{4} \, (\lambda - C_7) \, \varepsilon \, e^{\lambda t} \, \varphi(x) \, h \owedge h \in K$$

at (x_0, t_0). Hence, if we choose $\lambda > C_7$, then we have

(8)

$$\begin{aligned}&Z(\tilde{v}_1, \tilde{v}_3, \tilde{v}_1, \tilde{v}_3) + Z(\tilde{v}_1, \tilde{v}_4, \tilde{v}_1, \tilde{v}_4) \\ &\quad + Z(\tilde{v}_2, \tilde{v}_3, \tilde{v}_2, \tilde{v}_3) + Z(\tilde{v}_2, \tilde{v}_4, \tilde{v}_2, \tilde{v}_4) - 2 \, Z(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) \\ &\quad - \tilde{Q}(\hat{S})(\tilde{v}_1, \tilde{v}_3, \tilde{v}_1, \tilde{v}_3) - \tilde{Q}(\hat{S})(\tilde{v}_1, \tilde{v}_4, \tilde{v}_1, \tilde{v}_4) \\ &\quad - \tilde{Q}(\hat{S})(\tilde{v}_2, \tilde{v}_3, \tilde{v}_2, \tilde{v}_3) - \tilde{Q}(\hat{S})(\tilde{v}_2, \tilde{v}_4, \tilde{v}_2, \tilde{v}_4) + 2 \, \tilde{Q}(\hat{S})(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) \\ &> 0\end{aligned}$$

at (x_0, t_0). The inequality (8) is inconsistent with (6) and (7). Consequently, we have $\hat{S}_{(x,t)} \in K$ for all points $(x, t) \in M \times (0, T)$. Since $\varepsilon > 0$ is arbitrary, it follows that $S_{(x,t)} \in K$ for all points $(x, t) \in M \times (0, T)$.

Proposition 9. *Suppose that $(M, g(t)) \times \mathbb{R}^2$ has nonnegative isotropic curvature for all $t \in (0, T)$. Moreover, we assume that*

$$\sup_{(x,t) \in M \times (\alpha, T)} \text{scal}(x, t) < \infty$$

for all $\alpha \in (0, T)$. Then, $S_{(x,t)} \in K$ for all $(x, t) \in M \times (0, T)$.

Proof. Fix a real number $\alpha \in (0, T)$. By assumption, we have

$$\sup_{(x,t) \in M \times (\alpha, T)} |\text{Rm}| < \infty.$$

Using Shi's interior derivative estimates, we obtain

$$\sup_{(x,t) \in M \times (\alpha, T)} |D^m \text{Rm}| < \infty$$

for $m = 1, 2, \ldots$ (see e.g., [**12**], Theorem 13.1). Hence, we can apply Proposition 8 to the metrics $g(t + \alpha)$, $t \in (0, T - \alpha)$. Taking the limit as $\alpha \to 0$, the assertion follows.

Theorem 1 is an immediate consequence of Proposition 9. To see this, we consider a point $(x,t) \in M \times (0,T)$ and vectors $v, w \in T_xM$. By Proposition 9, we have

$$\begin{aligned}&S(\tilde{v}_1, \tilde{v}_3, \tilde{v}_1, \tilde{v}_3) + S(\tilde{v}_1, \tilde{v}_4, \tilde{v}_1, \tilde{v}_4) \\ &+ S(\tilde{v}_2, \tilde{v}_3, \tilde{v}_2, \tilde{v}_3) + S(\tilde{v}_2, \tilde{v}_4, \tilde{v}_2, \tilde{v}_4) - 2\,S(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) \geq 0\end{aligned}$$

for all vectors $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4 \in T_{(x,t)}(M \times (0,T))$. Hence, if we put

$$\tilde{v}_1 = \frac{\partial}{\partial t} + v, \qquad \tilde{v}_2 = 0, \qquad \tilde{v}_3 = w, \qquad \tilde{v}_4 = 0,$$

then we obtain

$$M(w,w) + 2\,P(v,w,w) + R(v,w,v,w) \geq 0.$$

This completes the proof of Theorem 1. In order to prove Corollary 2, we take the trace over w. This yields

$$\Delta\text{scal} + 2\,|\text{Ric}|^2 + \frac{1}{t}\,\text{scal} + 2\,\partial_i\text{scal}\,v^i + 2\,\text{Ric}(v,v) \geq 0.$$

Hence, Corollary 2 follows from the identity $\frac{\partial}{\partial t}\text{scal} = \Delta\text{scal} + 2\,|\text{Ric}|^2$.

5. The equality case in the Harnack inequality

In this section, we analyze the equality case in the Harnack inequality. Let $(M, g(t))$, $t \in (0,T)$, be a family of complete Riemannian manifolds evolving under Ricci flow. As above, we assume that $(M, g(t)) \times \mathbb{R}^2$ has nonnegative isotropic curvature for all $t \in (0,T)$. Moreover, we require that

$$\sup_{(x,t) \in M \times (\alpha, T)} \text{scal}(x,t) < \infty$$

for all $\alpha \in (0,T)$.

Let E be the tangent bundle of $M \times (0,T)$. We denote by P the total space of the vector bundle $E \oplus E \oplus E \oplus E$. The connection $\tilde{D}$ defines a horizontal distribution on P. Hence, the tangent bundle of P splits as a direct sum $TP = \mathbb{H} \oplus \mathbb{V}$, where $\mathbb{H}$ and $\mathbb{V}$ denote the horizontal and vertical distributions, respectively.

Let π be the projection from P to $M \times (0,T)$. For each $t \in (0,T)$, we denote by $P_t = \pi^{-1}(M \times \{t\})$ the time t slice of P. We define a function $u : P \to \mathbb{R}$ by

$$\begin{aligned}u : (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) \mapsto\ &S(\tilde{v}_1, \tilde{v}_3, \tilde{v}_1, \tilde{v}_3) + S(\tilde{v}_1, \tilde{v}_4, \tilde{v}_1, \tilde{v}_4) \\ &+ S(\tilde{v}_2, \tilde{v}_3, \tilde{v}_2, \tilde{v}_3) + S(\tilde{v}_2, \tilde{v}_4, \tilde{v}_2, \tilde{v}_4) \\ &- 2\,S(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4).\end{aligned}$$

By Proposition 9, u is a nonnegative function on P. Let $F = \{u = 0\}$ be the zero set of the function u. We claim that F is invariant under parallel transport:

Proposition 10. *Fix a real number $t_0 \in (0, T)$, and let $\tilde{\gamma} : [0, 1] \to P_{t_0}$ be a smooth horizontal curve such that $\tilde{\gamma}(0) \in F$. Then, $\tilde{\gamma}(s) \in F$ for all $s \in [0, 1]$.*

Proof. Without loss of generality, we may assume that the projected path $\pi \circ \tilde{\gamma} : [0, 1] \to M \times \{t_0\}$ is contained in a single coordinate chart. Let $\Omega \subset M \times (0, T)$ be a coordinate chart such that $\pi(\tilde{\gamma}(s)) \in \Omega$ for all $s \in [0, 1]$. We can find smooth vector fields $X_1, \ldots, X_n$ on Ω such that

$$\sum_{k=1}^n X_k \otimes X_k = \sum_{i,j=1}^n g^{ij} \, \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}.$$

Moreover, we define a vector field Y on Ω by

$$Y = \frac{\partial}{\partial t} + \sum_{k=1}^n \tilde{D}_{X_k} X_k.$$

Let $\tilde{X}_1, \ldots, \tilde{X}_n, \tilde{Y}$ be the horizontal lifts of $X_1, \ldots, X_n, Y$. At each point $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) \in \pi^{-1}(\Omega)$, we have

$$\begin{aligned}
&\tilde{Y}(u) - \sum_{k=1}^n \tilde{X}_k(\tilde{X}_k(u)) \\
&= \Big(\tilde{D}_{\frac{\partial}{\partial t}} S - \tilde{\Delta} S\Big)(\tilde{v}_1, \tilde{v}_3, \tilde{v}_1, \tilde{v}_3) + \Big(\tilde{D}_{\frac{\partial}{\partial t}} S - \tilde{\Delta} S\Big)(\tilde{v}_1, \tilde{v}_4, \tilde{v}_1, \tilde{v}_4) \\
&\quad + \Big(\tilde{D}_{\frac{\partial}{\partial t}} S - \tilde{\Delta} S\Big)(\tilde{v}_2, \tilde{v}_3, \tilde{v}_2, \tilde{v}_3) + \Big(\tilde{D}_{\frac{\partial}{\partial t}} S - \tilde{\Delta} S\Big)(\tilde{v}_2, \tilde{v}_4, \tilde{v}_2, \tilde{v}_4) \\
&\quad - 2 \Big(\tilde{D}_{\frac{\partial}{\partial t}} S - \tilde{\Delta} S\Big)(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4).
\end{aligned}$$

Using Proposition 3, we obtain

$$\begin{aligned}
&\tilde{Y}(u) - \sum_{k=1}^n \tilde{X}_k(\tilde{X}_k(u)) - \frac{2}{t} \, u \\
&= \tilde{Q}(S)(\tilde{v}_1, \tilde{v}_3, \tilde{v}_1, \tilde{v}_3) + \tilde{Q}(S)(\tilde{v}_1, \tilde{v}_4, \tilde{v}_1, \tilde{v}_4) \\
&\quad + \tilde{Q}(S)(\tilde{v}_2, \tilde{v}_3, \tilde{v}_2, \tilde{v}_3) + \tilde{Q}(S)(\tilde{v}_2, \tilde{v}_4, \tilde{v}_2, \tilde{v}_4) \\
&\quad - 2 \, \tilde{Q}(S)(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4)
\end{aligned}$$

for all points $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) \in \pi^{-1}(\Omega)$. Moreover, it follows from the calculations in Section 3 that

$$\begin{aligned}
&\tilde{Q}(S)(\tilde{v}_1, \tilde{v}_3, \tilde{v}_1, \tilde{v}_3) + \tilde{Q}(S)(\tilde{v}_1, \tilde{v}_4, \tilde{v}_1, \tilde{v}_4) \\
&\quad + \tilde{Q}(S)(\tilde{v}_2, \tilde{v}_3, \tilde{v}_2, \tilde{v}_3) + \tilde{Q}(S)(\tilde{v}_2, \tilde{v}_4, \tilde{v}_2, \tilde{v}_4) - 2 \, \tilde{Q}(S)(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) \\
&\geq C \inf_{\xi \in \mathbb{V}, \, |\xi| \leq 1} (D^2 u)(\xi, \xi)
\end{aligned}$$

for all points $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) \in \pi^{-1}(\Omega)$. Here, $D^2 u$ denotes the Hessian of u in vertical direction. Putting these facts together, we obtain

$$\tilde{Y}(u) - \sum_{k=1}^n \tilde{X}_k(\tilde{X}_k(u)) - \frac{2}{t}\,u \geq C \inf_{\xi \in \mathbb{V},\, |\xi| \leq 1} (D^2 u)(\xi, \xi)$$

on $\pi^{-1}(\Omega)$. Hence, the assertion follows from J.M. Bony's version of the strong maximum principle (see [**1**] or [**3**], Proposition 4).

For each point $(x,t) \in M \times (0,T)$, we denote by $\mathcal{N}_{(x,t)}$ the set of all vectors of the form $\tilde{v} = \frac{\partial}{\partial t} + v \in T_{(x,t)}(M \times (0,T))$, where $v \in T_x M$ satisfies

$$\frac{\partial}{\partial t}\text{scal} + \frac{1}{t}\,\text{scal} + 2\,\partial_i \text{scal}\, v^i + 2\,\text{Ric}(v,v) = 0.$$

In view of Theorem 1, we can characterize the set $\mathcal{N}_{(x,t)}$ as follows:

$$\begin{aligned}
&\frac{\partial}{\partial t} + v \in \mathcal{N}_{(x,t)} \\
&\iff \quad \frac{\partial}{\partial t}\text{scal} + \frac{1}{t}\,\text{scal} + 2\,\partial_i \text{scal}\, v^i + 2\,\text{Ric}(v,v) = 0 \\
&\iff \quad M(w,w) + 2\,P(v,w,w) + R(v,w,v,w) = 0 \quad \text{for all } w \in T_x M \\
&\iff \quad \Big(\frac{\partial}{\partial t} + v, 0, w, 0\Big) \in F \quad \text{for all } w \in T_x M.
\end{aligned}$$

By Proposition 10, the set F is invariant under parallel transport. Therefore, we can draw the following conclusion:

Corollary 11. *Fix a smooth path $\gamma : [0,1] \to M \times \{t_0\}$. We denote by $\tilde{P}_\gamma : T_{\gamma(0)}(M \times (0,T)) \to T_{\gamma(1)}(M \times (0,T))$ the parallel transport along γ with respect to the connection $\tilde{D}$. If $\tilde{v} \in \mathcal{N}_{\gamma(0)}$, then $\tilde{P}_\gamma \tilde{v} \in \mathcal{N}_{\gamma(1)}$.*

Proposition 12. *Let $(M, g(t))$, $t \in (0,T)$, be a family of complete Riemannian manifolds evolving under Ricci flow. For each $t \in (0,T)$, we assume that $(M, g(t)) \times \mathbb{R}^2$ has nonnegative isotropic curvature and $(M, g(t))$ has positive Ricci curvature. Moreover, suppose that there exists a point $(x_0, t_0) \in M \times (0,T)$ such that*

$$t_0 \cdot \text{scal}(x_0, t_0) = \sup_{(x,t) \in M \times (0,T)} t \cdot \text{scal}(x,t).$$

Then there exists a smooth vector field $V = V^j \frac{\partial}{\partial x^j}$ such that

$$D_{\frac{\partial}{\partial x^i}} V = \text{Ric}_i^j \frac{\partial}{\partial x^j} + \frac{1}{2t} \frac{\partial}{\partial x^i}$$

for all $(x,t) \in M \times \{t_0\}$. In particular, $(M, g(t_0))$ is an expanding Ricci soliton.

Proof. Since $(M, g(t))$ has positive Ricci curvature, there exists a unique vector field $V = V^j \frac{\partial}{\partial x^j}$ such that $\partial_i \text{scal} + 2\,\text{Ric}_{ij}\, V^j = 0$. We

claim that

$$\mathcal{N}_{(x,t)} \subset \Big\{ \frac{\partial}{\partial t} + V_{(x,t)} \Big\} \tag{9}$$

for all points $(x,t) \in M \times (0,T)$. In order to prove this, we consider an arbitrary vector $\tilde{v} \in \mathcal{N}_{(x,t)}$. The vector $\tilde{v}$ can be written in the form $\tilde{v} = \frac{\partial}{\partial t} + v$, where $v \in T_x M$ satisfies

$$\frac{\partial}{\partial t}\text{scal} + \frac{1}{t}\,\text{scal} + 2\,\partial_i \text{scal}\, v^i + 2\,\text{Ric}(v,v) = 0.$$

Using Corollary 2, we conclude that $\partial_i \text{scal} + 2\,\text{Ric}_{ij}\, v^j = 0$. Since $(M, g(t))$ has positive Ricci curvature, it follows that $v = V_{(x,t)}$. This completes the proof of (9). In particular, the set $\mathcal{N}_{(x,t)}$ contains at most one element.

By assumption, the function $t \cdot \text{scal}(x,t)$ attains its global maximum at (x_0, t_0). This implies

$$\frac{\partial}{\partial t}\text{scal} + \frac{1}{t}\,\text{scal} = 0$$

at (x_0, t_0). Consequently, the set $\mathcal{N}_{(x_0,t_0)}$ is non-empty. Hence, it follows from Corollary 11 that the set $\mathcal{N}_{(x,t)}$ is non-empty for all points $(x,t) \in M \times \{t_0\}$. Using (9), we obtain

$$\mathcal{N}_{(x,t)} = \Big\{ \frac{\partial}{\partial t} + V_{(x,t)} \Big\} \tag{10}$$

for all points $(x,t) \in M \times \{t_0\}$. Hence, by Corollary 11, we have

$$\tilde{P}_\gamma \Big(\frac{\partial}{\partial t} + V_{\gamma(0)} \Big) = \frac{\partial}{\partial t} + V_{\gamma(1)}$$

for every smooth path $\gamma : [0,1] \to M \times \{t_0\}$. Thus, we conclude that

$$\tilde{D}_{\frac{\partial}{\partial x^i}} \Big(\frac{\partial}{\partial t} + V \Big) = 0$$

for all points $(x,t) \in M \times \{t_0\}$. From this, the assertion follows.

6. Ancient solutions to the Ricci flow

In this final section, we consider ancient solutions to the Ricci flow. In this case, we are able to remove the $1/t$ terms in the Harnack inequality:

Proposition 13. *Let $(M, g(t))$, $t \in (-\infty, T)$, be a family of complete Riemannian manifolds evolving under Ricci flow. We assume that $(M, g(t)) \times \mathbb{R}^2$ has nonnegative isotropic curvature for all $t \in (-\infty, T)$. Moreover, we assume that*

$$\sup_{(x,t) \in M \times (\alpha, T)} \text{scal}(x,t) < \infty$$

for all $\alpha \in (-\infty, T)$. Then we have

$$\frac{\partial}{\partial t}\text{scal} + 2\,\partial_i\text{scal}\,v^i + 2\,\text{Ric}(v,v) \geq 0$$

for all points $(x,t) \in M \times (-\infty, T)$ and all vectors $v \in T_xM$.

Proof. We employ an argument due to R. Hamilton [**11**]. To that end, we fix a real number $\alpha \in (-\infty, T)$, and apply Corollary 2 to the metrics $g(t+\alpha)$, $t \in (0, T-\alpha)$. This implies

$$\frac{\partial}{\partial t}\text{scal} + \frac{1}{t-\alpha}\,\text{scal} + 2\,\partial_i\text{scal}\,v^i + 2\,\text{Ric}(v,v) \geq 0$$

for all points $(x,t) \in M \times (\alpha, T)$ and all $v \in T_xM$. Taking the limit as $\alpha \to -\infty$, the assertion follows.

Our last result generalizes Theorem 1.1 in [**11**]:

Proposition 14. *Let $(M, g(t))$, $t \in (-\infty, T)$, be a family of complete Riemannian manifolds evolving under Ricci flow. For each $t \in (-\infty, T)$, we assume that $(M, g(t)) \times \mathbb{R}^2$ has nonnegative isotropic curvature and $(M, g(t))$ has positive Ricci curvature. Moreover, suppose that there exists a point $(x_0, t_0) \in M \times (-\infty, T)$ such that*

$$\text{scal}(x_0, t_0) = \sup_{(x,t) \in M \times (-\infty, T)} \text{scal}(x,t).$$

Then there exists a smooth vector field $V = V^j \frac{\partial}{\partial x^j}$ such that

$$D_{\frac{\partial}{\partial x^i}} V = \text{Ric}_i^j \frac{\partial}{\partial x^j}$$

for all $(x,t) \in M \times \{t_0\}$. In particular, $(M, g(t_0))$ is a steady Ricci soliton.

The proof of Proposition 14 is analogous to the proof of Proposition 12 above. The details are left to the reader.

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Stanford University
450 Serra Mall, Bldg 380
Stanford, CA 94305

E-mail address: brendle@math.stanford.edu