Pricing European Equity Options Based On Vasicek Interest Rate Model

First Authors: Yueran Ma, Hao Wen

Second Author: Wan Duan

High School Affiliated to Renmin University of China

September 2009

Abstract

Options is an important part of global financial market，with great influence on national economies. While most classic option pricing models are based on the assumption of a constant interest rate, economic data show that interest rates in reality frequently fluctuated under the influence of varying economic performances and monetary policies. As interest rate fluctuation is closely related to the value and expected return of options, it is worth discussing option pricing under stochastic interest rate models. Since 1990s, scholars home and abroad have been conducting researches on this topic and have formulated price formulas for some types of options. However, because the pricing process involves two stochastic variables, the majority of previous studies employed sophisticated methods. As a result, their price formulas were too complicated to provide straightforward explanations of the parameters' influence on option prices, unable to offer investors direct assistance.

This paper selects Vasicek interest rate model to describe interest rate's stochastic movement, and discusses the pricing of European equity options whose underlying asset's price follows Geometric Brownian Motions in a complete market. The paper's value and innovation lie in the following aspects: ① It improves and simplifies the pricing methods for options under stochastic interest rate models, applies comparatively primary mathematical methods, and attains concise price formulas; ② it conducts in-depth analysis of major parameters’ financial significance, which helps investors to make better investment decisions by estimating the variations in option prices corresponding to different parameters.

Key words: Pricing European Equity Options; Expected Return; Vasicek Model; Black-Scholes Equation; Geometric Brownian Motions

Contents

I. Introduction

1. Motivation for the Research

The options market has had enormous developments since 1970s. In 1973, Fisher Black and Myron Scholes proposed the ground-breaking Black-Scholes Model for option pricing, which, under the assumption that equity prices follow logarithmic normal distributions, provided the unique no-arbitrage solution to option prices. In the following decades, this model has been widely applied and further developed. One of the important assumptions of the classic Black-Scholes Model is a constant interest rate. However, interest rates in reality often fluctuate. For example, following the turbulences caused by the Financial Crisis in 2008, central banks, notably the Federal Reserve, have repeatedly lowered interest rates to spur the economy, resulting in a downward trend of interest rates. This particular phenomenon inspired us to focus on how interest rates' variations affect option pricing. Referring to historical data of Federal Reserve's bench-mark interest rates in the recent decade (shown below), we clearly see that interest rates fluctuated significantly in different times, which confirms the necessity to discuss option pricing under stochastic interest models.

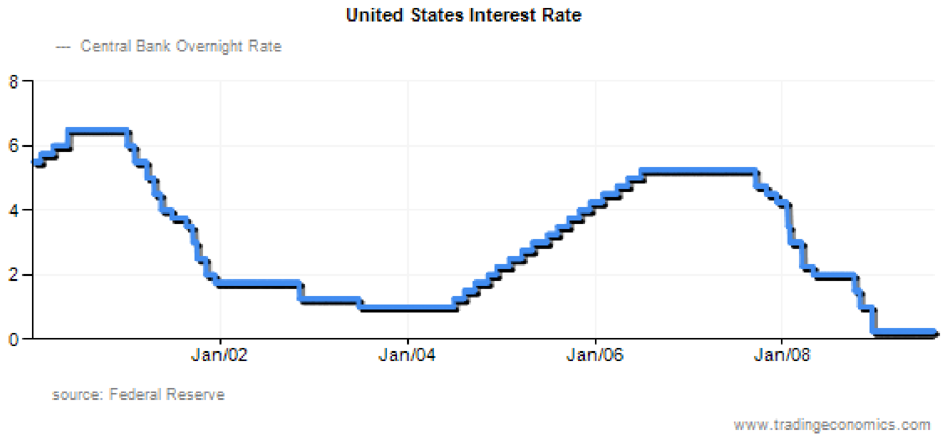


Chart 1 Central Bank Overnight Rate from 2002-2008

Table1 Federal Reserve Bench-mark Interest Rates from Jan.2006-Jan.2009

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2009	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25				
2008	3.00	3.00	2.25	2.00	2.00	2.00	2.00	2.00	2.00	1.00	1.00	0.25
2007	5.25	5.25	5.25	5.25	5.25	5.25	5.25	5.25	4.75	4.50	4.50	4.25
2006	4.25	4.50	4.50	4.75	4.75	5.00	5.25	5.25	5.25	5.25	5.25	5.25

2. The value, innovation and potential extension of this paper

(1) The practical significance for improving option pricing

Options are among the most traded products in financial markets. They are also the major tools investors use to reduce risks and make profits. Since options have rich varieties and sufficient mobility, their roles are becoming increasingly important. Improving option pricing has great importance for risk investment, cooperative securities pricing, financial engineering, investment optimization, mergers, and policy making.

(2) A brief review of previous studies

In 1990, John Hull and Alan White first proposed the pricing formula of European bond options under stochastic interest rate models. Their research initiated researches in this field. Since then, many scholars have applied different methods to calculate options prices under stochastic interest rate models. However, previous researches have limitations in several aspects. ①Most studies so far have focused on bond options, but only a few have discussed equity options. Since equity options' underlying assets are stocks, their pricing involves two stochastic processes. This characteristic makes the pricing of equity options more complex than that of bond options. ② The majority of previous studies employed complicated methods, such as Fourier Transforms, Quasi-Martingales Transforms, Extended Wiener Theorem and Ito Integral. These complex methods bring difficulties to analyzing the properties of the pricing formula.

(3) The value and significance of this paper

The proofs and deductions of theorems, lemmas and partial derivatives are all conducted by ourselves; the methods and conclusions are original.

This paper's significance and innovation lie in the following areas:

①This paper further discusses the pricing of European equity options; it directly describes the movement of equity prices under the framework of Geometric Brownian Motions, combines and simplifies the two stochastic processes with stochastic analysis techniques, uses comparatively primary mathematical methods, and formulates analytical pricing formulas;

② Based on the analytical price formulas and with the help of partial derivatives, this paper analyzes the influence on option prices corresponding to different parameters, and discusses their financial meanings. This analysis helps investors to estimate the changes of option prices once the parameters' values change;

③ Besides formulating price formulas, this paper also provides the option's expected returns, helping investors evaluate potential profits.

(4) The extension value of this paper's methods

The methods this paper applies can also simplify the pricing of other types of options under one-variable affine interest rate models and option pricing problems in markets that have more than one stochastic variable. Meanwhile, the methods may also be extended to European option pricing under multi-variable affine interest rate models.

II. Summary of Methods

1. Introduction of the models

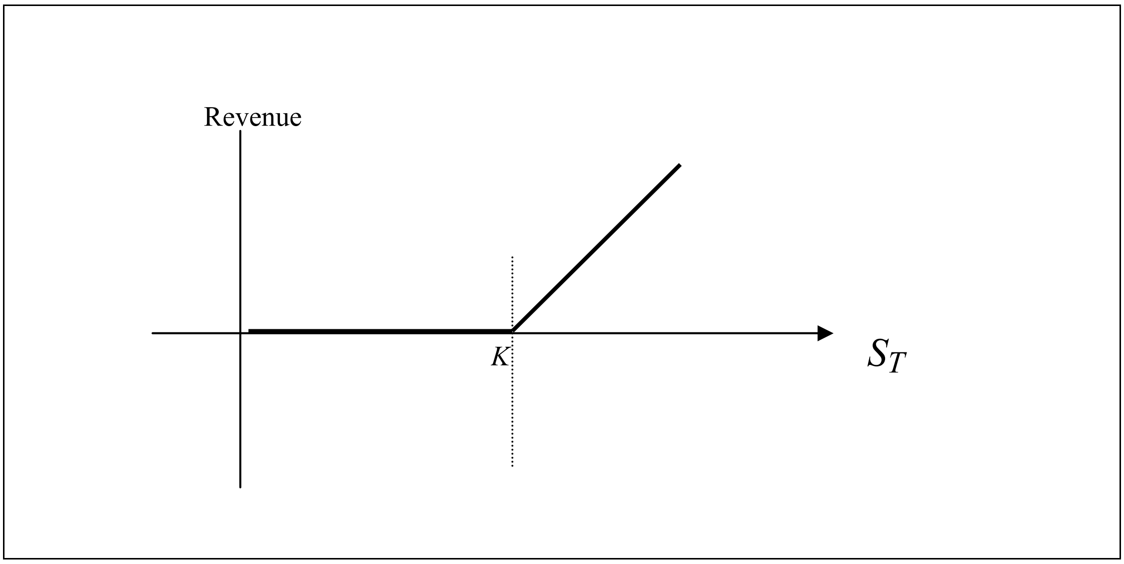
This paper aims to formulate the pricing formula of European options under Vasicek interest rate model. Generally, factors that influence option prices are values of the underlying assest (including its original value, mean and volatility), option's exercise time, strike price, interest rate and etc. Therefore, this paper needs to establish three major models: ①Asset Price Model, which describes the movement of underlying asset's prices, ② Interest Rate Model, which describes the movement of interest rates, and ③Option Pricing Model. The major theories included are Probability Theory, Geometric Brownian Motion, Black-Scholes Model, Vasicek Interest-rate Model Martingale Theory, Ordinary Differential Equation, and Stochastic Differential Equation.

2. The Asset Price Model

(1) European Equity Options

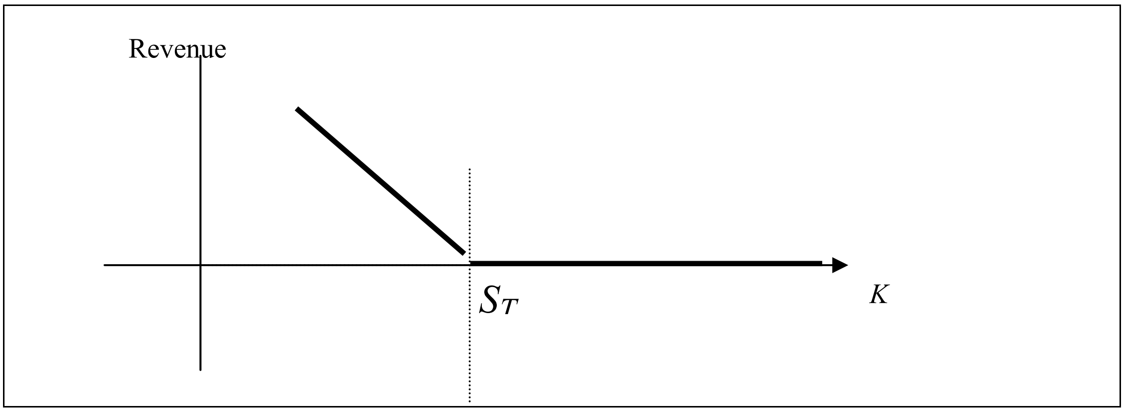
A European Equity Option is a contract between a buyer and a seller that gives the buyer the right—but not the obligation—to buy or to sell equities (whose prices are denoted by S_t) at a fixed time in the future (T) at a fixed price (Strike Price K). In return, the seller collects a payment (the premium) from the buyer.

For call options, if asset price S_T at time T is greater than the strike price K, buyers can choose to exercise the contract to buy equities at strike price K while selling at market price, and then profit from the difference, $(S_T - K)$, between the two prices. If asset price at T is less than or equal to K, buyers can choose not to exercise the contract, and contract will be of no value. Therefore, the return of the option at exercise time T is $(S_T - K)^+$ (as shown below).



Graph 2 Revenue of Call Options

Similarly, for put options, if asset price at T is less than K, buyers can choose to exercise the contract to sell equities at strike price K while buying at market price, then profit from the difference between the two prices $(K - S_T)$. If asset price S_T at time T is greater than or equal to strike price K, buyers can choose not to exercise. Therefore, the return of option at exercise time is $(K - S_T)^+$. (as shown below)



Graph 3 Revenue of Put Options

(2) Stock Price Processes

Geometric Brownian Motions is a widely applied model to describe stock price behavior. The definition for the basic Geometric Brownian Motions is the following: if the ratio of an asset's price at time *t* in the future to its present price is independent of previous prices and obeys logarithmic normal distribution with mean of μt and deviation of $\sigma^2 t$, the asset price process satisfies Geometric Brownian Motions. If stock price follows Geometric Brownian Motions, it satisfies the following stochastic differential equation: $dS_t = \mu S_t dt + \sigma S_t dB_t$

where μ stands for average yield, and σ stands for volatility.

Besides, pricing models under different risk probability measures are different. This paper includes Real Probability Measure and Risk-neutral Measure. The following paragraph briefly introduces the two situations.

Real Probability Measure is the probability space of the random variable used to describe stock prices. Risk-neutral Probability Measure is introduced to calculate the pricing of financial derivatives. When the market is assumed to be complete, there exists the unique Risk-neutral Probability Measure. Under Risk-neutral Probability Measure, the expected yield of all assets equals to the interest rate. Therefore, the price of financial derivatives can be obtained by calculating the expectation of risk-adjusted return in this probability space. Risk-neutral Probability Measure is equivalent to Objective Probability Measure; their relations can be obtained by calculating the market's risk prices.

①Under real probability measure P, stock price S_t satisfies the following stochastic differential equation:

$$dS_t = \mu_t S_t dt + \sigma S_t dB_t$$

in which μ_t is the expected yield, a deterministic function, and σ stands for volatility.

Usually, the following inequality holds,

$$\mu_t \geq E^P[r_t]$$

This inequality means that the expected yield of stocks should be greater than those of bank deposits. For an investor, if he holds a stock, this implies that he bears higher risks than depositing money, and his excepted yield, in return, is also higher than that of bank deposits.

②Under Risk-neutral Probability Measure Q， stock prices *St* satisfies:

$$dS_t = r_t S_t dt + \sigma S_t dB_t$$

It should be clearly stated that the option's expected return is calculated under Real Probability Measure while its price is calculated under

Risk-neutral Probability Measure.

3. Selection of Interest Rate Models

The most widely applied one-variable affine interest models are Merton, Vasicek and CIR models, which are all homogeneous affine models. In this section, we make a brief comparison of these interest rate models.

(1) Merton Model

Definition: In Merton Model, short-term interest rates can be described by the following stochastic differential equation:

$$dr_t = \alpha dt + \beta dW_t$$

where α and β are constants, denoting the drift and volatility of instantaneous interest rate, and W_t is a standard Weiner process, obeying a normal distribution with mean of 0 and variance of *T*.

Instantaneous interest rate at time T can also be expressed as

$$r_t = r_0 + \alpha t + \beta W_t$$

According to Merton, instantaneous interest rate r_t obeys normal distribution, with

$$E\left[r_t\right] = r_0 + \alpha t$$

Remarks:

Compared to Vasicek and CIR models, Merton Model is the simplest, favoring the analytical formulation of pricing formula. However, Merton Model is comparatively rough, with only the drift coefficient, and not as close to market behavior as the other models. Besides, the normal property of Merton Model doesn't obviate the possibilities of negative interest rates, which conflicts with real world economic situations.

(2) The Vasicek Model

Definition：Under Vasicek Model，interest rates can be described as:

$$dr_t = k\left(\theta - r_t\right)dt + \beta dW_t$$

where k and θ are positive constants.

Instantaneous interest rate at a future time follows

$$r_t = e^{-kt} r_0 + \theta\left(1 - e^{-kt}\right) + \int_0^t \beta e^{-k(t-u)} dW_u$$

where r_t is the interest rate at time t and r_0 is the present interest rate.

According to Vasicek, instantaneous interest rate process is also a normal process, satisfying normal distribution with mean

$$E\left[r_t\right] = \theta + \left(r_0 - \theta\right)e^{-kt}$$

and variance

$$Var\left[r_t\right] = \frac{\beta^2}{2k}\left(1 - e^{-2kt}\right)$$

Remarks

It is apparent from the definition that Vasicek Interest Rate Model has only one random variable, which means that the stochastic factors of every point along the return curve are absolutely related. The model is also an affine model, which means that the drift and volatility coefficients are 1 with regard to time (t).

Meanwhile, according to Vasicek, short-term interest rates have the property of average reversion; that is, the short-term interest rates tend to approach a reversion level representing people's expectation, the central bankers' target level or the long-term interest rate level. If $r_0 > \theta$, the drift coefficient is less than zero, and r_t will decrease. On the opposite, if $r_0<\theta$, the drift coefficient is greater than zero, and r_t will increase. The reversion property makes the model close to the movement of interest rates in the real-world; it also allows investors to calculate option prices under particular expectations by designing θ. In the fourth chapter, we will further discuss how θ affects the result of option prices.

Nevertheless, Vasicek Model has its drawbacks. First, during tiny time periods, changes of two different term structures are completely related. Secondly, as Vasicek Model is a normal model, nominal interest rates can also be negative. In spite of these limitations, Vasicek Model is still the most widely applied stochastic interest-rate model.

(3) CIR Model

In CIR model, instantaneous interest rate can be described as:

$$dr_t = k\left(\theta - r_t\right)dt + \beta\sqrt{r_t}\,dW_t$$

where k and θ are positive constants.

Remarks

The major differences between CIR model and Vasicek model are:

①Volatility in CIR model is inversely related to interest rate levels; that is, when interest rates are low, volatilities are low, and vice versa;

②Short-term interest rates are always positive in CIR model. However, interest rates in CIR model cannot be directly denoted by other variables but rely on the calculations of its expectation and variance. Since instantaneous interest-rates cannot be directly described, CIR model often makes the formulation of price formula very hard.

(4) Conclusion:

To get closest to real world market conditions, and at the same time obtain analytical price formulas, this paper applies Vasicek Model to describe the stochastic changes of interest-rate variation. The important properties of the Vasicek Model are: 1) average reversion tendencies and 2) the stochastic property.

4. The Option Pricing Model

The major models and methods applied to option pricing problems include: ①Traditional Option Pricing Model, ② Black-Scholes Option Pricing Model, ③ Binary Tree Model, ④ Monte-Carlo Simulations, ⑤ Finite Difference Method, ⑥Arbitrage Pricing Method and ⑦

Interval Pricing Method.

Having compared the methods above, we select Black-Scholes Model as our pricing model for three main reasons: ① it assumes that underlying asset's prices satisfy Geometric Brownian Motions (or logarithmic normal distribution), which agrees with the behavior of European equity option's underlying asset, stocks; ② Black-Scholes Model is straightforward and favors multiple transformations; ③ Black-Scholes Model is the most widely applied pricing method, and option pricing extension based on it can be easily accepted by investors.

The classic Black-Scholes Model has 7 important assumptions.

① Stock prices follow logarithmic normal distribution;

② During validity period, zero-risk interest rates and return of financial assets, μ, is constant;

③ The market is free of frictions, such as taxes and transaction costs; all securities can be perfectly divided;

④ The option is a European Option;

⑤ There is no zero-risk arbitrage.;

⑥ Stock transactions are constant;

⑦ Investors can get loans at zero-risk interest rates.

The classic Black-Scholes formula is:

$$\text{Price}^C = S(0)N(d_1) - e^{-rt}KN(d_2)$$

where $d_1 = \dfrac{rT + \frac{1}{2}\sigma^2 T + \ln\frac{S(0)}{K}}{\sigma\sqrt{T}}; d_2 = d_1 - \sigma\sqrt{T} = \dfrac{rT - \frac{1}{2}\sigma^2 T + \ln\frac{S(0)}{K}}{\sigma\sqrt{T}}$

$Price^C$ is the price of European call options, S_0 is stock price at present time, T is the exercise time and $N(x)$ is the density function of standard normal distribution.

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \quad ; \quad N(y) = \int_{-\infty}^{y} \Phi(x)dx = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

III. Formulas of Prices and Expected Returns

1. Illustration of Methods

This chapter is the main part of the paper and is divided into 3 major parts.

The first part proves several important identical equations under Vasicek Interest Rate Model. Using Fubini Theorem, stochastic differential theories, and transformations, we simplify the discount rate factor and other quantities related to interests, facilitating the calculation of expected return and price's formulas.

The second part deducts the formula of option's expected return under real probability measure P. First, according to the stochastic differential equation of stock prices under Real Probability Measure P, we get the analytic expression of stock prices at time *T*. Next, with the help of an indicator random variable *I*, we transform the sub-function $(S_T - K)^+$ into easier forms. Then, using integrals, we get the formulas for European call and put options' expected returns, as well as the call-put relation formula for expected returns.

The third part deducts options' price formulas under Risk-neutral Probability Measure Q. This part is similar to but more complicated than the second part. First, according to the stochastic differential equation of stick prices under Risk-neutral Probability Measure Q, we get the analytic expression of stock prices at time *T*. Next, with the help of an indicator random variable *I*, we transform the sub-function $(S_T - K)^+$ into easier forms. The conditions of *I*'s values under Q, however, are more complex, for two stochastic processes are involved in the inequalities. Then, using integrals, we get the price formulas of European call and put options, and call-put parity for option prices.

2. Important identical equations under Vasicek Interest Rate model.

According to the definition of the Vasicek Model,

$$dr_t = k\left(\theta - r_t\right)dt + \beta dW_t.$$

$$r_t = \mathrm{e}^{-kt} r_0 + \theta\left(1 - \mathrm{e}^{-kt}\right) + \int_0^t \beta \mathrm{e}^{-k(t-u)} \mathrm{d}W_u.$$

where r_t represents the instantaneous interest-rate at t, r_0 represents present interest-rate, θ represents the long-term interest-rate level or expected interest-rate level; k describes the speed at which r_t adjust to θ, β represents the volatility of interest rate, and W_t represents standard Wiener Motions, which is independent to the Brownian motion in stock price model. To facilitate the deductions of the main formulas, we start with proofs of two conclusions about Vasicek interest model

Theorem 1. Stochastic interest rate's integral to time is

$$\int_0^T r_t \mathrm{d}t = \theta T + \frac{r_0 - \theta}{k}\left(1 - \mathrm{e}^{-kT}\right) + \int_0^T \frac{\beta}{k}\left(1 - \mathrm{e}^{-k(T-u)}\right)\mathrm{d}W_u .$$

Proof：Using the integral of interest rate, we get,

$$\begin{aligned}
\int_0^T r_t \mathrm{d}t &= \int_0^T \left[\mathrm{e}^{-kt} r_0 + \theta\left(1 - \mathrm{e}^{-kt}\right) + \int_0^t \beta \mathrm{e}^{-k(t-u)} \mathrm{d}W_u\right]\mathrm{d}t \\
&= \int_0^T \left[\mathrm{e}^{-kt} r_0 + \theta\left(1 - \mathrm{e}^{-kt}\right)\right]\mathrm{d}t + \int_0^T \int_0^t \beta \mathrm{e}^{-k(t-u)} \mathrm{d}W_u \mathrm{d}t \\
&= \int_0^T \left[\mathrm{e}^{-kt} r_0 + \theta\left(1 - \mathrm{e}^{-kt}\right)\right]\mathrm{d}t + \int_0^T \int_u^T \beta \mathrm{e}^{-k(t-u)} \mathrm{d}t \mathrm{d}W_u \\
&= \theta T + \frac{r_0 - \theta}{k}\left(1 - \mathrm{e}^{-kT}\right) + \int_0^T \frac{\beta}{k}\left(1 - \mathrm{e}^{-k(T-u)}\right)\mathrm{d}W_u .
\end{aligned}$$

Explanation: The third line of the above proof uses Fubini Formula and exchanges two integrals.

Theorem 2. The expectations of zero-risk asset's expected yield and interest rate's discount factor:

$$E[\mathrm{e}^{\int_0^T r_t dt}] = \mathrm{e}^{\theta T + \frac{r_0 - \theta}{k}\left(1 - \mathrm{e}^{-kT}\right) + \frac{\beta^2 T}{2k^2} - \frac{\beta^2}{4k^3}\left(3 - 4\mathrm{e}^{-kT} + \mathrm{e}^{-2kT}\right)}.$$

$$E[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t}] = \mathrm{e}^{-\theta T - \frac{r_0 - \theta}{k}\left(1 - \mathrm{e}^{-kT}\right) + \frac{\beta^2 T}{2k^2} - \frac{\beta^2}{4k^3}\left(3 - 4\mathrm{e}^{-kT} + \mathrm{e}^{-2kT}\right)}.$$

Proof: According to theories of exponent martingale, we get,

$$E[\mathrm{e}^{\int_0^s f(u) \mathrm{d}W_u}] = \mathrm{e}^{\frac{1}{2}\int_0^s f(u)^2 \mathrm{d}u}.$$

Using this formula, we obtain the expectation of zero-risk asset's expected yield:

$$E[\mathrm{e}^{\int_0^T r_t \mathrm{d}t}] = E[\mathrm{e}^{\theta T + \frac{r_0-\theta}{k}\left(1-\mathrm{e}^{-kT}\right)}]E[\mathrm{e}^{\int_0^T \frac{\beta}{k}\left(1-\mathrm{e}^{-k(T-u)}\right)\mathrm{d}W_u}]$$

$$= \mathrm{e}^{\theta T + \frac{r_0-\theta}{k}\left(1-\mathrm{e}^{-kT}\right)}\mathrm{e}^{\frac{\beta^2}{2k^2}\int_0^T \left(1-\mathrm{e}^{-k(T-u)}\right)^2 \mathrm{d}u}$$

$$= \mathrm{e}^{\theta T + \frac{r_0-\theta}{k}\left(1-\mathrm{e}^{-kT}\right)+\frac{\beta^2 T}{2k^2}-\frac{\beta^2}{4k^3}\left(3-4\mathrm{e}^{-kT}+\mathrm{e}^{-2kT}\right)}.$$

Similarly, we obtain expectation of interest rate's discount factor:

$$E[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t}] = E[\mathrm{e}^{-\theta T - \frac{r_0-\theta}{k}\left(1-\mathrm{e}^{-kT}\right)}]E[\mathrm{e}^{-\int_0^T \frac{\beta}{k}\left(1-\mathrm{e}^{-k(T-u)}\right)\mathrm{d}W_u}]$$

$$= \mathrm{e}^{-\theta T - \frac{r_0-\theta}{k}\left(1-\mathrm{e}^{-kT}\right)}\mathrm{e}^{\frac{\beta^2}{2k^2}\int_0^T \left(1-\mathrm{e}^{-k(T-u)}\right)^2 \mathrm{d}u}$$

$$= \mathrm{e}^{-\theta T - \frac{r_0-\theta}{k}\left(1-e^{-kT}\right)+\frac{\beta^2 T}{2k^2}-\frac{\beta^2}{4k^3}\left(3-4\mathrm{e}^{-kT}+\mathrm{e}^{-2kT}\right)}.$$

To simplify calculation, we define：

$$H = \theta T + \frac{r_0 - \theta}{k}\left(1 - \mathrm{e}^{-kT}\right).$$

$$G = \frac{\beta^2}{k^2}\int_0^T \left(1 - \mathrm{e}^{-k(T-u)}\right)^2 \mathrm{d}u = \frac{\beta^2 T}{k^2} - \frac{\beta^2}{2k^3}\left(3 - 4\mathrm{e}^{-kT} + \mathrm{e}^{-2kT}\right).$$

$$X_G = \frac{1}{\sqrt{G}}\int_0^T \frac{\beta}{k}\left(1 - \mathrm{e}^{-k(T-u)}\right)\mathrm{d}W_u. \qquad Y_T = \frac{1}{\sqrt{T}}B_T.$$

According to the stochastic differential equation formula, we have

$$\int_0^T \frac{\beta}{k}\left(1 - \mathrm{e}^{-k(T-u)}\right)\mathrm{d}W_u \sim N\left(0, \int_0^T \left(1 - \mathrm{e}^{-k(T-u)}\right)^2 \mathrm{d}u\right).$$

$$B_T \sim N\left(0, T\right).$$

$$Var\left(X_G\right) = \frac{1}{G}\int_0^T \left(1 - \mathrm{e}^{-k(T-u)}\right)^2 \mathrm{d}u = 1.$$

$$Var(Y_T) = \frac{1}{T}T = 1.$$

In above deductions, we have proved that X_G and $\frac{1}{\sqrt{G}}\int_0^T \frac{\beta}{k}\left(1-\mathrm{e}^{-k(T-u)}\right)\mathrm{d}W_u$ have the same distribution, we use X_G to replace $\frac{1}{\sqrt{G}}\int_0^T \frac{\beta}{k}\left(1-\mathrm{e}^{-k(T-u)}\right)\mathrm{d}W_u$

Therefore, Theorems 1 and 2 can be re-written as:

Theorem 1'.

$$\int_0^T r_t \mathrm{d}t = H + \sqrt{G}\, X_G .$$

Theorem2'.

$$E[\mathrm{e}^{\int_0^T r_t \mathrm{d}t}] = \mathrm{e}^{H+\frac{1}{2}G}, \quad E[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t}] = \mathrm{e}^{-H+\frac{1}{2}G}.$$

3. Option's Expected Return under Real Probability Measure P

Theorem 3. Under Real Probability Measure P, stock prices satisfy:

$$S_T = S_0 \mathrm{e}^{\int_0^T \mu_t \mathrm{d}t + \sigma B_T - \frac{1}{2}\sigma^2 T}. \qquad (3.1)$$

$$E^P[S_T] = S_0 \mathrm{e}^{\int_0^T \mu_t \mathrm{d}t}. \qquad (3.2)$$

Proof：Formula(3.1) is obtained from the stochastic differential equation of stick prices under Real Probability Measure P,

$$\mathrm{d}S_t = \mu_t S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t .$$

and Itô Formula. It is frequently applied to financial engineering.

Since the proof of (3.1) is not the emphasis of this paper, the proof is omitted here. We will only prove (3.2) under the assumption of (3.1).

According to the formula of stock price S_T given by （3.1）, we get

$$
\begin{aligned}
E^P[S_T] &= E^P[S_0 \mathrm{e}^{\int_0^T \mu_t \mathrm{d}t + \sigma B_T - \frac{1}{2}\sigma^2 T}] \\
&= S_0 \mathrm{e}^{-\frac{1}{2}\sigma^2 T} \mathrm{e}^{\int_0^T \mu_t \mathrm{d}t} E^P[\mathrm{e}^{\sigma\sqrt{T} Y_T}] \\
&= S_0 \mathrm{e}^{-\frac{1}{2}\sigma^2 T} \mathrm{e}^{\int_0^T \mu_t \mathrm{d}t} \mathrm{e}^{\frac{1}{2}\sigma^2 T} \\
&= S_0 \mathrm{e}^{\int_0^T \mu_t \mathrm{d}t}.
\end{aligned}
$$

Theorem 4 The expected return of European Call Options is

$$
R^C = \mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t} \left(S_0 N(w) - K \mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t} N(w - \sigma\sqrt{T}) \right).
$$

where $w = \dfrac{\int_0^T \mu_t \mathrm{d}t + \frac{1}{2}\sigma^2 T + \ln \frac{S_0}{K}}{\sigma\sqrt{T}}$.

Proof：First of all, we introduce I, which is an indicator random variable，satisfying

$$
I = \begin{cases} 1, & S_T > K. \\ 0, & S_T \le K. \end{cases}
$$

Then, we analyze the conditions in which I equals 1.

$$
I = \begin{cases} 1, & S_T > \mathrm{K} \Leftrightarrow \int_0^T \mu_t \mathrm{d}t + \sigma\sqrt{T} Y_T - \frac{1}{2}\sigma^2 \mathrm{T} > \ln\frac{K}{S_0} \Leftrightarrow Y_T > \dfrac{\ln\frac{K}{S_0} + \frac{1}{2}\sigma^2 T - \int_0^T \mu_t \mathrm{d}t}{\sigma\sqrt{T}} = \sigma\sqrt{T} - w. \\ 0, & S_T \le \mathrm{K} \Leftrightarrow Y_T \le \sigma\sqrt{T} - w. \end{cases}
$$

Based on the above formula, we get,

$$
\begin{aligned}
R^C &= E^P[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t} I(S_T - K)] \\
&= E^P[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t}]\left(E^P[IS_T] - E^P[IK]\right) \\
&= E^P[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t}]\left(S_0 \mathrm{e}^{\int_0^T \mu_t \mathrm{d}t} E^P[I\mathrm{e}^{\sigma\sqrt{T}Y_T - \frac{1}{2}\sigma^2 T}] - KE^P[I]\right) \\
&= \mathrm{e}^{-H+\frac{1}{2}G} \mathrm{e}^{\int_0^T \mu_t \mathrm{d}t}\left(S_0 \int_{\sigma\sqrt{T}-w}^{+\infty} \mathrm{e}^{\sigma\sqrt{T}y - \frac{1}{2}\sigma^2 T} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}} \mathrm{d}y - K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t} \int_{\sigma\sqrt{T}-w}^{+\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}} \mathrm{d}y\right) \\
&= \mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t dt}\left(S_0 \int_{\sigma\sqrt{T}-w}^{+\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{\left(y-\sigma\sqrt{T}\right)^2}{2}} \mathrm{d}y - K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t} \int_{\sigma\sqrt{T}-w}^{+\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}} \mathrm{d}y\right) \\
&= \mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(S_0 \int_{-w}^{+\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}} \mathrm{d}y - K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t} \int_{\sigma\sqrt{T}-w}^{+\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}} \mathrm{d}y\right) \\
&= \mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(S_0 \int_{-\infty}^{w} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}} \mathrm{d}y - K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t} \int_{-\infty}^{w-\sigma\sqrt{T}} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}} \mathrm{d}y\right) \\
&= \mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(S_0 N(w) - K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t} N(w-\sigma\sqrt{T})\right).
\end{aligned}
$$

Theorem 5. Call-Put Relation Formula of Expected Returns

$$
R^C - R^P = \mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(S_0 - K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t}\right).
$$

Proof: From the financial definition of call and put options,

$$
R^C = E^P[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t} I(S_T - K)], \quad R^P = E^P[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t} (I-1)(S_T - K)]
$$

Therefore,

$$
\begin{aligned}
R^C - R^P &= E^P[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t} I(S_T - K)] - E^P[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t} (I-1)(S_T - K)] \\
&= E^P[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t} (S_T - K)] = E^P[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t}]\left(E^P[S_T] - K\right) \\
&= \mathrm{e}^{-H+\frac{1}{2}G}\left(S_0 \mathrm{e}^{\int_0^T \mu_t \mathrm{d}t} - K\right) = \mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(S_0 - K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t}\right).
\end{aligned}
$$

This formula establishes the connection between a call option and a put option with the same exercise time and strike price. It is very helpful for formulating the expected returns of European put options.

Theorem 6 The Expected Return of Put Options

$$R^P = \mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t} N(\sigma\sqrt{T}-w) - S_0 N(-w)\right).$$

Proof: From Theorems 4 and 5, we get

$$\begin{aligned}
R^P &= R^C - \left(R^C - R^P\right) \\
&= \mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(S_0 N(w) - K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t} N(w-\sigma\sqrt{T})\right) - \mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(S_0 - K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t}\right) \\
&= \mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t}\left(1 - N(w-\sigma\sqrt{T})\right) - S_0\left(1 - N(w)\right)\right) \\
&= \mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t} N(\sigma\sqrt{T}-w) - S_0 N(-w)\right).
\end{aligned}$$

In this section, we have obtained the formulas for the expected returns of European call and put options and call-put relation formula of expected return.

4. Price Formula under Risk-neutral Probability Measure Q

In this section, we discuss option pricing under Vasicek interest rate model.

Theorem 7 Under Risk-neutral Probability Measure, stock price satisfies:

$$S_T = S_0 \mathrm{e}^{\int_0^T r_t \mathrm{d}t + \sigma B_T - \frac{1}{2}\sigma^2 T}. \qquad (7.1)$$

$$E^Q[S_T] = S_0 \mathrm{e}^{H+\frac{1}{2}G}. \qquad (7.2)$$

For the definition of H and G, please refer to the proof of Theorem 1.

Proof:

The proofs of Theorems 7 and 3 are similar; they are only slightly different in the forms of differential equations.

Under Risk-neutral Probability Measure Q, stock prices follow

$$\mathrm{d}S_t = r_t S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t.$$

(7.1) can be obtained with the equation above and theories of stochastic differential equation.

(7.2) can be proved on the basis of (7.1) in the following way:

$$\begin{aligned} E^Q[S_T] &= E^Q[S_0 \mathrm{e}^{\int_0^T r_t \mathrm{d}t + \sigma\sqrt{T}Y_T - \frac{1}{2}\sigma^2 T}] \\ &= S_0 \mathrm{e}^{-\frac{1}{2}\sigma^2 T} E^Q[\mathrm{e}^{\int_0^T r_t \mathrm{d}t}] E^Q[\mathrm{e}^{\sigma\sqrt{T}Y_T}] \\ &= S_0 \mathrm{e}^{-\frac{1}{2}\sigma^2 T} \mathrm{e}^{H + \frac{1}{2}G} \mathrm{e}^{\frac{1}{2}\sigma^2 T} \\ &= S_0 \mathrm{e}^{H + \frac{1}{2}G}. \end{aligned}$$

To deduct the price formula of European call options (see Theorem 10), we need to first prove theorems 8 and 9.

Theorem 8

$$E^Q[I\mathrm{e}^{\sigma\sqrt{T}Y_T - \frac{1}{2}\sigma^2 T}] = \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right)\mathrm{d}x.$$

Here $F = \ln\frac{K}{S_0} + \frac{1}{2}\sigma^2 T - H$.

Proof:

Define I as an indicator random variable，satisfying

$$I = \begin{cases} 1, & S_T > K. \\ 0, & S_T \le K. \end{cases}$$

Under Risk-neutral Probability Measure Q, since there are two stochasic processes in the inequalities, the values of I are more complicated than those under Real Probability Measure.

$$I=\begin{cases}1, & S_T>K\Leftrightarrow H-\frac{1}{2}\sigma^2T+\sqrt{G}X_G+\sigma\sqrt{T}Y_T>\ln\frac{K}{S_0}\Leftrightarrow\sqrt{G}X_G+\sigma\sqrt{T}\,Y_T>F.\\ 0, & S_T<K\Leftrightarrow\sqrt{G}X_G+\sigma\sqrt{T}Y_T\le F.\end{cases}$$

$$\begin{aligned}
E^Q[I\mathrm{e}^{\sigma\sqrt{T}Y_T-\frac{1}{2}\sigma^2T}] &= \int_y\int_x P(Y_T=y)P(X_G=x)(I\mathrm{e}^{\sigma\sqrt{T}Y_T-\frac{1}{2}\sigma^2T}\big|_{X_G=x,Y_T=y})\\
&= \int_y P(Y_T=y)\mathrm{e}^{\sigma\sqrt{T}y-\frac{1}{2}\sigma^2T}\int_x (I\big|_{X_G=x,Y_T=y})P(X_G=x)\\
&= \int_y P(Y_T=y)\mathrm{e}^{\sigma\sqrt{T}y-\frac{1}{2}\sigma^2T}\left(\int_{-\infty}^{\frac{F-\sigma\sqrt{T}y}{\sqrt{G}}}0\Phi(x)\mathrm{d}x+\int_{\frac{F-\sigma\sqrt{T}y}{\sqrt{G}}}^{+\infty}1\Phi(x)\mathrm{d}x\right)\\
&= \int_y P(Y_T=y)\mathrm{e}^{\sigma\sqrt{T}y-\frac{1}{2}\sigma^2T}\int_{\frac{F-\sigma\sqrt{T}y}{\sqrt{G}}}^{+\infty}\Phi(x)\mathrm{d}x\\
&= \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}\mathrm{e}^{\sigma\sqrt{T}y-\frac{1}{2}\sigma^2T}\int_{-\infty}^{\frac{\sigma\sqrt{T}y-F}{\sqrt{G}}}\Phi(x)\mathrm{d}x\mathrm{d}y\\
&= \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(y-\sigma\sqrt{T})^2}{2}}N\left(\frac{\sigma\sqrt{T}y-F}{\sqrt{G}}\right)\mathrm{d}y\\
&= \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(y-\sigma\sqrt{T})^2}{2}}N\left(\frac{\sigma\sqrt{T}(y-\sigma\sqrt{T})+\sigma^2T-F}{\sqrt{G}}\right)\mathrm{d}y\\
&= \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}N\left(\frac{\sigma\sqrt{T}y+\sigma^2T-F}{\sqrt{G}}\right)\mathrm{d}y\\
&= \int_{-\infty}^{+\infty}\Phi(y)N\left(\frac{\sigma\sqrt{T}y+\sigma^2T-F}{\sqrt{G}}\right)\mathrm{d}y.
\end{aligned}$$

Define $x=\frac{\sigma\sqrt{T}y+\sigma^2T-F}{\sqrt{G}},\ y=\frac{\sqrt{G}x+F-\sigma^2T}{\sigma\sqrt{T}}.$

Notice: x here is not the same variable as used previous equations but a new mark introduced, however, y is still the same. Based on the top left equation that defines x, we get the top right equation expressing y with x.

Therefore,

$$E^Q[I\mathrm{e}^{\sigma\sqrt{T}Y_T-\frac{1}{2}\sigma^2T}]=\int_{-\infty}^{+\infty}\Phi(y)N\left(\frac{\sigma\sqrt{T}y+\sigma^2T-F}{\sqrt{G}}\right)\mathrm{d}y$$

$$=\int_{-\infty}^{+\infty}N\left(\frac{\sigma\sqrt{T}y+\sigma^2T-F}{\sqrt{G}}\right)\mathrm{d}\left(N(y)\right)$$

$$=N\left(\frac{\sigma\sqrt{T}y+\sigma^2T-F}{\sqrt{G}}\right)N(y)\Big|_{-\infty}^{+\infty}-\int_{-\infty}^{+\infty}N(y)\mathrm{d}\left(N\left(\frac{\sigma\sqrt{T}y+\sigma^2T-F}{\sqrt{G}}\right)\right)$$

$$=1-\int_{-\infty}^{+\infty}N\left(\frac{\sqrt{G}x+F-\sigma^2T}{\sigma\sqrt{T}}\right)\mathrm{d}\left(N(x)\right)$$

$$=N(x)\Big|_{-\infty}^{+\infty}-\int_{-\infty}^{+\infty}N\left(\frac{\sqrt{G}x+F-\sigma^2T}{\sigma\sqrt{T}}\right)\mathrm{d}\left(N(x)\right)$$

$$=\int_{-\infty}^{+\infty}\left(1-N\left(\frac{\sqrt{G}x+F-\sigma^2T}{\sigma\sqrt{T}}\right)\right)\mathrm{d}\left(N(x)\right)$$

$$=\int_{-\infty}^{+\infty}N\left(\frac{\sigma^2T-F-\sqrt{G}x}{\sigma\sqrt{T}}\right)\mathrm{d}\left(N(x)\right)$$

$$=\int_{-\infty}^{+\infty}\Phi(x)N\left(\frac{\sigma^2T-F-\sqrt{G}x}{\sigma\sqrt{T}}\right)\mathrm{d}x.$$

Theorem 9

$$E^Q[I\mathrm{e}^{-H-\sqrt{G}X_G}]=\mathrm{e}^{-H+\frac{1}{2}G}\int_{-\infty}^{+\infty}\Phi(x)N(\frac{\sqrt{G}x-G-F}{\sigma\sqrt{T}})\mathrm{d}x.$$

Proof： With the same method, we get,

$$
\begin{aligned}
E^{Q}[I\mathrm{e}^{-H-\sqrt{G}X_G}] &= \int_x\int_y P(X_G=x)P(Y_T=y)(I\mathrm{e}^{-H-\sqrt{G}X_G}\big|_{X_G=x,Y_T=y}) \\
&= \int_x P(X_G=x)\mathrm{e}^{-H-\sqrt{G}x}\int_y (I\big|_{X_G=x,Y_T=y})P(Y_T=y) \\
&= \int_x P(X_G=x)\mathrm{e}^{-H-\sqrt{G}x}\left(\int_{-\infty}^{\frac{F-\sqrt{G}x}{\sigma\sqrt{T}}} 0\Phi(y)\mathrm{d}y+\int_{\frac{F-\sqrt{G}x}{\sigma\sqrt{T}}}^{+\infty} 1\Phi(y)\mathrm{d}y\right) \\
&= \int_{-\infty}^{+\infty}\Phi(x)\mathrm{e}^{-H-\sqrt{G}x}\left(\int_{\frac{F-\sqrt{G}x}{\sigma\sqrt{T}}}^{+\infty}\Phi(y)\mathrm{d}y\right)\mathrm{d}x \\
&= \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}\mathrm{e}^{-H-\sqrt{G}x}\left(\int_{-\infty}^{\frac{\sqrt{G}x-F}{\sigma\sqrt{T}}}\Phi(y)\mathrm{d}y\right)\mathrm{d}x \\
&= \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\left(x+\sqrt{G}\right)^2}{2}}\mathrm{e}^{-H+\frac{G}{2}}N\left(\frac{\sqrt{G}x-F}{\sigma\sqrt{T}}\right)\mathrm{d}x \\
&= \mathrm{e}^{-H+\frac{G}{2}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\left(x+\sqrt{G}\right)^2}{2}}N\left(\frac{\sqrt{G}(x+\sqrt{G})-G-F}{\sigma\sqrt{T}}\right)\mathrm{d}x \\
&= \mathrm{e}^{-H+\frac{G}{2}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}N\left(\frac{\sqrt{G}x-G-F}{\sigma\sqrt{T}}\right)\mathrm{d}x \\
&= \mathrm{e}^{-H+\frac{G}{2}}\int_{-\infty}^{+\infty}\Phi(x)N\left(\frac{\sqrt{G}x-G-F}{\sigma\sqrt{T}}\right)\mathrm{d}x.
\end{aligned}
$$

Theorem 10 Price Formula of European Call Options under Vasicek Interest Rate Model

$$
\mathrm{Price}^{\mathrm{C}}=S_0\int_{-\infty}^{+\infty}\Phi\left(x\right)N(\frac{\sigma^2T-F-\sqrt{G}x}{\sigma\sqrt{T}})\mathrm{d}x-K\mathrm{e}^{-H+\frac{1}{2}G}\int_{-\infty}^{+\infty}\Phi\left(x\right)N(\frac{\sqrt{G}x-G-F}{\sigma\sqrt{T}})\mathrm{d}x.
$$

Proof: According to financial theories, under Risk-neutral Probability, the expectation of an option's discounted returns equal to its price, which is $\mathrm{Price}^{\mathrm{C}}=E^{Q}[\mathrm{e}^{-\int_0^T r_t\mathrm{d}t}\left(S_T-K\right)^{+}]$.

According to Theorems 8 and 9,

$$
\begin{aligned}
\text{Price}^{\text{C}} &= E^{Q}[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t}\left(S_T-K\right)^{+}] = E^{Q}[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t} I\left(S_T-K\right)] \\
&= E^{Q}[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t} IS_T] - E^{Q}[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t} IK] \\
&= E^{Q}[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t} IS_0 \mathrm{e}^{\int_0^T r_t \mathrm{d}t + \sigma\sqrt{T}Y_T - \frac{1}{2}\sigma^2 T}] - E^{Q}[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t} IK] \\
&= S_0 E^{Q}[I\mathrm{e}^{\sigma\sqrt{T}Y_T - \frac{1}{2}\sigma^2 T}] - KE^{Q}[\mathrm{e}^{-H-\sqrt{G}X_G} I] \\
&= \int_{-\infty}^{+\infty} \Phi\left(x\right) N\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right)\mathrm{d}x - K\mathrm{e}^{-H+\frac{1}{2}G}\int_{-\infty}^{+\infty} \Phi\left(x\right) N\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right)\mathrm{d}x.
\end{aligned}
$$

Theorem 11. **Call-Put Relation Formula under Vasicek Interest-rate Model**

$$
\text{Price}^{\text{C}} - \text{Price}^{\text{P}} = S_0 - K\mathrm{e}^{-H+\frac{1}{2}G}.
$$

Proof：According to the definition of options, we get,

$$
\text{Price}^{\text{C}} = E^{Q}[\mathrm{e}^{-\int_0^T r_t dt}\left(S_T-K\right)^{+}]. \quad \text{Price}^{\text{P}} = E^{Q}[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t}\left(K-S_T\right)^{+}].
$$

$$
\begin{aligned}
\text{Price}^{\text{C}} - \text{Price}^{\text{P}} &= E^{Q}[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t}\left(S_T-K\right)^{+}] - E^{Q}[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t}\left(K-S_T\right)^{+}] \\
&= E^{Q}[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t} I\left(S_T-K\right)] - E^{Q}[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t}(1-I)\left(K-S_T\right)] \\
&= E^{Q}[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t}\left(S_T-K\right)] = E^{Q}[S_0 \mathrm{e}^{\sigma\sqrt{T}Y_G - \frac{1}{2}\sigma^2 T}] - E^{Q}[K\mathrm{e}^{-H-\sqrt{G}X_G}] \\
&= S_0 - K\mathrm{e}^{-H+\frac{1}{2}G}.
\end{aligned}
$$

This formula establishes the connection between a call option and a put option with the same exercise time and strike price. It is very helpful for formulating the price formula of European put options.

Theorem 12. **Price Formula of European Put Options under Vasicek Interest Rate Model**

$$
\text{Price}^{\text{P}} = K\mathrm{e}^{-H+\frac{1}{2}G}\int_{-\infty}^{+\infty} \Phi\left(x\right) N\left(\frac{G+F-\sqrt{G}x}{\sigma\sqrt{T}}\right)\mathrm{d}x - S_0\int_{-\infty}^{+\infty} \Phi\left(x\right) N\left(\frac{\sqrt{G}x+F-\sigma^2 T}{\sigma\sqrt{T}}\right)\mathrm{d}x.
$$

Proof: According to Theorems 10 and 11, we get,

$$\begin{aligned}
\mathrm{Price}^{\mathrm{P}} &= \mathrm{Price}^{\mathrm{C}} - \left(\mathrm{Price}^{\mathrm{C}} - \mathrm{Price}^{\mathrm{P}}\right) \\
&= S_0 \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right) \mathrm{d}x - S_0 \\
&\qquad + K\mathrm{e}^{-H+\frac{1}{2}G} - K\mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right) \mathrm{d}x \\
&= K\mathrm{e}^{-H+\frac{1}{2}G} \left[1 - \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right) \mathrm{d}x\right] \\
&\qquad - S_0 \left[1 - \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right) \mathrm{d}x\right] \\
&= K\mathrm{e}^{-H+\frac{1}{2}G} \left[N(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} N\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right) \mathrm{d}\left(N(x)\right)\right] \\
&\qquad - S_0 \left[N(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} N\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right) \mathrm{d}\left(N(x)\right)\right] \\
&= K\mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \left[1 - N\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right)\right] \mathrm{d}\left(N(x)\right) \\
&\qquad - S_0 \int_{-\infty}^{+\infty} \left[1 - N\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right)\right] \mathrm{d}\left(N(x)\right) \\
&= K\mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} N\left(\frac{G + F - \sqrt{G}x}{\sigma\sqrt{T}}\right) \mathrm{d}\left(N(x)\right) \\
&\qquad - S_0 \int_{-\infty}^{+\infty} N\left(\frac{F + \sqrt{G}x - \sigma^2 T}{\sigma\sqrt{T}}\right) \mathrm{d}\left(N(x)\right) \\
&= K\mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{G + F - \sqrt{G}x}{\sigma\sqrt{T}}\right) \mathrm{d}x \\
&\qquad - S_0 \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{F + \sqrt{G}x - \sigma^2 T}{\sigma\sqrt{T}}\right) \mathrm{d}x.
\end{aligned}$$

So far, we have obtained the price formulas for European call and put options and the call-put parity formula.

IV. Analysis of Financial Significance

This chapter uses partial derivatives to analyze the risk indexes of the formulas for expected returns and option prices, and explains how the changes of the parameters' values influence the options' expected returns and prices. This analysis will help investors adjust their investment strategies to new market situations.

1. Risk Index of Expected Returns

To facilitate deduction, we first prove Theorem 13.

Theorem 13. $S_0\Phi(w) = \mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t} K\Phi(w-\sigma\sqrt{T})$.

Proof:

$$\begin{aligned}
S_0\Phi(w) &= S_0 \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{w^2}{2}} \\
&= S_0 \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{(w-\sigma\sqrt{T})^2}{2}} \mathrm{e}^{-w\sigma\sqrt{T}+\frac{1}{2}\sigma^2 T} \\
&= \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{(w-\sigma\sqrt{T})^2}{2}} S_0 \mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t - \frac{1}{2}\sigma^2 T - \ln\frac{S_0}{K} + \frac{1}{2}\sigma^2 T} \\
&= \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{(w-\sigma\sqrt{T})^2}{2}} \mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t} S_0 \mathrm{e}^{-\ln\frac{S_0}{K}} \\
&= \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{(w-\sigma\sqrt{T})^2}{2}} \mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t} K \\
&= \mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t} K\Phi(w-\sigma\sqrt{T}).
\end{aligned}$$

Theorem 13 will be very helpful in later partial derivative calculations, and will be repeatedly referred to.

Theorem 14.

The call option's expected return R^C decreases as present interest rate r_0 increases; the put option's expected return R^P decreases as present interest rate r_0 increases, which is,

$$\frac{\partial R^C}{\partial r_0} < 0 \quad , \quad \frac{\partial R^P}{\partial r_0} < 0 .$$

Proof：Calculating *R*'s partial derivative of r, we get,

$$\begin{aligned}\frac{\partial R^C}{\partial r_0} &= R^C \frac{\partial(-H)}{\partial r_0} + \frac{\partial w}{\partial r_0} \mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t} \left(S_0 \Phi(w) - K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t} \Phi(w-\sigma\sqrt{T}) \right) \\ &= -\frac{\partial H}{\partial r_0} R^C = -\frac{1}{k}\left(1-\mathrm{e}^{-kT}\right) R^C < 0.\end{aligned}$$

$$\begin{aligned}\frac{\partial R^P}{\partial r_0} &= \frac{\partial R^C}{\partial r_0} - \frac{\partial\left(R^C - R^P\right)}{\partial r_0} = -R^C \frac{\partial H}{\partial r_0} - \frac{\partial\left(\mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t} \left(S_0 - K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t} \right) \right)}{\partial r_0} \\ &= -\frac{\partial H}{\partial r_0} R^C + \frac{\partial H}{\partial r_0}\left(R^C - R^P\right) = -\frac{\partial H}{\partial r_0} R^P < 0.\end{aligned}$$

Theorem 15

The call option's expected return R^C decreases as the long-term interest rate θ increases; the put option's expected return R^P decreases as the long-term interest rate θ increases, which is,

$$\frac{\partial R^C}{\partial \theta} < 0, \qquad \frac{\partial R^P}{\partial \theta} < 0 .$$

Proof：Calculating R's partial derivative of θ, we get,

$$\frac{\partial H}{\partial \theta}=T-\frac{1}{k}\left(1-\mathrm{e}^{-kT}\right)=T+\frac{1}{k}\left(\mathrm{e}^{-kT}-1\right)>T+\frac{1}{k}\left(1-kT-1\right)=0.$$

$$\begin{aligned}\frac{\partial R^C}{\partial \theta}&=R^C\frac{\partial(-H)}{\partial \theta}+\frac{\partial w}{\partial \theta}\mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t dt}\left(S_0\Phi(w)-K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t}\Phi(w-\sigma\sqrt{T})\right)\\&=-\frac{\partial H}{\partial \theta}R^C=-\left(T-\frac{1}{k}\left(1-\mathrm{e}^{-kT}\right)\right)R^C<0.\end{aligned}$$

$$\begin{aligned}\frac{\partial R^P}{\partial \theta}&=\frac{\partial R^C}{\partial \theta}-\frac{\partial\left(R^C-R^P\right)}{\partial \theta}=-R^C\frac{\partial H}{\partial \theta}-\frac{\partial\left(e^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(S_0-K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t}\right)\right)}{\partial \theta}\\&=-\frac{\partial H}{\partial \theta}R^C+\frac{\partial H}{\partial \theta}\left(R^C-R^P\right)=-\frac{\partial H}{\partial \theta}R^P<0.\end{aligned}$$

Theorem 16.

The call option' expected return R^C increases as the present stock price S_0 increases; the put option's expected return R^P decreases as the present stock price S_0 increases, which is,

$$\frac{\partial R^C}{\partial S_0}>0,\qquad \frac{\partial R^P}{\partial S_0}<0.$$

Proof：Calculating R's partial derivative of S_0, we get,

$$\begin{aligned}\frac{\partial R^C}{\partial S_0}&=e^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(N(w)+S_0\Phi(w)\frac{\partial w}{\partial S_0}-K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t}\Phi(w-\sigma\sqrt{T})\frac{\partial w}{\partial S_0}\right)\\&=\mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}N(w)>0.\end{aligned}$$

$$\begin{aligned}\frac{\partial R^P}{\partial S_0}&=\frac{\partial R^C}{\partial S_0}-\frac{\partial\left(R^C-R^P\right)}{\partial S_0}\\&=\mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}N(w)-\frac{\partial\left(\mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(S_0-K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t}\right)\right)}{\partial S_0}\\&=\mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}N(w)-\mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}=-\mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}N(-w)<0.\end{aligned}$$

Theorem 17.

The call option' expected return R^C decreases as the option's strike price K increases; the put option's expected return R^P increases as the option's strike price K increases, which is,

$$\frac{\partial R^C}{\partial K} < 0, \qquad \frac{\partial R^P}{\partial K} > 0.$$

Proof：Calculating R's partial derivative of K, we get,

$$\frac{\partial R^C}{\partial K} = \mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(S_0 \Phi(w)\frac{\partial w}{\partial K} - K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t}\Phi(w-\sigma\sqrt{T})\frac{\partial w}{\partial K} - \mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t} N(w-\sigma\sqrt{T})\right)$$

$$= -\mathrm{e}^{-H+\frac{1}{2}G} N(w-\sigma\sqrt{T}) < 0.$$

$$\frac{\partial R^P}{\partial K} = \frac{\partial R^C}{\partial K} - \frac{\partial\left(R^C - R^P\right)}{\partial K}$$

$$= -\mathrm{e}^{-H+\frac{1}{2}G} N(w-\sigma\sqrt{T}) - \frac{\partial\left(\mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(S_0 - K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t}\right)\right)}{\partial K}$$

$$= -\mathrm{e}^{-H+\frac{1}{2}G} N(w-\sigma\sqrt{T}) + \mathrm{e}^{-H+\frac{1}{2}G} = \mathrm{e}^{-H+\frac{1}{2}G} N(\sigma\sqrt{T} - w) > 0.$$

Theorem 18.

The call option' expected return R^C increases as the option's volatility σ increases; the put option's expected return R^P increases as the option's volatility σ increases, which is,

$$\frac{\partial R^C}{\partial \sigma} > 0, \qquad \frac{\partial R^P}{\partial \sigma} > 0.$$

Proof：Calculating R's partial derivative of K, we get,

$$\frac{\partial R^C}{\partial \sigma} = \mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(S_0\Phi(w)\frac{\partial w}{\partial \sigma} - K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t}\Phi(w-\sigma\sqrt{T})\frac{\partial\left(w-\sigma\sqrt{T}\right)}{\partial \sigma}\right)$$

$$= \mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t}\Phi(w-\sigma\sqrt{T})\frac{\partial \sigma\sqrt{T}}{\partial \sigma}\right)$$

$$= K\sqrt{T}\mathrm{e}^{-H+\frac{1}{2}G}\Phi(w-\sigma\sqrt{T}) > 0.$$

$$\frac{\partial R^P}{\partial K} = \frac{\partial R^C}{\partial K} - \frac{\partial\left(R^C - R^P\right)}{\partial K}$$

$$= K\sqrt{T}\mathrm{e}^{-H+\frac{1}{2}G}\Phi(w-\sigma\sqrt{T}) - \frac{\partial\left(\mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(S_0 - K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t}\right)\right)}{\partial \sigma}$$

$$= K\sqrt{T}\mathrm{e}^{-H+\frac{1}{2}G}\Phi(w-\sigma\sqrt{T}) > 0.$$

Theorem 19. The call option' expected return R^C increases as the option's exercise time T increases, which is,

$$\frac{\partial R^C}{\partial T} > 0.$$

Proof ： From the inequality mentioned in chapter 2, $\frac{\partial\left(\int_0^T \mu_t \mathrm{d}t\right)}{\partial T} = \mu_T \geq E^P[r_T] = \frac{\partial H}{\partial T}$ and $\frac{\partial G}{\partial T} = \frac{\beta^2}{k^2}\left(1 - 2\mathrm{e}^{-kT} + \mathrm{e}^{-2kT}\right) = \frac{\beta^2}{k^2}\left(1-\mathrm{e}^{-kT}\right)^2 > 0$, we get,

$$\frac{\partial R^C}{\partial T} = R^C\frac{\partial\left(-H + \frac{1}{2}G + \int_0^T \mu_t \mathrm{d}t\right)}{\partial T}$$

$$+\mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(S_0\Phi(w)\frac{\partial w}{\partial T} - K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t}\Phi(w-\sigma\sqrt{T})\frac{\partial\left(w-\sigma\sqrt{T}\right)}{\partial T} + \mu_T K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t}N(w-\sigma\sqrt{T})\right)$$

$$= R^C\left(\frac{1}{2}\frac{\partial G}{\partial T} + \mu_T - \frac{\partial H}{\partial T}\right) + \mathrm{e}^{-H+\frac{1}{2}G+\int_0^T \mu_t \mathrm{d}t}\left(K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t}\Phi(w-\sigma\sqrt{T})\frac{\partial \sigma\sqrt{T}}{\partial T} + \mu_T K\mathrm{e}^{-\int_0^T \mu_t \mathrm{d}t}N(w-\sigma\sqrt{T})\right)$$

$$\geq \frac{1}{2}\frac{\partial G}{\partial T}R^C + K\mathrm{e}^{-H+\frac{1}{2}G}\left(\Phi(w-\sigma\sqrt{T})\frac{\sigma}{2\sqrt{T}} + \mu_T N(w-\sigma\sqrt{T})\right) > 0.$$

However, the partial derivative of European put option's expected return R^P of time T cannot be proved to be positive or negative; it is related to the values of other parameters. The partial derivative of European put option's expected return to time *T* is:

$$\frac{\partial R^P}{\partial T}=\frac{\partial R^C}{\partial T}-\frac{\partial\left(R^C-R^P\right)}{\partial T}$$

$$=R^C\left(\frac{1}{2}\frac{\partial G}{\partial T}+\mu_t-\frac{\partial H}{\partial T}\right)+K\mathrm{e}^{-H+\frac{1}{2}G}\left(\Phi(w-\sigma\sqrt{T})\frac{\partial\sigma\sqrt{T}}{\partial T}+\mu_T N(w-\sigma\sqrt{T})\right)$$

$$-\left(R^C-R^P\right)\frac{\partial\left(-H+\frac{1}{2}G+\int_0^T\mu_t\mathrm{d}t\right)}{\partial T}-\mathrm{e}^{-H+\frac{1}{2}G+\int_0^T\mu_t\mathrm{d}t}\mu_T K\mathrm{e}^{-\int_0^T\mu_t\mathrm{d}t}$$

$$=R^P\left(\frac{1}{2}\frac{\partial G}{\partial T}+\mu_t-\frac{\partial H}{\partial T}\right)+K\mathrm{e}^{-H+\frac{1}{2}G}\left(\Phi(w-\sigma\sqrt{T})\frac{\sigma}{2\sqrt{T}}-\mu_T N(\sigma\sqrt{T}-w)\right).$$

In conclusion, the relations between option's expected return and its parameters can be summarized as :

Table 2 Financial Parameters' Influences on Options' Expected Returns

As the value of the parameters rises	Expected Return of Call Options (R^C)	Expected Return of Put Options (R^P)
Present IR r_0	↓ Decreases	↓ Decreases
Long-term IR θ	↓ Decreases	↓Decreases
Present Stock Price S_0	↑ Increases	↓ Decreases
Strike Price K	↓ Decreases	↑ Increases
Volatility σ	↑ Increases	↑ Increases
Exercise Time T	↑ Increases	—

Analysis

①About r_0 and θ：

Under Real Probability Measure P, the fluctuations of interest rate and stocks are not related. If either r_0 or θ increases, the aggregate interest rate from t_0 to T becomes larger, and the discount factor gets larger. Therefore, the present value of an option's expected returns becomes smaller.

②About S_0 and K：

For call options, when the present stock price S_0, increases, the revenue of call option's expectation $(S_T\text{-}K)^+$ increases. Since $R^C = E^P[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t} I(S_T - K)]$，call option's expected return increases. When the strike price K increases, the call option's expectation $(S_T\text{-}K)^+$ decreases, and the call option's price decreases. Similarly, put option's price is inversely related to S_0 and directly related to K.

③About the stock's volatility σ:

When the stock's volatility σ increases, the risk of investing in stocks gets higher, but, according to（7. 2）, the expectation of stock prices doesn't change in such conditions. However, the stock's volatility σ is closely related to options' expected returns. Take call options as an example, when stock price goes up, the more it rises, the more the option buyer profits, and there is no limit to the maximum profit; when stock price goes down, no matter how much it drops, the option buyer loses no more than the premium. Therefore, stock price's ups and downs have uneven influences on option buyers. From the reasoning above, it is clear that when stock's volatility σ increases, call option's expected return also increases. Similarly, when stock's volatility σ increases, put option's expected return decreases.

④About Exercise Time *T*:

As the calculation of partial derivative showed above, the expected return of call options rises as *T* increases, but put options don't have a definite relation to *T*. This phenomenon can be explained by noting that the value of interest rate--the discount rate factor, is non-linear; the speed at which interest rate changes is fast at the beginning and slow at the end. For call options, the speed at which expected return increases is always greater than the discount rate, so that R^C has a constantly positive partial derivative to *T*. For put options, there are no definite results.

Besides，expected return's second partial derivative to present stock price S_0 also has important financial significance. It can be easily calculated that this partial derivative is constantly positive. Since it is a second partial derivative, it is not listed in the table above.

The first partial derivatives of the other two variables in expected return's formula, approximation rate k and interest rate volatility β，can also be calculated. However, since the calculations are very complex and these two variables have limited financial significance (for their values can hardly be obtained in practice), this paper does not further discuss them.

2. Risk Index of Option Price

To facilitate deduction, we start with the proof of theorems 20 and 21.

Theorem 20.

$$S_0\int_{-\infty}^{+\infty}\Phi(x)\Phi\left(\frac{\sigma^2T-F-\sqrt{G}x}{\sigma\sqrt{T}}\right)\mathrm{d}x=K\mathrm{e}^{-H+\frac{1}{2}G}\int_{-\infty}^{+\infty}\Phi(x)\Phi\left(\frac{\sqrt{G}x-G-F}{\sigma\sqrt{T}}\right)\mathrm{d}x. \tag{20.1}$$

Proof of Theorem 20：

Lemma: when neither a nor b is a function of x，

$$\int_{-\infty}^{+\infty}\Phi(x)\Phi(ax+b)\mathrm{d}x=\frac{1}{\sqrt{2\pi(a^2+1)}}\mathrm{e}^{-\frac{b^2}{2(a^2+1)}}. \tag{20. 2}$$

Proof of Lemma:

Define $x=\dfrac{u\sqrt{a^2+1}-ab}{a^2+1}$.

$$\begin{aligned}\int_{-\infty}^{+\infty}\Phi(x)\Phi(ax+b)\mathrm{d}x&=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(ax+b)^2}{2}}\mathrm{d}x\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(a^2+1)\left(x+\frac{ab}{a^2+1}\right)^2}{2}-\frac{b^2}{2(a^2+1)}}\mathrm{d}x\\&=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{b^2}{2(a^2+1)}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(a^2+1)\left(x+\frac{ab}{a^2+1}\right)^2}{2}}\mathrm{d}x\\&=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{b^2}{2(a^2+1)}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{u^2}{2}}\mathrm{d}\left(\frac{u\sqrt{a^2+1}-ab}{a^2+1}\right)\\&=\frac{1}{\sqrt{2\pi(a^2+1)}}\mathrm{e}^{-\frac{b^2}{2(a^2+1)}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{u^2}{2}}\mathrm{d}u\\&=\frac{1}{\sqrt{2\pi(a^2+1)}}\mathrm{e}^{-\frac{b^2}{2(a^2+1)}}.\end{aligned}$$

According to the formula of square difference,

$$-\frac{\left(\sigma^2T-F\right)^2}{2\left(G+\sigma^2T\right)}+\frac{\left(G+F\right)^2}{2\left(G+\sigma^2T\right)}=\frac{\left(G+\sigma^2T\right)\left(G+2F-\sigma^2T\right)}{2\left(G+\sigma^2T\right)}=F+\frac{1}{2}G-\frac{1}{2}\sigma^2T.$$

With the two formulas above, we get,

$$\begin{aligned}
S_0\int_{-\infty}^{+\infty}\Phi(x)\Phi\left(\frac{\sigma^2T-F-\sqrt{G}x}{\sigma\sqrt{T}}\right)\mathrm{d}x &= S_0\frac{1}{\sqrt{2\pi\left(\frac{G}{\sigma^2T}+1\right)}}\mathrm{e}^{-\frac{\left(\frac{\sigma^2T-F}{\sigma\sqrt{T}}\right)^2}{2\left(\frac{G}{\sigma^2T}+1\right)}}\\
&= S_0\sqrt{\frac{\sigma^2T}{2\pi\left(G+\sigma^2T\right)}}\mathrm{e}^{-\frac{\left(\sigma^2T-F\right)^2}{2\left(G+\sigma^2T\right)}}\\
&= S_0\sqrt{\frac{\sigma^2T}{2\pi\left(G+\sigma^2T\right)}}\mathrm{e}^{F+\frac{1}{2}G-\frac{1}{2}\sigma^2T}\mathrm{e}^{-\frac{\left(G+F\right)^2}{2\left(G+\sigma^2T\right)}}\\
&= S_0\sqrt{\frac{\sigma^2T}{2\pi\left(G+\sigma^2T\right)}}\mathrm{e}^{\ln\frac{K}{S_0}-H+\frac{1}{2}G}\mathrm{e}^{-\frac{\left(G+F\right)^2}{2\left(G+\sigma^2T\right)}}\\
&= K\mathrm{e}^{-H+\frac{1}{2}G}\sqrt{\frac{\sigma^2T}{2\pi\left(G+\sigma^2T\right)}}\mathrm{e}^{-\frac{\left(G+F\right)^2}{2\left(G+\sigma^2T\right)}}\\
&= K\mathrm{e}^{-H+\frac{1}{2}G}\frac{1}{\sqrt{2\pi\left(\frac{G}{\sigma^2T}+1\right)}}\mathrm{e}^{-\frac{\left(\frac{G+F}{\sigma\sqrt{T}}\right)^2}{2\left(\frac{G}{\sigma^2T}+1\right)}}\\
&= K\mathrm{e}^{-H+\frac{1}{2}G}\int_{-\infty}^{+\infty}\Phi(x)\Phi\left(\frac{\sqrt{G}x-G-F}{\sigma\sqrt{T}}\right)\mathrm{d}x.
\end{aligned}$$

Theorem 21. $1-\int_{-\infty}^{+\infty}\Phi(x)N(y)\mathrm{d}x=\int_{-\infty}^{+\infty}\Phi(x)N(-y)\mathrm{d}x.$

Proof:

$$\begin{aligned}
\int_{-\infty}^{+\infty}\Phi(x)N(-y)\mathrm{d}x+\int_{-\infty}^{+\infty}\Phi(x)N(y)\mathrm{d}x &= \int_{-\infty}^{+\infty}\Phi(x)\left(N(-y)+N(y)\right)\mathrm{d}x\\
&= \int_{-\infty}^{+\infty}\Phi(x)\mathrm{d}x=1.
\end{aligned}$$

Theorems 20 and 21 will be very helpful in later calculations of partial derivatives. They will be repeatedly referred to.

Theorem 22.

The call option' price $Price^C$ increases as the present interest rate r_0 increases; the put option's price Price^P decreases as the present interest rate r_0 increases, which is,

$$\frac{\partial \mathrm{Price}^C}{\partial r_0} > 0, \qquad \frac{\partial \mathrm{Price}^P}{\partial r_0} < 0.$$

Proof: Calculating the Price's partial derivative of r_0, we get,

$$\begin{aligned}
\frac{\partial \mathrm{Price}^C}{\partial r_0} &= S_0 \int_{-\infty}^{+\infty} \Phi(x)\,\Phi\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right) \frac{\partial\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right)}{\partial r_0} \mathrm{d}x \\
&\quad - K\mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi(x)\,\Phi\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right) \frac{\partial\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right)}{\partial r_0} \mathrm{d}x \\
&\quad - K\mathrm{e}^{-H+\frac{1}{2}G} \frac{\partial\left(-H+\frac{1}{2}G\right)}{\partial r_0} \int_{-\infty}^{+\infty} \Phi(x)\, N\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right) \mathrm{d}x \\
&= \frac{1}{\sigma\sqrt{T}} \frac{\partial F}{\partial r_0} \left(S_0 \int_{-\infty}^{+\infty} \Phi(x)\,\Phi\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right) \mathrm{d}x - K\mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi(x)\,\Phi\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right) \mathrm{d}x \right) \\
&\quad + K\mathrm{e}^{-H+\frac{1}{2}G} \frac{\partial H}{\partial r_0} \int_{-\infty}^{+\infty} \Phi(x)\, N\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right) \mathrm{d}x \\
&= \frac{1}{k}\left(1 - \mathrm{e}^{-kT}\right) K\mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi(x)\, N\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right) \mathrm{d}x > 0.
\end{aligned}$$

$$\frac{\partial \text{Price}^{\text{P}}}{\partial r_0} = \frac{\partial \text{Price}^{\text{C}}}{\partial r_0} - \frac{\partial\left(\text{Price}^{\text{C}} - \text{Price}^{\text{P}}\right)}{\partial r_0}$$

$$= \frac{\partial H}{\partial r_0} K\mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi\left(x\right) N\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right)\mathrm{d}x - \frac{\partial\left(S_0 - K\mathrm{e}^{-H+\frac{1}{2}G}\right)}{\partial r_0}$$

$$= \frac{\partial H}{\partial r_0} K\mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi\left(x\right) N\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right)\mathrm{d}x - K\mathrm{e}^{-H+\frac{1}{2}G} \frac{\partial H}{\partial r_0}$$

$$= -\frac{\partial H}{\partial r_0} K\mathrm{e}^{-H+\frac{1}{2}G} \left(1 - \int_{-\infty}^{+\infty} \Phi\left(x\right) N\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right)\mathrm{d}x\right)$$

$$= -\frac{1}{k}\left(1 - \mathrm{e}^{-kT}\right) K\mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi\left(x\right) N\left(\frac{G + F - \sqrt{G}x}{\sigma\sqrt{T}}\right)\mathrm{d}x < 0.$$

Theorem 23.

The call option' price $Price^C$ increases as the long-term interest rate θ increases; the put option's price $Price^P$ decreases as the long-term interest rate θ increases, which is,

$$\frac{\partial \text{Price}^{\text{C}}}{\partial \theta} > 0, \quad \frac{\partial \text{Price}^{\text{P}}}{\partial \theta} < 0.$$

Proof：Calculating the Price's partial derivative of θ, we get,

$$\frac{\partial H}{\partial \theta} = T - \frac{1}{k}\left(1 - \mathrm{e}^{-kT}\right) = T + \frac{1}{k}\left(\mathrm{e}^{-kT} - 1\right) > T + \frac{1}{k}\left(1 - kT - 1\right) = 0.$$

$$\frac{\partial \mathrm{Price}^{\mathrm{C}}}{\partial\theta}=S_0\int_{-\infty}^{+\infty}\Phi(x)\Phi\left(\frac{\sigma^2T-F-\sqrt{G}x}{\sigma\sqrt{T}}\right)\frac{\partial\left(\frac{\sigma^2T-F-\sqrt{G}x}{\sigma\sqrt{T}}\right)}{\partial\theta}\mathrm{d}x$$

$$-K\mathrm{e}^{-H+\frac{1}{2}G}\int_{-\infty}^{+\infty}\Phi(x)\Phi\left(\frac{\sqrt{G}x-G-F}{\sigma\sqrt{T}}\right)\frac{\partial\left(\frac{\sqrt{G}x-G-F}{\sigma\sqrt{T}}\right)}{\partial\theta}\mathrm{d}x$$

$$-K\mathrm{e}^{-H+\frac{1}{2}G}\frac{\partial\left(-H+\frac{1}{2}G\right)}{\partial\theta}\int_{-\infty}^{+\infty}\Phi(x)N(\frac{\sqrt{G}x-G-F}{\sigma\sqrt{T}})\mathrm{d}x$$

$$=\frac{1}{\sigma\sqrt{T}}\frac{\partial \mathrm{F}}{\partial\theta}\left(S_0\int_{-\infty}^{+\infty}\Phi(x)\Phi\left(\frac{\sigma^2T-F-\sqrt{G}x}{\sigma\sqrt{T}}\right)\mathrm{d}x-K\mathrm{e}^{-H+\frac{1}{2}G}\int_{-\infty}^{+\infty}\Phi(x)\Phi\left(\frac{\sqrt{G}x-G-F}{\sigma\sqrt{T}}\right)\mathrm{d}x\right)$$

$$+K\mathrm{e}^{-H+\frac{1}{2}G}\frac{\partial H}{\partial\theta}\int_{-\infty}^{+\infty}\Phi(x)N(\frac{\sqrt{G}x-G-F}{\sigma\sqrt{T}})\mathrm{d}x$$

$$=\frac{\partial H}{\partial\theta}K\mathrm{e}^{-H+\frac{1}{2}G}\int_{-\infty}^{+\infty}\Phi(x)N(\frac{\sqrt{G}x-G-F}{\sigma\sqrt{T}})\mathrm{d}x>0.$$

$$\frac{\partial \mathrm{Price}^{\mathrm{P}}}{\partial\theta}=\frac{\partial \mathrm{Price}^{\mathrm{C}}}{\partial\theta}-\frac{\partial\left(\mathrm{Price}^{\mathrm{C}}-\mathrm{Price}^{\mathrm{P}}\right)}{\partial\theta}$$

$$=\frac{\partial H}{\partial\theta}K\mathrm{e}^{-H+\frac{1}{2}G}\int_{-\infty}^{+\infty}\Phi(x)N(\frac{\sqrt{G}x-G-F}{\sigma\sqrt{T}})\mathrm{d}x-\frac{\partial\left(S_0-K\mathrm{e}^{-H+\frac{1}{2}G}\right)}{\partial\theta}$$

$$=\frac{\partial H}{\partial\theta}K\mathrm{e}^{-H+\frac{1}{2}G}\int_{-\infty}^{+\infty}\Phi(x)N(\frac{\sqrt{G}x-G-F}{\sigma\sqrt{T}})\mathrm{d}x-K\mathrm{e}^{-H+\frac{1}{2}G}\frac{\partial H}{\partial\theta}$$

$$=-\frac{\partial H}{\partial\theta}K\mathrm{e}^{-H+\frac{1}{2}G}\left(1-\int_{-\infty}^{+\infty}\Phi(x)N(\frac{\sqrt{G}x-G-F}{\sigma\sqrt{T}})\mathrm{d}x\right)$$

$$=-\frac{\partial H}{\partial\theta}K\mathrm{e}^{-H+\frac{1}{2}G}\int_{-\infty}^{+\infty}\Phi(x)N(\frac{G+F-\sqrt{G}x}{\sigma\sqrt{T}})\mathrm{d}x<0.$$

Theorem 24.

The call option' price $Price^C$ increases as the stock's present price S_0 increases; the put option's price $Price^P$ decreases as the stock's present price S_0 increases, which is,

$$\frac{\partial \mathrm{Price}^{\mathrm{C}}}{\partial S_0} > 0, \qquad \frac{\partial \mathrm{Price}^{\mathrm{P}}}{\partial S_0} < 0.$$

Proof:

$$\begin{aligned}
\frac{\partial \mathrm{Price}^{\mathrm{C}}}{\partial S_0} &= S_0 \int_{-\infty}^{+\infty} \Phi(x) \Phi\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right) \frac{\partial\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right)}{\partial S_0} \mathrm{d}x \\
&\quad - K\mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi(x) \Phi\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right) \frac{\partial\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right)}{\partial S_0} \mathrm{d}x \\
&\quad + \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right) \mathrm{d}x \\
&= -\frac{1}{\sigma\sqrt{T}} \frac{\partial \mathrm{F}}{\partial S_0} S_0 \int_{-\infty}^{+\infty} \Phi(x) \Phi\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right) \mathrm{d}x \\
&\quad + \frac{1}{\sigma\sqrt{T}} \frac{\partial \mathrm{F}}{\partial S_0} K\mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi(x) \Phi\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right) \mathrm{d}x \\
&\quad + \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right) \mathrm{d}x \\
&= \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right) \mathrm{d}x > 0.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathrm{Price}^{\mathrm{P}}}{\partial S_0} &= \frac{\partial \mathrm{Price}^{\mathrm{C}}}{\partial S_0} - \frac{\partial\left(\mathrm{Price}^{\mathrm{C}} - \mathrm{Price}^{\mathrm{P}}\right)}{\partial S_0} \\
&= \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right) \mathrm{d}x - \frac{\partial\left(S_0 - K\mathrm{e}^{-H+\frac{1}{2}G}\right)}{\partial S_0} \\
&= \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right) \mathrm{d}x - 1 \\
&= -\int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{F + \sqrt{G}x - \sigma^2 T}{\sigma\sqrt{T}}\right) \mathrm{d}x < 0.
\end{aligned}$$

Theorem 25.

The call option' price $Price^C$ decreases as the option's strike price K increases; the put option's price $Price^P$ increases as the option's strike price K increases, which is,

$$\frac{\partial \text{Price}^{\text{C}}}{\partial K} < 0, \quad \frac{\partial \text{Price}^{\text{P}}}{\partial K} > 0.$$

Proof: Calculating the Price's partial derivative of K, we get,

$$\begin{aligned}
\frac{\partial \text{Price}^{\text{C}}}{\partial K} &= S_0 \int_{-\infty}^{+\infty} \Phi(x) \Phi\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right) \frac{\partial \left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right)}{\partial K} \mathrm{d}x \\
&\quad - K\mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi(x) \Phi\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right) \frac{\partial \left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right)}{\partial K} \mathrm{d}x \\
&\quad - \mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right) \mathrm{d}x \\
&= -\frac{1}{\sigma\sqrt{T}} \frac{\partial \text{F}}{\partial K} S_0 \int_{-\infty}^{+\infty} \Phi(x) \Phi\left(\frac{\sigma^2 T - F - \sqrt{G}x}{\sigma\sqrt{T}}\right) \mathrm{d}x \\
&\quad + \frac{1}{\sigma\sqrt{T}} \frac{\partial \text{F}}{\partial K} K\mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi(x) \Phi\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right) \mathrm{d}x \\
&\quad - \mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right) \mathrm{d}x \\
&= -\mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right) \mathrm{d}x < 0.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \text{Price}^{\text{P}}}{\partial K} &= \frac{\partial \text{Price}^{\text{C}}}{\partial K} - \frac{\partial \left(\text{Price}^{\text{C}} - \text{Price}^{\text{P}}\right)}{\partial K} \\
&= -\mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right) \mathrm{d}x - \frac{\partial \left(S_0 - K\mathrm{e}^{-H+\frac{1}{2}G}\right)}{\partial K} \\
&= -\mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{\sqrt{G}x - G - F}{\sigma\sqrt{T}}\right) \mathrm{d}x + \mathrm{e}^{-H+\frac{1}{2}G} \\
&= \mathrm{e}^{-H+\frac{1}{2}G} \int_{-\infty}^{+\infty} \Phi(x) N\left(\frac{G + F - \sqrt{G}x}{\sigma\sqrt{T}}\right) \mathrm{d}x > 0.
\end{aligned}$$

In conclusion, the relations between option price and its parameters can be summarized as:

Table 3 Financial Parameters' Influence on Option Prices

As the value of the parameters rises	Call Option $Price^{C}$	Put Option $Price^{P}$
Present IR r_0	↑ Increases	↓ Decreases
Long-term IR θ	↑ Increases	↓ Decreases
Present Stock Price S_0	↑Increases	↓ Decreases
Strike Price K	↓ Decreases	↑ Increases

While the partial derivatives of call option's expected returns to r_0 and θ u are negative, those of its price are positive. These two contrary results can be explained with the following analysis.

Under Risk Neutral Probability Measure, interest rate not only serves as the discount factor, but also as the drift term of stock prices. Its influence on stock prices as the drift term far exceeds its influence as the discount factor. When r_0 increases, interest rate served as the drift term accelerates the rise of stock prices, while as the discount factor, it reduces the present value of the return. However, as the former is much greater than the latter, option price is a decreasing function with respect to r_0. Under Real Probability Measure P, r_0 only serves as the discount factor and always reduces the present value of expected return. The more r_0 increases, the more the present value of expected return reduces. Θ's influence can be similarly explained.

For put options, both its expected return and its price are inversely related to r_0 and θ. When r_0 increases, interest rate as the drift term accelerates the increase of stock price, which, in turn, reduces the revenue of put options. Meanwhile, r_0's increase increases the discount rate and further reduces the present value of put option's price. The two effects work in line so that put option's price is inversely related to interest rate level. Θ's influence can be similarly explained.

Under Risk-neutral Probability Measure Q, there are no capital flows among different financial markets, which means capital does not leave the stock market when interest rate increases, nor does it come when interest rate decreases. According to the analysis in the previous paragraph, the price of European call options is positive correlated to the interest rate, while the put option is negative correlated.

V. Conclusion

This paper first elaborates the significance and background of option pricing under stochastic interest rate models, as well as the development and present status of related pricing theories. Second, it introduces the major models for the description of underlying asset's price movement, interest rate movement and option pricing. Based on these models, it deducts the formulas of European equity option's prices and expected returns under stochastic interest rate models. Then, it analyses the major variables' influences on options' prices and expected returns, providing straightforward guidance for investment decisions.

VI. Learning Experiences and Acknowledgement

1. Experiences：Expanding Knowledge, Welcoming Challenges

For us, Financial Mathematics is an uncharted sea with myriad surprises and challenges. Every day in the past year, we have been digging into new knowledge for solutions of "alien" problems. During research, our path forward was repeatedly blocked by difficulties, from the selection of models and methods to the calculations of specific formulas, especially in the realm of stochastic integral. We then dived into piles of references to triumph in the long and hard "conquest" of option pricing under stochastic interest rate models. Along the rough journey to destination, we stumbled for nearly 10 times, gathering ourselves once and again after failures and blunders. Numerous modifications of methods and approaches finally lead us to quite ideal results with concise and precise reasoning processes. Absorbing new knowledge and conquering a rich diversity of problems construct the most fascinating experience of our research, maturing both our minds and our option contracts.

2. Wishes but not Whims：Understanding Finance, Applying Finance, and Making the World a Better Place

Finance marks the most dynamic yet controversial frontier of the present era. We chose to set off in this rough water for not only do we love math and finance, but more importantly, we wish to explore such an influential field of economic development, social progress, and individual happiness from our high school years. The break out of this “once-in-a-lifetime” financial crisis further reminded us the crucial importance of the understanding and appropriate application of this powerful yet risky tool. This research is out first attempt to “mine the gold” of finance, and “the work goes on, the cause endures and the dreams shall never die.”[1] In the future, we will work hard to develop deeper insight into this intriguing field, with the hope and faith of making it the fountain of joys rather than the cause of sorrows.

3. Acknowledgement

During our research, Assistant Researcher Mingyu Xu, PhD, from Academy of Mathematics and System Sciences, Chinese Academy of Sciences has always provided us with great inspiration, illumination. She has also helped revised this paper with her penetrating insight and in-depth guidance. We are very grateful for her generous help and the knowledge from her talented brain. We are also thankful for our families’ support during this project; they helped us open this gateway which greatly expanded our academic perspective and enriched our lives with challenges and delight. We endeavored to do the best throughout this

project, but, limited by our knowledge and experience, this paper may still have deficiencies and insufficiencies. We will be very grateful if our careful readers may point them out for further revision. Thank you!

VII. References

【1】 Wang Anxing. *Interest Rate Models.* Shanghai University of Finance and Economics Press

【2】 Jiang Lishang. *Modeling and Methods of Option Pricing (Second Edition)).* Higher Education Press

【2】 Capinsk, Zastawniak. *Mathematics for finance--an introduction to financial engineering* Renmin University of China Press

【3】 Gong Guanglu. *Introduction to Stochastic Differential Equations (2nd Edition).* Peking University Press

【4】 Gong Guanglu. *Stochastic Differential Equations and General Application.* Tsinghua University Press

【5】 M. Ross, Sheldon. *An Elementary Introduction to Mathematical Finance (2nd Edition).* China Machine Press

【6】 Wang Yawei, Li Suoping, Jiang Hong（2008）. *Study on European Contingent Claims under Vasicek Interest Rate Model with the Function Coefficients.* Journal of Gansu Sciences, Vol. 20, No. 1 Mar 2008.

【7】 Zhang Juan. Research of Option Pricing under Stochastic Interest Rate Models.

【8】 He Xiaoguang. The Development and Application of Option Pricing Theories. Economic Forum, 2009，457（9）.

【9】 Liu Hailong, Wu Chongfeng. Summary of Option Pricing Methods. Journal of Management Sciences. 2002，5（2）.

【10】 M. Kennedy, Edward. Confession Speech at Democratic National Convention 1980.