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ON THE GENUS-ONE GROMOV-WITTEN INVARIANTS OF COMPLETE INTERSECTIONS

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Abstract

We state and prove a long-elusive relation between genus-one Gromov-Witten of a complete intersection and twisted Gromov-Witten invariants of the ambient projective space. As shown in a previous paper, certain naturally arising cones of holomorphic vector bundle sections over the main component $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ of the moduli space of stable genus-one holomorphic maps into $\mathbb{P}^n$ have a well-defined euler class. In this paper, we extend this result to moduli spaces of perturbed, in a restricted way, J-holomorphic maps. This extension is used to show that these cones are the correct genus-one analogues of the vector bundles relating genus-zero Gromov-Witten invariants of a complete intersection to those of the ambient projective space. A relationship for higher-genus invariants is conjectured as well.

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1. Introduction

1.1. Gromov-Witten invariants and complete intersections. Gromov-Witten invariants of symplectic manifolds have been a subject of much research over the past two decades. A great deal of attention has been devoted in particular to Calabi-Yau manifolds. These manifolds play a prominent role in theoretical physics, and as a result physicists have made a number of important predictions concerning CY-manifolds. Some of these predictions have been verified mathematically; others have not.

If Y is a compact Kähler submanifold of the complex projective space $\mathbb{P}^n$, one could try to compute GW-invariants of Y by relating them to GW-invariants of $\mathbb{P}^n$. For example, suppose Y is a hypersurface in $\mathbb{P}^n$ of degree a. In other words, if $\gamma \longrightarrow \mathbb{P}^n$ is the tautological line bundle and $\mathfrak{L}=\gamma^{*\otimes a} \longrightarrow \mathbb{P}^n$, then

$$Y = s^{-1}(0),$$

for some $s\in H^0(\mathbb{P}^n;\mathfrak{L})$ such that s is transverse to the zero set. If g, k, and d are nonnegative integers, let $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)$ and $\overline{\mathfrak{M}}_{g,k}(Y,d)$ denote the moduli spaces of stable J_0-holomorphic degree-d maps from genus-g Riemann surfaces with k marked points to $\mathbb{P}^n$ and Y, respectively. These moduli spaces determine the genus-g degree-d GW-invariants of $\mathbb{P}^n$ and Y.

By definition, the moduli space $\overline{\mathfrak{M}}_{g,k}(Y,d)$ is a subset of the moduli space $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)$. In fact,

$$\overline{\mathfrak{M}}_{g,k}(Y,d) = \big\{[\mathcal{C},u]\in\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)\colon s\circ u = 0 \in H^0\big(\mathcal{C};u^*\mathfrak{L}\big)\big\}. \tag{1.1}$$

Here $[\mathcal{C},u]$ denotes the equivalence class of the holomorphic map $u:\mathcal{C}\longrightarrow\mathbb{P}^n$ from a genus-g curve $\mathcal{C}$ with k marked points. The relationship (1.1) can be restated more globally as follows. Let

$$\pi^d_{g,k}\colon \mathfrak{U}_{g,k}(\mathbb{P}^n,d) \longrightarrow \overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)$$

be the universal family and let

$$\mathrm{ev}^d_{g,k}\colon \mathfrak{U}_{g,k}(\mathbb{P}^n,d) \longrightarrow \mathbb{P}^n$$

be the natural evaluation map. In other words, the fiber of $\pi^d_{g,k}$ over $[\mathcal{C},u]$ is the curve $\mathcal{C}$ with k marked points, while

$$\mathrm{ev}^d_{g,k}\big([\mathcal{C},u;z]\big) = u(z) \qquad \text{if} \quad z\in\mathcal{C}.$$

We define a section $s^d_{g,k}$ of the sheaf $\pi^d_{g,k*}\mathrm{ev}^{d*}_{g,k}\mathfrak{L}\longrightarrow\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)$ by

$$s^d_{g,k}\big([\mathcal{C},u]\big)=[s\circ u].$$

By (1.1), $\overline{\mathfrak{M}}_{g,k}(Y,d)$ is the zero set of this section.

The previous paragraph suggests that it should be possible to relate the genus-g degree-d GW-invariants of the hypersurface Y to the moduli space $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)$ in general and to the sheaf

$$\pi^d_{g,k*}\mathrm{ev}^{d*}_{g,k}\mathfrak{L}\longrightarrow\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)$$

in particular. In fact, it can be shown that

$$\begin{aligned}\mathrm{GW}^Y_{0,k}(d;\psi)&\equiv\big\langle\psi,\big[\overline{\mathfrak{M}}_{0,k}(Y,d)\big]^{vir}\big\rangle\\&=\big\langle\psi\cdot e\big(\pi^d_{0,k*}\mathrm{ev}^{d*}_{0,k}\mathfrak{L}\big),\big[\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)\big]\big\rangle\end{aligned}\tag{1.2}$$

for all $\psi\in H^*(\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d);\mathbb{Q})$; this was observed early on by Beauville [**2**], for example. The moduli space $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)$ is a smooth orbivariety and

$$\pi^d_{0,k*}\mathrm{ev}^{d*}_{0,k}\mathfrak{L}\longrightarrow\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)\tag{1.3}$$

is a locally free sheaf, i.e. a vector bundle. The right-hand side of (1.2) can be computed via the classical Atiyah-Bott localization theorem [**1**], though the complexity of this computation increases rapidly with the degree d.

A hyperplane property, i.e. a relationship such as (1.2), for positive-genus GW-invariants has been elusive since the early days of the Gromov-Witten theory. If $g>0$, the sheaf

$$\pi^d_{g,k*}\mathrm{ev}^{d*}_{g,k}\mathfrak{L}\longrightarrow\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)$$

is not locally free and need not define an euler class. Thus, the right-hand side of (1.2) may not even make sense if 0 is replaced by $g>0$. Instead one might try to generalize (1.2) as

$$\begin{aligned}&\mathrm{GW}^Y_{g,k}(d;\psi)\equiv\big\langle\psi,\big[\overline{\mathfrak{M}}_{g,k}(Y,d)\big]^{vir}\big\rangle\\&\stackrel{?}{=}\big\langle\psi\cdot e\big(R^0\pi^d_{g,k*}\mathrm{ev}^{d*}_{g,k}\mathfrak{L}-R^1\pi^d_{g,k*}\mathrm{ev}^{d*}_{g,k}\mathfrak{L}\big),\big[\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)\big]^{vir}\big\rangle,\end{aligned}\tag{1.4}$$

where $R^i\pi^d_{g,k*}\mathrm{ev}^{d*}_{g,k}\mathfrak{L}\longrightarrow\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)$ is the i-th direct image sheaf. The right-hand side of (1.4) can be computed via the virtual localization theorem of Graber-Pandharipande [**10**]. However,

$$\begin{aligned}N_1(d)&\equiv\mathrm{GW}^Y_{1,0}(d;1)\\&\neq\big\langle e\big(R^0\pi^d_{1,0*}\mathrm{ev}^{d*}_{1,0}\mathfrak{L}-R^1\pi^d_{1,0*}\mathrm{ev}^{d*}_{1,0}\mathfrak{L}\big),\big[\overline{\mathfrak{M}}_1(\mathbb{P}^4,d)\big]^{vir}\big\rangle,\end{aligned}$$

according to Graber-Pandharipade [**11**] and Katz [**12**] for a quintic threefold $Y\subset\mathbb{P}^4$.

In this paper we prove a hyperplane property for genus-one GW-invariants. We denote by

$$\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) \subset \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$$

the closure in $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$, either in the stable-map or Zariski topology, of the subspace

$$\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d) = \big\{[\mathcal{C},u]\in\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)\colon \mathcal{C} \text{ is smooth}\big\}.$$

If $Y\subset\mathbb{P}^n$ is a hypersurface as above, let

$$\overline{\mathfrak{M}}^0_{1,k}(Y,d) = \overline{\mathfrak{M}}_{1,k}(Y,d) \cap \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d).$$

Since $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ is a unidimensional orbi-variety, it carries a fundamental class. By [**28**, Corollary 1.6], $\overline{\mathfrak{M}}^0_{1,k}(Y,d)$ carries a virtual fundamental class. It can be used to define reduced genus-one Gromov-Witten invariants:

$$\mathrm{GW}^{0;Y}_{1,k}(d;\psi) \equiv \big\langle \psi, \big[\overline{\mathfrak{M}}^0_{1,k}(Y,d)\big]^{vir}\big\rangle \in \mathbb{Q},$$

where ψ is a tautological (cohomology) class on $\overline{\mathfrak{M}}^0_{1,k}(Y,d)$; see below. We show in this paper that the reduced genus-one GW-invariants satisfy a natural analogue of (1.2).

Theorem 1.1. *Suppose d and a are positive integers, k is a nonnegative integer, $\mathfrak{L}=\gamma^{*\otimes a}\longrightarrow\mathbb{P}^n$,*

$$\pi^d_{1,k}\colon \mathfrak{U}_{1,k}(\mathbb{P}^n,d) \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) \quad \textit{and} \quad \mathrm{ev}^d_{1,k}\colon \mathfrak{U}_{1,k}(\mathbb{P}^n,d) \longrightarrow \mathbb{P}^n$$

are the universal family and the natural evaluation map, respectively. If $Y\subset\mathbb{P}^n$ is a smooth degree-a hypersurface, then

$$\mathrm{GW}^{0;Y}_{1,k}(d;\psi) = \big\langle \psi\cdot e(\pi^d_{1,k*}\mathrm{ev}^{d*}_{1,k}\mathfrak{L}), \big[\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)\big]\big\rangle \tag{1.5}$$

for every tautological class ψ on $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$.

The tautological classes on $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$ are certain natural cohomology classes. They include all geometric classes defined in Subsection 1.3. We describe the space of all cohomology classes ψ to which Theorem 1.1 applies in Subsection 2.2.

Implicit in the statement of Theorem 1.1 is that the euler class of the sheaf

$$\pi^d_{1,k*}\mathrm{ev}^{d*}_{1,k}\mathfrak{L} \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) \tag{1.6}$$

is well-defined, even though it is not locally free. This is the case by [**27**, Theorem 1.1].

The right-hand side of (1.5) should in principle be computable via localization directly. However, since the space $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ is not smooth and the sheaf (1.6) is not locally free, the classical localization theorem [**1**] is not immediately applicable. A desingularization of the space $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$, i.e. a smooth orbivariety $\widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ and a map

$$\tilde{\pi}\colon \widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d) \longrightarrow \overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d),$$

which is biholomorphic onto $\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d)$, is constructed in [**22**]. This desingularization of $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ comes with a desingularization of the sheaf (1.6), i.e. a vector bundle

$$\tilde{\mathcal{V}}_{1,k}^d \longrightarrow \widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d) \qquad \text{s.t.} \qquad \tilde{\pi}_*\tilde{\mathcal{V}}_{1,k}^d = \pi_{1,k*}^d \mathrm{ev}_{1,k}^{d*}\mathfrak{L}.$$

In particular,

$$\begin{aligned} &\big\langle \psi\cdot e\big(\pi_{1,k*}^d\mathrm{ev}_{1,k}^{d*}\mathfrak{L}\big), \big[\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)\big]\big\rangle \\ &\qquad\qquad = \big\langle \tilde{\pi}^*\psi\cdot e(\tilde{\mathcal{V}}_{1,k}^d), \big[\widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)\big]\big\rangle. \end{aligned} \tag{1.7}$$

Since a group action on $\mathbb{P}^n$ induces actions on $\widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ and on $\tilde{\mathcal{V}}_{1,k}^d$, the classical localization theorem is directly applicable to the right-hand side of (1.7), for a natural cohomology class ψ.

By itself, Theorem 1.1 does not provide a way of computing the standard genus-one GW-invariants of Y. However, the reduced genus-one GW-invariants capture the contribution of $\overline{\mathfrak{M}}_{1,k}^0(Y,d)$ to the standard genus-one GW-invariants. Thus, the difference between the two invariants is completely determined by the genus-zero invariants of Y; see [**31**, Subsection 1.2]. We give explicit formulas in some special cases in Subsection 1.3 below.

Remark 1: Theorem 1.1 generalizes to arbitrary smooth complete intersections in projective spaces. More precisely, if

$$\mathfrak{L} = \gamma^{*\otimes a_1}\oplus\ldots\oplus\gamma^{*\otimes a_m} \longrightarrow \mathbb{P}^n,$$

with $a_1,\ldots,a_m\in\mathbb{Z}^+$, $s\in H^0(\mathbb{P}^n;\mathfrak{L})$ is transverse to the zero set in $\mathfrak{L}$, and $Y=s^{-1}(0)$, then

$$\mathrm{GW}_{1,k}^{0;Y}(d;\psi) = \big\langle \psi\cdot e(\pi_{1,k*}^d\mathrm{ev}_{1,k}^{d*}\mathfrak{L}), \big[\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)\big]\big\rangle, \tag{1.8}$$

for every geometric cohomology class ψ on $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$.

Remark 2: In turn, Remark 1 generalizes as follows. Suppose (X,ω,J) is a compact almost Kähler manifold,

$$A\in H_2(X;\mathbb{Z})^*\equiv H_2(X;\mathbb{Z})-\{0\},$$

$(\mathfrak{L},\nabla)\longrightarrow X$ is a complex vector bundle with connection, and s is a ∇-holomorphic section of $\mathfrak{L}$; see Subsections 1.2 and 2.2 for terminology. If J is genus-one A-regular in the sense of [**26**, Definition 1.4], s is transverse to the zero set in $\mathfrak{L}$, and $(\mathfrak{L},\nabla)$ splits into line bundles that are (ω,A)-positive in the sense of Definition 1.2 below, then

$$\begin{aligned} \mathrm{GW}^{0;Y}_{1,k}(A;\psi) &= \big\langle \psi\cdot e(\mathcal{V}^A_{1,k}), \big[\overline{\mathfrak{M}}^0_{1,k}(X,A;J)\big]^{vir}\big\rangle \\ &\equiv \big\langle \psi, \mathrm{PD}_{\overline{\mathfrak{M}}^0_{1,k}(X,A;J)} e(\mathcal{V}^A_{1,k})\big\rangle, \end{aligned} \tag{1.9}$$

where $Y = s^{-1}(0)$, ψ is a tautological class, and the cone

$$\mathcal{V}^A_{1,k} \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(X,A;J)$$

is the geometric analogue of the sheaf $\pi^d_{1,k*}\mathrm{ev}^{d*}_{1,k}\mathfrak{L}$. It consists of ∇-holomorphic sections of the vector bundle $\mathfrak{L}$, as defined in Subsection 1.2 below. By Corollary 1.4, the Poincare dual of its euler class is well defined as long as $(\mathfrak{L},\nabla)$ is a direct sum of (ω,A)-positive line bundles.

Theorem 1.1 and Remarks 1 and 2 have a natural, but rather speculative, generalization to higher-genus invariants. Suppose that the main component

$$\overline{\mathfrak{M}}^0_{g,k}(X,A;J) \subset \overline{\mathfrak{M}}_{g,k}(X,A;J)$$

is well defined and carries a virtual fundamental class. If so, it determines reduced genus-g GW-invariants $\mathrm{GW}^{0;Y}_{g,k}(A;\psi)$. Suppose further that (the Poincare dual of) the euler of the cone

$$\mathcal{V}^A_{g,k} \longrightarrow \overline{\mathfrak{M}}^0_{g,k}(X,A;J)$$

corresponding to the vector bundle $(\mathfrak{L},\nabla)\longrightarrow X$ is well defined. If constructions of these objects are direct generalizations of the corresponding constructions in Subsection 1.2 and in [**26**]-[**28**], then the proof of Theorem 1.1 can be generalized to show that

$$\mathrm{GW}^{0;Y}_{g,k}(A;\psi) = \big\langle \psi\cdot e(\mathcal{V}^d_{g,k}), \big[\overline{\mathfrak{M}}^0_{g,k}(X,A;J)\big]^{vir}\big\rangle, \tag{1.10}$$

provided appropriate generalizations of the assumptions in Remark 2 hold. Along with an equally speculative generalization of [**28**, Theorem 1.1] stated in [**28**, Subsection 1.2], (1.10) would, if true, provide an algorithm for computing arbitrary-genus GW-invariants of complete intersections.

From the point of view of algebraic geometry as described by Behrend-Fantechi [**3**] and Li-Tian [**16**], the genus-g degree-d Gromov-Witten invariant $\mathrm{GW}^Y_{g,k}(d;\psi)$ is the evaluation of ψ on the virtual fundamental class $[\overline{\mathfrak{M}}_{g,k}(Y,d)]^{vir}$. Using the more concrete point of view of symplectic topology as described by Fukaya-Ono [**6**] and Li-Tian [**15**], $\mathrm{GW}^Y_{g,k}(d;\psi)$ can be interpreted as the euler class of a vector bundle, albeit of an infinite-rank vector bundle over a space of the "same" dimension. As in the finite-dimensional case, this euler class is the number of zeros, counted with appropriate multiplicities, of a transverse (multivalued, generic) section. It is shown by Li-Tian [**17**] and Siebert [**21**] that the two approaches are equivalent. In this paper, we take the latter point of view. Similarly, we view the euler class of the sheaf (1.6) as the zero set of a generic section of its geometric analogue $\mathcal{V}_{1,k}^d$ defined in Subsection 1.2.

Theorem 1.1 and Remark 1 are special cases of Remark 2, which is the same as Theorem 2.3. It is proved in Subsection 2.2 by showing that the zero sets of two bundle sections whose cardinalities are the two expressions in (1.9) are the same set. In fact, Theorem 2.3, just like its genus-zero analogue, follows easily from definitions of the two sides in (1.9), once it is established that these definitions are well-posed.

1.2. Cones of holomorphic bundle sections. Let (X,ω,J) be a compact almost Kähler manifold. In other words, (X,ω) is a symplectic manifold and J is an almost complex structure on X tamed by ω, i.e.

$$\omega(v,Jv)>0 \qquad \forall\, v\in TX-X.$$

If g,k are nonnegative integers and $A\in H_2(X;\mathbb{Z})$, let $\overline{\mathfrak{M}}_{g,k}(X,A;J)$ denote the moduli space of (equivalence classes of) stable J-holomorphic maps from genus-g Riemann surfaces with k marked points in the homology class A. Let $\overline{\mathfrak{M}}_{1,k}^0(X,A;J)$ be the main component of the moduli space $\overline{\mathfrak{M}}_{1,k}(X,A;J)$ described by [**26**, Definition 1.1]; see also Definition 2.2 below. This closed subspace of $\overline{\mathfrak{M}}_{1,k}(X,A;J)$ contains the subspace $\mathfrak{M}_{1,k}^0(X,A;J)$ consisting of the stable maps $[\Sigma,u]$ such that the domain Σ is a smooth Riemann surface. If J is sufficiently regular, $\overline{\mathfrak{M}}_{1,k}^0(X,A;J)$ is the closure of $\mathfrak{M}_{1,k}^0(X,A;J)$ in $\overline{\mathfrak{M}}_{1,k}(X,A;J)$.

Suppose $\mathfrak{L}\longrightarrow X$ is a complex line bundle and ∇ is a connection in $\mathfrak{L}$. If (Σ,j) is a Riemann surface and $u\colon\Sigma\longrightarrow X$ is a smooth map, let

$$\nabla^u\colon \Gamma(\Sigma;u^*\mathfrak{L})\longrightarrow \Gamma\big(\Sigma;T^*\Sigma\otimes u^*\mathfrak{L}\big)$$

be the pull-back of ∇ by u. If $b=(\Sigma,j;u)$, we define the corresponding $\bar\partial$-operator by

$$\begin{gathered}\bar\partial_{\nabla,b}\colon \Gamma(\Sigma;u^*\mathfrak{L})\longrightarrow \Gamma\big(\Sigma;\Lambda^{0,1}_{\mathfrak{i},j}T^*\Sigma\otimes u^*\mathfrak{L}\big),\\ \bar\partial_{\nabla,b}\xi=\frac{1}{2}\big(\nabla^u\xi+\mathfrak{i}\nabla^u\xi\circ j\big),\end{gathered} \tag{1.11}$$

where $\mathfrak{i}$ is the complex multiplication in the bundle $u^*\mathfrak{L}$ and

$$\Lambda^{0,1}_{\mathfrak{i},j}T^*\Sigma\otimes u^*\mathfrak{L}=\big\{\eta\in\mathrm{Hom}(T\Sigma,u^*\mathfrak{L})\colon \eta\circ j=-\mathfrak{i}\eta\big\}.$$

The kernel of $\bar\partial_{\nabla,b}$ is necessarily a finite-dimensional complex vector space.

We denote by $\mathfrak{X}_{1,k}(X,A)$ the space of all degree-A stable smooth maps from genus-one Riemann surfaces with k marked points into X and by

$$\mathcal{V}_{1,k}^A\longrightarrow\mathfrak{X}_{1,k}(X,A)$$

the cone, or the bundle of (orbi-)vector spaces, such that

$$\mathcal{V}_{1,k}^{A}\big|_{[b]} = \ker\bar{\partial}_{\nabla,b}\big/\mathrm{Aut}(b) \qquad \forall\ [b]\in\mathfrak{X}_{1,k}(X,A).$$

The spaces $\mathfrak{X}_{1,k}(X,A)$ and $\mathcal{V}_{1,k}^{A}$ have natural topologies; see Subsection 2.1 below. By [**27**, Theorem 1.1], if (X,ω,J) is the complex projective space $(\mathbb{P}^n,\omega_0,J_0)$ with its standard Kähler structure and $(\mathfrak{L},\nabla)$ is a positive power of the hyperplane line bundle, i.e. the dual of the tautological line bundle, γ^* with its standard connection, then the euler class of

$$\mathcal{V}_{1,k}^{A}\longrightarrow\overline{\mathfrak{M}}_{1,k}^{0}(X,A;J)$$

and its Poincare dual are well defined. By [**27**, Theorem 1.2], this is also the case if J is an almost complex structure on $\mathbb{P}^n$ sufficiently close to J_0.

The argument in [**27**] easily generalizes to all (X,ω,J), $(\mathfrak{L},\nabla)$, and A such that $(\mathfrak{L},\nabla)$ is a split positive vector bundle with connection and J satisfies a certain regularity condition. This regularity condition, which is described by [**26**, Definition 1.4], implies that $\overline{\mathfrak{M}}_{1,k}^{0}(X,A;J)$ has the expected topological structure of a unidimensional orbivariety. In this paper, we show that the *Poincare dual of the euler class* of $\mathcal{V}_{1,k}^{A}$ over $\overline{\mathfrak{M}}_{1,k}^{0}(X,A;J)$ is well defined without any condition on J, as long as $(\mathfrak{L},\nabla)$ satisfies the requirement of Definition 1.2; see Corollary 1.4 below.

Definition 1.2. Suppose (X,ω) is a symplectic manifold and $A\in H_2(X;\mathbb{Z})$. A complex line bundle $\mathfrak{L}\longrightarrow X$ is (ω,A)-positive if

$$\langle c_1(\mathfrak{L}),B\rangle>0 \qquad \forall\ B\in H_2(X;\mathbb{Z})^*\ \text{s.t.}\ B=A\ \text{or}\ \langle\omega,B\rangle<\langle\omega,A\rangle.$$

We note that $\mathcal{V}_{1,k}^{A}\longrightarrow\overline{\mathfrak{M}}_{1,k}^{0}(X,A;J)$ is *not* a vector bundle, as the fibers of $\mathcal{V}_{1,k}^{A}$ are of two possible dimensions. In [**27**, Subsections 1.2,1.3], the Poincare dual of the euler class of $\mathcal{V}_{1,k}^{A}$ is defined as the zero set of a generic multisection φ of $\mathcal{V}_{1,k}^{A}$ over $\overline{\mathfrak{M}}_{1,k}^{0}(X,A;J)$. This zero set determines a homology class in $\overline{\mathfrak{M}}_{1,k}^{0}(X,A;J)$ if φ is sufficiently regular. In [**27**, Section 3], it is shown that $\mathcal{V}_{1,k}^{A}$ contains a vector subbundle of a sufficiently high rank over a neighborhood of every stratum of $\overline{\mathfrak{M}}_{1,k}^{0}(X,A;J)$. The existence of such subbundles implies that regular sections of $\mathcal{V}_{1,k}^{A}$ exist; see [**27**, Subsection 3.1].

If J does not satisfy the regularity condition of [**26**, Definition 1.4], the moduli space $\overline{\mathfrak{M}}_{1,k}^{0}(X,A;J)$ itself need not carry a fundamental class. In this case, we cannot define the Poincare dual of the euler class of $\mathcal{V}_{1,k}^{A}$ as the zero set of a section of $\mathcal{V}_{1,k}^{A}$ over $\overline{\mathfrak{M}}_{1,k}^{0}(X,A;J)$. On the other hand, in [**28**], the definition of

$$\overline{\mathfrak{M}}_{1,k}^{0}(X,A;J)\subset\overline{\mathfrak{M}}_{1,k}(X,A;J)$$

given in [**26**] is generalized to define the main component $\overline{\mathfrak{M}}^0_{1,k}(X,A;J,\nu)$ of the moduli space $\overline{\mathfrak{M}}_{1,k}(X,A;J,\nu)$ of (J,ν)-holomorphic maps for an *effectively supported perturbation* ν of the $\bar{\partial}_J$-operator; see Definitions 2.1 and 2.2 below. By [**28**, Theorem 1.5], if ν is sufficiently small and generic, $\overline{\mathfrak{M}}^0_{1,k}(X,A;J,\nu)$ determines a rational homology class in a small neighborhood of $\overline{\mathfrak{M}}^0_{1,k}(X,A;J)$ in $\mathfrak{X}_{1,k}(X,A)$. This rational homology class is independent of the choice of ν. We will define the Poincare dual of the euler class of $\mathcal{V}^A_{1,k}$ as the zero set of a generic multisection of $\mathcal{V}^A_{1,k}$ over $\overline{\mathfrak{M}}^0_{1,k}(X,A;J,\nu)$. Tietze Extension Theorem will be used to show that $\mathcal{V}^A_{1,k}$ admits sections that are sufficiently nice for this purpose.

If J is an almost complex structure on X and $\underline{J}\equiv(J_t)_{t\in[0,1]}$ is a family of almost complex structures on X, we denote by

$$\mathfrak{G}^{\mathrm{es}}_{1,k}(X,A;J) \qquad \text{and} \qquad \mathfrak{G}^{\mathrm{es}}_{1,k}(X,A;\underline{J})$$

the spaces of effectively supported perturbations of the $\bar{\partial}_J$-operator on $\mathfrak{X}_{1,k}(X,A)$ and of effectively supported families of perturbations of the $\bar{\partial}_{J_t}$-operators on $\mathfrak{X}_{1,k}(X,A)$; see Subsection 2.1 for details. If

$$\bar{\nu}\equiv(\nu_t)_{t\in[0,1]} \in \mathfrak{G}^{\mathrm{es}}_{1,k}(X,A;\underline{J}),$$

we put

$$\overline{\mathfrak{M}}^0_{1,k}(X,A;\underline{J},\underline{\nu}) = \big\{(t,b)\in[0,1]\times\mathfrak{X}_{1,k}(X,A)\colon b\in\overline{\mathfrak{M}}^0_{1,k}(X,A;J_t,\nu_t)\big\}.$$

We denote by $\bar{\mathbb{Z}}^+$ the set of nonnegative integers. Let

$$\begin{aligned}\dim_{1,k}(X,A;\mathfrak{L}) &= \dim_{1,k}(X,A) - 2\langle c_1(\mathfrak{L}),A\rangle \\ &= 2\big(\langle c_1(TX)-c_1(\mathfrak{L}),A\rangle + k\big).\end{aligned}$$

Theorem 1.3. *Suppose (X,ω,J) is a compact almost Kähler manifold, $A\in H_2(X;\mathbb{Z})^*$, $k\in\bar{\mathbb{Z}}^+$, $(\mathfrak{L},\nabla)\longrightarrow X$ is an (ω,A)-positive line bundle with connection, $\mathcal{V}^A_{1,k}\longrightarrow\mathfrak{X}_{1,k}(X,A)$ is the corresponding cone, and W is a neighborhood of $\overline{\mathfrak{M}}^0_{1,k}(X,A;J)$ in $\mathfrak{X}_{1,k}(X,A)$. If $\nu\in\mathfrak{G}^{\mathrm{es}}_{1,k}(X,A;J)$ is sufficiently small and generic and φ is a generic multisection of $\mathcal{V}^A_{1,k}$ over $\overline{\mathfrak{M}}^0_{1,k}(X,A;J,\nu)$, then $\varphi^{-1}(0)$ determines a rational homology class in W. Furthermore, if $\underline{J}=(J_t)_{t\in[0,1]}$ is a family of ω-tamed almost complex structures on X, such that $J_0=J$ and J_t is sufficiently close to J for all t, ν_0 and ν_1 are sufficiently small generic effectively supported perturbations of $\bar{\partial}_{J_0}$ and $\bar{\partial}_{J_1}$, and φ_0 and φ_1 are generic multisections of $\mathcal{V}^A_{1,k}$ over $\overline{\mathfrak{M}}^0_{1,k}(X,A;J_0,\nu_0)$ and $\overline{\mathfrak{M}}^0_{1,k}(X,A;J_1,\nu_1)$, then there exist homotopies*

$$\underline{\nu}=(\nu_t)_{t\in[0,1]} \in \mathfrak{G}^{\mathrm{es}}_{1,k}(X,A;\underline{J}), \qquad \Phi \in \Gamma\big(\overline{\mathfrak{M}}^0_{1,k}(X,A;\underline{J},\underline{\nu});\mathcal{V}^A_{1,k}\big)$$

between ν_0 *and* ν_1 *and between* φ_0 *and* φ_1 *such that* $\Phi^{-1}(0)$ *determines a chain in* W *and*

$$\partial\Phi^{-1}(0) = \varphi_1^{-1}(0) - \varphi_0^{-1}(0).$$

Corollary 1.4. *If* (X,ω,J)*,* A*,* k*, and* $(\mathfrak{L},\nabla)$ *are as in Theorem* 1.3*, the cone* $\mathcal{V}_{1,k}^A \longrightarrow \mathfrak{X}_{1,k}(X,A)$ *corresponding to* $(\mathfrak{L},\nabla)$ *determines a well-defined homology class*

$$\mathrm{PD}_{\overline{\mathfrak{M}}_{1,k}^0(X,A;J)} e(\mathcal{V}_{1,k}^A) \in H_{\dim_{1,k}(X,A;\mathfrak{L})}\big(\overline{\mathfrak{M}}_{1,k}^0(X,A;J);\mathbb{Q}\big).$$

This class is an invariant of (X,ω) *and* $(\mathfrak{L},\nabla)$*.*

As in [**27**], we will describe the local structure of the cone $\mathcal{V}_{1,k}^A$. In contrast to [**27**], we will not construct a high-rank vector subbundle of $\mathcal{V}_{1,k}^A$ over a neighborhood of every stratum of $\overline{\mathfrak{M}}_{1,k}^0(X,A;J,\nu)$. Instead, we will use Tietze Extension Theorem to construct a sufficiently regular multisection of $\mathcal{V}_{1,k}^A$. Its zero set determines a homology class in a small neighborhood of $\overline{\mathfrak{M}}_{1,k}^0(X,A;J)$ in the space $\mathfrak{X}_{1,k}(X,A)$.

For a generic ν, $\overline{\mathfrak{M}}_{1,k}^0(X,A;J,\nu)$ can be stratified by orbifolds $\mathcal{U}_\alpha$ of even dimensions; see Subsection 3.4 and Remark 1 at the end of Subsection 3.3. The main stratum of $\overline{\mathfrak{M}}_{1,k}^0(X,A;J,\nu)$,

$$\mathfrak{M}_{1,k}^0(X,A;J,\nu) \equiv \overline{\mathfrak{M}}_{1,k}^0(X,A;J,\nu) \cap \mathfrak{X}_{1,k}^0(X,A),$$

is of dimension $\dim_{1,k}(X,A)$, where

$$\mathfrak{X}_{1,k}^0(X,A) \subset \mathfrak{X}_{1,k}(X,A)$$

is the subspace of stable maps with smooth domains. In Subsection 3.5, we describe a subcone $\mathcal{W}_{1,k}^A$ of $\mathcal{V}_{1,k}^A$ such that $\mathcal{W}_{1,k}^A|_{\mathcal{U}_\alpha}$ is a smooth vector bundle for every stratum $\mathcal{U}_\alpha$. By analyzing the obstruction to extending holomorphic bundle sections from singular to smooth domains in Section 4, we show that $\mathcal{W}_{1,k}^A$ is a regular obstruction-free cone in the sense of Definition 3.3. By Proposition 3.6, for a generic multisection φ of $\mathcal{W}_{1,k}^A \subset \mathcal{V}_{1,k}^A$ over $\overline{\mathfrak{M}}_{1,k}^0(X,A;J,\nu)$, $\varphi|_{\mathcal{U}_\alpha}$ is then transverse to the zero set in $\mathcal{W}_{1,k}^A|_{\mathcal{U}_\alpha}$. By the rank statements of Proposition 3.9, $\varphi^{-1}(0)$ is stratified by smooth orbifolds of even dimensions. Furthermore, the main stratum of $\varphi^{-1}(0)$ is of dimension $\dim_{1,k}(X,A;\mathfrak{L})$ and is contained in $\mathfrak{M}_{1,k}^0(X,A;J,\nu)$. We can then choose an arbitrarily small neighborhood U of the boundary of $\varphi^{-1}(0)$ such that

$$H_l(U;\mathbb{Q}) = \{0\} \qquad \forall\ l \geq \dim_{1,k}(X,A;\mathfrak{L})-1.$$

Since $\varphi^{-1}(0)-U$ is compact, via the pseudocycle construction of [**19**, Chapter 7] and [**20**, Section 1], $\varphi^{-1}(0)$ determines a homology class

$$\begin{aligned}\big[\varphi^{-1}(0)\big] \in H_{\dim_{1,k}(X,A;\mathfrak{L})}&(W,U;\mathbb{Q})\\ &\approx H_{\dim_{1,k}(X,A;\mathfrak{L})}(W;\mathbb{Q});\end{aligned}$$

see also [**29**]. The second part of Theorem 1.3 is a parametrized version of this construction. Corollary 1.4 is an immediate consequence of Theorem 1.3; see also Remark 2 in [**28**, Subsection 1.3] and the comments at the end of [**28**, Subsection 1.4].

The statement of Corollary 1.4 is not needed to show that the expressions on the right-hand sides of (1.5) and (1.8) are well defined, as this is the case by [**27**, Theorem 1.1]. However, the detailed statement of Theorem 1.3 is useful for proving Theorem 1.1 and its generalizations in Remarks 1 and 2 whenever Y is not a Fano complete intersection. If Y is Fano, Theorem 1.1 can be obtained from [**27**] by working just with J-holomorphic, instead of (J,ν)-holomorphic, maps.

Remark: If $\mathfrak{L}$ is a direct sum of (ω,A)-positive line bundles, the Poincare dual of the euler class of the corresponding cone is defined to be the intersection product of the Poincare duals of the euler classes of the cones corresponding to the component line bundles. The intersection product can be defined by intersecting pseudocycle representatives for the above homology classes; see [**27**, Subsection 1.2].

1.3. Some special cases. By [**28**, Proposition 3.1], the difference between the standard and reduced genus-one invariants of a symplectic manifold (Y,ω) is a combination of the genus-zero invariants of Y. The exact form of this combination in general is determined in [**31**].

If (Y,ω,J) is an almost Kähler manifold, for each $l=1,\ldots,k$ let

$$\mathrm{ev}_l\colon \overline{\mathfrak{M}}_{g,k}(Y,A;J)\longrightarrow Y,\qquad \big[\Sigma,y_1,\ldots,y_k;u\big]\longrightarrow u(y_l),$$

be the evaluation map at the l-th marked point. We will call a cohomology class ψ on $\overline{\mathfrak{M}}_{g,k}(Y,A;J)$ geometric if ψ is a product of the classes $\mathrm{ev}_l^*\mu_l$ for $\mu_l\in H^*(Y;\mathbb{Z})$. By [**28**, Theorem 1.1], if $A\in H_2(Y;\mathbb{Z})^*$, then

$$\begin{aligned}&\mathrm{GW}^Y_{1,k}(A;\psi)-\mathrm{GW}^{0;Y}_{1,k}(A;\psi)\\ &\qquad\qquad=\begin{cases}0, & \text{if } \dim_{\mathbb{R}}Y=4;\\ \frac{2-\langle c_1(TY),A\rangle}{24}\mathrm{GW}^Y_{0,k}(A;\psi), & \text{if } \dim_{\mathbb{R}}Y=6,\end{cases}\end{aligned}\tag{1.12}$$

for every geometric cohomology class ψ on $\overline{\mathfrak{M}}_{1,k}(Y,A;J)$.

In the rest of this subsection, we discuss some implications of Theorem 1.1 and Remarks 1 and 2, combined with (1.12), focusing on Calabi-Yau complete-intersection threefolds. We note that if Y is a Calabi-Yau threefold, then the expected dimension of the moduli space $\overline{\mathfrak{M}}_{g,0}(Y,A;J)$ is zero for every g and A.

With notation as in Theorem (1.3), if $a=5$, Y is a quintic threefold. It can be easily seen that $c_1(TY)=0$. Let

$$N_g(d)=\mathrm{GW}^Y_{g,0}(d;1).$$

Theorem 1.3 and equation (1.12) then give the following corollary.

d	1	2	3	4
$\langle\ldots\rangle$	0	$\frac{2{,}875}{32}$	$\frac{49{,}355{,}000}{81}$	$\frac{952{,}691{,}384{,}375}{256}$
$N_1(d)$	$\frac{2{,}875}{12}$	$\frac{407{,}125}{8}$	$\frac{243{,}388{,}750}{9}$	$\frac{366{,}163{,}353{,}125}{16}$
$n_1(d)$	0	0	609,250	3,721,431,625

Table 1. Low-degree GW-invariants of a quintic threefold.

Corollary 1.5. *Suppose d is a positive integer, $\mathfrak{L}=\gamma^{*\otimes 5}\longrightarrow\mathbb{P}^4$, and*

$$\pi_1^d\colon \mathfrak{U}_1(\mathbb{P}^4,d)\longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d) \quad \text{and} \quad \mathrm{ev}_1^d\colon \mathfrak{U}_1(\mathbb{P}^4,d)\longrightarrow \mathbb{P}^4$$

are the universal family and the natural evaluation map, respectively. If $Y\subset\mathbb{P}^4$ is a smooth quintic threefold,

$$N_1(d)=\frac{1}{12}N_0(d)+\big\langle e\big(\pi_{1*}^d\mathrm{ev}_1^{d*}\mathfrak{L}\big),\big[\overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d)\big]\big\rangle. \tag{1.13}$$

The middle number in (1.13) can be computed using (1.2). This has been done for every d in [**5**], [**7**], [**9**], [**13**], and [**14**]. As mentioned in Subsection 1.1, the last number in (1.13) can be computed, for each given d, via the classical localization theorem of [**1**]. Similarly to the genus-zero case, the complexity of computing the last term in (1.13) increases rapidly with the degree d, but this has been fully carried out in [**30**], finally confirming the genus-one prediction of [**4**] for a quintic threefold. A few low-degree values are shown in the second row of Table 1 (these numbers were obtained by a direct localization computation and predate the desingularization construction of [**22**] and the complete computation of [**30**]). The numbers $n_1(d)$ that appear in the last row of this table are defined by

$$N_0(d)=\sum_{k|d}\frac{n_0(d/k)}{k^3},$$

$$N_1(d)=\frac{1}{12}\sum_{k|d}\frac{n_0(d/k)}{k}+\sum_{k|d}\frac{\sigma(k)}{k}n_1(d/k),\quad \sigma(k)=\sum_{r|k}r.$$

The numbers $n_0(d)$ and $n_1(d)$ are of importance in theoretical physics. Conjecturally, $n_g(d)$ is a count of J-holomorphic degree-d genus-g curves in Y for a generic almost complex structure J on Y.

With notation as in Remark 1 in Subsection 1.1, if

$$a_1+\ldots+a_m=n+1$$

and Y is a corresponding complete intersection, then Y is a Calabi-Yau threefold. Let

$$N_g^Y(d) = \mathrm{GW}_{g,0}^Y(d;1).$$

The identities in Remark 1 and in (1.12) then give

$$N_1^Y(d) = \frac{1}{12}N_0^Y(d) + \big\langle e\big(\pi_{1*}^d \mathrm{ev}_1^{d*}\mathfrak{L}\big), \big[\overline{\mathfrak{M}}_1^0(\mathbb{P}^n,d)\big]\big\rangle.$$

Once again, both terms on the right-hand side are computable via (1.2) and the classical localization theorem.

In the more general case of Remark 2 in Subsection 1.1, Y is a Calabi-Yau threefold if

$$c_1(\mathfrak{L}) - c_1(TX) = 0 \qquad \text{and} \qquad \dim_{\mathbb{R}} X - 2\mathrm{rk}_{\mathbb{C}}\mathfrak{L} = 6.$$

In such a case,

$$N_1^Y(A) = \frac{1}{12}N_0^Y(A) + \big\langle e(\mathcal{V}_{1,k}^A), \big[\overline{\mathfrak{M}}_1^0(X,A;J)\big]\big\rangle,$$

where $\mathcal{V}_{1,k}^A$ is the cone of "holomorphic sections" corresponding to $(\mathfrak{L},\nabla)$ and $N_g^Y(A) = \mathrm{GW}_{g,0}^Y(A;1)$.

Two completely different approaches to computing positive-genus GW-invariants of complete intersections have been proposed by Gathmann [**8**] and Maulik-Pandharipande [**18**]. Both approaches use degenerations and relative Gromov-Witten invariants. The first approach can be used to compute the genus-one and -two GW-invariants of a quintic threefold. The latter can in principle be used to compute arbitrary-genus GW-invariants of a quintic threefold as well as of some other low-degree low-dimensional complete intersections. In contrast, Theorem 1.1 above and [**28**, Proposition 3.1] are at the present restricted to genus-one GW-invariants only, but are applicable to arbitrary complete intersections.

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2. Hyperplane property for genus-one GW-invariants

2.1. Review of definitions. Suppose X is a compact manifold, $A \in H_2(X;\mathbb{Z})$, and $g,k \in \bar{\mathbb{Z}}^+$. Let $\mathfrak{X}_{g,k}(X,A)$ denote the space of equivalence classes of stable smooth maps $u\colon \Sigma \longrightarrow X$ from genus-g Riemann surfaces with k marked points, which may have simple nodes, to X of degree A, i.e.

$$u_*[\Sigma] = A \in H_2(X;\mathbb{Z}).$$

The spaces $\mathfrak{X}_{g,k}(X,A)$ are topologized using L_1^p-convergence on compact subsets of smooth points of the domain and certain convergence requirements near the nodes; see [**15**, Section 3]. Here and throughout the rest of the paper, p denotes a real number greater than two. The spaces $\mathfrak{X}_{g,k}(X,A)$ can be stratified by the smooth infinite-dimensional orbifolds $\mathfrak{X}_{\mathcal{T}}(X)$ of stable maps from domains of the same geometric type and with the same degree distribution between the components of the domain; see Subsections 3.1 and 3.2. The closure of the main stratum, $\mathfrak{X}^0_{g,k}(X,A)$, is $\mathfrak{X}_{g,k}(X,A)$.

If J is an almost complex structure on X, let

$$\Gamma^{0,1}_{g,k}(X,A;J)\longrightarrow\mathfrak{X}_{g,k}(X,A)$$

be the bundle of (TX,J)-valued $(0,1)$-forms. In other words, the fiber of $\Gamma^{0,1}_{g,k}(X,A;J)$ over a point $[b]=[\Sigma,j;u]$ in $\mathfrak{X}_{g,k}(X,A)$ is the space

$$\Gamma^{0,1}_{g,k}(X,A;J)\big|_{[b]}=\Gamma^{0,1}(b;J)\big/\mathrm{Aut}(b),\qquad\text{where}$$
$$\Gamma^{0,1}(b;J)=\Gamma\big(\Sigma;\Lambda^{0,1}_{J,j}T^*\Sigma\otimes u^*TX\big).$$

Here j is the complex structure on Σ, the domain of the smooth map u. The bundle $\Lambda^{0,1}_{J,j}T^*\Sigma\otimes u^*TX$ over Σ consists of (J,j)-antilinear homomorphisms:

$$\Lambda^{0,1}_{J,j}T^*\Sigma\otimes u^*TX=\big\{\eta\in\mathrm{Hom}(T\Sigma,u^*TX)\colon J\circ\eta=-\eta\circ j\big\}.$$

The total space of the bundle $\Gamma^{0,1}_{g,k}(X,A;J)\longrightarrow\mathfrak{X}_{g,k}(X,A)$ is topologized using L^p-convergence on compact subsets of smooth points of the domain and certain convergence requirements near the nodes. The restriction of $\Gamma^{0,1}_{g,k}(X,A;J)$ to each stratum $\mathfrak{X}_{\mathcal{T}}(X)$ is a smooth vector orbibundle of infinite rank.

We define a continuous section of the bundle

$$\Gamma^{0,1}_{g,k}(X,A;J)\longrightarrow\mathfrak{X}_{g,k}(X,A)\qquad\text{by}$$
$$\bar\partial_J\big([\Sigma,j;u]\big)=\bar\partial_{J,j}u=\frac{1}{2}\big(du+J\circ du\circ j\big).$$

By definition, the zero set of this section is $\overline{\mathfrak{M}}_{g,k}(X,A;J)$. The restriction of $\bar\partial_J$ to each stratum of $\mathfrak{X}_{g,k}(X,A)$ is smooth. The section $\bar\partial_J$ is Fredholm, i.e. the linearization of its restriction to every stratum $\mathfrak{X}_{\mathcal{T}}(X)$ has finite-dimensional kernel and cokernel at every point of $\bar\partial_J^{-1}(0)\cap\mathfrak{X}_{\mathcal{T}}(X)$. The index of the linearization of $\bar\partial_J$ at an element of $\mathfrak{M}^0_{g,k}(X,A;J)$ is the expected dimension of the moduli space $\overline{\mathfrak{M}}_{g,k}(X,A;J)$,

$$\dim_{g,k}(X,A)\equiv2\big(\langle c_1(TX),A\rangle+(1-g)(n-3)+k\big),$$

where $2n=\dim_{\mathbb{R}}X$. This is the dimension of the cycle

$$\overline{\mathfrak{M}}_{g,k}(X,A;J,\nu)\equiv\big\{\bar\partial_J+\nu\big\}^{-1}(0)$$

for a small generic multivalued perturbation

$$\nu\in\mathfrak{G}^{0,1}_{g,k}(X,A;J)\equiv\Gamma\big(\mathfrak{X}_{g,k}(X,A),\Gamma^{0,1}_{g,k}(X,A;J)\big)$$

of $\bar{\partial}_J$, where $\mathfrak{G}^{0,1}_{g,k}(X,A;J)$ is the space of all continuous multisections ν of $\Gamma^{0,1}_{g,k}(X,A;J)$ such that the restriction of ν to each stratum $\mathfrak{X}_{\mathcal{T}}(X)$ is smooth. (Our term multisection, or multivalued section, corresponds to the notion of locally liftable multi-section in [**6**, Section 3].) Since the moduli space $\overline{\mathfrak{M}}_{g,k}(X,A;J)$ is compact, so is $\overline{\mathfrak{M}}_{g,k}(X,A;J,\nu)$ if ν is sufficiently small.

An element $[\Sigma;u]$ of $\mathfrak{X}_{1,k}(X,A)$ is an equivalence class of pairs consisting of a prestable genus-one Riemann surface Σ and a smooth map $u\colon\Sigma\longrightarrow X$. The prestable surface Σ is a union of the principal component(s) Σ_P, which is either a smooth torus or a circle of spheres, and trees of rational bubble components, which together will be denoted by Σ_B. Let

$$\mathfrak{X}^{\{0\}}_{1,k}(X,A)=\big\{[\Sigma;u]\in\mathfrak{X}_{1,k}(X,A)\colon u_*[\Sigma_P]\neq 0\in H_2(X;\mathbb{Z})\big\}.$$

Suppose

$$[\Sigma;u]\in\mathfrak{X}_{1,k}(X,A)-\mathfrak{X}^{\{0\}}_{1,k}(X,A), \tag{2.1}$$

i.e. the degree of $u|_{\Sigma_P}$ is zero. Let $\chi^0(\Sigma;u)$ be the set of components Σ_i of Σ such that for every bubble component Σ_h that lies between Σ_i and Σ_P, including Σ_i itself, the degree of $u|_{\Sigma_h}$ is zero. The set $\chi^0(\Sigma;u)$ includes the principal component(s) of Σ. We give an example of the set $\chi^0(\Sigma;u)$ in Figure 1. In this figure, we show the domain Σ of the stable map $(\Sigma;u)$ and shade the components of the domain on which the degree of the map u is not zero. Let

$$\Sigma^0_u=\bigcup_{i\in\chi^0(\Sigma;u)}\Sigma_i.$$

Every bubble component $\Sigma_i\subset\Sigma_B$ is a sphere and has a distinguished singular point, which will be called the *attaching node of* Σ_i. This is the node of Σ_i that lies either on Σ_P or on a bubble Σ_h that lies between Σ_i and Σ_P. We denote by $\chi(\Sigma;u)$ the set of bubble components Σ_i such that the attaching node of Σ_i lies on Σ^0_u and the degree of $u|_{\Sigma_i}$ is not zero.

Definition 2.1. Suppose (X,ω) is a compact symplectic manifold and $\underline{J}\equiv(J_t)_{t\in[0,1]}$ is a C^1-continuous family of ω-tamed almost structures on X. A continuous family of multisections $\underline{\nu}\equiv(\nu_t)_{t\in[0,1]}$, with $\nu_t\in\mathfrak{G}^{0,1}_{1,k}(X,A;J_t)$ for all $t\in[0,1]$, is effectively supported if for every element

$$b\equiv[\Sigma;u]\in\mathfrak{X}_{1,k}(X,A)-\mathfrak{X}^{\{0\}}_{1,k}(X,A)$$

there exists a neighborhood $\mathcal{W}_b$ of Σ^0_u in a semi-universal family of deformations for b such that

$$\nu_t(\Sigma';u')\big|_{\Sigma'\cap\mathcal{W}_b}=0\qquad\forall\ [\Sigma';u']\in\mathfrak{X}_{1,k}(X,A),\ t\in[0,1].$$

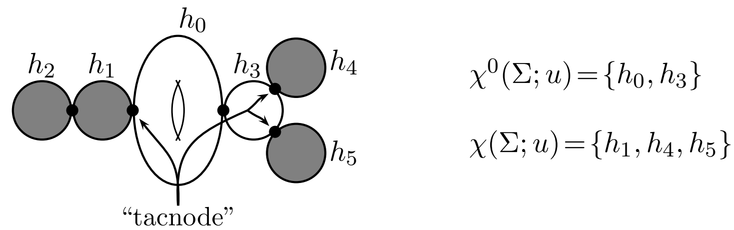

Figure 1. An illustration of Definition 2.2.

If $b=[\Sigma;u]$ is an element of $\mathfrak{X}_{1,k}(X,A)$, a semi-universal universal family of deformations for b is a fibration

$$\sigma_b\colon \tilde{\mathcal{U}}_b \longrightarrow \Delta_b$$

such that $\Delta_b/\mathrm{Aut}(b)$ is a neighborhood of b in $\mathfrak{X}_{1,k}(X,A)$ and the fiber of σ_b over a point $[\Sigma';u']$ is Σ'. If $\underline{J}\equiv(J_t)_{t\in[0,1]}$ is a continuous family of ω-tamed almost structures on X, we denote the space of effectively supported families $\underline{\nu}$ as in Definition 2.1 by $\mathfrak{G}^{\mathrm{es}}_{1,k}(X,A;\underline{J})$. Similarly, if J is an almost complex structure on X, let $\mathfrak{G}^{\mathrm{es}}_{1,k}(X,A;J)$ be the subspace of elements ν of $\mathfrak{G}^{0,1}_{1,k}(X,A;J)$ such that the family $\nu_t=\nu$ is effectively supported.

Suppose $\nu\in\mathfrak{G}^{\mathrm{es}}_{1,k}(X,A;J)$ and $[\Sigma;u]$ is an element of $\overline{\mathfrak{M}}_{1,k}(X,A;J,\nu)$ as in Definition 2.1. Since $\Sigma_i\subset\Sigma_B$ is a sphere, we can represent this element by a pair $(\Sigma;u)$ such that the attaching node of every bubble component $\Sigma_i\subset\Sigma_B$ is the south pole, or the point $\infty=(0,0,-1)$, of $S^2\subset\mathbb{R}^3$. Let $e_\infty=(1,0,0)$ be a nonzero tangent vector to S^2 at the south pole. If $i\in\chi(\Sigma;u)$, we put

$$\mathcal{D}_i(\Sigma;u)=d\big\{u|_{\Sigma_i}\big\}\big|_\infty e_\infty\in T_{u|_{\Sigma_i}(\infty)}X.$$

We note that $u|_{\Sigma^0_u}$ is a degree-zero holomorphic map and thus constant. Thus, u maps the attaching nodes of all elements of $\chi(\Sigma;u)$ to the same point in X.

Definition 2.2. Suppose (X,ω,J) is a compact almost Kähler manifold, $A\in H_2(X;\mathbb{Z})^*$, and $k\in\bar{\mathbb{Z}}^+$. If $\nu\in\mathfrak{G}^{\mathrm{es}}_{1,k}(X,A;J)$ is an effectively supported perturbation of the $\bar\partial_J$-operator, the main component of the space $\overline{\mathfrak{M}}_{1,k}(X,A;J,\nu)$ is the subset $\overline{\mathfrak{M}}^0_{1,k}(X,A;J,\nu)$ consisting of the elements $[\Sigma;u]$ of $\overline{\mathfrak{M}}_{1,k}(X,A;J,\nu)$ such that

(a) the degree of $u|_{\Sigma_P}$ is not zero, or

(b) the degree of $u|_{\Sigma_P}$ is zero and

$$\dim_{\mathbb{C}}\mathrm{Span}_{(\mathbb{C},J)}\{\mathcal{D}_i(\Sigma;u)\colon i\in\chi(\Sigma;u)\}<|\chi(\Sigma;u)|.$$

By [**28**, Theorem 1.4], $\overline{\mathfrak{M}}^0_{1,k}(X,A;J,\nu)$ is a compact space if ν is effectively supported and sufficiently small. For a generic effectively supported ν, $\overline{\mathfrak{M}}^0_{1,k}(X,A;J,\nu)$ determines a homology class of the expected dimension in a small neighborhood of $\overline{\mathfrak{M}}^0_{1,k}(X,A;J)$ in $\mathfrak{X}_{1,k}(X,A)$ which is independent of ν and J; see [**28**, Theorem 1.5, Corollary 1.6].

If X, A, g, and k are as above and $(\mathfrak{L},\nabla)\longrightarrow X$ is a vector bundle with connection, we denote by

$$\Gamma_{g,k}(\mathfrak{L},A)\longrightarrow\mathfrak{X}_{g,k}(X,A)$$

the cone such that the fiber of $\Gamma_{g,k}(\mathfrak{L},A)$ over $[b]=[\Sigma;u]$ in $\mathfrak{X}_{g,k}(X,A)$ is the Banach space

$$\Gamma_{g,k}(\mathfrak{L},A)\big|_{[b]}=\Gamma(b;\mathfrak{L})\big/\mathrm{Aut}(b),\qquad\text{where}\qquad\Gamma(b;\mathfrak{L})=L^p_1(\Sigma;u^*\mathfrak{L}).$$

The topology on the total space of $\Gamma_{g,k}(\mathfrak{L},A)$ is defined analogously to the topology on $\Gamma_{g,k}(TX,A)$ of [**15**, Section 3]. Let

$$\begin{aligned}\mathcal{V}^A_{g,k}=\big\{[b,\xi]\in\Gamma_{g,k}(\mathfrak{L},A)\colon[b]\in\mathfrak{X}_{g,k}(X,A);\ \xi\in\ker\bar\partial_{\nabla,b}\subset\Gamma_{g,k}(b;\mathfrak{L})\big\}\\ \subset\Gamma_{g,k}(\mathfrak{L},A).\end{aligned}$$

The cone $\mathcal{V}^A_{g,k}\longrightarrow\mathfrak{X}_{g,k}(X,A)$ inherits its topology from $\Gamma_{g,k}(\mathfrak{L},A)$.

2.2. Statement and proof of hyperplane property. We will call a cohomology class ψ on $\mathfrak{X}_{1,k}(X,A)$ tautological if there exists a vector bundle

$$\mathcal{W}\longrightarrow\mathfrak{X}_{1,k}(X,A)$$

such that $\mathcal{W}|_{\mathfrak{X}_{\mathcal{T}}(X)}$ is smooth for every stratum $\mathfrak{X}_{\mathcal{T}}(X)$ of $\mathfrak{X}_{1,k}(X,A)$ and $\psi=e(\mathcal{W})$.

If (X,J) is an almost complex manifold and $(\mathfrak{L},\nabla)\longrightarrow X$ is a complex vector bundle with connection, we will call a section s of $\mathfrak{L}$ ∇-holomorphic if

$$\bar\partial_\nabla s\equiv\frac{1}{2}\big(\nabla s+\mathfrak{i}\nabla s\circ J\big)=0.$$

Theorem 2.3. *Suppose (X,ω,J) is a compact almost Kähler manifold, $A\in H_2(X;\mathbb{Z})^*$, $k\in\mathbb{Z}^+$, $(\mathfrak{L},\nabla)\longrightarrow X$ is a complex vector bundle with connection, and s is a ∇-holomorphic section of $\mathfrak{L}$ such that J is genus-one A-regular in the sense of* [**26**, Definition 1.4]*, s is transverse to the zero set in $\mathfrak{L}$, and $(\mathfrak{L},\nabla)$ splits into (ω,A)-positive line bundles. If $\mathcal{V}^A_{1,k}\longrightarrow\mathfrak{X}_{1,k}(X,A)$ is the cone corresponding to $(\mathfrak{L},\nabla)$ and $Y=s^{-1}(0)$,*

$$\mathrm{GW}^{0;Y}_{1,k}(A;\psi)=\big\langle\psi,\mathrm{PD}_{\overline{\mathfrak{M}}^0_{1,k}(X,A;J)}e(\mathcal{V}^A_{1,k})\big\rangle\tag{2.2}$$

for every tautological class ψ on $\mathfrak{X}_{1,k}(X,A)$.

Proof. Since J is genus-one A-regular, $\overline{\mathfrak{M}}^0_{1,k}(X,A;J)$ has the expected structure of a topological orbivariety. By a generalization of the proof of the regularity statement of [**26**, Theorem 1.6] analogous to [**28**, Subsection 2.5], for all $\nu\in\mathfrak{G}^{\mathrm{es}}_{1,k}(X,A;J)$ sufficiently small $\overline{\mathfrak{M}}^0_{1,k}(X,A;J,\nu)$ also has the expected structure of a topological orbivariety. In particular, it is stratified by smooth orbifolds of even dimensions as described in Subsection 3.4 below. We will call $\nu\in\mathfrak{G}^{\mathrm{es}}_{1,k}(X,A;J)$ (∇,s)-compatible if

$$\nabla s|_u\circ\nu(\Sigma;u)=0 \qquad \forall\,[\Sigma;u]\in\mathfrak{X}_{1,k}(X,A).$$

We note that if ν is (∇,s)-compatible, then the map

$$(\Sigma;u)\longrightarrow s^A_{1,k}(\Sigma;u)\equiv s\circ u\in\Gamma(\Sigma;u^*\mathfrak{L})$$

defines a continuous section of the cone $\mathcal{V}^A_{1,k}$ over $\overline{\mathfrak{M}}^0_{1,k}(X,A;J)$.

Since the ∇-holomorphic section s is transverse to the zero set in $\mathfrak{L}$, the $(\mathfrak{i},J)$-linear map

$$\nabla s\colon TX\longrightarrow\mathfrak{L}$$

is surjective along $Y=s^{-1}(0)$. Let U_s be a small neighborhood of Y in X such that ∇s is surjective over U_s. The kernel of ∇s over U_s is then a complex subbundle of $(TX,J)|_{U_s}$, which restricts to TY along Y. We denote this subbundle by $\tilde{T}Y$. If $\nu\in\mathfrak{G}^{\mathrm{es}}_{1,k}(X,A;J)$ is such that for all $[\Sigma;u]\in\mathfrak{X}_{1,k}(X,A)$

$$\nu(\Sigma;u)\begin{cases}\in\Gamma(\Sigma;\Lambda^{0,1}_{J,j}T^*\Sigma\otimes\tilde{T}Y), & \text{if } u(\Sigma)\subset U_s;\\ =0, & \text{otherwise,}\end{cases}$$

then ν is (∇,s)-compatible. Thus, every element $\nu_Y\in\mathfrak{G}^{\mathrm{es}}_{1,k}(Y,A;J)$ can extended to a (∇,s)-compatible element ν of $\mathfrak{G}^{\mathrm{es}}_{1,k}(X,A;J)$. Furthermore, if ν_Y is a small, then ν can also be chosen to be small.

For a small generic $\nu_Y\in\mathfrak{G}^{\mathrm{es}}_{1,k}(Y,A;J)$, $\overline{\mathfrak{M}}^0_{1,k}(Y,A;J,\nu_Y)$ is stratified by smooth orbifolds of even dimensions so that the largest-dimensional stratum is $\mathfrak{M}^0_{1,k}(Y,A;J,\nu_Y)$ and

$$\dim\mathfrak{M}^0_{1,k}(Y,A;J,\nu_Y)=\dim_{1,k}(Y,A).$$

Let ν be an extension of ν_Y to a small (∇,s)-compatible element of $\mathfrak{G}^{\mathrm{es}}_{1,k}(X,A;J)$. Suppose

$$\mathcal{W}\longrightarrow\mathfrak{X}_{1,k}(X,A)$$

is a complex vector bundle of rank $\dim_{1,k}(Y,A)/2$ as in the first paragraph of this subsection. Choose a section f of $\mathcal{W}$ over $\overline{\mathfrak{M}}^0_{1,k}(X,A;J,\nu)$ such that $f|_{\mathcal{U}_\alpha}$ is transverse to the zero set in $\mathcal{W}|_{\mathcal{U}_\alpha}$ for every stratum $\mathcal{U}_\alpha$ of $\overline{\mathfrak{M}}^0_{1,k}(X,A;J,\nu)$ and of $\overline{\mathfrak{M}}^0_{1,k}(Y,A;J,\nu_Y)$. Then,

$$f^{-1}(0)\cap\overline{\mathfrak{M}}^0_{1,k}(Y,A;J,\nu_Y)\subset\mathfrak{M}^0_{1,k}(Y,A;J,\nu_Y) \qquad \text{and}$$

$$\mathrm{GW}^{0;Y}_{1,k}(A;\psi)={}^{\pm}\big|f^{-1}(0)\cap\overline{\mathfrak{M}}^0_{1,k}(Y,A;J,\nu_Y)\big|. \tag{2.3}$$

On the other hand, since ν is (∇, s)-compatible, $s^A_{1,k}$ is a section of

$$\mathcal{V}^A_{1,k} \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(X,A;J,\nu).$$

Furthermore,

$$\begin{aligned} \{s^A_{1,k}\}^{-1}(0) = \overline{\mathfrak{M}}^0_{1,k}(X,A;J,\nu)\cap\mathfrak{X}_{1,k}(Y,A) \equiv \overline{\mathfrak{M}}^0_{1,k}(Y,A;J,\nu_Y) \quad \Longrightarrow \\ f^{-1}(0)\cap\{s^A_{1,k}\}^{-1}(0) = f^{-1}(0)\cap\overline{\mathfrak{M}}^0_{1,k}(Y,A;J,\nu_Y) \\ \subset \mathfrak{M}^0_{1,k}(X,A;J,\nu). \end{aligned} \tag{2.4}$$

Note that if $[b]=[\Sigma;u]\in\mathfrak{M}^0_{1,k}(Y,A;J,\nu_Y)$,

$$\begin{aligned} \ker\nabla s^A_{1,k}|_b &= \{\xi\in\ker D_{J,\nu;b} \colon \nabla s|_u\circ\xi=0\} \\ &= \ker D_{J,\nu;b}\cap\Gamma(\Sigma;u^*TY) = \ker D_{J|_Y,\nu_Y;b}, \end{aligned} \tag{2.5}$$

where $D_{J,\nu;b}$ and $D_{J|_Y,\nu_Y;b}$ are the linearizations of the sections $\bar\partial_J+\nu$ and $\bar\partial_{J|_Y}+\nu_Y$ at b. The second equality above is immediate from the transversality of s. By (2.5),

$$\begin{aligned} \dim_{\mathbb{R}}\operatorname{Im}\nabla s^A_{1,k}|_{(\Sigma;u)} &= \dim\ker D_{J,\nu;b} - \dim\ker D_{J|_Y,\nu_Y;b} \\ &= \dim_{1,k}(X,A) - \dim_{1,k}(Y,A) \\ &= 2\langle c_1(\mathfrak{L}),A\rangle = \dim_{\mathbb{R}}\mathcal{V}^A_{1,k}|_{[b]}. \end{aligned} \tag{2.6}$$

The second equality above follows from our assumption that the operators $D_{J,\nu;b}$ and $D_{J|_Y,\nu_Y;b}$ are surjective; the last equality is a consequence of the $(\omega,\mathfrak{L})$-positivity assumption. By (2.6), $s^A_{1,k}$ is transverse to the zero set in $\mathcal{V}^A_{1,k}$ along $\mathfrak{M}^0_{1,k}(X,A;J,\nu)$. Since f is transverse to the zero set in $\mathcal{W}$ along $\mathfrak{M}^0_{1,k}(X,A;J,\nu_Y)$, it then follows from (2.4) that

$$\begin{aligned} \big\langle\psi,\mathrm{PD}_{\overline{\mathfrak{M}}^0_{1,k}(X,A;J)}e(\mathcal{V}^A_{1,k})\big\rangle &\equiv {}^{\pm}\big|f^{-1}(0)\cap\{s^A_{1,k}\}^{-1}(0)\big| \\ &= {}^{\pm}\big|f^{-1}(0)\cap\overline{\mathfrak{M}}^0_{1,k}(Y,A;J,\nu_Y)\big|. \end{aligned} \tag{2.7}$$

Theorem 2.3 follows from (2.3) and (2.7). q.e.d.

3. Ingredients in proof of Theorem 1.3

3.1. Notation: genus-zero maps. In this subsection we describe our detailed notation for bubble maps from genus-zero Riemann surfaces and for related objects. In general, moduli spaces of stable maps can stratified by the dual graph. However, in the present situation, it is more convenient to make use of *linearly ordered sets*.

Definition 3.1. (1) A finite nonempty partially ordered set I is a linearly ordered set if for all $i_1,i_2,h\in I$ such that $i_1,i_2<h$, either $i_1\le i_2$ or $i_2\le i_1$.
(2) A linearly ordered set I is a rooted tree if I has a unique minimal element, i.e. there exists $\hat 0\in I$ such that $\hat 0\le i$ for all $i\in I$.

If I is a linearly ordered set, let $\hat{I}$ be the subset of the non-minimal elements of I. For every $h\in\hat{I}$, denote by $\iota_h\in I$ the largest element of I which is smaller than h, i.e. $\iota_h=\max\{i\in I : i<h\}$.

We identify $\mathbb{C}$ with $S^2-\{\infty\}$ via the stereographic projection mapping the origin in $\mathbb{C}$ to the north pole, or the point $(0,0,1)$, in S^2. Let M be a finite set. A genus-zero X-valued bubble map with M-marked points is a tuple

$$b=\big(M,I;x,(j,y),u\big),$$

where I is a rooted tree, and

$$\begin{aligned} &x\colon \hat{I}\longrightarrow\mathbb{C}=S^2-\{\infty\}, \qquad j\colon M\longrightarrow I,\\ &y\colon M\longrightarrow\mathbb{C}, \quad \text{and} \quad u\colon I\longrightarrow C^\infty(S^2;X) \end{aligned} \tag{3.1}$$

are maps such that $u_h(\infty)=u_{\iota_h}(x_h)$ for all $h\in\hat{I}$. We associate such a tuple with Riemann surface

$$\begin{aligned} &\Sigma_b=\Big(\bigsqcup_{i\in I}\Sigma_{b,i}\Big)\Big/\sim, \qquad \text{where}\\ &\Sigma_{b,i}=\{i\}\times S^2, \quad (h,\infty)\sim(\iota_h,x_h)\ \forall h\in\hat{I}, \end{aligned} \tag{3.2}$$

with marked points

$$y_l(b)\equiv(j_l,y_l)\in\Sigma_{b,j_l} \qquad \text{and} \qquad y_0(b)\equiv(\hat{0},\infty)\in\Sigma_{b,\hat{0}},$$

and continuous map $u_b\colon\Sigma_b\longrightarrow X$, given by $u_b|_{\Sigma_{b,i}}=u_i$ for all $i\in I$. The general structure of bubble maps is described by tuples $\mathcal{T}=(M,I;j,\underline{A})$, where

$$A_i=u_{i*}[S^2]\in H_2(X;\mathbb{Z}) \qquad \forall i\in I.$$

We call such tuples bubble types. Let $\tilde{\mathfrak{X}}_{\mathcal{T}}(X)$ be the space of all bubble maps of type $\mathcal{T}$. For $l\in\{0\}\sqcup M$, let

$$\mathrm{ev}_l\colon\tilde{\mathfrak{X}}_{\mathcal{T}}(X)\longrightarrow X$$

be the evaluation map corresponding to the marked point y_l.

With notation as above, suppose

$$b\equiv\big(M,I;x,(j,y),u\big)\in\tilde{\mathfrak{X}}_{\mathcal{T}}(X).$$

In particular, I is a linearly ordered set with minimal element $\hat{0}$ and the special marked point is the point

$$y_0(b)=(\hat{0},\infty)\in\Sigma_{b,\hat{0}}.$$

Let $\chi^0(b)$ be the set of components $\Sigma_{b,i}$ of Σ_b such that for every component $\Sigma_{b,h}$ that lies between Σ_i and $\Sigma_{b,\hat{0}}$, including $\Sigma_{b,i}$ and $\Sigma_{b,\hat{0}}$, the degree of $u|_{\Sigma_{b,h}}$ is zero. The set $\chi^0(b)$ is empty if and only if the degree of the restriction of u_b to the component containing the special marked point is not zero. Let

$$\Sigma_b^0 = \{(\hat{0},\infty)\} \cup \bigcup_{i\in\chi^0(b)} \Sigma_{b,i}.$$

We denote by

$$\chi(b)=\chi(\mathcal{T}) \subset I$$

the set of components $\Sigma_{b,i}$ of Σ_b such that $\Sigma_{b,i}$ has a point in common with Σ_b^0 and the degree of $u_b|_{\Sigma_{b,i}}$ is not zero, i.e. $\Sigma_{b,i}$ is not an element of $\chi^0(b)$.

If $b=(\Sigma_b;u_b)$ is a bubble map with a special marked point as above and $i\in\chi(b)$, we put

$$\mathcal{D}_i b = du_{b,i}\big|_{\infty} e_\infty \in T_{u_{b,i}(\infty)}X,$$

where $u_{b,i}=u_b|_{\Sigma_{b,i}}$. Similarly, if $(\mathfrak{L},\nabla)$ is a complex line bundle with connection over X and $\xi=(\xi_h)_{h\in I}$ is an element of $\Gamma(b;\mathfrak{L})$, we put

$$\mathfrak{D}_{b,i}\xi = \nabla^{u_b}_{e_\infty}\xi_i\big|_{\infty} \in \mathfrak{L}_{u_{b,i}(\infty)}.$$

Note that if $\xi\in\ker\bar{\partial}_{\nabla,b}$,

$$\nabla^{u_b}_{c\cdot e_\infty}\xi_i\big|_{\infty} = c\cdot\mathfrak{D}_{b,i}\xi \qquad \forall\, c\in\mathbb{C}. \tag{3.3}$$

If in addition $\mathfrak{L}$ is $(\omega,u_{b*}[\Sigma_b])$-positive, then the linear operator $\bar{\partial}_{\nabla,b}$ and the linear map

$$\bar{\partial}^{y_0(b)}_{\nabla,b}\colon \ker\bar{\partial}_{\nabla,b} \longrightarrow \mathfrak{L}_{\mathrm{ev}_0(b)}, \qquad \xi \longrightarrow \xi\big(y_0(b)\big),$$

are surjective. This can be seen by an argument similar to [**24**, Subsection 6.2].

3.2. Notation: genus-one maps. We next set up notation for maps from genus-one Riemann surfaces. In this case, in contrast to the genus-zero case, we also need to specify the structure of the principal component. We describe it by *enhanced linearly ordered sets.*

Definition 3.2. An enhanced linearly ordered set is a pair $(I,\aleph)$, where I is a linearly ordered set, $\aleph$ is a subset of $I_0\times I_0$, and I_0 is the subset of minimal elements of I, such that if $|I_0|>1$,

$$\aleph = \{(i_1,i_2),(i_2,i_3),\ldots,(i_{n-1},i_n),(i_n,i_1)\}$$

for some bijection $i\colon\{1,\ldots,n\}\longrightarrow I_0$.

An enhanced linearly ordered set can be represented by an oriented connected graph. In Figure 2, the dots denote the elements of I. The arrows outside the loop, if there are any, specify the partial ordering of the linearly ordered set I. In fact, every directed edge outside of the loop connects a non-minimal element h of I with ι_h. Inside of the loop, there is a directed edge from i_1 to i_2 if and only if $(i_1,i_2)\in\aleph$.

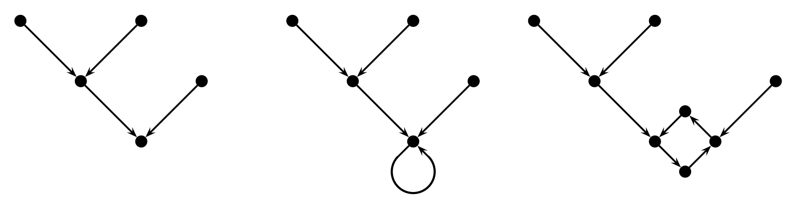

Figure 2. Some enhanced linearly ordered sets.

The subset $\aleph$ of $I_0\times I_0$ will be used to describe the structure of the principal curve of the domain of stable maps in a stratum of $\mathfrak{X}_{1,M}(X,A)$. If $\aleph=\emptyset$, and thus $|I_0|=1$, the corresponding principal curve Σ_P is a smooth torus, with some complex structure. If $\aleph\neq\emptyset$, the principal components form a circle of spheres:

$$\Sigma_P=\Big(\bigsqcup_{i\in I_0}\{i\}\times S^2\Big)\Big/\sim, \quad \text{where} \quad (i_1,\infty)\sim(i_2,0) \text{ if } (i_1,i_2)\in\aleph.$$

A genus-one X-valued bubble map with M-marked points is a tuple

$$b=\big(M,I,\aleph;S,x,(j,y),u\big),$$

where S is a smooth Riemann surface of genus one if $\aleph=\emptyset$ and the circle of spheres Σ_P otherwise. The objects x, j, y, u, and (Σ_b,u_b) are as in (3.1) and (3.2), except the sphere $\Sigma_{b,\hat{0}}$ is replaced by the genus-one curve $\Sigma_{b;P}\equiv S$. Furthermore, if $\aleph=\emptyset$, and thus $I_0=\{\hat{0}\}$ is a single-element set, $u_{\hat{0}}\in C^\infty(S;X)$ and $y_l\in S$ if $j_l=\hat{0}$. In the genus-one case, the general structure of bubble maps is encoded by the tuples of the form $\mathcal{T}=(M,I,\aleph;j,\underline{A})$. Similarly to the genus-zero case, we denote by $\tilde{\mathfrak{X}}_{\mathcal{T}}(X)$ be the space of all bubble maps of type $\mathcal{T}$. Let

$$\mathfrak{X}_{\mathcal{T}}(X)=\big\{[b]\in\mathfrak{X}_{1,M}(X,A)\colon b\in\tilde{\mathfrak{X}}_{\mathcal{T}}(X)\big\}.$$

If ν is an element of $\mathfrak{G}^{\text{es}}_{1,M}(X,A)$, we put

$$\mathcal{U}_{\mathcal{T},\nu}(X;J)=\big\{[b]\in\mathfrak{X}_{\mathcal{T}}(X)\colon\{\bar{\partial}_J+\nu\}(b)=0\big\}.$$

If $\mathcal{T}=(M,I,\aleph;j,\underline{A})$ is a bubble type such that $A_i=0$ for all minimal elements i of I and $[\Sigma;u]$ is an element of $\mathcal{U}_{\mathcal{T},\nu}(X;J)$, the map $u|_{\Sigma_P}$ is constant. Let

$$\text{ev}_P\colon\mathcal{U}_{\mathcal{T},\nu}(X;J)\longrightarrow X$$

be the map sending each element $[\Sigma;u]$ of $\mathcal{U}_{\mathcal{T},\nu}(X;J)$ to the image of the principal component Σ_P of Σ, i.e. the point $u(\Sigma_P)$ in X. We note that the map

$$\tilde{\mathfrak{X}}_{\mathcal{T}}(X)\longrightarrow 2^I, \qquad b\longrightarrow\chi(b),$$

is constant. We denote its value by $\chi(\mathcal{T})$.

Suppose $b=(\Sigma_b;u_b)$ is an element of $\mathfrak{X}_{1,k}(X,A)$ as above and $(\mathfrak{L},\nabla)$ is a complex vector bundle with connection over X. If $\xi=(\xi_h)_{h\in I}$ is an element of $\Gamma(b;\mathfrak{L})$ and $i\in I-I_0$, similarly to the genus-zero case, we put

$$\mathfrak{D}_{b,i}\xi = \nabla^{u_b}_{e_\infty}\xi_i\big|_\infty \in \mathfrak{L}_{u_{b,i}(\infty)},$$

where $u_{b,i}=u_b|_{\Sigma_{b,i}}$.

Finally, all vector orbi-bundles we encounter will be assumed to be normed. Some will come with natural norms; for others, we implicitly choose a norm once and for all. If $\pi_{\mathfrak{F}}\colon \mathfrak{F}\longrightarrow\mathfrak{X}$ is a normed vector bundle and $\delta\colon\mathfrak{X}\longrightarrow\mathbb{R}$ is any function, possibly constant, let

$$\mathfrak{F}_\delta = \big\{\upsilon\in\mathfrak{F}\colon |\upsilon|<\delta(\pi_{\mathfrak{F}}(\upsilon))\big\}.$$

If Ω is any subset of $\mathfrak{F}$, we take $\Omega_\delta=\Omega\cap\mathfrak{F}_\delta$.

3.3. Topology. In this subsection we prove a general topological statement, Proposition 3.6. For the sake of clarity, we state and prove it only in the manifold/section category, but Proposition 3.6 and its proof carry over easily to the orbifold/multisection category.

Similarly to Subsection 1.2, by a cone $\pi\colon\mathcal{W}\longrightarrow\overline{\mathfrak{M}}$ we mean a continuous map between two topological spaces such that $\mathcal{W}_x\equiv\pi^{-1}(x)$ is vector space for each $x\in\overline{\mathfrak{M}}$ and the vector space operations induce continuous functions on $\mathcal{W}\times_{\overline{\mathfrak{M}}}\mathcal{W}$ and $\mathbb{C}\times\mathcal{W}$.

Definition 3.3. (1) A cone $\pi\colon\mathcal{W}\longrightarrow\overline{\mathfrak{M}}$ is regular if for every $b\in\overline{\mathfrak{M}}$, there exist a neighborhood U_b of b in $\overline{\mathfrak{M}}$, $n_b\in\mathbb{Z}^+$, and a bundle map

$$\varphi_b\colon \mathcal{W}|_{U_b}\longrightarrow U_b\times\mathbb{C}^{n_b}$$

over U_b such that φ_b is a homeomorphism onto its image and the restriction of φ_b to each fiber is linear.
(2) A cone $\pi\colon\mathcal{W}\longrightarrow\overline{\mathfrak{M}}$ is obstruction-free if for every $b\in\overline{\mathfrak{M}}$, $\xi\in\mathcal{W}_b$, and a sequence $b_r\in\overline{\mathfrak{M}}$ converging to b, there exists a sequence $\xi_r\in\mathcal{W}_{b_r}$ converging to ξ in $\mathcal{W}$.

If $\mathcal{W}\longrightarrow\overline{\mathfrak{M}}$ is a cone, for each $r\in\mathbb{Z}$, let

$$\overline{\mathfrak{M}}_r(\mathcal{W}) = \big\{b\in\overline{\mathfrak{M}}\colon \mathrm{rk}\,\mathcal{W}_b=r\big\}.$$

Note that if $\mathcal{W}$ is obstruction-free, then the set $\bigcup_{r\le q}\overline{\mathfrak{M}}_r(\mathcal{W})$ is closed in $\overline{\mathfrak{M}}$.

Lemma 3.4. *Suppose $\overline{\mathfrak{M}}$ is a compact Hausdorff space that has a countable basis at each point, A is a closed subset of $\overline{\mathfrak{M}}$, $\mathcal{W}\longrightarrow\overline{\mathfrak{M}}$ is a regular obstruction-free cone, and s is a section of $\mathcal{W}$ over A. If*

$$r_A = \min\big\{\mathrm{rk}\,\mathcal{W}_b\colon b\in\overline{\mathfrak{M}}-A\big\},$$

s extends to a continuous section $\tilde{s}$ of $\mathcal{W}$ over $A\cup\overline{\mathfrak{M}}_{r_A}(\mathcal{W})$.

Remark 1: It is enough to assume that $\overline{\mathfrak{M}}$ is a paracompact (Hausdorff) space that has a countable basis at each point.

Remark 2: An immediate corollary of this lemma is that s extends to a continuous section of $\mathcal{W}$ over $\overline{\mathfrak{M}}$.

Proof. Let $\{U_b\}_{b\in\mathcal{A}}$ be a finite open cover of the compact set $A\cup\overline{\mathfrak{M}}_{r_A}(\mathcal{W})$ by open subspaces of $\overline{\mathfrak{M}}$ as in (1) of Definition 3.3. Since $\overline{\mathfrak{M}}$ is normal, we can choose an open cover $\{U_b'\}_{b\in\mathcal{A}}$ of $A\cup\overline{\mathfrak{M}}_{r_A}(\mathcal{W})$ such that $\bar{U}_b'\subset U_b$ for all $b\in\mathcal{A}$. Since $\bar{U}_b'$ is normal for each $b\in\mathcal{A}$, by Tietze Extension Theorem the continuous section $f\equiv\varphi_b\circ s$ of $U_b\times\mathbb{C}^{n_b}$ over $A\cap\bar{U}_b'$ extends to a continuous section $\tilde{f}$ over $\bar{U}_b'$. Let

$$\pi_b^-: U_b\times\mathbb{C}^{n_b}\longrightarrow \operatorname{Im}\varphi_b$$

be the orthogonal projection map. We will show in the next paragraph that the section $\pi_b^-\circ\tilde{f}$ is continuous over $(A\cup\overline{\mathfrak{M}}_{r_A}(\mathcal{W}))\cap\bar{U}_b'$. Since $A\cup\overline{\mathfrak{M}}_{r_A}(\mathcal{W})$ is normal, we can choose a partition of unity $\{\eta_b\}_{b\in\mathcal{A}}$ subordinate to $\{U_b'\}_{b\in\mathcal{A}}$. The section

$$\tilde{s}=\sum_{b\in\mathcal{A}}\eta_b\cdot\big(\varphi_b^{-1}\circ\pi_b^-\circ\tilde{f}\big)$$

is continuous over $A\cup\overline{\mathfrak{M}}_{r_A}(\mathcal{W})$. Since $f(x)\in\operatorname{Im}\varphi_b$ for all $x\in A\cap\bar{U}_b'$,

$$\pi_b^-\tilde{f}(x)=\pi_b^-f(x)=f(x)\qquad\forall\, x\in A\cap\bar{U}_b'.$$

Thus, $\tilde{s}|_A=s$ as required.

It remains to show that $\pi_b^-\circ\tilde{f}$ is continuous over $(A\cup\overline{\mathfrak{M}}_{r_A}(\mathcal{W}))\cap\bar{U}_b'$. Since

$$\pi_b^-\tilde{f}|_{A\cap\bar{U}_b'}=f|_{A\cap\bar{U}_b'},$$

$\pi_b^-\tilde{f}$ is continuous along the closed subset $A\cap\bar{U}_b'$ of $\bar{U}_b'$. Thus, we need to show that if

$$x_r\in\overline{\mathfrak{M}}_{r_A}(\mathcal{W})\cap\bar{U}_b'$$

is a sequence converging to $x\in\bar{U}_b'$, then $\pi_b^-\tilde{f}(x_r)$ converges to $\pi_b^-\tilde{f}(x)$. Suppose first

$$x\in\overline{\mathfrak{M}}_{r_A}(\mathcal{W})\cap\bar{U}_b'.$$

We will show that $\pi_b^-|_{\overline{\mathfrak{M}}_{r_A}(\mathcal{W})\cap\bar{U}_b'}$ is continuous at x. Let $\{\xi_i\}_{i\in[r_A]}$ be an orthonormal basis for $\operatorname{Im}\varphi_b|_x$. By (2) of Definition 3.3, for each $i\in[r_A]$ there exists a sequence $\xi_{i;r}\in\operatorname{Im}\varphi_b|_{x_r}$ converging to ξ_i. Since $b,b_r\in\overline{\mathfrak{M}}_{r_A}(\mathcal{W})$, $\{\xi_{i;r}\}_{i\in[r_A]}$ is basis for $\operatorname{Im}\varphi_b|_{x_r}$ for all r sufficiently large. Since $\xi_{i;r}\longrightarrow\xi_i$ for all $i\in[r_A]$ and $\{\xi_i\}_{i\in[r_A]}$ is an orthonormal basis,

$$\lim_{r\longrightarrow\infty}\langle\xi_{i;r},\xi_{j;r}\rangle=\begin{cases}1, & \text{if } i=j;\\ 0, & \text{if } i\neq j.\end{cases}$$

Thus, applying the Gram-Schmidt normalization procedure, we can find an orthonormal basis $\{\tilde{\xi}_{i;r}\}_{i\in[r_A]}$ for $\operatorname{Im}\varphi_b|_{x_r}$ such that $\tilde{\xi}_{i;r}\longrightarrow\xi_i$ for all

$i \in [r_A]$. It follows that $\pi_b^-|_{x_r} \longrightarrow \pi_b^-|_x$ as needed. On the other hand, suppose that $x \in A \cap \bar{U}_b'$. We will view f, $\tilde{f}$, and $\pi_b^- \tilde{f} = \pi_x \tilde{f}$ as $\mathbb{C}^{n_b}$-valued functions. By (2) of Definition 3.3, there exists a sequence $\xi_r \in \operatorname{Im} \varphi_b|_{x_r}$ converging to $f(x)$. Since $\pi_{x_r}\xi_r = \xi_r$,

$$\begin{aligned}\big|\pi_x \tilde{f}(x) - \pi_{x_r}\tilde{f}(x_r)\big| &= \big|\tilde{f}(x) - \pi_{x_r}\tilde{f}(x_r)\big| \\ &\le \big|\tilde{f}(x) - \xi_r\big| + \big|\pi_{x_r}\xi_r - \pi_{x_r}\tilde{f}(x)\big| + \big|\pi_{x_r}\tilde{f}(x) - \pi_{x_r}\tilde{f}(x_r)\big| \\ &\le 2\big|f(x) - \xi_r\big| + \big|\tilde{f}(x) - \tilde{f}(x_r)\big|.\end{aligned}$$

The last two terms above approach 0 by the assumption on ξ_r and the continuity of $\tilde{f}_r$. q.e.d.

Remark: The projection π_b^- is *not* continuous over U_b unless the rank of $\mathcal{W}$ is constant over U_b. Similarly, the section $\pi_b^- \circ \tilde{f}$ may not be continuous over $\bar{U}_b'$ unless the rank of $\mathcal{W}$ is constant over $\bar{U}_b' - A$.

Definition 3.5. If $\overline{\mathfrak{M}}$ is a topological space and $(\mathcal{A}, \prec)$ is a finite partially ordered set, a collection $\{\mathcal{U}_\alpha\}_{\alpha\in\mathcal{A}}$ of subspaces of $\overline{\mathfrak{M}}$ is a stratification of $\overline{\mathfrak{M}}$ if $\mathcal{U}_\alpha$ is a smooth manifold for all $\alpha \in \mathcal{A}$,

$$\partial\bar{\mathcal{U}}_\alpha \equiv \bar{\mathcal{U}}_\alpha - \mathcal{U}_\alpha \subset \bigcup_{\beta\prec\alpha} \mathcal{U}_\beta \qquad \forall\, \alpha \in \mathcal{A}, \qquad \text{and} \qquad \overline{\mathfrak{M}} = \bigsqcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha.$$

Proposition 3.6. *Suppose $\overline{\mathfrak{M}}$ is a compact Hausdorff space that has a countable basis at each point, $\{\mathcal{U}_\alpha\}_{\alpha\in\mathcal{A}}$ is a stratification of $\overline{\mathfrak{M}}$, and $\mathcal{W} \longrightarrow \overline{\mathfrak{M}}$ is a regular obstruction-free cone. If $\mathcal{W}|_{\mathcal{U}_\alpha} \longrightarrow \mathcal{U}_\alpha$ is a smooth vector bundle for all $\alpha \in \mathcal{A}$, the cone $\mathcal{W}$ admits a continuous section s over $\overline{\mathfrak{M}}$ such that $s|_{\mathcal{U}_\alpha}$ is smooth and transverse to the zero set in $\mathcal{W}|_{\mathcal{U}_\alpha}$ for all $\alpha \in \mathcal{A}$.*

Proof. Choose an ordering $<$ on the partially ordered set $(\mathcal{A}, \prec)$ such that for all $\alpha, \beta \in \mathcal{A}$,

$$\begin{aligned} &\operatorname{rk}\mathcal{W}|_{\mathcal{U}_\beta} < \operatorname{rk}\mathcal{W}|_{\mathcal{U}_\alpha} &&\Longrightarrow &&\beta < \alpha; \\ &\operatorname{rk}\mathcal{W}|_{\mathcal{U}_\beta} = \operatorname{rk}\mathcal{W}|_{\mathcal{U}_\alpha},\ \beta \prec \alpha &&\Longrightarrow &&\beta < \alpha. \end{aligned} \tag{3.4}$$

Since $\mathcal{W}$ is obstruction-free, $\bigcup_{r\le q}\overline{\mathfrak{M}}_r(\mathcal{W})$ is closed in $\overline{\mathfrak{M}}$, and

$$\partial\bar{\mathcal{U}}_\alpha \subset \bigcup_{\beta<\alpha} \mathcal{U}_\beta \qquad \forall\, \alpha \in \mathcal{A}, \tag{3.5}$$

by the closure condition of Definition 3.5. Suppose $\alpha \in \mathcal{A}$ and we have defined a continuous section s of $\mathcal{W}$ over the closed set

$$A \equiv \bigcup_{\beta<\alpha} \mathcal{U}_\beta$$

such that $s|_{\mathcal{U}_\beta}$ is smooth and transverse to the zero set in $\mathcal{W}|_{\mathcal{U}_\beta}$ for all $\beta < \alpha$. By (3.4) and (3.5),

$$r_A \equiv \min\big\{\operatorname{rk}\mathcal{W}_b : b \in \overline{\mathfrak{M}} - A\big\} = \operatorname{rk}\mathcal{W}|_{\mathcal{U}_\alpha}.$$

Thus, by Lemma 3.4, s extends to a continuous section $\tilde{s}$ over

$$A\cup\mathcal{U}_\alpha \subset A\cup\overline{\mathfrak{M}}_{r_A}(\mathcal{W}).$$

Perturbing $\tilde{s}$ over $\mathcal{U}_\alpha$, without changing it over A, we obtain a continuous section s over $A\cup\mathcal{U}_\alpha$ such that $s|_{\mathcal{U}_\beta}$ is smooth and transverse to the zero set in $\mathcal{W}|_{\mathcal{U}_\beta}$ for all $\beta\leq\alpha$. This construction implies Proposition 3.6.

q.e.d.

Remark 1: In the orbifold/multisection category as needed for the purposes of this paper, a stratum $\mathcal{U}_\alpha$ locally is a union of finitely many smooth suborbifolds of a smooth orbifold $\mathfrak{X}_\alpha$. We will still call such unions smooth orbifolds. The bundle $\mathcal{W}|_{\mathcal{U}_\alpha}$ is the restriction of a smooth orbibundle over $\mathfrak{X}_\alpha$.

Remark 2: The cone $\mathcal{V}_{1,k}^A$ is not obstruction-free, but is regular. In Subsection 3.5, we describe a subcone $\mathcal{W}_{1,k}^A\subset\mathcal{V}_{1,k}^A$ which is obstruction-free and sufficiently large for the purposes of Theorem 1.3.

3.4. The structure of the moduli space $\overline{\mathfrak{M}}_{1,k}^0(X,A;J,\nu)$.

In this subsection, we describe the strata of the moduli space $\overline{\mathfrak{M}}_{1,k}^0(X,A;J,\nu)$ for a small generic element ν of $\mathfrak{G}_{1,k}^{\mathrm{es}}(X,A;J)$. If $k\in\mathbb{Z}$, we denote by $[k]$ the set of positive integers that do not exceed k. Let $2n=\dim_{\mathbb{R}}X$.

Lemma 3.7. *Suppose (X,ω,J), A, k, and ν are as in Theorem 1.3. If*

$$\mathcal{T}=([k],I,\aleph;j,\underline{A})$$

is a bubble type such that $\sum_{i\in I}A_i=A$ and $A_i\neq0$ for some minimal element i of I, then $\mathcal{U}_{\mathcal{T},\nu}(X;J)$ is a smooth orbifold and

$$\dim\mathcal{U}_{\mathcal{T},\nu}(X;J)=\dim_{1,k}(X,A)-2\big(|\aleph|+|\hat{I}|\big).$$

The statement that $\mathcal{U}_{\mathcal{T},\nu}(X;J)$ is smooth should be interpreted as in Remark 1 at the end of the previous subsection. The branches of $\mathcal{U}_{\mathcal{T},\nu}(X;J)$ correspond to the branches of ν. For a generic ν, the linearization $D_{J,\nu;b}$ of the bundle section $\bar\partial_J+\nu$ at $[b]$ is surjective for every element b in $\overline{\mathfrak{M}}_{1,k}^0(X,A;J,\nu)$ such that $u_b|_{\Sigma_{b;P}}$ is not constant. Thus, Lemma 3.7 is obtained by a standard Contraction Principle argument, such as in [**19**, Chapter 3].

Lemma 3.8. *Suppose (X,ω,J), A, k, and ν are as in Theorem 1.3. If*

$$\mathcal{T}=([k],I,\aleph;j,\underline{A})$$

is a bubble type such that $\sum_{i\in I}A_i=A$ and $A_i=0$ for all minimal elements i of I, then for each $m\in\mathbb{Z}^+$,

$$\mathcal{U}_{\mathcal{T},\nu;1}^m(X;J)\equiv\Big\{[b]\in\mathcal{U}_{\mathcal{T},\nu}(X;J):\ \dim_{\mathbb{C}}Span_{(\mathbb{C},J)}\{\mathcal{D}_ib\colon i\in\chi(\mathcal{T})\}=|\chi(\mathcal{T})|-m\Big\}$$

is a smooth orbifold and

$$\dim \mathcal{U}^m_{\mathcal{T},\nu;1}(X;J) = \dim_{1,k}(X,A) - 2\big(|\aleph|+|\hat{I}| - n + (m+n-|\chi(\mathcal{T})|)m\big).$$

Proof. If $\mathcal{T}$ is as in Lemma 3.8 and $b\equiv(\Sigma_b;u_b)\in\mathcal{U}_{\mathcal{T},\nu}(X;J)$, $u_b|_{\Sigma_{b;P}}$ is constant. Let

$$\Gamma_B(b) = \big\{\zeta\in\Gamma(\Sigma_b;u_b^*TX)\colon \zeta|_{\Sigma_{b;P}}=0\big\}\,;$$
$$\Gamma^{0,1}_B(b;J) = \Big\{\eta\in\Gamma(\Sigma_b;\Lambda^{0,1}_{J,j}T^*\Sigma_b\otimes u_b^*TX)\colon \eta|_{\Sigma_{b;P}}=0\Big\}\,.$$

If ν is genetic, the operator

$$D^B_{J,\nu;b}\colon \Gamma_B(b)\longrightarrow \Gamma^{0,1}_B(b;J)$$

induced by $D_{J,\nu;b}$ is surjective. Thus, the space $\mathcal{U}_{\mathcal{T},\nu}(X;J)$ is a smooth orbifold of dimension

$$\dim \mathcal{U}^m_{\mathcal{T},\nu;1}(X;J) = \dim_{1,k}(X,A) - 2\big(|\aleph|+|\hat{I}|\big) + 2n.$$

We note that

$$\text{(3.6)}\quad \dim \mathcal{U}^m_{\mathcal{T},\nu;1}(X;J)\neq\emptyset \quad\Longrightarrow\quad \max(1,|\chi(\mathcal{T})|-n)\le m\le|\chi(\mathcal{T})|.$$

As at the end of [**27**, Subsection 2.3], we can construct a vector bundle F over $\mathcal{U}_{\mathcal{T},\nu}(X;J)$ of rank $|\chi(\mathcal{T})|$, a vector bundle V over

$$\pi_m\colon \mathrm{Gr}_m F\longrightarrow \mathcal{U}^m_{\mathcal{T},\nu;1}(X;J)$$

of rank mn, and a transverse section $\mathcal{D}_m$ of V such that

$$\pi_m\colon \mathcal{D}_m^{-1}(0)\longrightarrow \bigcup_{m'\ge m}\mathcal{U}^{m'}_{\mathcal{T},\nu;1}(X;J)$$

is surjective, and the restriction of $\pi_m|_{\mathcal{D}_m^{-1}(0)}$ to the preimage of the space $\mathcal{U}^m_{\mathcal{T},\nu;1}(X;J)$ is an embedding. This observation implies the dimension claim of Lemma 3.8. q.e.d.

The spaces $\mathcal{U}_{\mathcal{T},\nu}(X;J)$ and $\mathcal{U}^m_{\mathcal{T},\nu;1}(X;J)$ of Lemmas 3.7 and 3.8 are disjoint. By Definition 2.2, their union is $\overline{\mathfrak{M}}^0_{1,k}(X,A;J,\nu)$. Let $\mathcal{A}^*$ be the set of equivalence classes of bubble types $\mathcal{T}$ as in Lemma 3.7 and $\mathcal{A}^0$ the set of equivalence classes of pairs $(\mathcal{T},m)$ consisting of a bubble type $\mathcal{T}$ as in Lemma 3.8 and an integer m as in (3.6). We define a partial ordering on the set $\mathcal{A}\equiv\mathcal{A}^*\sqcup\mathcal{A}^0$ as follows. Suppose

$$\mathcal{T}=([k],I,\aleph;j,\underline{A})\qquad\text{and}\qquad \mathcal{T}'=([k],I',\aleph';j',\underline{A}')$$

are two bubble types as in Lemma 3.7 and/or in Lemma 3.8. We write

$$\mathcal{T}'\prec\mathcal{T}\qquad\Longleftrightarrow\qquad |\hat{I}'|\ge|\hat{I}|,\ |\aleph'|\ge|\aleph|,\ |\hat{I}'|+|\aleph'|>|\hat{I}|+|\aleph|.$$

If $\mathcal{T}\in\mathcal{A}^*$ and $(\mathcal{T}',m')\in\mathcal{A}^0$, we define

$$\mathcal{T}\prec(\mathcal{T}',m')\iff \mathcal{T}\prec\mathcal{T}';\qquad (\mathcal{T}',m')\prec\mathcal{T}\iff \mathcal{T}'\prec\mathcal{T}.$$

Finally, if $(\mathcal{T},m),(\mathcal{T}',m')\in\mathcal{A}^0$, we define

$$(\mathcal{T}',m')\prec(\mathcal{T},m) \qquad\Longleftrightarrow\qquad \mathcal{T}'\prec\mathcal{T} \quad\text{OR}\quad \mathcal{T}'=\mathcal{T},\ m'>m.$$

By definition of the stable-map topology,

$$\partial\bar{\mathcal{U}}_{\mathcal{T},\nu}(X;J)\cap\mathcal{U}_{\mathcal{T}',\nu}(X;J)\neq\emptyset \qquad\Longrightarrow\qquad \mathcal{T}'\prec\mathcal{T}.$$

Thus, the closure requirement of Definition 3.5 follows from the continuity of the maps $\mathcal{D}_i$ on $\tilde{\mathfrak{X}}_{\mathcal{T}}(X)$ with $\mathcal{T}$ as in Lemma 3.8.

3.5. The structure of the cone $\mathcal{V}_{1,k}^A$. In this subsection, we describe an obstruction-free subcone $\mathcal{W}_{1,k}^A$ of the cone

$$\mathcal{V}_{1,k}^A\longrightarrow\overline{\mathfrak{M}}_{1,k}^0(X,A;J,\nu).$$

The cone $\mathcal{V}_{1,k}^A$ over $\mathfrak{X}_{1,k}^A$ can be shown to be regular by standard arguments; see Remark 2 at the end of Subsection 4.3. Thus, $\mathcal{W}_{1,k}^A$ is regular as well.

If $\mathcal{T}$ is a bubble type as in Lemma 3.8 and $[b]\in[\Sigma_b,u_b]\in\mathcal{U}_{\mathcal{T},\nu}(X;J)$, let

$$\begin{aligned} F_b^1&=\Big\{(w_i)_{i\in\chi(\mathcal{T})}\in\mathbb{C}^{\chi(b)}\colon \sum_{i\in\chi(\mathcal{T})}w_i\cdot_J\mathcal{D}_ib=0\Big\};\\ \tilde{\Gamma}_-(b;\mathfrak{L})&=\Big\{\xi\in\ker\bar{\partial}_{\nabla,b}\colon \sum_{i\in\chi(\mathcal{T})}w_i\,\mathfrak{D}_{b,i}\xi=0\ \forall\,(w_i)_{i\in\chi(\mathcal{T})}\in F_b^1\Big\};\\ \mathcal{W}_{1,k}^A|_{[b]}&=\Big\{[(b,\xi)]\in\mathcal{V}_{1,k}^A|_{[b]}\colon\xi\in\tilde{\Gamma}_-(b;\mathfrak{L})\Big\}. \end{aligned}$$

By (3.3), the subspace $\mathcal{W}_{1,k}^A|_{[b]}$ of $\mathcal{V}_{1,k}^A|_{[b]}$ is well-defined. If $\mathcal{T}$ is a bubble type as in Lemma 3.7 and $[b]\in\mathcal{U}_{\mathcal{T},\nu}(X;J)$, let

$$\mathcal{W}_{1,k}^A|_{[b]}=\mathcal{V}_{1,k}^A|_{[b]}.$$

We take

$$\mathcal{W}_{1,k}^A=\bigcup_{[b]\in\overline{\mathfrak{M}}_{1,k}^0(X,A;J,\nu)}\mathcal{W}_{1,k}^A|_{[b]}\ \subset\mathcal{V}_{1,k}^A.$$

Proposition 3.9. *If (X,ω,J), $(\mathfrak{L},\nabla)$, A, k, and ν are as in Theorem 1.3, the cone*

$$\mathcal{W}_{1,k}^A\longrightarrow\overline{\mathfrak{M}}_{1,k}^0(X,A;J,\nu)$$

is regular and obstruction-free. If $\mathcal{T}$ is a bubble type as in Lemma 3.7, then $\mathcal{W}_{1,k}^A|_{\mathcal{U}_{\mathcal{T},\nu}(X;J)}$ is a smooth vector orbibundle and

$$\operatorname{rk}\mathcal{W}_{1,k}^A|_{\mathcal{U}_{\mathcal{T},\nu}(X;J)}=\langle c_1(\mathfrak{L}),A\rangle.$$

If $\mathcal{T}$ is a bubble type as in Lemma 3.8 *and m is an integer as in (*3.6*), then $\mathcal{W}^A_{1,k}|_{\mathcal{U}^m_{\mathcal{T},\nu;1}(X;J)}$ is a smooth vector orbibundle and*

$$\begin{aligned}\operatorname{rk}\mathcal{W}^A_{1,k}|_{\mathcal{U}^m_{\mathcal{T},\nu;1}(X;J)} &= \langle c_1(\mathfrak{L}),A\rangle-(m-1)\\ &> \frac{1}{2}\dim\mathcal{U}^m_{\mathcal{T},\nu;1}(X;J)-\frac{1}{2}\dim_{1,k}(X,A;\mathfrak{L}).\end{aligned}$$

If $\mathcal{T}$ is a bubble type as in Lemma 3.7 and $[b]\in\mathcal{U}_{\mathcal{T},\nu}(X;J)$, the operator $\bar{\partial}_{\nabla,b}$ is surjective, by the positivity assumption on the bundle $\mathfrak{L}$ and the same argument as in [**24**, Subsection 6.2]. In particular,

$$\dim\mathcal{V}^A_{1,k}|_{[b]}=\operatorname{ind}\bar{\partial}_{\nabla,b}=\langle c_1(\mathfrak{L}),A\rangle\qquad\forall\,[b]\in\mathcal{U}_{\mathcal{T},\nu}(X;J).$$

By standard arguments, the surjectivity of $\bar{\partial}_{\nabla,b}$ for every $[b]\in\mathcal{U}_{\mathcal{T},\nu}(X;J)$ implies that

$$\mathcal{W}^A_{1,k}|_{\mathcal{U}_{\mathcal{T},\nu}(X;J)}\equiv\mathcal{V}^A_{1,k}|_{\mathcal{U}_{\mathcal{T},\nu}(X;J)}$$

is a smooth vector bundle and that the restriction of $\mathcal{V}^A_{1,k}$ to a neighborhood of $\mathcal{U}_{\mathcal{T},\nu}(X;J)$ in $\mathfrak{X}_{1,k}(X,A)$ is a vector bundle. Local trivializations can be constructed using the homomorphisms $R_{\upsilon,\zeta}$ as in Subsection 4.2. In particular, the cone $\mathcal{W}^A_{1,k}$ satisfies the requirements of (1) and (2) of Definition 3.3 for every $[b]\in\mathcal{U}_{\mathcal{T},\nu}(X;J)$.

If $\mathcal{T}$ is a bubble type as in Lemma 3.8 and $[b]\equiv[\Sigma_b;u_b]\in\mathcal{U}_{\mathcal{T},\nu}(X;J)$, $u_b|_{\Sigma_{b;P}}$ is constant. Let

$$\begin{aligned}\Gamma_B(b;\mathfrak{L})&=\left\{\xi\in\Gamma(\Sigma_b;u_b^*\mathfrak{L})\colon\xi|_{\Sigma_{b;P}}=\text{const}\right\};\\ \Gamma^{0,1}_B(b;\mathfrak{L})&=\left\{\eta\in\Gamma(\Sigma_b;\Lambda^{0,1}_{\mathfrak{i},j}T^*\Sigma_b\otimes u_b^*\mathfrak{L})\colon\eta|_{\Sigma_{b;P}}=0\right\}.\end{aligned}$$

By the positivity assumption on the bundle $\mathfrak{L}$ and the same argument as in [**24**, Subsection 6.2], the operator

$$\bar{\partial}^B_{\nabla,b}\colon\Gamma_B(b;\mathfrak{L})\longrightarrow\Gamma^{0,1}_B(b;\mathfrak{L})$$

induced by $\bar{\partial}_{\nabla,b}$ is surjective. In particular,

$$\dim\mathcal{V}^A_{1,k}|_{[b]}=\dim\ker\bar{\partial}_{\nabla,b}=\dim\ker\bar{\partial}^B_{\nabla,b}=\operatorname{ind}\bar{\partial}^B_{\nabla,b}=\langle c_1(\mathfrak{L}),A\rangle+1$$

for all $[b]\in\mathcal{U}_{\mathcal{T},\nu}(X;J)$. Thus, $\mathcal{V}^A_{1,k}|_{\mathcal{U}_{\mathcal{T},\nu}(X;J)}$ is a smooth vector bundle. Similarly to [**27**, Subsection 3.3], for every m as in (3.6) we can construct a vector bundle $F^1\longrightarrow\mathcal{U}^m_{\mathcal{T},\nu;1}(X;J)$ of rank m and a surjective bundle homomorphism

$$\mathfrak{D}\colon\mathcal{V}^A_{1,k}\longrightarrow\operatorname{Hom}(F^1,\operatorname{ev}_P^*\mathfrak{L})$$

over $\mathcal{U}^m_{\mathcal{T},\nu;1}(X;J)$ such that the kernel of $\mathfrak{D}$ is $\mathcal{W}^A_{1,k}|_{\mathcal{U}^m_{\mathcal{T},\nu;1}(X;J)}$. Thus, $\mathcal{W}^A_{1,k}|_{\mathcal{U}^m_{\mathcal{T},\nu;1}(X;J)}$ is a smooth vector bundle of the claimed rank.

4. Proof of Proposition 3.9

4.1. Outline. In this section we prove a generalization of Proposition 3.9. It implies that the Poincare dual of the euler class of $\mathcal{V}_{1,k}^A$ defined as the zero set of a generic section of $\mathcal{V}_{1,k}^A$ over $\overline{\mathfrak{M}}_{1,k}^0(X,A;J,\nu)$ is independent of J and ν.

Suppose (X,ω) is a compact symplectic manifold, $(\mathfrak{L},\nabla)$ is an (ω,A)-positive line bundle with connection over X, $A\in H_2(X;\mathbb{Z})^*$, $k\in\bar{\mathbb{Z}}^+$, $\underline{J}\equiv(J_t)_{t\in[0,1]}$ is a continuous family of ω-tamed almost complex structures on X, and

$$\underline{\nu}\equiv(\nu_t)_{t\in[0,1]}\in\mathfrak{G}_{1,k}^{\mathrm{es}}(X,A;\underline{J})$$

is a family of sufficiently small perturbations of the $\bar{\partial}_{J_t}$-operators on $\mathfrak{X}_{1,k}(X,A)$. Let t_r and $[b_r]$ be sequences of elements in $[0,1]$ and in $\overline{\mathfrak{M}}_{1,k}^0(X,A;J_{t_r},\nu_{t_r})$ such that

$$\lim_{r\longrightarrow\infty}t_r=0\quad\text{and}\quad\lim_{r\longrightarrow\infty}[b_r]=[b]\in\overline{\mathfrak{M}}_{1,k}^0(X,A;J_0,\nu_0).$$

We need to show that for every $\xi\in\mathcal{W}_{1,k}^A|_{[b]}$ there exists a sequence $\xi_r\in\mathcal{W}_{1,k}^A|_{[b_r]}$ converging to ξ. By the paragraph following Proposition 3.9, it is sufficient to assume that $[b]$ is an element of $\mathcal{U}_{\mathcal{T},\nu_0}(X;J_0)$ for a bubble type

$$\mathcal{T}=\big([k],I,\aleph;j,\underline{A}\big)$$

such that $A_i=0$ for all minimal elements $i\in I$.

We can also assume that for some bubble type

$$\mathcal{T}'=\big([k],I',\aleph';j',\underline{A}'\big)$$

$[b_r]\in\mathcal{U}_{\mathcal{T}',\nu_{t_r}}(X;J_{t_r})$ for all r. We note that by Definition 1.2, for every map $u\colon\mathbb{P}^1\longrightarrow X$ such that

$$\langle\omega,u_*[\mathbb{P}^1]\rangle<\langle\omega,A\rangle$$

the linear operators

$$\bar{\partial}_{\nabla,u}\colon\Gamma(\mathbb{P}^1;u^*\mathfrak{L})\longrightarrow\Gamma\big(\mathbb{P}^1;\Lambda_{\mathfrak{i},j}^{0,1}T^*\mathbb{P}^1\otimes u^*\mathfrak{L}\big),$$
$$\ker\bar{\partial}_{\nabla,u}\longrightarrow\mathfrak{L}_{u(\infty)},\quad\xi\longrightarrow\xi(\infty),$$

are surjective. Thus, it is sufficient to consider two possibilities for $\mathcal{T}'$:

(1) $A_i'=0$ for all $i\in I_0'$ and $\{i\in I'\colon A_i'\neq0\}=\chi(\mathcal{T}')$;

(2) $A_i'\neq0$ for some $i\in I_0'$ and $\hat{I}'=\emptyset$,

where I_0' is the subset of minimal elements of I'. In the first case, for every $[b_r]\in\mathcal{U}_{\mathcal{T}',\nu_{t_r}}(X;J_{t_r})$, the map u_{b_r} is constant on the principal component $\Sigma_{b_r;P}$ of Σ_{b_r}, and thus so is every element $\xi\in\ker\bar{\partial}_{\nabla,b_r}$. In this case, the question of existence of a sequence ξ_r as above is an issue concerning the behavior of holomorphic bundle sections for genus-zero (J,ν)-holomorphic maps, for a certain class of perturbations ν of the $\bar{\partial}_J$-operator. This class is induced from the class of effectively supported perturbations of Definition 2.1 and is described in Definition 4.3 at the end of this subsection. The existence of a desired sequence in case (1) follows from Lemma 4.1 below. In the second case, $\Sigma_{b_r;P}=\Sigma_{b_r}$ is either a smooth torus or a circle of spheres, depending on whether $\aleph'$ is empty or not. There are no bubble components. In this case, the desired result follows from Lemma 4.2.

Lemma 4.1. *Suppose (X,ω) is a compact symplectic manifold, $(\mathfrak{L},\nabla)$ is an (ω,A)-positive line bundle with connection over X, $A\in H_2(X;\mathbb{Z})^*$, M is a finite set, $\underline{J}\equiv(J_t)_{t\in[0,1]}$ is a continuous family of ω-tamed almost complex structures on X, and*

$$\underline{\nu}\equiv(\nu_t)_{t\in[0,1]}\in\mathfrak{G}^{\text{es}}_{0,\{0\}\sqcup M}(X,A;\underline{J})$$

is a family of sufficiently small perturbations of the $\bar\partial_{J_t}$-operators on $\mathfrak{X}_{0,\{0\}\sqcup M}(X,A)$. Let t_r and $[b_r]$ be sequences of elements in $[0,1]$ and in $\mathfrak{M}^0_{0,\{0\}\sqcup M}(X,A;J_{t_r},\nu_{t_r})$ such that

$$\lim_{r\longrightarrow\infty}t_r=0\quad\textit{and}\quad\lim_{r\longrightarrow\infty}[b_r]=[b]\in\overline{\mathfrak{M}}_{0,\{0\}\sqcup M}(X,A;J_0,\nu_0).$$

Then there exist $(c_{r,i})_{i\in\chi(b)}\in(\mathbb{C}^*)^{\chi(b)}$, $\epsilon_r\in\mathbb{R}^+$, and isomorphisms

$$R_{b_r,b}\colon\ker\bar\partial_{\nabla,b}\longrightarrow\ker\bar\partial_{\nabla,b_r}$$

such that

$$\Big|\mathcal{D}_{\hat0}b_r-\sum_{i\in\chi(b)}c_{r,i}\cdot_{J_0}\mathcal{D}_ib\Big|\le\epsilon_r\sum_{i\in\chi(b)}|c_{r,i}|\,,\tag{4.1}$$

$$\Big|\mathfrak{D}_{b_r,\hat0}R_{b_r,b}\xi-\sum_{i\in\chi(b)}c_{r,i}\mathfrak{D}_{b,i}\xi\Big|\le\epsilon_r\sum_{i\in\chi(b)}|c_{r,i}|\cdot\|\xi\|\quad\forall\,\xi\in\ker\bar\partial_{\nabla,b}\,,\tag{4.2}$$

for all $r\in\mathbb{Z}^+$ and

$$\begin{gathered}\lim_{r\longrightarrow\infty}[R_{b_r,b}\xi]=[\xi]\in\mathcal{V}^A_{0,M}|_{[b]}\quad\forall\,\xi\in\ker\bar\partial_{\nabla,b},\\ \lim_{r\longrightarrow\infty}\epsilon_r=0,\quad|c_{r,i}|\le1\ \forall\,i\in\chi(b).\end{gathered}\tag{4.3}$$

Lemma 4.2. *Suppose (X,ω), $(\mathfrak{L},\nabla)$, $\underline{J}$, A, and M are as in Lemma 4.1 and*

$$\underline{\nu}\equiv(\nu_t)_{t\in[0,1]}\in\mathfrak{G}^{\text{es}}_{1,M}(X,A;\underline{J})$$

is a family of sufficiently small perturbations of the $\bar\partial_{J_t}$-operators on $\mathfrak{X}_{1,M}(X,A)$. Let t_r and $[b_r]$ be sequences of elements in $[0,1]$ and in $\overline{\mathfrak{M}}^0_{1,M}(X,A;J_{t_r},\nu_{t_r})$ such that

$$\Sigma_{b_r;P}=\Sigma_{b_r}\ \forall\,r,\quad\lim_{r\longrightarrow\infty}t_r=0,\quad\lim_{r\longrightarrow\infty}[b_r]=[b]\in\overline{\mathfrak{M}}^0_{1,M}(X,A;J_0,\nu_0).$$

If $b=(\Sigma_b;u_b)$ *is such that the degree of* $u_b|_{\Sigma_{b;P}}$ *is zero, there exist* $(w_i)_{i\in\chi(b)}\in(\mathbb{C}^*)^{\chi(b)}$ *such that*

$$\sum_{i\in\chi(b)} w_i\cdot_{J_0}\mathcal{D}_i b=0$$

and a subsequence of $\{b_r\}$*, which we still denote by* $\{b_r\}$*, such that*

$$\xi_r\in\ker\bar\partial_{\nabla,b_r},\quad \lim_{r\longrightarrow\infty}[\xi_r]=[\xi]\in\mathcal{V}_{1,M}^A|_{[b]}\quad\Longrightarrow\quad \sum_{i\in\chi(b)} w_i\,\mathfrak{D}_{b,i}\xi=0.$$

Suppose $\{b_r\}$ and b are as in Lemma 4.2, $\xi\in\tilde\Gamma_-(b;\mathfrak{L})$, and there exists no sequence

$$\xi_r\in\ker\bar\partial_{\nabla,b_r}\qquad\text{s.t.}\qquad \lim_{r\longrightarrow\infty}[\xi_r]=[\xi]. \tag{4.4}$$

After passing to a subsequence of $\{b_r\}$ if necessary, we can assume that

$$[\xi]\not\in\overline{\bigcup_{r=1}^{\infty}\mathcal{V}_{1,M}^A|_{[b_r]}}\subset\mathcal{V}_{1,M}^A. \tag{4.5}$$

Choose $\underline{w}\equiv(w_i)_{i\in\chi(b)}\in(\mathbb{C}^*)^{\chi(b)}$ and a subsequence of $\{b_r\}$, which we still denote by $\{b_r\}$, as at the end of Lemma 4.2. Let $\{\xi_r^j\}$ be a basis for $\ker\bar\partial_{\nabla,b_r}$ which is orthonormal with respect to a regularization at b as in (1) of Definition 3.3 for example. After passing to a subsequence if necessary, for some linearly independent $\xi^j\in\ker\bar\partial_{\nabla,b}$,

$$\lim_{r\longrightarrow\infty}[\xi_r^j]=[\xi^j]\in\mathcal{V}_{1,M}^A|_{[b]}\qquad\forall\,j.$$

By Lemma 4.2,

$$\xi^j\in\Gamma_-(b;\mathfrak{L};[\underline{w}])\equiv\Big\{\xi\in\ker\bar\partial_{\nabla,b}\colon \sum_{i\in\chi(b)} w_i\,\mathfrak{D}_{b,i}\xi=0\Big\}.$$

By the positivity assumption of Definition 1.2,

$$\dim\ker\bar\partial_{\nabla,b_r}=\operatorname{ind}\bar\partial_{\nabla,b_r}=\langle c_1(\mathfrak{L}),A\rangle\,,$$
$$\dim\Gamma_-(b;\mathfrak{L};[\underline{w}])=\dim\ker\bar\partial_{\nabla,b}-1=\operatorname{ind}\bar\partial^B_{\nabla,b}-1=\langle c_1(\mathfrak{L}),A\rangle.$$

Thus, $\{\xi^j\}$ is a basis for $\Gamma_-(b;\mathfrak{L};[\underline{w}])$. Since $\tilde\Gamma_-(b;\mathfrak{L})\subset\Gamma_-(b;\mathfrak{L};[\underline{w}])$, there exists a sequence $\xi_r\in\ker\bar\partial_{\nabla,b_r}$ as in (4.4), with $\{b_r\}$ replaced by a subsequence. However, this contradicts (4.5).

An element $[b_r]\in\mathcal{U}_{\mathcal{T}',\nu_{t_r}}(X;J_{t_r})$, with $\mathcal{T}'$ as in the first case above, corresponds to the genus-one curve $\Sigma_{b_r,0}=\Sigma^0_{b_r}$ and genus-zero maps $\{b_{r,h}\}_{h\in\chi(\mathcal{T}')}$ such that $\Sigma_{b_{r,h}}=\mathbb{P}^1$. If $[b]$ is the limit of the sequence $\{b_r\}$, b corresponds to a genus-one curve $\Sigma_{b,0}\subset\Sigma_b^0$ and genus-zero maps $\{b_h\}_{h\in\chi(\mathcal{T}')}$ such that

$$\lim_{r\longrightarrow\infty}\Sigma_{b_r,0}=\Sigma_{b,0}\qquad\text{and}\qquad\lim_{r\longrightarrow\infty}[b_{r,h}]=[b_h]\ \forall\,h\in\chi(\mathcal{T}').$$

Similarly, $\underline{\xi}_r \in \ker \bar{\partial}_{\nabla,b_r}$ and $\xi \in \ker \bar{\partial}_{\nabla,b}$ correspond to $\xi_{r,h} \in \ker \bar{\partial}_{\nabla,b_{r,h}}$ and $\xi_h \in \ker \bar{\partial}_{\nabla,b_h}$, with $h \in \chi(\mathcal{T}')$, such that

$$\xi_{r,h}(y_0(b_{r,h})) = \xi_{r,h'}(y_0(b_{r,h'})) \quad \text{and} \quad \xi_h(y_0(b_h)) = \xi_{h'}(y_0(b_{h'}))$$

for all $h, h' \in \chi(\mathcal{T}')$. Furthermore,

$$\lim_{r\longrightarrow\infty}[\underline{\xi}_r] = [\underline{\xi}] \in \mathcal{V}^A_{1,k}|_{[b]} \iff \lim_{r\longrightarrow\infty}[\xi_{r,h}] = [\xi_h] \in \mathcal{V}^A_{1,M_h}|_{[b_h]}$$

for $h \in \chi(\mathcal{T}')$, where M_h is the index set for the marked points of $b_{r,h}$ and b_h. We will assume that $[b_r] \in \mathcal{U}^{m'}_{\mathcal{T}',\nu_{t_r}}(X; J_{t_r})$ for some $m' \in \mathbb{Z}^+$ and for all r.

With $(c_{r,i})_{i\in\chi(b_h)} \in (\mathbb{C}^*)^{\chi(b_h)}$ for each $h \in \chi(\mathcal{T}')$ as in Lemma 4.1, let

$$\alpha_r \colon \mathbb{C}^{\chi(\mathcal{T}')} \longrightarrow \mathbb{C}^{\chi(b)} \equiv \prod_{h\in\chi(\mathcal{T}')} \mathbb{C}^{\chi(b_h)}$$

be the injective homomorphism defined by

$$\alpha_r\big((w_h)_{h\in\chi(\mathcal{T}')}\big) = (c_{r,i}w_h)_{i\in\chi(b_h),h\in\chi(\mathcal{T}')}.$$

We denote by $\tilde{F}^1_r \subset \mathbb{C}^{\chi(b)}$ the image of

$$F^1_{b_r} \equiv \Big\{ (w_h)_{h\in\chi(\mathcal{T}')} \in \mathbb{C}^{\chi(\mathcal{T}')} \colon \sum_{h\in\chi(\mathcal{T}')} w_h \cdot_{J_0} \mathcal{D}_h b_r \equiv \sum_{h\in\chi(\mathcal{T}')} w_h \cdot_{J_0} \mathcal{D}_{\hat{0}} b_{r,h} = 0 \Big\}$$

under α_r. By our assumption on b_r, $\dim \tilde{F}^1_r = m'$ for all r. Let

$$\Big\{ (v^l_{r,i})_{i\in\chi(b)} \Big\}_{l\in[m']} \qquad \text{and} \qquad \Big\{ \underline{\xi}^j_r \equiv (\xi^j_{r,h})_{h\in\chi(\mathcal{T}')} \Big\}$$

be orthonormal bases for $\tilde{F}^1_r$ and for

$$\tilde{\Gamma}_-(b_r;\mathfrak{L}) \equiv \Big\{ \underline{\xi} \equiv (\xi_h)_{h\in\chi(\mathcal{T}')} \in \ker \bar{\partial}_{\nabla,b_r} \colon \sum_{h\in\chi(\mathcal{T}')} w_h \mathfrak{D}_{b_r,h} \underline{\xi} \equiv \sum_{h\in\chi(\mathcal{T}')} w_h \mathfrak{D}_{b_{r,h},\hat{0}} \xi_h = 0 \ \forall (w_h)_{h\in\chi(\mathcal{T}')} \in F^1_{b_r} \Big\},$$

respectively. After passing to a subsequence if necessary, we can find

$$\underline{w}^l \equiv (w^l_i)_{i\in\chi(b)} \in \mathbb{C}^{\chi(b)} \qquad \text{and} \qquad \underline{\xi}^j \equiv (\xi^j_h)_{h\in\chi(\mathcal{T}')} \in \ker \bar{\partial}_{\nabla,b}$$

such that

$$\lim_{r\longrightarrow\infty} (v^l_{r,i})_{i\in\chi(b)} = \underline{w}^l \in \mathbb{C}^{\chi(b)} \quad \forall\, l \qquad \text{and}$$
$$\lim_{r\longrightarrow\infty} [\underline{\xi}^j_r] = [\underline{\xi}^j] \in \mathcal{V}^A_{1,k}|_{[b]} \quad \forall\, j.$$

Each of the sets $\{\underline{w}^l\}$ and $\{\underline{\xi}^j\}$ is orthonormal and thus linearly independent. By Lemma 4.1,

$$\Big|\sum_{i\in\chi(b)} v^l_{r,i}\cdot_{J_0}\mathcal{D}_i b\Big| \equiv \Big|\sum_{h\in\chi(\mathcal{T}')}\sum_{i\in\chi(b_h)} v^l_{r,i}\cdot_{J_0}\mathcal{D}_i b_h\Big| \le \tilde{\epsilon}_r \qquad \forall\, l \qquad \text{and}$$

$$\Big|\sum_{i\in\chi(b)} v^l_{r,i}\mathfrak{D}_{b,i}\underline{\xi}^j\Big| \equiv \Big|\sum_{h\in\chi(\mathcal{T}')}\sum_{i\in\chi(b_h)} v^l_{r,i}\mathfrak{D}_{b_h,i}\xi^j_h\Big| \le \tilde{\epsilon}_r \qquad \forall\, j,$$

for some sequence $\tilde{\epsilon}_r$ converging to 0. Thus,

$$\underline{w}^l \in F^1_b \equiv \Big\{(w_i)_{i\in\chi(b)}\in\mathbb{C}^{\chi(b)}\colon \sum_{i\in\chi(b)} w_i\cdot_{J_0}\mathcal{D}_i b=0\Big\},$$

$$\underline{\xi}^j \in \Gamma_-\big(b;\mathfrak{L};\{\underline{w}^l\}_{l\in[m']}\big) \equiv \Big\{\xi\in\ker\bar{\partial}_{\nabla,b}\colon \sum_{i\in\chi(b)} w^l_i\,\mathfrak{D}_{b,i}\xi=0\ \forall\, l\in[m']\Big\}$$

for all l,j. By Definition 1.2,

$$\begin{aligned}\dim\Gamma_-\big(b;\mathfrak{L};\{\underline{w}^l\}_{l\in[m']}\big) &= \dim\ker\bar{\partial}_{\nabla,b} - m'\\ &= \dim\ker\bar{\partial}_{\nabla,b_r} - m' = \dim\tilde{\Gamma}_-(b_r;\mathfrak{L}).\end{aligned}$$

Thus, $\{\underline{\xi}^j\}$ is a basis for $\Gamma_-(b;\mathfrak{L};\{\underline{w}^l\}_{l\in[m']})$. Since $\underline{w}^l\in F^1_b$ for all l,

$$\tilde{\Gamma}_-(b;\mathfrak{L}) \subset \Gamma_-\big(b;\mathfrak{L};\{\underline{w}^l\}_{l\in[m']}\big), \qquad \mathcal{W}^A_{1,k}|_{[b]} \subset \overline{\bigcup_{r=1}^{\infty}\mathcal{V}^A_{1,M}|_{[b_r]}} \subset \mathcal{V}^A_{1,M}.$$

As in the first paragraph after Lemma 4.2, this implies that for every $\xi\in\tilde{\Gamma}_-(b;\mathfrak{L})$ there exists a sequence $\xi_r\in\tilde{\Gamma}_-(b_r;\mathfrak{L})$ such that

$$\lim_{r\longrightarrow\infty}[\xi_r] = [\xi] \in \mathcal{V}^A_{1,k}|_{[b]}.$$

Remark: In (4.1) and (4.2), the differences are taken via a parallel transport along the shortest geodesic, with respect to a metric on X, between $\mathrm{ev}_0(b_r)$ and $\mathrm{ev}_0(b)$.

Lemmas 4.1-4.2 are proved in the next two subsections by extending the gluing constructions of [**28**, Subsections 2.4,2.5] from J-holomorphic maps to holomorphic bundle sections. These extensions parallel constructions in [**27**, Subsections 4.2,4.3]. In the rest of this subsection we recall the definition of the type of perturbations ν of the $\bar{\partial}_J$-operator on space of genus-zero stable maps that appears in Lemma 4.1; see [**28**, Subsection 2.1] for details.

Definition 4.3. Suppose (X,ω) is a compact symplectic manifold, $\underline{J}\equiv(J_t)_{t\in[0,1]}$ is a continuous family of ω-tamed almost structures on X, $A\in H_2(X;\mathbb{Z})^*$, and M is a finite set. A continuous family of multisections $\underline{\nu}\equiv(\nu_t)_{t\in[0,1]}$, with $\nu_t\in\mathfrak{G}^{0,1}_{0,\{0\}\sqcup M}(X,A;J_t)$ for all $t\in[0,1]$, is effectively supported if for every element b of $\mathfrak{X}_{0,\{0\}\sqcup M}(X,A)$ there exists a neighborhood $\mathcal{W}_b$ of Σ_b^0 in a semi-universal family of deformations for b such that

$$\nu_t(b')\big|_{\Sigma_{b'}\cap\mathcal{W}_b}=0 \qquad \forall\ [b']\in\mathfrak{X}_{0,\{0\}\sqcup M}(X,A),\ t\in[0,1].$$

We denote the space of effectively supported families $\underline{\nu}$ as in Definition 4.3 by $\mathfrak{G}^{\mathrm{es}}_{0,\{0\}\sqcup M}(X,A;\underline{J})$. If $\nu\in\mathfrak{G}^{\mathrm{es}}_{0,\{0\}\sqcup M}(X,A;\underline{J})$, $t\in[0,1]$, $[b]$ is an element of

$$\overline{\mathfrak{M}}_{0,\{0\}\sqcup M}(X,A;J_t,\nu_t)\equiv\big\{\bar\partial_J+\nu_t\big\}^{-1}(0),$$

and $i\in\chi(b)$, then $u_b|_{\Sigma_{b,i}}$ is J_t-holomorphic on a neighborhood of ∞ in $\Sigma_{b,i}$ and $\mathbb{C}\cdot_{J_t}\mathcal{D}_i b$ is determined by b, just as in Subsection 2.1. Furthermore, in this case $u_b|_{\Sigma_b^0}$ is a degree-zero holomorphic map and thus is constant. Thus, u_b maps the attaching nodes of all elements of $\chi(b)$ to the same point in X, as in the genus-one case of Subsection 2.1.

4.2. Proof of Lemma 4.1. In this subsection we review the genus-zero gluing construction of [**28**, Subsection 2.4] and extend it to holomorphic bundle sections in a manner similar to [**27**, Subsection 4.2]. This construction essentially constitutes the first step of the two-step gluing construction described in Subsection 4.3. Throughout this subsection we assume that M is a finite set, $A\in H_2(X;\mathbb{Z})$, and $\mathcal{T}=(M,I;j,\underline{A})$ is a bubble type such that $\hat 0$ is the minimal element of I,

$$\sum_{i\in I}A_i=A \qquad\text{and}\qquad \langle\omega,A_i\rangle\ge0\ \forall\, i\in I.$$

Let $(\mathfrak{L},\nabla)\longrightarrow X$ be an (ω,A)-positive line bundle with connection.

We put

$$\tilde{\mathfrak{X}}_{\mathcal{T};B}(X)=\big\{(\Sigma_b;u_b)\in\tilde{\mathfrak{X}}_{\mathcal{T}}(X)\colon u_b|_{\Sigma_b^0}=\text{const}\big\}.$$

We denote by

$$\widetilde{\mathcal{F}}\equiv\tilde{\mathfrak{X}}_{\mathcal{T};B}(X)\times\mathbb{C}^{\hat I}$$

the bundle of smoothing parameters and by $\widetilde{\mathcal{F}}^{\emptyset}$ the subset of $\widetilde{\mathcal{F}}$ consisting of the elements with all components nonzero. For each

$$b\in\tilde{\mathfrak{X}}_{\mathcal{T};B}(X),\qquad i\in\chi(\mathcal{T})\equiv\chi(b),\qquad \upsilon\equiv(b,v)=\big(b,(v_h)_{h\in\hat I}\big)\in\widetilde{\mathcal{F}},$$

we put

$$\rho_i(\upsilon)=\prod_{\hat0<h\le i}v_h\in\mathbb{C}\qquad\text{and}\qquad x_i(\upsilon)=\sum_{\hat0<i'\le i}\Big(x_{i'}(b)\prod_{\hat0<h<i'}v_h\Big)\ \in\ \mathbb{C},$$

where $x_i(b)$ is the point of Σ_{b,ι_i} to which the bubble $\Sigma_{b,i}$ is attached; see (3.2) and Figure 3.

For each sufficiently small element $\upsilon=(b,v)$ of $\widetilde{\mathcal{F}}^{\emptyset}$, let

$$q_\upsilon\colon\Sigma_\upsilon\longrightarrow\Sigma_b$$

be the basic gluing map constructed in [**25**, Subsection 2.2]. In this case, Σ_υ is the projective line $\mathbb{P}^1$ with $|M|+1$ marked points. The map q_υ collapses $|\hat{I}|$ circles on Σ_υ. It induces a metric g_υ on Σ_υ such that $(\Sigma_\upsilon, g_\upsilon)$ is obtained from Σ_b by replacing the $|\hat{I}|$ nodes of Σ_b by thin necks.

We put

$$u_\upsilon = u_b \circ q_\upsilon, \qquad b(\upsilon) = (\Sigma_\upsilon; u_\upsilon), \quad \text{and} \quad \bar{\partial}_{\nabla,\upsilon} = \bar{\partial}_{\nabla,b(\upsilon)}.$$

Fix a metric g on X and denote the corresponding Levi-Civita connection by ∇^X. By the same construction as in [**25**, Subsection 3.3], the map q_υ induces weighted L^p_1-norms $\|\cdot\|_{\upsilon,p,1}$ on the spaces

$$\big\{\zeta \in \Gamma(\Sigma_\upsilon; u_\upsilon^* TX) \colon \zeta(\infty)=0\big\} \qquad \text{and} \qquad \Gamma(\Sigma_\upsilon; u_\upsilon^*\mathfrak{L})$$

and a weighted L^p-norm $\|\cdot\|_{\upsilon,p}$ on the space $\Gamma(\Sigma_\upsilon; \Lambda^{0,1}_{\mathfrak{i},j} T^*\Sigma_\upsilon \otimes u_\upsilon^*\mathfrak{L})$. We denote the corresponding completions by $\Gamma(\upsilon)$, $\Gamma(\upsilon;\mathfrak{L})$, and $\Gamma^{0,1}(\upsilon;\mathfrak{L})$. The norms $\|\cdot\|_{\upsilon,p,1}$ and $\|\cdot\|_{\upsilon,p}$ are analogous to the ones used in [**15**, Section 3] for the bundle TX. We put

$$\begin{aligned} \Gamma_-(\upsilon;\mathfrak{L}) &= \big\{\xi \circ q_\upsilon \colon \xi \in \ker \bar{\partial}_{\nabla,b}\big\} \subset \Gamma(\upsilon;\mathfrak{L}); \\ \Gamma_+(\upsilon;\mathfrak{L}) &= \big\{\xi \in \Gamma(\upsilon;\mathfrak{L}) \colon \ \xi(\infty)=0; \\ &\qquad \langle\!\langle \xi, \xi' \rangle\!\rangle_{\upsilon,2} = 0 \ \forall\, \xi' \in \Gamma_-(\upsilon;\mathfrak{L}) \text{ s.t. } \xi'(\infty)=0\big\}. \end{aligned} \tag{4.6}$$

By the construction of the map q_υ in [**25**, Subsection 2.2],

$$\big\|\bar{\partial}_{\nabla,\upsilon}\xi\big\|_{\upsilon,p} \le C(b)|\upsilon|^{1/p}\|\xi\|_{\upsilon,p,1} \qquad \forall\, \xi \in \Gamma_-(\upsilon;\mathfrak{L}). \tag{4.7}$$

On the other hand, for the same reasons as in [**15**, Section 3], for some $\delta, C \in C(\tilde{\mathfrak{X}}_{\mathcal{T};B}(X);\mathbb{R}^+)$ and for all $\upsilon=(b,v)\in \tilde{\mathcal{F}}^{\emptyset}_{\delta}$,

$$C(b)^{-1}\big\|\xi\big\|_{\upsilon,p,1} \le \big\|\bar{\partial}_{\nabla,\upsilon}\xi\big\|_{\upsilon,p} \le C(b)\big\|\xi\big\|_{\upsilon,p,1} \quad \forall\, \xi \in \Gamma_+(\upsilon;\mathfrak{L}); \tag{4.8}$$

see [**25**, Subsections 5.3,5.4]. In particular, the operators $\bar{\partial}_{\nabla,\upsilon}$ are surjective, since $\bar{\partial}_{\nabla,b}$ and $\bar{\partial}^{y_0(b)}_{\nabla,b}$ are.

If $\zeta \in \Gamma(\upsilon)$, we set

$$b(\upsilon,\zeta) = \big(\Sigma_\upsilon; \exp_{u_\upsilon}\zeta\big) \qquad \text{and} \qquad u_{\upsilon,\zeta} = \exp_{u_\upsilon}\zeta,$$

where exp is the exponential map with respect to the connection ∇^X. Let

$$\Pi_{\upsilon,\zeta} \colon \Gamma(\Sigma_\upsilon; u_\upsilon^*\mathfrak{L}) \longrightarrow \Gamma(\Sigma_\upsilon; u_{\upsilon,\zeta}^*\mathfrak{L})$$

be the isomorphism induced by the ∇-parallel transport along the ∇^X-geodesics $\tau \longrightarrow \exp_\upsilon \tau\zeta$ with $\tau \in [0,1]$. By a direct computation,

$$\big\|\Pi_{\upsilon,\zeta}^{-1}\bar{\partial}_{\nabla,b(\upsilon,\zeta)}\Pi_{\upsilon,\zeta}\xi - \bar{\partial}_{\nabla,\upsilon}\xi\big\|_{\upsilon,p} \le C(b)\|\zeta\|^2_{\upsilon,p,1}\|\xi\|_{\upsilon,p,1} \tag{4.9}$$

for all $\xi\in\Gamma(\upsilon;\mathfrak{L})$ and $\zeta\in\Gamma(\upsilon)$; see the proof of [**23**, Corollary 2.3]. By (4.7)-(4.9), for every $\zeta\in\Gamma(\upsilon)$ sufficiently small and $\xi\in\ker\bar{\partial}_{\nabla,b}$, there exists a unique

$$\xi_{\upsilon,\zeta}(\xi)\in\Gamma_+(\upsilon;\mathfrak{L}) \qquad \text{s.t.} \qquad \Pi_{\upsilon,\zeta}\big(\xi\circ q_\upsilon+\xi_{\upsilon,\zeta}(\xi)\big)\in\ker\bar{\partial}_{\nabla,b(\upsilon,\zeta)}.$$

Furthermore,

$$\big\|\xi_{\upsilon,\zeta}(\xi)\big\|_{\upsilon,p,1}\le C(b)\big(|\upsilon|^{1/p}+\|\zeta\|^2_{\upsilon,p,1}\big)\|\xi\|_{\upsilon,p,1} \quad \forall\ \xi\in\ker\bar{\partial}_{\nabla,b}. \tag{4.10}$$

We define the isomorphism

$$R_{\upsilon,\zeta}\colon\ker\bar{\partial}_{\nabla,b}\longrightarrow\ker\bar{\partial}_{\nabla,b(\upsilon,\zeta)} \quad \text{by} \quad R_{\upsilon,\zeta}\xi=\Pi_{\upsilon,\zeta}\big(\xi\circ q_\upsilon+\xi_{\upsilon,\zeta}(\xi)\big).$$

We will use a convenient family of connections in the vector bundles $u^*\mathfrak{L}\longrightarrow\Sigma$, which is provided by Lemma 4.4 below. First, if $b=(\Sigma_b;u_b)$ is a stable bubble map, g_b is a Hermitian metric in the vector bundle $u_b^*\mathfrak{L}\longrightarrow\Sigma_b$, and ∇^b is a connection in $u_b^*\mathfrak{L}$, we will call the pair (g,∇)-admissible if

($g\nabla1$) ∇^b is g_b-compatible and $\bar{\partial}_{\nabla,b}$-compatible;

($g\nabla2$) $g_b=g_{u_b}$ and $\nabla^b=\nabla^{u_b}$ on Σ_b^0,

where g_{u_b} is the Hermitian metric in $u_b^*\mathfrak{L}$ induced from the standard metric in $\mathfrak{L}$. The second condition in ($g\nabla1$) means that

$$\bar{\partial}_{\nabla,b}\equiv\frac{1}{2}\big(\nabla^{u_b}+\mathfrak{i}\nabla^{u_b}\circ j\big)=\frac{1}{2}\big(\nabla^b+\mathfrak{i}\nabla^b\circ j\big),$$

where ∇^{u_b} is as in Subsection 1.2.

If b is any genus-zero bubble map and $\delta\in\mathbb{R}^+$, we put

$$\Sigma_b^0(\delta)=\Sigma_b^0\cup\bigcup_{i\in\chi(b)}A_{b,i}(\delta), \qquad \text{where}$$
$$A_{b,i}(\delta)=\big\{(i,z)\colon|z|\ge\delta^{-1/2}/2\big\}\subset\Sigma_{b',i}\approx S^2. \tag{4.11}$$

If $\upsilon=(b,v)$ is a gluing parameter such that the map q_υ is defined, let

$$\Sigma_\upsilon^0(\delta)=q_\upsilon^{-1}\big(\Sigma_b^0(\delta)\big). \tag{4.12}$$

Lemma 4.4. *If* (X,ω), $(\mathfrak{L},\nabla)$, A, M, $\mathcal{T}$, $\tilde{\mathfrak{X}}_{\mathcal{T};B}(X)$, $\widetilde{\mathcal{F}}$ *are as above, there exist*

$$\delta,C\in C(\tilde{\mathfrak{X}}_{\mathcal{T};B}(X);\mathbb{R}^+)$$

with the following property. For every

$$b\in\tilde{\mathfrak{X}}_{\mathcal{T};B}(X), \quad \upsilon=(b,v)\in\widetilde{\mathcal{F}}^\emptyset_\delta, \quad \textit{and} \quad \zeta\in\Gamma(\upsilon)\ \text{s.t.}\ \|\zeta\|_{\upsilon,p,1}\le\delta(b)$$

there exist metrics g_b and $g_{(\upsilon,\zeta)}$ and connections ∇^b and $\nabla^{(\upsilon,\zeta)}$ in the vector bundles

$$u_b^*\mathfrak{L}\longrightarrow\Sigma_b \qquad \textit{and} \qquad u_{\upsilon,\zeta}^*\mathfrak{L}\longrightarrow\Sigma_\upsilon$$

such that

(1) all pairs (g_b,∇^b) and $(g_{(\upsilon,\zeta)},\nabla^{(\upsilon,\zeta)})$ are admissible;

(2) the curvatures of ∇^b and $\nabla^{(\upsilon,\zeta)}$ vanish on $\Sigma_b^0(2\delta)$ and $\Sigma_\upsilon^0(2\delta)$,

respectively;

(3) for every $\xi\in\Gamma(b;\mathfrak{L})$,

$$\|\Pi_{\upsilon,\zeta}^{-1}\nabla^{(\upsilon,\zeta)}(\Pi_{\upsilon,\zeta}\xi\circ q_\upsilon)-\nabla^b\xi\circ dq_\upsilon\|_{\upsilon,p}\le C\big(|\upsilon|^{1/p}+\|\zeta\|_{\upsilon,p,1}\big)\|\xi\|_{b,p,1}\,;$$

(4) the map $b\longrightarrow(g_b,\nabla^b)$ *is continuous.*

This lemma is proved by exactly the same argument as [**27**, Lemma 4.1].

If $b\in\tilde{\mathfrak{X}}_{\mathcal{T};B}(X)$ and $i\in\chi(b)$, let w_i be the standard holomorphic coordinate centered at the point ∞ in $\Sigma_{b,i}=S^2$. If $m\in\mathbb{Z}^+$ and $\xi=(\xi_i)_{i\in I}\in\Gamma(b;\mathfrak{L})$, we put

$$\mathfrak{D}_{b,i}^{(m)}\xi=\frac{1}{m!}\frac{d^m}{dw_i^m}\xi_i(w_i)\Big|_{w_i=0}\in\mathfrak{L}_{\mathrm{ev}_0(b)},$$

where the derivatives are taken with respect to the connection ∇^b. Similarly, for all

$$\upsilon=(b,v)\in\widetilde{\mathcal{F}}_\delta^\emptyset,\quad \zeta\in\Gamma(\upsilon)\text{ s.t. }\|\zeta\|_{\upsilon,p,1}\le\delta(b),\text{ and }\xi\in\Gamma(\Sigma_\upsilon;u_{\upsilon,\zeta}^*\mathfrak{L}),$$

let

$$\mathfrak{D}_{(\upsilon,\zeta),\hat{0}}^{(m)}\xi=\frac{1}{m!}\frac{d^m}{dw^m}\xi(w)\Big|_{w=0}\in\mathfrak{L}_{\mathrm{ev}_0(b)},$$

where w is the standard holomorphic coordinate centered at the point ∞ in $\Sigma_\upsilon\approx S^2$ and the derivatives are taken with to the connection $\nabla^{(\upsilon,\zeta)}$. We note that

$$\begin{aligned}\mathfrak{D}_{b,i}^{(1)}\xi&=\mathfrak{D}_{b,i}\xi\quad\forall\, i\in\chi(b),\ \xi\in\Gamma(b;\mathfrak{L}),\\ \mathfrak{D}_{(\upsilon,\zeta),\hat{0}}^{(1)}\xi&=\mathfrak{D}_{b(\upsilon,\zeta),\hat{0}}\xi\quad\forall\,\xi\in\Gamma(\Sigma_\upsilon;u_{\upsilon,\zeta}^*\mathfrak{L}),\end{aligned}\tag{4.13}$$

by the second condition in $(g\nabla 2)$ above.

A key step in understanding the obstruction to extending holomorphic bundle sections from singular to smooth domains is the following power series expansion. For every

$$i\in\chi(b),\qquad k\in\mathbb{Z}^+,\qquad \upsilon=(b,v)\in\widetilde{\mathcal{F}}_\delta^\emptyset,\quad\text{and}\quad\zeta\in\Gamma(\upsilon),$$

there exist

$$\varepsilon_i^{(k)}(\upsilon,\zeta)\in\mathrm{Hom}\big(\ker\bar\partial_{\nabla,b},\mathfrak{L}_{\mathrm{ev}_0(b)}\big)$$

such that for all $\xi\in\ker\bar\partial_{\nabla,b}$

$$\begin{aligned}&\mathfrak{D}_{b(\upsilon,\zeta),\hat{0}}^{(m)}R_{\upsilon,\zeta}\xi\\ &\qquad=\sum_{k=1}^{k=m}\binom{m-1}{k-1}\sum_{i\in\chi(b)}x_i^{m-k}(\upsilon)\rho_i^k(\upsilon)\Big\{\mathfrak{D}_{b,i}^{(k)}\xi+\varepsilon_i^{(k)}(\upsilon,\zeta)\xi\Big\}\end{aligned}\tag{4.14}$$

and

$$\big|\varepsilon_i^{(k)}(\upsilon,\zeta)\xi\big|\le C\delta^{-k/2}\big(|\upsilon|^{1/p}+\|\zeta\|_{\upsilon,p,1}\big)\|\xi\|_{b,p,1}.\tag{4.15}$$

The expansion (4.14) is obtained by exactly the same integration-by-parts argument as the expansion in [**25**, Theorem 2.8]; see also the

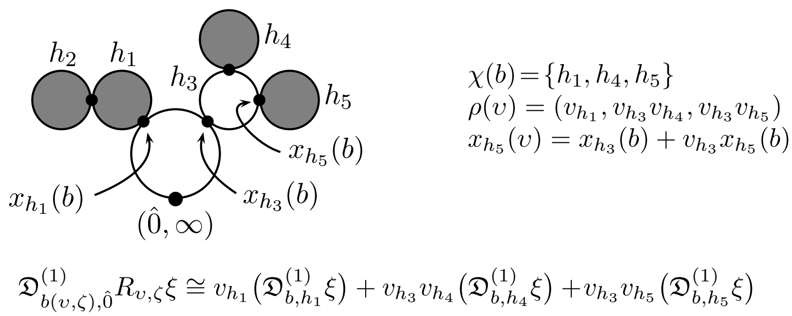

Figure 3. An example of the estimates (4.14) and (4.15).

paragraph following [**27**, Lemma 4.2]. We point out that $\varepsilon_i^{(k)}$ is independent of m. The $m=1$ case of the estimates (4.14) and (4.15) is illustrated in Figure 3.

Let t_r and b_r be as in Lemma 4.1. Since the sequence $[b_r]$ converges to $[b]$, for all r sufficiently large there exist

$$b'_r\in\tilde{\mathfrak{X}}_{\mathcal{T};B}(X), \qquad \upsilon_r=(b'_r,v_r)\in\widetilde{\mathcal{F}}^{\emptyset}, \qquad \text{and} \qquad \zeta_r\in\Gamma(\upsilon_r)$$

such that

$$\lim_{r\longrightarrow\infty}b'_r=b, \qquad \lim_{r\longrightarrow\infty}|v_r|=0, \qquad \lim_{r\longrightarrow\infty}\|\zeta_r\|_{\upsilon_r,p,1}=0, \qquad \text{and} \qquad b_r\equiv\big(\Sigma_{b_r};u_{b_r}\big)=\big(\Sigma_{\upsilon_r};\exp_{u_{\upsilon_r}}\zeta_r\big). \tag{4.16}$$

The last equality holds for a representative b_r for $[b_r]$. By [**28**, (2.12)] and (4.16),

$$\Big|\mathcal{D}_{\hat{0}}b_r-\sum_{i\in\chi(b)}\rho_i(\upsilon_r)\big(\mathcal{D}_ib'_r\big)\Big|\le\epsilon_r\sum_{i\in\chi(b)}\big|\rho_i(\upsilon_r)\big| \tag{4.17}$$

for a subsequence ϵ_r converging to 0. Furthermore, by the $m=1$ case of (4.14) and (4.15),

$$\Big|\mathfrak{D}_{b,\hat{0}}R_{\upsilon_r,\zeta_r}\xi-\sum_{i\in\chi(b)}\rho_i(\upsilon_r)\mathfrak{D}_{b'_r,i}\xi\Big|\le\epsilon_r\sum_{i\in\chi(b)}|\rho_i(\upsilon_r)|\cdot\|\xi\|_{b,p,1} \tag{4.18}$$

for all $\xi\in\ker\bar{\partial}_{\nabla,b'_r}$. For the purposes of Lemma 4.1, we take

$$R_{b_r,b}\colon\ker\bar{\partial}_{\nabla,b}\longrightarrow\ker\bar{\partial}_{\nabla,b_r}$$

to be the composition of

$$R_{\upsilon_r,\zeta_r}\colon\ker\bar{\partial}_{\nabla,b'_r}\longrightarrow\ker\bar{\partial}_{\nabla,b_r}$$

with an isomorphism

$$R_{b'_r,b}\colon\ker\bar{\partial}_{\nabla,b}\longrightarrow\ker\bar{\partial}_{\nabla,b'_r} \quad \text{s.t.} \quad \lim_{r\longrightarrow\infty}R_{b'_r,b}\xi=\xi\ \ \forall\,\xi\in\ker\bar{\partial}_{\nabla,b}.$$

We take $c_{r,i}=\rho_i(\upsilon_r)$. It is immediate that the requirements (4.3) of Lemma 4.1 are satisfied. Since

$$\lim_{r\longrightarrow\infty}\mathcal{D}_i b'_r=\mathcal{D}_i b\,,\ \lim_{r\longrightarrow\infty}\mathfrak{D}_{b'_r,i}R_{b'_r,b}\xi=\mathfrak{D}_{b,i}\xi\quad\forall\,\xi\in\ker\bar{\partial}_{\nabla,b},\ i\in\chi(b),$$

the requirements (4.1) and (4.2) are satisfied as well. This concludes the proof of Lemma 4.1.

Remark: A regularization φ_b of the cone $\mathcal{V}^A_{0,M}$ near b, as in Definition 3.3, can be constructed using the isomorphisms $R_{\upsilon,\zeta}$ as above and a description of open subsets in $\mathfrak{X}_{0,M}(X,A)$ as in [**15**, Section 3]. In this case, φ_b is a homeomorphism.

4.3. Proof of Lemma 4.2. In this subsection, we review the two-step gluing construction used in [**28**, Subsection 2.5] and extend it to holomorphic bundle sections in a way similar to [**27**, Subsection 4.2]. We assume that

$$\mathcal{T}=(M,I,\aleph;j,\underline{A})$$

is a bubble type such that

$$\sum_{i\in I}A_i=A,\quad\langle\omega,A_i\rangle\geq 0\ \forall\,i\in I,\quad A_i=0\ \forall\ i\in I_0,\ \text{and}\ I_0\subsetneq I,$$

where I_0 is the subset of minimal elements of I as before. Let $(\mathfrak{L},\nabla)\longrightarrow X$ be an (ω,A)-positive vector bundle with connection. Throughout this subsection we focus on the case

$$[b_r]\in\mathfrak{M}^0_{1,M}(X,A;J,\nu),$$

i.e. $\Sigma_{b_r;P}=\Sigma_{b_r}$ is a smooth torus for all r.

Similarly to Subsection 4.2, we put

$$\tilde{\mathfrak{X}}_{\mathcal{T};B}(X)=\big\{(\Sigma_b;u_b)\in\tilde{\mathfrak{X}}_{\mathcal{T}}(X):\,u_b|_{\Sigma^0_b}=\text{const}\big\}.$$

Let

$$I_1=\big\{h\in\hat{I}:\,\iota_h\in I_0\big\}\qquad\text{and}\qquad k_0=|I_1|+\big|\{l\in M:\,j_l\in I_0\}\big|.$$

We denote by $\overline{\mathcal{M}}_{1,k_0}$ the moduli space of genus-one curves with k_0 marked points and by

$$\pi_P:\tilde{\mathfrak{X}}_{\mathcal{T};B}(X)\longrightarrow\overline{\mathcal{M}}_{1,k_0},\qquad b\longrightarrow[\Sigma_{b;P}],$$
$$\text{ev}_P:\tilde{\mathfrak{X}}_{\mathcal{T};B}(X)\longrightarrow X,\qquad b\longrightarrow u_b(\Sigma_{b;P}),$$

the maps sending each element b of $\tilde{\mathfrak{X}}_{\mathcal{T};B}(X)$ to the equivalence class of the principal component(s) $\Sigma_{b;P}$ of its domain and to the image of $\Sigma_{b;P}$ in X. Let

$$\mathbb{E}\longrightarrow\overline{\mathcal{M}}_{1,k_0}$$

be the Hodge line bundle, i.e. the line bundle of holomorphic differentials.

Let

$$\widetilde{\mathcal{F}}\longrightarrow\tilde{\mathfrak{X}}_{\mathcal{T};B}(X)$$

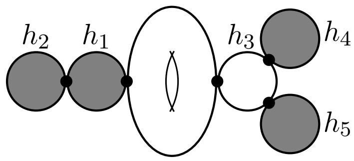

Figure 4. An example of $\rho(\upsilon)$.

be the bundle of gluing parameters. It has three distinguished components,

$$\widetilde{\mathcal{F}}=\widetilde{\mathcal{F}}_{\aleph}\oplus\widetilde{\mathcal{F}}_0\oplus\widetilde{\mathcal{F}}_1,$$

where

$$\widetilde{\mathcal{F}}_{\aleph}=\tilde{\mathfrak{X}}_{\mathcal{T};B}(X)\times\mathbb{C}^{\aleph},\quad \widetilde{\mathcal{F}}_1=\tilde{\mathfrak{X}}_{\mathcal{T};B}(X)\times\mathbb{C}^{\hat{I}-I_1},$$
$$\widetilde{\mathcal{F}}_0=\bigoplus_{h\in I_1}\widetilde{\mathcal{F}}_h,\quad\text{and}\quad \widetilde{\mathcal{F}}_h\big|_b=T_{x_h(b)}\Sigma_{b;P}\ \forall\ b\in\tilde{\mathfrak{X}}_{\mathcal{T};B}(X).$$

The total space of $\widetilde{\mathcal{F}}_h$ has a natural topology; see [**26**, Subsection 2.2]. We denote by $\widetilde{\mathcal{F}}^{\emptyset}$ the subset of $\tilde{\mathcal{F}}$ consisting of the elements with all components nonzero. If $i\in\hat{I}$, let $h(i)\in I_1$ be the unique element such that $h(i)\le i$. For each $\upsilon=(b,v)$, where $b\in\tilde{\mathfrak{X}}_{\mathcal{T};B}(X)$ and $v=(v_i)_{i\in\aleph\sqcup\hat{I}}$, and $i\in\chi(b)$, we put

$$\upsilon_0=\big(b,(v_i)_{i\in\aleph\sqcup I_1}\big),\qquad \upsilon_1=\big(b,(v_i)_{i\in\hat{I}-I_1}\big),$$
$$\tilde{\rho}_i(\upsilon)=\prod_{h(i)<h\le i}v_h\in\mathbb{C},\quad \rho_i(\upsilon)=\tilde{\rho}_i(\upsilon)\cdot v_{h(i)}\in T_{x_{h(i)}(b)}\Sigma_{b;P},$$
$$\text{and}\qquad \rho(\upsilon)=\big(b,\big(\rho_i(\upsilon)\big)_{i\in\chi(b)}\big)\in\widetilde{\mathfrak{F}}\equiv\bigoplus_{i\in\chi(b)}\mathcal{F}_{h(i)}.$$

The component υ_1 of υ consists of the smoothings of the nodes of Σ_b that lie away from the principal component. In the case of Figure 4, these are the attaching nodes of the bubbles h_2, h_4, and h_5. For each element $\tilde{\upsilon}=(b,(\tilde{\upsilon}_i)_{i\in\chi(b)})$ of $\widetilde{\mathfrak{F}}$, we define the linear map

$$\mathfrak{D}_{\tilde{\upsilon}}\colon\ker\bar{\partial}_{\nabla,b}\longrightarrow\mathbb{E}^*_{\pi_P(b)}\otimes\mathfrak{L}_{\mathrm{ev}_P(b)}\qquad\text{by}$$
$$\big\{\mathfrak{D}_{\tilde{\upsilon}}\xi\big\}(\psi)=\sum_{i\in\chi(b)}\psi_{x_{h(i)}(b)}\big(\tilde{\upsilon}_i\big)\cdot\mathfrak{D}_{b,i}\xi,$$

if $\psi\in\mathbb{E}_{\pi_P(b)}$.

For each sufficiently small element $\upsilon=(b,v)$ of $\widetilde{\mathcal{F}}^{\emptyset}$, let

$$q_{\upsilon_1}\colon\Sigma_{\upsilon_1}\longrightarrow\Sigma_b$$

be the basic gluing map constructed in [**25**, Subsection 2.2]. In this case, the principal component $\Sigma_{\upsilon_1;P}$ of Σ_{υ_1} is the same as principal component $\Sigma_{b;P}$ of Σ_b, and Σ_{υ_1} has $|I_1|$ bubble components $\Sigma_{\upsilon_1,h}$, with $h\in I_1$, attached directly to $\Sigma_{\upsilon_1;P}$. The map q_{υ_1} collapses $|\hat{I}-I_1|$ circles on the bubble components of Σ_{υ_1}. It induces a metric g_{υ_1} on Σ_{υ_1} such that $(\Sigma_{\upsilon_1},g_{\upsilon_1})$ is obtained from Σ_b by replacing $|\hat{I}-I_1|$ nodes by thin necks. Let

$$u_{\upsilon_1}=u_b\circ q_{\upsilon_1}, \qquad b(\upsilon_1)=(\Sigma_{\upsilon_1};u_{\upsilon_1}), \quad \text{and} \quad \bar\partial_{\nabla,\upsilon_1}=\bar\partial_{\nabla,b(\upsilon_1)}.$$

The map q_{υ_1} induces weighted L^p_1-norms $\|\cdot\|_{\upsilon_1,p,1}$ on the spaces

$$\Big\{\zeta\in\Gamma(\Sigma_{\upsilon_1};u_{\upsilon_1}^*TX)\colon \zeta|_{\Sigma_{\upsilon_1;P}}=0\Big\}, \quad \Big\{\xi\in\Gamma(\Sigma_{\upsilon_1};u_{\upsilon_1}^*\mathfrak{L})\colon \xi|_{\Sigma_{\upsilon_1;P}}=0\Big\}$$

and a weighted L^p-norm $\|\cdot\|_{\upsilon_1,p}$ on

$$\Big\{\eta\in\Gamma(\Sigma_{\upsilon_1};\Lambda^{0,1}_{\mathfrak{i},j}T^*\Sigma_{\upsilon_1}\otimes u_{\upsilon_1}^*\mathfrak{L})\colon \eta|_{\Sigma_{\upsilon_1;P}}=0\Big\};$$

see [**25**, Subsection 3.3] and the first remark in [**28**, Subsection 2.5]. We denote the corresponding completions by $\Gamma_B(\upsilon_1)$, $\Gamma_B(\upsilon_1;\mathfrak{L})$, and $\Gamma^{0,1}_B(\upsilon_1;\mathfrak{L})$.

For each $\zeta\in\Gamma_B(\upsilon_1)$, let

$$u_{\upsilon_1,\zeta}=\exp_{u_{\upsilon_1}}\zeta \qquad \text{and} \qquad b(\upsilon_1,\zeta)=(\Sigma_{\upsilon_1};u_{\upsilon_1,\zeta}).$$

For $\delta\in C(\tilde{\mathfrak{X}}_{\mathcal{T};B}(X);\mathbb{R}^+)$ sufficiently small,

$$\upsilon\equiv(b,v)\in\tilde{\mathcal{F}}^{\emptyset}_{\delta}, \qquad \text{and} \qquad \zeta\in\Gamma_B(\upsilon_1) \text{ s.t. } \|\zeta\|_{\upsilon_1,p,1}\le\delta(b),$$

the isomorphisms $R_{\upsilon,\zeta}$ of Subsection 4.2 corresponding to the restriction of $b(\upsilon_1,\zeta)$ to $\Sigma_{\upsilon_1,h}$, with $h\in I_1$, induce an isomorphism

$$R_{\upsilon_1,\zeta}\colon \ker\bar\partial_{\nabla,b}\longrightarrow\ker\bar\partial_{\nabla,b(\upsilon_1,\zeta)}$$

such that

$$\big\|R_{\upsilon_1,\zeta}\xi\big\|_{\upsilon_1,p,1}\le 2\|\xi\|_{b,p,1}.$$

Furthermore, by the $m=1$ case of (4.14) and (4.15),

$$\begin{aligned}\Big|\mathfrak{D}_{b(\upsilon_1,\zeta),h}R_{\upsilon_1,\zeta}\xi-\sum_{i\in\chi(b),h(i)=h}\tilde\rho_i(\upsilon)\mathfrak{D}_{b,i}\xi\Big|&\\ \le C(b)\big(|\upsilon|^{1/p}+\|\zeta\|_{\upsilon_1,p,1}\big)&\sum_{i\in\chi(b),h(i)=h}|\tilde\rho_i(\upsilon)|\cdot\|\xi\|_{b,p,1},\end{aligned} \tag{4.19}$$

for all $h\in I_1$ and $\xi\in\ker\bar\partial_{\nabla,b}$. Let $\nabla^{\upsilon_1,\zeta}$ and $g_{\upsilon_1,\zeta}$ be the connection and metric in the line bundle $u_{\upsilon_1,\zeta}^*\mathfrak{L}$ induced by the connections and metrics of Lemma 4.4. For each $h\in I_1$ and $\delta\in\mathbb{R}^+$, let

$$A_{\upsilon_1,h}(\delta)=q_{\upsilon}^{-1}\big(A_{b,h}(\delta)\big).$$

From the estimates (4.14) and (4.15), we find that

$$\begin{aligned}(4.20)\qquad &\big\|\nabla^{\upsilon_1,\zeta}R_{\upsilon_1,\zeta}\xi\big\|_{C^0(A_{\upsilon_1,h}(\delta(b))),g_{\upsilon_1}}\\ &\qquad\le C(b)\big(|\upsilon|^{1/p}+\|\zeta\|_{\upsilon_1,p,1}\big)\sum_{i\in\chi(b),h(i)=h}|\tilde\rho_i(\upsilon)|\cdot\|\xi\|_{b,p,1},\end{aligned}$$

for all $h\in I_1$ and $\xi\in\ker\bar\partial_{\nabla,b}$.

Fix a smooth function $\epsilon:\tilde{\mathfrak{X}}_{\mathcal{T};B}(X)\longrightarrow\mathbb{R}^+$ such that for every $b\in\tilde{\mathfrak{X}}_{\mathcal{T};B}(X)$ and $h\in I_1$ the disk of radius of $8\epsilon_b$ in $\Sigma_{b;P}$ around the node $x_h(b)$ contains no other special, i.e. singular or marked, point of Σ_b. For each

$$\upsilon\equiv(b,v)\equiv\big(b,(v_h)_{h\in\aleph\sqcup\hat I}\in\widetilde{\mathcal{F}}^{\emptyset}$$

sufficiently small, let

$$q_{\upsilon_0;2}:\Sigma_\upsilon\longrightarrow\Sigma_{\upsilon_1}\qquad\text{and}\qquad\tilde q_{\upsilon_0;2}:\Sigma_\upsilon\longrightarrow\Sigma_{\upsilon_1}$$

be the basic gluing map of [**25**, Subsection 2.2] corresponding to the gluing parameter v_0 and the modified basic gluing map defined in the middle of [**26**, Subsection 4.2] with the collapsing radius ϵ_b. In this case, Σ_υ is a smooth genus-one curve. For each $h\in I_1$, the maps $q_{\upsilon_0;2}$ and $\tilde q_{\upsilon_0;2}$ collapse the circles of radii $|v_h|^{1/2}$ and ϵ_b, respectively, around the point $x_h(b)\in\Sigma_{\upsilon_1;P}$. As before, the map

$$q_\upsilon\equiv q_{\upsilon_0;2}\circ q_{\upsilon_1}:\Sigma_\upsilon\longrightarrow\Sigma_b$$

induces a metric g_υ on Σ_υ such that $(\Sigma_\upsilon,g_\upsilon)$ is obtained from Σ_b by replacing all nodes by thin necks. Let

$$I_1^*=I_1-\big\{h\in I_1:A_i=0\ \forall\, i\ge h\big\}.$$

The map $\tilde q_{\upsilon_0;2}$ is biholomorphic outside $|\aleph|$ thin necks $A_{\upsilon,h}$, with $h\in\aleph$, of $(\Sigma_\upsilon,g_\upsilon)$ and the $|I_1|$ annuli

$$\tilde{\mathcal{A}}_{b,h}\equiv\tilde{\mathcal{A}}_{b,h}^-\cup\tilde{\mathcal{A}}_{b,h}^+,$$

with $h\in I_1$, where

$$\tilde{\mathcal{A}}_{b,h}^{\pm}\equiv\tilde{\mathcal{A}}_{b,h}^{\pm}\big(\delta(b)\big)\subset\Sigma_{b;P}\approx\Sigma_\upsilon$$

are annuli independent of υ. In addition,

$$(4.21)\qquad\begin{aligned}u_{\upsilon_1,\zeta}|_{\tilde q_{\upsilon_0;2}(A_{\upsilon,h})}&=\text{const}\qquad\forall\,h\in\aleph,\\ u_{\upsilon_1,\zeta}|_{\tilde q_{\upsilon_0;2}(\tilde{\mathcal{A}}_{b,h})}&=\text{const}\qquad\forall\,h\in I_1-I_1^*,\\ u_{\upsilon_1,\zeta}|_{\tilde q_{\upsilon_0;2}(\tilde{\mathcal{A}}^+_{b,h})}&=\text{const}\qquad\forall\,h\in I_1^*;\end{aligned}$$

$$(4.22)\qquad\left.\begin{aligned}\tilde q_{\upsilon_0;2}(\tilde{\mathcal{A}}^-_{b,h})&\subset A_{\upsilon_1,h}\big(|v_h|^2/\delta(b)\big),\\ \big\|d\tilde q_{\upsilon_0;2}\big\|_{C^0(\tilde{\mathcal{A}}^-_{b,h})}&\le C(b)|v_h|\end{aligned}\right.\qquad\forall\,h\in I_1^*,$$

if the C^0-norm of $d\tilde{q}_{\upsilon 0;2}$ is computed with respect to the metrics g_υ on Σ_υ and g_{υ_1} on Σ_{υ_1}. Furthermore,

$$\big\|d\tilde{q}_{\upsilon 0;2}\big\|_{C^0} \le C(b). \tag{4.23}$$

If $\upsilon=(b,v)\in\widetilde{\mathcal{F}}^{\emptyset}$ is sufficiently small and $\zeta\in\Gamma_B(\upsilon_1)$, we put

$$u_{\upsilon,\zeta}=u_{\upsilon_1,\zeta}\circ\tilde{q}_{\upsilon 0;2} \qquad \text{and} \qquad b(\upsilon,\zeta)=\big(\Sigma_\upsilon;u_{\upsilon,\zeta}\big).$$

The map $q_{\upsilon 0;2}$ induces weighted L^p_1-norms $\|\cdot\|_{\upsilon,p,1}$ on the spaces

$$\Gamma(\Sigma_\upsilon;u^*_{\upsilon,\zeta}TX) \qquad \text{and} \qquad \Gamma(\Sigma_\upsilon;u^*_{\upsilon,\zeta}\mathfrak{L})$$

and a weighted L^p-norm $\|\cdot\|_{\upsilon,p}$ on the space

$$\Gamma(\Sigma_\upsilon;\Lambda^{0,1}_{\mathfrak{i},j}T^*\Sigma_\upsilon\otimes u^*_{\upsilon,\zeta}TX).$$

Let $\Gamma(\upsilon,\zeta)$, $\Gamma(\upsilon,\zeta;\mathfrak{L})$, and $\Gamma^{0,1}(\upsilon,\zeta;\mathfrak{L})$ be the corresponding completions. We put

$$\Gamma_-(\upsilon,\zeta;\mathfrak{L})=\big\{R_{\upsilon,\zeta}\xi\equiv R_{\upsilon_1,\zeta}\xi\circ\tilde{q}_{\upsilon 0;2}:\xi\in\ker\bar{\partial}_{\nabla,b}\big\}\subset\Gamma(\upsilon,\zeta;\mathfrak{L}).$$

By (4.20)-(4.23), for all $\xi\in\ker\bar{\partial}_{\nabla,b}$

$$\begin{aligned}\big\|\bar{\partial}_{\nabla,b(\upsilon,\zeta)}R_{\upsilon,\zeta}\xi\big\|_{\upsilon,p} &\le C(b)\sum_{h\in I_1^*}|v_h|\sum_{i\in\chi(b),h(i)=h}|\tilde{\rho}_i(\upsilon)|\cdot\|\xi\|_{b,p,1}\\ &= C(b)\big|\rho(\upsilon)\big|\cdot\|\xi\|_{b,p,1}.\end{aligned} \tag{4.24}$$

Let $\Gamma_+(\upsilon,\zeta;\mathfrak{L})$ be the L^2-orthogonal complement of $\Gamma_-(\upsilon,\zeta;\mathfrak{L})$ in $\Gamma(\upsilon,\zeta;\mathfrak{L})$. We denote by

$$\pi_{\upsilon,\zeta;-}:\Gamma(\upsilon,\zeta;\mathfrak{L})\longrightarrow\Gamma_-(\upsilon,\zeta;\mathfrak{L})$$

the L^2-projection map. For the same reasons as before,

$$C(b)^{-1}\big\|\xi\big\|_{\upsilon,p,1}\le\big\|\bar{\partial}_{\nabla,b(\upsilon,\zeta)}\xi\big\|_{\upsilon,p}\le C(b)\big\|\xi\big\|_{\upsilon,p,1} \tag{4.25}$$

for all $\xi\in\Gamma_+(\upsilon,\zeta;\mathfrak{L})$, if $\upsilon=(b,v)\in\mathcal{F}^{\emptyset}$ and $\zeta\in\Gamma(\upsilon_1)$ are sufficiently small. Let $\Gamma_+^{0,1}(\upsilon,\zeta;\mathfrak{L})$ be the image of $\Gamma_+(\upsilon,\zeta;\mathfrak{L})$ under $\bar{\partial}_{\nabla,b(\upsilon,\zeta)}$.

The operator $\bar{\partial}_{\nabla,b_{\upsilon,\zeta}}$ is not surjective. We next describe its cokernel. Since the operator $\bar{\partial}^B_{\nabla,b}$ is surjective, the cokernel of $\bar{\partial}_{\nabla,b}$ can be identified with the vector space

$$\Gamma_-^{0,1}(b;\mathfrak{L})\equiv\mathcal{H}_{b;P}\otimes\mathfrak{L}_{\mathrm{ev}_P(b)}\approx\mathbb{E}^*_{\pi_P(b)}\otimes\mathfrak{L}_{\mathrm{ev}_P(b)},$$

where $\mathcal{H}_{b;P}$ is the space of harmonic antilinear differentials on the main component $\Sigma_{b;P}$ of Σ_b. As in [**27**, Subsection 4.3], there exist isomorphisms

$$R^{0,1}_{\upsilon;P}:\mathcal{H}_{b;P}\longrightarrow\mathcal{H}_{\upsilon;P}\equiv\mathcal{H}_{b(\upsilon,\zeta);P}, \qquad \upsilon=(b,v)\in\widetilde{\mathcal{F}}_\delta,$$

such that the family of induced homomorphisms

$$\mathcal{H}_{b;P}\longrightarrow\Gamma^{0,1}(\upsilon;\mathbb{C})^*,$$

$$\{R^{0,1}_{\upsilon;P}\eta\}(\eta')=\langle\!\langle R^{0,1}_{\upsilon;P}\eta,\eta'\rangle\!\rangle_2 \quad \forall\,\eta\in\mathcal{H}_{b;P},\ \eta'\in\Gamma^{0,1}(\upsilon;\mathbb{C}),$$

is continuous on $\widetilde{\mathcal{F}}_\delta$, and

$$R^{0,1}_{\upsilon;P}|_b = \mathrm{id} \qquad \forall\, b\in\tilde{\mathfrak{X}}_{\mathcal{T};B}(X). \tag{4.26}$$

Let $\beta\colon \mathbb{R}^+ \longrightarrow [0,1]$ be a smooth function such that

$$\beta(t) \in \begin{cases} 0, & \text{if } t\le 1;\\ 1, & \text{if } t\ge 2. \end{cases}$$

If $r\in\mathbb{R}^+$, let $\beta_r(t)=\beta(t/\sqrt{r})$. We define $\beta_b\in C^\infty(\Sigma_b;\mathbb{R})$ by

$$\beta_b(z) = \begin{cases} 1, & \text{if } z\in\Sigma_{b,i},\ i\in\chi^0(b);\\ 1-\beta_{\delta(b)}(r(z)), & \text{if } z\in\Sigma_{b,i},\ i\in\chi(b);\\ 0, & \text{otherwise,} \end{cases}$$

where $r(z) = |q_S^{-1}(z)|$ if $q_S : \mathbb{C} \longrightarrow S^2$ is the stereographic projection mapping the origin to the south pole of S^2. In other words, $\beta_b = 1$ on $\Sigma^0_b(\delta(b)/2)$ and vanishes outside of $\Sigma^0_b(2\delta(b))\subset\Sigma_b$. Let $\beta_\upsilon=\beta_b\circ q_\upsilon$. If $z\in\Sigma^0_\upsilon(2\delta(b))$, we denote by $\Pi^{(\upsilon,\zeta)}_z$ the parallel transport in the line bundle $u^*_{\upsilon,\zeta}\mathfrak{L}$ along a path from $x\in\tilde{q}^{-1}_{\upsilon_0;2}(\Sigma_{\upsilon_1;P})$ to z in $\Sigma^0_\upsilon(2\delta(b))$ with respect to the connection $\tilde{q}^*_{\upsilon_0;2}\nabla^{(\upsilon_1,\zeta)}$. For each

$$\upsilon = (b,v)\in\widetilde{\mathcal{F}}^{\emptyset}_\delta \qquad\text{and}\qquad \eta\in\Gamma^{0,1}_-(b;\mathfrak{L}), \tag{4.27}$$

let $R^{0,1}_{\upsilon,\zeta}\eta\in\Gamma^{0,1}(\upsilon,\zeta;\mathfrak{L})$ be given by

$$\{R^{0,1}_{\upsilon,\zeta}\eta\}_z w = \beta_\upsilon(z)\,\Pi^{\upsilon,\zeta}_z\eta_z(w) \in \mathfrak{L}_{u_{\upsilon,\zeta}(z)} \qquad z\in\Sigma_\upsilon,\ w\in T_z\Sigma_\upsilon.$$

Since the curvature of $\tilde{q}^*_{\upsilon_0;2}\nabla^{(\upsilon_1,\zeta)}$ vanishes over $\Sigma^0_\upsilon(2\delta(b))$, $\{R^{0,1}_{\upsilon,\zeta}\eta\}_z w$ is independent of the choice of x and path from x to z above.

If $\eta\in\Gamma^{0,1}_-(b;\mathfrak{L})$, we put

$$\|\eta\| = \sum_{h\in I_1^*} |\eta|_{x_h(b)},$$

where $|\eta|_{x_h(b)}$ is the norm of $\eta|_{x_h(b)}$ with respect to the metric $g_{\pi_P(b)}$ on $\Sigma_{b;P}$. If υ and η are as in (4.27) and $\|\eta\|=1$, we define by

$$\pi^{0,1}_{\upsilon,\zeta;-}\colon \Gamma^{0,1}(\upsilon,\zeta;\mathfrak{L}) \longrightarrow \Gamma^{0,1}_-(b;\mathfrak{L}) \qquad \text{by}$$
$$\pi^{0,1}_{\upsilon,\zeta;-}(\eta') = \langle\!\langle \eta', R^{0,1}_{\upsilon,\zeta}\eta\rangle\!\rangle_2\,\eta \quad \forall\,\eta'\in\Gamma^{0,1}(\upsilon,\zeta;\mathfrak{L}).$$

Since the space $\Gamma^{0,1}_-(b;\mathfrak{L})$ is one-dimensional, $\pi^{0,1}_{\upsilon,\zeta;-}$ is independent of the choice of η. We note that since $p>2$, by Holder's inequality

$$\big\|\pi^{0,1}_{\upsilon,\zeta;-}\eta'\big\| \le C(b)\|\eta'\|_{\upsilon,p} \qquad \forall\,\eta'\in\Gamma^{0,1}(\upsilon,\zeta;\mathfrak{L}). \tag{4.28}$$

Furthermore, by the proof of [**25**, Lemma 2.2],

$$\big\|\pi^{0,1}_{\upsilon,\zeta;-}\bar{\partial}_{\nabla,b(\upsilon,\zeta)}\xi\big\| \le C(b)\big|\rho(\upsilon)\big|\|\xi\|_{\upsilon,p,1} \qquad \forall\,\xi\in\Gamma(\upsilon,\zeta;\mathfrak{L}). \tag{4.29}$$

With the same restriction on the homomorphisms $R^{0,1}_{\upsilon;P}$ and identification of gluing parameters as described in [**26**, Subsection 4.2], we also have

$$\pi^{0,1}_{\upsilon,\zeta;-}\bar{\partial}_{\nabla,b(\upsilon,\zeta)}R_{\upsilon,\zeta}\xi = -2\pi\mathfrak{i}\,\mathfrak{D}_{\rho(\upsilon)}\xi \qquad \forall\,\xi\in\ker\bar{\partial}_{\nabla,b}, \tag{4.30}$$

by the proof of [**25**, Proposition 4.4].

For each $\upsilon=(b,v)\in\mathcal{F}^{\emptyset}_{\delta}$, $\zeta\in\Gamma_B(\upsilon_1)$, and $\zeta'\in\Gamma(\upsilon,\zeta)$, we put

$$u_{\upsilon,\zeta,\zeta'} = \exp_{u_{\upsilon,\zeta}}\zeta' \qquad \text{and} \qquad b(\upsilon,\zeta,\zeta') = \big(\Sigma_\upsilon; u_{\upsilon,\zeta,\zeta'}\big).$$

We denote by $\Pi_{\zeta'}$ the isomorphisms

$$\Gamma(\Sigma_\upsilon; u^*_{\upsilon,\zeta}\mathfrak{L}) \longrightarrow \Gamma(\Sigma_\upsilon; u^*_{\upsilon,\zeta,\zeta'}\mathfrak{L}) \qquad \text{and}$$
$$\Gamma(\Sigma_\upsilon; \Lambda^{0,1}_{\mathfrak{i},j}T^*\Sigma_\upsilon\otimes u^*_{\upsilon,\zeta}\mathfrak{L}\big) \longrightarrow \Gamma(\Sigma_\upsilon; \Lambda^{0,1}_{\mathfrak{i},j}T^*\Sigma_\upsilon\otimes u^*_{\upsilon,\zeta,\zeta'}\mathfrak{L}\big)$$

induced by the ∇-parallel transport along the geodesics $\tau\longrightarrow\exp_{u_{\upsilon,\zeta}}\tau\zeta'$ with $\tau\in[0,1]$. Let

$$L_{\upsilon,\zeta,\zeta'} = \Pi^{-1}_{\zeta'}\bar{\partial}_{\nabla,b(\upsilon,\zeta,\zeta')}\Pi_{\zeta'} - \bar{\partial}_{\nabla,b(\upsilon,\zeta)}\colon \Gamma(\upsilon,\zeta;\mathfrak{L}) \longrightarrow \Gamma^{0,1}(\upsilon,\zeta;\mathfrak{L}\big).$$

Similarly to (4.9),

$$\big\|L_{\upsilon,\zeta,\zeta'}\xi\big\|_{\upsilon,p} \le C(b)\|\zeta'\|^2_{\upsilon,p,1}\|\xi\|_{\upsilon,p,1} \qquad \forall\;\xi\in\Gamma(\upsilon,\zeta;\mathfrak{L}). \tag{4.31}$$

Let J be an almost complex structure on X. With notation as at the beginning of this subsection and in Subsection 3.2, we define the linear bundle map

$$\mathcal{D}_{J;\mathcal{T}}\colon \widetilde{\mathfrak{F}} \longrightarrow \pi^*_P\mathbb{E}^*\otimes_{\mathbb{C}}\mathrm{ev}^*_P(TX,J)$$

over $\tilde{\mathfrak{X}}_{\mathcal{T};B}(X)$ by

$$\big\{\mathcal{D}_{J;\mathcal{T}}\big(b,(\tilde{v}_i)_{i\in\chi(b)}\big)\big\}(\psi) = \sum_{i\in\chi(b)}\psi_{x_{h(i)}(b)}(\tilde{v}_i)\cdot_J\mathcal{D}_i b \in T_{\mathrm{ev}_P(b)}X.$$

Suppose t_r and b_r are as in Lemma 4.2 and $\Sigma_{b_r;P}=\Sigma_{b_r}$ is a smooth torus. Since the sequence $[b_r]$ converges to $[b]$, by [**28**, Subsection 2.5] there exist $C\in\mathbb{R}^+$ and for all r sufficiently large

$$\begin{gathered}\epsilon_r\in\mathbb{R}^+, \quad b'_r\in\tilde{\mathfrak{X}}_{\mathcal{T};B}(X), \quad \upsilon_r=(b'_r,v_r)\in\widetilde{\mathcal{F}}^{\emptyset}_{\delta}, \\ \zeta_r\in\Gamma(\upsilon_r), \quad \text{and} \quad \zeta'_r\in\Gamma(\upsilon_r,\zeta_r)\end{gathered} \tag{4.32}$$

such that

$$\begin{gathered}\lim_{r\longrightarrow\infty}b'_r=b, \qquad \lim_{r\longrightarrow\infty}|v_r|=0, \\ \lim_{r\longrightarrow\infty}\|\zeta_r\|_{\upsilon_r;1,p,1}=0, \quad \|\zeta'_r\|_{\upsilon_r,p,1}\le C\big|\rho(\upsilon)\big|,\end{gathered} \tag{4.33}$$

$$\begin{gathered}\lim_{r\longrightarrow\infty}|\epsilon_r|=0, \qquad \big|\mathcal{D}_{J_0;\mathcal{T}}\rho(\upsilon)\big|\le\epsilon_r\big|\rho(\upsilon)\big|, \\ \text{and} \qquad b_r\equiv\big(\Sigma_{b_r};u_{b_r}\big)=\big(\Sigma_{\upsilon_r},\exp_{u_{\upsilon_r,\zeta_r}}\zeta'_r\big).\end{gathered} \tag{4.34}$$

The last equality holds for a representative b_r for $[b_r]$.

By (4.31) and the last inequality in (4.33), for some $C \in \mathbb{R}^+$ and for all r sufficiently large

$$\big\|L_{\upsilon_r,\zeta_r,\zeta'_r}\xi\big\|_{\upsilon_r,p} \le C\big|\rho(\upsilon_r)\big|^2\|\xi\|_{\upsilon_r,p,1} \qquad \forall\, \xi\in\Gamma(\upsilon_r,\zeta_r;\mathfrak{L}). \tag{4.35}$$

Thus, by (4.24) and (4.25),

$$\big\|\pi_{\upsilon_r,\zeta_r;-}\Pi^{-1}_{\zeta'_r}\xi-\Pi^{-1}_{\zeta'_r}\xi\big\|_{\upsilon_r,p,1} \le C\big|\rho(\upsilon_r)\big|\|\xi\|_{\upsilon_r,p,1} \tag{4.36}$$

for all $\xi\in\ker\bar\partial_{\nabla,b_r}$. Since

$$\big\{\bar\partial_{\nabla,b(\upsilon_r,\zeta_r)}+L_{\upsilon_r,\zeta_r,\zeta'_r}\big\}\Pi^{-1}_{\zeta'_r}\xi=0 \qquad \forall\,\xi\in\ker\bar\partial_{\nabla,b_r},$$

by (4.28)-(4.30), (4.35), and (4.36),

$$\big|\mathfrak{D}_{\rho(\upsilon)}\xi\big| \le C\big|\rho(\upsilon_r)\big|^2\cdot\|\xi\|_{b'_r,p,1} \tag{4.37}$$

for all $R_{\upsilon_r,\zeta_r}\xi\in\pi_{\upsilon_r,\zeta_r;-}\Pi^{-1}_{\zeta'_r}\ker\bar\partial_{\nabla,b_r}$.

After passing to a subsequence if necessary, let

$$\underline{w}'\equiv(w'_i)_{i\in\chi(b)}\in\widetilde{\mathfrak{F}}_b-\{0\}$$

be such that

$$\lim_{r\longrightarrow\infty}\big[\rho(\upsilon_r)\big]=\big[\underline{w}'\big]\in\mathbb{P}\widetilde{\mathfrak{F}}_b.$$

Since $\mathcal{D}_ib'_r\longrightarrow\mathcal{D}_ib$ for all $i\in\chi(b)$,

$$\sum_{i\in\chi(b)}\psi_{x_{h(i)}(b)}(w'_i)\cdot_{J_0}\mathcal{D}_ib=0\in T_{\mathrm{ev}_P(b)}X \tag{4.38}$$

for all $\psi\in\mathbb{E}_{\pi_P(b)}$ by (4.34). If $\xi_r\in\ker\bar\partial_{\nabla,b_r}$ and $\xi\in\ker\bar\partial_{\nabla,b}$ are such that

$$\lim_{r\longrightarrow\infty}[\xi_r]=[\xi]\in\mathcal{V}^A_{1,M}|_{[b]},$$

then by (4.36) and (4.37),

$$\big|\{\mathfrak{D}_{\underline{w}'}\xi\}\psi\big|\equiv\bigg|\sum_{i\in\chi(b)}\psi_{x_{h(i)}(b)}(w'_i)\cdot\mathcal{D}_{b,i}\xi\bigg|\le\tilde\epsilon_r\big|\underline{w}'\big|\|\psi\| \tag{4.39}$$

for all $\psi\in\mathbb{E}_{\pi_P(b)}$, for a sequence $\tilde\epsilon_r$ converging to zero. Thus, by (4.38) and (4.39), for the purposes of Lemma 4.2 we can take

$$w_i=\psi_{x_{h(i)}(b)}(w'_i)\in\mathbb{C}^*,$$

where $\psi\in\mathbb{E}_{\pi_P(b)}$ is any nonzero element.

Remark 1: If $\Sigma_{b_r;P}=\Sigma_{b_r}$ is a circle of spheres, i.e. $\aleph'\neq\emptyset$ in the notation of Subsection 4.1, the proof of Lemma 4.2 is formally the same, but some details change in a way analogous to [**25**, Subsection 3.9]. In particular, in (4.32),

$$\upsilon_r\in\widetilde{\mathcal{F}}^{\aleph_0}\equiv\big\{\big(b,(v_h)_{h\in\aleph\sqcup\hat I}\big)\in\widetilde{\mathcal{F}}\colon v_h=0\iff h\in\aleph_0\big\},$$

for a nonempty subset $\aleph_0$ of $\aleph$. If $\upsilon\in\widetilde{\mathcal{F}}_{\delta}^{\aleph_0}$, Σ_υ is a circle of spheres with nodes $\aleph_0$. If in addition $\zeta\in\Gamma(\upsilon_1)$, $\Gamma(\upsilon,\zeta)$ consists of the vector fields on the $|\aleph_0|$ components of Σ_υ that agree at the nodes of Σ_υ. Similarly, $\Gamma(\upsilon,\zeta;\mathfrak{L})$ consists of the sections of $u_{\upsilon,\zeta}^*\mathfrak{L}$ over the components of Σ_υ that agree at the nodes. If $\eta\in\Gamma_-^{0,1}(b;\mathfrak{L})$, the $u_{\upsilon,\zeta}^*\mathfrak{L}$-valued $(0,1)$-form $R_{\upsilon,\zeta}^{0,1}\eta$ has poles at the nodes of Σ_υ with residues that add up to zero at each node. In particular, $R_{\upsilon,\zeta}^{0,1}\eta$ is not an element of $\Gamma^{0,1}(\upsilon,\zeta;\mathfrak{L})$, but the homomorphism $\pi_{\upsilon,\zeta;-}^{0,1}$ is well defined and still satisfies (4.28)-(4.30). Finally, the argument of [**28**, Subsection 2.5] easily generalizes to show that $(\upsilon_r,\zeta_r,\zeta_r',\epsilon_r)$ as in (4.32)-(4.34) exist in this situation.

Remark 2: A regularization φ_b of the cone $\mathcal{V}_{1,M}^A$ near b, as in Definition 3.3, can be constructed using the description of open subsets in $\mathfrak{X}_{1,M}(X,A)$ of [**15**, Section 3] and the corresponding analogues of the isomorphisms $R_{\upsilon,\zeta}$ and the injective homomorphisms $\pi_{\upsilon,\zeta;-}\Pi_{\zeta'}^{-1}$ as above.

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