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EXTENSION OF TWISTED HODGE METRICS FOR KÄHLER MORPHISMS

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Abstract

Let $f : X \longrightarrow Y$ be a holomorphic map of complex manifolds, which is proper, Kähler, and surjective with connected fibers, and which is smooth over $Y \setminus Z$ the complement of an analytic subset Z. Let E be a Nakano semi-positive vector bundle on X. In our previous paper, we discussed the Nakano semi-positivity of $R^q f_*(K_{X/Y} \otimes E)$ for $q \geq 0$ with respect to the so-called Hodge metric, when the map f is smooth. Here we discuss the extension of the induced metric on the tautological line bundle $\mathcal{O}(1)$ on the projective space bundle $\mathbb{P}(R^q f_*(K_{X/Y} \otimes E))$ "over $Y \setminus Z$" as a singular Hermitian metric with semi-positive curvature "over Y". As a particular consequence, if Y is projective, $R^q f_*(K_{X/Y} \otimes E)$ is weakly positive over $Y \setminus Z$ in the sense of Viehweg.

1. Introduction

The subject in this paper is the positivity of direct image sheaves of adjoint bundles $R^q f_* \left(K_{X/Y} \otimes E\right)$, for a Kähler morphism $f : X \longrightarrow Y$ endowed with a Nakano semi-positive holomorphic vector bundle (E, h) on X. In our previous paper [**28**], generalizing a result in [**2**] in case $q = 0$, we obtained the Nakano semi-positivity of $R^q f_* \left(K_{X/Y} \otimes E\right)$ with respect to a canonically attached metric, the so-called Hodge metric, under the assumption that f is smooth. However the smoothness assumption on f is rather restrictive, and it is desirable to remove it. This is the aim of this paper.

To state our result precisely, let us fix notations and recall basic facts. Let $f : X \longrightarrow Y$ be a holomorphic map of complex manifolds. A real d-closed $(1,1)$-form ω on X is said to be *a relative Kähler form* for f, if for every point $y \in Y$, there exists an open neighbourhood W of y and a smooth plurisubharmonic function ψ on W such that $\omega + f^*(\sqrt{-1}\partial\overline{\partial}\psi)$ is a Kähler form on $f^{-1}(W)$. A morphism f is said to be *Kähler*, if there exists a relative Kähler form for f ([**35**, 6.1]), and $f : X \longrightarrow Y$ is said to be a *Kähler fiber space*, if f is proper, Kähler, and surjective with connected fibers.

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Set up 1.1. (General global setting.) (1) Let X and Y be complex manifolds of $\dim X = n+m$ and $\dim Y = m$, and let $f : X \longrightarrow Y$ be a Kähler fiber space. We do not fix a relative Kähler form for f, unless otherwise stated. The *discriminant locus* of f is the minimum closed analytic subset $\Delta \subset Y$ such that f is smooth over $Y \setminus \Delta$.

(2) Let (E, h) be a Nakano semi-positive holomorphic vector bundle on X. Let q be an integer with $0 \leq q \leq n$. By Kollár [**22**] and Takegoshi [**35**], $R^q f_* \left(K_{X/Y} \otimes E\right)$ is torsion free on Y, and moreover it is locally free on $Y \setminus \Delta$ ([**28**, 4.9]). In particular we can let $S_q \subset \Delta$ be the minimum closed analytic subset of $\operatorname{codim}_Y S_q \geq 2$ such that $R^q f_* \left(K_{X/Y} \otimes E\right)$ is locally free on $Y \setminus S_q$. Let $\pi : \mathbb{P}(R^q f_* \left(K_{X/Y} \otimes E\right)|_{Y \setminus S_q}) \longrightarrow Y \setminus S_q$ be the projective space bundle, and let $\pi^*(R^q f_*(K_{X/Y} \otimes E)|_{Y \setminus S_q}) \longrightarrow \mathcal{O}(1)$ be the universal quotient line bundle.

(3) Let ω_f be a relative Kähler form for f. Then we have the Hodge metric g on the vector bundle $R^q f_* \left(K_{X/Y} \otimes E\right)|_{Y \setminus \Delta}$ with respect to ω_f and h ([**28**, §5.1]). By the quotient $\pi^*(R^q f_* \left(K_{X/Y} \otimes E\right)|_{Y \setminus \Delta}) \longrightarrow \mathcal{O}(1)|_{\pi^{-1}(Y \setminus \Delta)}$, the metric $\pi^* g$ gives the quotient metric $g^\circ_{\mathcal{O}(1)}$ on $\mathcal{O}(1)|_{\pi^{-1}(Y \setminus \Delta)}$. The Nakano, even weaker Griffiths, semi-positivity of g (by [**2**, 1.2] for $q = 0$, and by [**28**, 1.1] for q general) implies that $g^\circ_{\mathcal{O}(1)}$ has a semi-positive curvature. $\square$

In these notations, our main result is as follows (see also §6.2 for some variants).

Theorem 1.2. *Let $f : X \longrightarrow Y$, (E, h) and $0 \leq q \leq n$ be as in Set up* 1.1.

(1) Unpolarized case. Then, for every relatively compact open subset $Y_0 \subset Y$, the line bundle $\mathcal{O}(1)|_{\pi^{-1}(Y_0 \setminus S_q)}$ on $\mathbb{P}(R^q f_* \left(K_{X/Y} \otimes E\right)|_{Y_0 \setminus S_q})$ has a singular Hermitian metric with semi-positive curvature, and which is smooth on $\pi^{-1}(Y_0 \setminus \Delta)$.

(2) Polarized case. Let ω_f be a relative Kähler form for f. Assume that there exists a closed analytic set $Z \subset \Delta$ of $\operatorname{codim}_Y Z \geq 2$ such that $f^{-1}(\Delta)|_{X \setminus f^{-1}(Z)}$ is a divisor and has a simple normal crossing support (or empty). Then the Hermitian metric $g^\circ_{\mathcal{O}(1)}$ on $\mathcal{O}(1)|_{\pi^{-1}(Y \setminus \Delta)}$ can be extended as a singular Hermitian metric $g_{\mathcal{O}(1)}$ with semi-positive curvature of $\mathcal{O}(1)$ on $\mathbb{P}(R^q f_* \left(K_{X/Y} \otimes E\right)|_{Y \setminus S_q})$.

If in particular in Theorem 1.2, $R^q f_* \left(K_{X/Y} \otimes E\right)$ is locally free and Y is a smooth projective variety, then the vector bundle $R^q f_* \left(K_{X/Y} \otimes E\right)$ is pseudo-effective in the sense of [**9**, §6]. The above curvature property of $\mathcal{O}(1)$ leads to the following algebraic positivity of $R^q f_* \left(K_{X/Y} \otimes E\right)$.

Theorem 1.3. *Let $f : X \longrightarrow Y$ be a surjective morphism with connected fibers between smooth projective varieties, and let (E, h) be a Nakano semi-positive holomorphic vector bundle on X. Then the torsion* *free sheaf* $R^q f_* \big(K_{X/Y} \otimes E\big)$ *is weakly positive over* $Y \setminus \Delta$ *(the smooth locus of* f*), in the sense of Viehweg* [**38**, 2.13].

Here is a brief history of the semi-positivity of direct image sheaves, especially in case the map $f : X \longrightarrow Y$ is not smooth. The origin is due to Fujita [**12**] for $f_* K_{X/Y}$ over a curve, in which he analyzed the singularities of the Hodge metric. After [**12**], there are a lot of works mostly in algebraic geometry to try to generalize [**12**], for example by Kawamata [**17**] [**18**] [**19**], Viehweg [**37**], Zucker [**39**], Nakayama [**30**], Moriwaki [**25**], Fujino [**11**], Campana [**5**]. Their methods heavily depend on the theory of a variation of Hodge structures. While Kollár [**22**] and Ohsawa [**33**, §3] reduce the semi-positivity to their vanishing theorems. We refer to [**10**] [**31**, V.§3] [**38**] for further related works. There are more recent related works from the Bergman kernel point of view, by Berndtsson-Păun [**3**] [**4**] and Tsuji [**36**]. Their interests are the positivity of a relative canonical bundle twisted with a line bundle with a singular Hermitian metric of semi-positive curvature, or its zero-th direct image, which are slightly different from ours in this paper. In [**3**] [**4**], they rely on [**2**] and an extension theorem of Ohsawa-Takegoshi type, and give a new perspective.

The position of this paper is rather close to the original work of Fujita. We work in the category of Kählerian geometry. We will prove that a Hodge metric defined over $Y \setminus \Delta$ can be extended across the discriminant locus Δ, which is a local question on the base. Because of the twist with a Nakano semi-positive vector bundle E which may not be semi-ample, one can not take (nor reduce a study to) the variation of Hodge structures approach. The algebraic approach quoted above only concludes that the direct image sheaves have algebraic semi-positivities, such as nefness, or weak positivity. It is like semi-positivity of integration of the curvature along subvarieties. These algebraic semi-positivities already requires a global property on the base, for example (quasi-)projectivity. In the algebraic approach, to obtain a stronger result, they sometimes pose a normal crossing condition of the discriminant locus $\Delta \subset Y$ of the map, and/or a unipotency of local monodromies. We are free from these conditions, but we must admit that our method does not tell local freeness nor nefness of direct images sheaves. We really deal with Hodge metrics, and we do not use the theory of a variation of Hodge structures, nor global geometry on the base, in contrast to the algebraic approach.

In connection with a moduli or a deformation theory, a direct image sheaf on a parameter space defines a canonically attached sheaf quite often, and then the curvature of the Hodge metric describes the geometry of the parameter space. Then, especially as a consequence of our previous paper [**28**], the Nakano semi-positivity of the curvature on which the family is smooth, is quite useful in practice. If there exists a reasonable compactification of the parameter space, our results in this paper can be applied to obtain boundary properties. There might be further applications in this direction, we hope. While the algebraic semi-positivity is more or less Griffiths semi-positivity, which has nice functorial properties but is not strong enough especially in geometry.

Our method of proof is to try to generalize the one in [**12**]. The main issue is to obtain a positive lower bound of the singularities of a Hodge metric g. It is like a uniform upper estimate for a family of plurisubharmonic functions $-\log g(u,u)$ around $\Delta \subset Y$, where u is any nowhere vanishing local section of $R^q f_* \left(K_{X/Y} \otimes E\right)$. In case $\dim Y = 1$ and arbitrary $q \geq 0$, we can obtain rather easily the results we have stated, by combining [**12**] and our previous work [**28**]. In case $\dim Y \geq 1$, a major difficulty arises. If the fibers of f are reduced, it is not difficult to apply again the method we took in case $\dim Y = 1$. However in general, a singular fiber is not a divisor anymore, and in addition it can be non-reduced. To avoid such an analytically uncomfortable situation, we employ a standard technique in algebraic geometry; a semi-stable reduction and an analysis of singularities which naturally appear in the semi-stable reduction process (§3). A Hodge metric after a semi-stable reduction would be better and would be handled by known techniques, because fibers become reduced. Then the crucial point in the metric analysis is a comparison of the original Hodge metric and a Hodge metric after a semi-stable reduction. As a result of taking a ramified cover and a resolution of singularities in a semi-stable reduction, we naturally need to deal with a degenerate Kähler form, and then we are forced to develop a theory of relative harmonic forms (as in [**35**]) with respect to the degenerate Kähler form (§4). After a series of these observations, we bound singularities of the Hodge metric, and obtain a uniform estimate to extend the Hodge metric (§5). The proof is not so simple to mention more details here, because we need to consider a uniform estimate, when a rank one quotient $\pi_L : R^q f_* \left(K_{X/Y} \otimes E\right) \longrightarrow L$ moves and a section of the kernel of π_L moves. There is a technical introduction [**29**], where we explain the case $\dim Y = 1$, or the case where the map f has reduced fibers.

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2. Hodge Metric

2.1. Definition of Hodge metric. Let us start by recalling basic definitions and facts. Let $f : X \longrightarrow Y$ be a Kähler fiber space as in

Set up 1.1. For a point $y \in Y \setminus \Delta$, we denote by $X_y = f^{-1}(y), \omega_y = \omega|_{X_y}, E_y = E|_{X_y}, h_y = h|_{X_y}$, and for an open subset $W \subset Y$, we denote by $X_W = f^{-1}(W)$. We set $\Omega^p_{X/Y} = \bigwedge^p(\Omega^1_X/(\operatorname{Im} f^*\Omega^1_Y))$ rather formally for the natural map $f^*\Omega^1_Y \longrightarrow \Omega^1_X$, because we will only deal with $\Omega^p_{X/Y}$ where f is smooth. For an open subset $U \subset X$ where f is smooth, and for a differentiable form $\sigma \in A^{p,0}(U, E)$, we say σ is *relatively holomorphic* and write $[\sigma] \in H^0(U, \Omega^p_{X/Y} \otimes E)$, if for every $x \in U$, there exists an coordinate neighbourhood W of $f(x) \in Y$ with a nowhere vanishing $\theta \in H^0(W, K_Y)$ such that $\sigma \wedge f^*\theta \in H^0(U \cap X_W, \Omega^{p+m}_X \otimes E)$ ([**28**, §3.1]).

We remind the readers of the following basic facts, which we will use repeatedly. See [**35**, 6.9] for more general case when Y may be singular, [**28**, 4.9] for (3), and also [**22**].

Lemma 2.1. *Let $f : X \longrightarrow Y$ and (E, h) be as in Set up* 1.1*. Let q be a non-negative integer. Then (1) $R^q f_*\left(K_{X/Y} \otimes E\right)$ is torsion free, (2) Grauert-Riemenschneider vanishing: $R^q f_*\left(K_{X/Y} \otimes E\right) = 0$ for $q > n$, and (3) $R^q f_*\left(K_{X/Y} \otimes E\right)$ is locally free on $Y \setminus \Delta$.*

Using Grauert-Riemenschneider vanishing, a Leray spectral sequence argument shows that $R^q f_*\left(K_{X/Y} \otimes E\right)$ does not depend on smooth bimeromorphic models of X. Choices of a smooth bimeromorphic model of X and of a relative Kähler form for the new model give rise to a Hermitian metric on the vector bundle $R^q f_*\left(K_{X/Y} \otimes E\right)|_{Y \setminus \Delta}$ as follows.

Definition 2.2. (Hodge metric [**28**, §5.1].) In Set up 1.1, assume that f is smooth, Y is Stein with $K_Y \cong \mathcal{O}_Y$ (with a nowhere vanishing $\theta_Y \in H^0(Y, K_Y)$), and X is Kähler. A choice of a Kähler form ω on X gives an injection $S_\omega := S^q_f : R^q f_*\left(K_{X/Y} \otimes E\right) \longrightarrow f_*(\Omega^{n-q}_{X/Y} \otimes E)$. Then for every pair $u_y, v_y \in R^q f_*\left(K_{X/Y} \otimes E\right)_y$, we define

$$g(u_y, v_y) = \int_{X_y} (c_{n-q}/q!)\ \omega_y^q \wedge S_\omega(u_y) \wedge h_y \overline{S_\omega(v_y)}.$$

Here $c_p = \sqrt{-1}^{p^2}$ for every integer $p \geq 0$. Since f is smooth, these pointwise inner products define a smooth Hermitian metric g on $R^q f_*(K_{X/Y} \otimes E)$, which we call the *Hodge metric* with respect to ω and h. $\square$

Details for the construction of the map S_ω will be provided in Step 2 in the proof of Proposition 4.4. In Definition 2.2, another choice of a Kähler form ω' on X gives another metric g' on $R^q f_*\left(K_{X/Y} \otimes E\right)$. However in case ω and ω' relate with $\omega|_{X_y} = \omega'|_{X_y}$ for any $y \in Y$, these metrics coincide $g = g'$ ([**28**, 5.2]). Thus a Hodge metric is defined for a polarized smooth Kähler fiber space in Set up 1.1. In case when $q = 0$, the Hodge metric does not depend on a relative Kähler form. In fact, it is given by

$$g(u_y, v_y) = \int_{X_y} c_n u_y \wedge h_y \overline{v_y}$$

for $u_y, v_y \in H^0(X_y, K_{X_y} \otimes E_y)$.

2.2. Localization. We consider the following local setting, around a codimension 1 general point of $\Delta \subset Y$ (possibly after taking a modification of X).

Set up 2.3. (Generic local, relative normal crossing setting.) Let $f : X \longrightarrow Y$, (E, h) and $0 \leq q \leq n$ be as in Set up 1.1. Let us assume further the following:

(1) The base Y is (biholomorphic to) a unit polydisc in $\mathbb{C}^m$ with coordinates $t = (t_1, \dots, t_m)$. Let $K_Y \cong \mathcal{O}_Y$ be a trivialization by a nowhere vanishing section $dt = dt_1 \wedge \dots \wedge dt_m \in H^0(Y, K_Y)$.

(1.i) f is flat, and the discriminant locus $\Delta \subset Y$ is $\Delta = \{t_m = 0\}$ (or $\Delta = \emptyset$),

(1.ii) the effective divisor $f^*\Delta$ has a simple normal crossing support, and

(1.iii) the morphism $\operatorname{Supp} f^*\Delta \longrightarrow \Delta$ is relative normal crossing (see below).

(2) $R^q f_* \left(K_{X/Y} \otimes E\right) \cong \mathcal{O}_Y^{\oplus r}$, i.e., globally free and trivialized of rank $r > 0$.

(3) X admits a Kähler form ω. Let g be the Hodge metric on $R^q f_* \left(K_{X/Y} \otimes E\right)|_{Y \setminus \Delta}$ with respect to ω and h.

We may replace Y by slightly smaller polydiscs, or may assume everything is defined over a slightly larger polydisc. $\square$

In the above, $\operatorname{Supp} f^*\Delta \longrightarrow \Delta$ is relative normal crossing means that, around every $x \in X$, there exists a local coordinate $(U; z = (z_1, \dots, z_{n+m}))$ such that $f|_U$ is given by $t_1 = z_{n+1}, \dots, t_{m-1} = z_{n+m-1}$, $t_m = z_{n+m}^{b_{n+m}} \prod_{j=1}^n z_j^{b_j}$ with non-negative integers b_j and b_{n+m}.

Then the following version of Theorem 1.2 (2) is our main technical statement.

Theorem 2.4. *Let $f : (X, \omega) \longrightarrow Y \subset \mathbb{C}^m$, (E, h) and $0 \leq q \leq n$ be as in Set up 2.3. The pull-back metric $\pi^* g$ of the Hodge metric g on $R^q f_* \left(K_{X/Y} \otimes E\right)|_{Y \setminus \Delta}$ with respect to ω and h gives the quotient metric $g^\circ_{\mathcal{O}(1)}$ on $\mathcal{O}(1)|_{\pi^{-1}(Y \setminus \Delta)}$. The smooth Hermitian metric $g^\circ_{\mathcal{O}(1)}$ extends as a singular Hermitian metric $g_{\mathcal{O}(1)}$ on $\mathcal{O}(1)$ with semi-positive curvature.*

We shall see Theorem 1.2, by taking Theorem 2.4 for granted in the rest of this section. For a general Kähler fiber space $f : X \longrightarrow Y$, we can reduce the study of a Hodge metric to the study which is local on Y as in Set up 2.3, possibly after taking blowing-ups of X.

Lemma 2.5. *Let $f : X \longrightarrow Y$, (E, h) and $0 \leq q \leq n$ be as in Set up* 1.1. *Let $Y_0 \subset Y$ be a relatively compact open subset. Let $Z_0 \subset \Delta$ be a closed analytic subset of* $\operatorname{codim}_Y Z_0 \geq 2$ *such that $\Delta \setminus Z_0$ is a smooth divisor (or empty). Possibly after restricting everything on a relatively compact open neighbourhood over Y_0, let $\mu : X' \longrightarrow X$ be a birational map from a complex manifold X', which is obtained by a finite number of blowing-ups along non-singular centers, and which is biholomorphic over $X \setminus f^{-1}(\Delta)$, such that $f^*(\Delta \setminus Z_0)$ is a divisor with simple normal crossing support on $X \setminus f^{-1}(Z_0)$. Let $\omega_{f'}$ be a relative Kähler form for $f' := f \circ \mu$ over Y_0. Then*

(1) there exist (i) a closed analytic subset $Z \subset \Delta$ of $\operatorname{codim}_Y Z \geq 2$, *(ii) an open covering $\{W_i\}_i$ of $Y_0 \setminus Z$, and (iii) a Kähler form ω_i on $X'_{W_i} = f'^{-1}(W_i)$ for every i, such that (a) for every i, W_i is biholomorphic to the unit polydisc, and the induced $f'_i : (X'_{W_i}, \omega_i) \longrightarrow W_i \subset \mathbb{C}^m$, $(\mu^*E, \mu^*h)|_{X'_{W_i}}$ and $0 \leq q \leq n$ satisfy all the conditions in Set up* 2.3, *and that (b) $\omega_i|_{X'_y} = \omega_{f'}|_{X'_y}$ for every i and $y \in W_i$. Moreover one can take $\{W_i\}_i$ so that, the same is true, even if one replaces all W_i by slightly smaller concentric polydiscs.*

(2) Via the isomorphism $R^q f_* \big(K_{X/Y} \otimes E\big) \cong R^q f'_*(K_{X'/Y} \otimes \mu^* E)$, the Hodge metric on $R^q f'_*(K_{X'/Y} \otimes \mu^* E)|_{Y_0 \setminus \Delta}$ with respect to $\omega_{f'}$ and $\mu^* h$ induces a smooth Hermitian metric g with Nakano semi-positive curvature on $R^q f_* \big(K_{X/Y} \otimes E\big)|_{Y_0 \setminus \Delta}$.

Proof. In general, a composition $f \circ \mu$ of f and a blow-up $\mu : X' \longrightarrow X$ along a closed complex submanifold of X, is only locally Kähler ([**35**, 6.2.i-ii]). (We do not know if $f \circ \mu$ is Kähler. This is the point, why we need to mention "on every relatively compact open subset $Y_0 \subset Y$" in Theorem 1.2 (1).) Hence our modification $f' : X' \longrightarrow Y$ is locally Kähler, and we can take a relative Kähler form $\omega_{f'}$ for f' over Y_0. As we explained before, we have $R^q(f \circ \mu)_*(K_{X'/Y} \otimes \mu^* E) = R^q f_* \big(K_{X/Y} \otimes E\big)$ by Lemma 2.1.

To see (1), we note Lemma 2.1 that $R^q f'_*(K_{X'/Y} \otimes \mu^* E)$ is locally free in codimension 1 on Y. We then take $Z \supset Z_0$ to be the union of all subvarieties along which one of (1) – (2) in Set up 2.3 fails for f'. Other items are almost clear (by construction). q.e.d.

The following is a more precise statement of Theorem 1.2 (1).

Proposition 2.6. *Let $f : X \longrightarrow Y$, (E, h) and $0 \leq q \leq n$ be as in Set up* 1.1. *Let $Y_0 \subset Y$ be a relatively compact open subset. After taking a modification $\mu : X' \longrightarrow X$ (on a neighbourhood of $X_0 = f^{-1}(Y_0)$) and a relative Kähler form $\omega_{f'}$ for $f' = f \circ \mu$ over Y_0 as in Lemma* 2.5, *the Hermitian metric g on $R^q f_* \left(K_{X/Y} \otimes E\right)|_{Y_0 \setminus \Delta}$ in Lemma* 2.5 *(2) induces the quotient metric $g^\circ_{\mathcal{O}(1)}$ on $\mathcal{O}(1)|_{\pi^{-1}(Y_0 \setminus \Delta)}$ with semi-positive curvature. Then the smooth Hermitian metric $g^\circ_{\mathcal{O}(1)}$ extends as a singular Hermitian metric $g_{\mathcal{O}(1)}$ on $\mathcal{O}(1)|_{\pi^{-1}(Y_0 \setminus S_q)}$ with semi-positive curvature.*

Proof of Theorem 1.2. (1) It is enough to show Proposition 2.6. We use the notations in Lemma 2.5. We apply Theorem 2.4 on each $W_i \subset Y_0 \setminus Z$. Then we see, at this point, the smooth Hermitian metric $g^\circ_{\mathcal{O}(1)}$ on $\mathcal{O}(1)|_{\pi^{-1}(Y_0 \setminus \Delta)}$ extends as a singular Hermitian metric $g'_{\mathcal{O}(1)}$ on $\mathcal{O}(1)|_{\pi^{-1}(Y_0 \setminus Z)}$ with semi-positive curvature. Then by Hartogs type extension, the singular Hermitian metric $g'_{\mathcal{O}(1)}$ on $\mathcal{O}(1)|_{\pi^{-1}(Y_0 \setminus Z)}$ extends as a singular Hermitian metric $g_{\mathcal{O}(1)}$ on $\mathcal{O}(1)|_{\pi^{-1}(Y_0)}$ with semi-positive curvature.

(2) We can find a closed analytic subset $Z' \subset \Delta$ of $\operatorname{codim}_Y Z' \geq 2$, containing Z, so that we can describe $f : X \setminus f^{-1}(Z') \longrightarrow Y \setminus Z'$ as a union of Set up 2.3 as in Lemma 2.5 without taking any modifications $\mu : X' \longrightarrow X$. Then we obtain the Hodge metric on $R^q f_* \left(K_{X/Y} \otimes E\right)|_{Y \setminus \Delta}$ with respect to ω_f and h. The rest of the proof is the same as (1).

q.e.d.

3. Semi-Stable Reduction

Now our aim is to show Theorem 2.4. We shall devote this and next two sections for the proof. Throughout these three sections, we shall discuss under Set up 2.3 and also §3.1 below.

3.1. Weakly semi-stable reduction. ([**20**, Ch. II], [**24**, §7.2], [**38**, §6.4].) Let

$$f^*\Delta = \sum_j b_j B_j$$

be the prime decomposition. Let Y' be another copy of a unit polydisc in $\mathbb{C}^m$ with coordinates $t' = (t'_1, \ldots, t'_{m-1}, t'_m)$. Let ℓ be the least common multiple of all b_j. Let $\tau : Y' \longrightarrow Y$ be a ramified covering given by $(t'_1, \ldots, t'_{m-1}, t'_m) \mapsto (t'_1, \ldots, t'_{m-1}, {t'_m}^{\ell})$, and $X^\circ = X \times_Y Y'$ be the fiber product. Let $\nu : X' \longrightarrow X^\circ$ be the normalization, and $\mu : X'' \longrightarrow X'$ be a resolution of singularities, which is biholomorphic on the smooth locus of X'.

$$\begin{array}{ccccccc}
X'' & \xrightarrow{\mu} & X' & \xrightarrow{\nu} & X^\circ = X \times_Y Y' & \xrightarrow{\tau^\circ} & X \\
{\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & {\scriptstyle f^\circ}\downarrow & & \downarrow{\scriptstyle f} \\
Y' & \xrightarrow[\mathrm{id}]{} & Y' & \xrightarrow[\mathrm{id}]{} & Y' & \xrightarrow[\tau]{} & Y
\end{array}$$

Then there are naturally induced objects: $\tau^\circ : X^\circ \longrightarrow X, \tau' : X' \longrightarrow X, \tau'' : X'' \longrightarrow X,\ f^\circ : X^\circ \longrightarrow Y', f' : X' \longrightarrow Y', f'' : X'' \longrightarrow Y',\ E^\circ =$

$\tau^{\circ *}E, E' = \tau'^{*}E, E'' = \tau''^{*}E$, and $h'' = \tau''^{*}h$ the induced Hermitian metric on E'' with Nakano semi-positive curvature. We denote by $j_{X^\circ} : X^\circ \subset X \times Y'$ the inclusion map, and by $p_X : X \times Y' \longrightarrow X$ and $p_{Y'} : X \times Y' \longrightarrow Y'$ the projections. We may also denote by

$$F = R^q f_* \left(K_{X/Y} \otimes E\right), \quad F' = R^q f''_* \left(K_{X''/Y'} \otimes E''\right),$$

where F is globally free (Set up 2.3), and F' is torsion free (Lemma 2.1). Let $\Delta' = \{t'_m = 0\} \subset Y'$. The discriminant loci of f°, f', f'' are contained in Δ'. We can write

$$f''^{*}\Delta' = \sum_j B''_j + B''_{exc},$$

where $\sum B''_j$ is the prime decomposition of the non-μ-exceptional divisors in $f''^{*}\Delta'$, and B''_{exc} is the sum of μ-exceptional divisors in $f''^{*}\Delta'$. As we will see in Lemma 3.2, all coefficients in $\sum B''_j$ are 1. (As in [**20**], $f''^{*}\Delta'$ may be semi-stable in codimension 1. However we do not need this stronger result for B''_{exc}.)

We add a remark on the choice of the smooth model X''. We can assume, possibly after replacing Y by a smaller polydisc, that X'' can be obtained in the following way. We take an embedded resolution $\delta : \widetilde{X \times Y'} \longrightarrow X \times Y'$ of X°, by a finite number of blowing-ups along smooth centers, which are biholomorphic outside $\operatorname{Sing} X^\circ$. Let us denote by $X'' \subset \widetilde{X \times Y'}$ the smooth model of X°, and by $\mu : X'' \longrightarrow X'$ the induced morphism. We may assume further that $\operatorname{Supp} f''^{*}\Delta'$ is simple normal crossing.

3.2. Direct image sheaves and analysis of singularities. We will employ algebraic arguments to compare direct image sheaves on Y and Y', and to study the singularities on X'. We start with an elementary remark.

Lemma 3.1. *The variety X' is smooth on $X' \setminus \tau'^{-1}(\operatorname{Sing} f^{-1}(\Delta))$, and the induced map $j_{X^\circ} \circ \nu : X' \longrightarrow X \times Y'$ is locally embedding around every point on $X' \setminus \tau'^{-1}(\operatorname{Sing} f^{-1}(\Delta))$.*

Proof. We take a smooth point x_0 of $f^{-1}(\Delta)$. If $x_0 \in B_j$ in $f^*\Delta = \sum b_j B_j$, the map f is given by $z = (z_1, \ldots, z_{n+m}) \mapsto t = (z_{n+1}, \ldots, z_{n+m-1}, z_{n+m}^{b_j})$ for an appropriate local coordinate $(U; z = (z_1, \ldots, z_{n+m}))$ around x_0. Then $U^\circ = U \times_Y Y'$ is defined by $U^\circ = \{(z, t') \in U \times Y';\ f(z) = \tau(t')\}$, namely $z_{n+1} = t'_1, \ldots, z_{n+m-1} = t'_{m-1}, z_{n+m}^{b_j} = {t'_m}^{\ell}$. We write $\ell = b_j c_j$ with a positive integer c_j. Let ε be a b_j-th primitive root of unity. Then U° is a union of

$$U_p^\circ = \left\{(z, t') \in U \times Y';\ z_{n+1} = t'_1, \ldots, z_{n+m-1} = t'_{m-1}, z_{n+m} = \varepsilon^p {t'_m}^{c_j}\right\}$$

for $p = 1, \ldots, b_j$. Each U_p° itself is smooth, and the normalization U' of U° is just a disjoint union $\amalg_{p=1}^{b_j} U_p^\circ$. q.e.d.

The normal variety X' is almost smooth. For example the following properties are known.

Lemma 3.2. [**24**, 7.23] ([**20**, Ch. II]). *(1) The canonical divisor $K_{X'}$ is Cartier, (2) X' has at most toric, abelian quotient singularities, (3) a pair $(X', 0)$ is canonical, and a pair (X', D') is log-canonical, where $D' = f'^*\Delta'$ which is reduced.*

Since canonical singularities are Cohen-Macaulay, combined with Lemma 3.2 (1), we see X' is Gorenstein (refer [**24**, §2.3] including definitions).

Lemma 3.3. (*cf.* [**37**, Lemma 3.2] [**31**, V.3.30].) *There exists a natural inclusion map*

$$\varphi : F' = R^q f''_* \left(K_{X''/Y'} \otimes E''\right) \longrightarrow \tau^* F = \tau^* R^q f_* \left(K_{X/Y} \otimes E\right),$$

which is isomorphic over $Y' \setminus \Delta'$.

Proof. Recall that dualizing sheaves when they exist are flat and compatible with any base change [**21**, (9)]. The morphism ν being finite, there exists a dualizing sheaf ω_{X'/X° such that $\nu_*\omega_{X'/X^\circ} = \mathcal{H}om_{X^\circ}(\nu_*\mathcal{O}_{X'}, \mathcal{O}_{X^\circ})$. By base change, $\omega_{X^\circ/Y'} = \tau^{\circ *}K_{X/Y}$ is an invertible dualizing sheaf for f°. Because Y' is smooth, $\omega_{X^\circ} = \omega_{X^\circ/Y'} \otimes f^{\circ *}K_{Y'}$ is an invertible dualizing sheaf for X°. In particular X° is Gorenstein, in fact X° is locally complete intersection. Now, by composition [**21**, (26.vii)], $\omega_{X'/Y'} = \omega_{X'/X^\circ} \otimes \nu^*\omega_{X^\circ/Y'}$ is a dualizing sheaf for f'. Because $\omega_{X^\circ/Y'}$ is locally free, the projection formula reads $\nu_*\omega_{X'/Y'} = (\nu_*\omega_{X'/X^\circ}) \otimes \omega_{X^\circ/Y'} = \mathcal{H}om_{X^\circ}(\nu_*\mathcal{O}_{X'}, \mathcal{O}_{X^\circ} \otimes \omega_{X^\circ/Y'}) = \mathcal{H}om_{X^\circ}(\nu_*\mathcal{O}_{X'}, \omega_{X^\circ/Y'})$. Then we have a natural homomorphism $\alpha : \nu_*\omega_{X'/Y'} \longrightarrow \omega_{X^\circ/Y'}$. Since X' is Gorenstein and canonical (Lemma 3.2), we have $K_{X''} = \mu^*K_{X'} + C$ for an effective μ-exceptional divisor C, and hence $\nu_*\mu_*K_{X''/Y'} = \nu_*\mu_*(\mu^*\omega_{X'/Y'} \otimes \mathcal{O}_{X''}(C)) = \nu_*\omega_{X'/Y'}$. Then the map α induces $\nu_*\mu_*K_{X''/Y'} \longrightarrow \omega_{X^\circ/Y'} = \tau^{\circ *}K_{X/Y}$. We apply $R^q f^\circ_*$ to obtain a map $R^q f^\circ_*(\nu_*\mu_*(K_{X''/Y'} \otimes E'')) \longrightarrow R^q f^\circ_*(\tau^{\circ *}(K_{X/Y} \otimes E))$.

Since $\nu \circ \mu : X'' \longrightarrow X^\circ$ is birational, we have $R^q(\nu \circ \mu)_* K_{X''} = 0$ for $q > 0$ ([**35**, 6.9]). Noting $E'' = (\nu \circ \mu)^* E^\circ$, we have $R^q f''_*(K_{X''} \otimes E'') = R^q f^\circ_*(R^0(\nu \circ \mu)_*(K_{X''} \otimes E''))$. This gives $R^q f^\circ_*(\nu_*\mu_*(K_{X''/Y'} \otimes E'')) = R^q f''_* \left(K_{X''/Y'} \otimes E''\right)$. On the other hand, since τ is flat, the base change map $\tau^* R^q f_* \left(K_{X/Y} \otimes E\right) \longrightarrow R^q f^\circ_*(\tau^{\circ *}(K_{X/Y} \otimes E))$ is isomorphic. Thus we obtain a sheaf homomorphism

$$\varphi : R^q f''_* \left(K_{X''/Y'} \otimes E''\right) \longrightarrow \tau^* R^q f_* \left(K_{X/Y} \otimes E\right).$$

It is not difficult to see φ is isomorphic over $Y' \setminus \Delta'$, and hence the kernel of φ is a torsion sheaf on Y'. The injectivity of φ is then a consequence of the torsion freeness of $R^q f''_* \left(K_{X''/Y'} \otimes E''\right)$, by Lemma 2.1. q.e.d.

As we saw in Lemma 3.2, the singularities of X' are mild. However we need informations not only on the canonical sheaf of X', but also on the sheaf of holomorphic p-forms on X'. There are two canonical choices of the definition on a normal variety. Fortunately both of them coincide for our X'. In the rest of this subsection, p denotes a non-negative integer.

Definition 3.4. For every p, we define the sheaf of holomorphic p-forms on X' by $\Omega^p_{X'} := j_*\Omega^p_{X'_{reg}}$, where $j : X'_{reg} \longrightarrow X'$ is the open immersion of the regular part.

Lemma 3.5. [**7**, 1.6] [**34**, 1.11]. *$\mu_*\Omega^p_{X''} = \Omega^p_{X'}$ holds.*

Due to [**7**, 1.6], this lemma is valid not only for our X' and X'' here, but also more general toric variety X' and any resolution of singularities $\mu : X'' \longrightarrow X'$. Our X' is not an algebraic variety, however at every point $x' \in X'$, there exists an affine toric variety Z with a point 0 such that $(X', x') \cong (Z, 0)$ as germs of complex spaces. Hence this lemma follows from [**7**, 1.6]. This is also implicitly contained in the proof of [**16**, Lemma 3.9].

Another key property which we will use, due to Danilov, is the following

Lemma 3.6. *The sheaf $\Omega^p_{X'}$ is Cohen-Macaulay (CM for short), i.e., at each point $x' \in X'$, the stalk $\Omega^p_{X',x'}$ is CM as a module over a noetherian local ring $(\mathcal{O}_{X',x'}, \mathfrak{m}_{X',x'})$.*

Proof. Let $x' \in X'$. Since X' has a toric singularity at x', there exists an affine toric variety Z with a point 0 such that $(X', x') \cong (Z, 0)$ as germs of complex spaces. Let σ be a cone in a finite dimensional vector space $N_{\mathbb{R}}$ corresponding Z (or a fan F in $N_{\mathbb{R}}$ corresponding Z). Since $(X', x') \cong (Z, 0)$ is an abelian quotient singularity (Lemma 3.2), the cone σ is simplicial ([**6**, 3.7]). Then by a result of Danilov ([**32**, 3.10]), $\Omega^p_{X'}$is CM. q.e.d.

Corollary 3.7. *Let $y' \in \Delta'$, and let $(t'_1, \ldots, t'_m)$ be (other) coordinates of Y' centered at y' such that $\Delta' = \{t'_m = 0\}$. Then the central fiber $X'_{y'} \subset X'$ defined by $f'^*t'_1 = \cdots = f'^*t'_m = 0$ as a complex subspace is pure n-dimensional and reduced ([**24**, 7.23 (1)]). Let $x' \in X'_{y'}$. Let $s_{m+1}, \ldots, s_{m+n} \in \mathfrak{m}_{X',x'} \subset \mathcal{O}_{X',x'}$ be a sequence of holomorphic functions such that $\dim_{x'}(X'_{y'} \cap \{s_{m+1} = \cdots = s_{m+k} = 0\}) = n - k$ for any $1 \le k \le n$. Then $f'^*t'_1, \ldots, f'^*t'_m, s_{m+1}, \ldots, s_{m+n}$ is an $\Omega^p_{X',x'}$-regular sequence.*

Proof. Since we already know that $\Omega^p_{X',x'}$ is CM, it is enough to check that

$$\dim_{x'} \operatorname{Supp} \left(\Omega^p_{X',x'}/(f'^*t'_1, \ldots, f'^*t'_m, s_{m+1}, \ldots, s_{m+n})\Omega^p_{X',x'}\right) = 0.$$

(cf. [**24**, 5.1 (1) iff (2)] [**1**, III.4.3].) This is clear by our choice of $s_{m+1}, \dots, s_{m+n}$. q.e.d.

3.3. Non-vanishing. Recall $f''^* \Delta' = \sum B''_j + B''_{exc}$, where $\sum B''_j$ is the prime decomposition of the non-μ-exceptional divisors in $f''^*\Delta'$, and B''_{exc} is the sum of μ-exceptional divisors.

Lemma 3.8. *Let $v \in H^0(X'', \Omega^{n+m-q}_{X''} \otimes E'')$. Let $y' \in \Delta'$ such that* $\operatorname{Supp} f''^*\Delta' \longrightarrow \Delta'$ *is relative normal crossing around y'. Assume that v does not vanish at y' as an element of an $H^0(Y', \mathcal{O}_{Y'})$-module, i.e., $f''_* v$ is non-zero in $f''_*(\Omega^{n+m-q}_{X''} \otimes E'')/(\mathfrak{m}_{Y',y'} f''_*(\Omega^{n+m-q}_{X''} \otimes E''))$. Then there exists a non-μ-exceptional component B''_j in $f''^*\Delta'$ such that v does not vanish identically along $B''_j \cap f''^{-1}(y')$.*

Proof. Let us denote by $p = n+m-q$. We have $\mu_* v \in H^0(X', (\mu_*\Omega^p_{X''}) \otimes E')$. Recalling Lemma 3.5 that $\mu_*\Omega^p_{X''} = \Omega^p_{X'}$, we then have $f''_* v \in H^0(Y', f''_*(\Omega^p_{X''} \otimes E'')) = H^0(Y', f'_*(\Omega^p_{X'} \otimes E'))$. Assume on the contrary that v does vanish identically along $B''_j \cap f''^{-1}(y')$ for all j. Then it is enough to show that $\mu_* v \in H^0(X', f'^{-1}\mathfrak{m}_{Y',y'} \cdot (\Omega^p_{X'} \otimes E'))$. In fact it implies that $f'_*(\mu_* v)$ vanishes at y', and gives a contradiction to that $f''_* v = f'_*(\mu_* v) \in H^0(Y', f''_*(\Omega^p_{X''} \otimes E''))$ does not vanish at y'. Let

$$\alpha := (\mu_* v)|_{X'_{y'}} \in H^0\left(X'_{y'}, \left(\Omega^p_{X'}/(f'^*t'_1, \dots, f'^*t'_m)\Omega^p_{X'}\right) \otimes E'\right).$$

Then, $\alpha = 0$ leads to a contradiction as we want.

We would like to show that the support of α is empty. Assume on the contrary that there is a point $x' \in X'_{y'}$ such that $d := \dim_{x'} \operatorname{Supp} \alpha \geq 0$. Noting that $\mu : X'' \longrightarrow X'$ is isomorphic around every point on $\operatorname{Reg} X'_{y'}$, we see $\operatorname{Supp} \alpha \subset \operatorname{Sing} X'_{y'}$, because of our assumption that v vanishes identically along $B''_j \cap f''^{-1}(y')$ for all j. In particular $d < n$. We take general $s_{m+1}, \dots, s_{m+n} \in \mathfrak{m}_{X',x'} \subset \mathcal{O}_{X',x'}$ such that $\dim_{x'}(X'_{y'} \cap \{s_{m+1} = \cdots = s_{m+k} = 0\}) = n - k$ for any $1 \leq k \leq n$, and $\dim_{x'}(\operatorname{Supp} \alpha \cap \{s_{m+1} = \cdots = s_{m+k} = 0\}) = d - k$ for any $1 \leq k \leq d$. By the CM property of $\Omega^p_{X',x'}$: Corollary 3.7, $f'^*t'_1, \dots, f'^*t'_m, s_{m+1}, \dots, s_{m+n}$ form an $\Omega^p_{X',x'} \otimes E'$-regular sequence.

Assume $d \geq 1$. We set $\Sigma_d := X'_{y'} \cap \{s_{m+1} = \cdots = s_{m+d} = 0\}$ around x' on which $s_{m+1}, \dots, s_{m+n}$ are defined, and consider $\alpha|_{\Sigma_d} \in H^0(\Sigma_d, (\Omega^p_{X'}/(f'^*t'_1, \dots, f'^*t'_m,\ s_{m+1}, \dots,\ s_{m+d})\Omega^p_{X'}) \otimes E')$. Then $\operatorname{Supp}(\alpha|_{\Sigma_d})$ is contained in the zero locus of the function s_{m+d+1} around x'. Since $\alpha|_{\Sigma_d}$ is non-zero, (some power of s_{m+d+1} and hence) s_{m+d+1} is a zero divisor for $(\Omega^p_{X',x'}/(f'^*t'_1, \dots, f'^*t'_m, s_{m+1}, \dots, s_{m+d})\Omega^p_{X',x'}) \otimes E'_{x'}$, see [**13**, §2.2] Rückert Nullstellensatz, cf. [**15**, II.Ex.5.6]. This gives a contradiction to the fact that $f'^*t'_1, \dots, f'^*t'_m,\ s_{m+1},\ \dots,\ s_{m+d+1}$ is an $\Omega^p_{X',x'} \otimes E'_{x'}$-regular sequence.

We also obtain a contradiction assuming $d = 0$, by a similar manner as above without cutting out by s_{m+1} and so on. q.e.d.

4. Hodge Metric on the Ramified Cover

We still discuss in Set up 2.3 and §3.1. To compare the Hodge metric g of $F = R^q f_* \big(K_{X/Y} \otimes E\big)$ on $Y \setminus \Delta$ and a Hodge metric of $F' = R^q f''_* \big(K_{X''/Y'} \otimes E''\big)$ on $Y' \setminus \Delta'$, we need to put an appropriate metric on X''. We can not take arbitrary Kähler metric on X'' of course. The problem is that the pull-back $\tau''^* \omega$ on X'' is not positive definite any more. To overcome this problem, we introduce a modified degenerate Kähler metric and a sequence of auxiliary Kähler metrics.

4.1. Degenerate Kähler metric. We consider a direct sum

$$\widetilde{\omega} := p_X^* \omega + p_{Y'}^* \sqrt{-1} \sum dt'_j \wedge \overline{dt'_j},$$

which is a Kähler form on $X \times Y'$. Via the map $j_{X^\circ} \circ \nu \circ \mu : X'' \longrightarrow X \times Y'$, we let

$$\omega'' := (j_{X^\circ} \circ \nu \circ \mu)^* \widetilde{\omega} = (\delta^* \widetilde{\omega})|_{X''} = \tau''^* \omega + f''^* \sqrt{-1} \sum dt'_j \wedge \overline{dt'_j}$$

be a d-closed semi-positive $(1,1)$-form on X'', which we may call a degenerate Kähler form. We do not take $\tau''^* \omega$ as a degenerate Kähler form on X'', because it may degenerate totally along $f''^{-1}(\Delta')$. While it is not the case for ω'', as we see in the next lemma. We will denote by $\operatorname{Exc} \mu \subset X''$ the exceptional locus of the map μ.

Lemma 4.1. *There exists a closed analytic subset $V'' \subset X''$ of $\operatorname{codim}_{X''} V'' \geq 2$ and $f''(V'') \subset \Delta'$ such that ω'' is a Kähler form on $X'' \setminus (V'' \cup \operatorname{Exc} \mu)$.*

Proof. We look at $V' = \tau'^{-1}(\operatorname{Sing} f^{-1}(\Delta))$ first, which is a closed analytic subset of X' of $\operatorname{codim}_{X'} V' \geq 2$ with $f'(V') \subset \Delta'$ and $V' \supset \operatorname{Sing} X'$ by Lemma 3.1. We can see that $(j_{X^\circ} \circ (\nu|_{X' \setminus V'}))^* \widetilde{\omega}$ is positive definite (i.e., a Kähler form) on $X' \setminus V'$ as follows. We continue the argument in the proof of Lemma 3.1, and use the notations there. On each U_p° in $U' = \amalg_{p=1}^{b_j} U_p^\circ \subset X'$, the $(1,1)$-from $(j_{X^\circ} \circ \nu|_{U'})^* \widetilde{\omega}$ is $\widetilde{\omega}|_{U_p^\circ}$, and needless to say it is Kähler. Then our assertion follows from this observation, because we can write $\mu^{-1}(V') \cup \operatorname{Exc} \mu = V'' \cup \operatorname{Exc} \mu$ for some $V'' \subset X''$ as in the statement. q.e.d.

The replacement of $\tau''^* \omega$ by ω'' may cause troubles when we compare Hodge metrics on $Y \setminus \Delta$ and $Y' \setminus \Delta'$. However it is not the case by the following isometric lemma.

Lemma 4.2. *Let $t \in Y \setminus \Delta$ and take one $t' \in Y' \setminus \Delta'$ such that $\tau(t') = t$, and let $\varphi_{t'} : F'_{t'} \longrightarrow (\tau^* F)_{t'} = F_t$ be the isomorphism of fibers in Lemma* 3.3*. The fiber F_t (resp. $F'_{t'}$) has a Hermitian inner product:*

the Hodge metric $g = g_\omega$ with respect to ω and h (resp. $g' = g_{\omega''}$ with respect to ω'' and h''). Then $\varphi_{t'}$ is an isometry with respect to these inner products.

Proof. We take a small coordinate neighbourhood W (resp. W') around t (resp. t') such that $\tau|_{W'} : W' \longrightarrow W$ is isomorphic, and that $f'' : X''_{W'} \longrightarrow W'$ and $f : X_W \longrightarrow W$ are isomorphic as fiber spaces over the identification $\tau|_{W'} : W' \xrightarrow{\sim} W$. The Hermitian vector bundle (E'', h'') is $\tau''^*(E, h)$ by definition. If we put a Hermitian inner product $g_{\tau''^*\omega}$ on $F_{t'}$ with respect to $\tau''^*\omega$ and h'', the map $\varphi_{t'} : (F'_{t'}, g_{\tau''^*\omega}) \longrightarrow (F_t, g_\omega)$ is an isometry. Although $\omega'' \neq \tau''^*\omega$, ω'' and $\tau''^*\omega$ are the same as relative Kähler forms over W', more concretely $\omega'' = \tau''^*\omega + f''^*\sqrt{-1}\sum dt_j \wedge d\overline{t'_j}$. Then we have $g_{\omega''} = g_{\tau''^*\omega}$, by a part of the definition of Hodge metrics [**28**, 5.2]. q.e.d.

Definition 4.3. Let g' be the Hodge metric on $R^q f''_*(K_{X''/Y'} \otimes E'')|_{Y' \setminus \Delta'}$ with respect to ω'' and h''.

4.2. Hodge metric with respect to the degenerate Kähler metric. We would like to develop Takegoshi's theory of "relative harmonic forms" with respect to the degenerate Kähler form ω'' on X''. The goal is the following

Proposition 4.4. (*cf.* [**35**, 5.2].) *There exist $H^0(Y', \mathcal{O}_{Y'})$-module homomorphisms*

$$*\mathcal{H} : H^0\left(Y', R^q f''_*\left(K_{X''/Y'} \otimes E''\right)\right) \longrightarrow H^0\left(X'', \Omega^{n+m-q}_{X''} \otimes E''\right),$$

$$L^q : H^0\left(X'', \Omega^{n+m-q}_{X''} \otimes E''\right) \longrightarrow H^0\left(Y', R^q f''_*\left(K_{X''/Y'} \otimes E''\right)\right)$$

such that $(c_{n+m-q}/q!)L^q \circ *\mathcal{H} = id$. Moreover for every $u \in H^0(Y', R^q f''_*\left(K_{X''/Y'} \otimes E''\right))$, there exists a relative holomorphic form $[\sigma_u] \in H^0(X'' \setminus f''^{-1}(\Delta'), \Omega^{n-q}_{X''/Y'} \otimes E'')$ such that

$$(*\mathcal{H}(u))|_{X'' \setminus f''^{-1}(\Delta')} = \sigma_u \wedge f''^* dt'.$$

Proof. Step 1: a sequence of Kähler forms. We take $\{W'_k;\ k = 1, 2, \ldots\}$ a fundamental system of neighbourhoods of Δ' in Y', such as $W'_k = \{t' \in Y';\ |t'_m| < 1/(k+1)\}$. Let k be a positive integer. Since $\delta : \widetilde{X \times Y'} \longrightarrow X \times Y'$ in §3.1 is a composition of blowing-ups along smooth centers laying over $\operatorname{Sing} X^\circ$, there exists a d-closed real $(1,1)$-form ξ_k on $\widetilde{X \times Y'}$ with $\operatorname{Supp} \xi_k \subset (p_{Y'} \circ \delta)^{-1}(W'_k)$ such that $c_k \delta^* \widetilde{\omega} + \xi_k > 0$ on $\widetilde{X \times Y'}$ for a large constant c_k (possibly after shrinking Y and Y'). Possibly after replacing c_k by a larger constant, we may assume $\|\xi_k\|_\infty / c_k \to 0$ as $k \to \infty$. Here $\|\xi_k\|_\infty$ is the sup-norm with respect to any fixed Hermitian metric on $\widetilde{X \times Y'}$ (possibly after shrinking

Y and Y'). Thus we obtain a sequence of Kähler forms

$$\left\{\widetilde{\omega}_k := \delta^*\widetilde{\omega} + c_k^{-1}\xi_k\right\}_k$$

on $\widetilde{X \times Y'}$ such that $\widetilde{\omega}_k = \delta^*\widetilde{\omega}$ on $\widetilde{X \times Y'} \setminus (p_{Y'} \circ \delta)^{-1} W'_k$, and $\widetilde{\omega}_k \to \delta^*\widetilde{\omega}$ uniformly on $\widetilde{X \times Y'}$ as $k \to \infty$. For every positive integer k, we let

$$\omega''_k := \widetilde{\omega}_k|_{X''}$$

be a Kähler form on X''.

Step 2: Relative hard Lefschetz type theorem. We first recall the theory of Takegoshi with respect to the Kähler forms ω''_k on X''. Let $W' \subset Y'$ be a Stein subdomain with a strictly plurisubhamonic exhaustion function ψ. We take a global frame $dt' = dt'_1 \wedge \ldots \wedge dt'_m$ of $K_{Y'}$. Recalling $R^q f''_* \left(K_{X''/Y'} \otimes E''\right) = K_{Y'}^{\otimes(-1)} \otimes R^q f''_*(K_{X''} \otimes E'')$, this trivialization of $K_{Y'}$ gives an isomorphism $R^q f''_* \left(K_{X''/Y'} \otimes E''\right) \cong R^q f''_*(K_{X''} \otimes E'')$ on Y'. Since W' is Stein, we have also a natural isomorphism $H^0(W', R^q f''_*(K_{X''} \otimes E'')) \cong H^q(X''_{W'}, K_{X''} \otimes E'')$, where $X'' = f''^{-1}(W')$. We denote by α^q the composed isomorphism

$$\alpha^q : H^0\left(W', R^q f''_*\left(K_{X''/Y'} \otimes E''\right)\right) \xrightarrow{\sim} H^q\left(X''_{W'}, K_{X''} \otimes E''\right).$$

Let k be a positive integer. With respect to the Kähler form ω''_k on X'' in Step 1, we denote by $*_k$ the Hodge $*$-operator, and by

$$L^q_k : H^0\left(X''_{W'}, \Omega^{n+m-q}_{X''} \otimes E''\right) \longrightarrow H^q\left(X''_{W'}, K_{X''} \otimes E''\right)$$

the Lefschetz homomorphism induced from $\omega''^q_k \wedge \bullet$. Also with respect to ω''_k and h'', we set

$$\begin{aligned}
&\mathcal{H}^{n+m,q}\left(X''_{W'}, \omega''_k, E'', f''^*\psi\right) \\
&\quad = \left\{u \in A^{n+m,q}(X''_{W'}, E'');\ \overline{\partial} u = \vartheta_{h''} u = 0,\ e(\overline{\partial}(f''^*\psi))^* u = 0\right\}
\end{aligned}$$

(see [**35**, 4.3 or 5.2.i]). By [**35**, 5.2.i], $\mathcal{H}^{n+m,q}(X''_{W'}, \omega''_k, E'', f''^*\psi)$ represents $H^q(X''_{W'}, K_{X''} \otimes E'')$ as an $H^0(Y', \mathcal{O}_{Y'})$-module, and there exists a natural isomorphism

$$\iota_k : \mathcal{H}^{n+m,q}\left(X''_{W'}, \omega''_k, E'', f''^*\psi\right) \xrightarrow{\sim} H^q\left(X''_{W'}, K_{X''} \otimes E''\right)$$

given by taking the Dolbeault cohomology class. We have an isomorphism $\mathcal{H}_k := \iota_k^{-1} \circ \alpha^q$;

$$\mathcal{H}_k : H^0\left(W', R^q f''_*\left(K_{X''/Y'} \otimes E''\right)\right) \xrightarrow{\sim} \mathcal{H}^{n+m,q}\left(X''_{W'}, \omega''_k, E'', f''^*\psi\right).$$

Also by [**35**, 5.2.i], the Hodge $*$-operator gives an injective homomorphism

$$*_k : \mathcal{H}^{n+m,q}\left(X''_{W'}, \omega''_k, E'', f''^*\psi\right) \longrightarrow H^0\left(X''_{W'}, \Omega^{n+m-q}_{X''} \otimes E''\right),$$

and induces a splitting $*_k \circ \iota_k^{-1} : H^q(X''_{W'}, K_{X''} \otimes E'') \longrightarrow H^0(X''_{W'}, \Omega^{n+m-q}_{X''} \otimes E'')$ for the homomorphism L^q_k such that $(c_{n+m-q}/q!) L^q_k \circ$

$*_k \circ \iota_k^{-1} = \mathrm{id}$. (The homomorphism δ^q in [**35**, 5.2.i] with respect to ω''_k and h'' is $*_k \circ \iota_k^{-1}$ times a universal constant.) In particular

$$(c_{n+m-q}/q!)\left((\alpha^q)^{-1} \circ L_k^q\right) \circ (*_k \circ \mathcal{H}_k) = \mathrm{id}.$$

All homomorphisms $\alpha^q, *_k, L_k^q, \iota_k, \mathcal{H}_k$ are as $H^0(W', \mathcal{O}_{Y'})$-modules.

Let $u \in H^0(W', R^q f''_* \left(K_{X''/Y'} \otimes E''\right))$. Then we have $*_k \circ \mathcal{H}_k(u) \in H^0(X''_{W'}, \Omega^{n+m-q}_{X''} \otimes E'')$, and then by [**35**, 5.2.ii]

$$*_k \circ \mathcal{H}_k(u)|_{X''_{W'} \setminus f''^{-1}(\Delta')} = \sigma_k \wedge f''^* dt'$$

for some $[\sigma_k] \in H^0(X''_{W'} \setminus f''^{-1}(\Delta'), \Omega^{n-q}_{X''/Y'} \otimes E'')$. It is not difficult to see $[\sigma_k] \in H^0(X''_{W'} \setminus f''^{-1}(\Delta'), \Omega^{n-q}_{X''/Y'} \otimes E'')$ does not depend on the particular choice of a frame dt' of $K_{Y'}$.

Step 3: Takegoshi's theory with respect to ω''. We then consider the theory for ω''. In case a Stein subdomain $W' \subset Y'$ as above is contained in $Y' \setminus \Delta'$, the theory is the same because ω'' is Kähler on $X'' \setminus f''^{-1}(\Delta')$ (see Lemma 4.1). Hence we explain, how to avoid the degeneracy of ω'' along a part of $f''^{-1}(\Delta')$.

Let k_1 and k_2 be any pair of positive integers. We take any Stein subdomain $W' \subset Y' \setminus (W'_{k_1} \cup W'_{k_2})$, which admits a smooth strictly plurisubharmonic exhaustion function ψ. Due to [**35**, 5.2.iv], there are two commutative diagrams for $i = 1, 2$:

$$\begin{array}{ccc}
H^q\left(X'', K_{X''} \otimes E''\right) & \xrightarrow{*_{k_i} \circ \iota_{k_i}^{-1}} & H^0\left(X'', \Omega^{n+m-q}_{X''} \otimes E''\right) \\
\big\downarrow & & \big\downarrow \\
H^q\left(X''_{W'}, K_{X''} \otimes E''\right) & \xrightarrow[*_{k_i} \circ \iota_{k_i}^{-1}]{} & H^0\left(X''_{W'}, \Omega^{n+m-q}_{X''} \otimes E''\right).
\end{array}$$

Here the vertical arrows are restriction maps. The bottom horizontal maps depend only on $\omega''_{k_i}|_{X''_{W'}}$. Recall that $\omega''_k = \omega''$ on $X'' \setminus f''^{-1}(W'_k)$. Because of $\omega'' = \omega''_{k_1} = \omega''_{k_2}$ on $X''_{W'}$, the bottom horizontal maps are independent of k_1 and k_2.

Let us take $u \in H^0(Y', R^q f''_* \left(K_{X''/Y'} \otimes E''\right))$. Then by the observation above, two holomorphic forms $*_{k_1} \circ \mathcal{H}_{k_1}(u), *_{k_2} \circ \mathcal{H}_{k_2}(u) \in H^0(X'', \Omega^{n+m-q}_{X''} \otimes E'')$ coincide on an open subset $X''_{W'}$, and hence $*_{k_1} \circ \mathcal{H}_{k_1}(u) = *_{k_2} \circ \mathcal{H}_{k_2}(u)$ on X''. (Note that it may happen that $\mathcal{H}_{k_1}(u) \neq \mathcal{H}_{k_2}(u)$ around $f''^{-1}(\Delta')$, because $\mathcal{H}_k(u) = (c_{n+m-q}/q!)\omega''_k \wedge (*_k \circ \mathcal{H}_k(u))$ and $\omega''_{k_1} \neq \omega''_{k_2}$ around there.) We denote by

$$*\mathcal{H}(u) \in H^0\left(X'', \Omega^{n+m-q}_{X''} \otimes E''\right)$$

instead of arbitrary $*_k \circ \mathcal{H}_k(u)$. Since ω'' may not be positive definite along a part of $f''^{-1}(\Delta')$, the operators $*$ and $\mathcal{H}$ with respect to ω'' may not be defined across $f''^{-1}(\Delta')$. However

$$*\mathcal{H} : H^0\left(Y', R^q f''_*\left(K_{X''/Y'} \otimes E''\right)\right) \longrightarrow H^0\left(X'', \Omega^{n+m-q}_{X''} \otimes E''\right)$$

is defined. Recalling $\mathcal{H}_k(u) = (c_{n+m-q}/q!)\omega''_k \wedge (*_k \circ \mathcal{H}_k(u))$ in $H^q(X'', K_{X''} \otimes E'')$, since $\mathcal{H}_{k_1}(u)$ and $\mathcal{H}_{k_2}(u)$ are in the same Dolbeault cohomology class $\alpha^q(u) \in H^q(X'', K_{X''} \otimes E'')$, we have $L^q_{k_1}(*_{k_1} \circ \mathcal{H}_{k_1}(u)) = L^q_{k_2}(*_{k_2} \circ \mathcal{H}_{k_2}(u))$. We put

$$L^q = (\alpha^q)^{-1} \circ L^q_k :$$

$$H^0\left(X'', \Omega^{n+m-q}_{X''} \otimes E''\right) \longrightarrow H^0\left(Y', R^q f''_*\left(K_{X''/Y'} \otimes E''\right)\right)$$

for one arbitrary fixed large k. A different choice of k will give a different L^q, however the relation $(c_{n+m-q}/q!)((\alpha^q)^{-1} \circ L^q_k) \circ (*_k \circ \mathcal{H}_k) = \mathrm{id}$ in Step 2 implies our assertion (1). Recall $(*_k \circ \mathcal{H}_k(u))|_{X'' \setminus f''^{-1}(\Delta')} = \sigma_k \wedge f''^* dt'$ for some $[\sigma_k] \in H^0(X'' \setminus f''^{-1}(\Delta'), \Omega^{n-q}_{X''/Y'} \otimes E'')$. Then we see, $[\sigma_k]$ is also independent of k, and hence $(*\mathcal{H}(u))|_{X'' \setminus f''^{-1}(\Delta')}$ can be written as

$$(*\mathcal{H}(u))|_{X'' \setminus f''^{-1}(\Delta')} = \sigma_u \wedge f''^* dt'$$

for some $[\sigma_u] \in H^0(X'' \setminus f''^{-1}(\Delta'), \Omega^{n-q}_{X''/Y'} \otimes E'')$. This is (2). q.e.d.

Remark 4.5. (1) We recall the definition of the Hodge metric g' of $R^q f''_*\left(K_{X''/Y'} \otimes E''\right)|_{Y' \setminus \Delta'}$ with respect to ω'' and h''. We remind that ω'' is Kähler on $X'' \setminus f''^{-1}(\Delta')$. We only mention it for a global section $u \in H^0(Y', R^q f''_*\left(K_{X''/Y'} \otimes E''\right))$. It is given by

$$g'(u,u)(t') = \int_{X''_{t'}} (c_{n-q}/q!)\left(\omega''^q \wedge \sigma_u \wedge h''\overline{\sigma_u}\right)|_{X''_{t'}}$$

at $t' \in Y' \setminus \Delta'$.

(2) This is only a side remark, which we will not use later. The Hodge metric g'_k of $R^q f''_*\left(K_{X''/Y'} \otimes E''\right)|_{Y' \setminus \Delta'}$ with respect to ω''_k and h'' is given, for $u \in H^0(Y', R^q f''_*\left(K_{X''/Y'} \otimes E''\right))$, by

$$\begin{aligned} g'_k(u,u)(t') &= \int_{X''_{t'}} (c_{n-q}/q!)\left(\omega''^q_k \wedge \sigma_{u_k} \wedge h''\overline{\sigma_{u_k}}\right)|_{X''_{t'}} \\ &= \int_{X''_{t'}} (c_{n-q}/q!)\left(\omega''^q_k \wedge \sigma_u \wedge h''\overline{\sigma_u}\right)|_{X''_{t'}} \end{aligned}$$

at $t' \in Y' \setminus \Delta'$. Since $\omega''_k \to \omega''$ uniformly as $k \to \infty$, we have $g'_k(u,u)(t') \to g'(u,u)(t')$ as $k \to \infty$, for any fixed $t' \in Y' \setminus \Delta'$. □

4.3. Uniform estimate of Fujita type. We will give a key estimate of the singularities of the Hodge metric g' on $R^q f''_* \left(K_{X''/Y'} \otimes E''\right)|_{Y' \setminus \Delta'}$ with respect to ω'' and h''. This is the main place where we use the fact that, by weakly semi-stable reduction, we achieve $f''^* \Delta'$ is reduced plus μ-exceptional.

In this subsection we pose the following genericity condition around a point of Δ'.

Assumption 4.6. The map $f' : X'' \longrightarrow Y'$, (E'', h'') and $F' = R^q f''_* \left(K_{X''/Y'} \otimes E''\right)$ satisfy the conditions (1)–(2) in Set up 2.3. $\square$

We then take a global frame $e'_1, \ldots, e'_r \in H^0(Y', F')$ of $F' \cong \mathcal{O}_{Y'}^{\oplus r}$. For a constant vector $s = (s_1, \ldots, s_r) \in \mathbb{C}^r$, we let $u_s = \sum_{i=1}^r s_i e'_i \in H^0(Y', F')$. We denote by $S^{2r-1} = \{s \in \mathbb{C}^r;\ |s| = (\sum |s_i|^2)^{1/2} = 1\}$ the unit sphere.

We note the following two things. Since $e'_1, \ldots, e'_r$ generate F' over Y', u_s is nowhere vanishing on Y' as soon as $s \neq 0$, namely u_s is non-zero in $F'/(\mathfrak{m}_{Y',y'} F')$ at any $y' \in Y'$. The map $\mathbb{C}^r \longrightarrow H^0(X'', \Omega_{X''}^{n+m-q} \otimes E'')$ given by $s \mapsto u_s \mapsto *\mathcal{H}(u_s) = \sum_{i=1}^r s_i (*\mathcal{H}(e'_i))$ is continuous, with respect to the standard topology of $\mathbb{C}^r$ and the topology of $H^0(X'', \Omega_{X''}^{n+m-q} \otimes E'')$ of uniform convergence on compact sets.

Lemma 4.7. (*cf.* [**12**, 1.11].) *Under Assumption* 4.6 *and notations above, let $y' \in \Delta'$ and let $s_0 \in S^{2r-1}$. Then there exist a neighbourhood $S(s_0)$ of s_0 in S^{2r-1}, a neighbourhood $W'_{y'}$ of y' in Y' and a positive number N such that $g'(u_s, u_s)(t') \geq N$ for any $s \in S(s_0)$ and any $t' \in W'_{y'} \setminus \Delta'$.*

Proof. (1) We first claim the following variant of Lemma 3.8. Let $u \in H^0(Y', F')$, and assume u does not vanish at y'. Then there exists a non-μ-exceptional component B''_j in $f''^* \Delta' = \sum B''_j + B''_{exc}$, such that $*\mathcal{H}(u) \in H^0(X'', \Omega_{X''}^{n+m-q} \otimes E'')$ does not vanish identically along $B''_j \cap f''^{-1}(y')$.

In fact, by Proposition 4.4, the image $*\mathcal{H}(H^0(Y', F'))$ is a direct summand of $H^0(Y', f''_*(\Omega_{X''}^{n+m-q} \otimes E''))$ as an $H^0(Y', \mathcal{O}_{Y'})$-module. In particular, $*\mathcal{H}(u) \in H^0(X'', \Omega_{X''}^{n+m-q} \otimes E'')$ does not vanish at $y' \in Y'$ as an element of an $H^0(Y', \mathcal{O}_{Y'})$-module. Then we apply Lemma 3.8.

(2) For our nowhere vanishing u_{s_0}, we take a non-μ-exceptional component

$$B'' = B''_j$$

in $f''^* \Delta'$ such that $*\mathcal{H}(u_{s_0})$ does not vanish identically along $B'' \cap f''^{-1}(y')$. We take a general point $x_0 \in B'' \cap f''^{-1}(y')$, and a local coordinate $(U; z = (z_1, \ldots, z_{n+m}))$ centered at $x_0 \in X''$ such that f'' is given by $t' = f''(z) = (z_{n+1}, \ldots, z_{n+m})$ on U. In particular $(f''^* \Delta')|_U = B''|_U = \{z_{n+m} = 0\}$. Over U, we may assume that the bundle E'' is also trivialized, i.e., $E''|_U \cong U \times \mathbb{C}^{r(E)}$, where $r(E)$ is the rank of E. Using these local trivializations on U, we have a constant $a > 0$ such that (i) $\omega'' \geq a\omega_{eu}$ on U, where $\omega_{eu} = \sqrt{-1}/2\sum_{i=1}^{n+m} dz_i \wedge d\overline{z_i}$ (recall ω'' is positive definite around x_0 by Lemma 4.1!!), and (ii) $h'' \geq a\mathrm{Id}$ on U as Hermitian matrixes. Here we regard $h''|_U(z)$ as a positive definite Hermitian matrix at each $z \in U$ in terms of $E''|_U \cong U \times \mathbb{C}^{r(E)}$, and here Id is the $r(E) \times r(E)$ identity matrix.

(3) Let $s \in S^{2r-1}$. By Proposition 4.4, there exists $\sigma_s \in A^{n-q,0}(X'' \setminus f''^{-1}(\Delta'), E'')$ such that $(*\mathcal{H}(u_s))|_{X''\setminus f''^{-1}(\Delta')} = \sigma_s \wedge f''^* dt'$. We write $\sigma_s = \sum_{I\in I_{n-q}} \sigma_{sI} dz_I + R_s$ on $U \setminus B''$. Here I_{n-q} is the set of all multi-indexes $1 \leq i_1 < \ldots < i_{n-q} \leq n$ of length $n-q$ (not including $n+1, \ldots, n+m$), $\sigma_{sI} = {}^t(\sigma_{sI,1}, \ldots, \sigma_{sI,r(E)})$ is a row vector valued holomorphic function with $\sigma_{sI,i} \in H^0(U \setminus B'', \mathcal{O}_{X''})$, and here $R_s = \sum_{k=1}^m R_{sk} \wedge dz_{n+k} \in A^{n-q,0}(U \setminus B'', E'')$. Then

$$\sigma_s \wedge f''^* dt' = \left(\sum_{I\in I_{n-q}} \sigma_{sI} dz_I\right) \wedge dz_{n+1} \wedge \ldots \wedge dz_{n+m}$$

on $U \setminus B''$. Since $\sigma_s \wedge f''^* dt' = (*\mathcal{H}(u_s))|_{X''\setminus f''^{-1}(\Delta')}$ and $*\mathcal{H}(u_s) \in H^0(X'', \Omega_{X''}^{n+m-q} \otimes E'')$, all σ_{sI} can be extended holomorphically on U. We still denote by the same latter $\sigma_{sI} = {}^t(\sigma_{sI,1}, \ldots, \sigma_{sI,r(E)})$ its extension.

At the point $s_0 \in S^{2r-1}$, since $*\mathcal{H}(u_{s_0})$ does not vanish identically along $B'' \cap f''^{-1}(y')$, and since $x_0 \in B'' \cap f''^{-1}(y')$ is general, we have at least one $\sigma_{s_0 J_0, i_0} \in H^0(U, \mathcal{O}_{X''})$ such that $\sigma_{s_0 J_0, i_0}(x_0) \neq 0$. We take such

$$J_0 \in I_{n-q} \text{ and } i_0 \in \{1, \ldots, r(E)\}.$$

(4) By the continuity of $s \mapsto u_s \mapsto *\mathcal{H}(u_s)$, we can take an ε-polydisc $U(\varepsilon) = \{z = (z_1, \ldots, z_{n+m}) \in U;\ |z_i| < \varepsilon \text{ for any } 1 \leq i \leq n+m\}$ centered at x_0 for some $\varepsilon > 0$, and a neighbourhood $S(s_0)$ of s_0 in S^{2r-1} such that

$$A := \inf\{|\sigma_{sJ_0,i_0}(z)|;\ s \in S(s_0),\ z \in U(\varepsilon)\} > 0.$$

We set $W'_{y'} := f''(U(\varepsilon))$, which is an open neighbourhood of $y' \in Y'$, since f'' is flat (in particular it is open). Then for any $s \in S(s_0)$ and any $t' \in W'_{y'} \setminus \Delta'$, we have

$$\begin{aligned}\int_{X''_{t'}} &(c_{n-q}/q!) \left(\omega''^q \wedge \sigma_s \wedge h''\overline{\sigma_s}\right)|_{X''_{t'}} \\ &\geq a \int_{X''_{t'}\cap U} (c_{n-q}/q!) \left(\omega''^q \wedge \sigma_s \wedge \overline{\sigma_s}\right)|_{X''_{t'}\cap U} \\ &= a^{q+1} \int_{z\in X''_{t'}\cap U} \sum_{I\in I_{n-q}} \sum_{i=1}^{r(E)} |\sigma_{sI,i}(z)|^2 dV_n \\ &\geq a^{q+1} \int_{z\in X''_{t'}\cap U(\varepsilon)} A^2 \, dV_n \;=\; a^{q+1} A^2 (\pi\varepsilon^2)^n.\end{aligned}$$

Here $dV_n = (\sqrt{-1}/2)^n \bigwedge_{i=1}^n dz_i \wedge d\overline{z_i}$ is the standard euclidean volume form on $\mathbb{C}^n$. q.e.d.

Lemma 4.8. (*cf.* [**12**, 1.12].) *Under Assumption* 4.6 *and notations after that, let $y' \in \Delta'$. Then there exist a neighbourhood $W'_{y'}$ of y' in Y' and a positive number N, such that $g'(u_s, u_s)(t') \geq N$ for any $s \in S^{2r-1}$ and any $t' \in W'_{y'} \setminus \Delta'$.*

Proof. Since S^{2r-1} is compact, this is clear from Lemma 4.7. q.e.d.

5. Plurisubharmonic Extension

We still discuss in Set up 2.3 and §3.1. We are ready to talk about, say "the plurisubharmonic extension" of the quotient metric $g^{\circ}_{\mathcal{O}(1)}$ of $\mathcal{O}(1)|_{\pi^{-1}(Y\setminus\Delta)}$ in Theorem 2.4. Since such an extension is a local question on $\mathbb{P}(F)$, we shall discuss around a fixed point $P \in \mathbb{P}(F)$. We take a quotient line bundle $F \longrightarrow L$ so that P corresponds to $F_{\pi(P)} \longrightarrow L_{\pi(P)}$. We also take a trivialization of F given by $e_1, \ldots, e_r \in H^0(Y, F)$, so that the kernel M of $F \longrightarrow L$ is generated by $e_1, \ldots, e_{r-1}$. A choice of a frame $e_1, \ldots, e_r$ also gives a trivialization $\mathbb{P}(F) \cong Y \times \mathbb{P}^{r-1}$. From now on, we identify $\mathbb{P}(F)$ and $Y \times \mathbb{P}^{r-1}$.

5.1. Quotient metric. We first describe the quotient metric $g^{\circ}_{\mathcal{O}(1)}$ around P. Let $[a] = (a_1 : \ldots : a_r)$ be the homogeneous coordinates of $\mathbb{P}^{r-1}$. Then $P = \pi(P) \times (0 : \ldots : 0 : 1)$ in $Y \times \mathbb{P}^{r-1}$. Let

$$U = Y \times \left\{[a] \in \mathbb{P}^{r-1};\ a_r \neq 0\right\}$$

be a standard open neighbourhood of P. This neighbourhood of P (or of $F_{\pi(P)} \longrightarrow L_{\pi(P)}$) is also described as follows. Let $a = (a_1, \ldots, a_{r-1}) \in \mathbb{C}^{r-1}$ (be an inhomogeneous coordinate of $\mathbb{P}^{r-1}$). We set $e_{ia} = e_i + a_i e_r \in H^0(Y, F)$ for every $1 \leq i \leq r-1$, and $e_{ra} = e_r$, and let M_a be the subbundle of F generated by $e_{1a}, \ldots, e_{r-1a}$, and let $L_a = F/M_a$ be the quotient line bundle on Y. Every point $t \times a \in U$ corresponds to a subspace $M_{at} \subset F_t$ generated by $e_1(t) + a_1 e_r(t), \dots, e_{r-1}(t) + a_{r-1} e_r(t)$ and hence the quotient space $L_{at} = F_t/M_{at}$. For every fixed $a \in \mathbb{C}^{r-1}$, we have a nowhere vanishing section

$$\widehat{e}_{ra} \in H^0(Y, L_a)$$

defined by $\widehat{e}_{ra} : t \in Y \mapsto \widehat{e}_{ra}(t) \in L_{at}$. Here $\widehat{e}_{ra}(t)$ is the image of $e_r(t) \in F_t$ under the quotient $F_t \longrightarrow L_{at}$. We have a canonical nowhere vanishing section

$$\widehat{e}_r \in H^0(U, \mathcal{O}(1))$$

defined by $\widehat{e}_r : t \times a \in U \mapsto \widehat{e}_{ra}(t) \in L_{at}$.

Let $a \in \mathbb{C}^{r-1}$. With respect to the global frame $\{e_{ia}\}_{i=1}^r$ of F, the Hodge metric g on $F|_{Y\setminus\Delta}$ is written as $g_{i\bar{j}a} := g(e_{ia}, e_{ja}) \in A^0(Y \setminus \Delta, \mathbb{C})$ for $1 \le i, j \le r$. At each point $t \in Y \setminus \Delta$, $(g_{i\bar{j}a}(t))_{1\le i,j\le r}$ is a positive definite Hermitian matrix, in particular $(g_{i\bar{j}a}(t))_{1\le i,j\le r-1}$ is also positive definite. We let $(g_a^{\bar{i}j}(t))_{1\le i,j\le r-1}$ be the inverse matrix. The pointwise orthogonal projection of e_r to $(M_a|_{Y\setminus\Delta})^{\perp}$ with respect to g is given by

$$P_a(e_r) = e_r - \sum_{i=1}^{r-1}\sum_{j=1}^{r-1} e_{ia} g_a^{\bar{i}j} g_{j\bar{r}a} \ \in A^0(Y \setminus \Delta, F).$$

Then the quotient metric g_{L_a} on the line bundle $L_a|_{Y\setminus\Delta}$ is described as

$$g_{L_a}(\widehat{e}_{ra}, \widehat{e}_{ra}) = g(P_a(e_r), P_a(e_r)).$$

Then, at each $t \times a \in U \setminus \pi^{-1}(\Delta) = (Y \setminus \Delta) \times \mathbb{C}^{r-1}$, we have

$$g^{\circ}_{\mathcal{O}(1)}(\widehat{e}_r, \widehat{e}_r)(t \times a) = g_{L_a}(\widehat{e}_{ra}, \widehat{e}_{ra})(t).$$

We already know that $-\log(g^{\circ}_{\mathcal{O}(1)}(\widehat{e}_r, \widehat{e}_r)|_{(Y\setminus\Delta)\times\mathbb{C}^{r-1}})$ is plurisubharmonic ([**2**, 1.2] [**28**, 1.1]). What we want to prove is

Lemma 5.1. *Let ε be a real number such that $0 < \varepsilon < (2(r-1))^{-2}$, and let $D_\varepsilon = \{a = (a_1, \dots, a_{r-1}) \in \mathbb{C}^{r-1};\ \sum_{i=1}^{r-1} |a_i|^2 < \varepsilon\}$. Then $-\log(g^{\circ}_{\mathcal{O}(1)}(\widehat{e}_r, \widehat{e}_r)|_{(Y\setminus\Delta)\times D_\varepsilon})$ extends as a plurisubharmonic function on $Y \times D_\varepsilon$.*

In case $r = 1$, this (as well as Lemma 5.2 and 5.4 below) should be read that $-\log(g^{\circ}_{\mathcal{O}(1)}(\widehat{e}_r, \widehat{e}_r)|_{Y\setminus\Delta}) = -\log(g(e_1, e_1)|_{Y\setminus\Delta})$ extends as a plurisubharmonic function on Y. Since $P \in \mathbb{P}(F)$ is arbitrary, this lemma implies Theorem 2.4.

5.2. Boundedness and reduction on the ramified cover. In Lemma 3.3, we have a natural inclusion $\varphi : F' \longrightarrow \tau^* F$, which is isomorphic over $Y' \setminus \Delta'$. We will reduce our study of F to that of F' via this φ. Let $L' \subset \tau^* L$ be the image of the composition $F' \longrightarrow \tau^* F \longrightarrow \tau^* L$, and let

M' be the kernel of the quotient $F' \longrightarrow L'$. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & F' & \longrightarrow & L' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \\ 0 & \longrightarrow & \tau^*M & \longrightarrow & \tau^*F & \longrightarrow & \tau^*L & \longrightarrow & 0. \end{array}$$

Here, horizontals are exact, verticals are injective. Since F', L' and M' are all torsion free $\mathcal{O}_{Y'}$-module sheaves, we can find a closed analytic subset $Z' \subset \Delta'$ of $\operatorname{codim}_{Y'} Z' \geq 2$ such that F', L' and M' are all locally free on $Y' \setminus Z'$. We may also assume that f'' is flat over $Y' \setminus Z'$, and $\operatorname{Supp} f''^*\Delta' \longrightarrow \Delta'$ is relative normal crossing over $\Delta' \setminus Z'$. We set $Z = \tau(Z') \subset \Delta$ a closed analytic subset of $\operatorname{codim}_Y(Z) \geq 2$. We then take an arbitrary point

$$y \in \Delta \setminus Z \text{ and let } y' = \tau^{-1}(y) \in \Delta' \setminus Z'.$$

Then Lemma 5.1 is reduced to the following

Lemma 5.2. *There exists a neighbourhood W_y of y in Y such that $g^\circ_{\mathcal{O}(1)}(\widehat{e}_r, \widehat{e}_r)$ is bounded from below by a positive constant on $(W_y \setminus \Delta) \times D_\varepsilon$, for D_ε in Lemma 5.1.*

In fact, since $y \in \Delta \setminus Z$ is arbitrary, by Riemann type extension, $-\log(g^\circ_{\mathcal{O}(1)}(\widehat{e}_r, \widehat{e}_r))$ becomes plurisubhamonic on $(Y \setminus Z) \times D_\varepsilon$, and then it is plurisubhamonic on $Y \times D_\varepsilon$ by Hartogs type extension.

To show Lemma 5.2, we need to analyze the map $\varphi : F' \longrightarrow \tau^*F$ and its inverse. We shall formulate and prove a quantitative version of Lemma 5.2 as Lemma 5.4.

Since our assertion in Lemma 5.2 is local around the point y (and y') and over there for $\pi : \mathbb{P}(F) \longrightarrow Y$, by replacing Y (resp. Y') by a small polydisc centered at y (resp. y'), we can also assume that $F' \cong \mathcal{O}_{Y'}^{\oplus r}$. In particular the assumption to use Lemma 4.7 and Lemma 4.8 is satisfied (remind also the choice of Z'). We take a global frame $e'_1, \ldots, e'_r \in H^0(Y', F')$ of F' such that $e'_1, \ldots, e'_{r-1}$ generate M' and the image $\widehat{e}'_r \in H^0(Y', L')$ of e'_r under $F' \longrightarrow L'$ generates L'. We still use (the restriction of) the same global frame $e_1, \ldots, e_r \in H^0(Y, F)$ of F, although the point $\pi(P)$ may not belong to the new Y any more.

In terms of those frames $\{\tau^*e_j\}$ and $\{e'_j\}$, we represent the bundle map $\varphi : F' \longrightarrow \tau^*F$ on Y'. For each j, we write $\varphi(e'_j) = \sum_{i=1}^r (\tau^*e_i)\varphi_{ij}$ for some $\varphi_{ij} \in H^0(Y', \mathcal{O}_{Y'})$. Then φ is given by $\Phi = (\varphi_{ij})_{1 \leq i,j \leq r}$ an $r \times r$-matrix valued holomorphic function on Y'. Since $\varphi(e'_j)$ for $1 \leq j \leq r-1$ belongs to $H^0(Y', \tau^*M)$, we have $\varphi_{r1} = \ldots = \varphi_{rr-1} \equiv 0$. We write

$$\Phi = \begin{pmatrix} & \Phi_0 & & \varphi_{*r} \\ 0 & \cdots & 0 & \varphi_{rr} \end{pmatrix},$$

accordingly so that $(\varphi(e'_1),\dots,\varphi(e'_r)) = (\tau^*e_1,\dots,\tau^*e_r)\Phi$. Here $\varphi_{*r} = {}^t(\varphi_{1r},\dots,\varphi_{r-1r})$, and the last part φ_{rr} represents the line bundle homomorphism $L' \longrightarrow \tau^*L$ on Y'.

By replacing Y and Y' by smaller polydiscs, we may assume that there exists a constant $C_{\Phi 1} > 0$ such that

$$|\varphi_{ij}(t')| < C_{\Phi 1}$$

for any pair $1 \le i,j \le r$ and any $t' \in Y'$. Since φ is isomorphic over $Y' \setminus \Delta'$, we can talk about the inverse there. Let $\Phi^{-1} = (\varphi^{ij})_{1\le i,j\le r}$ be the inverse on $Y' \setminus \Delta'$. Then $\Phi_0^{-1} = (\varphi^{ij})_{1\le i,j\le r-1}$, $\varphi^{r1} = \ldots = \varphi^{rr-1} \equiv 0$, $\varphi^{rr} = \varphi_{rr}^{-1}$, and $\varphi^{ir} = -(\sum_{j=1}^{r-1}\varphi^{ij}\varphi_{jr})\varphi_{rr}^{-1}$:

$$\Phi^{-1} = \begin{pmatrix} \Phi_0^{-1} & -\Phi_0^{-1}\varphi_{*r}\varphi_{rr}^{-1} \\ 0 \ \cdots \ 0 & \varphi_{rr}^{-1} \end{pmatrix}.$$

Needless to say, $(\tau^*e_1,\dots,\tau^*e_r) = (\varphi(e'_1),\dots,\varphi(e'_r))\Phi^{-1}$.

Lemma 5.3. *Assume $r > 1$. Let $\Psi := \Phi_0{}^t\overline{\Phi_0} \in A^0(Y', M(r-1,\mathbb{C}))$ be a matrix valued smooth function on Y'. Then there exists a constant $C_{\Phi 2} > 0$ such that*

$$\lambda_1(\Psi^{-1}(t')) \ge 1/C_{\Phi 2}$$

for any $t' \in Y' \setminus \Delta'$, where $\lambda_1(\Psi^{-1}(t'))$ is the smallest eigenvalue of the Hermitian matrix $\Psi^{-1}(t')$.

Proof. (1) At each $t' \in Y'$, $\Psi(t')$ is a Hermitian matrix which is semi-positive. Moreover it is positive definite for any $t' \in Y' \setminus \Delta'$, since Φ_0 is non-singular on it. All entries of Ψ are also bounded by a constant on Y', namely if $\Psi = (\psi_{ij})_{1\le i,j\le r}$ with $\psi_{ij} \in A^0(Y',\mathbb{C})$, then $|\psi_{ij}(t')| < (r-1)C_{\Phi 1}^2$ for any pair $1 \le i,j \le r$ and any $t' \in Y'$. In particular, as we will see below (2), there exists a constant $C_{\Phi 2} = (r-1)^2C_{\Phi 1}^2 > 0$ such that $\lambda_{r-1}(\Psi(t')) \le C_{\Phi 2}$ for any $t' \in Y'$, where $\lambda_{r-1}(\Psi(t'))$ is the biggest eigenvalue of the matrix $\Psi(t')$. On $Y' \setminus \Delta'$, we have the inverse Ψ^{-1}, whose pointwise matrix value $\Psi^{-1}(t')$ is also positive definite at each $t' \in Y' \setminus \Delta'$. Then $\lambda_1(\Psi^{-1}(t')) = 1/\lambda_{r-1}(\Psi(t')) \ge 1/C_{\Phi 2}$ for any $t' \in Y' \setminus \Delta'$.

(2) We consider in general, a non-zero matrix $A = (a_{ij}) \in M(n,\mathbb{C})$. Let $C = \max\{|a_{ij}| \ ; \ 1 \le i,j \le n\}$. Then we have $|\lambda| \le nC$ for any eigenvalue λ of A as follows. Let $v = {}^t(v_1,\dots,v_n)$ be a non-zero vector such that $Av = \lambda v$, and take p such that $|v_p| = \max\{|v_j| \ ; \ 1 \le j \le n\} > 0$. Then $\lambda v_p = \sum_{j=1}^n a_{pj}v_j$, and $|\lambda||v_p| \le \sum_{j=1}^n |a_{pj}||v_j| \le nC|v_p|$. Hence $|\lambda| \le nC$. q.e.d.

We set $C_\Phi = \max\{C_{\Phi 1}, C_{\Phi 2}, 1\}$.

5.3. Final uniform estimate. The following is a quantitative version of Lemma 5.2:

Lemma 5.4. *Let $y \in \Delta \setminus Z$ and $y' \in \Delta' \setminus Z'$ as above in Lemma 5.2. Let $W'_{y'}$ be a neighbourhood of y' and N be a positive number as in Lemma 4.8, and set $W_y = \tau(W'_{y'})$ a neighbourhood of y. Then $g_{L_a}(\widehat{e}_{ra}, \widehat{e}_{ra})(t) \geq N(2C_\Phi)^{-2}$ for any $t \in W_y \setminus \Delta$ and any $a \in D_\varepsilon$.*

Proof. We take arbitrary $t \in W_y \setminus \Delta$ and $a \in D_\varepsilon$, and take one $t' \in W'_{y'}$ such that $\tau(t') = t$. In case $r = 1$, we have $g^\circ_{\mathcal{O}(1)}(\widehat{e}_r, \widehat{e}_r)(t) = g(e_1, e_1)(t) = g'(e'_1, e'_1)(t')|\varphi_{11}^{-1}(t')|^2$. While $g'(e'_1, e'_1)(t')|\varphi_{11}^{-1}(t')|^2 \geq NC_{\Phi 1}^{-2}$ by Lemma 4.8. These prove Lemma 5.2 in case $r = 1$. For the rest, we consider in case $r > 1$.

(1) We reduce an estimate on g_{L_a} to that on g' as follows. We set $\sigma_{ia} = \sum_{j=1}^{r-1} g_a^{\bar{i}j} g_{j\bar{r}a}$ for $1 \leq i \leq r-1$ and $\sigma_{ra} = 1 - \sum_{i=1}^{r-1} \sigma_{ia} a_i$, which are in $A^0(Y \setminus \Delta, \mathbb{C})$. We can write as $P_a(e_r) = \sigma_{ra} e_r - \sum_{i=1}^{r-1} \sigma_{ia} e_i$ on $Y \setminus \Delta$. Then $\tau^* P_a(e_r) = \sigma_{ra}\varphi_{rr}^{-1}\varphi(e'_r) + \sum_{i=1}^{r-1}(\sigma_{ra}\varphi^{ir} - \sum_{j=1}^{r-1}\sigma_{ja}\varphi^{ij})\varphi(e'_i)$, and

$$\varphi^{-1}\tau^* P_a(e_r) = \sigma_{ra}\varphi_{rr}^{-1} e'_r + \sum_{i=1}^{r-1}\left(\sigma_{ra}\varphi^{ir} - \sum_{j=1}^{r-1}\sigma_{ja}\varphi^{ij}\right)e'_i$$

on $Y' \setminus \Delta'$. Recall $g_{L_a}(\widehat{e}_{ra}, \widehat{e}_{ra})(t) = g(P_a(e_r), P_a(e_r))(t)$, and then $g(P_a(e_r), P_a(e_r))(t) = g'(\varphi_{t'}^{-1}\tau^* P_a(e_r), \varphi_{t'}^{-1}\tau^* P_a(e_r))(t')$ by Lemma 4.2. We set $s_r = \sigma_{ra}(t')\varphi_{rr}^{-1}(t')$ and $s_i = \sigma_{ra}(t')\varphi^{ir}(t') - \sum_{j=1}^{r-1}\sigma_{ja}(t')\varphi^{ij}(t')$ for $1 \leq i \leq r-1$. We obtain a non-zero vector $s = (s_1, \dots, s_r) \in \mathbb{C}^r$. Then $\varphi_{t'}^{-1}\tau^* P_a(e_r) = u_s(t') = \sum_{i=1}^r s_i e'_i(t')$ at the t'. Hence it is enough to show $g'(u_s, u_s)(t') \geq N(2C_\Phi)^{-2}$.

(2) We claim that $|s|^2 := \sum_{i=1}^r |s_i|^2 \geq (2C_\Phi)^{-2}$. This claim, combined with Lemma 4.8, implies that $g'(u_s, u_s)(t') = |s|^2 g'(u_{s/|s|}, u_{s/|s|})(t') \geq |s|^2 N \geq N(2C_\Phi)^{-2}$.

(3) We prove the claim in (2). By using the formula on Φ^{-1}, we have

$$s_i = -\sum_{j=1}^{r-1}\left\{\sigma_{ra}(t')\varphi_{rr}^{-1}(t')\varphi_{jr}(t') + \sigma_{ja}(t')\right\}\varphi^{ij}(t')$$

for $1 \leq i \leq r-1$. We set $v_j = \sigma_{ra}(t')\varphi_{rr}^{-1}(t')\varphi_{jr}(t') + \sigma_{ja}(t')$ for $1 \leq j \leq r-1$. Then ${}^t(s_1, \dots, s_{r-1}) = -\Phi_0^{-1}(t') \cdot {}^t(v_1, \dots, v_{r-1})$, and $\sum_{i=1}^{r-1}|s_i|^2 = \langle \Phi_0^{-1}(t')v, \Phi_0^{-1}(t')v\rangle = \langle \Psi(t')^{-1}v, v\rangle$. Here $v = {}^t(v_1, \dots, v_{r-1})$, and the bracket $\langle\ \rangle$ is the standard Hermitian inner product on $\mathbb{C}^{r-1}$, and recall $\Psi = \Phi_0{}^t\overline{\Phi_0}$. Then $\langle \Psi(t')^{-1}v, v\rangle \geq \sum_{i=1}^{r-1}|v_i|^2/C_\Phi$ by Lemma 5.3.

In case $|s_r| \geq (2C_\Phi)^{-1}$, our claim in (2) is clear. Hence we assume $|s_r| < (2C_\Phi)^{-1}$, namely $|\sigma_{ra}(t')||\varphi_{rr}^{-1}(t')| < (2C_\Phi)^{-1}$. Then $|1 - \sum_{i=1}^{r-1}\sigma_{ia}(t')a_i| = |\sigma_{ra}(t')| < |\varphi_{rr}(t')|(2C_\Phi)^{-1} < 1/2$. We have at least one $1 \leq j \leq r-1$ such that $|\sigma_{ja}(t')||a_j| > 1/(2(r-1))$. In particular $|\sigma_{ja}(t')| > 1/(2(r-1)|a_j|) > 1/(2(r-1)\sqrt{\varepsilon})$. Then for such j, $|v_j| = |\sigma_{ja}(t') + \sigma_{ra}(t')\varphi_{rr}^{-1}(t')\varphi_{jr}(t')| \geq |\sigma_{ja}(t')| - |s_r||\varphi_{jr}(t')| > 1/(2(r-1)\sqrt{\varepsilon}) - (2C_\Phi)^{-1}C_\Phi$. Using $\varepsilon < (2(r-1))^{-2}$, we have $v_j > 1/\sqrt{2}$. Then we have $\sum_{i=1}^{r-1}|s_i|^2 > \sum_{i=1}^{r-1}|v_i|^2/C_\Phi > (2C_\Phi)^{-1}$, and hence our claim in (2). q.e.d.

Thus we have proved all Lemma 5.2, Lemma 5.1, and hence Theorem 2.4.

6. Proof of Theorem 1.3 and Variants

6.1. Proof of Theorem 1.3. The projectivity assumption on Y is only used to define the weakly positivity of sheaves. As we will see in the proof below, it is enough to assume that $f : X \longrightarrow Y$ is a Kähler fiber space over a smooth projective variety Y.

After obtaining Theorem 1.2, the proof is standard and classical. A minor difficulty in analytic approach will be that the sheaf $R^q f_*(K_{X/Y} \otimes E)$ may not be locally free in general.

Let F be, in general, a torsion free coherent sheaf on a smooth projective variety Y, and let Y_1 be the maximum Zariski open subset of Y on which F is locally free. Let Y_0 be a Zariski open subset of Y, which is contained in Y_1. The sheaf F is said to be *weakly positive* over Y_0 in the sense of Viehweg [**38**, 2.13], if for any given ample line bundle A on Y and any given positive integer a, there exists a positive integer b such that $\widehat{S}^{ab}(F) \otimes A^{\otimes b}$ is generated by global sections $H^0(Y, \widehat{S}^{ab}(F) \otimes A^{\otimes b})$ over Y_0. Here $\widehat{S}^m(F)$ is the double dual of the m-th symmetric tensor product $\mathrm{Sym}^m(F)$. We note [**38**, 2.14] that this condition does not depend on the choice of A. We refer also [**31**, V.3.20].

Now we turn to our situation in Theorem 1.3. Let us denote by $F = R^q f_*\left(K_{X/Y} \otimes E\right)$ which is a torsion free sheaf on Y. Then by [**38**, 2.14], it is enough to show that there exists an ample line bundle A on Y with the following property: for any positive integer a, there exists a positive integer b such that $\widehat{S}^{ab}(F) \otimes A^{\otimes b}$ is generated by $H^0(Y, \widehat{S}^{ab}(F) \otimes A^{\otimes b})$ over $Y \setminus \Delta$.

Associated to F on Y, we have a scheme

$$\mathbb{P}(F) = \mathrm{Proj}\Big(\bigoplus_{m \geq 0} \mathrm{Sym}^m(F)\Big)$$

over Y, say $\pi : \mathbb{P}(F) \longrightarrow Y$, and the tautological line bundle $\mathcal{O}(1)$ on $\mathbb{P}(F)$. Let $\mathbb{P}'(F) \longrightarrow \mathbb{P}(F)$ be the normalization of the component of $\mathbb{P}(F)$ containing $\pi^{-1}(Y \setminus S_q)$, and let $Z' \longrightarrow \mathbb{P}'(F)$ be a birational morphism from a smooth projective variety that is an isomorphism over $Y \setminus S_q$ ([**31**, V.§3.c]). In particular $\mathbb{P}(F) \setminus \pi^{-1}(S_q)$ is a Zariski open subset of a smooth projective variety Z', in particular it admits a complete

Kähler metric [**8**, 0.2]. We denote by $Z = \mathbb{P}(F) \setminus \pi^{-1}(S_q)$, and take a complete Kähler form ω_Z on Z. The volume form will be denoted by dV.

We take a very ample line bundle A on Y such that $A \otimes K_Y^{-1} \otimes (\widehat{\det} F)^{-1}$ is ample, where $\widehat{\det} F$ is the double dual of $\bigwedge^r F$ and r is the rank of F. Let h_{K_Y} (resp. $h_{\widehat{\det} F}$) be a smooth Hermitian metric on K_Y (resp. $\widehat{\det} F$), and let h_A be a smooth Hermitian metric on A with positive curvature, and such that $h_A h_{K_Y}^{-1} h_{\widehat{\det} F}^{-1}$ has positive curvature too. Let a be a positive integer. Then, noting that $\widehat{S}^{ab}(F) \otimes A^{\otimes b}$ is reflexive, it is enough to show that the restriction map

$$H^0\big(\mathbb{P}(F) \setminus \pi^{-1}(S_q), \mathcal{O}(ab) \otimes \pi^* A^{\otimes b}\big) \longrightarrow H^0\big(\mathbb{P}(F_y), (\mathcal{O}(ab) \otimes \pi^* A^{\otimes b})|_{\mathbb{P}(F_y)}\big)$$

is surjective for any $y \in Y \setminus \Delta$ and any integer $b > m + 1$, where $m = \dim Y$. We now fix $y \in Y \setminus \Delta$ and $b > m + 1$.

We take general members $s_1, \ldots, s_m \in H^0(Y, A)$ such that the zero divisors $(s_1)_0, \ldots, (s_m)_0$ are smooth, and intersect transversally, and such that y is isolated in $\bigcap_{i=1}^m (s_i)_0$. Let $W_y \subset Y \setminus \Delta$ be an open neighbourhood of y, which is biholomorphic to a ball in $\mathbb{C}^m$ of radius 2, $W_y \cap \bigcap_{i=1}^m (s_i)_0 = \{y\}$, and $F|_{W_y}$ is trivialized. Let $\rho \in A^0(Y, \mathbb{R})$ be a cut-off function around y such that $0 \leq \rho \leq 1$ on W_y, $\operatorname{Supp} \rho \subset W_y$, and $\rho \equiv 1$ on W'_y the ball of radius 1 in W_y. Let $\phi = \log(\sum_{i=1}^m h_A(s_i, s_i))^m \in L^1_{loc}(Y, \mathbb{R})$. Then $h_A^m e^{-\phi}$ is a singular Hermitian metric on $A^{\otimes m}$ with semi-positive curvature.

We set

$$L := \mathcal{O}(ab + r)|_Z \otimes \pi^* \left(A^{\otimes b} \otimes K_Y^{-1} \otimes (\widehat{\det} F)^{-1}\right)|_Z.$$

We note $(\mathcal{O}(ab) \otimes \pi^* A^{\otimes b})|_Z = K_Z \otimes L$. By Theorem 1.2, $\mathcal{O}(1)|_Z$ has a singular Hermitian metric $g_{\mathcal{O}(1)}$ with semi-positive curvature. Then the line bundle L over Z has a singular Hermitian metric

$$g_L := g_{\mathcal{O}(1)}^{ab+r} \pi^* \left(h_A^{b-m-1} \cdot h_A^m e^{-\phi} \cdot h_A h_{K_Y}^{-1} h_{\widehat{\det} F}^{-1}\right)$$

of semi-positive curvature. Let h_L be a smooth Hermitian metric on L. Then g_L can be written as $g_L = h_L e^{-\psi}$ for a function $\psi \in L^1_{loc}(Z, \mathbb{R})$, which is a sum of a smooth function and a plurisubharmonic function around every point of Z. Let $\sqrt{-1}\partial\overline{\partial}\psi = \sqrt{-1}(\partial\overline{\partial}\psi)_c + \sqrt{-1}(\partial\overline{\partial}\psi)_s$ be the Lebesgue decomposition into the absolute continuous part $\sqrt{-1}(\partial\overline{\partial}\psi)_c$ and the singular part $\sqrt{-1}(\partial\overline{\partial}\psi)_s$. We set $c(L, \psi) = \overline{\partial}\partial \log h_L + (\partial\overline{\partial}\psi)_c$. Then $\sqrt{-1}c(L, \psi)$ is a semi-positive $(1,1)$-current, because it is the absolute continuous part of the curvature current of g_L. We also note that $\sqrt{-1}c(L, \psi) \geq (b - m - 1)\sqrt{-1}\,\overline{\partial}\partial \log(\pi^* h_A)$.

We take a section $\sigma \in H^0(\mathbb{P}(F_y), (\mathcal{O}(ab) \otimes \pi^* A^{\otimes b})|_{\mathbb{P}(F_y)})$, and take a local extension $\sigma' \in H^0(\mathbb{P}(F|_{W_y}), \mathcal{O}(ab) \otimes \pi^* A^{\otimes b})$. We consider $u := \overline{\partial}((\pi^*\rho)\sigma') = \overline{\partial}(\pi^*\rho) \cdot \sigma'$, which can be seen as an L-valued $(p, 1)$-form on Z, where $p = \dim Z = m + r - 1$. At each point $z \in Z$, we set $|u|^2_{c(L,\psi)}(z) = \inf\{\alpha \in \mathbb{R}_{\geq 0} \cup \{+\infty\};\ |(u,\beta)|^2 \leq \alpha^2(\sqrt{-1}c(L,\psi)\Lambda\beta,\beta)$ for any $\beta \in \Omega^{p,1}_{Z,z} \otimes L_z\}$ (see [**8**, p. 468]). Here $(\ ,\ )$ is the Hermitian inner product of $\Omega^{p,1}_Z \otimes L$ with respect to ω_Z and h_L, and Λ is the adjoint of the Lefschetz operator $\omega_Z \wedge \bullet$. Assume for the moment that $\int_Z |u|^2_{c(L,\psi)} e^{-\psi} dV < \infty$. Then by [**8**, 5.1], for u with $\overline{\partial} u = 0$ and $\int_Z |u|^2_{c(L,\psi)} e^{-\psi} dV < \infty$, there exists $v \in L^2_{p,0}(Z, L, loc)$ (an L-valued $(p,0)$-form on Z with locally square integrable coefficients) such that $\overline{\partial} v = u$ and $\int_Z |v|^2 e^{-\psi} dV \leq \int_Z |u|^2_{c(L,\psi)} e^{-\psi} dV$. Since $u \equiv 0$ on $\pi^{-1}(W'_y)$, v is holomorphic on $\pi^{-1}(W'_y)$. The integrability $\int_Z |v|^2 e^{-\psi} dV < \infty$, in particular $\int_{\pi^{-1}(W_y)} |v|^2 e^{-\pi^*\phi} dV < \infty$ ensures $v|_{\mathbb{P}(F_y)} \equiv 0$. (In a modern terminology, the multiplier ideal sheaf $\mathcal{I}(\pi^{-1}(W_y), e^{-\psi})$ is the defining ideal sheaf $\mathcal{I}_{\mathbb{P}(F_y)}$ of the fiber.) Then $\widetilde{\sigma} := (\pi^*\rho)\sigma' - v \in H^0(Z, K_Z \otimes L)$ and $\widetilde{\sigma}|_{\mathbb{P}(F_y)} = \sigma'|_{\mathbb{P}(F_y)} = \sigma$.

Let us see the integrability $\int_Z |u|^2_{c(L,\psi)} e^{-\psi} dV < \infty$. Because $\operatorname{Supp} u \subset \pi^{-1}(W_y \setminus W'_y)$, and ψ is smooth on $\pi^{-1}(W_y \setminus W'_y)$, it is enough to check that $|u|^2_{c(L,\psi)} < \infty$ on $\pi^{-1}(W_y \setminus W'_y)$. Let us take $z_0 \in Z$ such that $y_0 = \pi(z_0) \in W_y \setminus W'_y$. Let $(U, (z^1, \ldots, z^p))$ be a local coordinate centered at z_0 such that $dz^1, \ldots, dz^p$ form an orthonormal basis of Ω^1_Z at z_0 so that $\omega_Z = \frac{\sqrt{-1}}{2} \sum_{i=1}^p dz^i \wedge d\overline{z}^i$ at z_0. Let $(y^1, \ldots, y^m)$ be a local coordinate centered at y_0. We will use indexes i, j (resp. k, ℓ) for $1, \ldots, p$ of z^i (resp. $1, \ldots, m$ of y^k). We have $\pi^*(dy^k) = \sum_{i=1}^p c^k_i dz^i$ at z_0, where $c^k_i = \frac{\partial y^k}{\partial z^i}(z_0)$, and $\pi^*(\overline{\partial}\rho) = \sum_i (\sum_k \rho_{\overline{k}} \overline{c^k_i}) d\overline{z}^i$ at z_0, where $\rho_{\overline{k}} = \frac{\partial \rho}{\partial \overline{y}^k}(y_0)$. The canonical bundle K_Z is trivialized by $dz = dz^1 \wedge \ldots \wedge dz^p$. We take a nowhere vanishing section $e \in H^0(U, L)$ such that $h_L(e,e)(z_0) = 1$. Then we can write as $u = \pi^*(\overline{\partial}\rho) \wedge s dz \otimes e$ with some $s \in H^0(U, \mathcal{O}_Z)$. We write the curvature form of h_A as $\sqrt{-1}\Theta_A = \frac{\sqrt{-1}}{2} \sum_{k,\ell} a_{k\overline{\ell}} dy^k \wedge d\overline{y}^\ell$ at y_0. Then $\sqrt{-1}\pi^*\Theta_A = \frac{\sqrt{-1}}{2} \sum_{i,j} (\sum_{k,\ell} a_{k\overline{\ell}} c^k_i \overline{c^\ell_j}) dz^i \wedge d\overline{z}^j$ at z_0. Let $\beta \in \Omega^{p,1}_{Z,z_0} \otimes L_{z_0}$, which is written as $\beta = (\sum_i b_i dz \wedge d\overline{z}^i) \otimes e$. We set $b^k = \sum_i c^k_i b_i$ for $1 \leq k \leq m$. Since $\sqrt{-1}c(L,\psi) \geq \sqrt{-1}\pi^*\Theta_A$, we have $(\sqrt{-1}c(L,\psi)\Lambda\beta,\beta) \geq (\sqrt{-1}\pi^*\Theta_A\Lambda\beta,\beta) = 2^{p+1} \sum_{k,\ell} a_{k\overline{\ell}} b^k \overline{b^\ell}$. Let $\lambda_1 > 0$ be the smallest eigenvalue of the positive matrix $(a_{k\overline{\ell}})_{k,\ell}$. Then $\sum_{k,\ell} a_{k\overline{\ell}} b^k \overline{b^\ell} \geq \lambda_1 \sum_k |b^k|^2$. On the other hand, we have $|(u,\beta)|^2 = |(\sum_i (\sum_k \rho_{\overline{k}} \overline{c^k_i}) d\overline{z}^i \wedge s dz \otimes e, (\sum_i b_i dz \wedge d\overline{z}^i) \otimes e)|^2 = (2^{p+1})^2 |s|^2 |\sum_k \rho_{\overline{k}} \overline{b^k}|^2$, and hence $|(u,\beta)|^2 \leq (2^{p+1})^2 |s|^2 \sum_k |\rho_{\overline{k}}|^2 \sum_k |\overline{b^k}|^2$. Then $|(u,\beta)|^2 \leq 2^{p+1} \lambda_1^{-1} |s|^2 \sum_k |b^k|^2 (\sqrt{-1}c(L,\psi)\Lambda\beta,\beta)$. We finally have

$$|u|^2_{c(L,\psi)}(z_0) \leq 2^{p+1} \lambda_1^{-1} |s|^2 \sum_k |b^k|^2 < \infty.$$

Then the proof is complete. q.e.d.

6.2. Variants. We shall give some variants of the results in the introduction. In Theorem 1.2 (1), we need to restrict ourselves on a relatively compact subset $Y_0 \subset Y$ (see the proof of Lemma 2.5 for the reason). We remove it in some cases.

Variant 6.1. *Let $f : X \longrightarrow Y$ be a proper surjective morphism with connected fibers between smooth algebraic varieties, and let (E, h) be a Nakano semi-positive holomorphic vector bundle on X. Then the line bundle $\mathcal{O}(1)$ for $\pi : \mathbb{P}(R^q f_* \left(K_{X/Y} \otimes E\right)|_{Y \setminus S_q}) \longrightarrow Y \setminus S_q$ has a singular Hermitian metric with semi-positive curvature, and which is smooth on $\pi^{-1}(Y \setminus \Delta')$ for a closed algebraic subset $\Delta' \subsetneq Y$.*

Proof. By Chow lemma [**15**, II.Ex.4.10], there exists a modification $\mu : X' \longrightarrow X$ from a smooth algebraic variety X' such that $f' := f \circ \mu : X' \longrightarrow Y$ becomes projective. Moreover by Hironaka, we may assume $\operatorname{Supp} f'^{-1}(\Delta')$ is simple normal crossing. Here $\Delta' \subset Y$ is the discriminant locus of f', which Δ' may be larger than Δ for f. Since a projective morphism is Kähler ([**35**, 6.2.i]), we can take a relative Kähler form $\omega_{f'}$ for f'. We then have a Hodge metric on $R^q f_* \left(K_{X/Y} \otimes E\right)|_{Y \setminus \Delta'}$ with respect to $\omega_{f'}$ and $\mu^* h$. The rest of the proof is the same as Theorem 1.2, after Proposition 2.6. q.e.d.

Variant 6.2. *Let $f : X \longrightarrow Y$ and (E, h) be as in Set up 1.1, and let $q = 0$. Then, the line bundle $\mathcal{O}(1)$ for $\pi : \mathbb{P}(f_*(K_{X/Y} \otimes E)|_{Y \setminus S_0}) \longrightarrow Y \setminus S_0$ has a singular Hermitian metric $g_{\mathcal{O}(1)}$ with semi-positive curvature, and whose restriction on $\pi^{-1}(Y \setminus \Delta)$ is the quotient metric $g^{\circ}_{\mathcal{O}(1)}$ of $\pi^* g$, where g is the Hodge metric with respect to h.*

Proof. In case $q = 0$, we have the Hodge metric g on $f_*(K_{X/Y} \otimes E)|_{Y \setminus \Delta}$ with respect to h, which does not depend on a relative Kähler form. This Hodge metric does not change, even if we take a modification $\mu : X' \longrightarrow X$ (more precisely, for any relatively compact open subset $Y_0 \subset Y$ and a modification $\mu : X'_0 \longrightarrow X_0 = f^{-1}(Y_0)$) which is biholomorphic over $X \setminus f^{-1}(\Delta)$. Once a global metric is obtained, the extension problem is a local issue. Hence it is reduced to see that on every small coordinate neighbourhood $Y_0 \subset Y$, $g^{\circ}_{\mathcal{O}(1)}|_{\pi^{-1}(Y_0 \setminus \Delta)}$ extends as a singular Hermitian metric on $\mathcal{O}(1)|_{\pi^{-1}(Y_0)}$ with semi-positive curvature. As we saw in the proof of Theorem 1.2, this is reduced to Theorem 2.4 (or Theorem 1.2 itself). q.e.d.

We have the following standard consequence of our theorems. Corollary 6.3 can be also formulated under other assumptions as in two variants above. We leave it for the readers.

Corollary 6.3. *Let $f : X \longrightarrow Y$, (E,h) and $0 \leq q \leq n$ be as in Set up* 1.1. *Let L be a holomorphic line bundle on Y with a surjection $R^q f_* \big(K_{X/Y} \otimes E\big)|_{Y\setminus Z} \longrightarrow L|_{Y\setminus Z}$ on the complement of a closed analytic subset $Z \subset Y$ of* $\operatorname{codim}_Y Z \geq 2$.

(1) Unpolarized case. For every relatively compact open subset $Y_0 \subset Y$, $L|_{Y_0}$ has a singular Hermitian metric with semi-positive curvature.

(2) Polarized case. Assume the simple normal crossing condition in Theorem 1.2 *(2), and let ω_f be a relative Kähler form for f. Then L has a singular Hermitian metric with semi-positive curvature, whose restriction on $Y \setminus \Delta$ is the quotient metric of the Hodge metric g on $R^q f_* \big(K_{X/Y} \otimes E\big)|_{Y\setminus\Delta}$ with respect to ω_f and h.*

Proof. (1) Denote by $F = R^q f_* \big(K_{X/Y} \otimes E\big)$. We put a Hermitian metric g on $F|_{Y_0\setminus\Delta}$ as in Proposition 2.6. Assume for the moment $S_q = Z = \emptyset$. Then the line bundle L corresponds to a section $s : Y \longrightarrow \mathbb{P}(F)$ of $\pi : \mathbb{P}(F) \longrightarrow Y$ such that $L \cong s^*\mathcal{O}(1)$. Moreover the Hodge metric g on $F|_{Y_0\setminus\Delta}$ induces a quotient metric g_L° (resp. $g_{\mathcal{O}(1)}^\circ$) of $L|_{Y_0\setminus\Delta}$ by quotient $F \longrightarrow L$ (resp. $\mathcal{O}(1)|_{\pi^{-1}(Y\setminus\Delta)}$ by $\pi^*F \longrightarrow \mathcal{O}(1)$), and $g_L^\circ = s^* g_{\mathcal{O}(1)}^\circ$ over $Y_0 \setminus \Delta$ by the definition. Let $g_{\mathcal{O}(1)}$ be the extension of $g_{\mathcal{O}(1)}^\circ$ as a singular Hermitian metric on $\mathcal{O}(1)|_{\pi^{-1}(Y_0)}$ with semi-positive curvature. Then $g_L = s^* g_{\mathcal{O}(1)}$ over Y_0 is a (unique) extension of g_L° with semi-positive curvature.

In case $S_q \cup Z$ may not be empty, by virtue of Hartogs type extension as in the proof of Theorem 1.2, we can extend further the singular Hermitian metric g_L on $L|_{Y_0\setminus(S_q\cup Z)}$ with semi-positive curvature as a singular Hermitian metric on $L|_{Y_0}$ with semi-positive curvature. (2) is similar. q.e.d.

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