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A BOUNDARY VALUE PROBLEM FOR MINIMAL LAGRANGIAN GRAPHS

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Abstract

Let Ω and $\tilde{\Omega}$ be uniformly convex domains in $\mathbb{R}^n$ with smooth boundary. We show that there exists a diffeomorphism $f : \Omega \to \tilde{\Omega}$ such that the graph $\Sigma = \{(x, f(x)) : x \in \Omega\}$ is a minimal Lagrangian submanifold of $\mathbb{R}^n \times \mathbb{R}^n$.

1. Introduction

Consider the product $\mathbb{R}^n \times \mathbb{R}^n$ equipped with the Euclidean metric. The product $\mathbb{R}^n \times \mathbb{R}^n$ has a natural complex structure, which is given by

$$J \frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k}, \quad J \frac{\partial}{\partial y_k} = -\frac{\partial}{\partial x_k}.$$

The associated symplectic structure is given by

$$\omega = \sum_{k=1}^n dx_k \wedge dy_k.$$

A submanifold $\Sigma \subset \mathbb{R}^n \times \mathbb{R}^n$ is called Lagrangian if $\omega|_\Sigma = 0$.

In this paper, we study a boundary value problem for minimal Lagrangian graphs in $\mathbb{R}^n \times \mathbb{R}^n$. To that end, we fix two domains $\Omega, \tilde{\Omega} \subset \mathbb{R}^n$ with smooth boundary. Given a diffeomorphism $f : \Omega \to \tilde{\Omega}$, we consider its graph $\Sigma = \{(x, f(x)) : x \in \Omega\} \subset \mathbb{R}^n \times \mathbb{R}^n$. We consider the problem of finding a diffeomorphism $f : \Omega \to \tilde{\Omega}$ such that Σ is Lagrangian and has zero mean curvature. Our main result asserts that such a map exists if Ω and $\tilde{\Omega}$ are uniformly convex:

Theorem 1.1. *Let Ω and $\tilde{\Omega}$ be uniformly convex domains in $\mathbb{R}^n$ with smooth boundary. Then there exists a diffeomorphism $f : \Omega \to \tilde{\Omega}$ such that the graph*

$$\Sigma = \{(x, f(x)) : x \in \Omega\}$$

is a minimal Lagrangian submanifold of $\mathbb{R}^n \times \mathbb{R}^n$.

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Minimal Lagrangian submanifolds were first studied by Harvey and Lawson [**6**], and have attracted considerable interest in recent years. Yuan [**14**] has proved a Bernstein-type theorem for minimal Lagrangian graphs over $\mathbb{R}^n$. A similar result was established by Tsui and Wang [**10**]. Smoczyk and Wang have used the mean curvature flow to deform certain Lagrangian submanifolds to minimal Lagrangian submanifolds (see [**8**], [**9**], [**13**]). In [**1**], the first author studied a boundary value problem for minimal Lagrangian graphs in $\mathbb{H}^2 \times \mathbb{H}^2$, where $\mathbb{H}^2$ denotes the hyperbolic plane.

In order to prove Theorem 1.1, we reduce the problem to the solvability of a fully nonlinear PDE. As above, we assume that Ω and $\tilde{\Omega}$ are uniformly convex domains in $\mathbb{R}^n$ with smooth boundary. Moreover, suppose that f is a diffeomorphism from Ω to $\tilde{\Omega}$. The graph $\Sigma = \{(x, f(x)) : x \in \Omega\}$ is Lagrangian if and only if there exists a function $u : \Omega \to \mathbb{R}$ such that $f(x) = \nabla u(x)$. In that case, the Lagrangian angle of Σ is given by $F(D^2 u(x))$. Here, F is a real-valued function on the space of symmetric $n \times n$ matrices which is defined as follows: if M is a symmetric $n \times n$ matrix, then $F(M)$ is defined by

$$F(M) = \sum_{k=1}^{n} \arctan(\lambda_k),$$

where $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of M.

By a result of Harvey and Lawson (see [**6**], Proposition 2.17), Σ has zero mean curvature if and only if the Lagrangian angle is constant; that is,

$$F(D^2 u(x)) = c \tag{1}$$

for all $x \in \Omega$. Hence, we are led to the following problem:

($\star$) *Find a convex function $u : \Omega \to \mathbb{R}$ and a constant $c \in (0, \frac{n\pi}{2})$ such that ∇u is a diffeomorphism from Ω to $\tilde{\Omega}$ and $F(D^2 u(x)) = c$ for all $x \in \Omega$.*

Caffarelli, Nirenberg, and Spruck [**3**] have obtained an existence result for solutions of (1) under Dirichlet boundary conditions. In this paper, we study a different boundary condition, which is analogous to the second boundary value problem for the Monge-Ampère equation.

In dimension 2, P. Delanoë [**4**] proved that the second boundary value problem for the Monge-Ampère equation has a unique smooth solution, provided that both domains are uniformly convex. This result was generalized to higher dimensions by L. Caffarelli [**2**] and J. Urbas [**11**]. In 2001, J. Urbas [**12**] described a general class of Hessian equations for which the second boundary value problem admits a unique smooth solution.

In Section 2, we establish a-priori estimates for solutions of ($\star$). In Section 3, we prove that all solutions of ($\star$) are non-degenerate (that is, the linearized operator is invertible). In Section 4, we use the continuity method to show that ($\star$) has at least one solution. From this, Theorem 1.1 follows. Finally, in Section 5, we prove a uniqueness result for ($\star$).

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2. A priori estimates for solutions of ($\star$)

In this section, we prove a-priori estimates for solutions of ($\star$).

Let Ω and $\tilde{\Omega}$ be uniformly convex domains in $\mathbb{R}^n$ with smooth boundary. Moreover, suppose that u is a convex function such that ∇u is a diffeomorphism from Ω to $\tilde{\Omega}$ and $F(D^2u(x))$ is constant. For each point $x \in \Omega$, we define a symmetric $n \times n$-matrix $A(x) = \{a_{ij}(x) : 1 \leq i,j \leq n\}$ by

$$A(x) = \big[I + (D^2u(x))^2\big]^{-1}.$$

Clearly, $A(x)$ is positive definite for all $x \in \Omega$.

Lemma 2.1. *We have*

$$\frac{n\pi}{2} - F(D^2u(x)) \geq \arctan\left(\frac{\mathrm{vol}(\Omega)^{1/n}}{\mathrm{vol}(\tilde{\Omega})^{1/n}}\right)$$

for all points $x \in \Omega$.

Proof. Since ∇u is a diffeomorphism from Ω to $\tilde{\Omega}$, we have

$$\int_\Omega \det D^2u(x)\,dx = \mathrm{vol}(\tilde{\Omega}).$$

Therefore, we can find a point $x_0 \in \Omega$ such that

$$\det D^2u(x_0) \leq \frac{\mathrm{vol}(\tilde{\Omega})}{\mathrm{vol}(\Omega)}.$$

Hence, if we denote by $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ the eigenvalues of $D^2u(x_0)$, then we have

$$\lambda_1 \leq \frac{\mathrm{vol}(\tilde{\Omega})^{1/n}}{\mathrm{vol}(\Omega)^{1/n}}.$$

This implies

$$\begin{aligned}\frac{n\pi}{2} - F(D^2u(x_0)) &= \sum_{k=1}^n \arctan\left(\frac{1}{\lambda_k}\right)\\ &\geq \arctan\left(\frac{1}{\lambda_1}\right)\\ &\geq \arctan\left(\frac{\mathrm{vol}(\Omega)^{1/n}}{\mathrm{vol}(\tilde{\Omega})^{1/n}}\right).\end{aligned}$$

Since $F(D^2u(x))$ is constant, the assertion follows. q.e.d.

Lemma 2.2. *Let x be an arbitrary point in Ω, and let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of $D^2u(x)$. Then*

$$\frac{1}{\lambda_1} \geq \tan\left[\frac{1}{n} \arctan\left(\frac{\mathrm{vol}(\Omega)^{1/n}}{\mathrm{vol}(\tilde{\Omega})^{1/n}}\right)\right].$$

Proof. Using Lemma 2.1, we obtain

$$\begin{aligned} n \arctan\left(\frac{1}{\lambda_1}\right) &\geq \sum_{k=1}^n \arctan\left(\frac{1}{\lambda_k}\right) \\ &= \frac{n\pi}{2} - F(D^2u(x)) \\ &\geq \arctan\left(\frac{\mathrm{vol}(\Omega)^{1/n}}{\mathrm{vol}(\tilde{\Omega})^{1/n}}\right). \end{aligned}$$

From this, the assertion follows easily. q.e.d.

By Proposition A.1, we can find a smooth function $h : \Omega \to \mathbb{R}$ such that $h(x) = 0$ for all $x \in \partial\Omega$ and

$$\sum_{i,j=1}^n \partial_i \partial_j h(x)\, w_i\, w_j \geq \theta\, |w|^2 \tag{2}$$

for all $x \in \Omega$ and all $w \in \mathbb{R}^n$. Similarly, there exists a smooth function $\tilde{h} : \tilde{\Omega} \to \mathbb{R}$ such that $\tilde{h}(y) = 0$ for all $y \in \partial\tilde{\Omega}$ and

$$\sum_{i,j=1}^n \partial_i \partial_j \tilde{h}(y)\, w_i\, w_j \geq \theta\, |w|^2 \tag{3}$$

for all $y \in \tilde{\Omega}$ and all $w \in \mathbb{R}^n$. For abbreviation, we choose a positive constant C_1 such that

$$C_1\, \theta\, \sin^2\left[\frac{1}{n} \arctan\left(\frac{\mathrm{vol}(\Omega)^{1/n}}{\mathrm{vol}(\tilde{\Omega})^{1/n}}\right)\right] = 1.$$

We then have the following estimate:

Lemma 2.3. *We have*

$$\sum_{i,j=1}^n a_{ij}(x)\, \partial_i \partial_j h(x) \geq \frac{1}{C_1}$$

for all $x \in \Omega$.

Proof. Fix a point $x_0 \in \Omega$, and let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of $D^2u(x_0)$. It follows from (2) that

$$\sum_{i,j=1}^n a_{ij}(x_0)\, \partial_i \partial_j h(x_0) \geq \theta \sum_{k=1}^n \frac{1}{1+\lambda_k^2} \geq \theta\, \frac{1}{1+\lambda_1^2}.$$

Using Lemma 2.2, we obtain

$$\frac{1}{1+\lambda_1^2} \geq \sin^2\left[\frac{1}{n}\arctan\left(\frac{\mathrm{vol}(\Omega)^{1/n}}{\mathrm{vol}(\tilde{\Omega})^{1/n}}\right)\right] = \frac{1}{C_1\,\theta}.$$

Putting these facts together, the assertion follows. q.e.d.

In the next step, we differentiate the identity $F(D^2u(x)) = \text{constant}$ with respect to x. To that end, we need the following well-known fact:

Lemma 2.4. *Let $M(t)$ be a smooth one-parameter family of symmetric $n \times n$ matrices. Then*

$$\frac{d}{dt}F(M(t))\Big|_{t=0} = \mathrm{tr}\big[(I + M(0)^2)^{-1}\,M'(0)\big].$$

Moreover, if $M(0)$ is positive definite, then we have

$$\frac{d^2}{dt^2}F(M(t))\Big|_{t=0} \leq \mathrm{tr}\big[(I + M(0)^2)^{-1}\,M''(0)\big].$$

Proof. The first statement follows immediately from the definition of F. To prove the second statement, we observe that

$$\begin{aligned}\frac{d^2}{dt^2}F(M(t))\Big|_{t=0} &= \mathrm{tr}\big[(I + M(0)^2)^{-1}\,M''(0)\big] \\ &\quad - 2\,\mathrm{tr}\big[M(0)\,(I + M(0)^2)^{-1}\,M'(0)\,(I + M(0)^2)^{-1}\,M'(0)\big].\end{aligned}$$

Since $M(0)$ is positive definite and $M'(0)$ is symmetric, we have

$$\mathrm{tr}\big[M(0)\,(I + M(0)^2)^{-1}\,M'(0)\,(I + M(0)^2)^{-1}\,M'(0)\big] \geq 0.$$

Putting these facts together, the assertion follows. q.e.d.

Proposition 2.5. *We have*

$$\sum_{i,j=1}^n a_{ij}(x)\,\partial_i\partial_j\partial_k u(x) = 0 \tag{4}$$

for all $x \in \Omega$. Moreover, we have

$$\sum_{i,j,k,l=1}^n a_{ij}(x)\,\partial_i\partial_j\partial_k\partial_l u(x)\,w_k\,w_l \geq 0 \tag{5}$$

for all $x \in \Omega$ and all $w \in \mathbb{R}^n$.

Proof. Fix a point $x_0 \in \Omega$ and a vector $w \in \mathbb{R}^n$. It follows from Lemma 2.4 that

$$0 = \frac{d}{dt}F\big(D^2u(x_0 + tw)\big)\Big|_{t=0} = \sum_{i,j,k=1}^n a_{ij}(x)\,\partial_i\partial_j\partial_k u(x_0)\,w_k.$$

Moreover, since the matrix $D^2u(x_0)$ is positive definite, we have

$$0 = \frac{d^2}{dt^2}F\big(D^2u(x_0 + tw)\big)\Big|_{t=0} \leq \sum_{i,j,k,l=1}^n a_{ij}(x)\,\partial_i\partial_j\partial_k\partial_l u(x_0)\,w_k\,w_l.$$

From this, the assertion follows. q.e.d.

Proposition 2.6. *Fix a smooth function $\Phi : \Omega \times \tilde{\Omega} \to \mathbb{R}$, and define $\varphi(x) = \Phi(x, \nabla u(x))$. Then*

$$\Big| \sum_{i,j=1}^n a_{ij}(x)\, \partial_i \partial_j \varphi(x) \Big| \leq C$$

for all $x \in \Omega$. Here, C is a positive constant that depends only on the second order partial derivatives of Φ.

Proof. The partial derivatives of the function $\varphi(x)$ are given by

$$\partial_i \varphi(x) = \sum_{k=1}^n \Big(\frac{\partial}{\partial y_k} \Phi \Big)(x, \nabla u(x))\, \partial_i \partial_k u(x) + \Big(\frac{\partial}{\partial x_i} \Phi \Big)(x, \nabla u(x)).$$

This implies

$$\begin{aligned} \partial_i \partial_j \varphi(x) &= \sum_{k=1}^n \Big(\frac{\partial}{\partial y_k} \Phi \Big)(x, \nabla u(x))\, \partial_i \partial_j \partial_k u(x) \\ &+ \sum_{k,l=1}^n \Big(\frac{\partial^2}{\partial y_k \partial y_l} \Phi \Big)(x, \nabla u(x))\, \partial_i \partial_k u(x)\, \partial_j \partial_l u(x) \\ &+ \sum_{k=1}^n \Big(\frac{\partial^2}{\partial x_j \partial y_k} \Phi \Big)(x, \nabla u(x))\, \partial_i \partial_k u(x) \\ &+ \sum_{l=1}^n \Big(\frac{\partial^2}{\partial x_i \partial y_l} \Phi \Big)(x, \nabla u(x))\, \partial_j \partial_l u(x) \\ &+ \Big(\frac{\partial^2}{\partial x_i \partial x_j} \Phi \Big)(x, \nabla u(x)). \end{aligned}$$

Using (4), we obtain

$$\begin{aligned} &\sum_{i,j=1}^n a_{ij}(x)\, \partial_i \partial_j \varphi(x) \\ &= \sum_{i,j,k,l=1}^n a_{ij}(x) \Big(\frac{\partial^2}{\partial y_k \partial y_l} \Phi \Big)(x, \nabla u(x))\, \partial_i \partial_k u(x)\, \partial_j \partial_l u(x) \\ &+ 2 \sum_{i,j,k=1}^n a_{ij}(x) \Big(\frac{\partial^2}{\partial x_j \partial y_k} \Phi \Big)(x, \nabla u(x))\, \partial_i \partial_k u(x) \\ &+ \sum_{i,j=1}^n a_{ij}(x) \Big(\frac{\partial^2}{\partial x_i \partial x_j} \Phi \Big)(x, \nabla u(x)). \end{aligned}$$

We now fix a point $x_0 \in \Omega$. Without loss of generality, we may assume that $D^2u(x_0)$ is a diagonal matrix. This implies

$$\begin{aligned}\sum_{i,j=1}^n a_{ij}(x_0)\,\partial_i\partial_j\varphi(x_0) &= \sum_{k=1}^n \frac{\lambda_k^2}{1+\lambda_k^2}\Big(\frac{\partial^2}{\partial y_k^2}\Phi\Big)(x_0,\nabla u(x_0)) \\ &\quad + 2\sum_{k=1}^n \frac{\lambda_k}{1+\lambda_k^2}\Big(\frac{\partial^2}{\partial x_k\partial y_k}\Phi\Big)(x_0,\nabla u(x_0)) \\ &\quad + \sum_{k=1}^n \frac{1}{1+\lambda_k^2}\Big(\frac{\partial^2}{\partial x_k^2}\Phi\Big)(x_0,\nabla u(x_0)),\end{aligned}$$

where $\lambda_k = \partial_k\partial_k u(x_0)$. Thus, we conclude that

$$\Big|\sum_{i,j=1}^n a_{ij}(x_0)\,\partial_i\partial_j\varphi(x_0)\Big| \leq C,$$

as claimed. q.e.d.

We next consider the function $H(x) = \tilde{h}(\nabla u(x))$. The following estimate is an immediate consequence of Proposition 2.6:

Corollary 2.7. *There exists a positive constant C_2 such that*

$$\Big|\sum_{i,j=1}^n a_{ij}(x)\,\partial_i\partial_j H(x)\Big| \leq C_2$$

for all $x \in \Omega$.

Proposition 2.8. *We have $H(x) \geq C_1C_2\,h(x)$ for all $x \in \Omega$.*

Proof. Using Lemma 2.3 and Corollary 2.7, we obtain

$$\sum_{i,j=1}^n a_{ij}(x)\,\partial_i\partial_j(H(x) - C_1C_2\,h(x)) \leq 0$$

for all $x \in \Omega$. Hence, the function $H(x) - C_1C_2\,h(x)$ attains its minimum on $\partial\Omega$. Thus, we conclude that $H(x) - C_1C_2\,h(x) \geq 0$ for all $x \in \Omega$. q.e.d.

Corollary 2.9. *We have*

$$\langle \nabla h(x), \nabla H(x)\rangle \leq C_1C_2\,|\nabla h(x)|^2$$

for all $x \in \partial\Omega$.

Proposition 2.10. *Fix a smooth function $\Phi : \Omega \times \tilde{\Omega} \to \mathbb{R}$, and define $\varphi(x) = \Phi(x,\nabla u(x))$. Then*

$$|\langle \nabla\varphi(x), \nabla\tilde{h}(\nabla u(x))\rangle| \leq C$$

for all $x \in \partial\Omega$. Here, C is a positive constant that depends only on C_1, C_2, and the first order partial derivatives of Φ.

Proof. A straightforward calculation yields

$$\begin{aligned}\langle \nabla\varphi(x), \nabla\tilde{h}(\nabla u(x))\rangle &= \sum_{k=1}^n \Big(\frac{\partial}{\partial x_k}\Phi\Big)(x, \nabla u(x))\,(\partial_k\tilde{h})(\nabla u(x)) \\ &\quad + \sum_{k=1}^n \Big(\frac{\partial}{\partial y_k}\Phi\Big)(x, \nabla u(x))\,\partial_k H(x)\end{aligned}$$

for all $x \in \Omega$. By Corollary 2.9, we have $|\nabla H(x)| \leq C_1 C_2\,|\nabla h(x)|$ for all points $x \in \partial\Omega$. Putting these facts together, the assertion follows. q.e.d.

Proposition 2.11. *We have*

$$\begin{aligned}0 &< \sum_{k,l=1}^n \partial_k\partial_l u(x)\,(\partial_k\tilde{h})(\nabla u(x))\,(\partial_l\tilde{h})(\nabla u(x)) \\ &\leq C_1 C_2\,\langle \nabla h(x), \nabla\tilde{h}(\nabla u(x))\rangle\end{aligned}$$

for all $x \in \partial\Omega$.

Proof. Note that the function H vanishes along $\partial\Omega$ and is negative in the interior of Ω. Hence, for each point $x \in \partial\Omega$, the vector $\nabla H(x)$ is a positive multiple of $\nabla h(x)$. Since u is convex, we obtain

$$\begin{aligned}0 &< \sum_{k,l=1}^n \partial_k\partial_l u(x)\,(\partial_k\tilde{h})(\nabla u(x))\,(\partial_l\tilde{h})(\nabla u(x)) \\ &= \langle \nabla H(x), \nabla\tilde{h}(\nabla u(x))\rangle \\ &= \frac{\langle \nabla h(x), \nabla H(x)\rangle}{|\nabla h(x)|^2}\,\langle \nabla h(x), \nabla\tilde{h}(\nabla u(x))\rangle\end{aligned}$$

for all $x \in \partial\Omega$. In particular, we have $\langle h(x), \nabla\tilde{h}(\nabla u(x))\rangle > 0$ for all points $x \in \partial\Omega$. The assertion follows now from Corollary 2.9. q.e.d.

Proposition 2.12. *There exists a positive constant* C_4 *such that*

$$\langle \nabla h(x), \nabla\tilde{h}(\nabla u(x))\rangle \geq \frac{1}{C_4}$$

for all $x \in \partial\Omega$.

Proof. We define a function $\chi(x)$ by

$$\chi(x) = \langle \nabla h(x), \nabla\tilde{h}(\nabla u(x))\rangle.$$

By Proposition 2.6, we can find a positive constant C_3 such that

$$\Big|\sum_{i,j=1}^n a_{ij}(x)\,\partial_i\partial_j\chi(x)\Big| \leq C_3$$

for all $x \in \Omega$. Using Lemma 2.3, we obtain

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\partial_i\partial_j(\chi(x) - C_1C_3\,h(x)) \leq 0$$

for all $x \in \Omega$. Hence, there exists a point $x_0 \in \partial\Omega$ such that

$$\inf_{x\in\Omega}(\chi(x) - C_1C_3\,h(x)) = \inf_{x\in\partial\Omega}\chi(x) = \chi(x_0).$$

It follows from Proposition 2.11 that $\chi(x_0) > 0$. Moreover, we can find a nonnegative real number μ such that

$$\nabla\chi(x_0) = (C_1C_3 - \mu)\,\nabla h(x_0).$$

A straightforward calculation yields

$$\begin{aligned}\langle\nabla\chi(x), \nabla\tilde{h}(\nabla u(x))\rangle &= \sum_{i,j=1}^{n} \partial_i\partial_j h(x)\,(\partial_i\tilde{h})(\nabla u(x))\,(\partial_j\tilde{h})(\nabla u(x)) \\ &+ \sum_{i,j=1}^{n}(\partial_i\partial_j\tilde{h})(\nabla u(x))\,\partial_i h(x)\,\partial_j H(x) \qquad (6)\end{aligned}$$

for all $x \in \partial\Omega$. Using (2), we obtain

$$\sum_{i,j=1}^{n} \partial_i\partial_j h(x)\,(\partial_i\tilde{h})(\nabla u(x))\,(\partial_j\tilde{h})(\nabla u(x)) \geq \theta\,|\nabla\tilde{h}(\nabla u(x))|^2$$

for all $x \in \partial\Omega$. Since $\nabla H(x)$ is a positive multiple of $\nabla h(x)$, we have

$$\sum_{i,j=1}^{n}(\partial_i\partial_j\tilde{h})(\nabla u(x))\,\partial_i h(x)\,\partial_j H(x) \geq 0$$

for all $x \in \partial\Omega$. Substituting these inequalities into (6) gives

$$\langle\nabla\chi(x), \nabla\tilde{h}(\nabla u(x))\rangle \geq \theta\,|\nabla\tilde{h}(\nabla u(x))|^2$$

for all $x \in \partial\Omega$. From this, we deduce that

$$\begin{aligned}(C_1C_3 - \mu)\,\chi(x_0) &= (C_1C_3 - \mu)\,\langle\nabla h(x_0), \nabla\tilde{h}(\nabla u(x_0))\rangle \\ &= \langle\nabla\chi(x_0), \nabla\tilde{h}(\nabla u(x_0))\rangle \\ &\geq \theta\,|\nabla\tilde{h}(\nabla u(x_0))|^2.\end{aligned}$$

Since $\mu \geq 0$ and $\chi(x_0) > 0$, we conclude that

$$\chi(x_0) \geq \frac{\theta}{C_1C_3}\,|\nabla\tilde{h}(\nabla u(x_0))|^2 \geq \frac{1}{C_4}$$

for some positive constant C_4. This completes the proof of Proposition 2.12. q.e.d.

Lemma 2.13. *Suppose that*

$$\sum_{k,l=1}^n \partial_k \partial_l u(x)\, w_k\, w_l \leq M\, |w|^2$$

for all $x \in \partial\Omega$ *and all* $w \in T_x(\partial\Omega)$. *Then*

$$\sum_{k,l=1}^n \partial_k \partial_l u(x)\, w_k\, w_l \leq M \left| w - \frac{\langle \nabla h(x), w\rangle}{\langle \nabla h(x), \nabla \tilde{h}(\nabla u(x))\rangle}\, \nabla \tilde{h}(\nabla u(x)) \right|^2 + C_1 C_2 C_4\, \langle \nabla h(x), w\rangle^2$$

for all $x \in \partial\Omega$ *and all* $w \in \mathbb{R}^n$.

Proof. Fix a point $x \in \partial\Omega$ and a vector $w \in \mathbb{R}^n$. Moreover, let

$$z = w - \frac{\langle \nabla h(x), w\rangle}{\langle \nabla h(x), \nabla \tilde{h}(\nabla u(x))\rangle}\, \nabla \tilde{h}(\nabla u(x)).$$

Clearly, $\langle \nabla h(x), z\rangle = 0$; hence $z \in T_x(\partial\Omega)$. This implies

$$\sum_{k,l=1}^n \partial_k \partial_l u(x)\, (\partial_k \tilde{h})(\nabla u(x))\, z_l = \langle \nabla H(x), z\rangle = 0.$$

From this we deduce that

$$\begin{aligned}&\sum_{k,l=1}^n \partial_k \partial_l u(x)\, w_k\, w_l - \sum_{k,l=1}^n \partial_k \partial_l u(x)\, z_k\, z_l \\ &= \frac{\langle \nabla h(x), w\rangle^2}{\langle \nabla h(x), \nabla \tilde{h}(\nabla u(x))\rangle^2} \sum_{k,l=1}^n \partial_k \partial_l u(x)\, (\partial_k \tilde{h})(\nabla u(x))\, (\partial_l \tilde{h})(\nabla u(x)).\end{aligned}$$

It follows from Proposition 2.11 and Proposition 2.12 that

$$\begin{aligned}&\frac{\langle \nabla h(x), w\rangle^2}{\langle \nabla h(x), \nabla \tilde{h}(\nabla u(x))\rangle^2} \sum_{k,l=1}^n \partial_k \partial_l u(x)\, (\partial_k \tilde{h})(\nabla u(x))\, (\partial_l \tilde{h})(\nabla u(x)) \\ &\leq C_1 C_2\, \frac{\langle \nabla h(x), w\rangle^2}{\langle \nabla h(x), \nabla \tilde{h}(\nabla u(x))\rangle} \leq C_1 C_2 C_4\, \langle \nabla h(x), w\rangle^2.\end{aligned}$$

Moreover, we have

$$\sum_{k,l=1}^n \partial_k \partial_l u(x)\, z_k\, z_l \leq M\, |z|^2$$

by definition of M. Putting these facts together, the assertion follows. q.e.d.

Proposition 2.14. *There exists a positive constant* C_9 *such that*

$$\sum_{k,l=1}^n \partial_k \partial_l u(x)\, w_k\, w_l \leq C_9\, |w|^2$$

for all $x \in \partial\Omega$ and all $w \in T_x(\partial\Omega)$.

Proof. Let

$$M = \sup \left\{ \sum_{k,l=1}^n \partial_k \partial_l u(x)\, z_k\, z_l : x \in \partial\Omega,\ z \in T_x(\partial\Omega),\ |z| = 1 \right\}.$$

By compactness, we can find a point $x_0 \in \partial\Omega$ and a unit vector $w \in T_{x_0}(\partial\Omega)$ such that

$$\sum_{k,l=1}^n \partial_k \partial_l u(x_0)\, w_k\, w_l = M.$$

We define a function $\psi : \Omega \to \mathbb{R}$ by

$$\psi(x) = \sum_{k,l=1}^n \partial_k \partial_l u(x)\, w_k\, w_l$$

for all $x \in \Omega$. Moreover, we define functions $\varphi_1 : \Omega \to \mathbb{R}$ and $\varphi_2 : \Omega \to \mathbb{R}$ by

$$\varphi_1(x) = \left| w - \frac{\langle \nabla h(x), w \rangle}{\eta(\langle \nabla h(x), \nabla \tilde{h}(\nabla u(x)) \rangle)}\, \nabla \tilde{h}(\nabla u(x)) \right|^2$$

and

$$\varphi_2(x) = \langle \nabla h(x), w \rangle^2$$

for all $x \in \Omega$. Here, $\eta : \mathbb{R} \to \mathbb{R}$ is a smooth cutoff function satisfying $\eta(s) = s$ for $s \geq \frac{1}{C_4}$ and $\eta(s) \geq \frac{1}{2C_4}$ for all $s \in \mathbb{R}$.

The inequality (5) implies that

$$\sum_{i,j=1}^n a_{ij}(x)\, \partial_i \partial_j \psi(x) \geq 0$$

for all $x \in \Omega$. Moreover, by Proposition 2.6, there exists a positive constant C_5 such that

$$\left| \sum_{i,j=1}^n a_{ij}(x)\, \partial_i \partial_j \varphi_1(x) \right| \leq C_5$$

and

$$\left| \sum_{i,j=1}^n a_{ij}(x)\, \partial_i \partial_j \varphi_2(x) \right| \leq C_5$$

for all $x \in \Omega$. Hence, the function

$$\begin{aligned} g(x) = \psi(x) &- M\, \varphi_1(x) - C_1 C_2 C_4\, \varphi_2(x) \\ &+ C_1 C_5\, (M + C_1 C_2 C_4)\, h(x) \end{aligned}$$

satisfies

$$\sum_{i,j=1}^n a_{ij}(x)\, \partial_i \partial_j g(x) \geq 0 \tag{7}$$

for all $x \in \Omega$.

It follows from Proposition 2.12 that

$$\varphi_1(x) = \left| w - \frac{\langle \nabla h(x), w \rangle}{\langle \nabla h(x), \nabla \tilde{h}(\nabla u(x)) \rangle} \, \nabla \tilde{h}(\nabla u(x)) \right|^2$$

for all $x \in \partial\Omega$. Using Lemma 2.13, we obtain

$$\psi(x) \leq M \, \varphi_1(x) + C_1 C_2 C_4 \, \varphi_2(x)$$

for all $x \in \partial\Omega$. Therefore, we have $g(x) \leq 0$ for all $x \in \partial\Omega$. Using the inequality (7) and the maximum principle, we conclude that $g(x) \leq 0$ for all $x \in \Omega$.

On the other hand, we have $\varphi_1(x_0) = 1$, $\varphi_2(x_0) = 0$, and $\psi(x_0) = M$. From this, we deduce that $g(x_0) = 0$. Therefore, the function g attains its global maximum at the point x_0. This implies $\nabla g(x_0) = \mu \, \nabla h(x_0)$ for some nonnegative real number μ. From this, we deduce that

$$\langle \nabla g(x_0), \nabla \tilde{h}(\nabla u(x_0)) \rangle = \mu \, \langle \nabla h(x_0), \nabla \tilde{h}(\nabla u(x_0)) \rangle \geq 0. \tag{8}$$

By Proposition 2.10, we can find a positive constant C_6 such that

$$|\langle \nabla \varphi_1(x), \nabla \tilde{h}(\nabla u(x)) \rangle| \leq C_6$$

for all $x \in \partial\Omega$. Hence, we can find positive constants C_7 and C_8 such that

$$\begin{aligned} \langle \nabla g(x), \nabla \tilde{h}(\nabla u(x)) \rangle &= \langle \nabla \psi(x), \nabla \tilde{h}(\nabla u(x)) \rangle \\ &\quad - M \, \langle \nabla \varphi_1(x), \nabla \tilde{h}(\nabla u(x)) \rangle \\ &\quad - C_1 C_2 C_4 \, \langle \nabla \varphi_2(x), \nabla \tilde{h}(\nabla u(x)) \rangle \\ &\quad + C_1 C_5 \, (M + C_1 C_2 C_4) \, \langle \nabla h(x), \nabla \tilde{h}(\nabla u(x)) \rangle \\ &\leq \langle \nabla \psi(x), \nabla \tilde{h}(\nabla u(x)) \rangle + C_7 \, M + C_8 \end{aligned} \tag{9}$$

for all $x \in \partial\Omega$. Combining (8) and (9), we conclude that

$$\langle \nabla \psi(x_0), \nabla \tilde{h}(\nabla u(x_0)) \rangle + C_7 \, M + C_8 \geq 0. \tag{10}$$

A straightforward calculation shows that

$$\begin{aligned} &\sum_{k,l=1}^n \partial_k \partial_l H(x_0) \, w_k \, w_l \\ &= \sum_{i,k,l=1}^n (\partial_i \tilde{h})(\nabla u(x_0)) \, \partial_i \partial_k \partial_l u(x_0) \, w_k \, w_l \\ &+ \sum_{i,j,k,l=1}^n (\partial_i \partial_j \tilde{h})(\nabla u(x_0)) \, \partial_i \partial_k u(x_0) \, \partial_j \partial_l u(x_0) \, w_k \, w_l. \end{aligned} \tag{11}$$

Since H vanishes along $\partial\Omega$, we have

$$\sum_{k,l=1}^{n} \partial_k \partial_l H(x_0)\, w_k\, w_l = -\langle \nabla H(x_0), I\!I(w,w)\rangle,$$

where $I\!I(\cdot,\cdot)$ denotes the second fundamental form of $\partial\Omega$ at x_0. Using the estimate $|\nabla H(x_0)| \leq C_1 C_2\, |\nabla h(x_0)|$, we obtain

$$\sum_{k,l=1}^{n} \partial_k \partial_l H(x_0)\, w_k\, w_l \leq C_1 C_2\, |\nabla h(x_0)|\, |I\!I(w,w)|.$$

Moreover, we have

$$\sum_{i,k,l=1}^{n} (\partial_i \tilde{h})(\nabla u(x_0))\, \partial_i \partial_k \partial_l u(x_0)\, w_k\, w_l = \langle \nabla\psi(x_0), \nabla\tilde{h}(\nabla u(x_0))\rangle.$$

Finally, it follows from (3) that

$$\begin{aligned} &\sum_{i,j,k,l=1}^{n} (\partial_i \partial_j \tilde{h})(\nabla u(x_0))\, \partial_i \partial_k u(x_0)\, \partial_j \partial_l u(x_0)\, w_k\, w_l \\ &\geq \theta \sum_{i,j,k,l=1}^{n} \partial_i \partial_k u(x_0)\, \partial_j \partial_l u(x_0)\, w_i\, w_j\, w_k\, w_l = \theta\, M^2. \end{aligned}$$

Substituting these inequalities into (11), we obtain

$$\begin{aligned} C_1 C_2\, |\nabla h(x_0)|\, |I\!I(w,w)| &\geq \sum_{k,l=1}^{n} \partial_k \partial_l H(x_0)\, w_k\, w_l \\ &\geq \langle \nabla\psi(x_0), \nabla\tilde{h}(\nabla u(x_0))\rangle + \theta\, M^2 \\ &\geq \theta\, M^2 - C_7\, M - C_8. \end{aligned}$$

Therefore, we have $M \leq C_9$ for some positive constant C_9. This completes the proof of Proposition 2.14. q.e.d.

Corollary 2.15. *There exists a positive constant C_{10} such that*

$$\sum_{k,l=1}^{n} \partial_k \partial_l u(x)\, w_k\, w_l \leq C_{10}\, |w|^2$$

for all $x \in \partial\Omega$ and all $w \in \mathbb{R}^n$.

Proof. It follows from Lemma 2.13 that

$$\begin{aligned} \sum_{k,l=1}^{n} \partial_k \partial_l u(x)\, w_k\, w_l \leq C_9 &\left| w - \frac{\langle \nabla h(x), w\rangle}{\langle \nabla h(x), \nabla\tilde{h}(\nabla u(x))\rangle}\, \nabla\tilde{h}(\nabla u(x)) \right|^2 \\ &+ C_1 C_2 C_4\, \langle \nabla h(x), w\rangle^2 \end{aligned}$$

for all $x \in \partial\Omega$ and all $w \in \mathbb{R}^n$. Hence, the assertion follows from Proposition 2.12. q.e.d.

Using Corollary 2.15 and (5), we obtain uniform bounds for the second derivatives of the function u:

Proposition 2.16. *We have*

$$\sum_{k,l=1}^{n} \partial_k \partial_l u(x)\, w_k\, w_l \leq C_{10}\, |w|^2$$

for all $x \in \Omega$ and all $w \in \mathbb{R}^n$.

Proof. Fix a unit vector $w \in \mathbb{R}^n$, and define

$$\psi(x) = \sum_{k,l=1}^{n} \partial_k \partial_l u(x)\, w_k\, w_l.$$

The inequality (5) implies that

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \partial_i \partial_j \psi(x) \geq 0$$

for all $x \in \Omega$. Using the maximum principle, we obtain

$$\sup_{x \in \Omega} \psi(x) = \sup_{x \in \partial\Omega} \psi(x) \leq C_{10}.$$

This completes the proof. q.e.d.

Once we have a uniform C^2 bound, we can show that u is uniformly convex:

Corollary 2.17. *There exists a positive constant C_{11} such that*

$$\sum_{k,l=1}^{n} \partial_k \partial_l u(x)\, w_k\, w_l \geq \frac{1}{C_{11}}\, |w|^2$$

for all $x \in \Omega$ and all $w \in \mathbb{R}^n$.

Proof. By assumption, the map $f(x) = \nabla u(x)$ is a diffeomorphism from Ω to $\tilde{\Omega}$. Let $g : \tilde{\Omega} \to \Omega$ denote the inverse of f. Then $Dg(y) = \big[Df(x)\big]^{-1}$, where $x = g(y)$. Since the matrix $Df(x) = D^2 u(x)$ is positive definite for all $x \in \Omega$, we conclude that the matrix $Dg(y)$ is positive definite for all $y \in \tilde{\Omega}$. Hence, there exists a convex function $v : \tilde{\Omega} \to \mathbb{R}$ such that $g(y) = \nabla v(y)$. The function v satisfies $F(D^2 v(y)) = \frac{n\pi}{2} - F(D^2 u(x))$, where $x = g(y)$. Since $F(D^2 u(x))$ is constant, it follows that $F(D^2 v(y))$ is constant. Applying Proposition 2.16 to the function v, we conclude that the eigenvalues of $D^2 v(y)$ are uniformly bounded from above. From this, the assertion follows. q.e.d.

In the next step, we show that the second derivatives of u are uniformly bounded in $C^\gamma(\overline{\Omega})$. To that end, we use results of G. Lieberman and N. Trudinger [**7**]. In the remainder of this section, we describe how the problem ($\star$) can be rewritten so as to fit into the framework of Lieberman and Trudinger.

We begin by choosing a smooth cutoff function $\eta : \mathbb{R} \to [0,1]$ such that

$$\begin{cases} \eta(s) = 0 & \text{for } s \leq 0 \\ \eta(s) = 1 & \text{for } \frac{1}{C_{11}} \leq s \leq C_{10} \\ \eta(s) = 0 & \text{for } s \geq 2C_{10}. \end{cases}$$

There exists a unique function $\psi : \mathbb{R} \to \mathbb{R}$ satisfying $\psi(1) = \frac{\pi}{4}$, $\psi'(1) = \frac{1}{2}$, and $\psi''(s) = -\frac{2s}{(1+s^2)^2}\,\eta(s) \leq 0$ for all $s \in \mathbb{R}$. Clearly, $\psi(s) = \arctan(s)$ for $\frac{1}{C_{11}} \leq s \leq C_{10}$. Moreover, it is easy to see that $\frac{1}{1+4C_{10}^2} \leq \psi'(s) \leq 1$ for all $s \in \mathbb{R}$. If M is a symmetric $n \times n$ matrix, we define

$$\Psi(M) = \sum_{k=1}^{n} \psi(\lambda_k),$$

where $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of M. Since $\psi''(s) \leq 0$ for all $s \in \mathbb{R}$, it follows that Ψ is a concave function on the space of symmetric $n \times n$ matrices.

We next rewrite the boundary condition. For each point $x \in \partial\Omega$, we denote by $\nu(x)$ the outward-pointing unit normal vector to $\partial\Omega$ at x. Similarly, for each point $y \in \partial\tilde{\Omega}$, we denote by $\tilde{\nu}(y)$ the outward-pointing unit normal vector to $\partial\tilde{\Omega}$ at y. By Proposition 2.12, there exists a positive constant C_{12} such that

$$\langle \nu(x), \tilde{\nu}(\nabla u(x)) \rangle \geq \frac{1}{C_{12}} \tag{12}$$

for all $x \in \partial\Omega$.

We define a subset $\Gamma \subset \partial\Omega \times \mathbb{R}^n$ by

$$\Gamma = \Big\{ (x,y) \in \partial\Omega \times \mathbb{R}^n : y + t\,\nu(x) \in \tilde{\Omega} \text{ for some } t \in \mathbb{R} \Big\}.$$

For each point $(x,y) \in \Gamma$, we define

$$\tau(x,y) = \sup \Big\{ t \in \mathbb{R} : y + t\,\nu(x) \in \tilde{\Omega} \Big\}$$

and

$$\Phi(x,y) = y + \tau(x,y)\,\nu(x) \in \partial\tilde{\Omega}.$$

If (x,y) lies on the boundary of the set Γ, then

$$\langle \nu(x), \tilde{\nu}(\Phi(x,y)) \rangle = 0.$$

We now define a function $G : \partial\Omega \times \mathbb{R}^n \to \mathbb{R}$ by

$$G(x,y) = \langle \nu(x), y \rangle - \chi\big(\langle \nu(x), \tilde{\nu}(\Phi(x,y)) \rangle\big)\,\big[\langle \nu(x), y \rangle + \tau(x,y)\big]$$

for $(x,y) \in \Gamma$ and

$$G(x,y) = \langle \nu(x), y \rangle$$

for $(x,y) \notin \Gamma$. Here, $\chi : \mathbb{R} \to [0,1]$ is a smooth cutoff function satisfying $\chi(s) = 1$ for $s \geq \frac{1}{C_{12}}$ and $\chi(s) = 0$ for $s \leq \frac{1}{2\,C_{12}}$. It is easy to see that G is smooth. Moreover, we have

$$G(x, y + t\,\nu(x)) = G(x,y) + t$$

for all $(x,y) \in \partial\Omega \times \mathbb{R}^n$ and all $t \in \mathbb{R}$. Therefore, G is oblique.

Proposition 2.18. *Suppose that $u : \Omega \to \mathbb{R}$ is a convex function such that ∇u is a diffeomorphism from Ω to $\tilde{\Omega}$ and $F(D^2u(x)) = c$ for all $x \in \Omega$. Then $\Psi(D^2u(x)) = c$ for all $x \in \Omega$. Moreover, we have $G(x, \nabla u(x)) = 0$ for all $x \in \partial\Omega$.*

Proof. It follows from Proposition 2.16 and Corollary 2.17 that the eigenvalues of $D^2u(x)$ lie in the interval $[\frac{1}{C_{11}}, C_{10}]$. This implies $\Psi(D^2u(x)) = F(D^2u(x)) = c$ for all $x \in \Omega$.

It remains to show that $G(x, \nabla u(x)) = 0$ for all $x \in \partial\Omega$. In order to verify this, we fix a point $x \in \partial\Omega$, and let $y = \nabla u(x) \in \partial\tilde{\Omega}$. By Proposition 2.11, we have $\langle \nu(x), \tilde{\nu}(y)\rangle > 0$. From this, we deduce that $(x,y) \in \Gamma$ and $\tau(x,y) = 0$. This implies $\Phi(x,y) = y$. Therefore, we have

$$G(x,y) = \langle \nu(x), y\rangle - \chi\big(\langle \nu(x), \tilde{\nu}(y)\rangle\big)\,\langle \nu(x), y\rangle.$$

On the other hand, it follows from (12) that $\chi(\langle \nu(x), \tilde{\nu}(y)\rangle) = 1$. Thus, we conclude that $G(x,y) = 0$. q.e.d.

In view of Proposition 2.18 we may invoke general regularity results of Lieberman and Trudinger. By Theorem 1.1 in [**7**], the second derivatives of u are uniformly bounded in $C^\gamma(\overline{\Omega})$ for some $\gamma \in (0,1)$. Higher regularity follows from Schauder estimates.

3. The linearized operator

In this section, we show that all solutions of $(\star)$ are non-degenerate. To prove this, we fix a real number $\gamma \in (0,1)$. Consider the Banach spaces

$$\mathcal{X} = \left\{ u \in C^{2,\gamma}(\overline{\Omega}) : \int_\Omega u = 0 \right\}$$

and

$$\mathcal{Y} = C^\gamma(\overline{\Omega}) \times C^{1,\gamma}(\partial\Omega).$$

We define a map $\mathcal{G} : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}$ by

$$\mathcal{G}(u,c) = \Big(F(D^2u) - c,\ (\tilde{h} \circ \nabla u)|_{\partial\Omega}\Big).$$

Hence, if $(u,c) \in \mathcal{X} \times \mathbb{R}$ is a solution of $(\star)$, then $\mathcal{G}(u,c) = (0,0)$.

Proposition 3.1. *Suppose that $(u, c) \in \mathcal{X} \times \mathbb{R}$ is a solution to $(\star)$. Then the linearized operator $D\mathcal{G}_{(u,c)} : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}$ is invertible.*

Proof. The linearized operator $\mathcal{B} = D\mathcal{G}_{(u,c)}$ is given by

$$\mathcal{B} : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}, \quad (w, a) \mapsto (Lw - a, Nw).$$

Here, the operator $L : C^{2,\gamma}(\overline{\Omega}) \to C^{\gamma}(\overline{\Omega})$ is defined by

$$Lw(x) = \operatorname{tr}\Big[\big(I + (D^2u(x))^2\big)^{-1} D^2w(x)\Big]$$

for $x \in \Omega$. Moreover, the operator $N : C^{2,\gamma}(\overline{\Omega}) \to C^{1,\gamma}(\partial\Omega)$ is defined by

$$Nw(x) = \langle \nabla w(x), \nabla\tilde{h}(\nabla u(x)) \rangle$$

for $x \in \partial\Omega$. Clearly, L is an elliptic operator. Since u is a solution of $(\star)$, Proposition 2.11 implies that $\langle \nabla h(x), \nabla\tilde{h}(\nabla u(x)) > 0$ for all $x \in \partial\Omega$. Hence, the boundary condition is oblique.

We claim that $\mathcal{B}$ is one-to-one. To see this, we consider a pair $(w, a) \in \mathcal{X} \times \mathbb{R}$ such that $\mathcal{B}(w, a) = (0, 0)$. This implies $Lw(x) = a$ for all $x \in \Omega$ and $Nw(x) = 0$ for all $x \in \partial\Omega$. Hence, the Hopf boundary point lemma (cf. [**5**], Lemma 3.4) implies that $w = 0$ and $a = 0$.

It remains to show that $\mathcal{B}$ is onto. To that end, we consider the operator

$$\tilde{\mathcal{B}} : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}, \quad (w, a) \mapsto (Lw, Nw + w + a).$$

It follows from Theorem 6.31 in [**5**] that $\tilde{\mathcal{B}}$ is invertible. Moreover, the operator

$$\tilde{\mathcal{B}} - \mathcal{B} : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}, \quad (w, a) \mapsto (a, w + a)$$

is compact. Since $\mathcal{B}$ is one-to-one, it follows from the Fredholm alternative (cf. [**5**], Theorem 5.3) that $\mathcal{B}$ is onto. This completes the proof. q.e.d.

4. Existence of a solution to $(\star)$

In this section, we prove the existence of a solution to $(\star)$. To that end, we employ the continuity method. Let Ω and $\tilde{\Omega}$ be uniformly convex domains in $\mathbb{R}^n$ with smooth boundary. By Proposition A.1, we can find a smooth function $h : \Omega \to \mathbb{R}$ with the following properties:

- h is uniformly convex
- $h(x) = 0$ for all $x \in \partial\Omega$
- If s is sufficiently close to $\inf_\Omega h$, then the sub-level set $\{x \in \Omega : h(x) \leq s\}$ is a ball.

Similarly, there exists a smooth function $\tilde{h} : \tilde{\Omega} \to \mathbb{R}$ such that:

- $\tilde{h}$ is uniformly convex
- $\tilde{h}(y) = 0$ for all $y \in \partial\tilde{\Omega}$

- If s is sufficiently close to $\inf_{\tilde{\Omega}} \tilde{h}$, then the sub-level set $\{y \in \tilde{\Omega} : \tilde{h}(y) \leq s\}$ is a ball.

Without loss of generality, we may assume that $\inf_\Omega h = \inf_{\tilde{\Omega}} \tilde{h} = -1$. For each $t \in (0, 1]$, we define

$$\Omega_t = \{x \in \Omega : h(x) \leq t - 1\}, \quad \tilde{\Omega}_t = \left\{y \in \tilde{\Omega} : \tilde{h}(y) \leq t - 1\right\}.$$

Note that Ω_t and $\tilde{\Omega}_t$ are uniformly convex domains in $\mathbb{R}^n$ with smooth boundary. We then consider the following problem (cf. [**1**]):

($\star_t$) *Find a convex function $u : \Omega \to \mathbb{R}$ and a constant $c \in (0, \frac{n\pi}{2})$ such that ∇u is a diffeomorphism from Ω_t to $\tilde{\Omega}_t$ and $F(D^2u(x)) = c$ for all $x \in \Omega_t$.*

If $t \in [0, 1)$ is sufficiently small, then Ω_t and $\tilde{\Omega}_t$ are balls in $\mathbb{R}^n$. Consequently, ($\star_t$) is solvable if $t \in (0, 1]$ is sufficiently small. In particular, the set

$$I = \{t \in (0, 1] : (\star_t) \text{ has at least one solution}\}$$

is non-empty. It follows from the a-priori estimates in Section 2 that I is a closed subset of $(0, 1]$. Moreover, Proposition 3.1 implies that I is an open subset of $(0, 1]$. Consequently, $I = (0, 1]$. This completes the proof of Theorem 1.1.

5. Proof of the uniqueness statement

In this final section, we show that the solution to ($\star$) is unique up to addition of constants. To that end, we use a trick that we learned from J. Urbas.

As above, let Ω and $\tilde{\Omega}$ be uniformly convex domains in $\mathbb{R}^n$ with smooth boundary. Moreover, suppose that (u, c) and $(\hat{u}, \hat{c})$ are solutions to ($\star$). We claim that the function $\hat{u} - u$ is constant.

Suppose this is false. Without loss of generality, we may assume that $\hat{c} \geq c$. (Otherwise, we interchange the roles of u and $\hat{u}$.) For each point $x \in \Omega$, we define a symmetric $n \times n$-matrix $B(x) = \{b_{ij}(x) : 1 \leq i, j \leq n\}$ by

$$B(x) = \int_0^1 \Big[I + \big(s\, D^2\hat{u}(x) + (1 - s)\, D^2u(x)\big)^2\Big]^{-1} ds.$$

Clearly, $B(x)$ is positive definite for all $x \in \Omega$. Moreover, we have

$$\begin{aligned} &\sum_{i,j=1}^n b_{ij}(x)\, (\partial_i\partial_j\hat{u}(x) - \partial_i\partial_j u(x)) \\ &= F(D^2\hat{u}(x)) - F(D^2u(x)) = \hat{c} - c \geq 0 \end{aligned}$$

for all $x \in \Omega$. By the maximum principle, the function $\hat{u} - u$ attains its maximum at a point $x_0 \in \partial\Omega$. By the Hopf boundary point lemma (see

[**5**], Lemma 3.4), there exists a real number $\mu > 0$ such that $\nabla \hat{u}(x_0) - \nabla u(x_0) = \mu \, \nabla h(x_0)$. Using Proposition 2.11, we obtain

$$\langle \nabla \hat{u}(x_0) - \nabla u(x_0), \nabla \tilde{h}(\nabla u(x_0)) \rangle = \mu \, \langle \nabla h(x_0), \nabla \tilde{h}(\nabla u(x_0)) \rangle > 0.$$

On the other hand, we have

$$\langle \nabla \hat{u}(x_0) - \nabla u(x_0), \nabla \tilde{h}(\nabla u(x_0)) \rangle \leq \tilde{h}(\nabla \hat{u}(x_0)) - \tilde{h}(\nabla u(x_0)) = 0$$

since $\tilde{h}$ is convex. This is a contradiction. Therefore, the function $\hat{u} - u$ is constant.

Appendix A. The construction of the boundary defining function

The following result is standard. We include a proof for the convenience of the reader.

Proposition A.1. *Let Ω be a uniformly convex domain in $\mathbb{R}^n$ with smooth boundary. Then there exists a smooth function $h : \Omega \to \mathbb{R}$ with the following properties:*

- *h is uniformly convex*
- *$h(x) = 0$ for all $x \in \partial\Omega$*
- *If s is sufficiently close to $\inf_\Omega h$, then the sub-level set $\{x \in \Omega : h(x) \leq s\}$ is a ball.*

Proof. Let x_0 be an arbitrary point in the interior of Ω. We define a function $h_1 : \Omega \to \mathbb{R}$ by

$$h_1(x) = \frac{d(x, \partial\Omega)^2}{4 \operatorname{diam}(\Omega)} - d(x, \partial\Omega).$$

Since Ω is uniformly convex, there exists a positive real number ε such that h_1 is smooth and uniformly convex for $d(x, \partial\Omega) < \varepsilon$. We assume that ε is chosen so that $d(x_0, \partial\Omega) > \varepsilon$. We next define a function $h_2 : \Omega \to \mathbb{R}$ by

$$h_2(x) = \frac{\varepsilon \, d(x_0, x)^2}{4 \operatorname{diam}(\Omega)^2} - \frac{\varepsilon}{2}.$$

For each point $x \in \partial\Omega$, we have $h_1(x) = 0$ and $h_2(x) \leq -\frac{\varepsilon}{4}$. Moreover, if $d(x, \partial\Omega) \geq \varepsilon$, then $h_1(x) \leq -\frac{3\varepsilon}{4}$ and $h_2(x) \geq -\frac{\varepsilon}{2}$.

Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying $\Phi''(s) \geq 0$ for all $s \in \mathbb{R}$ and $\Phi(s) = |s|$ for $|s| \geq \frac{\varepsilon}{16}$. We define a function $h : \Omega \to \mathbb{R}$ by

$$h(x) = \frac{h_1(x) + h_2(x)}{2} + \Phi\Big(\frac{h_1(x) - h_2(x)}{2}\Big).$$

If x is sufficiently close to $\partial\Omega$, then we have $h(x) = h_1(x)$. In particular, we have $h(x) = 0$ for all $x \in \partial\Omega$. Moreover, we have $h(x) = h_2(x)$ for $d(x, \partial\Omega) \geq \varepsilon$. Hence, the function h is smooth and uniformly convex for $d(x, \partial\Omega) \geq \varepsilon$.

We claim that the function h is smooth and uniformly convex on all of Ω. To see this, we consider a point x with $d(x, \partial\Omega) < \varepsilon$. The Hessian of h at the point x is given by

$$\begin{aligned}
&\partial_i \partial_j h(x) \\
&= \frac{1}{2} \Big[1 + \Phi'\Big(\frac{h_1(x) - h_2(x)}{2}\Big)\Big]\, \partial_i \partial_j h_1(x) \\
&+ \frac{1}{2} \Big[1 - \Phi'\Big(\frac{h_1(x) - h_2(x)}{2}\Big)\Big]\, \partial_i \partial_j h_2(x) \\
&+ \frac{1}{4}\, \Phi''\Big(\frac{h_1(x) - h_2(x)}{2}\Big)\, (\partial_i h_1(x) - \partial_i h_2(x))\, (\partial_j h_1(x) - \partial_j h_2(x)).
\end{aligned}$$

Note that $|\Phi'(s)| \leq 1$ and $\Phi''(s) \geq 0$ for all $s \in \mathbb{R}$. Since h_1 and h_2 are uniformly convex, it follows that h is uniformly convex.

It remains to verify the last statement. The function h attains its minimum at the point x_0. Therefore, we have $\inf_\Omega h = -\frac{\varepsilon}{2}$. Suppose that s is a real number satisfying

$$-\frac{\varepsilon}{2} < s < \frac{\varepsilon\, (d(x_0, \partial\Omega) - \varepsilon)^2}{4\, \mathrm{diam}(\Omega)^2} - \frac{\varepsilon}{2}.$$

Then we have $\{x \in \Omega : h(x) \leq s\} = \{x \in \Omega : h_2(x) \leq s\}$. Consequently, the set $\{x \in \Omega : h(x) \leq s\}$ is a ball. This completes the proof of Proposition A.1. q.e.d.

References

[1] S. Brendle, *Minimal Lagrangian diffeomorphisms between domains in the hyperbolic plane,* J. Differential Geom. **80**, 1–22 (2008), MR 2434257, Zbl 1154.53034.

[2] L. Caffarelli, *Boundary regularity of maps with convex potentials, II,* Ann. of Math. **144**, 453–496 (1996), MR 1426885, Zbl 0916.35016.

[3] L. Caffarelli, L. Nirenberg & J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations, III: functions of the eigenvalues of the Hessian,* Acta Math. **155**, 261–301 (1985), MR 0806416, Zbl 0654.35031.

[4] P. Delanoë, *Classical solvability in dimension two of the second boundary-value problem associated with the Monge-Ampère operator,* Ann. Inst. H. Poincaré **8**, 443–457 (1991), MR 1136351, Zbl 0778.35037.

[5] D. Gilbarg & N. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Springer-Verlag, 2001, MR 1814364, Zbl 1042.35002.

[6] R. Harvey & H.B. Lawson Jr., *Calibrated geometries,* Acta Math. **148**, 47–157 (1982), MR 0666108, Zbl 0584.53021.

[7] G. Lieberman & N. Trudinger, *Nonlinear oblique boundary value problems for nonlinear elliptic equations,* Trans. Amer. Math. Soc. **295**, 509–546 (1986), MR 0833695, Zbl 0619.35047.

[8] K. Smoczyk, *Angle theorems for the Lagrangian mean curvature flow,* Math. Z. **240**, 849–883 (2002), MR 1922733, Zbl 1020.53045.

[9] K. Smoczyk & M.T. Wang, *Mean curvature flows of Lagrangian submanifolds with convex potentials,* J. Differential Geom. **62**, 243–257 (2002), MR 1988504, Zbl 1070.53042.

[10] M.P. Tsui & M.T. Wang, *A Bernstein type result for special Lagrangian submanifolds,* Math. Res. Lett. **9**, 529–535 (2002), MR 1928873, Zbl 1027.53102.

[11] J. Urbas, *On the second boundary value problem for equations of Monge-Ampère type,* J. Reine Angew. Math. **487**, 115–124 (1997), MR 1454261, Zbl 0880.35031.

[12] J. Urbas, *The second boundary value problem for a class of Hessian equations,* Comm. PDE **26**, 859–882 (2001), MR 1843287, Zbl pre01658453.

[13] M.T. Wang, *Mean curvature flow of surfaces in Einstein four-manifolds,* J. Differential Geom. **57**, 301–338 (2001), MR 1879229, Zbl 1035.53094.

[14] Y. Yuan, *A Bernstein problem for special Lagrangian equations,* Invent. Math. **150**, 117–125 (2002), MR 1930884, Zbl 1027.53055.

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