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Roman Domination

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Abstract

In his article "Defend the Roman Empire!" (1999), Ian Stewart discussed a strategy of Emperor Constantine for defending the Roman Empire. Motivated by this article, Cockayne et al. (2004) introduced the notion of Roman domination in graphs.

Let $G = (V, E)$ be a graph. A Roman dominating function of G is a function $f : V \rightarrow \{0, 1, 2\}$ such that every vertex v for which $f(v) = 0$ has a neighbor u with $f(u) = 2$. The weight of a Roman dominating function f is $w(f) = \sum_{v \in V} f(v)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of all possible Roman dominating functions.

This paper introduces a quantity $R(xy)$ for each pair of non-adjacent vertices $\{x, y\}$ in G, called the Roman dominating index of $\{x, y\}$, which is defined by $R(xy) = \gamma_R(G) - \gamma_R(G + xy)$. We prove that $0 \leq R(xy) \leq 1$ and give a necessary and sufficient condition on $\{x, y\}$ for which $R(xy) = 1$.

This paper also introduces the Roman-critical graph. We call $G = (V, E)$ Roman-critical if $\gamma_R(G - e) > \gamma_R(G), \forall e \in E$. It is proved that a Roman-critical graph can only be a star graph whose order is not equal to 2, or the union of such graphs.

In addition, this paper shows that for each connected graph G of order $n \geq 3$, $2 \leq \gamma_R(G) \leq \left\lfloor \frac{4n}{5} \right\rfloor$. A family of graphs for which the respective equality holds is also provided.

Finally, this paper finds the lower bound of the Roman domination number for 3-regular graphs and the exact value of the Roman domination number for $C_{\frac{n}{2}} \times P_2$ and 3-regular circulant graphs.

Contents

Note: *Additional information*, a separate document, offers solutions to problems mentioned in section 3.

1. History and Motivation

About 1700 years ago, the Roman Empire was under attack, and Emperor Constantine had to decide where to station his four field army units to protect eight regions. His trick was to place the army units so that every region was either secured by its own army (one or two units) or was securable by a neighbor with two army units, one of which can be sent to the undefended region directly if a conflict breaks out.

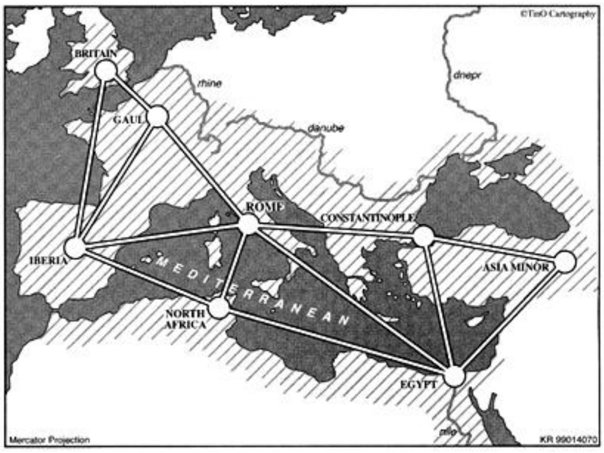

Figure 1: map showing the regions and the steps between the regions (courtesy of American Mathematics Association)

Constantine chose to place two army units in Rome and two at his new capital, Constantinople. This meant only Britain could not be reached in one step. As it happened, Constantine's successors lost control of Britain. The causes were surely more complex than anything that can be explained by this simple model. Nevertheless, Stewart (1) is right in arguing that if Constantine had been a better mathematician, the Roman Empire might have lasted a little longer than it did.

Indeed, there are six ways to improve on Constantine's deployment. These results were obtained through a form of zero-one integer programming by ReVelle and Rosing (2).

Besides placing of Roman army units, the same sort of mathematics can also be used for optimizing the location of the declining number of British Fleets at the end of the 19th century or American Military Units during the Cold War (2). In addition to army placement, the same sort of mathematics is also useful when people want to know the best place in town to put a new hospital, fire station, or fast-food restaurant. Many times such optimization problems can be modeled by Roman domination or its variants.

2. Definitions and existing results

2.1. General definitions in graph theory

The following are some basic definitions in graph theory, many of which are adopted from *Introduction to Graph Theory: H3 Mathematics* (4).

A **graph** G consists of a non-empty finite set $V(G)$ of vertices together with a finite set $E(G)$ (possibly empty) of edges such that:

1. each edge joins two *distinct* vertices in $V(G)$ and
2. any two distinct vertices in $V(G)$ are joined by *at most one* edge.

The number of vertices in G, denoted by $v(G)$, is called the **order** of G.

Let u, v be any two vertices in G. They are said to be **adjacent** if they are joined by an edge, say, e in G. We also write $e = uv$ or $e = vu$ (the ordering of u and v in the expression is immaterial), and we say that

1. u is a **neighbor** of v and vice versa,
2. the edge e is **incident with** the vertex u (and v) and
3. u and v are the two **ends** of e.

The set of all neighbors of v in G is denoted by $N(v)$; that is,

$$N(v) = \{x|x \text{ is a neighbor of } v\}.$$

The **degree** of v in G, denoted by $d(v)$, is defined as the number of edges incident with v. The vertex v is called an **end-vertex** if $d(v) = 1$.

A **path** in a graph G is an alternating sequence of vertices and edges beginning and ending at vertices:

$$v_0e_0v_1e_1v_2 \dots v_{k-1}e_{k-1}v_k$$

where $k \geq 1$, e_i is incident with v_i and v_{i+1}, for each $i = 0, 1, \dots, k-1$, and the vertices v_i's and edges e_i's need to be distinct. The **length** of the path above is defined as k, which is the number of occurrences of edges in the sequence.

A graph G is said to be **connected** if every two vertices in G are joined by a path, and **disconnected** if it is not connected.

The **distance** from u to v, denoted by $d(u, v)$, is defined as the *smallest length* of all $u - v$ paths in G. (Note that $d(v)$ denotes the degree of v in G.)

Let P_n denote a **path** of n vertices, $P_n = v_1 v_2 \dots v_n$, and C_n a **cycle** of n vertices, $C_n = v_1 v_2 \dots v_n v_1$.

Notice that we have two definitions for path. What a 'path' really means should be clear from the context when it is mentioned.

A graph H is called a **subgraph** of graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of a graph G is said to be **spanning** if $V(H) = V(G)$.

A **bipartite graph** is a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent.

2.2. Roman domination defined

Let $G = (V, E)$ be a graph. A **Roman dominating function** is a function $f : V \to \{0, 1, 2\}$ such that every vertex v for which $f(v) = 0$ has a neighbor u with $f(u) = 2$.

The **weight** of a Roman dominating function f is $w(f) = \sum_{v \in V} f(v)$. This corresponds to the total number of army units required under a specific deployment scheme.

We are interested in finding Roman dominating function(s) of minimum weight for a particular graph. It makes sense in the army placement context, because we want to minimize the number of army units needed to secure a particular set of given regions.

A Roman dominating function of minimum weight among all the possible Roman dominating functions is called a $\boldsymbol{\gamma_R}$**-function**. The **Roman domination number** of a graph *G*, denoted by $\gamma_R(G)$, is the weight of a γ_R-function – the minimum weight of all possible Roman dominating functions.

2.3. Existing results

The decision problem corresponding to computing $\gamma_R(G)$ is NP-complete. (3)

We use $\lceil x \rceil$ to denote the smallest integer larger than or equal to x while $\lfloor x \rfloor$ to denote the largest integer smaller than or equal to x.

The following result was proved by Dreyer (3).

Proposition 1: For path P_n and cycle C_n of order n,

$$\gamma_R(P_n) = \gamma_R(C_n) = \left\lceil \frac{2n}{3} \right\rceil.$$

The next observation follows readily from the definition.

Proposition 2: If H is a spanning subgraph of a graph G, then $\gamma_R(H) \geq \gamma_R(G)$.

Sections 3 and 4 offer a comprehensive study on how adding or deleting an edge will affect the Roman domination number of a graph and how from the change in Roman domination number we can deduce about some properties of the graph and its Roman dominating function.

3. Roman dominating index

In practice, armies, utility operators, etc are concerned about where to build a new road, a pipeline and others so as to reduce the size of the army or reap the most economic benefits. As such, I would like to introduce a new concept called the Roman dominating index. It will be useful in simplifying some Roman domination problems like the one shown in Section 3.2.

Let G be a graph and x, y two non-adjacent vertices in G. The **Roman dominating index** of $\{x, y\}$, denoted by $R(x, y)$, is defined by $R(xy) = \gamma_R(G) - \gamma_R(G + xy)$.

As G is a spanning subgraph of $(G + xy)$, by Proposition 2, $R(xy) \geq 0$. In what follows, we shall show that this quantity is always bounded above by 1.

3.1. The bounds of the Roman dominating index

Proposition 3: Let G be a graph. For any pair of non-adjacent vertices $\{x, y\}$ in G, $0 \leq R(xy) \leq 1$.

Proof: We need only to prove that $R(xy) \leq 1$. Let $G^{'} = G + xy$ and f' be a γ_R-function of G'. There are two cases to consider.

Case 1: $\{f^{'}(x), f^{'}(y)\} = \{0,2\}$.

Without loss of generality, assume that $f'(x) = 0$ and $f'(y) = 2$, and define $f : V \rightarrow \{0, 1, 2\}$:

$$f(v) = \begin{cases} f'(v), v \neq x, \\ \quad 1, v = x. \end{cases}$$

Then f is a Roman dominating function of G as removing edge xy only raises the possibility that vertex x may be unprotected, if f' for G' is to be used for G. Simply adding one more army to this vertex x will resolve the issue – in this way, all vertices are again protected, with an increase of one in Roman domination number.

Clearly, $w(f) = \gamma_R(G^{'}) + 1$. Thus $\gamma_R(G) \leq w(f) = \gamma_R(G^{'}) + 1$. It follows that $R(xy) = \gamma_R(G) - \gamma_R(G^{'}) \leq 1$.

Case 2: The negation of case 1.

Neither of the two vertices x and y is protected by the other. Thus, the existence of the edge xy in G' does not help protect either vertex x or y. Hence when edge xy is removed from G' to get G, $f^{'}$ is still a Roman dominating function for G. Hence $\gamma_R(G) \leq w(f^{'}) = \gamma_R(G')$. So $R(xy) = \gamma_R(G) - \gamma_R(G^{'}) \leq 0$. Hence $R(xy) = 0$.

Summing up the aforementioned two cases of discussion on the upper bound, we have $R(xy) \leq 1$. ■

Remark: Both the lower and upper bounds are reachable.

To show the lower bound is reachable, the Roman dominating index of an edge which joins the two ends of a path of order three P_3 together to form a cycle C_3 is 0.

To show the upper bound is achievable, the Roman dominating index of any edge that joins two non-neighboring vertices in cycle C_4 is 1.

Proposition 4: Let $\{x, y\}$ be a pair of non-adjacent vertices in a graph G. Then $R(xy) = 1$ if and only if there exists a γ_R-function f of G such that $\{f(x), f(y)\} = \{1,2\}$.

Proof:

Sufficiency: We may assume that $f(x) = 1 \text{ and } f(y) = 2$ for G. Define f' on $G^{'}$ as follows:

$$f'(v) = \begin{cases} f(v), v \neq x, \\ 0, v = x. \end{cases}$$

f' is a Roman dominating function as x is protected by y in G'.

Now that $w(f^{'}) = \gamma_R(G) - 1$, we have $\gamma_R(G^{'}) \leq w(f^{'}) = \gamma_R(G) - 1$. So $R(xy) = \gamma_R(G) - \gamma_R(G^{'}) \geq 1$. As $R(xy) \leq 1$, we have $R(xy) = 1$.

Necessity: Assume $R(xy) = 1$. As shown in the proof for Proposition 3, there exists a γ_R-function f' of G' such that

$$\{f'(x), f^{'}(y)\} = \{0,2\}$$

Assume $f'(x) = 0, f'(y) = 2$, then we have a Roman dominating function f for G as defined by

$$f(v) = \begin{cases} f'(v), v \neq x, \\ 1, v = x. \end{cases}$$

Note that $w(f) = \gamma_R(G^{'}) + 1$. Thus $w(f) = \gamma_R(G^{'}) + 1 = \gamma_R(G) - R(xy) + 1 = \gamma_R(G)$. By definition, f is a γ_R-function for G, with $f(x) = 1, f(y) = 2$. ■

3.2. An application of Proposition 3

Problem 1: Given a path P_n of order $n \geq 3$, are there pairs of non-adjacent vertices x, y in P_n such that $R(xy) = 1$? If yes, which pairs?

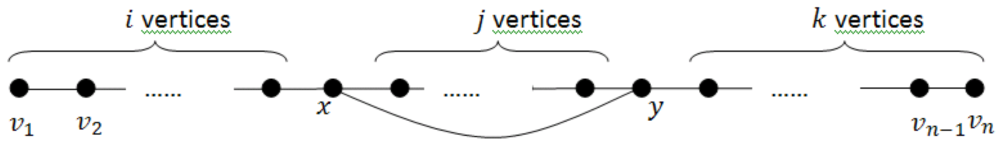

Figure 2: Can $R(xy)$ be 1?

Solution: Let x, y be two vertices in P_n as shown in Figure 2.

Dreyer (3) showed that if $n \equiv 0 (\text{mod } 3)$, no vertices in P_n are mapped to 1 in any γ_R-function. According to Proposition 4, $R(xy) = 0$.

Dreyer (3) showed that if $n \equiv 1 (\text{mod } 3)$ where $n \neq 1$, there exists for any γ_R-function f of P_n a vertex mapped to 1 and vertices mapped to 2. Thus $\max\,[R(xy)] = 1$. To find the exact vertices to connect to obtain this maximum value, we just need to find the possible value-1 and value-2 vertices in f. Without loss of generality, let $f(x) = 1$ and $f(y) = 2$. As the positions of value-1 and value-2 vertices in f follow a simple pattern, it is easy to show that the Roman dominating index of 1 can be achieved if and only if $i \equiv 0, j \equiv 1, k \equiv 1 \ (\text{mod } 3)$.

Similarly, if $n \equiv 2 (\text{mod } 3)$ $(n \neq 2)$, $\max\,[R(xy)] = 1$. Let f be a γ_R-function of P_n and without loss of generality assume $f(x) = 1$ and $f(y) = 2$. The necessary and sufficient condition for which $R(xy) = 1$ is

$$i \equiv 0, j \equiv 1, k \equiv 2 \ (\text{mod } 3), \text{or}$$
$$i \equiv 1, j \equiv 1, k \equiv 1 \ (\text{mod } 3), \text{or}$$
$$i \equiv 0, j \equiv 2, k \equiv 1 \ (\text{mod } 3). \ \blacksquare$$

Remark 1: By similar arguments, we can determine the condition to achieve a Roman dominating index of 1 for some other classes of graphs.

For cycle C_n of order n, if $n \equiv 0 (\text{mod} 3)$, $R(xy) = 0$. If $n \equiv 1 \text{ or } 2 (\text{mod } 3)$, $\max\,[R(xy)] = 1$.

For two disjoint paths/cycles or a path and a cycle, where both orders of the two components are not multiples of three, by joining the two disjoint components, we can have a Roman dominating edge with $R(xy) = 1$.

The exact positions of the vertices to connect can be determined as before by finding possible value-1 and value-2 vertices in a γ_R-function of the graph.

Remark 2: Without using Proposition 3 and 4, it may be much more tedious to solve Problem 1 as in some previous (successful) attempts by the author. For details, refer to *Additional information*.

3.3. Discussion on adding successive new edges to a path

Problem 2: Given a path P_n of order $n \geq 3$, a positive integer m with $m \leq n$, and a vertex v not in P_n, how do we add m new edges to join v and m vertices in P_n so that the resulting graph G has the largest $\gamma_R(G)$? What is the value of this largest $\gamma_R(G)$? What about the smallest one?

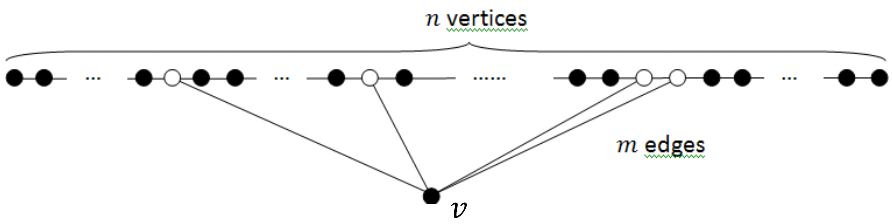

Figure 3: adding successive new edges to a path

Detailed solution is available in *Additional information*.

Result:

Largest:

If $m \leq \left\lfloor \frac{n+1}{3} \right\rfloor + 1$, then $\gamma_R(G) = \left\lceil \frac{2n+2}{3} \right\rceil$, and $f(v) = \begin{cases} 1, if\ n \equiv 0\ or\ 1\ (mod3), \\ 0, if\ n \equiv 2\ (mod3). \end{cases}$

If $m \geq \left\lfloor \frac{n+1}{3} \right\rfloor + 1$, then $\gamma_R(G) = n - m + 2$, and $f(v) = 2$.

Smallest:

If $m \leq 3$, then $\gamma_R(G) = \left\lceil \frac{2n}{3} \right\rceil$, and $f(v) = 0$.

If $m \geq 3$, then $\gamma_R(G) = \left\lceil \frac{2}{3}(n - m) \right\rceil + 2$, and $f(v) = 2$. ■

Remark 1: This problem can model the transition from a segmented, line-like distribution system of gas/water/heat, to a centralized, star-like one.

Remark 2: Following the trend of adding an edge between two disjoint graph in sections 3.2 and adding successive edges in section 3.3, a direction for further research is to combine these two cases and look into the effect of adding successive edges between two disjoint graphs.

4. Roman-critical graphs

Let $G = (V, E)$ be a graph. We call G **Roman-critical** if $\gamma_R(G - xy) > \gamma_R(G)$ for all $xy \in E$.

The **star graph** S_n of order n is a tree on n vertices with one vertex having degree $(n-1)$ and the other $(n-1)$ having vertex degree 1. Note that S_1 is a single vertex. A **galaxy** is a union of star graphs.

Now we will characterize Roman-critical graphs.

Lemma: Let $G = (V, E)$ be a graph, $\{x, y\}$ a pair of adjacent vertices in G, and f a γ_R-function of G defined on V. If $\gamma_R(G - xy) = \gamma_R(G) + 1$, then $\{f(x), f(y)\} = \{0,2\}$.

Proof:

We shall prove the contrapositive of the given proposition, i.e. If $\{f(x), f(y)\} \neq \{0,2\}$, then $\gamma_R(G - xy) = \gamma_R(G)$.

Checking the cases where $\{f(x), f(y)\} \neq \{0,2\}$, we find that f will still be a Roman dominating function for $(G - xy)$.

Thus, $w(f) \geq \gamma_R(G - xy) \geq \gamma_R(G) = w(f)$. So $\gamma_R(G - e) = \gamma_R(G)$. ■

Proposition 5: G is a Roman-critical graph if and only if G is a galaxy without S_2 as a component.

Proof:

Sufficiency: Let f be the γ_R-function of G. $f(v)=\begin{cases}2, d(v)\geq 2,\\ 0, d(v)=1.\end{cases}$

Let $d(x)=1$. Then any γ_R-function f' of $(G-xy)$ must have

$$f'(v)=\begin{cases}1, v=x,\\ f(v), v\neq x.\end{cases}$$

Note that $f(x)=0$. Thus $\gamma_R(G-xy)-\gamma_R(G)=1$. This implies that G is Roman-critical.

Necessity: By lemma, G is Roman-critical implies that $\{f(x),f(y)\}=\{0,2\}$ for all γ_R-functions f of G and for all pairs of adjacent vertices x,y. We may assume that $f(x)=0, f(y)=2$.

Assume that $d(x)>1$. Let any neighbor of x other than y be z. Since x does not need z's protection (if any) and z is not protected by x, $\gamma_R(G-xz)=\gamma_R(G)$. Thus, G is not Roman-critical, a contradiction.

Assume now that $d(y)=1$. Let f' be a function from $V(G-xy)$ to $\{0,1,2\}$ as follows:

$$f'(v)=\begin{cases}1, v=x,y,\\ f(v), v\neq x \text{ or } y.\end{cases}$$

Clearly, f' is a Roman dominating function for $(G-xy)$. Note that

$$\gamma_R(G-xy)\leq w(f')=w(f)=\gamma_R(G).$$

Hence G is not Roman-critical.

Thus $d(x)=1$ and $d(y)\geq 2$. Hence, G is a galaxy without S_2 as a component. ■

5. Bound of Roman domination number

The **diameter** of a graph G, denoted by $D(G)$, is defined as

$$D(G) = \max\{d(u,v) | u, v \text{ are in } V\}.$$

Note that $\lfloor a+b \rfloor \geq \lfloor a \rfloor + \lfloor b \rfloor$ for any real number a and b. We now establish the following result.

Proposition 6: For any tree T of order $n \geq 3$, $2 \leq \gamma_R(T) \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

Proof: The lower bound is trivial as no matter how large the order is, a star always has a Roman domination number of 2.

I will prove the upper bound by **mathematical induction** on the diameter of the tree, $D(T)$.

Base cases: If $D(T) = 2, 3, \text{or } 4$, $\gamma_R(T) \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

Case 1: $D(T) = 2$. Obviously $\gamma_R(T) = 2 \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

Case 2: $D(T) = 3$. Find the path $v_0 e_0 v_1 e_1 v_2 e_2 v_3$ which maximizes $d(v_2)$.

If $d(v_1) > 2$ and $d(v_2) > 2$, we can remove e_1 and thus get two isolated trees T_1 and T_2 of diameter 2. $\gamma_R(T) \leq \gamma_R(T_1) + \gamma_R(T_2) \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

Otherwise, let $f(v_2) = 2$ and $f(v_0) = 1 \Rightarrow \gamma_R(T) = 3 \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

Case 3: $D(T) = 4$. Find a path $v_0 e_0 v_1 e_1 v_2 e_2 v_3 e_3 v_4$ which maximizes $d(v_3)$.

If $d(v_3) > 2$, we can remove it together with all its neighboring end vertices as a tree of diameter 2. Repeat this until the tree decreases in diameter to some cases previously discussed, or becomes a tree T' where $d(v_1) = d(v_3) = 2$ and $d(v_2) \geq 2$. The former is handled by previously discussed trees of diameter 2 or 3. As to the latter, construct a Roman dominating function f such that $f(v_2) = 2$ and $f(v) = 1$ for all end-vertices v in T'. Thus we have $\gamma_R(T') \leq \frac{2+d(v_2)}{2d(v_2)+1} n' \leq \left\lfloor \frac{4n'}{5} \right\rfloor$, where n' is the order of T'.

Inductive hypothesis: If $\gamma_R(T) \le \left\lfloor \frac{4n}{5} \right\rfloor$ for any tree T where $k-3 \le D(T) \le k-1$, then for any tree T of $D(T) = k$, $\gamma_R(T) \le \left\lfloor \frac{4n}{5} \right\rfloor$. To show this:

1. Given a tree T where $D(T) = k$, find its longest path $v_0 e_0 v_1 e_1 v_2 \dots v_{k-3} e_{k-3} v_{k-2} e_{k-2} v_{k-1} e_{k-1} v_k$.
2. Remove edge e_{k-3}. Since there is only one path linking a vertex to another in any tree, removing an edge means that these two vertices are no longer linked by edges or vertices. Thus two disjoint trees T_{b1} and T_1 result.

   T_{b1} contains path $v_{k-2} e_{k-2} v_{k-1} e_{k-1} v_k$. $d(v_{k-2}, v_k) = 2$ implies that $D(T_{b1}) \ge 2$. Since we chose the longest path in T, $D(T_{b1}) \le 4$. Thus $2 \le D(T_{b1}) \le 4$ and T_{b1} falls in base cases aforementioned. Let $v(T_{b1})$ denote the order of T_{b1}. We have $\gamma_R(T_{b1}) \le \left\lfloor \frac{4v(T_{b1})}{5} \right\rfloor$.

   T_1 contains path $v_0 e_0 v_1 e_1 v_2 \dots v_{k-4} e_{k-4} v_{k-3}$. $d(v_0, v_{k-3}) = k-3$ implies that $D(T_1) \ge k-3$. In addition, $D(T_1) \le D(T) = k$. Thus $k-3 \le D(T_1) \le k$. If $D(T_1) = k$, do note that there are fewer paths of length k in T_1 than in T as path $v_0 e_0 v_1 e_1 v_2 \dots v_{k-3} e_{k-3} v_{k-2} e_{k-2} v_{k-1} e_{k-1} v_k$ and possibly others no longer exist in T_1.
3. If $D(T_1) = k$, repeat steps 1 and 2. At the i^{th} repetition of steps 1 and 2, we divide T_{i-1} into T_{bi} and T_i. As the number of path of length k is finite and this number decreases each time we apply step 1 and 2, we are certain that after m repeats $D(T_m)$ will for the first time be smaller than k. So we have $k-3 \le D(T_m) \le k-1$.

   $\gamma_R(T_m) \le \left\lfloor \frac{4v(T_m)}{5} \right\rfloor$ by the inductive hypothesis. Thus, for T whose $D(T) = k$

we have

$$\gamma_R(T) \le \gamma_R\left(\sum_{i=1}^{m} T_{bi} + T_m\right) = \sum_{i=1}^{m} \gamma_R(T_{bi}) + \gamma_R(T_m) \le \sum_{i=1}^{m} \left\lfloor \frac{4v(T_{bi})}{5} \right\rfloor + \left\lfloor \frac{4v(T_m)}{5} \right\rfloor$$

$$\le \left\lfloor \frac{4\sum_1^m v(T_{bi}) + 4v(T_m)}{5} \right\rfloor = \left\lfloor \frac{4v(T)}{5} \right\rfloor \qquad \blacksquare$$

Remark 1: This bound is achievable by constructing trees of the following structures.

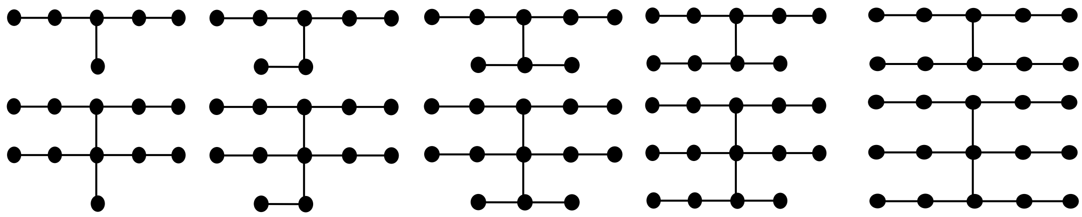

Figure 4

Remark 2: Given a tree T of order $n \geq 3$, $\gamma_R(T) = \frac{4n}{5}$ if and only if T has a structure like the right most ones shown in Figure 4.

Proof: Sufficiency is shown directly by Proposition 6. For necessity, we need a closer examination of the proof for Proposition 6. We find that given $2 \leq D(T) \leq 4$, only $\gamma_R(P_5) = \frac{4n}{5}$. Only when $T_{bi} = P_5$ for all $1 \leq i \leq m$ will we have $\gamma_R(T) = \frac{4n}{5}$. ■

Corollary: For any connected graph G of order $n \geq 3$, $2 \leq \gamma_R(G) \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

Proof: Proof for lower bound is trivial while the one for upper bound follows immediately from Proposition 2 and 6. ■

6. Roman domination in 3-regular graphs

A graph is called a **3-regular graph** if the degree of all vertices are 3. The order of any 3-regular graph can only be even.

For a graph $G = (V, E)$, let f be a function from V to the set $\{0,1,2\}$, and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V | f(v) = i\}$ for $i = 0, 1, 2$. There is a one-to-one correspondence between the functions $f: V \rightarrow \{0,1,2\}$ and the ordered partitions (V_0, V_1, V_2) of V. Thus, we will write $f = (V_0, V_1, V_2)$.

For any vertex $v \in V$, the **open neighborhood** of v is the set $N(v) = \{u \in V | uv \in E\}$ and the **closed neighborhood** is the set $N[v] = N(v) \cup \{v\}$.

Proposition 7: If G is a 3-regular graph of order n, then

$$\gamma_R(G) \geq \begin{cases} \frac{n}{2}, n \equiv 0 \ (\text{mod}4), \\ \frac{n}{2} + 1, n \equiv 2 (\text{mod}4). \end{cases}$$

Proof: Because in a 3-regular graph every vertex has a degree of three, a vertex in V_2 can protect its closed neighborhood, namely itself and its three neighbors. The most efficient protection occurs when there is no intersection between the closed neighborhood of any vertex in V_2.

Thus, when $n = 4m$, $\gamma_R(G) \geq \frac{4m}{3+1} \times 2 = 2m = \frac{n}{2}$.

When $n = 4m + 2$, the two extra vertices will increase the Roman domination number by 2.

Thus, when $n = 4m + 2$, $\gamma_R(G) \geq \frac{4m}{3+1} \times 2 + 2 = 2m + 2 = \frac{n}{2} + 1$. ■

Remark: the lower bounds are achievable as shown in Proposition 8.

6.1. Roman domination in $C_{\frac{n}{2}} \times P_2$

For graph G and H, the **Cartesian product** of G and H, denoted by $G \times H$, is the graph with vertex set $\{(u, v) | u \in V(G), v \in V(H)\}$. Two vertices (u_1, v_1) and (u_2, v_2) in $G \times H$ are adjacent if and only if one of the following is true: $u_1 = u_2$ and v_1 is adjacent to v_2 in H; or $v_1 = v_2$ and u_1 is adjacent to u_2 in G.

Proposition 8: If $G = C_{\frac{n}{2}} \times P_2$, then

$$\gamma_R(G) = \begin{cases} \frac{n}{2}, n \equiv 0 \ (\text{mod}8), \\ \frac{n}{2} + 1, n \equiv 2, 4, 6 \ (\text{mod}8). \end{cases}$$

Proof:

$C_{\frac{n}{2}} \times P_2$ is a 3-regular graph.

Let $V\left(C_{\frac{n}{2}}\right) = \{0,1,2,\dots,\frac{n}{2}-1\}$ and $uv \in E(C_{\frac{n}{2}})$ if and only if $u - v \equiv -1 \text{ or } 1 \ (\text{mod } \frac{n}{2})$. Let $V(P_2) = \{1,2\}$.

Case 1: $n = 8m\ (m \geq 1)$. Let

$$V_2 = \{(4k,1)|0 \leq k \leq m-1\} \cup \{(4l+2,2)|0 \leq l \leq m-1\},$$
$$V_1 = \emptyset,$$
$$V_0 = V - V_2.$$

$f = (V_0, V_1, V_2)$ is a Roman dominating function for G with weight $4m$. By Proposition 7, $\gamma_R(G) \geq 4m$. So $\gamma_R(G) = 4m = \frac{n}{2}$.

Case 2: $n = 8m + 2\ (m \geq 1)$. Let

$$V_2 = \{(4k,1)|0 \leq k \leq m\} \cup \{(4l+2,2)|0 \leq l \leq m-1\},$$
$$V_1 = \emptyset,$$
$$V_0 = V - V_2.$$

$f = (V_0, V_1, V_2)$ is a Roman dominating function for G with weight $(4m+2)$. By Proposition 7, $\gamma_R(G) \geq 4m+2$. So $\gamma_R(G) = 4m+2 = \frac{n}{2}+1$.

Case 3: $n = 8m + 4\ (m \geq 1)$. By Proposition 7, $\gamma_R(G) \geq 4m+2$. We will show by a coloring method that the equality cannot be reached. Since G is a bipartite graph. Let its partite sets be X and Y. Color each vertex in X black and each vertex in Y white. Each vertex in one set is adjacent to exactly 3 vertices in the other set.

Assume for contradiction that $\gamma_R(G) = 4m+2$ can be reached. Thus, $2|V_2| + |V_1| = 4m+2$. Since a vertex in V_2 can protect at most 4 vertices, $4|V_2| + |V_1| \geq 8m+4$. Eliminate $|V_2|$ from the previous two expressions, we have $|V_1| \leq 0$. Hence, we have $|V_2| = 2m+1, V_1 = \emptyset, V_0 = V - V_2$, and there must be no intersection between the closed neighborhoods of any two vertices in V_2.

In V_2, we assume that there exist s black vertices and t white vertices. Then V_2 protects $(s+3t)$ black vertices and $(3s+t)$ white vertices. Since V_2 should protect all vertices of G without overlapping, $(s+3t)$ is the number of black vertices in G and $(s+3t)$ is the number of white vertices in G. Hence both $(s+3t)$ and $(3s+t)$ are even.

But $s+t=|V_2|=2m+1$ is odd. Thus $s+3t=(s+t)+2t$ and $3s+t=(s+t)+2s$ are both odd, a contradiction. Thus $\gamma_R(G)=4m+2$ cannot be reached.

We will show $\gamma_R(G)=4m+3$. Let

$$V_2=\{(4k,1)|0\le k\le m\}\cup\{(2+4l,2)|0\le l\le m-1\},$$
$$V_1=\{(4m+1,2)\},$$
$$V_0=V-V_1-V_2.$$

$f=(V_0,V_1,V_2)$ is a Roman dominating function for G with weight $(4m+3)$. So $\gamma_R(G)=4m+3=\frac{n}{2}+1$.

Case 4: $n=8m+6$ $(m\ge 0)$. Let

$$V_2=\{(4k,1),(4k+2,2)|0\le k\le m\},$$
$$V_1=\emptyset,$$
$$V_0=V-V_2.$$

$f=(V_0,V_1,V_2)$ is a Roman dominating function for G with weight $(4m+4)$. By Proposition 7, $\gamma_R(G)\ge 4m+4$. So $\gamma_R(G)=4m+4=\frac{n}{2}+1$. ■

6.2. Roman domination in 3-regular circulant graph

A circulant graph $C_n\langle a_1,a_2,\dots,a_k\rangle$ with n vertices $0,1,2,\dots,n-1$ refers to a simple graph whose vertex i is adjacent to $i\pm a_1, i\pm a_2,\dots,i\pm a_k$ (take the remainder r mod n, $0\le r\le n-1$), where $a_1,a_2,\dots,a_k$ are positive integers and $0<a_i<\frac{n+1}{2}, a_i\ne a_j$ $(i\ne j,\ i,j=1,2,\dots,k)$.

The necessary and sufficient condition for a circulant graph $C_n\langle a_1,a_2,\dots,a_k\rangle$ to be connected is that the greatest common divisor of $(n,a_1,a_2,\dots,a_k)$, denoted by $\gcd(n,a_1,a_2,\dots,a_k)$ is 1. A 3-regular circulant graph must be $C_n\langle a,\frac{n}{2}\rangle$, where n is an even number larger than 2, $1\le a\le\frac{n}{2}-1$. If $G=C_n\langle a,\frac{n}{2}\rangle$ is a connected 3-regular circulant graph, then $\gcd(n,a)=1$ or 2. In addition, if $\gcd(n,a)=1$, then $C_n\langle a,\frac{n}{2}\rangle$ is isomorphic to $C_n\langle 1,\frac{n}{2}\rangle$. If $\gcd(n,a)=2$, then $\frac{n}{2}$ must be odd and $C_n\langle a,\frac{n}{2}\rangle$ is isomorphic to $C_{\frac{n}{2}}\times P_2$.

Proposition 9: If $G = C_n\langle a, \frac{n}{2}\rangle$ is a 3-regular circulant graph and $\gcd(n, a) = 1$, then

$$\gamma_R(G) = \begin{cases} \frac{n}{2}, \text{if } n \equiv 4 \ (\text{mod} 8), \\ \frac{n}{2} + 1, \text{if } n \equiv 0, 2, 6 \ (\text{mod} 8). \end{cases}$$

Proof: $G \cong C_n\langle 1, \frac{n}{2}\rangle$.

Case 1: $n = 8m + 4 \ (m \geq 0)$. Let

$$V_2 = \{4k | k = 0, 1, \dots, 2m\},$$
$$V_1 = \emptyset,$$
$$V_0 = V - V_2.$$

Odd number vertices are obviously neighbors of vertices in V_2. Even number vertices are either in V_2 or equal to $2 + 4l \ (l = 0, 1, \dots, m - 1)$. The vertices $(2 + 4l)$ are also neighbors of vertices in V_2 as

$(2 + 4l) + \frac{n}{2} = (2 + 4l) + (4m + 2) = 4(l + m + 1) \equiv 4k (\text{mod}\, n)$.

Thus, $f = (V_0, V_1, V_2)$ is a Roman dominating function for G.

So $\gamma_R(G) \leq 4m + 2$. However, by Proposition 7, $\gamma_R(G) \geq 4m + 2$. So $\gamma_R(G) = 4m + 2 = \frac{n}{2}$.

Case 2: $n = 8m + 6 \ (m \geq 0)$. Let

$$V_2 = \{4k | k = 0,1, \dots, m\} \cup \left\{\frac{n}{2} + 2 + 4l | l = 0,1, \dots, m\right\},$$
$$V_1 = \emptyset,$$
$$V_0 = V - V_2.$$

Similar to case 1, we can check that $f = (V_0, V_1, V_2)$ is a Roman dominating function for G. By Proposition 7, $\gamma_R(G) \geq 4m + 4$. So $\gamma_R(G) = 4m + 4 = \frac{n}{2} + 1$.

Case 3: $n = 8m \ (m \geq 1)$. By Proposition 7, $\gamma_R(G) \geq 4m$. We will show by contradiction that the equality cannot be reached. Suppose $\gamma_R(G) = 4m$. Then $|V_2| = 2m, V_1 = \emptyset, V_0 = V - V_2$. Because G is a 3-regular graph and there are $8m$ vertices in G, there should be no intersection between the closed neighborhoods of any two vertices in V_2. Without loss of generality, let $0 \in V_2$. Thus $1, 2 \in V_0$. If

$3 \in V_2$, consider $(\frac{n}{2}+1)$ in $V(G)$. This vertex can only be protected by vertex $\frac{n}{2}, \frac{n}{2}+1$ or $\frac{n}{2}+2$. Thus, one of these three vertices belongs to V_2. But its closed neighborhood will intersect with the closed neighborhood of 0 or 3, a contradiction. Hence, $3 \notin V_2$. Then, $4 \in V_2$; otherwise 2 and 3 have to be covered by vertices $(\frac{n}{2}+2)$ and $(\frac{n}{2}+3)$, but their closed neighborhoods intersect. Similarly, 5, 6 and 7 do not belong to V_2 but 8 does. Generally, $4k-1, 4k-2$ and $4k-3$ do not belong to V_2 but $4k$ does. Because $\frac{n}{2} = 4m$, $\frac{n}{2}$ belongs to V_2; but now it intersects with the closed neighborhood of vertex 0, a contradiction. Hence it is proved that $\gamma_R(G) > 4m$.

On the other hand, let

$$V_2 = \{4k | k = 0, 1, \dots, m-1\} \cup \left\{\frac{n}{2} + 2 + 4l \middle| l = 0, 1, \dots, m-1\right\},$$
$$V_1 = \{4m-1\},$$
$$V_0 = V - V_2 - V_1.$$

$f = (V_0, V_1, V_2)$ is a Roman dominating function for G. Thus, $\gamma_R(G) \leq 4m+1$ and hence $\gamma_R(G) = 4m+1 = \frac{n}{2}+1$.

Case 4: $n = 8m+2$ $(m \geq 1)$. Let

$$V_2 = \{4k | k = 0,1, \dots, m\} \cup \left\{\frac{n}{2} + 2 + 4l | l = 0,1, \dots, m-1\right\},$$
$$V_1 = \emptyset,$$
$$V_0 = V - V_2.$$

Similar to case 1, we can check that $f = (V_0, V_1, V_2)$ is a Roman dominating function for G with weight $(4m+2)$. By Proposition 7, $\gamma_R(G) \geq 4m+2$. So $\gamma_R(G) = 4m+2 = \frac{n}{2}+1$. ■

Proposition 10: If $G = C_n \langle a, \frac{n}{2} \rangle$ is a 3-regular circulant graph, $(n, a) = 2$, then $\gamma_R(G) = \frac{n}{2}+1$.

Proof: The result follows from Proposition 8. ■

7. Areas for future research

One limitation of Roman domination is that when there are too many regions in the neighborhood of a region with two armies, when multiple attacks are launched simultaneously on the neighborhood, the two armies may not be enough to defend them. Thus we devise a new kind of Roman domination called k-Roman domination.

Let $G = (V, E)$ be a graph, $f: V \to \{0,1,2,\dots,k\}$ and $V_i = \{v \in V | f(v) = i\} (i = 0,1,2,\dots,k)$. If $V_0 \subseteq N[\cup V_i]$ $(i \geq 1)$ and for all $v \in V_i$ $(i \geq 1)$, $|N(v) \cap V_0| \leq i$, then we call f a k-Roman dominating function for G.

In addition, since some roads between regions may be one-way only, we can apply Roman domination to directed graphs. We may redefine Roman dominating function as $f : V \to \{0, 1, 2\}$ such that every vertex v for which $f(v) = 0$ has a neighbor u with $f(u) = 2$ and there exists an arc uv.

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Additional Information for Roman Domination

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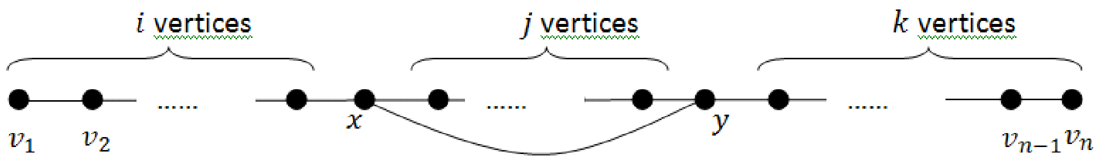
Contents

1. Alternative solution for the problem in Section 3.2

Note: The author discovered the following alternative solution before proving Propositions 3 and 4, the two propositions which enable the shorter solution presented in Section 3.2 on the page 10 of the Report.

Problem 1: Given a path P_n of order $n \geq 3$, are there pairs of non-adjacent vertices x, y in P_n such that $R(xy) = 1$? If yes, which pairs?



The alternative solution:

We denote the path of order n by P_n, and the new graph formed by adding an extra edge by $P_n{}'$.

Case 1: Neither vertices x nor y is assigned 2 under f'_{γ_R}, γ_R-function for $P_n{}'$.

It is obvious that graph G and G' will have the same Roman domination number and thus the Roman dominating index of edge xy will always be zero. According to Proposition 1,

$$\gamma_R(P_n{}') = \left\lceil \frac{2n}{3} \right\rceil, f(x) \neq 2 \text{ and } f(y) \neq 2$$

Case 2: One of vertices x and y is assigned 2 by function f'_{γ_R}.

Without loss of generality, let $f'_{\gamma_R}(x) = 0$ and $f'_{\gamma_R}(y) = 2$. As shown in Figure 1.1, i is the number of vertices on the left of x, j between x and y (not inclusive) and k on the right of y.

Numbers assigned to vertices x and y are already fixed (0 and 2 respectively). In addition, vertex y can protect its three neighbors in G'. The remaining is to find the Roman domination number for three paths, of order $i, (j-1)$, and $(k-1)$, which can be easily done using Formula 1,

$$\gamma_R(P_n{}') = \left\lceil \frac{2i}{3} \right\rceil + \left\lceil \frac{2(j-1)}{3} \right\rceil + \left\lceil \frac{2(k-1)}{3} \right\rceil + 2, f(x) = 2 \text{ or } f(y) = 2$$

$$n = i + j + k + 2$$

Combining the two cases, we have

$$\gamma_R(P_n{}') = \min\left[\left\lceil\frac{2n}{3}\right\rceil, \left\lceil\frac{2i}{3}\right\rceil + \left\lceil\frac{2(j-1)}{3}\right\rceil + \left\lceil\frac{2(k-1)}{3}\right\rceil + 2\right],$$

where $n = i + j + k + 2$,

and $\min[a, b] =$ the smaller value of a and b (if $a = b, \min[a, b] = a = b$)

$$\therefore R(xy) = \left\lceil\frac{2n}{3}\right\rceil - \min\left[\left\lceil\frac{2n}{3}\right\rceil, \left\lceil\frac{2i}{3}\right\rceil + \left\lceil\frac{2(j-1)}{3}\right\rceil + \left\lceil\frac{2(k-1)}{3}\right\rceil + 2\right]$$

The following result can be checked:

cases	$i(\text{mod}3)$	$j(\text{mod}3)$	$k(\text{mod}3)$	$R(xy)$
1	0	0	0	0
2	0	0	1	0
3	0	0	2	0
4	0	1	0	0
5	0	1	1	1
6	0	1	2	1
7	0	2	0	0
8	0	2	1	1
9	0	2	2	0
10	1	0	0	0
11	1	0	1	0
12	1	0	2	0
13	1	1	0	0
14	1	1	1	1
15	1	1	2	0
16	1	2	0	0
17	1	2	1	0
18	1	2	2	0
19	2	0	0	0
20	2	0	1	0
21	2	0	2	0
22	2	1	0	0
23	2	1	1	0
24	2	1	2	0
25	2	2	0	0
26	2	2	1	0
27	2	2	2	0

Thus the conclusion follows that, for P_n:

If $n \equiv 0(\mathrm{mod}3)$, $\max[R(xy)] = 0$.

If $n \equiv 1(\mathrm{mod}3), \max[R(xy)] = 1$. Refer to case 5 for which two vertices to connect.

If $n \equiv 2(\mathrm{mod}3), \max[R(xy)] = 1$. Refer to cases 6, 8 and 14 for which two vertices to connect. ■

Remark: Similar problems on other classes of graph as mentioned in Remark 1 of Section 3.2 can also be solved in the same way whereby graphs of unknown Roman domination number is transformed to some classes of graphs of known Roman domination number such as a path.

2. Detailed discussion on adding successive new edges

Note: This section corresponds to Section 3.3 on page 11 of the Report. For brevity, only results are presented in Section 3.3; for completeness, the detailed derivations are presented below.

Problem 2: Given a path P_n of order $n \geq 3$, a positive integer m with $m \leq n$, and a vertex v not in P_n, how do we add m new edges to join v and m vertices in P_n so that the resulting graph G has the largest $\gamma_R(G)$? What is the value of this largest $\gamma_R(G)$? What about the smallest one?

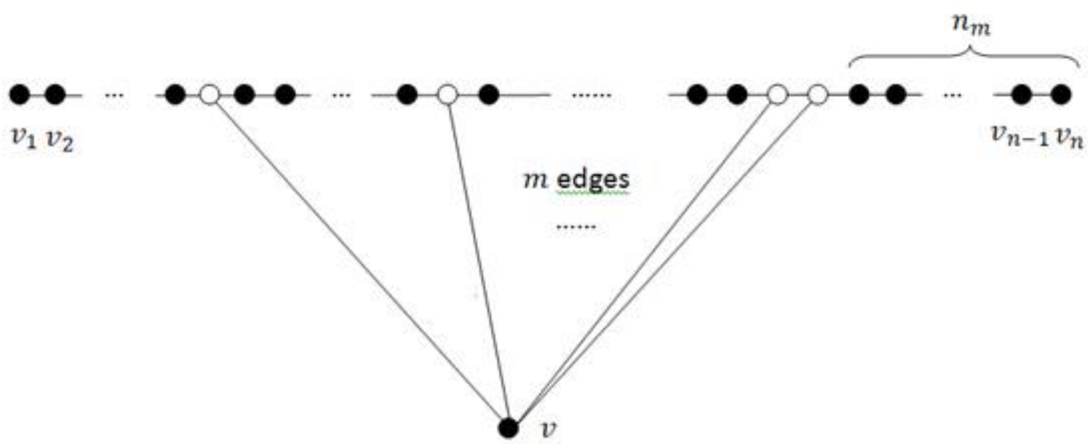


Figure 1: Adding successive new edges in detail

Solution:

Largest $\gamma_R(G)$:

As shown in Figure 1, let the new vertex be v and the vertices on the path $v_1, v_2, \dots, v_n$.

We compare minimum weight under the two cases below to find the largest Roman domination number.

Case 1: v is mapped to 2.

Sub-case 1.1: $m \geq \left\lfloor \frac{n}{3} \right\rfloor$, where $\lfloor x \rfloor$ is floor function.

By connecting $v_{3i}, 1 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor$ to v, we have $n_i \leq 2$, for $0 \leq i \leq m$. By mapping to 1 any vertex on path that is not connected to v, and v to 2, we have

$$\text{minimum weight} = n - m + 2.$$

Sub-case 1.2: $m < \left\lfloor \frac{n}{3} \right\rfloor$.

We connect the m edges to vertices $v_{3i}, 1 \leq i \leq m$.

This maximizes weight as, for any integer x and y,

$$\gamma_R(P_{x+y-2}) + \gamma_R(P_2) = \left\lceil \frac{2}{3}(x+y-2) \right\rceil + \left\lceil \frac{2}{3} \times 2 \right\rceil$$
$$\geq \left\lceil \frac{2}{3}x \right\rceil + \left\lceil \frac{2}{3}y \right\rceil = \gamma_R(P_x) + \gamma_R(P_y).$$

By mapping v_{3i-2} and v_{3i-1} to 1 and applying Proposition 1 to the $(n-3m)$ consecutive vertices, we have,

$$\text{minimum weight} = 2m + \left\lceil \frac{2}{3}(n-3m) \right\rceil + 2 = \left\lceil \frac{2}{3}n \right\rceil + 2.$$

Case 2: v is not mapped to 2.

Sub-case 2.1: $n \equiv 0 \pmod 3$. We must map vertices v_{3i-1} $(1 \leq i \leq \frac{n}{3})$ to 2 while others on the path to 0.

When $m \leq \frac{2n}{3}$, we connect the new edges to vertices others than v_{3i-1}. Then, v must be mapped to 1. Thus,

$$\text{minimum weight} = \frac{2n}{3} + 1, m \leq \frac{2n}{3}.$$

When $m > \frac{2n}{3}$, v will be connected to some v_{3i-1}. We map v to 0. Thus,

$$\text{minimum weight} = \frac{2n}{3}, m > \frac{2n}{3}.$$

In summary,

$$\text{minimum weight} = \begin{cases} \frac{2n}{3} + 1, m \leq \frac{2n}{3} \\ \frac{2n}{3}, m > \frac{2n}{3} \end{cases}.$$

Sub-case 2.2: $n \equiv 1 \pmod 3$.

If $f(v_n) = 1$, vertices v_{3i-1} $(1 \leq i \leq \frac{n-1}{3})$ will be mapped to 2 while others on the path to 0. If $f(v_1) = 1$, vertices v_{3i} will be mapped to 2 while others on the path to 0. Only vertices v_{3i-2} $(1 \leq i \leq \frac{n+2}{3})$ will never be mapped to 2.

Thus when $m \leq \frac{n+2}{3}$, we connect the new edges to vertices v_{3i-2}. Then, v must be mapped to 1. Thus,

$$\text{minimum weight} = \left\lceil \frac{2n}{3} \right\rceil + 1 = \frac{2n+4}{3}, m \leq \frac{n+2}{3}.$$

When $m > \frac{n+2}{3}$, v will be connected to some v_{3i-1} or v_{3i} $(1 \leq i \leq \frac{n-1}{3})$. Then, v will be mapped to 0. Thus,

$$\text{minimum weight} = \left\lceil \frac{2n}{3} \right\rceil = \frac{2n+1}{3}, m > \frac{n+2}{3}.$$

In summary,

$$\text{minimum weight} = \begin{cases} \dfrac{2n+4}{3}, m \leq \dfrac{n+2}{3} \\ \dfrac{2n+1}{3}, m > \dfrac{n+2}{3} \end{cases}.$$

Sub-case 2.3: $n \equiv 2 \pmod 3$.

If $f(v_{n-1}) = f(v_n) = 1$, vertices v_{3i-1} $(1 \leq i \leq \frac{n-2}{3})$ will be mapped to 2 while others on the path to 0. If $f(v_1) = f(v_n) = 1$, vertices v_{3i} will be mapped to 2 while others on the path to 0. If $f(v_1) = f(v_2) = 1$, vertices v_{3i+1} will be mapped to 2 while others on the path to 0.

v_1 can also be mapped to 2: $f(v_1) = 2, f(v_2) = 0, f(v_{3i+1}) = 2$, all other vertices being mapped to 0. By symmetry, v_n can also be mapped to 2.

Thus,

$$\text{minimum weight} = \left\lceil \frac{2n}{3} \right\rceil = \frac{2n+2}{3}$$

Now we have obtained the minimum weight possible under the two cases, we can compare them and determine which one offers the smaller value. We will have three cases.

Case 1: When $n \equiv 0 \pmod 3$,

$$\gamma_R(G) = \min\left[\begin{cases} \frac{2n}{3}+2, m < \frac{n}{3} \\ n-m+2, m \geq \frac{n}{3} \end{cases}, \begin{cases} \frac{2n}{3}+1, m \leq \frac{2n}{3} \\ \frac{2n}{3}, m > \frac{2n}{3} \end{cases}\right]$$

When $0 \leq m < \frac{n}{3}$, $\gamma_R(G) = \frac{2n}{3}+1$, v mapped to 1.

When $\frac{n}{3} \leq m \leq \frac{2n}{3}$, we compare the values $(n-m+2)$ and $(\frac{2n}{3}+1)$. When $m \leq \frac{n}{3}+1$, $n-m+2 \geq \frac{2n}{3}+1$; when $m \geq \frac{n}{3}+1$, $n-m+2 \leq \frac{2n}{3}+1$.

Thus, when $\frac{n}{3} \leq m \leq \frac{n}{3}+1$, $\gamma_R(G) = \frac{2n}{3}+1$, v mapped to 1; when $\frac{n}{3}+1 \leq m \leq \frac{2n}{3}, \gamma_R(G) = n-m+2$, v mapped to 2.

When $\frac{2n}{3} < m \leq n$, we compare the values $(n-m+2)$ and $\frac{2n}{3}$. When $m \leq \frac{n}{3}+2$, $n-m+2 \geq \frac{2n}{3}$; when $m \geq \frac{n}{3}+2$, $n-m+2 \leq \frac{2n}{3}$.

As $n \geq 3$, we have $m \geq \frac{2n}{3}+1 \geq \frac{n}{3}+2$. Thus, $\gamma_R(G) = n-m+2$, v mapped to 2.

To conclude for the case where $n \equiv 0 \pmod 3$,

When $0 \leq m \leq \frac{n}{3}+1$, $\gamma_R(G) = \frac{2n}{3}+1$, v mapped to 1.

When $\frac{n}{3}+1 \leq m \leq n, \gamma_R(G) = n-m+2$, v mapped to 2.

Case 2: When $n \equiv 1 \pmod 3$,

$$\gamma_R(G) = \min\left[\begin{cases} \frac{2n+1}{3}+2, m < \frac{n-1}{3} \\ n-m+2, m \geq \frac{n-1}{3} \end{cases}, \begin{cases} \frac{2n+4}{3}, m \leq \frac{n+2}{3} \\ \frac{2n+1}{3}, m > \frac{n+2}{3} \end{cases}\right]$$

When $0 \leq m < \frac{n-1}{3}$, $\gamma_R(G) = \frac{2n+4}{3}$, v mapped to 1.

When $\frac{n-1}{3} \leq m \leq \frac{n+2}{3}$, we have only two possible integer values for m:

When $m = \frac{n-1}{3}$, $n-m+2 = \frac{2}{3}n + \frac{7}{3} > \frac{2n+4}{3}$, When $m=\frac{n+2}{3}$, $n-m+2=\frac{2}{3}n+\frac{4}{3}=\frac{2n+4}{3}$.

Thus when $\frac{n-1}{3}\le m\le\frac{n+2}{3}$, $\gamma_R(G)=\frac{2n+4}{3}$, v mapped to 1.

When $\frac{n+2}{3}<m\le n$, we have $n-m+2\le\frac{2n+1}{3}$, with equality holds only when $m=\frac{n+5}{3}$. Thus $\gamma_R(G)=n-m+2$, v mapped to 2.

To conclude for the case where $n\equiv 1 \pmod 3$,

When $0\le m\le\frac{n+2}{3}$, $\gamma_R(G)=\frac{2n+4}{3}$, v mapped to 1.

When $\frac{n+2}{3}\le m\le n$, $\gamma_R(G)=n-m+2$, v mapped to 2.

Case 3: When $n\equiv 2 \pmod 3$ or in another word $n=3k_n+2$, Roman domination number for the resultant graph is

$$\min\left[\begin{cases}\frac{2n+2}{3}+2, m<\frac{n-2}{3}\\ n-m+2, m\ge\frac{n-2}{3}\end{cases}, \frac{2n+2}{3}\right].$$

When $0\le m<\frac{n-2}{3}$, $\gamma_R(G)=\frac{2n+2}{3}$, v mapped to 0.

When $\frac{n-2}{3}\le m\le\frac{n+4}{3}$, $n-m+2\ge\frac{2n+2}{3}$, $\gamma_R(G)=\frac{2n+2}{3}$, v mapped to 0.

When $\frac{n+4}{3}\le m\le n$, $n-m+2\le\frac{2n+2}{3}$, $\gamma_R(G)=n-m+2$, v mapped to 2.

Result: We summarize the three cases:

If $m\le\left\lfloor\frac{n+1}{3}\right\rfloor+1$,

$$\gamma_R(G)=\left\lceil\frac{2n+2}{3}\right\rceil,$$

$$f(v)=\begin{cases}1, \text{if } n\equiv 0 \text{ or } 1 \pmod 3\\ 0, \text{if } n\equiv 2 \pmod 3\end{cases}.$$

If $m\ge\left\lfloor\frac{n+1}{3}\right\rfloor+1$,

$$\gamma_R(G)=n-m+2,$$

$$f(v)=2.$$

Smallest $\gamma_R(G)$:

To find G with the smallest $\gamma_R(G)$, we have two cases.

Case 1: v is mapped to 2. G ought to have as many $n_i \equiv 0 \pmod 3$ as possible. One simple way to do that is to connect e_1 to the first vertex on the path, e_2 the second vertex, e_3 the third vertex and so on.

$$\text{minimum weight} = \left\lceil \frac{2}{3}(n-m) \right\rceil + 2.$$

Case 2: v is not mapped to 2. We can always connect v to the path such that it is adjacent to a value-2 vertex. Thus,

$$\text{minimum weight} = \left\lceil \frac{2n}{3} \right\rceil.$$

Comparing the two cases, the minimum Roman domination number is

$$\min\left[\left\lceil \frac{2}{3}(n-m) \right\rceil + 2, \left\lceil \frac{2n}{3} \right\rceil\right].$$

By property of ceiling function,

$$\left\lceil \frac{2}{3}(n-m) \right\rceil + 2 = \left\lceil \frac{2}{3}(n-m) + 2 \right\rceil = \left\lceil \frac{2}{3}n + (2 - \frac{2}{3}m) \right\rceil.$$

Thus,

If $m \leq 3$,

$$\gamma_R(G) = \left\lceil \frac{2n}{3} \right\rceil,$$

$$f(v) = 0;$$

If $m \geq 3$,

$$\gamma_R(G) = \left\lceil \frac{2}{3}(n-m) \right\rceil + 2,$$

$$f(v) = 2. \quad \blacksquare$$