

**KNOTS, SUTURES, AND EXCISION**

PETER KRONHEIMER &amp; TOMASZ MROWKA

**Abstract**

We develop monopole and instanton Floer homology groups for balanced sutured manifolds, in the spirit of [12]. Applications include a new proof of Property P for knots.

## CONTENTS

1. Introduction	301
2. Background on monopole Floer homology	302
3. Floer’s excision theorem	317
4. Monopole Floer homology for sutured manifolds	323
5. Knot homology	330
6. Fibered knots	337
7. Instantons	347
References	363

**1. Introduction**

Floer homology for sutured manifolds is an invariant  $SFH(M, \gamma)$  of “balanced sutured 3-manifolds”  $(M, \gamma)$ , introduced by Juhász in [12, 13]. It incorporates the knot Floer homology of Ozsváth-Szabó and Rasmussen [26, 28] as a special case, and it provides a framework in which to adapt the arguments of Ghiggini and Ni [11, 23, 25] to reprove, for example, that knot Floer homology detects fibered knots.

The construction that forms the basis of Juhász’s invariant is an adaptation of Ozsváth and Szabó’s Heegaard Floer homology for 3-manifolds. The purpose of the present paper is to show how something very similar can be done using either monopole Floer homology [18] or instanton Floer homology [4] in place of the Heegaard version. We will define an invariant of balanced sutured manifolds by gluing them up, with some extra pieces, to form a closed manifold and then applying ordinary Floer

---

The work of the first author was supported by the National Science Foundation through NSF grant number DMS-0405271. The work of the second author was supported by NSF grants DMS-0206485, DMS-0244663, and DMS-0805841.

Received 8/18/08.

homology, of either variety, to this closed manifold. Many of the theorems and constructions of Ghiggini, Ni, and Juhász can be repeated in this context. In particular, our construction leads to candidates for “monopole knot homology” and “instanton knot homology”: the monopole and instanton counterparts of the Heegaard knot homology groups. Adapting the arguments of [11] and [23], we shall also prove that fibered knots can be characterized using either of these invariants.

The definition of instanton knot homology which arises in this way, motivated by Juhász’s sutured manifold framework, is not new. It turns out to be exactly the same as an earlier instanton homology for knots, defined by Floer twenty years ago [8]. We conjecture that, over a field of characteristic zero, the knot homology groups of Ozsváth-Szabo and Rasmussen are isomorphic to Floer’s instanton knot homology.

Monopole Floer homology for balanced sutured manifolds is defined in section 4, and the definition is adapted to the instanton case in section 7. The same definition could be applied with Heegaard Floer homology: it is not clear to the authors whether the resulting invariant of sutured manifolds would be the same as Juhász’s invariant, but we would conjecture that this is the case. It seems, at least, that the construction recaptures Heegaard knot homology, as has been communicated to the authors by Rasmussen [27]. Some things are missing however. Our construction leads to knot homology groups which lack (a priori) the homological  $\mathbb{Z}$ -grading as well as the additional structures that are present in the theory developed in [26] and [28].

In the setting of instanton homology, we obtain new non-vanishing theorems. Among other applications, the non-vanishing theorems lead to a new proof of Property P for knots. In contrast to the proof in [17], the argument presented here is independent of the work of Feehan and Leness [6] concerning Witten’s conjecture, and does not require any tools from contact or symplectic topology. As a related matter, we show that instanton homology captures the Thurston norm on an irreducible 3-manifold, answering a question raised in [15].

**Acknowledgments.** The authors would like to thank András Juhász, Jake Rasmussen, and the referee for helpful comments and corrections to earlier versions of this paper.

## 2. Background on monopole Floer homology

**2.1. Monopole Floer homology recalled.** We follow the notation of [18] for monopole Floer homology. Thus, to a closed, connected, oriented 3-manifold  $Y$  equipped with a  $\text{spin}^c$  structure  $\mathfrak{s}$ , we associate three varieties of Floer homology groups with integer coefficients,

$$\widetilde{HM}_\bullet(Y, \mathfrak{s}), \quad \widehat{HM}_\bullet(Y, \mathfrak{s}), \quad \overline{HM}_\bullet(Y, \mathfrak{s}).$$

The notation using  $\bullet$  in place of the more familiar  $*$  was introduced in [18] to denote that, in general, there is a completion involved in the definition. In all that follows, the distinction between  $\widehat{HM}_\bullet$  and  $\widehat{HM}_*$  does not arise, but we preserve the former notation as a visual clue. Unless  $c_1(\mathfrak{s})$  is torsion, these groups are not  $\mathbb{Z}$ -graded, but they always have a canonical  $\mathbb{Z}/2$  grading.

The three varieties are related by a long exact sequence

$$\cdots \rightarrow \overline{HM}_\bullet(Y, \mathfrak{s}) \xrightarrow{i} \widetilde{HM}_\bullet(Y, \mathfrak{s}) \xrightarrow{j} \widehat{HM}_\bullet(Y, \mathfrak{s}) \xrightarrow{p} \overline{HM}_\bullet(Y, \mathfrak{s}) \rightarrow \cdots .$$

If  $c_1(\mathfrak{s})$  is not torsion, then  $\overline{HM}_\bullet(Y, \mathfrak{s})$  is zero and  $\widetilde{HM}_\bullet(Y, \mathfrak{s})$  and  $\widehat{HM}_\bullet(Y, \mathfrak{s})$  are canonically isomorphic, via  $j$ . In this case, we simply write  $HM_\bullet(Y, \mathfrak{s})$  for either  $\widetilde{HM}_\bullet(Y, \mathfrak{s})$  or  $\widehat{HM}_\bullet(Y, \mathfrak{s})$ . All these groups can be non-zero only for finitely many  $\text{spin}^c$  structures on a given  $Y$ : we write

$$\widetilde{HM}_\bullet(Y) = \bigoplus_{\mathfrak{s}} \widetilde{HM}_\bullet(Y, \mathfrak{s})$$

for the total Floer homology, taking the sum over all isomorphism classes of  $\text{spin}^c$  structure, with similar notation for the  $\widehat{HM}$  and  $\overline{HM}$  cases.

**2.2. Local coefficients.** We can also define a version of Floer homology with a local system of coefficients. The following definition is adapted from [18, section 22.6]. Let  $\mathcal{R}$  denote any commutative ring with 1 supplied with an “exponential map,” a group homomorphism

$$(1) \quad \exp : \mathbb{R} \rightarrow \mathcal{R}^\times .$$

We will use polynomial notation for the exponential map, writing

$$t = \exp(1)$$

and so writing  $\exp(n)$  as  $t^n$ . Let  $\mathcal{B}(Y, \mathfrak{s})$  denote the Seiberg-Witten configuration space for a  $\text{spin}^c$  structure  $\mathfrak{s}$  on  $Y$ ; that is,  $\mathcal{B}(Y, \mathfrak{s})$  is the space of gauge equivalence classes  $[A, \Phi]$  consisting of a  $\text{spin}^c$  connection  $A$  and a section  $\Phi$  of the spin bundle. Given a smooth 1-cycle  $\eta$  in  $Y$  with real coefficients, we can associate to each path  $z : [0, 1] \rightarrow \mathcal{B}(Y, \mathfrak{s})$  a real number  $r(z)$  by

$$r(z) = \frac{i}{2\pi} \int_{[0,1] \times \eta} \text{tr} F_{A_z} ,$$

where  $A_z$  is the 4-dimensional connection on  $[0, 1] \times Y$  arising from the path  $z$ . Now define a local system  $\Gamma_\eta$  on  $\mathcal{B}(Y, \mathfrak{s})$  by declaring its fiber at every point to be  $\mathcal{R}$  and declaring the map  $\mathcal{R} \rightarrow \mathcal{R}$  corresponding to a path  $z$  to be multiplication by  $t^{r(z)}$ . Following [18, section 22], we obtain Floer homology groups with coefficients in  $\Gamma_\eta$ ; they will be  $\mathcal{R}$ -modules denoted

$$\widetilde{HM}_\bullet(Y; \Gamma_\eta), \quad \widehat{HM}_\bullet(Y; \Gamma_\eta), \quad \overline{HM}_\bullet(Y; \Gamma_\eta).$$

These still admit a direct sum decomposition by isomorphism classes of  $\text{spin}^c$  structures. The following is essentially proposition 32.3.1 of [18]:

**Proposition 2.1.** *If there is an integer cohomology class that evaluates as 1 on  $[\eta]$ , and if  $t - t^{-1}$  is invertible in  $\mathcal{R}$ , then  $\overline{HM}_\bullet(Y; \Gamma_\eta)$  is zero; thus we again have an isomorphism  $j$  between  $\widetilde{HM}_\bullet(Y; \Gamma_\eta)$  and  $\widehat{HM}_\bullet(Y; \Gamma_\eta)$ .*

In the situation of the proposition, we once more drop the decorations and simply write

$$HM_\bullet(Y; \Gamma_\eta) = \bigoplus_{\mathfrak{s}} HM_\bullet(Y, \mathfrak{s}; \Gamma_\eta)$$

for this  $\mathcal{R}$ -module.

**2.3. Cobordisms.** Cobordisms between 3-manifolds give rise to maps between their Floer homology groups. More precisely, if  $W$  is a compact, oriented cobordism from  $Y_1$  to  $Y_2$ , equipped with a homology-orientation in the sense of [18], then  $W$  gives rise to a map

$$\widetilde{HM}(W) : \widetilde{HM}_\bullet(Y_1) \rightarrow \widetilde{HM}_\bullet(Y_2)$$

with similar maps on  $\widehat{HM}_\bullet$  and  $\overline{HM}_\bullet$ . If  $\eta_1$  and  $\eta_2$  are 1-cycles in  $Y_1$  and  $Y_2$ , respectively, then to obtain a maps between the Floer groups with local coefficients, we need an additional piece of data: a 2-chain  $\nu$  in  $W$  with  $\partial\nu = \eta_2 - \eta_1$ . In this case, we obtain a map which we denote by

$$\widetilde{HM}(W; \Gamma_\nu) : \widetilde{HM}_\bullet(Y_1; \Gamma_{\eta_1}) \rightarrow \widetilde{HM}_\bullet(Y_2; \eta_2).$$

The map  $\widetilde{HM}(W)$  and its relatives are defined by taking a sum over all  $\text{spin}^c$  structures on  $W$ . In the case of  $\widetilde{HM}(W; \Gamma_\nu)$ , the  $\text{spin}^c$  contributions are weighted according to the pairing of the curvature of the connection with the cycle  $\nu$ . There is a corresponding invariant for a closed 4-manifold  $X$  with  $b_+ \geq 2$  containing a closed 2-cycle  $\nu$ . In [18], this invariant of  $(X, \nu)$  is denoted by  $\mathfrak{m}(X, \nu)$  (or  $\mathfrak{m}(X, [\nu])$ , because only the homology class of  $\nu$  matters); it is an element of  $\mathcal{R}$  defined by

$$(2) \quad \mathfrak{m}(X, [\nu]) = \sum_{\mathfrak{s}} \mathfrak{m}(X, \mathfrak{s}) t^{\langle c_1(\mathfrak{s}), [\nu] \rangle},$$

where  $\mathfrak{m}(X, \mathfrak{s})$  denotes the ordinary Seiberg-Witten invariant for a  $\text{spin}^c$  structure  $\mathfrak{s}$ .

**2.4. Adjunction inequalities and non-vanishing theorems.** Monopole Floer homology detects the Thurston norm of a 3-manifold  $Y$ . We recall from [18] what lies behind this slogan. Let  $F \subset Y$  be a closed, oriented, connected surface in our closed, oriented 3-manifold  $Y$ . We shall suppose  $F$  is not sphere. Then we have a vanishing theorem [18, corollary 40.1.2], which states that

$$HM_\bullet(Y, \mathfrak{s}) = 0$$

for all  $\text{spin}^c$  structures  $\mathfrak{s}$  satisfying

$$\langle c_1(\mathfrak{s}), [F] \rangle > 2 \text{genus}(F) - 2.$$

(Note that this condition implies that  $c_1(\mathfrak{s})$  is not torsion.) This vanishing theorem is usually referred to as the “adjunction inequality.” Accompanying this result is a rather deeper *non*-vanishing theorem, which we state (for the sake of simplicity) in the case that the genus of  $F$  is at least 2. In this case, the non-vanishing theorem asserts that if  $F$  is genus-minimizing in its homology class, then there exists a  $\text{spin}^c$  structure  $\mathfrak{s}_c$  with

$$HM_\bullet(Y, \mathfrak{s}_c) \neq 0$$

and

$$\langle c_1(\mathfrak{s}_c), [F] \rangle = 2 \text{genus}(F) - 2.$$

Slightly more specifically, Gabai’s theorem from [9] tells us that  $Y$  admits a taut foliation having  $F$  as a compact leaf. A foliation in turn determines a  $\text{spin}^c$  structure on  $Y$ . The non-vanishing result holds for any  $\text{spin}^c$  structure  $\mathfrak{s}_c$  arising in this way. This result appears as corollary 41.4.2 in [18]. The techniques of this paper provide an alternative proof, which we will explain in the context of instanton homology in section 7.8 below.

It is convenient to introduce the following shorthand. We denote the set of isomorphism classes of  $\text{spin}^c$  structures on a closed oriented manifold  $Y$  by  $\mathcal{S}(Y)$ . If  $F \subset Y$  is a closed, connected oriented surface of genus  $g \geq 2$ , then we write  $\mathcal{S}(Y|F)$  for the set of isomorphism classes of  $\text{spin}^c$  structures  $\mathfrak{s}$  on  $Y$  satisfying the constraint

$$(3) \quad \langle c_1(\mathfrak{s}), [F] \rangle = 2 \text{genus}(F) - 2,$$

and we write

$$HM_\bullet(Y|F) \subset HM_\bullet(Y)$$

for the subgroup

$$(4) \quad HM_\bullet(Y|F) = \bigoplus_{\mathfrak{s} \in \mathcal{S}(Y|F)} HM_\bullet(Y, \mathfrak{s}).$$

Note again that all the  $\text{spin}^c$  structures in  $\mathcal{S}(Y|F)$  have non-torsion first Chern class. When a local system  $\Gamma_\eta$  is given, we define  $HM_\bullet(Y|F; \Gamma_\eta)$  similarly. If  $F$  is a surface with more than one component, we define

$$\mathcal{S}(Y|F) = \bigcap_{F_i \subset F} \mathcal{S}(Y|F_i)$$

where the  $F_i$  are the components, and we define  $HM_\bullet(Y|F)$  accordingly.

As a special case, we have:

**Lemma 2.2.** *Let  $F$  be a closed, connected, oriented surface of genus at least 2, and let  $Y = F \times S^1$ . Regard  $F$  as a surface  $F \times \{p\}$  in  $Y$ . Then we have*

$$HM_{\bullet}(Y|F) = \mathbb{Z}.$$

*Indeed, if  $F$  is given a metric of constant negative curvature and  $Y$  is given the product metric, then the complex that computes  $HM_{\bullet}(Y|F)$  has a single generator, corresponding to a single, non-degenerate solution of the Seiberg-Witten equations.*

*Proof.* This is standard. The  $\text{spin}^c$  structure that contributes is the product  $\text{spin}^c$  structure, which corresponds to the 2-plane field tangent to the fibers of the map  $Y \rightarrow S^1$ . The unique gauge-equivalence class of solutions to the equations is a pair  $[A, \Phi]$  with  $\Phi$  covariantly constant. q.e.d.

**Corollary 2.3.** *Let  $Y$  be the product  $F \times S^1$ , as in the previous lemma. Then for any local coefficient system  $\Gamma_{\eta}$ , we have*

$$HM_{\bullet}(Y|F; \Gamma_{\eta}) = \mathcal{R},$$

where  $\mathcal{R}$  is the coefficient ring.

*q.e.d.*

**2.5. Disconnected 3-manifolds, part I.** So far, following [18], we have discussed connected 3-manifolds and connected 4-dimensional cobordisms between them. Because of the special role played by reducible connections, one must be careful when generalizing; but there are simple situations where the discussion can be carried over without difficulty to the case of 3-manifolds with several components. The analysis of the Seiberg-Witten equations on a manifold with cylindrical ends is carried out in [18] for an arbitrary number of ends, and our task here is just to package the resulting information.

Let  $W$  be cobordism from  $Y_1$  to  $Y_2$ , and suppose that each of these has components

$$Y_1 = Y_{1,1} \cup \cdots \cup Y_{1,r},$$

$$Y_2 = Y_{2,1} \cup \cdots \cup Y_{2,s}.$$

Although we label them this way, no ordering of the components need be chosen at this point. We may allow either  $r$  or  $s$  (or both) to be zero, and we do not require  $W$  to be connected. If  $W$  has any closed components, we insist that each such component has  $b_+ \geq 2$ .

As a simple way to avoid reducible connections, let us give a closed, oriented surface  $F_1 \subset Y_1$  and  $F_2 \subset Y_2$ . We will suppose that each component of  $Y_1$  contains a component of  $F_1$  and that all components of  $F_1$  have genus 2 or more. Thus we have non-empty surfaces

$$\begin{aligned} F_{1,i} &= F_1 \cap Y_{1,i} \\ &\subset Y_{1,i}. \end{aligned}$$

We make a similar hypothesis for  $F_2$ . We can regard the union  $F_1 \cup F_2$  as a subset of  $W$ , and we suppose that we are given a surface  $F_W \subset W$  which contains  $F_1 \cup F_2$  in addition perhaps to other components. The notation we previously used for  $\text{spin}^c$  structures with constraints on  $c_1$  can be extended to this case: we write  $\mathcal{S}(W|F_W)$  for the set of  $\text{spin}^c$  structures  $\mathfrak{s}$  on  $W$  such that (3) holds for every component of  $F_W$ .

We can define  $HM_\bullet(Y_1|F_1)$  by taking a product over the components of  $Y_1$ . That is, we should define the configuration space  $\mathcal{B}(Y_1)$  as the product of the  $\mathcal{B}(Y_{1,i})$ , and we should construct  $HM_\bullet(Y_1|F_1)$  as the Floer homology of the Chern-Simons-Dirac functional on the components of this product space which belong to the appropriate  $\text{spin}^c$  structures. The only slight twist here is in understanding the orientations of moduli spaces that are needed to fix the signs. There are various issues here. The first is to simply verify that (as expected) it is possible to define orientations so that the Floer homology of a disconnected manifold (and the maps induced by cobordisms) do make sense with  $\mathbb{Z}$  coefficients (rather than merely mod 2). The second issue is to compare the orientations arising in the case of a disconnected 3-manifold with the “product orientations” coming from the individual components: this is a similar issue to one that would arise when considering the Künneth theorem in the setting of Morse theory, with a Morse function on a product space obtained as the sum of Morse functions coming from each factor. A third issue is a little different, and concerns the signs that arise when two components of the boundary of a 4-manifold with boundary are identified by an orientation-reversing diffeomorphism, so as to form a new 4-manifold with fewer boundary components. We take up each of these issues in turn.

**Orientations in the disconnected case.** For a cobordism such as  $W$  above, perhaps with several components, we define a 2-element set  $\Lambda(W)$  of homology orientations of  $W$  as follows. Attach cylindrical ends to the incoming and outgoing ends to get a complete manifold  $W^+$ , and let  $t$  be a function which agrees with the cylindrical coordinate on the ends. The function  $t$  tends to  $+\infty$  on the outgoing ends and  $-\infty$  on the incoming ends. Consider the linearized anti-self-duality operator  $\delta = d^* \oplus d^+$  acting on the weighted Sobolev spaces

$$\delta : L_{1,\epsilon}^2(i\Lambda^1) \rightarrow L_\epsilon^2(i\Lambda^0 \oplus i\Lambda^+),$$

where  $L_{k,\epsilon}^2 = e^{-\epsilon t} L_k^2$ . Fix a  $\text{spin}^c$  structure on  $W$  and let  $D_{A_0}^+$  be the Dirac operator, for a  $\text{spin}^c$  connection  $A_0$  that is constant on the ends. We consider  $D_A$  acting on weighted Sobolev spaces of the same sort, and we write

$$P = \delta + D_{A_0}^+.$$

These are the linearized Seiberg-Witten equations on  $W$ , with Coulomb gauge fixing, at a configuration where the spinor is zero. There exists

$\epsilon_0 > 0$  such that the operator  $P$  is Fredholm for all  $\epsilon$  in the interval  $(0, \epsilon_0)$ . We define  $\Lambda(W)$  to be the set of orientations of the determinant line of  $P$ , for any  $\epsilon$  in this range. Using weighted Sobolev spaces here is equivalent to using ordinary Sobolev spaces and replacing  $P$  by a zeroth-order perturbation which on the ends has the form

$$P - \epsilon\Theta$$

where  $\Theta$  is obtained from applying the symbol of  $P$  to the vector field  $\partial/\partial t$  along the cylinder. The Dirac operator is irrelevant at this point because it is complex and its real determinant is therefore canonically oriented; so we could use the operator  $\delta$  instead.

Now suppose that  $\alpha_1$  and  $\alpha_2$  are gauge-equivalence classes corresponding to non-degenerate critical points of the Chern-Simons-Dirac functional on  $Y_1$  and  $Y_2$ , respectively. Let  $\gamma = (A, \Phi)$  be any configuration on  $W^+$  that is asymptotic to these gauge-equivalence classes on the ends. Let  $P_\gamma$  be the corresponding operator (acting on the Sobolev spaces without weights). Define  $\Lambda(W; \alpha_1, \alpha_2)$  to be the set of orientations of the determinant of  $P_A$ . This is independent of the choice of  $\gamma$ , in a canonical manner. If  $\Lambda_1$  and  $\Lambda_2$  are two 2-element sets, we use the notation  $\Lambda_1\Lambda_2$  to denote the 2-element set formed by the obvious ‘‘multiplication’’ (the set of bijections from  $\Lambda_1$  to  $\Lambda_2$ ). With this in mind, we define

$$\Lambda(\alpha_1, \alpha_2) = \Lambda(W)\Lambda(W; \alpha_1, \alpha_2).$$

An excision argument makes this independent of  $W$ . Given now a 3-manifold  $Y$  (with several components) and a non-degenerate critical point  $\alpha$ , we choose a cobordism  $X$  from the empty set to  $Y$  and we define

$$\Lambda(\alpha) = \Lambda(\emptyset, \alpha).$$

We then have

$$\Lambda(\alpha_1, \alpha_2) = \Lambda(\alpha_1)\Lambda(\alpha_2).$$

What this last equality means in practice is the following. If we are given a cobordism  $W$  with a choice of homology orientation in  $\Lambda(W)$  and a moduli space  $M = M(W; \alpha_1, \alpha_2)$ , then a choice of orientation of  $M$  is the same as a choice of bijection from  $\Lambda(\alpha_1)$  to  $\Lambda(\alpha_2)$ . In the case that  $Y_1 = Y_2$ , the cylindrical cobordism has a canonical homology-orientation because the operator  $P$  is invertible; so in this case  $\Lambda(\alpha_1)\Lambda(\alpha_2)$  orients the moduli spaces. The appropriate definition of  $HM_\bullet(Y_1, \mathfrak{s})$  for  $\text{spin}^c$  structures  $\mathfrak{s}$  that are non-torsion on each component is therefore to take the complex to be

$$C_\bullet(Y_1, \mathfrak{s}) = \bigoplus_{\alpha_1} \mathbb{Z}\Lambda(\alpha_1)$$

and to define the differential using the corresponding orientation of the moduli spaces. In this way, we construct  $HM_\bullet(Y_1|F_1)$  and  $HM_\bullet(Y_2|F_2)$ .



If we supply  $W$  with a homology orientation in the above sense, then  $W$  defines a map

$$(5) \quad HM(W|F_W) : HM_\bullet(Y_1|F_1) \rightarrow HM_\bullet(Y_2|F_2).$$

The notation  $HM(W|F_W)$  is meant to imply that we use only the  $\text{spin}^c$  structures from  $\mathcal{S}(W|F_W)$ .

**Products and the Künneth theorem.** The complex  $C_\bullet(Y_1, \mathfrak{s})$  just defined can be considered as a tensor product over the connected components of  $Y_1$ :

$$(6) \quad C_\bullet(Y_1, \mathfrak{s}) = \bigotimes_i C_\bullet(Y_{1,i}, \mathfrak{s}_i),$$

but there are some choices involved. Let us pick an ordering of the components. For each  $i$ , let  $X_i$  be a cobordism from the empty set to  $Y_{1,i}$ . Using the standard convention for the orientation of a direct sum, we can then identify

$$\Lambda(X) = \Lambda(X_1)\Lambda(X_2) \cdots \Lambda(X_r),$$

and similarly with  $\Lambda(X, \alpha_1)$ . In this way, we can specify an isomorphism

$$\mathbb{Z}\Lambda(\alpha_1) \rightarrow \mathbb{Z}\Lambda(\alpha_{1,1}) \otimes \cdots \otimes \mathbb{Z}\Lambda(\alpha_{1,r}).$$

This allows us to identify the complexes on the left and right in (6) as groups. Ordering issues mean that there will be the expected alternating signs appearing when we compare the differentials on the left and right. As usual with products in homology, what results from this is a split short exact sequence

$$(7) \quad 0 \rightarrow \bigotimes_i HM_\bullet(Y_{1,i}|F_{1,i}) \rightarrow HM_\bullet(Y_1|F_1) \rightarrow T \rightarrow 0$$

where  $T$  is a torsion group. If  $W$  is closed and has more than one component, the invariant is a product of the contributions from each component.

**Orientations and graded traces.** There is another sign issue to discuss. Consider the case of a 3-manifold  $Y$  with non-torsion  $\text{spin}^c$  structure  $\mathfrak{s}$ . Let  $Z$  be the 4-manifold  $S^1 \times Y$ . We can pull back the  $\text{spin}^c$  structure to  $Z$ , and we still call it  $\mathfrak{s}$ . For clarity, suppose that  $b_+(Z)$  is bigger than 1, so that  $\mathfrak{m}(Z, \mathfrak{s})$  is defined. To fix the sign of  $\mathfrak{m}(Z, \mathfrak{s})$ , we need a homology orientation of  $Z$ ; but a product such as  $Z$  has a preferred homology orientation. To define it, we must specify an orientation for the determinant of  $P$  on  $Z$ . The operator  $P - \epsilon\Theta$  is invertible for small  $\epsilon$ , and we use this to orient the determinant. Now let  $\alpha$  be non-degenerate critical point for the (possibly perturbed) Chern-Simons-Dirac functional on  $(Y, \mathfrak{s})$ . This pulls back to an isolated, non-degenerate solution on  $Z$  to the 4-dimensional Seiberg-Witten equations, say,  $\hat{\alpha}$ . This solution contributes either  $+1$  or  $-1$  to the invariant  $\mathfrak{m}(Z, \mathfrak{s})$ . We have the following lemma.

**Lemma 2.4.** *The solution  $\hat{\alpha}$  contributes  $+1$  or  $-1$  to the invariant  $\mathfrak{m}(Z, \mathfrak{s})$  according as the critical point  $\alpha$  has odd or even grading in  $C_\bullet(Y, \mathfrak{s})$ , for the canonical  $\mathbb{Z}/2$  grading.*

*Proof.* We have two operators differing by zeroth-order terms

$$\begin{aligned} P_0 &= P - \epsilon\Theta, \\ P_1 &= P_\gamma. \end{aligned}$$

Let  $P_s$  be a homotopy between them. We have a determinant line for this family of operators over the interval  $[0, 1]$ , and the invertibility of  $P_0$  and  $P_1$  at the two ends gives the determinant line a canonical orientation at the two ends. The sign with which  $\hat{\alpha}$  contributes is, by definition,  $+1$  or  $-1$  according as these two orientations at  $s = 0, 1$  are homotopic.

On the other hand, we can write  $P_s$  as

$$\frac{d}{dt} + L_s$$

on  $S^1 \times Y$ , where  $L_s$  is a self-adjoint elliptic operator perturbed by a bounded term, and the canonical mod 2 grading of  $\alpha$  is determined, by definition, by the parity of the spectral flow of the family of operators  $L_s$  from  $s = 0$  to  $s = 1$ .

So we must see that the parity of the spectral flow of the operators  $L_s$  determines whether the invertible operators  $P_0$  and  $P_1$  provide the same orientation. This is a general fact about families of self-adjoint Fredholm operators. What we have here are two non-trivial homomorphisms

$$H_1(S) \rightarrow \mathbb{Z}/2,$$

where  $S$  is a suitable space of self-adjoint operators. One can argue as in [18, section 33.1], following [1], that one may take  $S$  to have the homotopy of  $U(\infty)/O(\infty)$ , at which point it is clear that these two are the same. q.e.d.

A consequence of the lemma is that the invariant  $\mathfrak{m}(Z, \mathfrak{s})$  is equal to the Euler characteristic of  $HM_\bullet(Y, \mathfrak{s})$ , computed using the canonical mod 2 grading. From the lemma and excision, we obtain similar results in other situations of the following sort. Consider again a cobordism  $W$  from  $Y_1$  to  $Y_2$  with surfaces  $F_W$ ,  $F_1$  and  $F_2$  as before. Suppose that one of the incoming boundary components is the same as one of the outgoing ones: say,

$$Y_{1,r} = Y_{2,s}.$$

We may form a new  $W^*$  from  $W$  by identifying these boundary components, so  $W^*$  has  $r - 1$  incoming and  $s - 1$  outgoing boundary components. The manifolds  $Y_{1,r}$  and  $Y_{2,s}$  may belong either to the same or to different components of  $W$ , but we treat these cases together. The surface  $F_W$  gives rise to a homeomorphic surface  $F_{W^*}$  in  $W^*$ . (We push  $F_{1,r}$  and  $F_{2,s}$  away from the boundary of  $W$  before gluing to  $Y_{1,r}$  to

$Y_{2,r}$ , to keep these surfaces disjoint, if necessary.) If is possible that this process has created a  $W^*$  which has one more closed component than  $W$ . This new closed component of  $W^*$  will have  $b_+$  at least 1; but we shall suppose that, if there is such a component, it has  $b_+$  at least 2. (The case of  $b_+ = 1$  will be discussed in a slightly different context in the next subsection.)

Under this hypothesis on  $b_+$  for the closed components, we now have a new map

$$(8) \quad HM(W^*|F_{W^*}) : HM_\bullet(Y_1^*|F_1^*) \rightarrow HM_\bullet(Y_2^*|F_2^*),$$

where  $Y_1^*$  is  $Y_1 \setminus Y_{1,r}$  and  $Y_2^*$  is defined similarly. The analysis from [18, chapter V] (in particular, section 18 of [18]) provides a “gluing theorem” which tells us that the map  $HM(W^*|F_{W^*})$  is obtained from  $HM(W|F_W)$  by a contraction. More precisely, at the chain level,  $(W, F_W)$  defines a chain map

$$\bigotimes_i C_\bullet(Y_{1,i}|F_{1,i}) \rightarrow \bigotimes_j C_\bullet(Y_{2,j}|F_{2,j}).$$

This map can be contracted by taking an alternating trace over

$$C_\bullet(Y_{1,r}|F_{1,r}) = C_\bullet(Y_{2,s}|F_{2,s}),$$

and the result of this contraction is a chain map which is chain-homotopic to the chain map defined by  $(W^*, F_{W^*})$ .

The cobordism  $W$  from  $Y_1$  to  $Y_2$  can also be regarded as a cobordism  $\tilde{W}$  from  $\tilde{Y}_1$  to  $\tilde{Y}_2$ , where

$$\tilde{Y}_1 = Y_1 \cup (-Y_{2,s})$$

and

$$\tilde{Y}_2 = Y_2 \setminus Y_{2,s}.$$

(That is, we regard the last outgoing component as an incoming component with the opposite orientation.) The relation between the maps defined by  $W$  and  $\tilde{W}$  can be put in the same context as the above gluing theorem. We first add an extra component  $Z$  to  $W$ , where  $Z$  is the cylinder  $[0, 1] \times Y_{2,s}$ , regarded as a cobordism from  $Y_{2,s} \cup (-Y_{2,s})$  to the empty set. The map defined by  $W \cup Z$  is a tensor product, at the chain level, and the cobordism  $\tilde{W}$  can be obtained by gluing an outgoing component of  $W$  to an incoming component of  $Z$ . All that is left is to understand the map defined by  $Z$ . Discounting torsion, this last map is the Poincaré duality pairing

$$HM_\bullet(-Y_{2,s}|F_{2,s}) \otimes HM_\bullet(Y_{2,s}|F_{2,s}) \rightarrow \mathbb{Z}.$$

As in [18], this pairing depends on a homology orientation of  $Y_{2,s}$ , which reappears as the need to choose a homology orientation for the extra component  $Z$ .

Let us pursue a simple application of this formalism. Let  $W$  be again a cobordism from  $Y_1$  to  $Y_2$  and let  $F_1$  and  $F_2$  be surfaces in these

boundary 3-manifolds as above. Suppose that  $W$  contains in its interior a product 3-manifold

$$Z = G \times S^1$$

where  $G$  is connected of genus at least 2. Regard  $G = G \times \{p\}$  also as a submanifold of  $W$ . Form a new cobordism  $W^\dagger$  from  $Y_1$  to  $Y_2$  by the following process. Cut  $W$  open along  $Z$  to obtain a manifold  $W'$  with two extra boundary components  $G \times S^1$ , then attach a copy of  $G \times D^2$  to each of these boundary components to obtain  $W^\dagger$ . Set

$$\begin{aligned} F_W &= (F_1 \cup F_2 \cup G) \subset W, \\ F_{W^\dagger} &= (F_1 \cup F_2 \cup G) \subset W^\dagger. \end{aligned}$$

Then we have:

**Proposition 2.5.** *The maps  $HM(W|F_W)$  and  $HM(W^\dagger|F_{W^\dagger})$  are equal, up to sign, as maps*

$$HM(Y_1|F_1) \rightarrow HM(Y_2|F_2).$$

*Proof.* Consider the manifold  $W'$  obtained from  $W$  by cutting open along  $Z$ . This is a cobordism from  $Y_1 \cup Z$  to  $Y_2 \cup Z$ . The manifold  $W$  or  $W^\dagger$  can be obtained from  $W'$  by gluing with  $[0, 1] \times Z$  or with  $(D^2 \amalg D^2) \times G$ , respectively. We can regard  $[0, 1] \times Z$  and  $(D^2 \amalg D^2) \times G$  as two different cobordisms from  $Z$  to  $Z$ , and they both induce maps

$$HM_\bullet(Z|G) \rightarrow HM_\bullet(Z|G).$$

The result follows from the glueing formalism as long as we know that these two maps on  $HM_\bullet(Z|G)$  are the same. Lemma 2.2 tells us that  $HM_\bullet(Z|G)$  is simply  $\mathbb{Z}$ . The product  $[0, 1] \times Z$  of course induces the identity map on this copy of  $Z$ . So it only remains to show that the invariant of manifold  $D^2 \times G$  in  $HM_\bullet(Z|G)$  is  $\pm 1$ . This can be seen directly by examining the solutions of the Seiberg-Witten equations; or one can see indirectly—on general grounds—that this must be so. Indeed, we know that the product cobordism,  $W_1 = [0, 1] \times Z$ , from  $Z$  to  $Z$  induces the identity map on  $HM_\bullet(Z|G)$ :

$$(9) \quad HM(W_1) : HM_\bullet(Z|G) \rightarrow HM_\bullet(Z|G).$$

Let  $W_2$  be the cobordism from  $Z \amalg Z$  to  $Z$  obtained from  $W_1$  by removing a neighborhood of a copy of  $G$  from the interior. The map (9) is obtained from the map

$$HM(W_2) : HM_\bullet(Z|G) \otimes HM_\bullet(Z|G) \rightarrow HM_\bullet(Z|G)$$

by evaluating on the invariant of  $D^2 \times G$  in the first factor. Since (9) is an isomorphism from  $\mathbb{Z}$  to  $\mathbb{Z}$ , it follows that the invariant of  $D^2 \times G$  is  $\pm 1$ . q.e.d.

**2.6. Disconnected 3-manifolds, part II.** In the previous subsection we discussed gluing results in a context where the boundary components of the cobordisms carried  $\text{spin}^c$  structures that had non-torsion first Chern classes. The non-torsion condition ensures that reducible solutions on the 3-manifolds play no role. A situation that is algebraically similar is when the boundary components  $Y$  carry 1-cycles  $\eta$  and we use local coefficients for which the vanishing theorem Proposition 2.1 applies. We can think of  $\widehat{HM}_\bullet(Y; \Gamma_\eta)$  as measuring the contribution of the reducible solutions; so in a situation where this group is zero, as in the proposition, we can expect simple gluing results. This expectation is confirmed in the case of connected 3-manifolds by the results of [18, section 32]. We will deal here with the simplest situation, in which the boundary components are 3-tori and local coefficients are used.

Let  $W$  be a compact oriented 4-manifold with boundary, and suppose the oriented boundary consists of a collection of 3-tori,

$$\partial W = T_1 \cup \cdots \cup T_r.$$

We do not need to suppose that  $W$  is connected, but we do require that every closed component of  $W$  has  $b_+$  at least 2. Let  $\nu \subset W$  be a 2-chain with

$$\partial \nu = \eta_1 + \cdots + \eta_r.$$

We suppose that each  $\eta_i$  is a 1-cycle in  $T_i$  satisfying the hypotheses of Proposition 2.1 and that our coefficient ring  $\mathcal{R}$  has  $t - t^{-1}$  invertible. We may take it that each  $\eta_i$  is a standard circle. For each  $i$ , the map

$$j : \widehat{HM}_\bullet(T_i; \Gamma_{\eta_i}) \rightarrow \widehat{HM}_\bullet(T_i; \Gamma_{\eta_i})$$

is an isomorphism according to the proposition, so we again just write

$$HM_\bullet(T_i; \Gamma_i)$$

for this group, using  $j$  to identify the two. According to [18, section 37], this group is a free  $\mathcal{R}$ -module of rank 1,

$$HM_\bullet(T_i; \Gamma_{\eta_i}) \cong \mathcal{R}.$$

(The proof in [18] was done in the case that  $\mathcal{R} = \mathbb{R}$ , but only the invertibility of  $t - t^{-1}$  is needed.) After choosing a basis element in  $HM_\bullet(T_i; \Gamma_{\eta_i})$ , we should expect  $W$  to have an invariant living in

$$\bigotimes_i HM_\bullet(T_i; \Gamma_{\eta_i}) = \mathcal{R}.$$

However, there is a short-cut to defining an  $\mathcal{R}$ -valued invariant of  $W$ , used in [7] and [18, section 38]. We now describe this shortcut. In the remainder of this subsection, we will leave aside the question of choosing homology-orientations to fix the sign of the invariants that arise. So a 4-manifold or a cobordism will have an invariant that is ambiguous in its overall sign.

Let  $E(1)$  be a rational elliptic surface and let  $\widehat{E(1)}$  be the complement of the neighborhood of a regular fiber, so that  $\partial\widehat{E(1)} = T^3$ . Let  $\nu_1$  be a 2-cycle in  $E(1)$  arising from a section meeting the neighborhood of the fiber transversely in a disk, and let  $\hat{\nu}_1$  be the corresponding 2-chain in  $\widehat{E(1)}$ . Let  $\bar{W}$  be the closed 4-manifold obtained by attaching  $r$  copies of  $\widehat{E(1)}$  to  $W$ , making the attachments in such a way that the 1-cycles in the boundary tori match up: thus the manifold

$$\bar{W} = W \cup_{T_1} \widehat{E(1)} \cdots \cup_{T_r} \widehat{E(1)}$$

contains a 2-cycle

$$\bar{\nu} = \nu \cup_{\eta_1} \hat{\nu}_1 \cdots \cup_{\eta_r} \hat{\nu}_1.$$

We can now compute a Seiberg-Witten invariant of the closed pair  $(\bar{W}, \bar{\nu})$ , and the result depends only on  $(W, \nu)$ , not on the choice of gluing. Thus we may make a definition:

**Definition 2.6.** Let  $W$  have boundary a collection of 3-tori, as above, let  $\nu$  be a 2-chain in  $W$ , and let  $(\bar{W}, \bar{\nu})$  be the closed manifold obtained by attaching copies of  $\widehat{E(1)}$ . Suppose that every component of  $\bar{W}$  has  $b_+ \geq 2$ . Then we write

$$\mathbf{m}(W, \nu) \in \mathcal{R}$$

for the invariant  $\mathbf{m}(\bar{W}, \bar{\nu})$  of the closed manifold, as defined at (2).  $\diamond$

There is a formal device that can be used to extend this definition to include the case that  $\bar{W}$  has closed components with  $b_+ = 1$ . Let  $E(n)$  denote the elliptic surface without multiple fibers and having Euler number  $n$ , and let  $\widehat{E(n)}$  be the complement of a fiber. There is a 2-chain  $\nu_n$  just as in the case  $n = 1$ . Instead of attaching  $\widehat{E(1)}$  to each  $T_i$  to form  $\bar{W}$ , we can similarly attach  $\widehat{E(n_i)}$  to  $T_i$ , for any  $n_i \geq 1$ . We still refer to the resulting closed manifold as  $\bar{W}$ . It contains a 2-cycle  $\bar{\nu}$  as before. By choosing  $n_i$  larger than 1 when needed, we can ensure that all components of  $\bar{W}$  have  $b_+$  least 2. We then define  $\mathbf{m}(W, \nu)$  by

$$(10) \quad \mathbf{m}(W, \nu) = (t - t^{-1})^{-\sum(n_i-1)} \mathbf{m}(\bar{W}, \bar{\nu}).$$

By the results of [18, section 38], the quantity on the right is independent of the choice of the  $n_i$ .

Suppose next that  $W$  contains in its interior another 3-torus  $T$  which intersects  $\nu$  transversely in a single circle  $\eta$  representing a primitive element of  $H_1(T)$ . We can then cut  $W$  open along  $T$  to obtain  $W'$ , a manifold whose boundary consists of  $(r + 2)$  tori. We can denote the two new boundary components by  $T_{r+1}$  and  $T_{r+2}$ . By cutting  $\nu$  also, we obtain a 2-chain  $\nu'$  in  $W'$  whose boundary has two new circles  $\eta_{r+1}$  and  $\eta_{r+2}$  in the new boundary components. We have the following glueing theorem. (The hypothesis that  $t - t^{-1}$  is invertible in  $\mathcal{R}$  remains in place.)

**Proposition 2.7.** *In the above situation, the invariants of  $(W, \nu)$  and  $(W', \nu')$  are equal: thus*

$$\mathbf{m}(W, \nu) = \mathbf{m}(W', \nu')$$

*in the ring  $\mathcal{R}$ .*

*Proof.* There are two cases, according as  $T$  is separating or not. The separating case is treated in [18, section 38]. We deal here with the non-separating case. The definitions mean that both sides are to be interpreted as invariants of suitable closed manifolds. Restating it in such terms, and throwing out the components that do not contain  $T$ , we arrive at the following. Let  $X$  be a closed, connected 4-manifold with  $b_+ \geq 2$ , and let  $T \subset X$  be a non-separating 3-torus. Let  $\nu$  be a 2-cycle in  $X$  meeting  $T$  transversely in a standard circle  $\eta$  with multiplicity 1. Let  $X'$  be cobordism from  $T$  to  $T$  obtained by cutting  $X$  open, and let  $\nu'$  be the resulting 2-chain in  $X'$ . Because of what we already know about the separating case, the proposition is equivalent to the following lemma, which we shall prove. q.e.d.

**Lemma 2.8.** *In the above situation, the map induced by the cobordism,*

$$\widehat{HM}_\bullet(X'; \Gamma_{\nu'}) : \widehat{HM}_\bullet(T; \Gamma_\eta) \rightarrow \widehat{HM}_\bullet(T; \Gamma_\eta),$$

*is given by multiplication by the element  $\mathbf{m}(X, \nu) \in \mathcal{R}$ .*

*Proof.* It is convenient to arrange first that  $X'$  has  $b_+$  at least 1. We can do this by choosing a standard 2-torus  $F$  near  $T$  intersecting  $\nu$  transversely and forming a fiber sum at  $F$  with an elliptic surface  $E(n)$ . From what we know about separating 3-tori, we can conclude that this modification multiplies both  $\widehat{HM}_\bullet(X'; \Gamma_{\nu'})$  and  $\mathbf{m}(X, \nu)$  by  $(t - t^{-1})^{n-1}$ .

We now perturb the Chern-Simons-Dirac functional on  $T$ , as in [18, section 37], so that there are only reducible critical points, and we stretch  $X$  at  $T$ , inserting a cylinder  $[-R, R] \times T$  and letting  $R$  increase to infinity as usual. We consider what happens to the zero-dimensional moduli spaces on  $X$  in the limit. Because  $b_+(X')$  is at least 1, we obtain in the limit only irreducible solutions on the cylindrical-end manifold obtained from  $X'$ . Furthermore, these irreducible solutions run from boundary-unstable critical points at the incoming end to boundary-stable critical points at the outgoing end. The weighted count of such solutions defines the map

$$\overrightarrow{HM}(X'; \Gamma_{\nu'}) : \widehat{HM}_\bullet(T; \Gamma_\eta) \rightarrow \widetilde{HM}_\bullet(T; \Gamma_\eta)$$

in the notation of [18, subsection 3.5]. We must also obtain in the limit some (possibly broken) trajectories on the cylindrical part, running from boundary-stable critical points to boundary-unstable critical points. For dimension-counting reasons, these trajectories must actually

be unbroken and must be boundary-obstructed. The weighted count of such trajectories defines the map

$$j : \widetilde{HM}_\bullet(T; \Gamma_\eta) \rightarrow \widehat{HM}_\bullet(T; \Gamma_\eta).$$

Thus  $\mathfrak{m}(X, \nu)$  is equal to the contraction by the Kronecker pairing of two chain maps which on homology define the composite

$$j \circ \overrightarrow{HM}_\bullet(X; \Gamma_{\nu'}) : \widehat{HM}_\bullet(T; \Gamma_\eta) \rightarrow \widehat{HM}_\bullet(T; \Gamma_\eta).$$

It follows that  $\mathfrak{m}(X, \nu)$  is the trace of this composite map. The composite is equal to  $\widehat{HM}_\bullet(X'; \Gamma_{\nu'})$ , and the Floer group here is a free  $\mathcal{R}$ -module of rank 1, so the result follows. q.e.d.

There is a straightforward modification of the above results in the case that  $W$  has some additional boundary components which are not 3-tori but contain surfaces  $F$  of genus 2 or more, as in the previous subsection. That is, we suppose that the boundary of  $W$  is a union of 3-tori  $T_1, \dots, T_r$  together with a pair of 3-manifolds  $-Y_1$  and  $Y_2$ , each of which may have several components. We suppose also that  $Y_1$  and  $Y_2$  contain surfaces  $F_1$  and  $F_2$  all of whose components have genus 2 or more. We also ask that each component of  $Y_i$  contains a component of  $F_i$ . We shall suppose that there is a 2-chain  $\nu$  in  $W$  whose boundary we write as

$$\partial\nu = -\zeta_1 + \zeta_2 + \eta_1 + \dots + \eta_r.$$

The  $\eta_i$  are to be standard circles, one in each torus  $T_i$  as before. The 1-cycles  $\zeta_1$  and  $\zeta_2$  will be in  $Y_1$  and  $Y_2$ , but we can allow these to be arbitrary (zero for example). We take  $F_W$  to be any closed surface in  $W$  consisting of  $F_1 \cup F_2$  together perhaps with additional components. We again suppose that any closed component of  $W$  has  $b_+ \geq 2$ . Then  $W$  should give rise to a map

$$(11) \quad HM_\bullet(W|F_W; \Gamma_\nu) : HM_\bullet(Y_1|F_1; \Gamma_{\zeta_1}) \rightarrow HM_\bullet(Y_2|F_2; \Gamma_{\zeta_2}).$$

To define this map, we can again attach  $(\widehat{E(1)}, \hat{\nu}_1)$  to each of the 3-tori, to obtain  $(\bar{W}, \bar{\nu})$  a cobordism from  $Y_1$  to  $Y_2$  containing a 2-chain  $\bar{\nu}$  and a surface  $F_W$ . The boundary of  $\bar{\nu}$  is just  $-\zeta_1 + \zeta_2$ . As in Definition 2.6, we take  $HM_\bullet(W|F_W; \Gamma_\nu)$  to be *defined* by the map given by the cobordism  $\bar{W}$ . In the event that  $\bar{W}$  has any closed components with  $b_+ = 1$ , we modify the construction by using elliptic surfaces  $E(n_i)$  as in (10). Proposition 2.7 then has the following variant.

**Proposition 2.9.** *Let  $W$  be as above, and let  $T \subset W$  be a 3-torus meeting  $\nu$  transversely in a standard circle with multiplicity 1. Let  $W'$  and  $\nu'$  be obtained from  $W$  and  $\nu$  by cutting along  $T$ . Suppose that  $F_W$  is disjoint from  $T$ , so that it becomes also a surface  $F_{W'}$  in  $W'$ . Assume*



as always that  $t - t^{-1}$  is invertible in  $\mathcal{R}$ . Then the maps

$$(12) \quad \begin{aligned} HM_{\bullet}(W|F_W; \Gamma_{\nu}) &: HM_{\bullet}(Y_1|F_1; \Gamma_{\zeta_1}) \rightarrow HM_{\bullet}(Y_2|F_2; \Gamma_{\zeta_2}) \\ HM_{\bullet}(W'|F_{W'}; \Gamma_{\nu'}) &: HM_{\bullet}(Y_1|F_1; \Gamma_{\zeta_1}) \rightarrow HM_{\bullet}(Y_2|F_2; \Gamma_{\zeta_2}) \end{aligned}$$

are equal up to sign. q.e.d.

A particular application of this setup will be used in the sequel, a version of Proposition 2.5. We formulate the result as the following corollary:

**Corollary 2.10.** *Let  $W$  be a cobordism from  $Y_1$  to  $Y_2$  containing a 2-chain  $\eta$  with boundary  $-\zeta_1 \cup \zeta_2$ . Let  $F_1, F_2$  and  $F_W$  be surfaces as above. Let  $T \subset W$  be a 3-torus disjoint from  $F_W$  and cutting  $\nu$  in a standard circle  $\eta \subset T$ . Form  $W^{\dagger}$  by cutting  $W$  along  $T$  and attaching two copies of  $D^2 \times T^2$  in such a way that  $\partial D^2 \times \{p\}$  is glued to  $\eta$  in both copies. Let  $\eta^{\dagger}$  be the 2-chain in  $W^{\dagger}$  obtained by attaching 2-disks  $D^2 \times \{p\}$ . Then, as maps from  $HM_{\bullet}(Y_1|F_1; \Gamma_{\zeta_1})$  to  $HM_{\bullet}(Y_2|F_2; \Gamma_{\zeta_2})$ , we have*

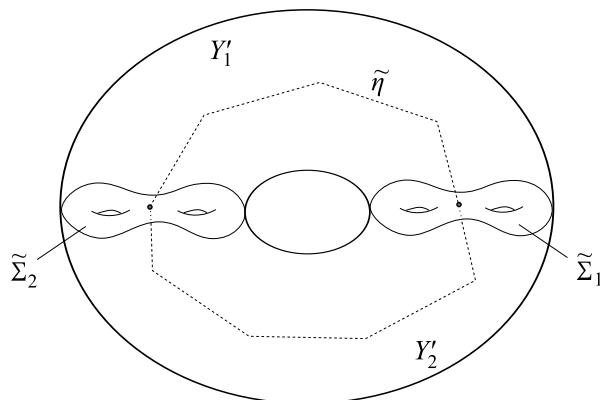
$$HM(W|F_W; \Gamma_{\nu}) = (t - t^{-1})HM(W^{\dagger}|F_{W^{\dagger}}; \Gamma_{\nu^{\dagger}}),$$

to within an overall sign.

*Proof.* Using Proposition 2.9, this can be proved with the same strategy that we applied to Proposition 2.5. That is, we consider two different cobordisms from  $T$  to  $T$ : first, the product cobordism, and second the (disconnected) cobordism formed from two copies of  $D^2 \times T^2$ . In each case, there is an obvious 2-chain whose boundary is the difference of the two copies of  $\eta$ . Each of these cobordisms has an invariant which lives in  $\mathcal{R}$ , according to Definition 2.6, or more accurately its correction at (10). In this sense, the product cobordism has invariant  $1 \in \mathcal{R}$ . The invariant of the other cobordism is  $(t - t^{-1})^{-1}$ , as can be deduced from the invariants of the elliptic surfaces. q.e.d.

### 3. Floer’s excision theorem

**3.1. The setup.** We shall need to understand how monopole Floer homology behaves under certain cutting and gluing operations on the underlying 3-manifold. A formula of the type that we need was first proved by Floer in the context of instanton homology. Floer’s “excision formula,” as he called it, applied only to cutting along tori; but in the monopole homology context one can equally well cut along surfaces of higher genus, as long as one restricts to  $\text{spin}^c$  structures that are of top degree on the surface where the cut is made. We give the proof in the monopole Floer homology context in this section: it is almost identical to Floer’s argument, as presented in [2]. Similar formulae have been proved in Heegaard Floer theory, by Ghiggini, Ni, and Juhász [11, 23, 25, 12, 13].



**Figure 1.** Forming a manifold  $\tilde{Y}$  from  $Y_1$  and  $Y_2$ , for the excision theorem.

The setup is the following. Let  $Y$  be a closed, oriented 3-manifold, of either one or two components. In the case of two components, we call the components  $Y_1$  and  $Y_2$ . Let  $\Sigma_1$  and  $\Sigma_2$  be closed oriented surfaces in  $Y$ , both of them connected and of equal genus. If  $Y$  has two components, then we suppose that  $\Sigma_i$  is a non-separating surface in  $Y_i$  for  $i = 1, 2$ . If  $Y$  is connected, then we suppose that  $\Sigma_1$  and  $\Sigma_2$  represent independent homology classes. In either case, we write  $\Sigma$  for  $\Sigma_1 \cup \Sigma_2$ . Fix an orientation-preserving diffeomorphism  $h : \Sigma_1 \rightarrow \Sigma_2$ . From this data, we construct a new manifold  $\tilde{Y}$  as follows. Cut each  $Y$  along  $\Sigma$  to obtain a manifold  $Y'$  with four boundary components: with orientations, we can write

$$\partial Y' = \Sigma_1 \cup (-\Sigma_1) \cup \Sigma_2 \cup (-\Sigma_2).$$

If  $Y$  has two components, then so does  $Y'$ , and we can write  $Y' = Y'_1 \cup Y'_2$ . Now form  $\tilde{Y}$  by gluing the boundary component  $\Sigma_1$  to the boundary component  $-\Sigma_2$  and gluing  $\Sigma_2$  to  $-\Sigma_1$ , using the chosen diffeomorphism  $h$  both times. See Figure 1 for a picture in the case that  $Y$  has two components. In either case,  $\tilde{Y}$  is connected. We write  $\tilde{\Sigma}_1$  for the image of  $\Sigma_1 = -\Sigma_2$  in  $\tilde{Y}$  and  $\tilde{\Sigma}_2$  for the image of  $\Sigma_2 = -\Sigma_1$ . So  $\tilde{Y}$  contains a surface  $\tilde{\Sigma} = \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$ .

If we wish to use local coefficients in Floer homology, we will need to augment this excision picture with 1-cycles  $\eta$ . Specifically, we take a 1-cycle  $\eta$  in  $Y$  that intersects each  $\Sigma_i$  transversely in a single point  $p_i$  ( $i = 1, 2$ ) with positive orientation. If  $Y$  has two components, then we may write  $\eta = \eta_1 + \eta_2$  for its two parts. We suppose that the diffeomorphism  $h$  is chosen so that  $h(p_1) = p_2$ . When this is done, the 1-cycle gives to a 1-cycle  $\tilde{\eta}$  in the new manifold  $\tilde{Y}$ , as shown, by cutting and gluing.

We begin with a statement of the excision theorem with integer coefficients, when the genus of  $\Sigma$  is two or more.

**Theorem 3.1.** *If  $\tilde{Y}$  is obtained from  $Y$  as above and the genus of  $\Sigma_1$  and  $\Sigma_2$  is at least 2, then there is an isomorphism of Floer groups with integer coefficients,*

$$HM_{\bullet}(Y|\Sigma) \rightarrow HM_{\bullet}(\tilde{Y}|\tilde{\Sigma}).$$

REMARK. In the case that  $Y$  has two components, the left-hand side is the homology of a tensor product of complexes. In this case, the statement of the theorem implies that there is a split short exact sequence

$$(13) \quad HM_{\bullet}(Y_1|\Sigma_1) \otimes HM_{\bullet}(Y_2|\Sigma_2) \rightarrow HM_{\bullet}(\tilde{Y}|\tilde{\Sigma}) \rightarrow \text{Tor}(HM_{\bullet}(Y_1|\Sigma_1), HM_{\bullet}(Y_2|\Sigma_2)).$$

Floer’s version of this theorem has  $\Sigma_1$  and  $\Sigma_2$  of genus 1, with  $Y = Y_1 \cup Y_2$ . It uses instanton Floer homology associated to an  $SO(3)$  bundle with non-zero Stiefel-Whitney class on  $\Sigma$ . To obtain a version in monopole Floer homology when  $\Sigma$  has genus 1, we need to use local coefficients. We present a version that is tailored to our later needs. We recall that  $\Gamma_{\eta}$  denotes a system of local coefficients with fiber  $\mathcal{R}$ , a commutative ring as in section 2.1. We suppose, as just discussed, that  $\eta$  meets  $\Sigma_1$  and  $\Sigma_2$  each in a single point so that we may form  $\tilde{\eta}$  as shown. Under these hypotheses, we expect there to be an isomorphism

$$HM_{\bullet}(Y; \Gamma_{\eta}) \rightarrow HM_{\bullet}(\tilde{Y}; \Gamma_{\tilde{\eta}}).$$

We shall not endeavor to prove this variant of Floer’s excision theorem here, because it involves considering reducible solutions on multiple boundary components. Instead, as in section 2.6, we introduce some auxiliary surfaces  $F$  and corresponding constraints on the  $\text{spin}^c$  structures, just to avoid reducibles.

Thus we suppose in addition that  $Y$  contains an oriented surface  $F$  meeting  $\Sigma = \Sigma_1 \cup \Sigma_2$  transversely, and that the diffeomorphism  $h : \Sigma_1 \rightarrow \Sigma_2$  carries the oriented intersection  $\Sigma_1 \cap F$  to  $\Sigma_2 \cap F$ . In this case, we can form an oriented surface  $\tilde{F}$  in the new 3-manifold  $\tilde{Y}$ , by cutting  $F$  and regluing. We suppose that neither  $F$  nor  $\tilde{F}$  contains a 2-sphere, and that every component of  $Y$  contains a component of  $F$  whose genus is at least 2.

**Theorem 3.2.** *Suppose  $\tilde{Y}$  and  $\tilde{F}$  are obtained from  $Y$  and  $F$  as above, with  $\Sigma_1$  and  $\Sigma_2$  both of genus 1. Let  $\tilde{\eta}$  be the 1-cycle in  $\tilde{Y}$  formed from the cycle  $\eta$  in  $Y$  as shown in Figure 1. Assume as usual that  $t - t^{-1}$  is invertible in the ring  $\mathcal{R}$ . Then there is an isomorphism:*

$$HM_{\bullet}(Y|F; \Gamma_{\eta}) \rightarrow HM_{\bullet}(\tilde{Y}|\tilde{F}; \Gamma_{\tilde{\eta}}).$$

REMARK. Note again that if  $Y$  has two components and  $\mathcal{R}$  is a field, then the left-hand side is the tensor product

$$HM_{\bullet}(Y_1|F_1; \Gamma_{\eta_1}) \otimes_{\mathcal{R}} HM_{\bullet}(Y_2|F_2; \Gamma_{\eta_2}).$$

There is also a simpler way in which local coefficients can enter into the excision theorem, when the cycle  $\eta$  does not intersect  $\Sigma$ . We state an adaptation of Theorem 3.1 of this sort.

**Theorem 3.3.** *Let  $\tilde{Y}$  be obtained from  $Y$  as in Theorem 3.1, with  $\Sigma$  of genus at least two. Let  $\eta_0$  be a 1-cycle in  $Y$ , disjoint from  $\Sigma$ . This becomes a cycle also in  $\tilde{Y}$ , which we denote by  $\tilde{\eta}_0$ . Then we have an isomorphism of  $\mathcal{R}$ -modules:*

$$HM_{\bullet}(Y|\Sigma; \Gamma_{\eta_0}) \rightarrow HM_{\bullet}(\tilde{Y}|\tilde{\Sigma}; \Gamma_{\tilde{\eta}_0}).$$

In Theorem 3.3, consider the case that  $Y = Y_1 \cup Y_2$  and  $\eta_0$  is contained in  $Y_1$ . In this case, the chain complex that computes the group  $HM_{\bullet}(Y|\Sigma; \Gamma_{\eta_0})$  on the left is

$$C_{\bullet}(Y_1|\Sigma_1; \Gamma_{\eta_0}) \otimes_{\mathbb{Z}} C_{\bullet}(Y_2|\Sigma_2),$$

(the tensor product of a complex of free  $\mathcal{R}$ -modules and a complex of free abelian groups, both finitely generated). By the Künneth theorem, if  $\mathcal{R}$  has no  $\mathbb{Z}$ -torsion and  $HM_{\bullet}(Y_1|\Sigma_1; \Gamma_{\eta_0})$  is a free  $\mathcal{R}$ -module, then the theorem provides an isomorphism

$$(14) \quad HM_{\bullet}(Y_1|\Sigma_1; \Gamma_{\eta_0}) \otimes HM_{\bullet}(Y_2|\Sigma_2) \rightarrow HM_{\bullet}(\tilde{Y}|\tilde{\Sigma}; \Gamma_{\tilde{\eta}_0}).$$

As a particular application of this result, we have:

**Corollary 3.4.** *Let  $\Sigma \subset Y$  be a closed, oriented surface whose components have genus at least 2 and let  $\eta$  be a 1-cycle in  $Y$  whose support lies in  $\Sigma$ . Suppose that  $\mathcal{R}$  has no  $\mathbb{Z}$ -torsion. Then*

$$HM_{\bullet}(Y|\Sigma; \Gamma_{\eta}) \cong HM_{\bullet}(Y|\Sigma) \otimes \mathcal{R}.$$

*Proof.* Apply the isomorphism of (14) with  $(Y_2, \Sigma_2) = (Y, \Sigma)$  and  $(Y_1, \Sigma_1) = (\Sigma \times S^1, \Sigma \times \{p\})$ . Take  $\eta_0$  in  $\Sigma \times S^1$  to be the cycle corresponding to  $\eta$ . By Proposition 2.3 we have

$$HM_{\bullet}(Y_1|\Sigma_1; \Gamma_{\eta_0}) = \mathcal{R}.$$

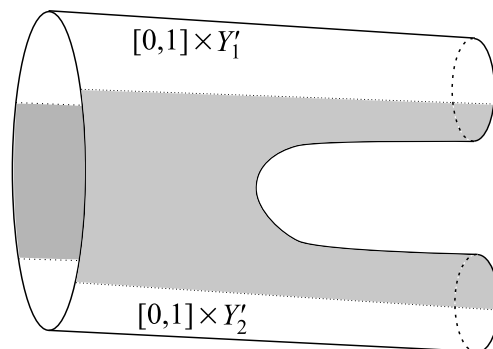
The manifold  $\tilde{Y}$  is another copy of the original  $Y$  and  $\tilde{\Sigma}$  is two parallel copies of  $\Sigma$ . The cycle  $\eta_0$  becomes now the original 1-cycle  $\eta$ , so

$$HM_{\bullet}(\tilde{Y}|\tilde{\Sigma}; \Gamma_{\tilde{\eta}_0}) = HM_{\bullet}(Y|\Sigma; \Gamma_{\eta}).$$

Thus (14) gives an isomorphism

$$\mathcal{R} \otimes HM_{\bullet}(Y|\Sigma) \rightarrow HM_{\bullet}(Y|\Sigma; \Gamma_{\eta}).$$

q.e.d.



**Figure 2.** A cobordism  $W$  from  $\tilde{Y}$  to  $Y = Y_1 \cup Y_2$ .

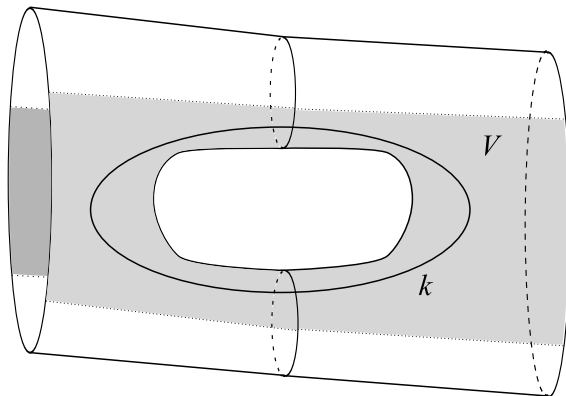
**3.2. Proof of the excision theorems.** The proof of Theorem 3.1 is very much the same as Floer's proof of his original excision theorem, as described in [2]. The first step (which is common to both Theorem 3.1 and Theorem 3.2) is to construct a cobordism  $W$  from  $\tilde{Y}$  to  $Y$ . In the case that  $Y$  is a disjoint union  $Y_1 \cup Y_2$ , the cobordism  $W$  admits a map  $\pi : W \rightarrow P$ , where  $P$  is a 2-dimensional pair-of-pants cobordism. This is shown schematically in figure 2. The 4-dimensional cobordism is the union of two pieces. The first piece is the product  $[0, 1] \times Y'$ , where  $Y'$  as before is obtained from  $Y$  by cutting open along  $\Sigma_1$  and  $\Sigma_2$ . (In the figure, this appears as the union of two pieces, corresponding to the decomposition of  $Y'$  as  $Y'_1 \cup Y'_2$ .) The second piece is the product of the closed surface  $\Sigma_1$  with a 2-manifold  $U$  with corners:  $U$  corresponds to the gray-shaded area in the figure. The two pieces are fitted together as shown, using the diffeomorphism  $h$ . If  $Y$  is connected, then the picture looks just the same in the neighborhood of the shaded region, but the product region  $[0, 1] \times Y'$  is connected; the cobordism  $W$  in this case does not admit a map to the pair of pants.

There is a very similar cobordism  $\bar{W}$  which goes the other way: Theorem 3.1 arises because the cobordisms  $W$  and  $\bar{W}$  give rise to mutually inverse maps (in the case of genus at least 2)

$$\begin{aligned} HM(W) &: HM_{\bullet}(\tilde{Y}|\tilde{\Sigma}) \rightarrow HM_{\bullet}(Y|\Sigma), \\ HM(\bar{W}) &: HM_{\bullet}(Y|\Sigma) \rightarrow HM_{\bullet}(\tilde{Y}|\tilde{\Sigma}), \end{aligned}$$

when the coefficients are a field.

To show that the cobordisms induce mutually inverse maps, let  $X$  be the cobordism from  $\tilde{Y}$  to  $\tilde{Y}$  formed as the union of  $W$  and  $\bar{W}$ . We must show that  $X$  gives rise to the identity map on  $HM_{\bullet}(\tilde{Y}|\tilde{\Sigma})$ . This will show that  $HM(\bar{W}) \circ HM(W) = 1$ , and there will be a similar argument for the other composite. Note that  $\Sigma$  and  $\tilde{\Sigma}$  are homologous in  $X$ , so



**Figure 3.** The composite cobordism  $X$  from  $\tilde{Y}$  to  $\tilde{Y}$ .

the map induced by  $X$  really does factor through  $HM_{\bullet}(Y|\Sigma)$ , not just  $HM_{\bullet}(Y)$ .

The manifold  $X$  is shown schematically in Figure 3 for the case that  $Y$  has two components,  $Y_1 \cup Y_2$ , in which case it admits a map  $\pi$  to the twice-punctured genus-1 surface, as drawn. Over the shaded region  $V$  it is a product,

$$\begin{aligned}\pi^{-1}(V) &= \Sigma_1 \times V \\ &= \Sigma_2 \times V.\end{aligned}$$

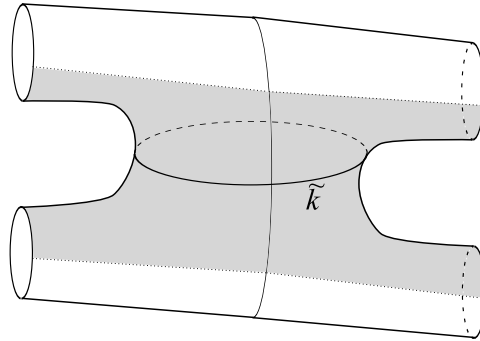
If  $Y$  is connected, the picture is essentially the same in the neighborhood of  $\pi^{-1}(V)$ . Let  $k$  be the closed curve in  $V$  that is shown, and let  $K$  be the inverse image

$$\begin{aligned}K &= \pi^{-1}(k) \\ &= \Sigma_1 \times k.\end{aligned}$$

(We continue to identify  $\Sigma_1$  with  $\Sigma_2$  via  $h$  in what follows.) Let  $X'$  be the manifold-with-boundary formed by cutting along  $K$ . Its boundary is two copies of  $K$ . Let  $X^*$  be the new cobordism from  $\tilde{Y}$  to  $\tilde{Y}$  obtained by attaching two copies of  $\Sigma_1 \times D^2$ , with  $\partial D^2$  being identified with  $k$ :

$$X^* = X' \cup (\Sigma_1 \times D^2) \cup (\Sigma_1 \times D^2).$$

Floer's proof hinges on the fact that the manifold  $X^*$  is just the product cobordism from  $\tilde{Y}$  to  $\tilde{Y}$ . This means that we only need show that  $X^*$  gives rise to the same map as  $X$ . This desired equality can be deduced from the formalism of section 2.5, for it is precisely Proposition 2.5. This concludes the proof that  $HM(\bar{W}) \circ HM(W) = 1$ . The picture for the composite of the two cobordisms in the other order is shown in Figure 4. The proof that this composite gives the identity is essentially the same: the relationship between  $Y$  and  $\tilde{Y}$  is a symmetric one, except that we have allowed only  $Y$  to have two components. Figure 4 shows the corresponding curve  $\tilde{k}$  in this case, along which one



**Figure 4.** The composite cobordism in the opposite order, from  $Y$  to  $Y$ , in the case that  $Y$  has two components.

must cut, just as we cut along  $k$  in the previous case. This completes the proof of Theorem 3.1.

The proof of Theorem 3.2 is very similar. The same cobordisms  $W$  and  $\bar{W}$  are used. In the cobordism  $W$ , there is a 2-chain  $\nu_W$  whose boundary is  $\eta - \tilde{\eta}$ . It consists of the product chain  $[0, 1] \times \eta'$  in part of  $W$  obtained from  $[0, 1] \times Y$ ; while over the shaded region  $U$  in Figure 2, the cycle  $\nu_W$  is a section  $\{p\} \times U$  of  $\Sigma_1 \times U$ . There is a similar 2-chain  $\nu_{\bar{W}}$  in  $\bar{W}$ , and these fit together to give a 2-chain  $\nu_X$  in the composite cobordism  $X$  (Figure 3). The 3-manifold  $K \subset X$  lying over the curve  $k$  is now a 3-torus, and  $K$  meets  $\nu_X$  transversely in a standard circle. The proof now proceeds as before, but using Corollary 2.10 in place of Proposition 2.5. We learn that the composite cobordism  $X$  gives a map which is  $(t - t^{-1})$  times the map arising from the trivial product cobordism  $X^\dagger$ . That is,

$$HM(\bar{W}|F_{\bar{W}}; \Gamma_{\nu_{\bar{W}}}) \circ HM(W|F_W; \Gamma_{\nu_W}) = (t - t^{-1}).$$

The same holds for the composite in the opposite order. Since  $t - t^{-1}$  is a unit in  $\mathcal{R}$ , this means that  $HM(W|F_W; \Gamma_{\nu_W})$  is an isomorphism, as required.

#### 4. Monopole Floer homology for sutured manifolds

In this section, we give the definition of the monopole homology groups for balanced sutured manifolds, which are the main object of study in this paper.

**4.1. Closing up sutured manifolds.** We recall Juhász’s definition of a balanced sutured manifold [12], a restricted version of Gabai’s notion of a sutured manifold [9]:

**Definition 4.1.** A *balanced sutured manifold*  $(M, \gamma)$  is a compact, oriented 3-manifold  $M$  with boundary, equipped with the following data:

- 1) a closed, oriented 1-manifold  $s(\gamma)$  in  $\partial M$ , i.e. a collection of disjoint oriented circles in the boundary, called the sutures;
- 2) a union  $A(\gamma)$  of annuli, which comprise a tubular neighborhood of  $s(\gamma)$  in  $\partial M$ ; the closure of  $\partial M \setminus A(\gamma)$  is called  $R(\gamma)$ .

These are required to satisfy the following conditions:

- 1)  $M$  has no closed components;
- 2) if the components of  $\partial A(\gamma)$  are oriented in the same sense as the sutures, then it should be possible to orient  $R(\gamma)$  so that its oriented boundary coincides with this given orientation of  $A(\gamma)$ ;
- 3)  $R(\gamma)$  has no closed components (which implies that the orientation in the previous item is unique); we call it the *canonical orientation*;
- 4) if we define  $R_+(\gamma)$  (and  $R_-(\gamma)$  also) as the subset of  $R(\gamma)$  where the canonical orientation coincides with the boundary orientation (or its opposite, respectively), then  $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$ .  $\diamond$

It is often helpful to consider sutured manifolds as manifolds with corners: the corners run along the circles  $\partial A(\gamma)$  and separate the flat annuli from the rest of the boundary. Note that  $M$  need not be connected. A model example is a *product sutured manifold*

$$([-1, 1] \times T, \delta).$$

Here  $T$  is an oriented surface with non-empty boundary and no closed components, and the sutures are

$$s(\delta) = \{0\} \times \partial T$$

with the boundary orientation. The annuli  $A(\delta)$  are  $[-1, 1] \times \partial T$ , and we have

$$\begin{aligned} R_+(\delta) &= \{1\} \times T, \\ R_-(\delta) &= \{-1\} \times T. \end{aligned}$$

Given a balanced sutured manifold  $(M, \gamma)$ , we form a closed, oriented manifold  $Y = Y(M, \gamma)$  as follows. The closed manifold is dependent on some choices, as we shall see. First, we choose an oriented connected surface  $T$  whose boundary components are in one-to-one correspondence with the components of  $s(\gamma)$ . We call  $T$  the *auxiliary surface*. From  $T$  we form the product sutured manifold  $([-1, 1] \times T, \delta)$  as just described. We then glue the annuli  $A(\delta)$  to the annuli  $A(\gamma)$ : this is done by a map

$$A(\delta) \rightarrow A(\gamma)$$

which is orientation-reversing with respect to the boundary orientations and which maps  $\partial R_+(\delta)$  to  $\partial R_+(\gamma)$ . The result of this step is a 3-manifold with exactly two boundary components,  $\bar{R}_+$  and  $\bar{R}_-$ , which are closed orientable surfaces of equal genus:

$$\begin{aligned} \bar{R}_+ &= R_+(\gamma) \cup \{1\} \times T, \\ \bar{R}_- &= R_-(\gamma) \cup \{-1\} \times T. \end{aligned}$$



We require  $T$  to be of sufficiently large genus (genus zero may suffice, and genus two always will) so that two conditions hold:

- C1) the genus of  $\bar{R}_\pm$  is at least 2;
- C2) the surface  $T$  contains a simple closed curve  $c$  such that  $\{1\} \times c$  and  $\{-1\} \times c$  are non-separating curves in  $\bar{R}_+$  and  $\bar{R}_-$ , respectively.

Finally, form  $Y(M, \gamma)$  by identifying  $\bar{R}_+$  with  $\bar{R}_-$  using any diffeomorphism which reverses the boundary orientations (i.e., preserves the canonical orientations),

$$h : \bar{R}_+ \rightarrow \bar{R}_-.$$

Inside  $Y$  is a closed, connected, non-separating surface  $\bar{R}$ , obtained from the identification of  $\bar{R}_+$  with  $\bar{R}_-$ . We can orient  $\bar{R}$  using the canonical orientation of  $R_+(\gamma)$ . As an oriented pair,  $(Y, \bar{R})$  depends only on two things, beyond  $(M, \gamma)$  itself: first, the choice of genus for  $T$ , and second the choice of diffeomorphism  $h$  used in the last step.

**Definition 4.2.** We call  $(Y, \bar{R})$  a *closure* of the balanced sutured manifold  $(M, \gamma)$  if it is obtained in this way, by attaching to  $(M, \gamma)$  a product region  $[-1, 1] \times T$  satisfying the above conditions and then attaching  $\bar{R}_+$  to  $\bar{R}_-$  by some  $h$ . ◇

**4.2. The definition.** Let  $Y = Y(M, \gamma)$  be formed from a sutured manifold  $(M, \gamma)$  as described in the previous subsection. Recall that  $Y$  contains a connected, oriented closed surface  $\bar{R}$ , by construction, whose genus is at least two. We make the following definition:

**Definition 4.3.** We define the monopole Floer homology of the sutured manifold  $(M, \gamma)$  to be the finitely generated abelian group

$$SHM(M, \gamma) := HM_\bullet(Y|\bar{R}),$$

where  $Y = Y(M, \gamma)$  is a closure of  $(M, \gamma)$  as described in Definition 4.2, and the notation on the right follows (4). ◇

As it stands, this definition appears to depend on the choice of genus,  $g$ , for the auxiliary surface  $T$ , as well as on the choice of gluing diffeomorphism  $h$ . In section 4.3 we shall prove:

**Theorem 4.4.** *The group  $SHM(M, \gamma)$  defined in 4.3 depends only on  $(M, \gamma)$ , not on the choice of genus  $g$  for the auxiliary surface  $T$  or the diffeomorphism  $h$ .*

There is a version of  $SHM$  with local coefficients that we shall use at some points along the way. Recall that  $T$  is required to contain a curve  $c$  that yields non-separating curves  $\{\pm 1\} \times c$  on  $\bar{R}_\pm$ . Let us choose the diffeomorphism  $h$  so that  $h$  maps  $\{1\} \times c$  to  $\{-1\} \times c$ , preserving orientation. Thus the surface  $\bar{R}$  in  $Y(M, \gamma)$  now contains a closed curve  $\bar{c}$ , the image of  $\{\pm 1\} \times c$ . Let  $c'$  be any dual curve on  $\bar{R}$ : a curve  $c'$  with  $\bar{c} \cdot c' = 1$  on  $\bar{R}$ .

**Definition 4.5.** We define the monopole Floer homology of the sutured manifold  $(M, \gamma)$  with local coefficients to be the  $\mathcal{R}$ -module

$$SHM(M, \gamma; \Gamma_\eta) := HM_\bullet(Y|\bar{R}; \Gamma_\eta),$$

where the closure  $Y = Y(M, \gamma)$  is constructed using a diffeomorphism  $h$  satisfying the constraint just described, and  $\eta$  is the 1-cycle in  $Y$  carried by the curve  $c'$  dual to  $\bar{c}$  as above.  $\diamond$

We shall see that this is independent of the choice of  $\eta$ . When using local coefficients in this way, we can relax the requirement that  $\bar{R}$  has genus 2 or more (condition C1 above) and allow closures in which  $\bar{R}$  has genus 1:

**Proposition 4.6.** *As long as  $t - t^{-1}$  is invertible in  $\mathcal{R}$ , the  $\mathcal{R}$ -module  $SHM(M, \gamma; \Gamma_\eta)$  defined in 4.5 depends only on  $(M, \gamma)$  and  $\mathcal{R}$ , not on the remaining choices. Furthermore, subject to the same condition on  $\mathcal{R}$ , one can relax the condition C1 above and allow  $\bar{R}$  to have genus 1 when using local coefficients.*

In the case that  $\bar{R}$  does have genus 2 or more, we shall also see that we can take  $\eta$  to be any non-separating curve on  $\bar{R}$ , rather than a curve dual to  $\bar{c}$ .

**4.3. Proof of independence.** We now prove Theorem 4.4: our definition of the monopole Floer homology of a balanced sutured manifold  $(M, \gamma)$  is independent of the choices made in its definition. The proof consists of several applications of Floer's excision theorem. We begin with an observation about mapping tori:

**Lemma 4.7.** *Let  $Y \rightarrow S^1$  be a fibered 3-manifold whose fiber  $R$  is a closed surface of genus at least 2. Then  $HM(Y|R) \cong \mathbb{Z}$ .*

*Proof.* In the case of the product fibration, we have already seen this in the previous section. If  $Y_h$  denotes the mapping torus of a diffeomorphism  $h : R \rightarrow R$ , then the excision theorem, Theorem 3.1, in the guise of (13), gives us an injective map

$$HM(Y_h|R) \otimes HM(Y_g|R) \rightarrow HM(Y_{gh}|R)$$

with cokernel the Tor term. When  $g = h^{-1}$ , the mapping tori  $Y_h$  and  $Y_g$  are orientation-reversing diffeomorphic, and  $HM(Y_g|R)$  is therefore isomorphic to  $HM(Y_h|R)$  as an abelian group. (This is for the same reason that the homology and cohomology of a finitely generated complex of free  $\mathbb{Z}$ -modules are isomorphic, as abelian groups.) So we obtain an injective map

$$HM(Y_h|R) \otimes HM(Y_h|R) \rightarrow \mathbb{Z}$$

whose cokernel is torsion. This forces  $HM(Y_h|R)$  to be  $\mathbb{Z}$ . q.e.d.

**Corollary 4.8.** *Let  $Y_1$  be a closed oriented 3-manifold containing a non-separating oriented surface  $\bar{R}$  of genus two or more. Let  $\tilde{Y}$  be obtained from  $Y_1$  by cutting along  $\bar{R}$  and re-gluing by an orientation-preserving diffeomorphism  $h$ . Then  $HM(Y_1|\bar{R})$  and  $HM(\tilde{Y}|\bar{R})$  are isomorphic.*

*Proof.* Apply the excision theorem, Theorem 3.1, with  $Y = Y_1 \cup Y_2$ , taking  $Y_2$  to be the mapping torus of  $h$  and  $\Sigma_1 = \Sigma_2 = \bar{R}$ . Lemma 4.7 tells us that  $HM(Y_2|\bar{R}) \cong \mathbb{Z}$ , so  $HM(Y_1|\bar{R}) \cong HM(\tilde{Y}|\bar{R})$  by the excision theorem. q.e.d.

Consider now the situation of Theorem 4.4. We have a closed 3-manifold  $Y = Y(M, \gamma)$  whose construction depends on a choice of genus  $g$  for  $T$  and a choice of diffeomorphism  $h$ . We are always supposing that  $Y$  has been constructed using an auxiliary surface  $T$  subject to the conditions C1 and C2. The above corollary tells us that  $HM(Y|\bar{R})$  is independent of the choice of  $h$ . So the group  $SHM(M, \gamma)$ , as we have defined it, depends only on the choice of  $g$ . Let us temporarily write it as

$$(15) \quad SHM^g(M, \gamma).$$

We can apply the same arguments with local coefficients: Theorem 3.3 can be used in place of Theorem 3.1 to see that

$$(16) \quad SHM^g(M, \gamma; \Gamma_\eta)$$

(as defined in Definition 4.5) depends at most on the choice of  $g$ , not on  $h$  (as long as conditions C1 and C2 hold). However, we can also relate (15) to (16) directly:

**Lemma 4.9.** *If the coefficient ring  $\mathcal{R}$  has no  $\mathbb{Z}$ -torsion, then we have*

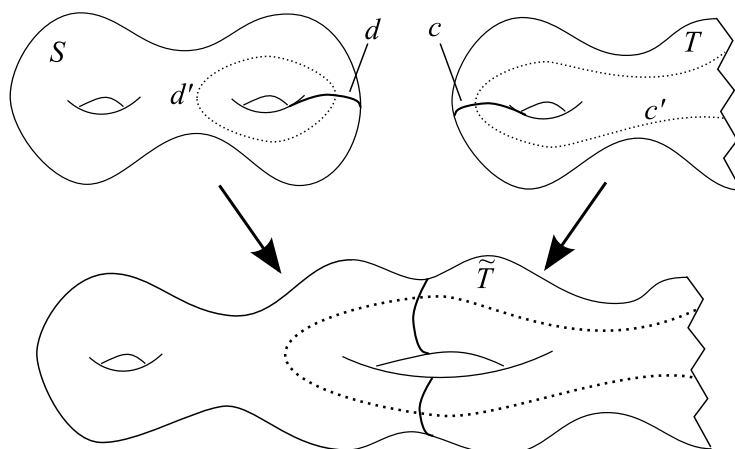
$$SHM^g(M, \gamma; \Gamma_\eta) = SHM^g(M, \gamma) \otimes \mathcal{R}.$$

*Proof.* In the definition of the local system  $\Gamma_\eta$ , the 1-cycle  $\eta$  is parallel to a curve lying on  $\bar{R}$ . The result therefore follows from the definitions and Corollary 3.4. q.e.d.

Because we already know that  $SHM^g(M, \gamma)$  is independent of  $h$ , the above lemma establishes that  $SHM^g(M, \gamma; \Gamma_\eta)$  is also independent of  $h$ , and that it is also independent of the choice of  $\eta$ . Next we prove:

**Proposition 4.10.** *If  $t - t^{-1}$  is invertible in the coefficient ring  $\mathcal{R}$  and  $\mathcal{R}$  has no  $\mathbb{Z}$ -torsion, then the Floer group with local coefficients,  $SHM^g(M, \gamma; \Gamma_\eta)$ , is independent of  $g$ .*

*Proof.* Fix  $g_1$  and let  $T$  be a surface of genus  $g_1$ . Let  $Y_1$  be the resulting closure of  $(M, \gamma)$ , and write  $\bar{R}_1$  for the surface it contains. Recall that we required  $T$  to contain a simple closed curve  $c$  such that  $\{1\} \times c$  and  $\{-1\} \times c$  are non-separating in  $\bar{R}_\pm$ . We can form a surface



**Figure 5.** Increasing the genus of  $T$  by 1.

$\tilde{T}$  of genus  $g_1 + 1$  by the following process. We take a closed surface  $S$  of genus 2 containing a non-separating closed curve  $d$ . We then cut  $T$  along  $c$  and cut  $S$  along  $d$ , and we reglue to form  $\tilde{T}$  as shown in Figure 5. The figure also shows curves  $c'$  and  $d'$  dual to  $c$  and  $d$ . The curve  $c'$  is supposed to be extended (out of the picture) to become a simple closed curve dual to  $c$  in the larger surface  $\bar{R}_+ = R_+(\gamma) \cup \{1\} \times T$ .

In forming the closure  $Y_1$  using  $T$ , we can arrange that the diffeomorphism  $h : \bar{R}_+ \rightarrow \bar{R}_-$  carries  $\{1\} \times c$  to  $\{-1\} \times c$  by the identity map on  $c$ . This is because any non-separating curve is equivalent to any other in an oriented surface. This implies that  $Y_1$  can be identified with the product  $S^1 \times T$  over some neighborhood of  $c$  in  $T$ . So  $Y_1$  contains a torus,  $S^1 \times c$ . The dual curve  $c'$  on  $\bar{R}_+$  becomes a curve (also called  $c'$ ) in  $Y_1$  which intersects the torus  $S^1 \times c$  once.

We will apply the second version of the excision theorem, Theorem 3.2, as follows. We take  $Y = Y_1 \cup Y_2$  with  $Y_1$  as given, and  $Y_2 = S^1 \times S$ . We take  $\Sigma_1$  to be the torus  $S^1 \times c$  inside the product region of  $Y_1$  and  $\Sigma_2$  to be  $S^1 \times d$ . We take  $\eta$  in  $Y$  to be  $\eta_1 + \eta_2$ , where the cycle  $\eta_1$  is  $c'$  and  $\eta_2$  is  $\{\text{point}\} \times d'$ . These 1-cycles intersect the respective tori once each, and  $\eta_1$  is of the sort required for the definition of  $SHM^{g_1}(M, \gamma; \Gamma_{\eta_1})$  in Definition 4.5. To play the role of the surface  $F = F_1 \cup F_2$  in Theorem 3.2 we take  $\bar{R}_1 \cup \bar{R}_2$ , where  $\bar{R}_2$  is the genus-2 surface  $\{\text{point}\} \times S$ .

The manifold  $\tilde{Y}$  obtained from  $Y = Y_1 \cup Y_2$  in the excision theorem is another closure of the original  $(M, \gamma)$ , using the auxiliary surface  $\tilde{T}$  of genus one larger than  $T$  and a diffeomorphism  $\tilde{h}$  obtained by extending  $h$  trivially over the extra handle. It contains a closed surface  $\tilde{R}$  whose genus is one larger than the genus of  $\bar{R}_1$ . This is the surface obtained

from  $\bar{R}_1$  and  $\bar{R}_2$  by cutting and gluing. So to prove the proposition, we must prove

$$(17) \quad HM(Y_1|\bar{R}_1; \Gamma_{\eta_1}) \cong HM(\tilde{Y}|\tilde{R}; \Gamma_{\tilde{\eta}}).$$

Theorem 3.2, provides an isomorphism

$$HM((Y_1 \cup Y_2)|(\bar{R}_1 \cup \bar{R}_2); \Gamma_{\eta}) \rightarrow HM(\tilde{Y}|\tilde{R}; \Gamma_{\tilde{\eta}}).$$

But  $HM(Y_2|\bar{R}_2; \Gamma_{\eta_2})$  is just  $\mathcal{R}$  by Corollary 2.3, because this manifold is a product, so (17) follows from the Künneth theorem. This completes the proof of the proposition. q.e.d.

REMARK. Although Figure 5 is drawn so as to make clear that the excision theorem is applicable, the topology can be described more simply. Let  $G$  denotes the genus-1 surface with one boundary component, obtained by cutting  $S$  open along  $d$  and then removing a neighborhood of  $d'$ . Then the operation of forming  $\tilde{T}$  as shown is the same as removing a neighborhood of the point  $x = c \cap c'$  and attaching  $G$  to the boundary so created: a connected sum in other words. The 3-manifold picture is obtained from this connected-sum picture by multiplying by  $S^1$ . That is, we drill out a neighborhood of  $S^1 \times \{x\}$  and glue in  $S^1 \times G$ .

Now we can complete the proof of the theorem:

*Proof of Theorem 4.4.* We have seen that there is no dependence on the choice of diffeomorphism  $h$ , and we have been considering the dependence on the genus  $g$ : we wish to show that  $SHM^g(M, \gamma)$  is independent of  $g$ . From Lemma 4.9 and Proposition 4.10, we learn that the  $\mathcal{R}$ -module

$$SHM^g(M, \gamma) \otimes \mathcal{R}$$

is independent of  $g$  whenever  $\mathcal{R}$  has no  $\mathbb{Z}$ -torsion and  $t - t^{-1}$  is invertible. But if  $A$  and  $B$  are finitely generated abelian groups and  $A \otimes \mathcal{R} \cong B \otimes \mathcal{R}$  as  $\mathcal{R}$ -modules for all such  $\mathcal{R}$ , then we must have  $A \cong B$ . For this one can take a universal example for  $\mathcal{R}$ , namely the ring obtained by inverting  $t - t^{-1}$  in  $\mathbb{Z}[\mathbb{R}]$ , the group ring of  $\mathbb{R}$ . q.e.d.

Finally, we turn to Proposition 4.6. Up until this point we have been assuming that  $\bar{R}$  has genus 2 or more. But the proof of Proposition 4.10 works just as well in the genus-1 case. Thus if  $Y_1$  is a closure formed with  $\bar{R}_1$  of genus 1 and  $(\tilde{Y}, \tilde{R})$  is formed as in the proof of Proposition 4.10 with  $\bar{R}$  of genus 2, then

$$HM_{\bullet}(Y_1|\bar{R}_1; \Gamma_{\eta_1}) \cong HM_{\bullet}(\tilde{Y}|\tilde{R}; \Gamma_{\tilde{\eta}}).$$

The group on the right is something we already know to be independent of other choices: we have therefore

$$(18) \quad HM_{\bullet}(Y_1|\bar{R}_1; \Gamma_{\eta_1}) = SHM(M, \gamma) \otimes \mathcal{R}.$$

This verifies Proposition 4.6. q.e.d.

## 5. Knot homology

Juhász showed in [12] that knot homology could be obtained as a special case of his (Heegaard) Floer homology of a sutured manifold. Specifically, given a knot  $K$  in a closed 3-manifold  $Z$ , one can form a sutured manifold  $(M, \gamma)$  by taking  $M$  to be the knot complement (with a torus boundary) and taking the sutures to be two oppositely-oriented meridians. In the monopole case we have at present no a priori notion of knot homology, but we are free to take Juhász's prescription as a *definition* of knot homology and pursue the consequences. Thus:

**Definition 5.1.** For a knot  $K$  in a closed, oriented 3-manifold  $Z$ , we define the monopole knot homology  $KHM(Z, K)$  to be the monopole homology of the sutured manifold  $(M, \gamma)$  associated to  $(Z, K)$  by Juhász's construction. That is,

$$KHM(Z, K) := SHM(M, \gamma)$$

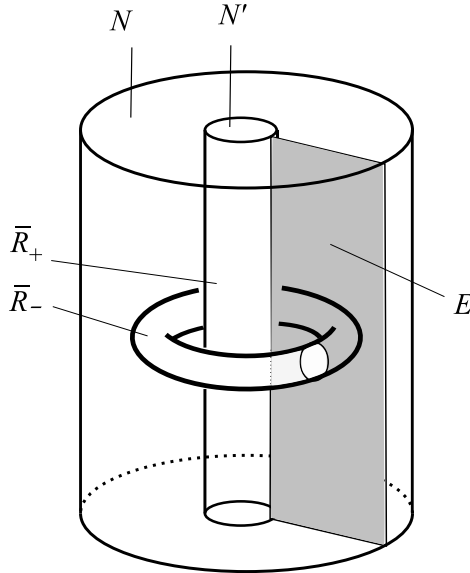
where  $M = Z \setminus N^\circ(K)$  is the knot complement and  $s(\gamma)$  consists of two oppositely oriented meridians.  $\diamond$

To understand what this definition leads to, we must construct a suitable closure of the sutured manifold.

**5.1. Closures of knot complements.** So let  $K$  be a knot in a closed manifold  $Z$ , and let  $(M, \gamma)$  be the knot complement, with two sutures as just described. We can describe a particularly simple closure of  $(M, \gamma)$  as follows, if we temporarily relax the rules and allow the auxiliary surface  $T$  to be an annulus. (The reason this is not a valid closure of  $(M, \gamma)$  for our purposes is that the resulting surfaces  $\bar{R}_\pm$  will have genus 1. We will correct this shortly, replacing the annulus by a surface of genus 1.) Let  $N$  be a closed tubular neighborhood of  $K$ , and let  $N' \subset N$  be a smaller one. Let  $m$  be a meridian of  $K$ , lying outside  $N'$  but inside  $N$ . We will consider  $M$  to be  $\bar{Z} \setminus \bar{N}'$ , and we take two meridional sutures  $s(\gamma)$  on the boundary of  $N'$ . If we take  $T$  to be an annulus and attach  $[-1, 1] \times T$  by gluing the two annuli  $[-1, 1] \times \partial T$  to the sutures  $A(\gamma)$ , then what results is a 3-manifold  $L$  with two tori as boundary components: we can identify it with the complement of a tubular neighborhood of  $m$  in  $M$ .

Figure 6 shows the part of  $L$  that lies inside the tubular neighborhood  $N$  of  $K$ . (The top and bottom are identified.) The figure shows a vertical solid torus  $N$  with a smaller vertical solid torus  $N'$  drilled out of it, as well as a neighborhood  $U$  of the meridian  $m$ , which has also been removed. The boundary of  $L$  consists of the inner vertical boundary (the boundary of  $N'$ ) and the boundary of the horizontal solid torus (the boundary of  $U$ ). These boundary components are  $\bar{R}_+$  and  $\bar{R}_-$ .

If we choose a framing of  $K$ , then we obtain a fibration of  $L \cap N$  by punctured annuli  $E$  (one of which is shown gray in the figure). We now form the closure  $Y_1 = Y(M, \gamma)$  using  $T$  as the auxiliary surface



**Figure 6.** The part of  $L$  lying inside the larger tubular neighborhood  $N$ .

by gluing  $\bar{R}_+$  to  $\bar{R}_-$ : on each punctured annulus  $E$ , we glue the circle  $E \cap \bar{R}_+$  to  $E \cap \bar{R}_-$ . This turns each annulus  $E$  into a genus-1 surface  $F$  with one boundary component. (The remaining boundary component of  $F$  lies on the outer torus,  $\partial N$ .) Thus we have seen:

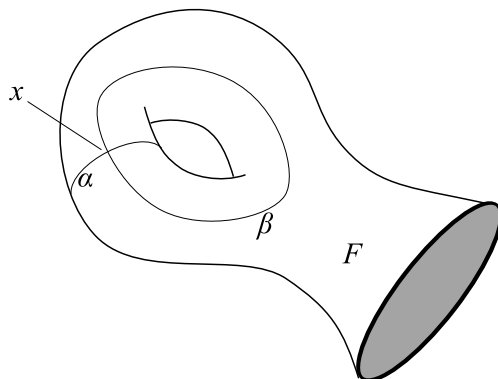
**Lemma 5.2.** *Using an annulus  $T$  as the auxiliary surface, a closure of the sutured manifold  $(M, \gamma)$  associated to a knot  $K$  in  $Z$  can be described by taking a surface  $F$  of genus 1, with one boundary component, and gluing  $F \times S^1$  to the knot complement  $\overline{Z \setminus N}$ . The gluing is done so that  $\{p\} \times S^1$  is attached to the meridian of  $K$  on  $\partial N$  and  $\partial F \times \{q\}$  is glued to any chosen longitude of  $K$  on  $\partial N$ . q.e.d.*

A shorter way to say what we have done is to that we have glued together two knot complements: for the knot  $K$  in  $Z$  and the standard circle “knot” in the 3-torus, using any chosen framing of the former and the standard framing of the latter, attaching longitudes to meridians and meridians to longitudes. We give a name to this closed manifold:

**Definition 5.3.** We write  $Y_1(Z, K)$  for the closed 3-manifold obtained from the framed knot  $K$  in  $Z$  by the construction just described.

◇

As we pointed out at the beginning of this subsection, we have described a closure of the sutured manifold  $(M, \gamma)$  that is illegitimate, because  $T$  is an annulus and  $\bar{R}$  has genus 1. We now describe how  $Y_1(Z, K)$  gets modified if we use a surface  $\tilde{T}$  of genus 1 (still with two



**Figure 7.** The closed surface  $F$  obtained by gluing two boundary components of  $E$ : a genus-1 surface with one boundary component. The curve  $\alpha$  is the gluing locus.

boundary components) instead of  $T$ . Figure 7 shows the surface  $F$ . The curve  $\alpha$  on  $F$  is the intersection of  $F$  with the torus  $\bar{R} \subset Y(Z, K)$  where  $\bar{R}_+$  and  $\bar{R}_-$  are glued. Thus  $\bar{R}$  is the torus

$$\begin{aligned} \bar{R} &= \alpha \times S^1 \\ &\subset F \times S^1 \\ &\subset Y_1(Z, K). \end{aligned}$$

The image of  $[-1, 1] \times T$  in  $Y_1(Z, K)$  is a copy of  $S^1 \times T$  and can be identified with the neighborhood of  $\beta \times S^1$ :

$$S^1 \times T = \text{nb}d(\beta) \times S^1 \subset F \times S^1,$$

where  $\text{nb}d(\beta) \subset F$  is an annular neighborhood of  $\beta$ . The identification of the various factors is as indicated: the  $S^1$  factor in  $S^1 \times T$  becomes the  $\beta$  factor on the right, and the core of the annulus  $T$  becomes the  $S^1$  factor on the right. Recalling the remark made at the end of section 4.3, we see that to effectively increase the genus of the auxiliary surface by 1, we should:

- 1) drill out a tubular neighborhood  $\beta \times D^2$  of the circle  $\beta \times \{q\} \subset F \times S^1$ ;
- 2) attach  $S^1 \times G$ , where  $G$  is a genus-1 surface with one boundary component, by a diffeomorphism

$$S^1 \times \partial G \rightarrow \beta \times \partial D^2$$

which preserves the order of the factors.

(In the second step, a framing of  $\{q\} \times \beta$  is needed, but we have a preferred one because  $\beta$  lies on  $\{q\} \times F$ .)



**Definition 5.4.** We write  $\tilde{Y}(Z, K)$  for the manifold obtained from  $Y_1(Z, K)$  by the two steps just described. It is a closure of the sutured manifold  $(M, \gamma)$  associated to the knot  $K$  in  $Z$  obtained using a genus-one auxiliary surface  $\tilde{T}$ ; and it depends only on a choice of framing for  $K$ .  $\diamond$

While the closure  $Y_1(Z, K)$  has a genus-1 surface  $\bar{R}$ , the closure  $\tilde{Y}(Z, K)$  has a genus-2 surface  $\tilde{R}$ . The latter is obtained from  $\bar{R} = \alpha \times S^1$  by removing a neighborhood of the point  $(x, q)$  in  $\alpha \times S^1$  and adding the genus-1 surface  $\{x\} \times G$ . To summarize this discussion, we have the following, essentially by definition now:

**Corollary 5.5.** *The monopole knot homology  $KHM(Z, K)$  can be computed as the ordinary monopole homology  $HM(\tilde{Y}|\tilde{R})$ , where  $\tilde{Y} = \tilde{Y}(Z, K)$  is as above. Any framing of  $K$  can be used in the construction of  $\tilde{Y}$ . q.e.d.*

REMARK. Both  $Y_1(Z, K)$  and  $\tilde{Y}(Z, K)$  can be described alternatively as follows. Let  $S$  be a closed surface of genus  $l$ , let  $c$  be a non-separating simple closed curve on  $S$ , and let  $\hat{c}$  be the curve  $\{p\} \times c$  in the 3-manifold  $S^1 \times S$ . Let  $N(\hat{c})$  be a tubular neighborhood. Let  $Y_l$  be the result of gluing the complement of  $\hat{c}$  to the complement of  $K$ :

$$(19) \quad Y_l = (S^1 \times S) \setminus N^\circ(\hat{c}) \cup_\phi Z \setminus N^\circ(K)$$

where  $\phi$  identifies the meridian curves of  $\hat{c}$  to the longitudes of  $K$  and vice versa. (We give  $\hat{c}$  the obvious framing, and we recall that a framing of  $K$  has been chosen.) Then the manifold  $Y_1$  is  $Y_1(Z, K)$ , and  $Y_2$  is  $\tilde{Y}(Z, K)$ .

We can also use the simpler manifold  $Y_1$  to compute monopole knot homology, as long as we switch to local coefficients. This is the content of the next lemma.

**Lemma 5.6.** *If  $t-t^{-1}$  is invertible in the coefficient ring  $\mathcal{R}$  and  $\mathcal{R}$  has no  $\mathbb{Z}$ -torsion, then the knot homology  $KHM(Z, K) \otimes \mathcal{R}$  can be computed as  $HM_\bullet(Y_1; \Gamma_{\hat{\alpha}})$ , where  $Y_1$  is the manifold described in Definition 5.3 and  $\hat{\alpha}$  is the curve  $\alpha \times \{p\}$  in  $F \times S^1 \subset Y_1$ , regarded as a 1-cycle.*

*Proof.* According to Proposition 4.6, we can use the closure  $Y_1$  to compute  $SHM(M, \gamma; \Gamma_\eta)$ . Together with Lemma 4.9, this tells us that

$$HM_\bullet(Y_1; \Gamma_{\hat{\alpha}}) \cong SHM(M, \gamma) \otimes \mathcal{R},$$

where  $(M, \gamma)$  is the sutured manifold obtained from the knot complement by Juhász's prescription. q.e.d.

**5.2. Properties of monopole knot homology.** Suppose that the knot  $K \subset Z$  is null-homologous, and let  $\Sigma$  be a Seifert surface for  $K$ : an oriented embedded surface in  $Z \setminus N^\circ$  with boundary a simple closed curve on  $\partial N$ . We can frame the knot  $K$  so that  $N$  is identified with  $K \times D^2$  and  $\partial\Sigma$  is  $K \times \{q'\}$  for some  $q' \in S^1$ . We can also regard  $\Sigma$  as a surface in the manifold  $Y_1(Z, K)$  (Definition 5.3). The union of  $\Sigma$  and  $F \times \{q'\}$  in  $Y_1(Z, K)$  is a closed oriented surface

$$\bar{\Sigma} = \Sigma \cup (F \times \{q'\}) \subset Y(Z, K).$$

Its genus is one more than the genus of  $\Sigma$ . The surface  $F \times \{q'\} \subset Y_1(Z, K)$  remains intact in the manifold  $\tilde{Y}(Z, K)$  for  $q' \neq q$  (Definition 5.4), so we can regard  $\bar{\Sigma}$  also as a closed surface in  $\tilde{Y} = \tilde{Y}(Z, K)$ . Using the surface  $\bar{\Sigma}$ , we can decompose  $KHM(Z, K)$  according to the first Chern class of the  $\text{spin}^c$  structure. We write

$$KHM(Z, K) = \bigoplus_{i \in \mathbb{Z}} KHM(Z, K, i)$$

where

$$KHM(Z, K, i) = \bigoplus_{\substack{\mathfrak{s} \in \mathcal{S}(\tilde{Y}|\tilde{R}) \\ \langle c_1(\mathfrak{s}), [\bar{\Sigma}] \rangle = 2i}} HM_\bullet(\tilde{Y}, \mathfrak{s}).$$

If  $Z$  is not a homology sphere, then the decomposition by  $\text{spin}^c$  structures may depend on the choice of the relative homology class for the Seifert surface  $\Sigma$ , in which case one should write

$$(20) \quad KHM(Z, K, [\Sigma], i)$$

for the summands.

Some familiar properties of the (Heegaard) knot homology of Ozsváth-Szabó and Rasmussen carry over to this monopole version.

**Lemma 5.7.** *The groups  $KHM(Z, K, i)$  and  $KHM(Z, K, -i)$  are isomorphic.*

*Proof.* The isomorphism arises from the general isomorphism (applicable to an arbitrary 3-manifold  $Y$ ), between  $HM_\bullet(Y, \mathfrak{s})$  and  $HM_\bullet(Y, \bar{\mathfrak{s}})$ , where  $\bar{\mathfrak{s}}$  is the conjugate  $\text{spin}^c$  structure. We cannot apply this directly to the group  $HM(\tilde{Y}|\tilde{R})$  that defines  $KHM(Z, K)$ , because the condition  $\mathfrak{s} \in \mathcal{S}(\tilde{Y}|\tilde{R})$  is not preserved when replacing  $\mathfrak{s}$  by  $\bar{\mathfrak{s}}$ . We can, however, apply it to the group  $HM_\bullet(Y_1; \Gamma_{\hat{\alpha}})$  that appears in Lemma 5.6; so we can conclude that

$$KHM(Z, K, i) \otimes \mathcal{R} \cong KHM(Z, K, -i)\mathcal{R}.$$

As in the proof of Theorem 4.4 on page 329, it follows that  $KHM(Z, K, i) \cong KHM(Z, K, -i)$ . q.e.d.

**Lemma 5.8.** *The group  $KHM(Z, K, i)$  is zero for  $|i|$  larger than the genus of  $\Sigma$ .*

*Proof.* The adjunction inequality tells us that  $HM_{\bullet}(Y, \mathfrak{s})$  is zero for  $\text{spin}^c$  structures  $\mathfrak{s}$  with  $c_1(\mathfrak{s})[\bar{\Sigma}]$  greater than  $2g(\bar{\Sigma}) - 2$ . The genus of  $\bar{\Sigma}$  is one larger than the genus of  $\Sigma$ . q.e.d.

**Lemma 5.9.** *For a classical knot  $K$  in  $S^3$  of genus  $g$ , the monopole knot homology group  $KHM(S^3, K, g)$  is non-zero.*

*Proof.* We use the description of  $\tilde{Y} = \tilde{Y}(K)$  as the manifold  $Y_2$ , where  $Y_l$  is the manifold described by (19). Let  $S$  be the genus-2 surface used there, let  $c$  be the closed curve on  $S$ , and let  $c'$  be a dual curve on  $S$  meeting  $c$  once. According to Gabai's results [9, 10], a Seifert surface  $\Sigma$  of  $K$  of genus  $g$  arises as a compact leaf of a taut foliation  $\mathcal{F}_K$  of  $S^3 \setminus N^\circ(K)$ , and we can ask that the leaves of  $\mathcal{F}_K$  meet  $\partial N(K)$  in parallel circles. On the other hand,  $S^1 \times S$  has a taut foliation  $\mathcal{F}_S$  which is transverse to the curve  $\hat{c} = \{p\} \times c$ . This foliation is obtained from the trivial product foliation by cutting along the torus  $S^1 \times c'$  and regluing with a small rotation of the  $S^1$  factor. Together, the foliations  $\mathcal{F}_K$  and  $\mathcal{F}_S$  define a foliation  $\mathcal{F}$  of  $\tilde{Y} = Y_2$ . The surface  $\bar{\Sigma}$  sits inside  $Y_2$  as the union of the Seifert surface  $\Sigma$  and the punctured torus  $(S^1 \times c') \setminus D^2$ . The  $\text{spin}^c$  structure  $\mathfrak{s}_c$  determined by  $\mathcal{F}$  has first Chern class of degree  $2g$  on  $\bar{\Sigma}$  and degree 2 on the genus-2 surface, so  $HM_{\bullet}(\tilde{Y}, \mathfrak{s}_c)$  is a summand of  $KHM(S^3, K, g)$  by definition. The non-vanishing theorem from section 2.4 tells us that this group is non-zero. q.e.d.

**Lemma 5.10.** *Let  $K$  be a classical knot and let  $\chi(K, i)$  denote the Euler characteristic of  $KHM(S^3, K, i)$ , computed using the canonical  $\mathbb{Z}/2$  grading on monopole Floer homology [18]. Then the finite Laurent series*

$$\sum_i \chi(K, i) T^i$$

*is the symmetrized Alexander polynomial,  $\Delta_K(T)$ , for the knot  $K$ , up to an overall sign.*

*Proof.* In different guise, this is essentially the same result as that of Fintushel-Stern [7] and Meng-Taubes [21]. Let  $\tilde{Y} = \tilde{Y}(S^3, K)$  be the usual closure of the sutured manifold associated to  $(S^3, K)$  as in Definition 5.4, let  $\tilde{R}$  be the genus-2 surface in  $\tilde{Y}$  and let  $\bar{\Sigma} \subset \tilde{Y}$  be the surface of genus  $g + 1$  formed from a Seifert surface  $\Sigma$  for  $K$  and the genus-1 surface  $F$ .

The Euler characteristic can be computed from the Seiberg-Witten invariants of the manifold  $S^1 \times \tilde{Y}$ . Specifically, regard both  $\bar{\Sigma}$  and  $\tilde{R}$  as surfaces in

$$X_K = S^1 \times \tilde{Y}.$$

Take  $\nu$  to be the 2-cycle in  $X_K$  defined by  $\bar{\Sigma}$  and consider the generating function  $\mathfrak{m}(X_K, [\nu])$  as in (2), but modified to use only  $\text{spin}^c$  structures

that are of top degree on  $\tilde{R}$ . We introduce the notation

$$\mathbf{m}'(X_K, [\nu]) = \sum_{\mathfrak{s} \in \mathcal{S}(X_K | \tilde{R})} \mathbf{m}(X_K, \mathfrak{s}) t^{(c_1(\mathfrak{s}), [\nu])}.$$

We then have

$$\sum_i \chi(K, i) t^{2i} = \mathbf{m}'(X_K, [\nu]).$$

Let  $X_0$  be the same type of 4-manifold as  $X_K$ , but formed using the unknot in place of  $K$ . The corresponding 3-manifold  $\tilde{Y}_0$  is  $S^1 \times S$ , where  $S$  has genus 2; so  $X_0$  is  $T^2 \times S$ . The remark following Corollary 5.5 explains that  $\tilde{Y}$  is formed from  $\tilde{Y}_0$  by drilling out a neighborhood of a curve  $\hat{c}$  and gluing in the knot complement  $M_K = S^2 \setminus N^\circ(K)$ . It follows that  $X_K$  is formed from  $X_0$  by a “knot surgery” in the sense of [7]. That is, one drills out a neighborhood of the torus  $S^1 \times \hat{c}$  and glues in  $S^1 \times M_K$ . In the formalism of section 2.6, we can therefore compute the ratio

$$\mathbf{m}'(X_K, [\nu]) / \mathbf{m}'(X_0, [\nu])$$

as the ratio of the invariants associated to  $(S^1 \times M_K, \nu_K)$  and  $(S^1 \times M_0, \nu_0)$ . Here  $M_0$  is the knot complement for the unknot, and  $\nu_K$  and  $\nu_0$  are the 2-chains defined by Seifert surfaces for  $K$  and the unknot, respectively. This ratio is precisely what is calculated in [7] (see also [18, section 42.5]), and it is equal to  $\Delta_K(t^2)$ . The lemma follows. q.e.d.

Given a null-homologous knot  $K$  in a 3-manifold  $Z$ , there is a rather more straightforward way to arrive at a sutured manifold than the one that leads to knot homology. We can simply choose a Seifert surface  $\Sigma$  for  $K$  and cut the knot complement  $Z \setminus N^\circ(K)$  open along  $\Sigma$ . The result is a sutured manifold  $(M_\Sigma, \delta)$  with a single suture and having  $R_+(\delta) = R_-(\delta) = \Sigma$ . The monopole Floer homology of this sutured manifold captures the top-degree part of the monopole knot homology:

**Proposition 5.11.** *In the above situation, let  $g$  be the genus of the Seifert surface  $\Sigma$ , and suppose  $g \neq 0$ . Then  $SHM(M_\Sigma, \delta)$  is isomorphic to  $KHM(Z, K, [\Sigma], g)$ .*

*Proof.* It is sufficient to prove that

$$SHM(M_\Sigma, \delta) \otimes \mathcal{R} \cong KHM(Z, K, [\Sigma], g) \otimes \mathcal{R}$$

when the coefficient ring  $\mathcal{R}$  has  $t - t^{-1}$  invertible and no  $\mathbb{Z}$ -torsion. Lemma 5.6 tells us that we can compute the right-hand side using the manifold  $Y_1$  as

$$KHM(Z, K, [\Sigma], g) \otimes \mathcal{R} = HM_\bullet(Y_1 | \bar{\Sigma}; \Gamma_{\hat{\alpha}})$$

where  $\bar{\Sigma}$  is the surface of genus  $g + 1$  in  $Y_1$ . On the other hand, the same manifold  $Y_1$  arises as a closure of  $(M_\Sigma, \delta)$  in the sense of section 4.1, so

we also have

$$\begin{aligned} SHM(M_\Sigma, \delta) \otimes \mathcal{R} &= SHM(M_\Sigma, \delta; \Gamma_\eta) \\ &= HM_\bullet(Y_1 | \bar{\Sigma}; \Gamma_{\hat{\alpha}}). \end{aligned}$$

This proves the proposition.

q.e.d.

## 6. Fibered knots

**6.1. Statement of the result.** In this section, we adapt the material from [23] to show that the monopole version of knot homology detects fibered knots. For the most part, the arguments of [23] carry over with little modification.

A balanced sutured manifold  $(M, \gamma)$  is a *homology product* if the inclusions  $R_+(\gamma) \rightarrow M$  and  $R_-(\gamma) \rightarrow M$  are both isomorphisms on integer homology groups. The main target is the following theorem.

**Theorem 6.1.** *Suppose that the balanced sutured manifold  $(M, \gamma)$  is taut and a homology product. Then  $(M, \gamma)$  is a product sutured manifold if and only if  $SHM(M, \gamma) = \mathbb{Z}$ .*

The application to fibered knots is a corollary:

**Corollary 6.2.** *If  $K \subset S^3$  is a knot of genus  $g$ , then  $K$  is fibered if and only if  $KHM(S^3, K, g) = \mathbb{Z}$ .*

*Proof of the corollary.* The “only if” direction is a straightforward matter: it follows from Lemma 4.7 and Proposition 5.11. The interesting direction is the “if” direction, and this can be deduced from Theorem 6.1 as follows.

Suppose that  $KHM(S^3, K, g) = \mathbb{Z}$ . From Lemma 5.10 we learn that the Alexander polynomial of  $K$  is monic and that its degree is  $g$ . Let  $\Sigma$  be a Seifert surface for  $K$  of genus  $g$ , and let  $(M_\Sigma, \delta)$  be the balanced sutured manifold obtained by cutting open the knot complement along  $\Sigma$ . As Ni observes in [23, section 3], the fact that the Alexander polynomial is monic tells us that  $(M_\Sigma, \delta)$  is a homology product. The group  $SHM(M_\Sigma, \delta)$  is isomorphic to  $KHM(S^3, K, g)$  by Proposition 5.11, so  $SHM(M_\Sigma, \delta) = \mathbb{Z}$ . Theorem 6.1 implies that  $(M_\Sigma, \delta)$  is a product sutured manifold, from which it follows that the knot complement is fibered. q.e.d.

We will prove Theorem 6.1 after some preliminary material on further properties of  $SHM$ .

**6.2. Spin<sup>c</sup> structures.** The following definition of relative spin<sup>c</sup> structures on sutured manifolds coincides with that of Juhász [13], in slightly different notation. If we regard  $(M, \gamma)$  as a manifold with corners, then it carries a preferred 2-plane field  $\xi_\partial$  on its boundary: on  $R_+(\gamma)$  and  $R_-(\gamma)$ , we take  $\xi_\partial$  to be the tangent planes to the boundary, with the

canonical orientation; and on each component of  $A(\gamma)$  we take  $\xi_\partial$  to have, as oriented basis, first the outward normal to  $M$  and second the direction parallel to the oriented suture. On a 3-manifold an oriented 2-plane field defines a  $\text{spin}^c$  structure; so  $\xi_\partial$  gives a  $\text{spin}^c$  structure in a neighborhood of the boundary. We define  $\mathcal{S}(M, \gamma)$  to be the set of extensions of  $\mathfrak{s}_\partial$  to a  $\text{spin}^c$  structure on all of  $M$ , up to isomorphisms which are 1 on  $\partial M$ . We refer to elements of  $\mathcal{S}(M, \gamma)$  as relative  $\text{spin}^c$  structures.

Consider the process of forming the closure  $Y = Y(M, \gamma)$ . When we attach  $[-1, 1] \times T$  to the annuli in  $\partial M$ , the 2-plane field  $\xi_\partial$  extends in the obvious way, as the tangents to  $\{p\} \times T$ . When we attach  $\bar{R}_+$  to  $\bar{R}_-$  using  $h$ , we obtain a 2-plane field on all of  $Y(M, \gamma)$  except the interior of the original  $M$ . On the surface  $\bar{R} \subset Y$ , this 2-plane field is the tangent plane field. So we obtain a natural map

$$(21) \quad \epsilon : \mathcal{S}(M, \gamma) \rightarrow \mathcal{S}(Y|\bar{R}).$$

**Lemma 6.3.** *Let  $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathcal{S}(M, \gamma)$  be relative  $\text{spin}^c$  structures whose difference element in  $H^2(M, \partial M)$  is not torsion. Then we can choose  $T$  and the diffeomorphism  $h$  so that  $\epsilon(\mathfrak{s}_1)$  and  $\epsilon(\mathfrak{s}_2)$  are  $\text{spin}^c$  structures in  $\mathcal{S}(Y|\bar{R})$  whose difference is still non-torsion.*

*Proof.* The statement only concerns the difference elements. The dual of  $H^2(M, \partial M; \mathbb{Q})$  is  $H^1(M; \mathbb{Q})$ , and what we must show is that given a non-zero element  $\alpha \in H^1(M)$ , we can choose  $T$  and  $h$  so that  $\alpha$  is in the image of the map

$$H^1(Y) \rightarrow H^1(M).$$

To do this, consider as an intermediate step the manifold  $Y'$  with boundary  $\bar{R}_+ \cup \bar{R}_-$  formed from  $M$  by attaching  $[-1, 1] \times T$ . The map  $H^1(Y') \rightarrow H^1(M)$  is surjective. Let  $\beta$  be a class in  $H^1(Y')$  which restricts to  $\alpha$ . Represent the dual of  $\beta$  by a closed surface  $(B, \partial B)$  in  $(Y', \partial Y')$ . By adding to  $B$  an annulus contained in the product region  $[-1, 1] \times T$  if necessary, we can be assured that  $\partial B$  intersects both  $\bar{R}_+$  and  $\bar{R}_-$  in a collection of curves representing a primitive, non-zero homology class. We can then modify  $B$  without changing its class so that  $\partial B$  consists of two circles: a non-separating curve in each of  $\bar{R}_+$  and  $\bar{R}_-$ . Finally, we choose the diffeomorphism  $h : \bar{R}_+ \rightarrow \bar{R}_-$  so as to match up these curves. In this way we obtain a closed surface  $\bar{B}$  in  $Y$  whose dual class in  $H^1(Y)$  maps to  $\alpha$  in  $H^1(M)$ . q.e.d.

The following corollary is the tool used by Ghiggini [11] in his proof of the original version of Corollary 6.2 for genus-1 knots.

**Corollary 6.4.** *Suppose that  $(M, \gamma)$  admits two taut foliations  $\mathcal{F}_1$  and  $\mathcal{F}_2$  such that the corresponding  $\text{spin}^c$  structures  $\mathfrak{s}_1$  and  $\mathfrak{s}_2$  have non-torsion difference element in  $H^2(M, \partial M)$ . Then  $\text{SHM}(M, \gamma)$  has rank at least 2.*

*Proof.* Choose the closure  $Y = Y(M, \gamma)$  so that  $\epsilon(\mathfrak{s}_1)$  and  $\epsilon(\mathfrak{s}_2)$  are different  $\text{spin}^c$  structures on  $Y$ , as Lemma 6.3 allows. The foliations  $\mathcal{F}_1$  and  $\mathcal{F}_2$  extend in an obvious way to foliations of  $Y$  belonging to the  $\text{spin}^c$  structures  $\epsilon(\mathfrak{s}_1)$  and  $\epsilon(\mathfrak{s}_2)$ . By the non-vanishing theorem described in section 2.4, the Floer groups  $HM_\bullet(Y, \epsilon(\mathfrak{s}_1))$  and  $HM_\bullet(Y, \epsilon(\mathfrak{s}_2))$  both have non-zero rank. Both of these Floer groups contribute to  $HM_\bullet(Y|\bar{R}) = SHM(M, \gamma)$ , because the  $\text{spin}^c$  structures  $\epsilon(\mathfrak{s}_i)$  belong to  $\mathcal{S}(Y|\bar{R})$ . So  $SHM(M, \gamma)$  has rank at least 2. q.e.d.

**6.3. Decomposition theorems.** The excision theorems, in addition to their role in showing that  $SHM(M, \gamma)$  is well-defined, can be used in a straightforward way to establish some decomposition theorems which relate the Floer homology of a sutured manifold  $(M, \gamma)$  to that of  $(M', \gamma')$ , obtained from  $(M, \gamma)$  by cutting along a surface. We record a few types of such decomposition theorem here. To avoid various circumlocutions involving tensor products and the Künneth theorem, we shall work over  $\mathbb{Q}$  instead of  $\mathbb{Z}$  here; and when using local coefficients we shall take  $\mathcal{R}$  to be a field of characteristic zero: either  $\mathbb{R}$  with the usual exponential map, or the field of fractions of the group ring  $\mathbb{Q}[\mathbb{R}]$ .

**Proposition 6.5.** *Suppose  $(M, \gamma)$  is a disjoint union  $(M_1, \gamma_1) \cup (M_2, \gamma_2)$  and that both pieces are balanced. Then*

$$SHM(M, \gamma; \mathbb{Q}) \cong SHM(M_1, \gamma_1; \mathbb{Q}) \otimes SHM(M_2, \gamma_2; \mathbb{Q}).$$

*Proof.* It will be sufficient to prove this for the local coefficient versions,  $SHM(M, \gamma; \Gamma_\eta)$ , because of Lemma 4.9. Form the closures  $(Y_1, \bar{R}_1)$  and  $(Y_2, \bar{R}_2)$  of  $(M_1, \gamma_1)$  and  $(M_2, \gamma_2)$  by attaching product regions  $[-1, 1] \times T_1$  and  $[-1, 1] \times T_2$ , respectively. Let  $c_1$  and  $c_2$  be non-separating curves on  $T_1$  and  $T_2$ . When forming the closures  $Y_1$  and  $Y_2$ , choose the diffeomorphisms  $h_1$  and  $h_2$  so that  $h_i$  maps  $\{1\} \times c_i$  to  $\{-1\} \times c_i$ , as in the proof of Proposition 4.10. Let  $\tilde{T}$  be the connected closed surface obtained from  $T_1$  and  $T_2$  by cutting open along  $c_1$  and  $c_2$  and reattaching, similarly to Figure 5. Let  $\tilde{h}$  be the diffeomorphism of  $\tilde{T}$  that arises from  $h_1$  and  $h_2$ , and let  $\tilde{Y}$  be the closure of  $(M, \gamma)$  that is obtained by attaching  $\tilde{T}$  to  $(M, \gamma)$  and gluing up using  $\tilde{h}$ . We now have a connected closure  $\tilde{Y}$  that is related to  $Y = Y_1 \cup Y_2$  by cutting and gluing along 2-tori  $S^1 \times c_i$ . So the excision theorem, Theorem 3.2, provides an isomorphism

$$HM_\bullet(Y|\bar{R}; \Gamma_\eta) \rightarrow HM_\bullet(\tilde{Y}|\tilde{R}; \Gamma_{\tilde{\eta}}),$$

and hence an isomorphism

$$HM_\bullet(Y|\bar{R}; \mathcal{R}) \rightarrow HM_\bullet(\tilde{Y}|\tilde{R}; \mathcal{R}).$$

Since  $\mathcal{R}$  is a field and  $Y$  is a disjoint union, the left-hand side is a tensor product, and the proposition follows. q.e.d.

Next we prove a version of Ni's "horizontal decomposition" formula. A *horizontal surface* in  $(M, \gamma)$  is a surface  $S$  with  $\chi(S) = \chi(R_+(\gamma))$  such that  $\partial S$  consists of one circle in each of the annuli comprising  $A(\gamma)$ ; it is required to represent the same relative homology class as  $R_\pm(\gamma)$  in  $H_2(M, A(\gamma))$  and should have  $[\partial S] = [s(\gamma)]$  in  $H_1(A(\gamma))$ . Cutting along a horizontal surface creates a new sutured manifold

$$(M', \gamma') = (M_1, \gamma_1) \cup (M_2, \gamma_2).$$

**Proposition 6.6** ([23, proposition 4.1]). *If  $(M', \gamma')$  is obtained from  $(M, \gamma)$  by cutting along a horizontal surface, then*

$$SHM(M, \gamma; \mathbb{Q}) = SHM(M', \gamma'; \mathbb{Q}).$$

*Proof.* This follows directly from Theorem 3.1 and Proposition 6.5. q.e.d.

We shall also need to decompose sutured manifolds by cutting along vertical surfaces. We prove a result along the lines of [23] and [13]. A *product annulus* in  $(M, \gamma)$  is an embedded annulus  $A = [-1, 1] \times d$  in  $(M, \gamma)$  such that the circle  $d_+ = \{1\} \times d$  lies in the interior of  $R_+(\gamma)$  and  $d_- = \{-1\} \times d$  lies in the interior of  $R_-(\gamma)$ .

**Proposition 6.7.** *Let  $(M', \gamma')$  be obtained from  $(M, \gamma)$  by cutting along a product annulus  $A$ . Then*

$$SHM(M, \gamma; \mathbb{Q}) = SHM(M', \gamma'; \mathbb{Q})$$

*if we are in either of the following two situations:*

- 1) *the curves  $d_+$  and  $d_-$  represent non-zero classes in the first homology of  $R_+(\gamma)$  and  $R_-(\gamma)$ , respectively; or*
- 2) *the curves  $d_+$  and  $d_-$  represent the zero class in  $H_1(R_+(\gamma))$  and  $H_1(R_-(\gamma))$ , respectively, at least one of them does not bound a disk, and the annulus  $A$  separates  $M$  into two parts,  $M_1 \cup M_2$ , one of which is disjoint from the annuli  $A(\gamma)$ .*

*Proof.* We begin with case 1 of the proposition. We shall construct closures  $(Y, \bar{R})$  and  $(\tilde{Y}, \tilde{R})$  for  $(M, \gamma)$  and  $(M', \gamma')$  which are related to each other as described in the excision theorem, Theorem 3.2, and the result will follow.

When we attach the product  $[-1, 1] \times T$  to  $(M, \gamma)$ , the curves  $d_+$  and  $d_-$  remain non-separating in the closed surfaces  $\bar{R}_\pm$ , because  $T$  is connected. By taking  $T$  to have non-zero genus, we can also ensure that there is a curve  $c$  in the interior of  $T$  which is non-separating in  $T$ . So after attaching the product region, we have two product annuli  $[-1, 1] \times d$  and  $[-1, 1] \times c$ , with independent non-separating curves  $d_+, c_+$  in  $\bar{R}_+$  in  $d_-, c_-$  in  $\bar{R}_-$ . We can close up the manifold using a diffeomorphism  $h: \bar{R}_+ \rightarrow \bar{R}_-$  such that  $h(d_+) = d_-$  and  $h(c_+) = c_-$ . The closure  $(Y, \bar{R})$



of  $(M, \gamma)$  that we arrive at in this way contains two tori,

$$\begin{aligned} \Sigma_1 &= S^1 \times c, \\ \Sigma_2 &= S^1 \times d. \end{aligned}$$

There is a 1-cycle  $\eta$  lying on  $\bar{R}$  that is transverse to both of these tori, so Theorem 3.2 is applicable. (This is an instance of that theorem where the manifold  $Y'$  obtained by cutting along  $\Sigma_1$  and  $\Sigma_2$  is connected.) The manifold  $(\tilde{Y}, \tilde{R})$  obtained from  $(Y, \bar{R})$  by cutting along  $\Sigma_1 \cup \Sigma_2$  and regluing is a closure of the  $(M', \gamma')$ , so we are done with case 1.

We turn to case 2. Without loss of generality, we suppose that  $M_1$  does not meet  $A(\gamma)$  and  $d_+$  does not bound a disk. Let  $R_{+,1}$  denote  $R_+(\gamma) \cap M_1$  and let  $R_{-,2}$  denote  $R_-(\gamma) \cap M_2$ . The surface  $R_{+,1}$  has genus at least 1 and its only boundary component is  $d_+$ . In [23], Ni uses the following observation. The union

$$R_{+,1} \cup A \cup R_{-,2}$$

is isotopic to a horizontal surface in  $(M, \gamma)$  to which Proposition 6.6 applies. By cutting along this horizontal surface, the pieces we get from  $(M, \gamma)$  are (up to diffeomorphism)

$$([-1, 1] \times R_{+,1}) \cup_{[-1,1] \times d} M_2$$

and

$$M_1 \cup_{[-1,1] \times d} ([-1, 1] \times R_{-,2}).$$

In this way, case 2 is reduced to the case that either  $M_1$  or  $M_2$  is a product.

If  $M_2$  is a product,  $[-1, 1] \times R_{-,2}$ , then the result is entirely straightforward: the surface  $R_{-,2}$  contains all the annuli  $A(\gamma)$ . A closure  $Y$  of  $(M, \gamma)$  using an auxiliary surface  $T$  can also be regarded as a closure of  $(M_1, \gamma_1)$  using the auxiliary surface  $R_{-,2} \cup T$ . So we have

$$SHM(M, \gamma) = SHM(M_1, \gamma_1).$$

On the other hand, because  $M_2$  is a product, we have  $SHM(M_1, \gamma_1) = SHM(M', \gamma')$  by Proposition 6.5. Finally, if  $M_1$  is a product, then we can cut  $M_1$  open along a non-separating annulus because  $R_{+,1}$  has positive genus, and this does not change  $SHM$ , by part 1 of the proposition. After cutting open  $M_1$  in this way, we arrive at a situation in which case 1 applies again, and the proof is complete. q.e.d.

**6.4. Proof of Theorem 6.1.** Those ingredients of Ni's proof from [23] which involve Heegaard Floer homology have all been replicated here in the context of monopole Floer homology, so the proof carries through with little change. We outline the argument, adapted from [23]. Let  $(M, \gamma)$  be a balanced sutured manifold satisfying the hypotheses of the theorem, and suppose  $(M, \gamma)$  is not a product sutured manifold. We shall show that  $SHM(M, \gamma)$  has rank at least 2.

Because of Proposition 6.5, it is sufficient to treat the case that  $M$  is connected. Similarly, because of Proposition 6.6, we may assume that  $(M, \gamma)$  is “vertically prime;” that is, every horizontal surface in  $(M, \gamma)$  is a parallel copy of either  $R_+(\gamma)$  or  $R_-(\gamma)$ . By attaching product regions to  $(M, \gamma)$  and appealing to Proposition 6.7, we are also free to suppose that  $(M, \gamma)$  has only one suture. We now consider a maximal product pair  $i : [-1, 1] \times E \hookrightarrow (M, \gamma)$  as in [23, 24] and the induced map

$$i_* : H_1([-1, 1] \times E) \rightarrow H_1(M).$$

There are two cases.

**Case 1:  $i_*$  is not surjective.** In this case, Ni establishes that  $(M, \gamma)$  admits two taut foliations  $\mathcal{F}_1$  and  $\mathcal{F}_2$  whose difference element is non-torsion in  $H^2(M, \partial M)$ . It then follows from Corollary 6.4 that  $SHM(M, \gamma)$  has rank 2 or more, as required.

**Case 2:  $i_*$  is surjective.** In this case, let  $(M', \gamma')$  be the complement of the maximal product pair. This is non-empty, because  $(M, \gamma)$  is not a product sutured manifold. Proposition 6.7 tells us that  $SHM(M, \gamma)$  and  $SHM(M', \gamma')$  have the same rank. Ni observes that the vertically prime condition on  $(M, \gamma)$  implies that  $M'$  is connected. Furthermore,  $(M', \gamma')$  is a homology product, and its top and bottom surfaces  $R_\pm(\gamma')$  are planar, because of the surjectivity of  $i_*$ . The (connected) surfaces  $R_\pm(\gamma')$  are not disks, so  $(M', \gamma')$  has at least two sutures. Let  $r \geq 2$  be the number of sutures in  $(M', \gamma')$ . Let  $S$  be a planar surface with  $r + 1$  boundary components, so that the product sutured manifold  $[-1, 1] \times S$  has  $r + 1$  sutures. Form a new sutured manifold  $(\tilde{M}, \tilde{\gamma})$  by gluing  $r$  of the annuli from  $[-1, 1] \times S$  to the annuli of  $(M', \gamma')$ . The resulting sutured manifold  $(\tilde{M}, \tilde{\gamma})$  has

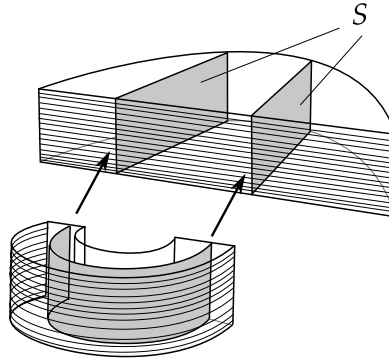
$$\text{rank } SHM(\tilde{M}, \tilde{\gamma}) = \text{rank } SHM(M', \gamma')$$

by Proposition 6.7. Furthermore,  $(\tilde{M}, \tilde{\gamma})$  is a homology product, and its maximal product pair is  $[-1, 1] \times S$  up to isotopy. The construction has been made so that the inclusion of the maximal product pair in  $(\tilde{M}, \tilde{\gamma})$  is not surjective on  $H_1$ , so we now have a situation which falls into Case 1 above. It follows that  $SHM(\tilde{M}, \tilde{\gamma})$  has rank at least 2, and so too therefore does  $SHM(M, \gamma)$ . This completes Ni’s proof.

**6.5. More decomposition theorems.** In [13], rather general sutured manifold decompositions are considered, and results of the following sort are obtained. Let  $(M, \gamma)$  be a balanced sutured manifold, and let  $S \subset M$  be a decomposing surface in the sense of [9]. There is a sutured manifold decomposition,

$$(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma'),$$

and we shall suppose that  $(M', \gamma')$  is also balanced (which implies that  $S$  has no closed components). Under some mild restrictions on  $S$ , Juhász



**Figure 8.** Adding product 1-handles to a sutured manifold containing a decomposing surface  $S$ .

proves in [13] that  $SFH(M', \gamma')$  is a direct summand of  $SFH(M, \gamma)$ . An entirely similar theorem can be proved in the context of monopole Floer homology, using  $SHM(M, \gamma)$  in place of  $SFH(M, \gamma)$ . The following is a restatement of theorem 1.3 of [13], though with less specific information about the  $\text{spin}^c$  structures that are involved behind the scenes. In the statement of the theorem, an oriented simple closed curve  $C$  in  $R(\gamma)$  is called *boundary coherent* if it either represents a non-zero class in  $H_1(R(\gamma))$  or it is the oriented boundary  $\partial R_1$  of a compact subsurface  $R_1 \subset R(\gamma)$  with its canonical orientation.

**Theorem 6.8** ([13, theorem 1.3]). *Let  $(M, \gamma)$  be a balanced sutured manifold and*

$$(M, \gamma) \xrightarrow{S} (M', \gamma')$$

*a sutured manifold decomposition. Suppose that the decomposing surface  $S$  has no closed components, and that for every component  $V$  of  $R(\gamma)$ , the set of closed components of  $S \cap V$  consists of parallel oriented boundary-coherent simple closed curves. Then the Heegaard Floer homology  $SFH(M', \gamma')$  is a direct summand of  $SFH(M, \gamma)$ . q.e.d.*

We have the following result.

**Proposition 6.9.** *Theorem 6.8 continues to hold with monopole Floer homology in place of Heegaard Floer homology. That is, with the same hypotheses,  $SHM(M', \gamma')$  is a direct summand of  $SHM(M, \gamma)$ .*

*Proof.* By lemma 4.5 of [13], Juhász reduces this to the special case of a “good” decomposing surface  $S$ , by which is meant a surface  $S$  such that every component of  $\partial S$  intersects both  $R_+(\gamma)$  and  $R_-(\gamma)$ .

Starting from a good decomposing surface  $S$ , we can pass to another special case as follows. Let  $C$  be a component of  $\partial S$ . By the definition of a good decomposing surface,  $C$  intersects the annuli  $A(\gamma)$  in

vertical arcs. The number of these arcs counted with sign is zero. Pair up these arcs accordingly; and for each pair attach a product 1-handle as shown in Figure 8. Repeat this with every other boundary component of  $\partial S$ . The result of this process is a new balanced sutured manifold  $(M_1, \gamma_1)$  containing a new decomposing surface  $S_1$ . We have  $SHM(M_1, \gamma_1) \cong SHM(M, \gamma)$ , because adding product handles has no effect. (The inverse operation to adding a product handle can also be described as removing a larger product region, by cutting along annuli in the original  $M$  parallel to the annular sutures where the handle is attached; so this operation is a special case of one we have seen before.) Furthermore, if  $(M'_1, \gamma'_1)$  is what we obtain from  $(M_1, \gamma_1)$  by sutured manifold decomposition along  $S_1$ , then  $(M'_1, \gamma'_1)$  is also related to  $(M', \gamma')$  by adding adding product 1-handles. It therefore suffices to prove that  $SHM(M'_1, \gamma'_1)$  is a direct summand of  $SHM(M_1, \gamma_1)$ .

Looking at  $(M_1, \gamma_1)$ , we now see that it is sufficient to prove the following lemma, which is a priori a special case of the proposition. q.e.d.

**Lemma 6.10.** *Let  $(M, \gamma)$  be a balanced sutured manifold and let*

$$(M, \gamma) \xrightarrow{S} (M', \gamma')$$

*be a sutured manifold decomposition. Suppose that  $S$  has no closed components and that the oriented boundary of  $\partial S$  consists of  $n$  simple closed curves  $C_1^+, \dots, C_n^+$  in  $R_+(\gamma)$  and  $n$  simple closed curves  $C_1^-, \dots, C_n^-$  in  $R_-(\gamma)$ . Suppose further that the homology classes of  $C_1^+, \dots, C_n^+$  are a collection of independent classes in  $H_1(R_+(\gamma))$ , and make a similar assumption for  $R_-(\gamma)$ . Then  $SHM(M', \gamma')$  is a direct summand of  $SHM(M, \gamma)$ .*

*Proof of the lemma.* Form the closure  $Y = Y(M, \gamma)$  by attaching a product region  $[-1, 1] \times T$  as usual and then choosing the diffeomorphism  $h$  in such a way that  $h(C_i^+) = C_i^-$  (with the opposite orientation) for all  $i$ . The result of this is that  $Y$  contains two closed surfaces: first the usual surface  $\bar{R}$ , and second a surface  $\bar{S}$  obtained from  $S$  by identifying  $C_i^+$  with  $C_i^-$  for all  $i$ . The intersection  $\bar{S} \cap \bar{R}$  consists of  $n$  circles,  $C_1, \dots, C_n$ . Let  $F$  be the oriented surface obtained from  $\bar{S} \cup \bar{R}$  by smoothing out the circles of double points, respecting orientations.

The same surface  $F \subset Y$  can be arrived at from a different direction. Start with  $(M', \gamma')$ . We can write  $A(\gamma')$  as a union of components

$$A(\gamma') = A(\gamma) \cup A_1,$$

where  $A(\gamma)$  are the annuli of the original sutured manifold  $(M, \gamma)$  and  $A_1$  are the new annuli. The new annuli can be written as  $[-1, 1] \times D_i^\pm$ , where the collection of curves  $D_i^\pm$  are in natural correspondence with the curves  $C_i^\pm$ . We now form a closure  $Y'$  of  $(M', \gamma')$  as follows. We attach

a product region  $[-1, 1] \times T'$  to  $(M', \gamma')$ , where  $T'$  is a (disconnected) surface

$$T' = T \cup T_1.$$

Here  $T$  is the surface used to close  $Y$  and  $T_1$  is a collection of  $n$  annuli

$$T_1 = T_{1,1} \cup \dots \cup T_{1,n}.$$

Although  $T'$  breaks the rules by being disconnected, we can still effectively use  $T'$  in constructing  $SHM(M', \gamma')$  because of the arguments of section 6.3. In attaching  $[-1, 1] \times T'$  to  $(M', \gamma')$ , we glue  $[-1, 1] \times \partial T$  to the annuli  $A(\gamma) \subset A(\gamma')$  as we did when closing  $(M, \gamma)$ , and we glue the two components  $[-1, 1] \times T_{1,i}$  to the two annuli  $[-1, 1] \times D_i^\pm$  belonging to  $A_1$ .

At this point, we have a manifold

$$(M', \gamma') \cup [-1, 1] \times (T \cup T_1)$$

with two boundary components  $\bar{R}'_\pm$ . The top surface  $\bar{R}'_+$  can be described as a union

$$\bar{R}'_+ = \bar{R}_+^\dagger \cup S_+ \cup \{+1\} \times T_1.$$

Here  $\bar{R}_+^\dagger$  is the surface with boundary obtained by cutting open  $\bar{R}_+$  along the circles  $C_i^+$ , and the annuli  $\{+1\} \times T_1$  are collars of half of the boundary components of  $\bar{R}_+^\dagger$ . The surface  $S_+$  is a copy of  $S$ . Up to diffeomorphism, we can forget these annular regions and write

$$\begin{aligned} \bar{R}'_+ &= \bar{R}_+^\dagger \cup S_+, \\ \bar{R}'_- &= \bar{R}_-^\dagger \cup S_-. \end{aligned}$$

That is,  $\bar{R}'_+$  is obtained from  $\bar{R}_+$  by cutting open along the circles  $C_i^+$  and inserting a copy of  $S$ . Finally, form the closure  $Y'$  by using a diffeomorphism

$$h' : \bar{R}'_+ \rightarrow \bar{R}'_-$$

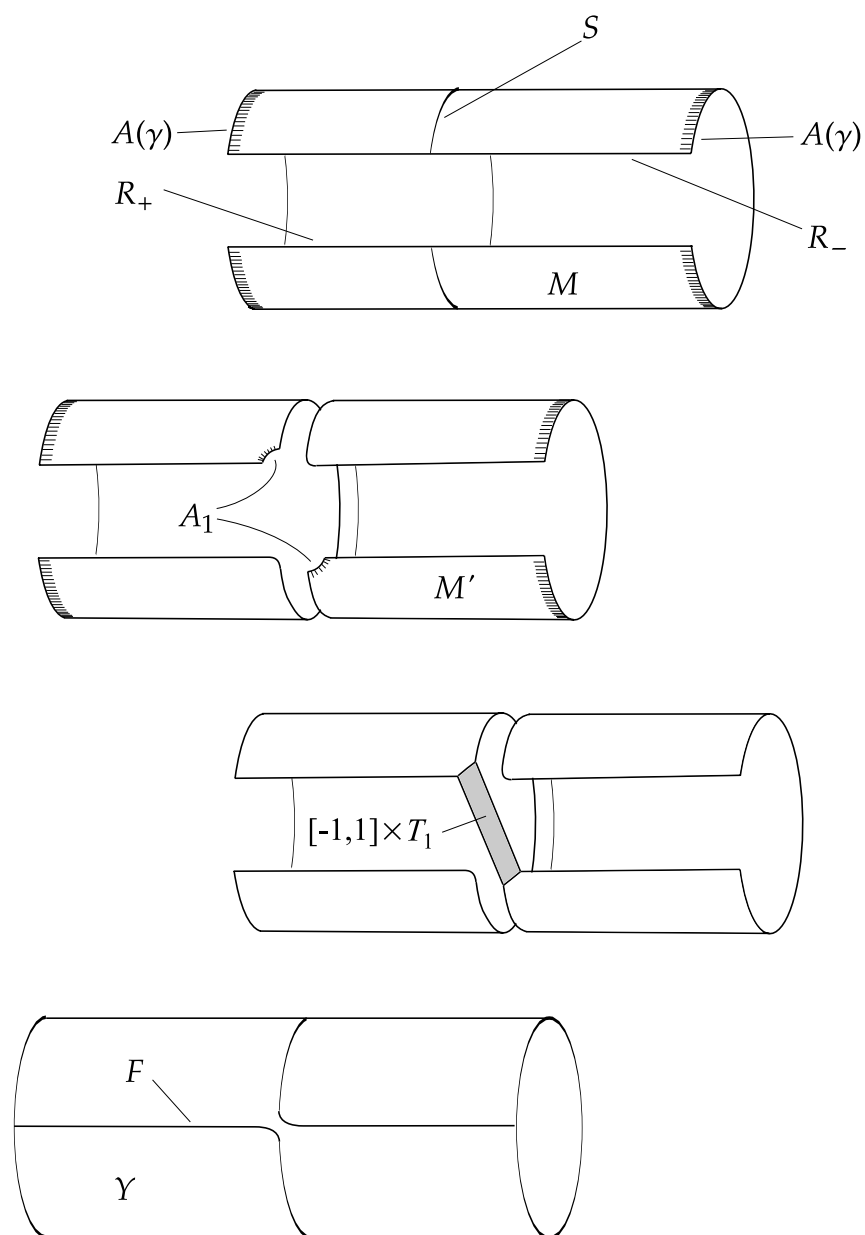
which is equal to  $h$  on  $\bar{R}_+^\dagger$  and equal to the identity on  $S$ .

The resulting closure  $Y'$  of  $(M', \gamma')$  is diffeomorphic to  $Y$ , and under this diffeomorphism, the surface  $\bar{R}' \subset Y'$  obtained from  $\bar{R}'_\pm$  becomes the surface  $F$ . (See Figure 9.) It follows that we can calculate  $SHM(M', \gamma')$  as

$$SHM(M', \gamma') = HM_\bullet(Y|F).$$

The homology class of  $F$  is the sum of the classes of  $\bar{R}$  and  $\bar{S}$ . Furthermore,  $\chi(F) = \chi(\bar{R}) + \chi(\bar{S})$ . It follows from the adjunction inequality that the only  $\text{spin}^c$  structures in  $\mathcal{S}(Y|F)$  which can have non-zero Floer homology are those in the intersection  $\mathcal{S}(Y|\bar{R}) \cap \mathcal{S}(Y|\bar{S})$ . So we have

$$SHM(M', \gamma') = \bigoplus_{\mathfrak{s} \in \mathcal{S}(\bar{R}) \cap \mathcal{S}(\bar{S})} HM_\bullet(Y, \mathfrak{s}),$$



**Figure 9.** Decomposing  $M$  along  $S$  and then closing up to get  $F$ . The collars of  $\partial M$  and  $\partial M'$  are marked with hatching near  $A(\gamma)$  and  $A(\gamma')$ . The product part  $[-1, 1] \times T$  is not shown in the figure, which is otherwise a faithful representation after multiplying by  $S^1$ .

while

$$SHM(M, \gamma) = \bigoplus_{\mathfrak{s} \in \mathcal{S}(\bar{R})} HM_{\bullet}(Y, \mathfrak{s}).$$

This shows that  $SHM(M', \gamma')$  is a direct summand of  $SHM(M, \gamma)$ , as the lemma asserts. q.e.d.

As is pointed out in [13], one can use Proposition 6.9 to give an alternative proof of the non-vanishing of  $SHM(M, \gamma)$  when  $(M, \gamma)$  is taut. One uses a sutured manifold hierarchy, starting at  $(M, \gamma)$  and ending at a product sutured manifold, whose (monopole) Floer homology we know to be  $\mathbb{Z}$ , so showing that  $\mathbb{Z}$  is a summand of  $SHM(M, \gamma)$ .

### 7. Instantons

Much of the contents of this paper can be adapted to the case of (Yang-Mills) instanton homology, instead of (Seiberg-Witten) monopole Floer homology. We present some of this material in this section. For background on instanton homology, we refer to [4].

**7.1. Instanton Floer homology.** When looking at the monopole Floer homology groups  $HM_{\bullet}(Y, \mathfrak{s})$  of a 3-manifold  $Y$ , we could avoid difficulties arising from reducible solutions by considering situations where only non-torsion  $\text{spin}^c$  structures  $\mathfrak{s}$  played a role. In instanton homology, reducibles can be avoided by using  $SO(3)$  bundles with non-zero  $w_2$ . We proceed as follows.

Fix a hermitian line bundle  $w \rightarrow Y$  such that  $c_1(w)$  has odd pairing with some integer homology class. Let  $E \rightarrow Y$  be a  $U(2)$  bundle with an isomorphism  $\theta : \Lambda^2 E \rightarrow w$ . Let  $\mathcal{C}$  be the space of  $SO(3)$  connections in  $\text{ad}(E)$  and let  $\mathcal{G}$  be the group of determinant-1 gauge transformations of  $E$  (the automorphisms of  $E$  that respect  $\theta$ ). The Chern-Simons functional on the space  $\mathcal{B} = \mathcal{C}/\mathcal{G}$  leads to a well-defined instanton homology group which we write as  $I_*(Y)_w$  [4]. It is also possible to use a slightly larger gauge group than  $\mathcal{G}$ . Fix a surface  $R \subset Y$  that has odd pairing with  $c_1(w)$ . Let  $\xi = \xi_R$  be a real line bundle with  $w_1(\xi)$  dual to  $R$ . The map  $E \mapsto E \otimes \xi$  gives rise to a map on the space of connections,

$$\iota_R : \mathcal{B} \rightarrow \mathcal{B},$$

without fixed points, and there is a quotient  $\mathcal{B}/\iota_R$ . This is the same as the quotient of  $\mathcal{C}$  by a gauge group which has  $\mathcal{G}$  as an index-2 subgroup. Let us temporarily write  $I_*(Y)_{w,R}$  for the resulting instanton homology group: it is the fixed space of an induced involution on  $I_*(Y)$ . As an example, in the case  $Y = T^3$ , we have  $I_*(T^3)_w = \mathbb{Z} \oplus \mathbb{Z}$ . The involution interchanges the two copies of  $\mathbb{Z}$ , and  $I_*(T^3)_{w,R} = \mathbb{Z}$  whenever  $w \cdot [R]$  is non-zero. In general,  $I_*(Y)_w$  is  $(\mathbb{Z}/8)$ -graded. The involution acts with degree 4, and the group  $I_*(Y)_{w,R}$  is  $(\mathbb{Z}/4)$ -graded.

Although these groups are defined with  $\mathbb{Z}$  coefficients, it will be convenient to work with a field of characteristic zero; and in what follows we will take that field to be  $\mathbb{C}$ . Thus we will take it that

$$I_*(T^3)_{w,R} = \mathbb{C}.$$

**7.2. The eigenspace decomposition.** The monopole Floer homology detects the Thurston norm of a 3-manifold (see section 2.4), but the formulation of this statement requires the decomposition of the monopole Floer homology according to the different  $\text{spin}^c$  structures. In order to relate *instanton* homology to the Thurston norm, one needs a decomposition of the instanton homology. As suggested in [15], such a decomposition arises from the eigenspaces of natural operators on the Floer groups.

Let  $Y$  be again a closed 3-manifold and  $w$  a line bundle as above. Given an oriented closed surface  $R$  in  $Y$ , there is a 2-dimensional cohomology class  $\mu(R)$  in  $\mathcal{B}$  (for which our conventions follow [5]) and hence an operation of degree  $-2$  on both  $I_*(Y)_w$  and  $I_*(Y)_{w,R}$ . There is also the class  $\mu(y)$ , for  $y$  a point in  $y$ , which acts with degree 4. The operators  $\mu(R)$  and  $\mu(y)$  commute, so one can look for simultaneous eigenvalues. In the special case that  $Y = S^1 \times \Sigma$ , with  $\Sigma$  a surface of positive genus, the eigenvalues of  $\mu(\Sigma)$  and  $\mu(y)$  were computed by Muñoz in [22]:

**Proposition 7.1** ([22, proposition 20]). *Let  $w \rightarrow S^1 \times \Sigma$  be the line bundle whose first Chern class is dual to the  $S^1$  factor. Then the simultaneous eigenvalues of the action of  $\mu(\Sigma)$  and  $\mu(y)$  on  $I_*(S^1 \times \Sigma)_w$  are the pairs of complex numbers*

$$(i^r(2k), (-1)^r 2)$$

for all the integers  $k$  in the range  $0 \leq k \leq g - 1$  and all  $r = 0, 1, 2, 3$ . Here  $i$  denotes  $\sqrt{-1}$ . q.e.d.

REMARK. In [22], the 2-dimensional class called  $\alpha$  corresponds to  $2\mu(\Sigma)$  here, and the class  $\beta$  corresponds to  $-4\mu(y)$ . Also, the group  $HF^*(S^1 \times \Sigma)$  that appears in [22] is our  $I_*(S^1 \times \Sigma)_{w,\Sigma}$ . Muñoz computes the spectrum in the case of  $I_*(S^1 \times \Sigma)_{w,\Sigma}$ , but the case of  $I_*(S^1 \times \Sigma)_w$  follows in a straightforward manner. Observe, in particular, that because  $\mu(\Sigma)$  is an operator of degree 2 on a  $(\mathbb{Z}/8)$ -graded vector space, the eigenspaces of eigenvalues  $\lambda$  and  $i\lambda$  will always be isomorphic.

As a corollary of this proposition, a similar result holds for a general 3-manifold  $Y$ .

**Corollary 7.2.** *Let  $R \subset Y$  be closed connected surface of positive genus, and let  $w$  have odd pairing with  $R$ . Then the eigenvalues of the action of the pair of operators  $\mu(R)$  and  $\mu(y)$  on  $I_*(Y)_w$  are a subset of*



the eigenvalues that occur in the case of the product manifold  $S^1 \times R$ . That is, they are pairs of complex numbers

$$(i^r(2k), (-1)^r 2)$$

for integers  $k$  in the range  $0 \leq k \leq g - 1$ .

*Proof.* Let  $R_0$  be a copy of  $R$  in the interior of the product cobordism  $W = [-1, 1] \times Y$ . The action of  $\mu(R)$  on  $I_*(Y)_w$  can be regarded as being defined by this copy of  $R$  in the 4-dimensional cobordism. Let  $W'$  be the cobordism from the disjoint union  $S^1 \times R$  and  $Y$  at the incoming end to  $Y$  at the outgoing end, obtained by removing an open tubular neighborhood of  $R_0$  from  $W$ . We have a map defined by  $W'$ ,

$$\psi_{W'} : I_*(S^1 \times R)_w \otimes I_*(Y)_w \rightarrow I_*(Y)_w.$$

The map is surjective, because one obtains the product cobordism by closing off the boundary component  $S^1 \times \Sigma$ . Furthermore, because  $R_0$  is homologous to surfaces in each of the three boundary components, we have, for example,

$$\psi_{W'}(\mu(R)a \otimes b) = \mu(R)\psi_{W'}(a \otimes b).$$

From this relation and the surjectivity of  $\psi_{W'}$ , it follows that the eigenvalues of  $\mu(R)$  on the outgoing end  $Y$  are a subset of the eigenvalues of the action of  $\mu(R)$  on  $S^1 \times R$ . We obtain the result of the corollary by applying a similar argument to  $\mu(y)$  and to  $\mu(R)^2 + \mu(y)$ . q.e.d.

We can now give a definition in instanton homology of something that will play the role that  $HM_\bullet(Y|R)$  played in the monopole theory.

**Definition 7.3.** Let  $Y$  be a closed, oriented 3-manifold,  $w$  a hermitian line bundle on  $Y$ , and  $R \subset Y$  a closed, connected, oriented surface on which  $c_1(w)$  is odd. Let  $g$  be the genus of  $R$ , which we require to be positive. We define

$$I_*(Y|R)_w$$

to be the simultaneous eigenspace for the operators  $\mu(R)$ ,  $\mu(y)$  for the pair of eigenvalues  $(2g - 2, 2)$ . ◇

REMARK. Except in the case that the genus is 1, we could define this more simply as just the  $(2g - 2)$ -eigenspace of  $\mu(R)$ , as can be seen from Corollary 7.2

Although Muñoz does not calculate the dimensions of the eigenspaces in general for  $S^1 \times \Sigma$ , one can readily read off from the proof of [22, proposition 20] that the dimension of the eigenspace belonging to the largest eigenvalue is 1. That is:

**Proposition 7.4.** *Let  $Y = S^1 \times R$  with  $\Sigma$  of positive genus, and let  $w$  be the line bundle dual to the  $S^1$  factor. Then*

$$I_*(Y|R)_w = \mathbb{C}.$$

*q. e. d.*

There is a simple extension of the above definition to the case that  $R$  has more than one component, as long as  $w$  is odd on each component. If the components are  $R_m$ , then the corresponding operators  $\mu(R_m)$  commute, and we may take the appropriate simultaneous eigenspace. In general, the action of  $\mu(R)$  on  $I_*(Y)_w$  is not diagonalizable, but one can read off from [22] that the eigenspace of  $\mu(R)$  belonging to the top eigenvalue  $2g - 2$  is simple when one restricts to the kernel of  $\mu(y) - 2$ . That is,

$$\ker(\mu(y) - 2) \cap \ker(\mu(R) - (2g - 2))^N = \ker(\mu(y) - 2) \cap \ker(\mu(R) - (2g - 2))$$

for all  $N \geq 1$ .

**Proposition 7.5.** *Given any  $Y$ ,  $R$  and  $w$  for which  $I_*(Y|R)_w$  is defined, and given any other surface  $\Sigma \subset Y$  of positive genus, the action of  $\mu(\Sigma)$  on  $I_*(Y|R)_w$  has eigenvalues belong to the set of even integers in the range from  $-(2g - 2)$  to  $2g - 2$ , where  $g$  is the genus of  $\Sigma$ .*

*Proof.* The action of  $\mu(\Sigma)$  on  $I_*(Y)_w$  commutes with  $\mu(R)$ , so the action of  $\mu(\Sigma)$  does preserve the subspace  $I_*(Y|R)_w \subset I_*(Y)_w$ . If  $w$  is odd on  $\Sigma$ , then the proposition follows from Corollary 7.2 together with the fact that  $\mu(y) - 2$  is zero on this subspace. If  $w$  is even on  $\Sigma$ , then one can consider a surface in the homology class of  $R + n\Sigma$  and use the additivity of  $\mu$ . *q. e. d.*

Because the actions of  $\mu(\Sigma_1)$  and  $\mu(\Sigma_2)$  commute for any pair of classes  $\Sigma_1$  and  $\Sigma_2$ , we have a decomposition of  $I_*(Y|R)_w$  by cohomology classes (as outlined in [15]):

**Corollary 7.6.** *There is a direct sum decomposition into generalized eigenspaces*

$$I_*(Y|R)_w = \bigoplus_s I_*(Y|R, s)_w$$

where the sum is over all homomorphisms

$$s : H_2(Y; \mathbb{Z}) \rightarrow 2\mathbb{Z}$$

subject to the constraints

$$|s([S])| \leq 2 \text{ genus}(S) - 2$$

for all connected surfaces  $S$  with positive genus and  $s([R]) = 2 \text{ genus}(R) - 2$ . The summand  $I_*(Y|R, s)_w$  is the simultaneous generalized eigenspace

$$I_*(Y|R, s)_w = \bigcap_{\sigma \in H_2(Y)} \bigcup_{N \geq 0} \ker(\mu(\sigma) - s(\sigma))^N.$$

*q. e. d.*

It will be convenient at a later point to have a notation for the sort of homomorphisms  $s$  that arise here. Choosing a notation reminiscent of our notation for  $\text{spin}^c$  structures, we write

$$\mathcal{H}(Y) = \text{Hom}(H_2(Y), 2\mathbb{Z}),$$

and for an embedded surface  $R \subset Y$  of genus  $g$  we write

$$(22) \quad \mathcal{H}(Y|R) = \{s \in \mathcal{H}(Y) \mid s([R]) = 2g - 2\}.$$

**7.3. Excision for instanton homology.** Let  $Y$  be a closed, oriented 3-manifold equipped with a line bundle  $w$ , and suppose  $\Sigma = \Sigma_1 \cup \Sigma_2$  is an oriented embedded surface with two connected components of equal genus, which we require to be positive. Suppose also that  $c_1(w)[\Sigma_1]$  and  $c_1(w)[\Sigma_2]$  are equal and odd. We allow that  $Y$  has either one or two components. In the latter case, we require one of the  $\Sigma_i$  to be in each component. In the former case, when  $Y$  is connected, we assume that  $\Sigma_1$  and  $\Sigma_2$  are not homologous. Choose a diffeomorphism  $h : \Sigma_1 \rightarrow \Sigma_2$ , and lift it to a bundle-isomorphism  $\hat{h}$  on the restrictions of the line bundle  $w$ . From this data, we form  $\tilde{Y}$  by cutting along the  $\Sigma_i$  and gluing up using  $h$  as before. The lift  $\hat{h}$  can be used to glue up the bundle also, giving us a bundle  $\tilde{w} \rightarrow \tilde{Y}$ . As before, we write  $\tilde{\Sigma} = \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$  for the surfaces in  $\tilde{Y}$ .

**Theorem 7.7.** *If  $(\tilde{Y}, \tilde{\Sigma})$  is obtained from  $(Y, \Sigma)$  as above, then there is an isomorphism*

$$I_*(Y|\Sigma)_w \cong I_*(\tilde{Y}|\tilde{\Sigma})_{\tilde{w}}.$$

*We interpret the left-hand side as a tensor product in the case that  $Y$  has two components.*

*Proof.* In the case that  $\Sigma$  has genus 1, this result is due to Floer [8, 2]. In Floer’s statement of the result,  $Y$  had two components, but the proof does not require it. It should also be said that the statement of Floer’s theorem in [2] involves  $I_*(Y)_w$  rather than  $I_*(Y|\Sigma)_w$ , which leads to an extra factor of 2 in the dimensions when  $Y$  has two components.

The case of genus 2 or more is essentially the same, once one knows that  $I_*(S^1 \times \Sigma_i|\Sigma_i)_w$  has rank 1, following as prototype the proof of Proposition 2.5, which was main step in the monopole case. q.e.d.

In the case of genus 1, note that passing from  $I_*(Y)_w$  to  $I_*(Y|\Sigma)_w$  can also be achieved by taking the +2 eigenspace of  $\mu(y)$ , for one point  $y$  in each component of  $Y$ .

Here are two particular applications of the excision theorem. They are both variants of Proposition 7.4, but involve different line bundles.

**Proposition 7.8.** *Let  $Y$  be the product  $S^1 \times \Sigma$ , with  $\Sigma$  a surface of genus 1 or more, and let  $w$  again be the line bundle dual to the  $S^1$  factor. Let  $u \rightarrow Y$  be a line bundle whose first Chern class is dual to a*

curve  $\gamma$  lying on  $\{\text{point}\} \times \Sigma$ , and write the tensor product line bundle as  $uw$ . Then we have

$$I_*(Y|\Sigma)_{uw} = \mathbb{C}.$$

*Proof.* Write  $B$  for the vector space  $I_*(Y|\Sigma)_{uw}$  and  $A$  for the vector space  $I_*(Y|\Sigma)_w$ . We apply the excision theorem in a setting where the incoming manifold is two copies of  $Y$  with the line bundle  $uw$  and the outgoing manifold is a single copy of  $Y$  with the line bundle  $u^2w$ . The latter gives the same Floer homology as for the line bundle  $w$ , so we learn that

$$B \otimes B \cong A.$$

We already know that  $A$  is 1-dimensional, and it follows that  $B$  is also 1-dimensional. q.e.d.

For the second application, we can dispense with  $w$ :

**Proposition 7.9.** *In the situation of Proposition 7.8, the eigenspace of the pair of operators  $(\mu(\Sigma), \mu(y))$  on  $I_*(Y)_u$  for the eigenvalues  $(2g - 2, 2)$  is also 1-dimensional.*

*Proof.* We can see more generally that for any  $\lambda$  the eigenspace for  $(\lambda, 2)$  on  $I_*(Y)_u$  is the same as the corresponding eigenspace in  $I_*(Y)_{wu}$ . For this, one can apply the excision theorem as follows. Let  $c$  be a closed curve on  $\Sigma$  so that the torus  $S^1 \times c$  intersects  $u$  once. Let  $Y_1$  be  $S^1 \times T^2$ , and let  $u_1, w_1$ , and  $c_1$  be similar there to  $u, w$ , and  $c$ . Apply the excision theorem with incoming manifold  $Y_1 \cup Y$  with the line bundles  $u_1w_1$  and  $uw$ , respectively, cutting along the tori  $S^1 \times c_1$  and  $S^1 \times c$ . The outgoing manifold is diffeomorphic to  $Y$ , with the line bundle  $uw^2$ , which gives the same homology as  $u$ . The excision theorem gives an isomorphism between the  $+2$  eigenspaces of  $\mu(y)$ , which we denote

$$\phi : I_*(Y)_{uw}^{(2)} \rightarrow I_*(Y)_u^{(2)}.$$

The map that gives rise to the isomorphism in the excision theorem intertwines (in this instance) the maps  $\mu(\Sigma)$  on the outgoing end with

$$\mu(T^2) \otimes 1 + 1 \otimes \mu(\Sigma)$$

on the incoming end. Since  $\mu(T^2)$  is zero on  $I_*(S^1 \times T^2)_{u_1w_1}$ , the map  $\phi$  actually commutes with  $\mu(\Sigma)$ . q.e.d.

REMARK. The Floer homology group  $I_*(S^1 \times \Sigma)_u$  is something that appears to be rather simpler than the more familiar  $I_*(S^1 \times \Sigma)_w$ . In particular, excision shows that it behaves “multiplicatively” in  $g - 1$ . The representation variety that is involved here is easy to identify: the critical point set of the Chern-Simons functional is two copies of a torus  $T^{2g-2}$ . The involution interchanges  $\iota_\Sigma$  interchanges the two copies. It seems likely that the Floer group  $I_*(S^1 \times \Sigma)_{w,\Sigma}$  can be identified with the homology of this torus.

**7.4. Instanton Floer homology for sutured manifolds.** Let  $(M, \gamma)$  be a balanced sutured manifold. Just as we did in the monopole case, we attach a connected product sutured manifold  $[-1, 1] \times T$  to  $(M, \gamma)$  to obtain a manifold  $Y'$  with boundary  $\bar{R}_+ \cup \bar{R}_-$ , a pair of diffeomorphic connected closed surfaces. As before, we require that there be a closed curve  $c$  in  $T$  such that  $\{-1\} \times c$  and  $\{1\} \times c$  are both non-separating in their respective boundary components. We also pick a marked point,  $t_0 \in T$ , which we did not need before. Now we glue  $\bar{R}_+$  to  $\bar{R}_-$  by a diffeomorphism. We require that  $h(t_0) = t_0$ , so that the resulting closed manifold  $Y = Y(M, \gamma)$  contains a standard circle running through  $t_0$ . This circle intersects once the closed surface  $\bar{R}$  obtained by identifying  $\bar{R}_\pm$ . We no longer require that  $\bar{R}$  has genus 2 or more: in the instanton case, genus 1 will suffice.

**Definition 7.10.** The instanton homology of the sutured manifold  $(M, \gamma)$  is the vector space

$$SHI(M, \gamma) := I_*(Y|\bar{R})_w,$$

where  $(Y, \bar{R})$  is obtained from  $(M, \gamma)$  by closing as just described, and  $w$  is the line bundle whose first Chern class is dual to the standard circle through  $t_0$ . ◇

REMARK. As an example, it follows from Proposition 7.4 that the instanton homology of a product sutured manifold is  $\mathbb{C}$ .

The proof that  $SHI(M, \gamma)$  is independent of the choice of genus for  $T$  and the choice of diffeomorphism  $h$  can be carried over almost verbatim from the monopole case, using the excision theorem. It is even somewhat easier to manage, because the case of genus 1 is no longer special. When showing that  $SHI$  is independent of the choice of genus, we used twisted coefficients  $HM_\bullet(Y|\bar{R}; \Gamma_\eta)$  as an intermediate step in the monopole case. The counterpart of twisted coefficients in the proof for the instanton case is the introduction of the auxiliary line bundle  $u$  that appears in Propositions 7.8 and 7.9 above. One applies excision along tori, following the same scheme as shown in Figure 5, to increase the genus by 1. On the components  $S^1 \times S$ , with  $S$  of genus 2 as shown, one should take the line bundle  $u$ , where  $u$  is the line bundle whose first Chern class is dual to the dotted curve  $d'$ . This argument shows that

$$I_*(Y|\bar{R})_{uw} = I_*(\tilde{Y}|\tilde{R})_{\tilde{u}\tilde{w}}$$

where  $Y$  and  $\tilde{Y}$  are closures of  $(M, \gamma)$  obtained using auxiliary surfaces  $T$  and  $\tilde{T}$  of genus  $g$  and  $g + 1$ . Another application of excision (cutting along copies of  $\bar{R}$  and using Proposition 7.8) shows that

$$\begin{aligned} I_*(Y|\bar{R})_{uw} &\cong I_*(Y|\bar{R})_w \\ &= SHI^g(M, \gamma). \end{aligned}$$

### 7.5. Decompositions of sutured manifolds and non-vanishing.

The proofs of the decomposition results of sections 6.3 and 6.5 carry over without change to the instanton setting also. In particular, Proposition 6.9 holds in the instanton case:

**Proposition 7.11.** *Let  $(M, \gamma)$  be a balanced sutured manifold and*

$$(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$$

*a sutured manifold decomposition satisfying the hypotheses of Theorem 6.8. Then  $SHI(M', \gamma')$  is a direct summand of  $SHI(M, \gamma)$ .*

*Proof.* The proof is the same as the proof of Proposition 6.9; but at the last step in Lemma 6.10, instead of using the decomposition into  $\text{spin}^c$  structures, one uses the generalized-eigenspace decomposition of Corollary 7.6. q.e.d.

As shown in [13] and mentioned above at the end of section 6.5, a result such as Proposition 7.11 gives a non-vanishing theorem for the case of taut sutured manifolds. We therefore have:

**Theorem 7.12.** *If the balanced sutured manifold  $(M, \gamma)$  is taut, then  $SHI(M, \gamma)$  is non-zero.* q.e.d.

The only alternative route known to the authors for proving a non-vanishing theorem for instanton homology is the strategy in [17], which draws on results from symplectic and contact topology, as well as on the partial proof of Witten's conjecture relating Donaldson invariants and Seiberg-Witten invariants of closed 4-manifolds [6]. We shall return to non-vanishing theorems for instanton homology in section 7.8.

**7.6. Floer's instanton homology for knots.** Just as we did for the monopole case in section 5, we can take Juhász's prescription as a definition of knot homology. Let  $K \subset Z$  be again a knot in a closed, oriented 3-manifold. Let  $(M, \gamma)$  be the sutured manifold obtained by taking  $M$  to be the knot complement  $Z \setminus N^\circ(K)$  and  $s(\gamma)$  a pair of oppositely oriented meridians on  $\partial K$ . In the instanton case, there is no need for  $\bar{R}$  to have genus 2 or more, so we may use the closure  $Y_1(Z, K)$  described in Definition 5.3. This is the closure of  $(M, \gamma)$  obtained using  $[-1, 1] \times T$ , where  $T$  is an annulus. It is also described in Lemma 5.2 as obtained from  $Z \setminus N^\circ(K)$  by attaching  $F \times S^1$ , where  $F$  has genus 1: the gluing is done so that  $\{p\} \times S^1$  is attached to a meridian of  $K$ . We summarize the construction of this instanton knot homology in the following definition. The definition is not new: it is the same "instanton homology for knots" that Floer defined in [8]. For the purposes of this paper, we call it  $KHI(Z, K)$ :

**Definition 7.13.** The instanton knot homology  $KHI(Z, K)$  of a knot  $K$  in  $Z$  is defined to be the instanton homology of the sutured manifold

$(M, \gamma)$  above, or equivalently, the instanton homology group  $I_*(Y_1|\bar{R})_w$ . Here  $Y_1$  is obtained from the knot complement by attaching  $F \times S^1$  as described, the surface  $\bar{R}$  is the torus  $\alpha \times S^1$  as shown in Figure 7, and  $w$  is the line bundle with  $c_1(w)$  dual to  $\beta \times \{p\} \subset F \times S^1$ .  $\diamond$

The only difference between this and Floer’s original definition is that we have used  $I_*(Y_1|\bar{R})_w$  in place of  $I_*(Y_1)_w$ . Since  $\bar{R}$  has genus 1, the former group can be characterized as the +2 eigenspace of  $\mu(y)$  acting on the latter group. The latter group is the sum of two subspaces of equal dimension, the eigenspaces for the eigenvalues 2 and  $-2$ .

For a classical knot  $K$  in  $S^3$ , we shall simply write  $KHI(K)$  for the instanton knot homology. To get a feel for what this invariant is, let us examine the set of critical points of the Chern-Simons functional on  $\mathcal{B}$ , or in other words the space of flat connections in the appropriate  $SO(3)$  bundle, modulo the determinant-1 gauge transformations. To do this, we start by looking at  $F \times S^1$ , where  $F$  is the genus-1 surface with one boundary component, and the line-bundle  $w$  with  $c_1(w)$  dual to  $\beta \times \{p\}$ . The appropriate representation variety can also be viewed as the space of flat  $SU(2)$  connections on the complement of the curve  $\beta \times \{p\}$  with the property that the holonomy around a small circle linking  $\beta \times \{p\}$  is the central element  $-1$ . Consider such a flat connection  $A$  and let  $J_1$  and  $J_2$  be the holonomies of  $A$  around respectively the curves  $\alpha \times \{q\}$  and  $a \times S^1$  in  $F \times S^1$ , where  $a$  is a point on  $\alpha \setminus \beta$ . The torus  $\alpha \times S^1$  intersects the circle  $\beta \times \{p\}$  once, so we have

$$[J_1, J_2] = -1$$

in  $SU(2)$ . Up to a gauge transformation, we must have

$$J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Let  $J_3$  be the holonomy around  $\beta' \times \{q\}$ , where  $\beta'$  is a parallel copy of  $\beta$ . The elements  $J_1$  and  $J_3$  must commute, so

$$J_3 = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

for some  $\theta$  in  $[0, 2\pi)$ . The angle  $\theta$  is now determined without ambiguity from the gauge-equivalence class of the connection  $A$ , and the matrices  $J_1, J_2$ , and  $J_3$  determine  $A$  entirely. We have proved:

**Lemma 7.14.** *The representation variety of flat  $SO(3)$  connections on  $F \times S^1$  for the given  $w$ , modulo the determinant-1 gauge group, is diffeomorphic to a circle  $S^1$ , via  $J_3$  as above. q.e.d.*

Let us examine the restriction of these representations to the boundary of  $F \times S^1$ . On this torus  $\partial F \times S^1$ , the flat connections can be regarded as  $SU(2)$  connections. The holonomy around the  $S^1$  factor

is  $J_2$ , which we have described above. The holonomy around the  $\partial F$  factor is given by the commutator

$$[J_3, J_1] = \begin{pmatrix} e^{2i\theta} & 0 \\ 0 & e^{-2i\theta} \end{pmatrix}.$$

So for the representation variety described in the lemma, the restriction to the boundary is a two-to-one map whose image is the space of connections having holonomy around the  $S^1$  factor given by

$$\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Finally, we can attach  $F \times S^1$  to the knot complement  $S^3 \setminus N^\circ(K)$ , and we obtain the following description of the representation variety.

**Lemma 7.15.** *Let  $K \subset S^3$  be a knot and let  $Y_1$  and  $w$  be as described in Definition 7.13. Then the representation variety given by the critical points of the Chern-Simons functional in the corresponding space of connections  $\mathcal{B}$  can be identified with a double cover of the space*

$$\mathcal{R}(K, \mathbf{i}) = \{ \rho : \pi_1(S^3 \setminus K) \rightarrow SU(2) \mid \rho(m) = \mathbf{i} \},$$

where  $m$  is a chosen meridian.

*q.e.d.*

Note that  $\mathcal{R}(K, \mathbf{i})$  is a space of homomorphisms, not a space of conjugacy classes of homomorphisms. The centralizer of  $\mathbf{i}$  (a circle subgroup) still acts on  $\mathcal{R}(K, \mathbf{i})$  by conjugation. There is always exactly one point of  $\mathcal{R}(K, \mathbf{i})$  which is fixed by the action of this circle, namely, the homomorphism  $\rho$  which factors through the abelianization  $H_1(S^3 \setminus K) = \mathbb{Z}$ . All other orbits are irreducible: they have stabilizer  $\pm 1$ , so they are circles. In a generic case,  $\mathcal{R}(K, \mathbf{i})$  consists of one isolated point corresponding to the abelian (reducible) representation, and finitely many circles, one for each conjugacy class of irreducible representations. In such a case, the representation variety described in the lemma above is a trivial double-cover of  $\mathcal{R}(K, \mathbf{i})$ . It therefore has two isolated points corresponding to the reducible, and two circles for each irreducible conjugacy class.

Because it comprises only the  $+2$  eigenspace of  $\mu(y)$ , the knot Floer homology  $KHI(K)$  has just half the dimension of  $I_*(Y_1)_w$  in Definition 7.13. Heuristically, we can think of each irreducible conjugacy class in  $\mathcal{R}(K, \mathbf{i})$  as contributing the homology of the circle,  $H_*(S^1; \mathbb{C})$ , to the complex that computes  $KHI(K)$ , while the reducible contributes a single  $\mathbb{C}$ . In any event, if there are only  $n$  conjugacy classes of irreducibles and the corresponding circles of critical points are non-degenerate in the Morse-Bott sense, then it will follow that the dimension of  $KHI(K)$  is bounded above by  $2n + 1$ .



For a knot  $K \subset Z$  supplied with a Seifert surface  $\Sigma$ , there is a decomposition of the instanton knot homology  $KHI(Z, K)$  as

$$KHI(Z, K) = \bigoplus_{i=-\text{genus}(\Sigma)}^{\text{genus}(\Sigma)} KHI(Z, K, [\Sigma], i).$$

The definition is the same as in the monopole case (20), but uses the generalized-eigenspace decomposition of Corollary 7.6 in place of the decomposition by  $\text{spin}^c$  structures. In particular, for a classical knot  $K \subset S^3$ , we can write

$$KHI(K) = \bigoplus_{i=-g}^g KHI(K, i),$$

where  $g$  is the genus of the knot. Just as in the monopole case, the top summand  $KHI(K, g)$  can be identified with the instanton Floer homology  $SHI(M, \gamma)$ , where  $(M, \gamma)$  is the sutured manifold obtained by cutting open the knot complement along a Seifert surface of genus  $g$ . (See Proposition 5.11.) From the non-vanishing theorem, Theorem 7.12, we therefore deduce a non-vanishing theorem for  $KHI$ .

**Proposition 7.16.** *Let  $K$  be a classical knot of genus  $g$ . Then the instanton knot homology group  $KHI(K, g)$  is non-zero. In particular, instanton knot homology detects the genus of a knot. q.e.d.*

This proposition provides an alternative proof for results from [17] and [16]. In particular, we have the following corollary:

**Corollary 7.17.** *If  $K \subset S^3$  is non-trivial knot, then there exists an irreducible homomorphism  $\rho : \pi_1(S^3 \setminus K) \rightarrow SU(2)$  which maps a chosen meridian  $m$  to the element  $\mathbf{i} \in SU(2)$ .*

*Proof.* If there is no such homomorphism, then  $\mathcal{R}(K, \mathbf{i})$  consists only of the reducible, which is always non-degenerate. The critical point set in  $\mathcal{B}$  then consists of two irreducible critical points, so the rank of  $I_*(Y_1)_w$  is at most 2, and the rank of  $KHI(K)$  is therefore at most 1. This is inconsistent with non-vanishing of  $KHI(K, g)$ , since  $KHI(K, g)$  is isomorphic to  $KHI(K, -g)$ . q.e.d.

**7.7. Instanton homology and fibered knots.** Instanton knot homology detects fibered knots, just as the other versions do. We state and prove this here. We need, however, an extra hypothesis on the Alexander polynomial. For Heegaard knot homology, and also in the monopole case, we know the Alexander polynomial is determined by the knot homology, and the extra hypothesis is not needed. It seems likely that the same holds in the instanton case, but we have not proved it.

We begin with a version of Theorem 6.1 for the instanton case.

**Theorem 7.18.** *Suppose that the balanced sutured manifold  $(M, \gamma)$  is taut and a homology product. Then  $(M, \gamma)$  is a product sutured manifold if and only if  $SHI(M, \gamma) = \mathbb{C}$ .*

*Proof.* Ni’s argument, as presented for monopole knot homology in the proof of Theorem 6.1, works just as well for  $SHI$  as it does for  $SHM$ , with one slight change (a change which is in the spirit of [13]). The key point occurs in Case 1 in the proof of Theorem 6.1 (section 6.4), where it is already assumed that  $(M, \gamma)$  has just one suture. We described this step using  $\text{spin}^c$  structures, but we can argue using homology instead.

Let  $N$  be obtained from  $(M, \gamma)$  by adding a product region  $[-1, 1] \times T$  to the single suture. The boundary of  $N$  is  $\bar{R}_+ \cup \bar{R}_-$ . In [23], Ni shows that if  $E \times I$  does not carry all the homology of  $(M, \gamma)$ , then one can find two decomposing surfaces  $S_1$  and  $S_2$  in  $N$  with the following properties. First, the boundaries of  $S_1$  and  $S_2$  are the same and consist of a pair of circles  $\omega_+$  and  $\omega_-$  which represent non-zero homology classes in  $\bar{R}_+$  and  $\bar{R}_-$ . Second, the sutured manifolds  $(M'_1, \gamma'_1)$  and  $(M'_2, \gamma'_2)$  obtained by decomposition of  $N$  along  $S_1$  and  $S_2$ , respectively, are both taut. Third, if  $Y$  is obtained from  $N$  by gluing  $\bar{R}_+$  to  $\bar{R}_-$  by a diffeomorphism  $h$  with  $h(\omega_+) = \omega_-$ , then the resulting closed surfaces  $\bar{S}_1, \bar{S}_2$ , and  $\bar{R}$  in  $Y$  satisfy the following conditions, for some  $m > 0$  and some closed surface  $\bar{S}_0$  with  $\chi(\bar{S}_0)$  non-zero:

$$\begin{aligned} [\bar{S}_1] &= m[\bar{R}] + [\bar{S}_0], \\ [\bar{S}_2] &= m[\bar{R}] - [\bar{S}_0], \end{aligned}$$

and

$$\begin{aligned} \chi(\bar{S}_1) &= \chi(\bar{S}_2) \\ &= m\chi(\bar{R}) + \chi(\bar{S}_0). \end{aligned}$$

These last conditions imply that  $\mathcal{H}(Y|\bar{R}) \cap \mathcal{H}(Y|\bar{S}_1)$  is disjoint from  $\mathcal{H}(Y|\bar{R}) \cap \mathcal{H}(Y|\bar{S}_2)$ . (The notation  $\mathcal{H}$  is introduced at (22).)

For  $i = 1, 2$ , let  $F_i$  be the surface in  $Y$  obtained by smoothing out the intersection of  $\bar{R}$  and  $\bar{S}_i$  (a single circle in both cases). The proof of Lemma 6.10 shows that

$$\begin{aligned} SHI(M'_i, \gamma'_i) &= \bigoplus_{s \in \mathcal{H}(Y|F_i)} I_*(Y|\bar{R}, s)_w \\ &= \bigoplus_{s \in \mathcal{H}(Y|\bar{R}) \cap \mathcal{H}(Y|\bar{S}_i)} I_*(Y|\bar{R}, s)_w \\ &\subset I_*(Y|\bar{R})_w \\ &= SHI(M, \gamma). \end{aligned}$$

The disjointness of the two indexing sets for  $s$  means that we have

$$SHI(M'_1, \gamma'_1) \oplus SHI(M'_2, \gamma'_2) \subset SHI(M, \gamma).$$

Finally, both summands on the right are non-zero because these sutured manifolds are taut. q.e.d.

**Corollary 7.19.** *Let  $K$  be a non-trivial knot in  $S^3$ . Suppose that the symmetrized Alexander polynomial  $\Delta_K(T)$  is monic and that its degree (by which we mean the highest power of  $T$  that appears) is  $g$ . Then  $K$  is fibered if and only if  $KHI(K, g)$  is 1-dimensional.*

*Proof.* The proof given for Corollary 6.2 (the monopole case) needs no alteration, except that the hypothesis on the Alexander polynomial has been explicitly included, rather than being deduced from Lemma 5.10. q.e.d.

**Corollary 7.20.** *Let  $K \subset S^3$  be a knot whose Alexander polynomial is monic of degree equal to the genus of the knot. Consider the irreducible homomorphisms  $\rho : \pi_1(S^3 \setminus K) \rightarrow SU(2)$  which map a chosen meridian  $m$  to the element  $\mathbf{i} \in SU(2)$ . If there is only one conjugacy class of such homomorphisms, and if these homomorphisms are non-degenerate, then  $K$  is fibered. q.e.d.*

**7.8. Non-vanishing theorems in the closed case.** Theorem 7.12 asserts the non-vanishing of instanton Floer homology for balanced sutured manifolds, but the theorem does not say anything directly about closed 3-manifolds  $Y$ . Nevertheless, with a little extra input, we obtain the following result as a corollary.

**Theorem 7.21.** *Let  $Y$  be a closed irreducible 3-manifold containing a closed, connected, oriented surface  $\bar{R}$  representing a non-zero class in second homology. Let  $w$  be a hermitian line bundle whose first Chern class has odd evaluation on  $[R]$ . Then  $I_*(Y|\bar{R})_w$  is non-zero.*

*Proof.* Let  $M$  be the manifold obtained by cutting  $Y$  open along  $R$ , and write the boundary of  $M$  as  $R_+ \cup R_-$ . We regard  $M$  as a sutured manifold, with an empty set of sutures. (The absence of sutures means that  $M$  fails to be balanced.) Let  $N$  be the double of  $M$ . We can regard  $R = R_+ \cup R_-$  as a surface in the closed manifold  $N$ . We can “double” the line bundle also; so we have a line bundle, also denoted by  $w$ , on  $N$ . By the excision theorem, it will be sufficient to show that  $I_*(N|R)_w$  is non-zero. Since  $R_-$  and  $R_+$  are homologous in  $N$  and of equal genus, we have

$$I_*(N|R)_w = I_*(N|R_+)_w,$$

so we could equally well deal with  $I_*(N|R_+)_w$  instead.

From the proof of theorem 3.13 of [9], we have a closed, oriented surface  $T \subset N$  with the following properties:

- the surface  $T$  meets  $R$  transversely in a non-empty set of circles;
- if we write  $T'$  be the surface obtained from  $T$  and  $R$  by smoothing these circles of double points, then this  $T'$  has the property that

by cutting  $N$  open along  $T'$  and then decomposing further along a non-empty collection of annuli  $J$ , we arrive at a *taut*, sutured manifold  $(N'', \delta'')$ .

(To clarify the nature of the annuli above, the reader should refer to figure 3.5 in [9] or the rather similar picture in Figure 9 above. The core circles of the annuli  $J$  are the circles or  $T \cap R$ .)

If  $T$  intersects both  $R_+$  and  $R_-$ , then  $(N'', \delta'')$  is balanced. If  $T$  intersects only  $R_+$ , say, then  $(N'', \delta'')$  fails to be balanced, because its boundary contains two copies of  $R_-$ : these are components of  $\partial N''$  which fail to meet  $A(\delta'')$ , contrary to the definition of balanced. If this is what happens, we re-attach these two copies of  $R_-$ . We rename the resulting manifold as our new  $N''$  and proceed. At this point,  $(N'', \delta'')$  is a balanced sutured manifold.

By Theorem 7.12, we know that  $SHM(N'', \delta'')$  is non-zero. We can regard the manifold  $N$  as a closure of  $(N'', \delta'')$ , but with an auxiliary surface that fails to be connected: the auxiliary surface is the collection of annuli  $J$ . But as we argued in the proof of Lemma 6.10, a disconnected auxiliary surface is as good as a connected one here. We can therefore compute  $SHM(N'', \delta'')$  as  $I_*(N|F)_w$ , where  $F$  is the surface in  $N$  formed from  $R_\pm(\delta'')$  when making the closure. Thus

$$I_*(N|F)_w \neq 0.$$

This surface  $F$  can be identified with  $T'$  in the case that  $T$  meets both  $R_+$  and  $R_-$ . In the case that  $T$  meets only  $R_+$ , then  $F$  is  $T' \setminus R_-$ . In other words,  $F$  is obtained by smoothing the circles of double points of either  $T \cup R$  or  $T \cup R_+$ . As in the proof Lemma 6.10, the Floer homology  $I_*(N|F)_w$  is a direct summand of  $I_*(N|R_+)_w$ . So the latter is non-zero, and we are done. q.e.d.

**Corollary 7.22.** *If  $Y$  is obtained from zero-surgery on a non-trivial knot  $K \subset S^3$ , then  $I_*(Y)_w$  is non-zero for an odd line bundle  $w$ . q.e.d.*

Essentially the same theorem and corollary are proved in [17]. But the present proof requires considerably less geometry and analysis. From Floer's surgery exact triangle, one obtains, as in [17]:

**Corollary 7.23.** *If  $Y_1$  is obtained as +1 surgery on  $K \subset S^3$ , then  $\pi_1(Y_1)$  admits a non-trivial homomorphism to  $SU(2)$ . In particular,  $Y_1$  is not a homotopy sphere. q.e.d.*

This provides a proof of the Property P conjecture that is independent of the work of Feehan and Leness in [6] and independent also of Perelman's proof of the Poincaré conjecture.

**7.9. Questions and conjectures.** There are various questions and conjectures which naturally arise. The most obvious of these is:

**Conjecture 7.24.** *For balanced sutured manifolds  $(M, \gamma)$ , the monopole and Heegaard groups  $SHM(M, \gamma)$  and  $SFH(M, \gamma)$  are isomorphic. When tensored with  $\mathbb{C}$ , they are both isomorphic to the instanton version,  $SHI(M, \gamma)$ .*

As a special case, we have:

**Conjecture 7.25.** *With complex coefficients, the knot homologies defined by Ozsváth-Szabó and Rasmussen are isomorphic to Floer’s instanton homology for knots,  $KHI(K)$ , as defined here and in [8].*

There are various more modest questions one should ask. We have not shown that the Alexander polynomial can be recovered from the instanton knot homology groups  $KHI(K, i)$ , but it is natural to conjecture that this is so, just as in the monopole and Heegaard theories. This may be only a matter of repeating [7] in the instanton context:

**Conjecture 7.26.** *The Euler characteristics of the instanton knot homology groups  $KHI(K, i)$ , for  $i = -g, \dots, g$ , are the coefficients of the symmetrized Alexander polynomial of  $K$ .*

If this conjecture is proved, then the hypothesis on the Alexander polynomial could be dropped from Corollary 7.19.

A loose end in our development of  $SHM(M, \gamma)$  is the lack of a complete accounting of  $\text{spin}^c$  structures. The material of section 6.2 is a step in the right direction. In [13], Juhász proves that his Heegaard Floer homology of sutured manifolds can be decomposed as a direct sum indexed by the set of relative  $\text{spin}^c$  structures  $\mathcal{S}(M, \gamma)$ , and it would be desirable to have a similar statement for the monopole and instanton cases.

Juhász [14] has considered an extension of the fibering theorem, which prompts naturally a conjecture in the instanton context. Motivated by this, we have:

**Conjecture 7.27** (cf. [14]). *Let  $K \subset S^3$  be a knot, and consider the irreducible homomorphisms  $\rho : \pi_1(S^3 \setminus K) \rightarrow SU(2)$  which map a chosen meridian  $m$  to the element  $\mathbf{i} \in SU(2)$ . Suppose that these homomorphisms are non-degenerate and that the number of conjugacy classes of such homomorphisms is less than  $2^{k+1}$ . Then the knot complement  $S^3 \setminus N^\circ(K)$  admits a foliation of depth at most  $2k$ , transverse to the torus boundary.*

The fact that  $I_*(Y|\Sigma)_w$  is of rank 1 in the case that  $Y$  is a surface bundle of  $S^1$  with fiber  $\Sigma$  is something that has other applications. For example, combined with Donaldson’s theorem on the existence of Lefschetz pencils [3], it yields a fairly direct proof that symplectic 4-manifolds have non-zero Donaldson invariants. Essentially the same strategy was used by Ozsváth and Szabó in the Heegaard context. What

the argument shows specifically is that if  $X \rightarrow S^2$  is a symplectic Lefschetz fibration whose fiber  $F$  has genus 2 or more, and if  $w$  is the line bundle dual to a section, then the Donaldson invariant  $D^w(F^n)$  is non-zero for all large enough  $n$  in the appropriate residue class mod 4.

Another matter is whether one can relate either the monopole or instanton knot homologies to the corresponding Floer homologies of the 3-manifolds obtained by surgery on the knot, particularly for large integer surgeries. This is how Heegaard knot homology arose in [28].

In a previous paper [19], the authors described another knot homology constructed using instantons. The definition there is distinctly different from the definition of  $KHI(K)$  given in this paper, because instantons with singularities in codimension-2 were involved. Nevertheless, both theories involve the same representation variety  $\mathcal{R}(K, \mathbf{i})$ . Various versions are defined in [19], but the one most closely related to  $KHI(K)$  is the “reduced” variant, called  $RI_*(K)$  in [19]. Like  $KHI(K)$ , the group  $RI_*(K)$  is a Floer homology group, constructed from a Chern-Simons functional whose set of critical points can be identified with  $\mathcal{R}(K, \mathbf{i})$ . The paper [19] develops its theory for the gauge group  $SU(N)$ , not just  $SU(2)$ , and it would be interesting to pursue a similar direction with  $SHI(M, \gamma)$  and  $KHI(K)$ .

The “hat” version of Heegaard Floer homology, for a closed 3-manifold  $Y$ , can also be recovered as a special case of Juhász’s  $SFH$ , as shown in [12]. The appropriate manifold  $M$  is the complement of a ball in  $Y$ , and one takes a single annular suture on the resulting 2-sphere boundary. One can take this as a definition of a “hat” version of monopole Floer homology. In the instanton case, this leads to essentially the same construction that was used in [19] to avoid reducibles: one replaces  $Y$  by  $Y \# T^3$  and takes  $w$  to be a line bundle that is trivial on  $Y$  and of degree 1 on a  $T^2$  in the  $T^3$ .

Finally, as we mentioned in the introduction, it is worth asking whether, in the Heegaard theory, the Floer homology of a balanced sutured manifold  $(M, \gamma)$ , as defined in [12], can also be recovered as the Heegaard Floer homology of a closed manifold  $Y = Y(M, \gamma)$ , of the sort that we have used here. If so, it would be interesting to know whether the existing proofs of the decomposition theorems in [23] and [13], for example, can be adapted to prove Floer’s excision theorem in the context of Heegaard Floer theory.

**Added in proof.** Since the first version of this paper was prepared, an affirmative answer to a question raised in the introduction has been given by Lekili [20, Theorem 21], who showed that the constructions of the present paper, when applied in the context of Heegaard Floer homology, do indeed reproduce Juhász’s invariant of sutured manifolds.

## References

- [1] M.F. Atiyah & I.M. Singer. *Index theory for skew-adjoint Fredholm operators*. Inst. Hautes Études Sci. Publ. Math. No., **37**:5–26, 1969, MR 0285033, Zbl 0194.55503.
- [2] P.J. Braam & S.K. Donaldson. *Floer's work on instanton homology, knots and surgery*. In *The Floer memorial volume*, volume **133** of Progr. Math., pages 195–256. Birkhäuser, Basel, 1995, MR 1362829, Zbl 0996.57516.
- [3] S.K. Donaldson. *Lefschetz pencils on symplectic manifolds*. J. Differential Geom., **53**(2):205–236, 1999, MR 1802722, Zbl 1040.53094.
- [4] S.K. Donaldson. *Floer homology groups in Yang-Mills theory*, volume **147** of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2002, MR 1883043, Zbl 0998.53057. With the assistance of M. Furuta and D. Kotschick.
- [5] S.K. Donaldson & P.B. Kronheimer. *The geometry of four-manifolds*. Oxford University Press, New York, 1990, MR 1079726, Zbl 0820.57002.
- [6] P.M.N. Feehan & T.G. Leness. *On Donaldson and Seiberg-Witten invariants*. In *Topology and geometry of manifolds (Athens, GA, 2001)*, volume **71** of Proc. Sympos. Pure Math., pages 237–248. Amer. Math. Soc., Providence, RI, 2003, MR 2024636, Zbl 1042.57017.
- [7] R. Fintushel & R.J. Stern. *Knots, links, and 4-manifolds*. Invent. Math., **134**(2):363–400, 1998, MR 1650308, Zbl 0914.57015.
- [8] A. Floer. *Instanton homology, surgery, and knots*. In *Geometry of low-dimensional manifolds, 1 (Durham, 1989)*, volume **150** of London Math. Soc. Lecture Note Ser., pages 97–114. Cambridge Univ. Press, Cambridge, 1990, MR 1171893, Zbl 0788.57008.
- [9] D. Gabai. *Foliations and the topology of 3-manifolds*. J. Differential Geom., **18**(3):445–503, 1983, MR 723813, Zbl 0539.57013.
- [10] D. Gabai. *Foliations and the topology of 3-manifolds. III*. J. Differential Geom., **26**(3):479–536, 1987, MR 910018, Zbl 0639.57008.
- [11] P. Ghiggini. *Knot Floer homology detects genus-one fibred knots*. Amer. J. Math., **130**(5):1151–1169, 2008, MR 2450204, Zbl 1149.57019.
- [12] A. Juhász. *Holomorphic discs and sutured manifolds*. *Algebr. Geom. Topol.*, **6**:1429–1457 (electronic), 2006, MR 2253454, Zbl 1167.57005.
- [13] A. Juhász. *Floer homology and surface decompositions*. *Geom. Topol.*, **12**(1):299–350, 2008, MR 2390347, Zbl 1167.57005.
- [14] A. Juhász. *The sutured Floer homology polytope*. Preprint, 2008.
- [15] P.B. Kronheimer & T.S. Mrowka. *Recurrence relations and asymptotics for four-manifold invariants*. Bull. Amer. Math. Soc. (N.S.), **30**(2):215–221, 1994, MR 1246469, Zbl 0815.57010.
- [16] P.B. Kronheimer & T.S. Mrowka. *Dehn surgery, the fundamental group and  $SU(2)$* . *Math. Res. Lett.*, **11**(5-6):741–754, 2004, MR 2106239, Zbl 1084.57006.
- [17] P.B. Kronheimer & T.S. Mrowka. *Witten's conjecture and property P*. *Geom. Topol.*, **8**:295–310 (electronic), 2004, MR 2023280, Zbl 1072.57005.
- [18] P.B. Kronheimer & T.S. Mrowka. *Monopoles and three-manifolds*. *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2007, MR 2388043, Zbl 1158.57002.

- [19] P.B. Kronheimer & T.S. Mrowka. *Knot homology groups from instantons*. Preprint, 2008.
- [20] Y. Lekili. *Heegaard Floer homology of broken fibrations over the circle*. Preprint, 2009.
- [21] G. Meng & C.H. Taubes.  $\underline{SW} = \text{Milnor torsion}$ . *Math. Res. Lett.*, **3**(5):661–674, 1996, MR 1418579, Zbl 1053.57022.
- [22] V. Muñoz. *Ring structure of the Floer cohomology of  $\Sigma \times \mathbf{S}^1$* . *Topology*, **38**(3):517–528, 1999, MR 1670396, Zbl 0927.57032.
- [23] Y. Ni. *Knot Floer homology detects fibred knots*. *Invent. Math.*, **170**(3):577–608, 2007, MR 2357503, Zbl 1138.57031.
- [24] Y. Ni. Erratum: *Knot Floer homology detects fibred knots*. *Invent. Math.*, **177**(1):235–238, 2009, MR 2507641, Zbl 1162.57300.
- [25] Y. Ni. *Heegaard Floer homology and fibred 3-manifolds*. *Am. J. Math.*, **131**(4):1047–1063, 2009, Zbl pre05598033.
- [26] P. Ozsváth & Z. Szabó. *Holomorphic disks and knot invariants*. *Adv. Math.*, **186**(1):58–116, 2004, MR 2065507, Zbl 1062.57019.
- [27] J. Rasmussen. Private communication.
- [28] J. Rasmussen. *Floer homology and knot complements*. PhD thesis, Harvard University, 2003.

HARVARD UNIVERSITY  
CAMBRIDGE MA 02138

*E-mail address:* kronheim@math.harvard.edu

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
CAMBRIDGE MA 02139

*E-mail address:* mrowka@math.mit.edu