J. DIFFERENTIAL GEOMETRY
84 (2010) 525-560

DEFORMATIONS OF GENERALIZED COMPLEX AND GENERALIZED KÄHLER STRUCTURES

Ryushi Goto

Abstract

In this paper we obtain a stability theorem of generalized Kähler structures with one pure spinor under small deformations of generalized complex structures. (This is analogous to the stability theorem of Kähler manifolds by Kodaira-Spencer.) We apply the stability theorem to a class of compact Kähler manifolds which admits deformations to generalized complex manifolds and obtain non-trivial generalized Kähler structures on Fano surfaces and toric Kähler manifolds. In particular, we show that every nonzero holomorphic Poisson structure on a Kähler manifold induces deformations of non-trivial generalized Kähler structures.

0. Introduction

A notion of generalized complex structures was introduced by Hitchin [**12**], which interpolates between complex and symplectic structures. An associated notion of generalized Kähler structures is developed by Gualtieri [**10**]. Examples of generalized Kähler structures have been constructed by the reduction [**3**, **20**] which is a generalization of the symplectic quotient construction. Hitchin gave an explicit construction of generalized Kähler structures on Del Pezzo surfaces by using holomorphic Poisson structures and suggested that generalized Kähler structures are related to holomorphic Poisson structures [**13**, **14**].

Kodaira and Spencer showed that Kähler structures on compact complex manifolds are stable under sufficiently small deformations of complex structures [**18**]. More precisely, if V_0 is a compact Kähler manifold, then any small deformation V_t of V_0 is also a Kähler manifold.

The purpose of this paper is to establish a stability theorem of generalized Kähler structures under small deformations of generalized complex structures. Applying the theorem, we shall obtain a systematic

The author would like to thank Professor Fujiki and Professor Namikawa for valuable discussions and suggestions. He wishes to thank Professor Hitchin for meaningful discussions. After he posted his paper to *arXiv,* he received a kind and sincere message from Professor Gualtieri. He is also grateful to Professor Yi Lin for his valuable message about the reduction.

Received June 5 2007.

construction of non-trivial generalized Kähler structures which arise as deformations of ordinary Kähler manifolds with holomorphic Poisson structures. The construction provides many examples by using both holomorphic Poisson structures and deformations of complex structures. In our construction, it is intriguing to solve the problem of obstructions to deformations of generalized Kähler structures. Note that there exists an obstruction to deformations of generalized complex structures in general. We assume that there exists a family of deformations of generalized complex structures on a generalized Kähler manifold X. Then we apply the method in [**8**] and show that every obstruction to corresponding deformations of generalized Kähler structures vanishes. The method is a generalization of the one in unobstructed theorem of Calabi-Yau manifolds by Bogomolov-Tian-Todorov [**24**], which is also applied to obtain unobstructed deformations and the local Torelli type theorem for Riemannian manifolds with special holonomy group [**7**]. For the more precise statement of the stability theorem, we explain generalized complex structures, generalized Kähler structures, and in particular, a relation to pure spinors.

The notion of generalized complex structures is based on an idea of replacing the tangent bundle T of a manifold with the direct sum of the tangent bundle T and the cotangent bundle T^*. The fibre bundle of the direct sum $T \oplus T^*$ admits an indefinite metric $\langle\, , \,\rangle$ by which we obtain the fibre bundle $\mathrm{SO}(T \oplus T^*)$ with fibre the special orthogonal group. An almost generalized complex structure $\mathcal{J}$ is defined as a section of the fibre bundle $\mathrm{SO}(T \oplus T^*)$ with $\mathcal{J}^2 = -\mathrm{id}$, which gives rise to the decomposition $(T \oplus T^*) \otimes \mathbb{C} = L_{\mathcal{J}} \oplus \overline{L}_{\mathcal{J}}$, where $L_{\mathcal{J}}$ is $-\sqrt{-1}$-eigenspace of $\mathcal{J}$ and $\overline{L}_{\mathcal{J}}$ denotes its complex conjugate. Almost generalized complex structures form an orbit of the action of the real Clifford group of the real Clifford algebra bundle CL with respect to $(T \oplus T^*, \langle\, , \,\rangle)$ (cf. [**6**]). A generalized complex structure is an almost generalized complex structure which is integrable with respect to the Courant bracket.

A generalized Kähler structure is a pair $(\mathcal{J}_0, \mathcal{J}_1)$ consisting of commuting generalized complex structures $\mathcal{J}_0$ and $\mathcal{J}_1$ which gives rise to a generalized metric $G := -\mathcal{J}_0\mathcal{J}_1$.

The direct sum $T \oplus T^*$ acts on differential forms on a manifold by the interior product and the exterior product. For a differential form ψ, we define a subspace L_ψ by $L_\psi := \{\, E \in (T \oplus T^*) \otimes \mathbb{C} \,|\, E \cdot \psi = 0 \,\}$. A non-degenerate pure spinor is a differential form ψ which gives a decomposition $(T \oplus T^*) \otimes \mathbb{C} = L_\psi \oplus \overline{L}_\psi$. Thus a non-degenerate pure spinor ψ induces an almost generalized complex structure $\mathcal{J}_\psi$. It turns out that if a non-degenerate pure spinor ψ is d-closed, then the induced structure $\mathcal{J}_\psi$ is integrable. For a Kähler form ω, the exponential $e^{\sqrt{-1}\omega}$ is a non-degenerate pure spinor which induces the generalized complex structure $\mathcal{J}_\omega$. From this point of view, we introduce *a generalized Kähler* *structure with one pure spinor* as a pair $(\mathcal{J}, \psi)$ consisting of a generalized complex structure $\mathcal{J}$ and a d-closed, non-degenerate pure spinor ψ which induces the generalized Kähler structure $(\mathcal{J}, \mathcal{J}_\psi)$. Then we obtain the following stability theorem.

Theorem 3.1 *Let $(\mathcal{J},\, \psi)$ be a generalized Kähler structure with one pure spinor on a compact manifold X. We assume that there exists an analytic family of generalized complex structures $\{\mathcal{J}_t\}_{t\in\triangle}$ on X with $\mathcal{J}_0 = \mathcal{J}$ parametrized by the complex one-dimensional open disk $\triangle$ containing the origin 0. Then there exists an analytic family of generalized Kähler structures with one pure spinor $\{\,(\mathcal{J}_t,\, \psi_t)\}_{t\in\triangle'}$ with $\psi_0 = \psi$ parametrized by a sufficiently small open disk $\triangle' \subset \triangle$ containing the origin.*

An analytic family of generalized complex structures is a family of generalized complex structures $\{\mathcal{J}_t\}$ which depend analytically on the parameter t in $\triangle$. If the space of obstructions to deformations of generalized complex structures vanishes, then infinitesimal deformations generate an analytic family of deformations of generalized complex structures. It is remarkable that a holomorphic Poisson structure on a compact Kähler manifold gives the analytic family of deformations of generalized complex structures which induces a family of deformations of non-trivial generalized Kähler structures.

In section 1, we present an exposition on generalized complex and generalized Kähler geometry. Preliminary results are collected in subsections 1-1 and 1-2 (cf. [**10**, **11**, **12**]). In subsection 1-3, we introduce a generalized Kähler structure with one pure spinor and construct a differential complex $(K^\bullet, d)$ which is a subcomplex of the de Rham complex. Applying the generalized Hodge decomposition [**11**], we obtain an injective map from the cohomology $H^*(K^\bullet)$ of the complex $(K^\bullet, d)$ to the de Rham cohomology group. In section 2 we discuss deformations of generalized complex structures from the view point of pure spinors. The Maurer-Cartan equation naturally arises as the integrability of almost generalized complex structures. Further we show that an analytic family of generalized complex structures $\{\mathcal{J}_t\}_{t\in\triangle}$ are described in terms of an analytic family of sections $a(t)$ of the real Clifford bundle CL^2 with respect to $(T \oplus T^*, \langle\,,\,\rangle)$ which is the Lie algebra of the Clifford group (conformal pin group). The exponential of sections $a(t)$ of CL^2 is the family of sections of the Clifford group which acts on $\mathcal{J}_0$ by the adjoint action, and we have

$$\mathcal{J}_t = \mathrm{Ad}_{e^{a(t)}} \mathcal{J}_0.$$

We prove the stability theorem in section 3 in the sense of formal power series. For the analytic family $a(t)$, we will construct a family of sections

$b(t)$ of CL^2 such that

$$d\,(e^{a(t)}\,e^{b(t)}\,\psi_0) = 0, \tag{1}$$

$$\mathrm{Ad}_{e^{b(t)}}\mathcal{J}_0 = \mathcal{J}_0. \tag{2}$$

It follows from the Campbell-Hausdorff formula [**23**] that we have a unique family $z(t) \in \mathrm{CL}^2$ with

$$e^{z(t)} = e^{a(t)}\,e^{b(t)}.$$

Then from (1), $e^{z(t)}\,\psi_0$ is a d-closed and non-degenerate pure spinor and we have

$$\mathrm{Ad}_{e^{z(t)}}\mathcal{J}_0 = \mathcal{J}_t,$$

from (2). Since almost generalized Kähler structures also form the orbit of the action of the Clifford group, it follows that $(\mathcal{J}_t,\ e^{z(t)}\,\psi)$ is a family of generalized Kähler structures with one pure spinor. When we try to solve the equations (1) and (2), we encounter the class of obstruction $[\widetilde{\mathrm{Ob}}_k] \in H^2(K^\bullet)$ for each $k > 0$. It turns out that each representative $\widetilde{\mathrm{Ob}}_k$ is a d-exact differential form. Since the cohomology group $H^2(K^\bullet)$ is embedded into the de Rham cohomology group, it follows that the class $[\widetilde{\mathrm{Ob}}_k]$ vanishes and we obtain a solution $b(t)$ of the equations (1) and (2) as the formal power series. Our solution $b(t)$ is not unique in general. A solution $b(t)$ together with $a(t)$ gives rise to a cohomology class of $H^1(K^\bullet)$ by the action on ψ_0. We show that there exists a family of solutions of the equations (1) and (2) which are locally parametrized by the first cohomology group $H^1(K^\bullet)$ of the complex $(K^\bullet, d)$.

Theorem 3.2 *Let $\{\mathcal{J}_t\}_{t\in\triangle}$ and ψ be as in theorem 3.1. Then there is an open set W in $H^1(K^\bullet)$ containing the origin such that there exists a family of generalized Kähler structures with one pure spinor $\{(\mathcal{J}_t,\ \psi_{t,s})\}$ with $\psi_{0,0} = \psi$ parametrized by $t \in \triangle'$ and $s \in W$ in $H^1(K^\bullet)$. Further if we denote by $[\psi_{t,s}]$ the de Rham cohomology class represented by $\psi_{t,s}$, then $[\psi_{t,s_1}] \neq [\psi_{t,s_2}]$ for $s_1 \neq s_2$.*

In section 4, we will prove that the formal power series $b(t)$ converges and finish the proof of the stability theorem. In section 5, we construct examples of generalized Kähler structures on compact Kähler manifolds such as Fano surfaces and toric manifolds. Since there is no obstruction to deformations of generalized complex structures on any Fano surface, we can count the dimensions of deformations of generalized complex and generalized Kähler structures, respectively. We show that a holomorphic Poisson structure induces many interesting generalized Kähler structures. If there is an action of a complex 2-dimensional commutative Lie group which gives a non-trivial holomorphic Poisson structure on a compact Kähler manifold, then we obtain a family of deformations of non-trivial generalized Kähler structures. It follows that every compact toric Kähler manifold admits non-trivial generalized Kähler structures.

There is a one-to-one correspondence between generalized Kähler structures and bihermitian structures [**10**]. Then by using the stability theorem, it is shown that there exists a family of non-trivial bihermitian structures on every compact Kähler manifold (X, ω) with a non-zero holomorphic Poisson structure β. Then we obtain an unobstructed deformations of complex structures whose infinitesimal deformation is given by $\beta \cdot \omega$, which is a $\overline{\partial}$-closed $T^{1,0}$-valued form of type $(0, 1)$ given by the contraction of β by ω. Thus we obtain:

Theorem 3.2 [**9**] *Let X be a compact Kähler manifold with a holomorphic Poisson structure β. The class $[\beta \cdot \omega] \in H^1(X, \Theta)$ gives rise to unobstructed deformations of complex structures.* (see section 3 in [**9**] for more detail).

1. Generalized complex and Kähler structures

1.1. Generalized complex structures. Let $T \oplus T^*$ be the direct sum of the tangent bundle TX and the cotangent bundle T^*X on a manifold X of real $2n$ dimension. Then there is a symmetric bilinear form $\langle\, , \rangle$ on $T \oplus T^*$ which is given by

$$\langle v + \theta, w + \eta \rangle = \frac{1}{2}\theta(w) + \frac{1}{2}\eta(v), \tag{1.1}$$

where $v, w \in TX$ and $\theta, \eta \in T^*X$. Then we have the fibre bundle $\mathrm{SO}(T \oplus T^*)$ with fibre the special orthogonal group with respect to $\langle\, , \rangle$. We define *an almost generalized complex structure* $\mathcal{J}$ as a section of the bundle $\mathrm{SO}(T \oplus T^*)$ with $\mathcal{J}^2 = -\mathrm{id}$. The direct sum $T \oplus T^*$ acts on the differential forms $\wedge^\bullet T^*X$ by the interior product and the exterior product,

$$(v + \theta) \cdot \alpha := i_v \alpha + \theta \wedge \alpha, \tag{1.2}$$

where $\alpha \in \wedge^\bullet T^*X$. Let CL be the real Clifford algebra bundle of $T \oplus T^*$ with respect to the bilinear form $\langle\, , \rangle$. Then from (1.1) and (1.2) we have the induced action of CL on differential forms $\wedge^\bullet T^*X$, which is the spin representation of CL. For a complex differential form ϕ we define a subspace L_ϕ of $(T \oplus T^*) \otimes \mathbb{C}$ by

$$L_\phi := \{\, E \in (T \oplus T^*) \otimes \mathbb{C} \,|\, E \cdot \phi = 0 \,\}. \tag{1.3}$$

A complex differential form ϕ is a (complex) *pure spinor* if L_ϕ is maximally isotropic, i.e., $2n$ dimensional. A (complex) pure spinor ϕ is *non-degenerate* if we have the decomposition of $(T \oplus T^*) \otimes \mathbb{C}$ into L_ϕ and its complex conjugate $\overline{L}_\phi$,

$$(T \oplus T^*) \otimes \mathbb{C} = L_\phi \oplus \overline{L}_\phi. \tag{1.4}$$

The decomposition (6) induces the almost generalized complex structure $\mathcal{J}_\phi$ which is defined by

$$\mathcal{J}_\phi(E) = \begin{cases} -\sqrt{-1}E, & (E \in L_\phi), \\ \sqrt{-1}E, & (E \in \overline{L}_\phi). \end{cases} \tag{1.5}$$

We call $\mathcal{J}_\phi$ the induced structure from the non-degenerate pure spinor ϕ.

Let $\mathcal{J}$ be an almost generalized complex structure with the $-\sqrt{-1}$-eigenspace $L_{\mathcal{J}}$. Then we have the decomposition $(T \oplus T^*) \otimes \mathbb{C} = L_{\mathcal{J}} \oplus \overline{L}_{\mathcal{J}}$. We denote by $\mathrm{CL}^{[i]}$ the subbundle of CL of degree i. Then we identify the Lie algebra bundle $so(T \oplus T^*)$ with $\mathrm{CL}^{[2]}$. Under the identification $so(T \oplus T^*) = \mathrm{CL}^{[2]}$, $\mathcal{J}$ acts on $\wedge^\bullet T^*X \otimes \mathbb{C}$ by the spin representation. Then we have the eigenspace decomposition of $\wedge^\bullet T^*X \otimes \mathbb{C}$,

$$\wedge^\bullet T^*X \otimes \mathbb{C} = U^{-n} \oplus U^{-n+1} \oplus \cdots \oplus U^{n-1} \oplus U^n, \tag{1.6}$$

where U^k denotes the eigenspace with eigenvalue $k\sqrt{-1}$. The space U^{-n} is a complex line bundle which we call the canonical line bundle of $\mathcal{J}$. (We also denote it by $K_{\mathcal{J}}$). Let $\wedge^k \overline{L}_{\mathcal{J}}$ be the kth exterior product of $\overline{L}_{\mathcal{J}}$. Then the eigenspace U^{-n+k} is given by the action of $\wedge^k \overline{L}_{\mathcal{J}}$ on $K_{\mathcal{J}}$,

$$U^{-n+k} = \wedge^k \overline{L}_{\mathcal{J}} \cdot K_{\mathcal{J}}. \tag{1.7}$$

We denote by $\{(U_\alpha, \phi_\alpha)\}$ a trivialization of the line bundle $K_{\mathcal{J}}$, where $\{U_\alpha\}$ is a covering of X. Each ϕ_α is a non-vanishing section of $K_{\mathcal{J}}|_{U_\alpha}$ which is a non-degenerate pure spinor with the induced structure $\mathcal{J}$. Let d be the exterior derivative and E an element of $\mathrm{CL}^{[1]} \otimes \mathbb{C} = (T \oplus T^*) \otimes \mathbb{C}$. Then the anti-commutator $\{d, E\} := dE + Ed$ acts on $\wedge^\bullet T^*X$. We have the derived bracket by the commutator of $\{d, E\}$ and F,

$$[E, F]_d := [\{d, E\}, F]. \tag{1.8}$$

By skew-symmetrization of the derived bracket, we construct the Courant bracket as

$$[E, F]_{\mathrm{co}} := \frac{1}{2}[\{d, E\}, F] - \frac{1}{2}[\{d, F\}, E]. \tag{1.9}$$

This is known as the derived bracket construction [**19**]. Note that if $E = v, F = w \in TX$, then the Courant bracket becomes the standard bracket of vector fields. If the subbundle $L_{\mathcal{J}}$ is involutive with respect to the Courant bracket, then $\mathcal{J}$ is *integrable*. A generalized complex structure is an almost generalized complex structure which is integrable. The integrability of $\mathcal{J}$ is also given in terms of the corresponding pure spinor. The following observation can be found in section 4.4 [**10**].

Lemma 1.1. *Let ϕ be a non-degenerate pure spinor with the induced structure $\mathcal{J}_\phi$. Then $\mathcal{J}_\phi$ is integrable if and only if there exists $E \in \mathrm{CL}^{[1]} \otimes \mathbb{C} = (T \oplus T^*) \otimes \mathbb{C}$ such that*

$$d\phi + E \cdot \phi = 0. \tag{1.10}$$

To make the paper self-contained, we will give a proof.

Proof. It suffices to show that $[E_1, E_2]_{\mathrm{co}} \in L_\phi$ for $E_1, E_2 \in L_\phi$. It follows that

$$[\{d, E_1\}, E_2]\phi = -E_2 E_1 d\phi. \tag{1.11}$$

If we have $d\phi + E \cdot \phi = 0$, then it follows that

$$[\{d, E_1\}, E_2]\phi = E_2 E_1 E\phi, \tag{1.12}$$

$$= \langle E_1, E\rangle E_2\phi = 0. \tag{1.13}$$

Hence from (1.9), we have $[E_1, E_2]_{\mathrm{co}}\phi = 0$. It implies that L_ϕ is involutive. Conversely, assume that $\mathcal{J}$ is integrable. From (1.6), $d\phi$ is decomposed into

$$d\phi = \sum_{k=-n}^{n} (d\phi)^{[k]}, \tag{1.14}$$

where $(d\phi)^{[k]} \in U^k$. Then it follows that if $(d\phi)^{[k]} \neq 0$ for $k > -n+1$, then there are E_1, E_2 such that $[\{d, E_1\}, E_2]\phi = -E_2 E_1 d\phi \neq 0$. Hence $d\phi \in U^{-n+1}$. It implies that $(d\phi) = -E \cdot \phi$ for $E \in \mathrm{CL}^{[1]} \otimes \mathbb{C}$. q.e.d.

If $\mathcal{J}$ is integrable, the image $d(U^k)$ is a subspace of the direct sum $U^{k-1} \oplus U^{k+1}$. Then d is decomposed into $\partial + \overline{\partial}$,

$$d\alpha = \partial\alpha + \overline{\partial}\alpha,$$

where $\partial\alpha \in U^{k-1}$ and $\overline{\partial}\alpha \in U^{k+1}$ for $\alpha \in U^k$. There is a natural filtration of the even part of the real Clifford bundle CL,

$$\mathrm{CL}^0 \subset \mathrm{CL}^2 \subset \cdots. \tag{1.15}$$

We also have a filtration of the odd part of the real Clifford bundle,

$$\mathrm{CL}^1 \subset \mathrm{CL}^3 \subset \cdots. \tag{1.16}$$

For instance, the first several ones are given by

$$\mathrm{CL}^0 = C^\infty(X), \quad \mathrm{CL}^1 = \mathrm{CL}^{[1]} = T \oplus T^*,$$
$$\mathrm{CL}^2 = \mathrm{CL}^0 \oplus \mathrm{CL}^{[2]}, \quad \mathrm{CL}^3 = \mathrm{CL}^{[1]} \oplus \mathrm{CL}^{[3]},$$

where $\mathrm{CL}^{[i]}$ denotes the skew-symmetric subspace of $(T \oplus T^*)$ in CL^i. The filtrations give rise to the filtration of bundles $\mathbf{E}^k$ given by the action of CL^{k+1} on the canonical line bundle $K_{\mathcal{J}}$,

$$\mathbf{E}^k := \mathrm{CL}^{k+1} \cdot K_{\mathcal{J}},$$

where $\mathbf{E}^k = \{0\}$ for $k < -1$. Note that $\mathbf{E}^k$ is the complex vector bundle since $K_{\mathcal{J}}$ is the complex line bundle. We change the degree of $\mathbf{E}^\bullet$.

For instance, $\mathbf{E}^{-1}$ is the canonical line bundle $K_{\mathcal{J}}$ and $\mathbf{E}^0$ and $\mathbf{E}^1$ are respectively written in the forms

$$\mathbf{E}^0 = \{\, E\cdot\phi \,|\, E \in \mathrm{CL}^1, \phi \in K_{\mathcal{J}}\,\}, \tag{1.17}$$

$$\mathbf{E}^1 = \{\, a\cdot\phi \,|\, a \in \mathrm{CL}^2, \phi \in K_{\mathcal{J}}\,,\}. \tag{1.18}$$

Then $\mathbf{E}^k$ is the direct sum in terms of $U^{-n+\bullet}$. The first four bundles are given by

$$\mathbf{E}^{-1} = U^{-n}, \tag{1.19}$$

$$\mathbf{E}^0 = U^{-n+1}, \tag{1.20}$$

$$\mathbf{E}^1 = U^{-n}\oplus U^{-n+2}, \tag{1.21}$$

$$\mathbf{E}^2 = U^{-n+1}\oplus U^{-n+3}. \tag{1.22}$$

Then U^{-n+k} is given as the quotient bundle,

$$U^{-n+k} = \mathbf{E}^{k-1}/\mathbf{E}^{k-3}.$$

It follows from $d = \partial + \overline{\partial}$ that $\mathbf{E}^\bullet$ is invariant under the action of d. Hence we have the differential complex $(\mathbf{E}^\bullet, d)$,

$$0 \xrightarrow{d} \mathbf{E}^{-1} \xrightarrow{d} \mathbf{E}^0 \xrightarrow{d} \mathbf{E}^1 \xrightarrow{d} \mathbf{E}^2 \xrightarrow{d} \cdots .$$

It is shown that the complex $(\mathbf{E}^\bullet, d)$ is elliptic in [**8**]. We denote by $H^k(\mathbf{E}^\bullet)$ the kth cohomology of the complex $(\mathbf{E}^\bullet, d)$.

1.2. Generalized Kähler structures. In this subsection, we use the same notation as in [**11**]. Let $(\mathcal{J}_0, \mathcal{J}_1)$ be a pair of commuting generalized complex structures. Then we define $\hat{G}$ by the composition

$$\hat{G} = -\mathcal{J}_0\mathcal{J}_1 = -\mathcal{J}_1\mathcal{J}_0.$$

The symmetric bilinear form G is given by $G(E_1, E_2) := \langle \hat{G}E_1, E_2\rangle$ for $E_1, E_2 \in T\oplus T^*$.

Definition 1.2. A pair $(\mathcal{J}_0, \mathcal{J}_1)$ consisting of commuting generalized complex structures is a generalized Kähler structure if the symmetric bilinear form G is positive-definite.

Let $U^p_{\mathcal{J}_i}$ be the eigenspace with respect to $\mathcal{J}_i$ for $i = 0, 1$. Because we have the commuting pair $(\mathcal{J}_0, \mathcal{J}_1)$, we have the simultaneous decomposition into eigenspaces,

$$\wedge^\bullet T^*X\otimes\mathbb{C} = \oplus_{p,q}U^{p,q},$$

where $U^{p,q} = U^p_{\mathcal{J}_0}\cap U^q_{\mathcal{J}_1}$. Then the image of $U^{p,q}$ by the exterior derivative d is decomposed into four components $U^{p+1,q+1}\oplus U^{p+1,q-1}\oplus U^{p-1,q-1}\oplus U^{p-1,q+1}$ which induces the decomposition of d,

$$d = \overline{\delta}_+ + \overline{\delta}_- + \delta_+ + \delta_-.$$

$$\begin{tikzcd}
U^{p-1,q+1} & & U^{p+1,q+1} \\
& U^{p,q} \arrow[ul, "\delta_-"] \arrow[ur, "\overline{\delta}_+"'] \arrow[dl, "\delta_+"'] \arrow[dr, "\overline{\delta}_-"] & \\
U^{p-1,q-1} & & U^{p+1,q-1}
\end{tikzcd}$$

Figure 1

1.3. Generalized Kähler structures with one pure spinor. We already see that a non-degenerate pure spinor ψ is a differential form which induces the almost generalized complex structure $\mathcal{J}_\psi$.

Definition 1.3. Let $(\mathcal{J}, \psi)$ be a pair consisting of generalized complex structure $\mathcal{J}$ and a non-degenerate pure spinor ψ with $d\psi = 0$. A pair $(\mathcal{J}, \psi)$ is a generalized Kähler structure with one pure spinor if the corresponding pair $(\mathcal{J}, \mathcal{J}_\psi)$ is a generalized Kähler structure.

We denote by K^1 the bundle $U^{0,-n+2}$ and define the graded left module $K^\bullet$ generated by K^1 over the Clifford algebra CL. We set $K^i = \{0\}$ for $i \leq 0$. Then it follows that

$$K^1 = U^{0,-n+2}, \tag{1.23}$$

$$K^2 = U^{1,-n+1} \oplus U^{-1,-n+1} \oplus U^{1,-n+3} \oplus U^{1,-n+3}. \tag{1.24}$$

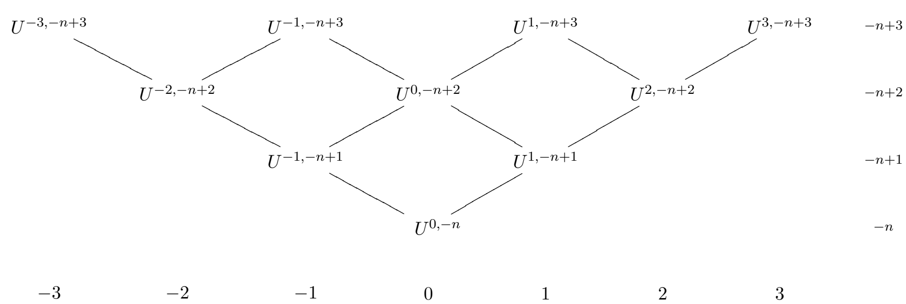

Figure 2

Then we have the following lemma from the decomposition of the exterior derivative d.

Lemma 1.4. *$(K^\bullet, d)$ is a differential complex.*

Let $(\mathcal{J}, \psi)$ be a generalized Kähler structure with one pure spinor. We denote by $a \cdot K_{\mathcal{J}}$ the action of $a \in \mathrm{CL}$ on the canonical line bundle $K_{\mathcal{J}}$. We define a bundle $\ker^i$ by

$$\ker^i = \{\, a \in \mathrm{CL}^{i+1} \,|\, a \cdot K_{\mathcal{J}} = 0 \,\}, \tag{1.25}$$

for $i = 0, 1, 2$. We also define $\widetilde{\ker}^i$ by using the filtration of CL and $\mathbf{E}^i := \mathrm{CL}^{i+1} \cdot K_{\mathcal{J}}$,

$$\widetilde{\ker}^i = \{\, a \in \mathrm{CL}^{i+1} \,|\, a \cdot K_{\mathcal{J}} \in \mathrm{CL}^{i-1} \cdot K_{\mathcal{J}} \,\}. \tag{1.26}$$

Then we have:

Lemma 1.5.

$$U^{0,-n} \oplus U^{0,-n+2} = \{\, a \cdot \psi \,|\, a \in \widetilde{\ker}^1 \,\}, \tag{1.27}$$

Proof. The real bundle $\widetilde{\ker}^1$ consists of linear combinations of the real part $E \cdot \overline{F}$ where $E \in L_{\mathcal{J}}$ and $\overline{F} \in \overline{L}_{\mathcal{J}}$. Since $E \cdot \overline{F}\psi \in U^{0,-n} \oplus U^{0,-n+2}$, it follows that $\widetilde{\ker}^1 \cdot \psi \in U^{0,-n} \oplus U^{0,-n+2}$. Conversely, it follows that $U^{0,-n} \oplus U^{0,-n+2}$ is generated by forms $(E \cdot \overline{F} + \overline{E} \cdot F)\psi$ and $\sqrt{-1}(E \cdot \overline{F} - \overline{E} \cdot F)\psi$ for $E \in L_{\mathcal{J}}$ and $\overline{F} \in \overline{L}_{\mathcal{J}}$. q.e.d.

The bundle K^2 is also described in terms of $\ker^2$ and $\widetilde{\ker}^2$:

Lemma 1.6.

$$\begin{aligned} K^2 &= \{\, b \cdot \psi \,|\, b \in \ker^2 \,\}, \\ &= \{\, b \cdot \psi \,|\, b \in \widetilde{\ker}^2 \,\}. \end{aligned} \tag{1.28}$$

Proof. We denote by $\widetilde{K}^2$ the bundle $\{\, b \cdot \psi \,|\, b \in \widetilde{\ker}^2 \,\}$. Since K^2 is generated by K^1, we see that

$$K^2 \subset \{\, b \cdot \psi \,|\, b \in \ker^2 \,\} \subset \widetilde{K}^2. \tag{1.29}$$

The space $U^{3,-n+3}$ is given by $\wedge^3 \overline{L}_{\mathcal{J}} \cdot \psi$. Let h be an element of $\wedge^3 \overline{L}_{\mathcal{J}}$. Then $h \cdot K_{\mathcal{J}} \in \mathrm{CL}^1 \cdot K_{\mathcal{J}}$ if and only if $h = 0$. Since $\ker^2$ is real, $\widetilde{K}^2$ does not contain the components $U^{3,-n+3}$ and $U^{-3,-n+3}$. Hence it follows from (1.24) that $K^2 = \widetilde{K}^2$. We have the result from (1.29). q.e.d.

Lemma 1.7. *$(K^\bullet, d)$ is an elliptic complex for $i = 1, 2$.*

Proof. We will show that the symbol complex of the complex $(K^\bullet, d)$ is exact. It is sufficient to prove that if $u \wedge \alpha = 0$ for non-zero one form $u \in T^*$ and $\alpha \in K^i$ then α is given by $\alpha = u \wedge \beta$ for a $\beta \in K^{i-1}$ for $i = 1, 2$. We have the commuting generalized complex structures $\mathcal{J}$ and $\mathcal{J}_\psi$ which act on $(T \oplus T^*) \otimes \mathbb{C}$. Then we have the simultaneous eigenspace decomposition

$$(T \oplus T^*) \otimes \mathbb{C} = \overline{L}_+ \oplus \overline{L}_- \oplus L_+ \oplus L_-, \tag{1.30}$$

where $\overline{L}_+ \oplus \overline{L}_-$ is $-\sqrt{-1}$-eigenspace with respect to $\mathcal{J}$ and $\overline{L}_+ \oplus L_-$ is $-\sqrt{-1}$-eigenspace with respect to $\mathcal{J}_\psi$. The non-zero element u is decomposed into

$$u = \overline{u}_+ + \overline{u}_- + u_+ + u_-, \tag{1.31}$$

where $\overline{u}_\pm \in \overline{L}_\pm$ and $u_\pm \in L_\pm$. Since $u \in T^*$, we have $\langle u, u\rangle = 0$. Hence

$$0 = \langle u, u\rangle = \langle u_+, \overline{u}_+\rangle + \langle u_-, \overline{u}_-\rangle. \tag{1.32}$$

The composition $\hat{G} = -\mathcal{J}\mathcal{J}_\psi = -\mathcal{J}_\psi\mathcal{J}$ defines the generalized metric. Since $\hat{G}(u_\pm + \overline{u}_\pm) = \pm(u_\pm + \overline{u}_\pm)$, we have $(\pm 1)\langle u_\pm, \overline{u}_\pm\rangle > 0$. In particular, it follows that

$$\langle u_\pm, \overline{u}_\pm\rangle \neq 0, \tag{1.33}$$

because the generalized metric is positive-definite. At first we consider the case $K^1 = U^{0,-n+2}$. We assume that $u \wedge \alpha = 0$ for non-zero $u \in T^*$ and $\alpha \in U^{0,-n+2}$. Then it follows from the decomposition (1.31) that

$$\overline{u}_\pm \cdot \alpha = 0, \quad u_\pm \cdot \alpha = 0. \tag{1.34}$$

Then we have

$$u_+ \cdot \overline{u}_+ \cdot \alpha = \langle u_+, \overline{u}_+\rangle\alpha = 0. \tag{1.35}$$

Since $\langle u_+, \overline{u}_+\rangle \neq 0$, we have $\alpha = 0$. In the case K^2, we assume that $u \wedge \alpha = 0$ for non-zero $u \in T^*$ and $\alpha \in K^2$. From (1.24), we see that $K^2 \subset U^{-n+1}_{\mathcal{J}_\psi} \oplus U^{-n+3}_{\mathcal{J}_\psi}$. Let $(\mathbf{E}_\psi, d)$ be the differential complex defined by the action of CL on the canonical line bundle $K_{\mathcal{J}_\psi}$. Since the complex $(\mathbf{E}_\psi, d)$ is elliptic, we have that there exists $\tilde{\beta} \in U^{-n+2}_{\mathcal{J}_\psi}$ such that

$$\alpha = u \wedge \tilde{\beta}. \tag{1.36}$$

We decompose $\tilde{\beta}$ by

$$\tilde{\beta} = \tilde{\beta}^{(2)} + \tilde{\beta}^{(0)} + \tilde{\beta}^{(-2)}, \tag{1.37}$$

where $\tilde{\beta}^{(i)} \in U^{i,-n+2}$. Then we define $\gamma^{(\pm 1)} \in U^{\pm 1,-n+1}$ by

$$\gamma^{(1)} = -\langle u_+, \overline{u}_+\rangle^{-1} u_+ \cdot \tilde{\beta}^{(2)}, \tag{1.38}$$

$$\gamma^{(-1)} = \langle u_-, \overline{u}_-\rangle^{-1} \overline{u}_- \cdot \tilde{\beta}^{(-2)}. \tag{1.39}$$

Then applying (1.32) and (1.36), we obtain that

$$\begin{aligned} u \wedge (u_- \cdot \gamma^{(1)}) &= (\overline{u}_+ + \overline{u}_-) \cdot u_- \cdot \gamma^{(1)} \\ &= -(\overline{u}_+ + \overline{u}_-) \cdot u_- \cdot \langle u_+, \overline{u}_+\rangle^{-1} u_+ \cdot \tilde{\beta}^{(2)} \\ &= (u_- + u_+) \cdot \tilde{\beta}^{(2)} \\ &= u \wedge \tilde{\beta}^{(2)}. \end{aligned}$$

We also apply a similar method to $\beta^{(-2)}$; then we have two equations

$$u \wedge (u_- \cdot \gamma^{(1)}) = u \wedge \tilde{\beta}^{(2)} \tag{1.40}$$

$$-u \wedge (\overline{u}_+ \cdot \gamma^{(-1)}) = u \wedge \tilde{\beta}^{(-2)}. \tag{1.41}$$

We define $\beta^{(0)} \in U^{0,-n+2}$ by

$$\beta^{(0)} = \tilde{\beta}^{(0)} + u_- \cdot \gamma^{(1)} - \overline{u}_+ \cdot \gamma^{(-1)}. \tag{1.42}$$

Then it follows from (1.40) and (1.41) that

$$u \wedge \beta^{(0)} = u \wedge \tilde{\beta}^{(0)} + u \wedge \tilde{\beta}^{(2)} + u \wedge \tilde{\beta}^{(-2)} \tag{1.43}$$

$$= u \wedge \beta = \alpha. \tag{1.44}$$

Hence the complex $(K^\bullet, d)$ is elliptic for $i = 1, 2$. q.e.d.

We denote by $H^i(K^\bullet)$ the ith cohomology group of the complex $(K^\bullet, d)$. The complex $(K^\bullet, d)$ is a subcomplex of the (full) de Rham complex $\{\cdots \overset{d}{\to} \wedge^\bullet T^*X \overset{d}{\to} \wedge^\bullet T^*X \overset{d}{\to} \cdots\}$. The cohomology group of the full de Rham complex is given by the full de Rham cohomology group $H_{dR}(X) := \oplus_{i=0}^{2n} H^i(X, \mathbb{C})$. Then we have the induced map $p_K^i : H^i(K^\bullet) \to H_{dR}(X)$.

Lemma 1.8. *The map* $p_K^i : H^i(K^\bullet) \to H_{dR}(X)$ *is injective for* $i = 1, 2$.

Proof. Our proof is based on the generalized Kähler identities (cf. proposition 2 in [**11**])

$$\overline{\delta}_+^* = -\delta_+, \quad \overline{\delta}_-^* = \delta_-, \tag{1.45}$$

where the exterior derivative d is given by

$$d = \overline{\delta}_+ + \overline{\delta}_- + \delta_+ + \delta_-, \tag{1.46}$$

and $\overline{\delta}_\pm^*$ is the adjoint operator of $\overline{\delta}_\pm$ with respect to the generalized Hodge star operator. Then the identities imply the equality of all available Laplacian,

$$\triangle_d = 2\triangle_{\overline{\partial}_\psi} = 4\triangle_{\overline{\delta}_\pm} = 4\triangle_{\delta_\pm}, \tag{1.47}$$

where $\overline{\partial}_\psi = \overline{\delta}_+ + \delta_-$. We obtain a (p, q) decomposition for the de Rham cohomology of any compact generalized Kähler manifold,

$$H^\bullet(X, \mathbb{C}) = \bigoplus_{\substack{|p+q| \le n \\ p+q \equiv n \ (\mathrm{mod} 2)}} \mathcal{H}^{p,q}, \tag{1.48}$$

where $\mathcal{H}^{p,q}$ are $\triangle_d$-harmonic forms in $U^{p,q}$. At first we consider the cohomology $H^1(K^\bullet)$. Let α be a d-closed element of K^1. Then from (1.46) we have

$$\overline{\delta}_\pm \alpha = 0, \quad \delta_\pm \alpha = 0. \tag{1.49}$$

Then if follows from the generalized Kähler identities (1.45) that

$$\overline{\delta}_+ \alpha = 0, \qquad \overline{\delta}_+^* \alpha = -\delta_+ \alpha = 0. \tag{1.50}$$

Hence we have

$$\triangle_{\overline{\delta}_+} \alpha = (\overline{\delta}_+ \overline{\delta}_+^* + \overline{\delta}_+^* \overline{\delta}_+)\alpha = 0. \tag{1.51}$$

Then from (1.47), α is $\triangle_d$-harmonic and we have

$$H^1(K^\bullet) \cong \mathcal{H}^{0,-n+2}. \tag{1.52}$$

Hence we have the injection $p_K^1 : H^1(K^\bullet) \to H_{dR}(X)$.

In the case $H^2(K^\bullet)$, we use the Green operators $G_{\overline{\delta}_\pm}$, $G_{\delta_\pm}$ and the Hodge decomposition of each $U^{p,q}$ by the elliptic operator $\triangle_{\overline{\delta}_\pm}$. We assume that $\alpha \in K^2$ is d-exact, i.e., $\alpha = d\beta$. Then it follows from $dd^{\mathcal{J}}$-lemma [**11**] that we have an element of $\tilde{\beta} \in U_{\mathcal{J}_\psi}^{-n+2}$ such that

$$\alpha = d\tilde{\beta}. \tag{1.53}$$

(See the discussion [**8**].) Then $\tilde{\beta}$ is decomposed into the form

$$\tilde{\beta} = \tilde{\beta}^{(2)} + \tilde{\beta}^{(0)} + \tilde{\beta}^{(-2)}, \tag{1.54}$$

where $\tilde{\beta}^{(i)} \in U^{i,-n+2}$. We define $\gamma^{(\pm 1)}$ by

$$\gamma^{(1)} = \delta_+ G_{\overline{\delta}_+} \tilde{\beta}^{(2)}, \tag{1.55}$$

$$\gamma^{(-1)} = \overline{\delta}_- G_{\delta_-} \tilde{\beta}^{(-2)}. \tag{1.56}$$

Then from the generalized Kähler identities (1.45) we have

$$d\delta_- \gamma^{(1)} = d\tilde{\beta}^{(2)}, \tag{1.57}$$

$$-d\overline{\delta}_+ \gamma^{(-1)} = d\tilde{\beta}^{(-2)}. \tag{1.58}$$

We define $\beta^{(0)}$ by

$$\beta^{(0)} = \tilde{\beta}^{(0)} + \delta_- \gamma^{(1)} - \overline{\delta}_+ \gamma^{(-1)}. \tag{1.59}$$

Then it follows from (1.57) and (1.58) that

$$d\beta^{(0)} = d\tilde{\beta}^{(0)} + d(\delta_- \gamma^{(1)}) - d(\overline{\delta}_+ \gamma^{(-1)}) \tag{1.60}$$

$$= d\tilde{\beta}^{(0)} + d\tilde{\beta}^{(2)} + d\tilde{\beta}^{(-2)} \tag{1.61}$$

$$= d\tilde{\beta} = \alpha. \tag{1.62}$$

Hence every d-exact element $\alpha \in K^2$ is written as

$$\alpha = d\beta^{(0)}, \tag{1.63}$$

for $\beta^{(0)} \in U^{0,-n+2} = K^1$. It implies that the map $p_K^2 : H^2(K^\bullet) \to H_{dR}(X)$ is injective. q.e.d.

2. Deformations of generalized complex structures

Let $\mathcal{J}$ be a generalized complex structure on a manifold X with the maximally isotropic subspace $L(= L_{\mathcal{J}})$ in $(T \oplus T^*) \otimes \mathbb{C}$. In the deformation theory of generalized complex structures developed in [**10**], we will deform L in the Grassmannian which consists of maximally isotropic subspaces. Then a small deformation of isotropic subspace is given by

$$L_\varepsilon := (1 + \varepsilon)L = \{E + [E, \varepsilon] \mid E \in L\}, \tag{2.1}$$

for sufficiently small $\varepsilon \in \wedge^2\overline{L}$. Then we have the decomposition $(T \oplus T^*) \otimes \mathbb{C}$ into L_ε and its complex conjugate $\overline{L}_\varepsilon$ which defines an almost generalized complex structure $\mathcal{J}_\varepsilon$ for ε. The integrability of $\mathcal{J}_\varepsilon$ is equivalent to the one of almost Dirac structures in [**21**].

Theorem 2.1. ([**21**]) *The structure $\mathcal{J}_\varepsilon$ is integrable if and only if ε satisfies the generalized Maurer-Cartan equation*

$$d_L\varepsilon + \frac{1}{2}[\varepsilon, \varepsilon]_L = 0, \tag{2.2}$$

where $d_L : \wedge^k\overline{L} \to \wedge^{k+1}\overline{L}$ denotes the exterior derivative of the Lie algebroid and $[\, ,\,]_L$ is the Lie algebroid bracket of $\overline{L}$, i.e., the Schouten bracket.

Let ϕ be a locally defined nowhere vanishing section of $K_{\mathcal{J}}$. Then ϕ is a non-degenerate pure spinor which induces the structure $\mathcal{J}$. The exponential e^ε acts on ϕ and we have the deformed non-degenerate pure spinor $e^\varepsilon \cdot \phi$ which induces $\mathcal{J}_\varepsilon$. We already show that $\mathcal{J}_\varepsilon$ is integrable if and only if the differential form $e^\varepsilon\phi$ satisfies

$$de^\varepsilon\,\phi + E_\varepsilon \cdot e^\varepsilon\,\phi = 0, \tag{2.3}$$

for $E_\varepsilon \in \mathrm{CL}^1 \otimes \mathbb{C}$. We will give another proof of theorem 2.1 from the viewpoint of pure spinors. Our proof is suitable for our argument in this paper.

Proof of theorem 2.1. We recall the decomposition of differential forms,

$$\wedge^\bullet T^*X \otimes \mathbb{C} = \bigoplus_{k=-n}^{n} U^k. \tag{2.4}$$

Let $\pi_{U^{-n+3}}$ be the projection to the component U^{-n+3}. Since $\mathcal{J}_\varepsilon$ is integrable, we have

$$de^\varepsilon\phi = -E_\varepsilon \cdot e^\varepsilon\phi. \tag{2.5}$$

Let $\hat{E}_\varepsilon$ be $e^{-\varepsilon}\,E\,e^\varepsilon \in \mathrm{CL}^1 \otimes \mathbb{C}$. Then by the left action of $e^{-\varepsilon}$, we have

$$e^{-\varepsilon}\,de^\varepsilon\,\phi = -\hat{E}_\varepsilon \cdot \phi. \tag{2.6}$$

We see that $e^{-\varepsilon}de^\varepsilon$ is a Clifford-Lie operator of order 3 (cf. definition 2.2 in [**8**]). It follows from definition that $e^{-\varepsilon}de^\varepsilon$ is locally given by the Clifford algebra valued Lie derivative,

$$e^{-\varepsilon}de^\varepsilon = \sum_i E_i\mathcal{L}_{v_i} + N_i,$$

where $\mathcal{L}_{v_i}$ is the Lie derivative by a vector filed v_i and $E_i \in \mathrm{CL}^1 \otimes \mathbb{C}$, $N_i \in \mathrm{CL}^3 \otimes \mathbb{C}$. Thus $e^{-\varepsilon}de^\varepsilon\phi$ is an element of $U^{-n+1} \oplus U^{-n+3}$. It implies that $\mathcal{J}_\varepsilon$ is integrable if and only if we have $\pi_{U^{-n+3}}\,(e^{-\varepsilon}de^\varepsilon\phi) = 0$. The operator $e^{-\varepsilon}de^{\varepsilon}\phi$ is written in the form of power series (cf. lemma 2-7 in [**8**])

$$e^{-\varepsilon}de^{\varepsilon}\phi = d\phi + [d,\varepsilon]\phi + \frac{1}{2!}[[d,\varepsilon],\varepsilon]\phi + \cdots . \tag{2.7}$$

We define $N(\varepsilon,\varepsilon)$ by

$$N(\varepsilon,\varepsilon) := [[d,\varepsilon],\varepsilon]. \tag{2.8}$$

Lemma 2.2. *The operator $N(\varepsilon,\varepsilon)$ linearly acts on $\wedge^{\bullet}T^*X$, which is not a differential operator.*

Proof. We will show that $[[d,\varepsilon_1],\varepsilon_2]f\alpha = f[[d,\varepsilon_1],\varepsilon_2]\alpha$ for $\alpha \in \wedge^* T^*$ and a function f, where $\varepsilon_1,\varepsilon_2 \in \wedge^2\overline{L}$. It follows that

$$\begin{aligned}
&[[d,\varepsilon_1],\varepsilon_2]f\alpha - f[[d,\varepsilon_1],\varepsilon_2]\alpha\\
&\quad = (df)\varepsilon_1\varepsilon_2 - \varepsilon_1(df)\varepsilon_2 - \varepsilon_2(df)\varepsilon_1 + \varepsilon_2\varepsilon_1(df)\\
&\quad = (df)\varepsilon_1\varepsilon_2 - [\varepsilon_1, df]\varepsilon_2 - [\varepsilon_2, df]\varepsilon_1 + \varepsilon_2[\varepsilon_1, df]\\
&\qquad\qquad - (df)\varepsilon_1\varepsilon_2 - (df)\varepsilon_2\varepsilon_1 + [\varepsilon_2,(df)]\varepsilon_1\\
&\qquad\qquad + (df)\varepsilon_2\varepsilon_1\\
&\qquad\qquad = [\varepsilon_2,[\varepsilon_1,(df)]].
\end{aligned}$$

Since $\varepsilon_i \in \wedge^2\overline{L}$, we have $[\varepsilon_i,(df)] \in \overline{L}$. Hence

$$[\varepsilon_i,[\varepsilon_j,(df)]] = 0,$$

for $i,j = 1,2$. Thus the result follows. q.e.d.

The higher order terms of (2.7) are given by the adjoint action of ε on $N(\varepsilon,\varepsilon)$ successively. We define $\mathrm{ad}^l_\varepsilon N(\varepsilon,\varepsilon)$ by

$$\mathrm{ad}^l_\varepsilon N(\varepsilon,\varepsilon) := [\mathrm{ad}^{l-1}_\varepsilon N(\varepsilon,\varepsilon),\varepsilon].$$

Hence we have

$$e^{-\varepsilon}de^{\varepsilon} = d\phi + [d,\varepsilon]\phi + \frac{1}{2!}N(\varepsilon,\varepsilon)\phi \tag{2.9}$$

$$+\sum_{l=1}^{\infty}\frac{1}{(l+2)!}\,\mathrm{ad}^l_\varepsilon N(\varepsilon,\varepsilon). \tag{2.10}$$

Since d_L is the exterior derivative of the Lie algebroid $\overline{L}$, we have the complex

$$\cdots \xrightarrow{d_L} \wedge^p\overline{L} \xrightarrow{d_L} \wedge^{p+1}\overline{L} \xrightarrow{d_L} \cdots .$$

Then $d_L\varepsilon \in \wedge^3\overline{L}$ for $\varepsilon \in \wedge^2\overline{L}$ is given by:

Lemma 2.3.

$$\pi_{U^{-n+3}}[d,\varepsilon]\phi = (d_L\varepsilon)\phi.$$

Proof. Since we have $d\phi + E\phi = 0$ for $E \in \overline{L}$, it follows that

$$\pi_{U^{-n+3}}(d+E)\varepsilon\phi = (d_L\varepsilon)\phi. \tag{2.11}$$

Then we have

$$[d,\varepsilon]\phi = d\varepsilon\phi - \varepsilon d\phi \tag{2.12}$$
$$= d\varepsilon\phi + \varepsilon E\phi \tag{2.13}$$
$$= d\varepsilon\phi + E\varepsilon\phi \tag{2.14}$$
$$= (d+E)\varepsilon\phi. \tag{2.15}$$

Thus it follows that

$$\pi_{U^{-n+3}}[d,\varepsilon]\phi = (d_L\varepsilon)\phi.$$

q.e.d.

Lemma 2.4. *The Schouten bracket* $[\varepsilon,\varepsilon]_L$ *is given by*

$$[\varepsilon,\varepsilon]_L = N(\varepsilon,\varepsilon).$$

Proof. Let E_i be a section of $T \oplus T^*$ for $i = 1,2,3,4$. In terms of the derived bracket $[E_i, E_j]_d = [\{d, E_i\}, E_j]$ in (1.8), the bracket $[[d,\varepsilon_1],\varepsilon_2]$ is written as

$$[[d,\varepsilon_1],\varepsilon_2] = -[E_1,E_3]_d E_2E_4 + [E_1,E_4]_d E_2E_3 \tag{2.16}$$
$$+ [E_2,E_3]_d E_1E_4 - [E_2,E_4]_d E_1E_3 \tag{2.17}$$

for $\varepsilon_1 = E_1E_2$ and $\varepsilon_2 = E_3E_4$. Then the result follows. q.e.d.

Note that lemma 2.4 can be extended to higher order terms (see appendix).
We also have

Lemma 2.5.

$$\mathrm{ad}_\varepsilon^l N(\varepsilon,\varepsilon) = 0,$$

for all $l \geq 1$.

Proof. Since $N(\varepsilon,\varepsilon) \in \wedge^3\overline{L}$, it follows that

$$[N(\varepsilon,\varepsilon),\varepsilon] = 0. \tag{2.18}$$

Similarly we have $\mathrm{ad}_\varepsilon^l N(\varepsilon,\varepsilon) = 0$. q.e.d.

Then it follows from lemma 2.3 and 2.4 that we have

$$\pi_{U^{-n+3}}\, e^{-\varepsilon} d e^{\varepsilon}\phi = d_L\varepsilon\phi + \frac{1}{2!}[\varepsilon,\varepsilon]_L\phi \tag{2.19}$$
$$= \left(d_L\varepsilon + \frac{1}{2}[\varepsilon,\varepsilon]_L\right)\phi. \tag{2.20}$$

Thus the equation

$$\pi_{U^{-n+3}}\, e^{-\varepsilon} d e^{\varepsilon}\phi = 0, \tag{2.21}$$

is equivalent to the Maurer-Cartan equation

$$\left(d_L\varepsilon + \frac{1}{2}[\varepsilon,\varepsilon]_L\right) = 0. \tag{2.22}$$

Hence we have the result. q.e.d.

Let $\varepsilon(t)$ be an analytic family of sections of $\wedge^2\overline{L}$. Then $\varepsilon(t)$ is written in the form of the power series in t,

$$\varepsilon(t) = \varepsilon_1 t + \varepsilon_2\frac{t^2}{2!} + \varepsilon_3\frac{t^3}{3!} + \cdots, \tag{2.23}$$

where t is a sufficiently small complex parameter. Then $\varepsilon(t)$ gives deformations of almost generalized complex structures $\mathcal{J}_{\varepsilon(t)}$ by (2.1). The set of almost generalized complex structures forms an orbit of the adjoint action of $\mathrm{SO}(T\oplus T^*)$. The Lie algebra of $\mathrm{SO}(T\oplus T^*)$ is identified with $\wedge^2(T\oplus T^*)$, which is the subspace $\mathrm{CL}^{[2]}$ of CL^2. Thus $\mathcal{J}_{\varepsilon(t)}$ is written as $\mathcal{J}_{\varepsilon(t)} = \mathrm{Ad}_{e^{a(t)}}\mathcal{J}$ for $a(t)\in\wedge^2(T\oplus T^*)$. We denote by $(\wedge^2\overline{L}\oplus\wedge^2 L)^{\mathbb{R}}$ the real part of the bundle $(\wedge^2\overline{L}\oplus\wedge^2 L)$ which is a subbundle of CL^2. Then we have:

Proposition 2.6. *There exists a unique analytic family $a(t)$ of sections of $(\wedge^2\overline{L}\oplus\wedge^2 L)^{\mathbb{R}}$ such that*

$$\mathcal{J}_{\varepsilon(t)} = \mathrm{Ad}_{e^{a(t)}}\mathcal{J} \tag{2.24}$$

where we take sufficiently small t if necessary.

Proof. The action of $e^{\varepsilon(t)}$ on the canonical line bundle $K_{\mathcal{J}}$ defines a line bundle $e^{\varepsilon(t)}\cdot K_{\mathcal{J}}$. We also have a line bundle $e^{a(t)}\cdot K_{\mathcal{J}}$ by the action of $a(t)\in\mathrm{CL}^2$. The condition $e^{\varepsilon(t)}\cdot K_{\mathcal{J}} = e^{a(t)}\cdot K_{\mathcal{J}}$ is equivalent to the condition $\mathcal{J}_{\varepsilon(t)} = \mathrm{Ad}_{e^{a(t)}}\mathcal{J}$. Thus it suffices to construct a section $a(t)\in(\wedge^2\overline{L}\oplus\wedge^2 L)^{\mathbb{R}}$ which satisfies

$$(e^{-\varepsilon(t)}e^{a(t)})\phi\in K_J, \quad \text{for all } \phi\in K_{\mathcal{J}}. \tag{2.25}$$

Given two differential forms α,β, if $\alpha-\beta\in K_{\mathcal{J}}$, then we write it by

$$\alpha\equiv\beta \pmod{K_{\mathcal{J}}}.$$

Then the equation (2.25) is written as

$$(e^{-\varepsilon(t)}e^{a(t)})\phi\equiv 0 \pmod{K_{\mathcal{J}}} \quad \text{for all } \phi\in K_{\mathcal{J}}.$$

We write $a(t)$ in the form of the power series in t,

$$a(t) = a_1 t + a_2\frac{t^2}{2!} + \cdots, \tag{2.26}$$

where a_k is a section of $(\wedge^2\overline{L}\oplus\wedge^2 L)^{\mathbb{R}}$. We denote by $(e^{-\varepsilon(t)}e^{a(t)})_{[k]}\phi$ the kth term in t. Then the equation (2.25) is reduced to infinitely many equations,

$$(e^{-\varepsilon(t)}e^{a(t)})_{[k]}\phi\in K_{\mathcal{J}}, \qquad \text{for all } \phi\in K_{\mathcal{J}}. \tag{2.27}$$

We will show that there exists a solution $a(t)$ by induction on k. For $k=1$, we have

$$(e^{-\varepsilon(t)}e^{a(t)})_{[1]}\phi = -\varepsilon_1\phi + a_1\phi \in K_{\mathcal{J}}. \tag{2.28}$$

Thus if we set $a_1 = \varepsilon_1 + \overline{\varepsilon}_1$, then $(e^{-\varepsilon(t)}e^{a(t)})_{[1]}\phi = 0 \in K_{\mathcal{J}}$. We assume that there are sections $a_1, \cdots, a_{k-1} \in (\wedge^2\overline{L} \oplus \wedge^2 L)^{\mathbb{R}}$ such that

$$\left(e^{-\varepsilon(t)}e^{a(t)}\right)_{[i]}\phi \in K_{\mathcal{J}}, \tag{2.29}$$

for $\forall i < k$. If follows from the Campbel-Hausdorff formula that there exists $z(t) \in \mathrm{CL}^2 \otimes \mathbb{C}$ such that $e^{-\varepsilon(t)}e^{a(t)} = e^{z(t)}$, where

$$z(t) = -\varepsilon(t) + a(t) - [\varepsilon(t), a(t)] + \cdots . \tag{2.30}$$

Thus our assumption (2.29) is $\left(e^{z(t)}\right)_{[i]} \cdot \phi \in K_{\mathcal{J}}$ for all $i < k$. Since the degree of $z(t)$ is greater than or equal to 1, we have $z(t)_{[1]} \cdot \phi \in K_{\mathcal{J}}$ and it successively follows from our assumption that $z(t)_{[i]} \cdot \phi \in K_{\mathcal{J}}$, $(\forall i < k)$. Then we have

$$(e^{z(t)})_{[k]} \cdot \phi \equiv z(t)_{[k]}\phi \pmod{K_{\mathcal{J}}} \quad \text{for all } \phi \in K_{\mathcal{J}}. \tag{2.31}$$

Hence from (2.30), there is a $H_k \in \mathrm{CL}^2 \otimes \mathbb{C}$ such that

$$(e^{z(t)})_{[k]} \cdot \phi \equiv \frac{1}{k!}a_k\phi - H_k\phi \pmod{K_{\mathcal{J}}} \quad \text{for all } \phi \in K_{\mathcal{J}} \tag{2.32}$$

where H_k is written in terms of $a_1, \cdots, a_{k-1}$ and $\varepsilon_1 \cdots, \varepsilon_k$. Then there is a $\hat{H}_k \in \wedge^2\overline{L}$ such that $\hat{H}_k\phi - H_k\phi \in K_{\mathcal{J}}$. Thus a_k is defined as the real part of $(k!)\hat{H}_k$ and we have

$$\frac{1}{k!}a_k\phi - H_k\phi \in K_{\mathcal{J}}. \tag{2.33}$$

Hence it follows that

$$(e^{z(t)})_{[k]} \cdot \phi = \left(e^{-\varepsilon(t)}e^{a(t)}\right)_{[k]}\phi \in K_{\mathcal{J}}. \tag{2.34}$$

Then we have a solution $a(t)$ as the formal power series. It follows that the $a(t)$ is a convergent series which is a smooth section. Thus $a(t)$ is a unique section of $(\wedge^2\overline{L} \oplus \wedge^2 L)^{\mathbb{R}}$ with $\mathcal{J}_{\varepsilon(t)} = \mathrm{Ad}_{e^{a(t)}}\mathcal{J}$ which depends analytically on t. q.e.d.

3. Stability theorem of generalized Kähler structures

We use the same notation as in sections 1 and 2.

Theorem 3.1. *Let $(\mathcal{J}, \psi)$ be a generalized Kähler structure with one pure spinor on a compact manifold X. We assume that there exists an analytic family of generalized complex structures $\{\mathcal{J}_t\}_{t\in\triangle}$ on X with $\mathcal{J}_0 = \mathcal{J}$ parametrized by the complex one-dimensional open disk $\triangle$ containing the origin 0. Then there exists an analytic family of generalized Kähler structures with one pure spinor $\{(\mathcal{J}_t, \psi_t)\}_{t\in\triangle'}$ with $\psi_0 = \psi$*

parametrized by a sufficiently small open disk $\triangle' \subset \triangle$ containing the origin.

Theorem 3.1 implies that generalized Kähler structures with one pure spinor are stable under deformations of generalized complex structures. Theorem 3.1 is a generalization of the so-called stability theorem of Kähler structures due to Kodaira-Spencer. We also obtain:

Theorem 3.2. *Let $\{\mathcal{J}_t\}_{t\in\triangle}$ and ψ be as in theorem 3.1. Then there is an open set W in $H^1(K^\bullet)$ containing the origin such that there exists a family of generalized Kähler structures with one pure spinor $\{(\mathcal{J}_t,\, \psi_{t,s})\}$ with $\psi_{0,0} = \psi$ parametrized by $t \in \triangle'$ and $s \in W$ in $H^1(K^\bullet)$. Further if we denote by $[\psi_{t,s}]$ the de Rham cohomology class represented by $\psi_{t,s}$, then $[\psi_{t,s_1}] \neq [\psi_{t,s_2}]$ for $s_1 \neq s_2$.*

This section is devoted to proving theorem 3.1 and theorem 3.2. Let $K_{\mathcal{J}_0}$ be the canonical line bundle with respect to $\mathcal{J}_0$. We take a trivialization $\{U_\alpha, \phi_\alpha\}$ of $K_{\mathcal{J}_0}$, where $\{U_\alpha\}$ is a covering of X and ϕ_α is a non-vanishing section of $K_{\mathcal{J}_0}|_{U_\alpha}$ which induces the generalized complex structure $\mathcal{J}_0$. Since $\mathcal{J}_0$ is integrable, we have $d\phi_\alpha + E_\alpha\phi_\alpha = 0$ for $E_\alpha \in \mathrm{CL}^1 \otimes \mathbb{C}|_{U_\alpha}$. It follows from section 2 that deformations $\{\mathcal{J}_t\}$ is given by an analytic family of global sections $a(t) \in \mathrm{CL}^2$ which is constructed from an analytic family of global sections $\varepsilon(t) \in \wedge^2\overline{L}$. Each section $a(t)$ gives the non-degenerate pure spinor $e^{a(t)}\phi_\alpha$ which induces the structure $\mathcal{J}_t$. Since $\mathcal{J}_t$ is integrable, we have

$$de^{a(t)}\phi_\alpha + E_\alpha(t)e^{a(t)}\phi_\alpha = 0. \tag{3.1}$$

It follows from the left action of $e^{-a(t)}$ that

$$e^{-a(t)}\, d\, e^{a(t)}\phi_\alpha + e^{-a(t)}E_\alpha(t)e^{a(t)}\phi_\alpha = 0. \tag{3.2}$$

We define $\tilde{E}_\alpha(t)$ by

$$\widetilde{E}_\alpha(t) = e^{-a(t)}E_\alpha(t)e^{a(t)} \in (T \oplus T^*)|_{U_\alpha} = (\mathrm{CL}^1)|_{U_\alpha}. \tag{3.3}$$

Then we have

$$e^{-a(t)}\, d\, e^{a(t)}\phi_\alpha + \widetilde{E}_\alpha(t)\phi_\alpha = 0. \tag{3.4}$$

Hence it follows that

$$(e^{-a(t)}\, d\, e^{a(t)})\phi_\alpha \in \mathbf{E}^0_{\mathcal{J}_0}|_{U_\alpha} = \{\, E \cdot \phi_\alpha \,|\, E \in \mathrm{CL}^1|_{U_\alpha}\,\}. \tag{3.5}$$

Since $e^{-a(t)}\, d\, e^{a(t)}$ is a Clifford-Lie operator of order 3 (cf. definition 2.2 in [**8**]), it follows that $e^{-a(t)}\, d\, e^{a(t)}$ is locally written in terms of the Lie derivative and the Clifford algebra,

$$e^{-a(t)}\, d\, e^{a(t)} = \sum_i E_i\mathcal{L}_{v_i} + N_i, \tag{3.6}$$

where $E_i \in \mathrm{CL}^1$, $v_i \in T$ and $N \in \mathrm{CL}^3$. Then we have:

Lemma 3.3. *There is a section* $a_i \in \mathrm{CL}^2$ *such that*

$$\mathcal{L}_{v_i}\phi_\alpha \equiv a_i \cdot \phi_a \bmod (K_{\mathcal{J}_0}), \tag{3.7}$$

$$\mathcal{L}_{v_i}\psi = a_i \cdot \psi, \tag{3.8}$$

for each vector field v_i*, where the equation* (3.7) *implies that*

$$\mathcal{L}_{v_i}\phi_\alpha - a_i \cdot \phi_a = \rho_\alpha \phi_\alpha$$

for a function ρ_a*.*

Proof. The set of almost generalized Kähler structures with one pure spinor forms an orbit under the diagonal action of the Clifford group whose Lie algebra is given by CL^2. Thus small deformations of the structures are given by the exponential action of CL^2. Let f_t be the one parameter subgroup of diffeomorphisms defined by the vector field v, i.e.,

$$\frac{d}{dt} f_t|_{t=0} = v.$$

Since the set of almost generalized Kähler structures with one pure spinor is invariant under the action of diffeomorphisms, there is a section $a(t) \in \mathrm{CL}^2$ with $a(0) = 0$ such that

$$(f_t^* \mathcal{J}_0,\ f_t^*\psi) = (\mathrm{Ad}_{e^{a(t)}}\mathcal{J}_0,\ e^{a(t)} \cdot \psi).$$

By differentiating with respect to t, we have

$$(\mathcal{L}_v \mathcal{J}_0, \mathcal{L}_v\psi) = ([a, \mathcal{J}_0],\ a \cdot \psi),$$

where $a = \frac{d}{dt}a(t)|_{t=0}$. Since $f_t^*\phi_a$ and $e^{a(t)}\phi_\alpha$ induce the same generalized complex structure $\mathrm{Ad}_{e^{a(t)}}\mathcal{J}_0$, we have

$$f_t^*\phi_\alpha = e^{\rho(t)} e^{a(t)} \phi_\alpha,$$

for a function $\rho(t)$ with $\rho(0) = 0$. Then we have

$$\mathcal{L}_v\phi_\alpha \equiv a \cdot \phi_a \bmod (K_{\mathcal{J}_0}), \tag{3.9}$$

$$\mathcal{L}_v\psi = a \cdot \psi. \tag{3.10}$$

q.e.d.

Hence it follows from (3.6) that there exists a section $h_\alpha \in \mathrm{CL}^3|_{U_\alpha}$ such that

$$(e^{-a(t)}\, d\, e^{a(t)})\phi_\alpha \equiv h_\alpha \cdot \phi_\alpha \bmod (\mathrm{CL}^1 \cdot K_{\mathcal{J}_0}), \tag{3.11}$$

$$(e^{-a(t)}\, d\, e^{a(t)})\psi = h_\alpha \cdot \psi. \tag{3.12}$$

Let $K^\bullet$ be the graded left module generated by $U^{0,-n+2}$ over the Clifford algebra CL, as in section 1.3. The exterior derivative d gives rise to the differential complex:

$$0 \to K^1 \to K^2 \to \cdots . \tag{3.13}$$

Then we see that K^2 is given by

$$K^2 = U^{1,-n+1} \oplus U^{-1,-n+1} \oplus U^{1,-n+3} \oplus U^{-1,-n+3}. \tag{3.14}$$

We define a vector bundle $\ker^i$ by

$$\ker^i = \{\, a \in \mathrm{CL}^{i+1} \,|\, a \cdot \phi_\alpha = 0 \,\}, \tag{3.15}$$

for $i = 1, 2$. In section 1, we define a bundle $\widetilde{\ker}^i$ by

$$\widetilde{\ker}^i = \{\, a \in \mathrm{CL}^{i+1} \,|\, a \cdot \phi_\alpha \in \mathrm{CL}^{i-1} \cdot K_{\mathcal{J}_0} \,\}. \tag{3.16}$$

The $\widetilde{\ker}^i$ gives the bundle

$$\tilde{K}^i = \{\, a \cdot \psi \,|\, a \in \widetilde{\ker}^i \,\}. \tag{3.17}$$

In section 1.3 we also have

$$\tilde{K}^1 = U^{0,-n} \oplus U^{0,-n+2}, \tag{3.18}$$

$$\tilde{K}^2 = K^2. \tag{3.19}$$

Hence K^1 is the subbundle of $\tilde{K}^1$,

$$K^1 \subset \tilde{K}^1. \tag{3.20}$$

Proposition 3.4.

$$e^{-a(t)}\, d\, e^{a(t)} \psi \in K^2.$$

Proof. It follows from (3.11) that there exists $h_\alpha \in \mathrm{CL}^3|_{U_\alpha}$ for each α such that

$$e^{-a(t)}\, d\, e^{a(t)} \phi_\alpha \equiv h_\alpha \cdot \phi_\alpha \mod (\mathrm{CL}^1 \cdot K_{\mathcal{J}_0}), \tag{3.21}$$

$$e^{-a(t)}\, d\, e^{a(t)} \psi = h_\alpha \cdot \psi, \tag{3.22}$$

where (3.21) implies that there is a section $F_\alpha \in T \oplus T^*$ such that $e^{-a(t)}\, d\, e^{a(t)} \phi_\alpha - h_\alpha \cdot \phi_\alpha = F_\alpha \cdot \phi_\alpha$. Since $\mathcal{J}_t$ is integrable, from (3.4) we have

$$e^{-a(t)}\, d\, e^{a(t)} \phi_\alpha = -\widetilde{E}_\alpha(t) \cdot \phi_\alpha \in \mathrm{CL}^1 \cdot K_{\mathcal{J}_0}|_{U_\alpha}. \tag{3.23}$$

Hence it follows that $h_\alpha \in \widetilde{\ker}^2$ and we have

$$e^{-a(t)}\, d\, e^{a(t)} \psi = h_\alpha \cdot \psi \in \widetilde{K}^2 = K^2. \tag{3.24}$$

q.e.d.

Proof of theorem 3.1 and 3.2. We will construct a smooth family $b(t)$ of sections of $\ker^1$ such that

$$d(\, e^{a(t)}\, e^{b(t)}\, \psi) = 0. \tag{3.25}$$

Then it follows from the Campbel-Haudorff formula that there exists $z(t) \in \mathrm{CL}^2$ such that

$$e^{z(t)} = e^{a(t)} e^{b(t)}. \tag{3.26}$$

Explicitly, the first five components of $z(t)$ are given by

$$z(t) = a(t) + b(t) + \frac{1}{2}[a(t), b(t)] \tag{3.27}$$

$$+ \frac{1}{12}[x, [x, y]] + \frac{1}{12}[y, [y, x]] + \cdots , \tag{3.28}$$

(cf. [**23**].) Since $b(t) \in \ker^1$, we have

$$e^{z(t)}\phi_\alpha = e^{a(t)}e^{b(t)}\phi_\alpha \tag{3.29}$$

$$= e^{a(t)}\phi_\alpha. \tag{3.30}$$

It implies that $e^{z(t)}\phi_\alpha$ induces the same deformations $\mathcal{J}_t$ as before and the pair $(\mathcal{J}_t,\, e^{z(t)}\psi)$ gives deformations of generalized Kähler structure with one pure spinor. Consequently the equation we must solve is that

$$d\, e^{a(t)}\, e^{b(t)}\, \psi = 0, \quad b(t) \in \ker^1 . \tag{eq}$$

The section $a(t)$ is written as the power series

$$a(t) = a_1 t + a_2\frac{t^2}{2!} + a_3\frac{t^3}{3!} + \cdots , \tag{3.31}$$

where $a_i \in \mathrm{CL}^2$. We shall construct a solution $b(t)$ as the formal power series

$$b(t) = b_1 t + b_2\frac{t^2}{2!} + b_3\frac{t^3}{3!} + \cdots , \tag{3.32}$$

where $b_i \in \ker^1$. The ith homogeneous part of the equation (eq) in t is denoted by

$$\Big(d\, e^{a(t)}\, e^{b(t)}\, \psi\Big)_{[i]} = 0, \quad b(t) \in \ker^1 . \tag{$\mathrm{eq}_{[i]}$}$$

Thus in order to obtain a solution $b(t)$, it suffices to determine $b_1, \cdots , b_i$ satisfying $(\mathrm{eq})_{[i]}$ by induction on i. In the case $i = 1$, we have

$$\big(\, d\, e^{a(t)}\, e^{b(t)}\, \big)_{[1]}\psi = da_1\psi + db_1\psi \tag{3.33}$$

$$= [d, a_1]\psi + db_1\psi = 0. \tag{3.34}$$

From proposition 3.4 we have $\big(e^{-a(t)}\, d\, e^{a(t)}\psi\big)_{[1]} = [d, a_1]\psi \in K^2$. Since $da_1\psi = [d, a_1]\psi \in K^2$ is a d-exact differential form, $da_1\psi$ defines a class of cohomology $[\widetilde{\mathrm{Ob}}_1]$ in $H^2(K^*)$ whose image vanishes in the de Rham cohomology group $H_{dR}(X)$. Since the map $p^2_K : H^2(K^\bullet) \to H_{dR}(X)$ is injective, it follows that $[\widetilde{\mathrm{Ob}}_1] = 0$. Thus we have a solution $b_1 \in \ker^1$ which is given by

$$b_1\psi = -d^*G_K(da_1\psi) \in K^1, \tag{3.35}$$

where d^* is the adjoint operator and G_K is the Green operator of the complex (K^*, d) with respect to a metric. Further for each representative s of the first cohomology group $H^1(K^\bullet)$, we have a solution $b_{1,s}$ which is defined by

$$b_{1,s}\psi = -d^* G_K(da_1\psi) + s. \tag{3.36}$$

Assume that we already have $b_1, \cdots, b_{k-1} \in \ker^1$ such that

$$\left(de^{a(t)}e^{b(t)}\psi\right)_{[i]} = 0, \tag{3.37}$$

for all $i < k$. From the Campbel-Hausdorff formula we have

$$e^{z(t)} = e^{a(t)}e^{b(t)}. \tag{3.38}$$

Hence it follows from our assumption (3.37) that

$$\begin{aligned}\left(e^{-z(t)}\, d\, e^{z(t)}\right)_{[k]}\psi &= \sum_{\substack{i+j=k\\ i,j\geq 0}} \left(e^{-z(t)}\right)_{[i]} \left(de^{z(t)}\right)_{[j]}\psi \\ &= \left(de^{z(t)}\right)_{[k]}\psi.\end{aligned} \tag{3.39}$$

Since $(e^{-z(t)}de^{z(t)})$ is given by

$$(e^{-z(t)}de^{z(t)}) = d + [d, z(t)] + \frac{1}{2!}\left[[d, z(t)], z(t)\right] + \cdots, \tag{3.40}$$

the left-hand side of (3.39) is written as

$$(e^{-z(t)}de^{z(t)})_{[k]}\psi = \frac{1}{k!}db_k\psi + \frac{1}{k!}da_k\psi + \mathrm{Ob}_k,$$

where Ob_k is the higher order term which is determined by $a_1, \cdots, a_{k-1}$, and $b_1, \cdots b_{k-1}$. We define $\widetilde{\mathrm{Ob}_k}$ by

$$\widetilde{\mathrm{Ob}_k} = \frac{1}{k!}da_k\,\psi + \mathrm{Ob}_k. \tag{3.41}$$

Thene $(\mathrm{eq})_{[k]}$ is reduced to

$$\frac{1}{k!}db_k\psi + \widetilde{\mathrm{Ob}_k} = 0, \qquad (b_k \in \ker^1).$$

From (3.4), we have

$$\begin{aligned}e^{-z(t)}\, d\, e^{z(t)}\phi_\alpha &= e^{-b(t)}\, e^{-a(t)}\, d\, e^{a(t)}\, e^{b(t)}\phi_\alpha \\ &= -\left(e^{-b(t)}\tilde{E}_\alpha(t)e^{b(t)}\right)\phi_\alpha \in \mathrm{CL}^1 \cdot K_{\mathcal{J}_0}.\end{aligned}$$

Thus it follows from the same argument as in proposition 3.3 that we have

$$(e^{-z(t)}de^{z(t)})\psi \in K^2. \tag{3.42}$$

It follows from (3.39) that $\widetilde{\mathrm{Ob}_k} \in K^2$ is d-exact. It implies that $\widetilde{\mathrm{Ob}_k}$ gives rise to the class of the cohomology $[\widetilde{\mathrm{Ob}_k}] \in H^2(K^*)$ with $p_K^2(([\widetilde{\mathrm{Ob}_k}]) = 0$. Since p_K^2 is injective from lemma 1.8, we have $[\widetilde{\mathrm{Ob}_k}] = 0$. Thus $b_k \in \ker^1$ is given by

$$\frac{1}{k!} b_k \psi = -d^* G_K(\widetilde{\mathrm{Ob}_k}) \in K^1, \tag{3.43}$$

where d^* is the adjoint operator and G_K is the Green operator of the complex $(K^\bullet, d)$. Hence it follows from the induction that we have the solution $b(t)$ of the equation (eq) as the formal power series. As we see from (3.36), we obtain the family of sections $b_{1,s}$ parametrized by $s \in H^1(K^\bullet)$ which gives rise to a family $b(t,s)$ of solutions. A family of non-degenerate pure spinor $\{\psi_{t,s}\}$ are constructed as $e^{b(t,s)} \cdot \psi_0$. Since the map $p_K^1 : H^1(K^\bullet) \to H_{dR}(X)$ is injective, we have $[\psi_{t,s_1}] \neq [\psi_{t,s_2}] \in H_{dR}(X)$ for $s_1 \neq s_2$. In section 4 we show that the formal power series $b(t)$ converges. q.e.d.

4. The convergence

This section is devoted to showing that both power series $b(t)$ and $z(t)$ in section 3 are convergent series. We will use a similar method to that in [**16**] which applies the elliptic estimate of the Green operator. However we must develop an estimate of the obstruction Ob in section 3 which includes the higher-order term. We will use the induction on the degree k. At first we will estimate the first terms b_1 and z_1 of power series $b(t)$ and $z(t)$. We assume that $b(t)$ and $z(t)$ satisfy the inequalities (4.16) and (4.17), respectively. Then we will show that $b(t)$ satisfies the inequality (4.6) and then obtain the inequality (4.7).

We shall fix our notation. We denote by $\|f\|_s = \|f\|_{C^{s,\alpha}}$ the Hölder norm of a section f of a bundle with respect to a metric. Then we have an inequality,

$$\|fg\|_s \leq C_s \|f\|_s \, \|g\|_s,$$

where f, g are sections and C_s is a constant. We have the elliptic complex $(K^\bullet, d)$ in section 1 and we use the Schauder estimates of the elliptic operators with respect to the complex $(K^\bullet, d)$ with a constant C_K. Let $P(t)$ be a formal power series in t. We denote by $(P(t))_{[k]}$ the kth coefficient of $P(t)$. Given two power series $P(t)$ and $Q(t)$, if $(P(t))_{[k]} < (Q(t))_{[k]}$ for all k, we denote it by

$$P(t) \ll Q(t).$$

For a positive integer k, if $(P(t))_{[i]} < (Q(t))_{[i]}$ for all $i \leq k$, then we write it by

$$P(t) \underset{k}{\ll} Q(t).$$

We also consider a formal power series $f(t)$ in t whose coefficients are sections of a bundle. Then we put $\|f(t)\|_s = \sum_i \| (f(t))_{[i]} \|_s t^i$. We define a convergent power series $M(t)$ by

$$M(t) = \sum_{\nu=1}^{\infty} \frac{1}{16c} \frac{(ct)^\nu}{\nu^2} = \sum_{\nu=1}^{\infty} M_\nu t^\nu.$$

In [**16**], it turns out that the series $M(t)$ satisfies:

Lemma 4.1.

$$M(t)^2 \ll \frac{1}{c} M(t).$$

We put $\lambda = \frac{1}{c}$. Then it follows from lemma 4.1 that

$$\frac{1}{l!} M(t)^l \ll \frac{1}{l!} \lambda^{l-1} M(t) = \frac{\lambda^l}{l!} \frac{1}{\lambda} M(t).$$

Hence we have:

Lemma 4.2.

$$e^{M(t)} \ll \frac{1}{\lambda} e^\lambda M(t).$$

As in section 3, the power series $z(t)$ is defined by the Campbel-Hausdorff formula,

$$e^{z(t)} = e^{a(t)} e^{b(t)},$$

where

$$z(t) = \sum_{l=0}^{\infty} \frac{t^l}{l!} z_k, \tag{4.1}$$

$$\begin{aligned} e^{z(t)} &= \sum_{j=0}^{\infty} \frac{1}{j!} z(t)^j \\ &= 1 + z(t) + \frac{1}{2!} z(t)^2 + \cdots . \end{aligned} \tag{4.2}$$

The power series $a(t)$ is the convergent series which induces deformations of generalized complex structures $\{\mathcal{J}_t\}$ defined in proposition 2.6. The norm of $a(t)$ is written as

$$\|a(t)\|_s = \sum_{l=1}^{\infty} \frac{1}{l!} \|a_k\|_s t^l.$$

Then we can assume that $\|a(t)\|_s$ satisfies

$$\|a(t)\|_s \ll K_1 M(t), \tag{4.3}$$

for a non-zero constant K_1 and λ if we take $a(t)$ sufficiently small. We will show that there exist constants K_1, K_2, and λ such that we have the following inequalities,

$$\|b(t)\|_s \ll K_2 M(t), \tag{4.4}$$

$$\|z(t)\|_s \ll M(t) \tag{4.5}$$

for sufficiently small $a(t)$. Note that K_1, K_2, and λ are determined by $a(t)$, $\mathcal{J}$, and ψ which do not depend on $b(t)$ and $z(t)$. The inequalities (4.4) and (4.5) are reduced to the infinitely many inequalities on degree k

$$\|b(t)\|_s \underset{k}{\ll} K_2 M(t), \tag{4.6}$$

$$\|z(t)\|_s \underset{k}{\ll} M(t) \tag{4.7}$$

We will show both inequalities (4.6) and (4.7) by the induction on k. In this section we denote by C_i constants which do not depend on $z(t)$, $b(t)$, and k but depend on $a(t)$, $\mathcal{J}$, and ψ. For $k = 1$, as in section 3, $b_1\psi$ satisfies the equation

$$db_1\psi + da_1\psi = 0, \qquad (b_1\psi \in K^1)$$

Then $b_1\psi$ is given by

$$b_1\psi = -d^* G_K(da_1\psi), \tag{4.8}$$

where d^* is the adjoint operator and G_K is the Green operator of the complex $(K^\bullet, d)$. It follows from the Schauder estimate of the elliptic operators that

$$\begin{aligned} \|b_1\psi\|_s &\le C_K \|a_1\psi\|_s \le C_K C_s \|a_1\|_s \|\psi\|_s \\ &\le \frac{1}{16} C_1 K_1, \end{aligned} \tag{4.9}$$

where $\|a_1\|_s \le K_1 M_1 = \frac{K_1}{16}$ and $C_1 = C_K C_s \|\psi\|_s$.

We can define b_1 as a section of the real part of $\overline{L}_+ L_-$. Then we have

$$\|b_1\|_s \le C_2 \|b_1\psi\|_s. \tag{4.10}$$

Substituting (4.9) into (4.10), we have

$$\|b_1\|_s \le \frac{1}{16} C_1 C_2 K_1 = M_1 C_1 C_2 K_1. \tag{4.11}$$

Thus if we take K_2 with $C_1 C_2 K_1 < K_2$, then we have

$$\|b_1\|_s \le K_2 M_1. \tag{4.12}$$

Since $z_1 = a_1 + b_1$, if we take K_1 and K_2 satisfying $K_1 + K_2 < 1$, we have

$$\|z_1\|_s \le \|a_1\|_s + \|b_1\|_s \tag{4.13}$$

$$\le M_1 K_1 + M_1 K_2 \tag{4.14}$$

$$= (K_1 + K_2) M_1 < M_1. \tag{4.15}$$

It follows from (4.12), (4.15) that we have inequalities (4.6) and (4.7) for $k = 1$. We assume that the following inequalities hold:

$$\|b(t)\| \underset{k-1}{\ll} K_2 M(t), \tag{4.16}$$

$$\|z(t)\| \underset{k-1}{\ll} M(t). \tag{4.17}$$

Let Ob_k be the higher order term in section 3. Then we have:

Lemma 4.3. $\mathrm{Ob}_k = \mathrm{Ob}_k(a_1, \cdots, a_{k-1}, b_1 \cdots, b_{k-1})$ *satisfies the inequality*

$$\|\mathrm{Ob}_k\|_{s-1} \le C(\lambda) M_k,$$

where $C(\lambda)$ *depends on* λ *and we have*

$$\lim_{\lambda \to 0} C(\lambda) = 0.$$

Proof. Since Ob_k is determined by the terms of order greater than or equal to 2,

$$\mathrm{Ob}_k = \sum_{l=2}^{k} \frac{1}{l!} (\mathrm{ad}^l_{z(t)}\, d)_{[k]} \psi.$$

We have

$$\| \,[d, z(t)]\psi\, \|_{s-1} \ll 2\|z(t)\psi\|_s.$$

Since $(\mathrm{ad}^l_{z(t)}\, d) = [\, \mathrm{ad}^{l-1}_{z(t)}\, d,\, z(t)\,]$, we find

$$\| \left(\mathrm{ad}^l_{z(t)} d\right)_{[k]} \psi\|_{s-1} \le 2(2C_s)^l (\|z(t)\|_s^l \, \|\psi\|_s)_{[k]}. \tag{4.18}$$

Hence it follows that

$$\|\mathrm{Ob}_k\|_{s-1} = \sum_{l=2}^{k} \frac{1}{l!} \| \left(\mathrm{ad}^l_{z(t)}\, d\right)_{[k]} \psi\|_{s-1} \tag{4.19}$$

$$\le \sum_{l=2}^{k} \frac{1}{l!} 2(2C_s)^l \left(\|z(t)\|_s^l \, \|\psi\|_s\right)_{[k]}. \tag{4.20}$$

Since the degree of $z(t)$ is greater than or equal to 1, it follows from our assumption (4.17) and $l \ge 2$ that we have

$$\left(\|z(t)\|_s^l\right)_{[k]} \le (M(t))^l_{[k]}. \tag{4.21}$$

(Note that $\left(\|z(t)\|_s^l\right)_{[k]}$ consists of the term $\|z_i\|_s$, for $i < k$.) Substituting (4.21) into (4.20) and using lemma 4.2, we obtain

$$\|\mathrm{Ob}_k\|_{s-1} \leq \sum_{l=2}^{k} \frac{1}{l!} 2(2C_s)^l \left(M(t)^l\right)_{[k]} \|\psi\|_s \tag{4.22}$$

$$\leq C_3 \sum_{l=2}^{k} \frac{1}{l!} (2C_s)^l \lambda^{l-1} M_k \tag{4.23}$$

$$\leq C_3 \lambda^{-1} (e^{2C_s\lambda} - 1 - 2C_s\lambda) M_k \tag{4.24}$$

$$= C(\lambda) M_k,$$

where $C_3 = 2\|\psi\|_s$. Then it follows that the constant $C(\lambda)$ satisfies

$$\lim_{t\to 0} C(\lambda) = 0.$$

q.e.d.

Lemma 4.4.

$$\|b(t)\|_s \underset{k}{\ll} K_2 M(t).$$

Proof. In section 3, b_k is defined as the solution of the equation

$$\frac{1}{k!} db_k\psi + \frac{1}{k!} da_k\psi + \mathrm{Ob}_k = 0 \tag{4.25}$$

In fact, $b_k\psi$ is given by

$$\frac{1}{k!} b_k\psi = -G_K d^*(\mathrm{Ob}_k) - G_K d^*(\frac{1}{k!} a_k\psi). \tag{4.26}$$

Thus it follows from (4.10) and the Schauder estimate that

$$\|\frac{1}{k!} b_k\|_s \leq C_2 C_K \|\mathrm{Ob}_k\|_{s-1} + C_2 C_K \|\frac{1}{k!} a_k\psi\|_s. \tag{4.27}$$

Applying lemma 4.3 and (4.3) to (4.27), we have

$$\|\frac{1}{k!} b_k\|_s \leq C_2 C_K C(\lambda) M_k + C_s C_2 C_K K_1 M_k \|\psi\|_s$$

$$\leq (C_4 C(\lambda) + C_5 K_1) M_k \tag{4.28}$$

where $C_4 = C_2 C_K$ and $C_5 = C_s C_2 \|\psi\|_s$. Then from (4.11) and (4.28) if we take K_2 as

$$K_2 := \max\{\, C_2 C_1 K_1, (C_4 C(\lambda) + C_5 K_1)\}, \tag{4.29}$$

then we have the inequality,

$$\|b(t)\|_s \underset{k}{\ll} K_2 M(t). \tag{4.30}$$

q.e.d.

Finally we estimate z_k. It follows that

$$(z(t))_{[k]} = \frac{1}{k!} z_k = \left(e^{z(t)} - 1 - \sum_{p=2}^{k} \frac{1}{p!} z(t)^p \right)_{[k]}.$$

Hence we have

$$\| \frac{1}{k!} z_k \|_s \leq \| (e^{z(t)} - 1)_{[k]} \|_s + \sum_{p=2}^{k} \frac{1}{p!} \| \left(z(t)^p \right)_{[k]} \|_s. \tag{4.31}$$

From our assumption and (4.30),

$$\|a(t)\|_s \ll K_1 M(t), \qquad \|b(t)\|_s \underset{k}{\ll} K_2 M(t).$$

Then it follows from lemma 4.1 and lemma 4.2 that

$$\|e^{a(t)} - 1\|_s \ll \frac{1}{\lambda}(e^{K_1 \lambda} - 1) M(t). \tag{4.32}$$

We also have

$$\|e^{b(t)} - 1\|_s \underset{k}{\ll} \frac{1}{\lambda}(e^{K_2 \lambda} - 1) M(t). \tag{4.33}$$

Then we obtain:

Lemma 4.5.

$$\|z(t)\|_s \underset{k}{\ll} M(t).$$

Proof. It follows from lemma 4.2 and lemma 4.3 that

$$\|e^{a(t)}\|_s \ll \frac{1}{\lambda} e^{K_1 \lambda} M(t).$$

Then from (4.32) and (4.33), we have

(4.34)

$$\|(e^{z(t)} - 1)\|_s \underset{k}{\ll} \|e^{a(t)}(e^{b(t)} - 1)\|_s + \|e^{a(t)} - 1\|_s$$

$$\underset{k}{\ll} \frac{1}{\lambda} e^{K_1 \lambda} M(t) \frac{1}{\lambda}(e^{K_2 \lambda} - 1) M(t) + \frac{1}{\lambda}(e^{K_1 \lambda} - 1) M(t) \tag{4.35}$$

(4.36)

Applying lemma 4.1 again, we have

$$\| \left(e^{z(t)} - 1 \right) \|_s \underset{k}{\ll} \left(e^{K_1 \lambda} \frac{1}{\lambda}(e^{K_2 \lambda} - 1) + \frac{1}{\lambda}(e^{K_1 \lambda} - 1) \right) M(t) \tag{4.37}$$

$$\underset{k}{\ll} C(K_1, K_2) M(t), \tag{4.38}$$

where $C(K_1, K_2)$ is a constant which depends only on K_1 and K_2. Since $(z(t))^p_{[k]}$ consists of terms z_i for $i < k$, it follows from our assumption of the induction that the second term of (4.31) satisfies

$$\sum_{p=2}^{k} \frac{1}{p!} \| (z(t)^p)_{[k]} \|_s \leq \sum_{p=2}^{k} \frac{1}{p!} ((C_s M(t))^p)_{[k]} \tag{4.39}$$

$$\leq \frac{1}{\lambda}(e^{C_s\lambda} - 1 - C_s\lambda)M_k \tag{4.40}$$

$$= C_1(\lambda)M_k, \tag{4.41}$$

where $\lim_{\lambda\to 0} C_1(\lambda) = 0$. Thus if we take K_1, K_2, λ which satisfy

$$C(K_1, K_2) + C_1(\lambda) \leq 1, \tag{4.42}$$

it follows from (4.31) that

$$\frac{1}{k!}\|z_k\|_s \leq (C(K_1, K_2) + C_1(\lambda))M_k \leq M_k. \tag{4.43}$$

Thus $\|z(t)\|_s \underset{k}{\ll} M(t)$. q.e.d.

If we take $a(t)$ sufficiently small, we can take K_1, K_2, and λ with $K_1 + K_2 < 1$ which satisfy (4.29) and (4.42). Hence by the induction, it turns out that $b(t)$ and $z(t)$ in section 3 are convergent series.

5. Applications

5.1. Generalized Kähler structures on Kähler manifolds. Let X be a compact Kähler manifold with the complex structure J and the Kähler form ω. Then we have the generalized Kähler structure $(\mathcal{J}, e^{\sqrt{-1}\omega})$ with one pure spinor on X. The deformations complex of generalized complex structures is given by the complex $(\wedge^\bullet \overline{L}, d_L)$. The complex $(\wedge^\bullet \overline{L}, d_L)$ is isomorphic to the complex $(U^{-n+\bullet} \otimes K_J^{-1}, \pi_\bullet \circ d_{E_0})$, where K_J^{-1} denotes the dual of the (usual) canonical line bundle of the complex manifold (X, J). In the case $(\mathcal{J}, e^{\sqrt{-1}\omega})$ on a Kähler manifold, we see that $U^{-n+\bullet}$ is written in terms of the (usual) complex forms of type (r, s),

$$U^{-n} = \wedge^{n,0}, \tag{5.1}$$

$$U^{-n+1} = \wedge^{n,1} \oplus \wedge^{n-1,0}, \tag{5.2}$$

$$U^{-n+2} = \wedge^{n,2} \oplus \wedge^{n-1,1} \oplus \wedge^{n-2,0}, \tag{5.3}$$

$$U^{-n+3} = \wedge^{n,3} \oplus \wedge^{n-1,2} \oplus \wedge^{n-2,1} \oplus \wedge^{n-3,0}. \tag{5.4}$$

We take an open cover $\{V_\alpha\}$ of X and Ω_α as a nowhere vanishing holomorphic n-form on V_α. Then $E_{\alpha,0} = 0$ and the operator $\pi_\bullet \circ d_{E_{\alpha,0}}$ is the (usual) $\overline{\partial}$ operator. It implies that the space of infinitesimal deformations of generalized complex structures on X is given by the direct sum of the K_J^{-1}-valued Dolbeault cohomology groups

$$H^{n,2}_{\overline{\partial}}(X, K_J^{-1}) \oplus H^{n-1,1}_{\overline{\partial}}(X, K_J^{-1}) \oplus H^{n-2,0}_{\overline{\partial}}(X, K_J^{-1}), \tag{5.5}$$

where the space $H^{n-1,1}_{\overline{\partial}}(X, K_J^{-1}) \cong H^1(X, \Theta)$ is the space of infinitesimal deformations of complex structures in Kodaira-Spencer theory. The space $H^{n,2}_{\overline{\partial}}(X, K_J^{-1})$ is given by the action of B-fields (2-forms) and the space $H^{n-2,0}_{\overline{\partial}}(X, K_J^{-1})$ is induced by the action of holomorphic 2-vector fields.

The space of the obstructions is given by

$$H^{n,3}_{\overline{\partial}}(X, K_J^{-1}) \oplus H^{n-1,2}_{\overline{\partial}}(X, K_J^{-1}) \oplus H^{n-2,1}_{\overline{\partial}}(X, K_J^{-1}) \oplus H^{n-3,0}_{\overline{\partial}}(X, K_J^{-1}). \tag{5.6}$$

Note that the description in equation (5.5) is related to that in [**10**]. Similarly we find that the first cohomology of the complex $(K^\bullet, d)$ is described as

$$H^1(K^\bullet) \cong H^{1,1}_{\overline{\partial}}(X). \tag{5.7}$$

Hence it follows from theorems 3.1 and 3.2 that we obtain

Theorem 5.1. *Let X be a compact Kähler manifold with the generalized Kähler structure $(\mathcal{J}, e^{\sqrt{-1}\omega})$. If the obstruction space*

$$\bigoplus_{i=0}^{3} H^{n-i,3-i}_{\overline{\partial}}(X, K_J^{-1})$$

vanishes, then we have the family of generalized Kähler structures $\{\mathcal{J}_t, \psi_{t,s}\}$ with $(\mathcal{J}_0, \psi_{0,0}) = (\mathcal{J}, e^{\sqrt{-1}\omega})$ which is parametrized by $(t, s) \in \triangle' \times W$, where $\triangle'$ is a small open set of

$$\bigoplus_{i=0}^{2} H^{n-i,2-i}_{\overline{\partial}}(X, K_J^{-1})$$

and W denotes a small open set of $H^{1,1}_{\overline{\partial}}(X)$ containing the origin.

There are no deformations of complex structures on the complex projective space $\mathbb{C}P^2$. However there is a family of deformations of generalized complex structures on $\mathbb{C}P^2$ which is parametrized by the space of holomorphic 2-vector fields $H^0(\mathbb{C}P^2, \wedge^2\Theta)$. Let $\{ V_\alpha\, , \Omega_\alpha\}$ be a trivialization of the canonical line bundle K. Let β be a holomorphic 2-vector field on $\mathbb{C}P^2$. Then it follows that the action of spin group on Ω_α,

$$e^{\beta t} \wedge \Omega_\alpha,$$

induces deformations of generalized complex structure on $\mathbb{C}P^2$. In fact, we take inhomogeneous coordinates (z_1^α, z_2^α) on each U_α with $\Omega_\alpha =$

$dz_1^\alpha \wedge dz_2^\alpha$, and β is written as

$$\beta = f \frac{\partial}{\partial z_1^\alpha} \wedge \frac{\partial}{\partial z_2^\alpha},$$

where f is a cubic function. Then

$$e^\beta \wedge \Omega_a = f + \Omega_\alpha.$$

Thus $e^\beta \wedge \Omega_a$ is a non-degenerate pure spinor which induces a generalized complex structure $\mathcal{J}_\beta$. The type of generalized complex structure $\mathcal{J}$ is defined as the minimal degree of differential forms (non-degenerate pure spinors) which induces $\mathcal{J}$. Thus the type of J_β is 0 on the complement of the zero set of β and the type of $\mathcal{J}_\beta$ is 2 at the zero set of β. Since we have $H^0(\mathbb{C}P^2, \wedge^2\Theta) \cong H^0(\mathbb{C}P^2, \mathcal{O}(3))$, it follows from the theorem of stability that we have a family of generalized Kähler structures on $\mathbb{C}P^2$ parametrized by $H^0(\mathbb{C}P^2, \mathcal{O}(3)) \oplus H^{1,1}_{\overline{\partial}}(X)$.

5.2. Generalized Kähler structures on Fano surfaces. Our theorem can be applied to Fano surfaces. Let S_n be a blown up $\mathbb{C}P^2$ at n points whose anti-canonical line bundle is ample ($n \leq 8$). Then it follows from the Kodaira vanishing theorem that the space of obstructions vanishes. Thus deformations of generalized complex structures are parametrized by an open set of $H^0(S_n, K^{-1}) \oplus H^1(S_n, \Theta)$, whose dimensions are given by

$$\dim H^1(S_n, \Theta) = \begin{cases} 2n - 8, & (n = 5, 6, 7, 8), \\ 0, & (n = 0, 1, 2, 3, 4) \end{cases}$$

$$\dim H^0(S_n, K^{-1}) = 10 - n.$$

It follows from the theorem of stability that we have the family of generalized Kähler structures on S_n which is parametrized by an open set of the direct sum

$$H^0(S_n, K^*) \oplus H^1(S_n, \Theta) \oplus H^{1,1}(S_n),$$

where $H^{1,1}(S_n)$ denotes the Dolbeault cohomology of type $(1,1)$ which coincides with the cohomology $H^1(K^\bullet)$ (see section 4),

$$\dim H^{1,1}(S_n) = 1 + n.$$

5.3. Poisson structures and generalized Kähler structures. In general, we have an obstruction to deformations of generalized complex structures and the space of infinitesimal deformations does not coincide with the space of actual deformations. However, the theorem of stability can be applied as long as we have a one-dimensional analytic family of deformations of generalized complex structures. Typical examples are constructed from holomorphic Poisson structures. Let X be a compact Kähler manifold with a holomorphic 2-vector field β. If β satisfies that

$$[\beta, \beta]_L = 0, \tag{5.8}$$

where the bracket denotes the Schouten bracket, then β is called a holomorphic Poisson structure on X. Since β is holomorphic, we find $d_L\beta = 0$. Hence β also satisfies the Maurer-Cartan equation and the adjoint action of $e^{\beta t}$ on $\mathcal{J}$ induces an analytic family of deformations of generalized complex structures. We write it by $\mathcal{J}_{t\beta} = \mathrm{Ad}_{e^{t\beta}}\mathcal{J}$. Hence we obtain from theorems 3.1 and 3.2:

Theorem 5.2. *Let β be a holomorphic Poisson structure on a compact Kähler manifold X. Then we have a family of generalized Kähler structures $\{\mathcal{J}_{t\beta}, \psi_t\}$.*

The rank of 2-vector β at x is r if $\beta_x^r \neq 0$ and $\beta_x^{r+1} = 0$ for a point $x \in X$. Then we denote it by $\mathrm{rank}\, \beta_x = r$. Since the type of generalized complex structure of $\mathcal{J}_\beta$ is defined as the minimal degree of differential form $e^\beta \cdot \Omega_\alpha$, where Ω_α denotes a non-zero holomorphic n-form. Thus we have

$$\mathrm{type}(\mathcal{J}_\beta)_x = n - 2\,\mathrm{rank}\,\beta_x. \tag{5.9}$$

This is concerned with the fact that the type $(\mathcal{J}_\beta)_x$ can jump, depending on a choice of $x \in X$. Let X be a Kähler manifold with an action of an l-dimensional complex commutative Lie group G ($l \geq 2$). We denote by $\{\xi_i\}_{i=1}^l$ a basis of the Lie algebra of G which induces the corresponding holomorphic vector fields $\{V_i\}_{i=1}^l$ on X. We take β as a linear combination of $V_i \wedge V_j$'s,

$$\beta = \sum_{i,j} \lambda_{i,j} V_i \wedge V_j, \tag{5.10}$$

where each $\lambda_{i,j}$ denotes a constant. Since $[V_i, V_j] = 0$, we have $[\beta, \beta]_L = 0$. Then we have a family of generalized Kähler structure on X. The type of $\mathcal{J}_\beta$ can change, according to the fixed points set of the action of G. Hence we have:

Theorem 5.3. *Let X be a compact Kähler manifold of dimension n. If we have an action of an l-dimensional complex commutative Lie group G with a non-trivial 2-vector β as in (5.10), then we have a family of deformations of non-trivial generalized Kähler structures on X.*

Since the type of $\mathcal{J}_\beta$ is given by $n - 2\,\mathrm{rank}\,\beta$ from (5.9), it follows that generalized Kähler structures in theorem 5.3 are not obtained by the action of B-fields (2-forms) from usual Kähler structures.

Theorems 5.1, 5.2, and 5.3 imply that there are many examples of deformations of generalized Kähler structures on Kähler manifolds, such as every toric Kähler manifold and the Grassmannians. On a complex surface, any holomorphic section of anti-canonical bundle gives the Poisson structure. There is a classification of holomorphic Poisson surfaces and we can count the dimensions of sections of anti-canonical bundles on a given holomorphic Poisson surfaces [**4**, **22**].

6. Appendix

Let $\mathcal{J}$ be a generalized complex structure on a manifold X. Then we have the decomposition

$$(T \oplus T^*) \otimes \mathbb{C} = L_{\mathcal{J}} \oplus \overline{L}_{\mathcal{J}}.$$

We denote by $|a|$ the degree of $a \in \wedge^p \overline{L}_{\mathcal{J}}$, that is, p. Then for $a \in \wedge^* \overline{L}_{\mathcal{J}}$, we define a graded bracket by

$$[d, a]_G = da - (-1)^{|a|} ad.$$

We also define a bracket $[a, b]_L$ by

$$[a, b]_L = [d, a]_G b - (-1)^{(|a|+1)|b|} b[d, a]_G.$$

There is the following explicit description.

Proposition 6.1. *$[a, b]_L$ is an element of $\wedge^{|a|+|b|-1}\overline{L}_{\mathcal{J}}$ which is given in terms of the derived bracket*

(6.1)

$$[E_1 \cdots E_n,\, F_1 \cdots F_m]_L = \sum_{i,j} (-1)^{i+j} E_1 \cdots \overset{\check{i}}{E_i} \cdots E_n [E_i, F_j]_d F_1 \cdots \overset{\check{j}}{F_j} \cdots F_m$$

for $E_i,\, F_j \in \overline{L}_J,\ i = 1, \cdots, n,\ j = 1, \cdots, m$.

Proof. The bracket $[a, b]_L$ is an operator acting on the differential forms $\wedge^* T^*$. Then it turns out that

$$[a, b]_L f\phi = f[a, b]_L \phi, \qquad \phi \in \wedge^* T^*$$

for a function f. Thus $[a, b]_L$ is not a differential operator but an element of $\wedge^* \overline{L}_{\mathcal{J}}$. Next we see that

$$[E, F_1 \cdots F_m]_L = [\{d, E\}, F_1 \cdots F_m]_L \tag{6.2}$$

$$= \sum_j (-1)^{j+1} [E, F_j]_d F_1 \cdots \overset{\check{j}}{F_j} \cdots F_m. \tag{6.3}$$

Further for $a, b \in \wedge^* \overline{L}_J$ and $E \in \overline{L}_{\mathcal{J}}$, we have

$$[E \wedge a,\, b]_L = a \wedge [E, b]_L - E[a, b]_L. \tag{6.4}$$

Then by the induction, we have the result. q.e.d.

References

[1] V. Apostolov, P. Gauduchon & G. Grantcharov, *Bihermitian structures on complex surfaces*, Pro. London Math. Soc. **79** (1999), 414–428, Corrigendum: **92**(2006), 200–202, MR 2192389, Zbl 1089.53503.

[2] S. Barannikov & M. Kontsevich, *Frobenius manifolds and formality of Lie algebras of polyvector fields* (1998), International Math. Res. Notices, (4), 201–215, MR 1609624, Zbl 0914.58004.

[3] H. Bursztyn, M Gualtieri & G. Cavalcanti, *Reduction of Courant algebroids and generalized complex structures*, Adv. Math, **211**(2)(2007), 726–765, MR 2323543, Zbl 1115.53056.

[4] C. Bartocci & E. Macrì, *Classification of Poisson surfaces*, math.AG/0402338, Commun. Contemp. Math. **7** (2005), no. 1, 89–95. MR 2129789, Zbl 1071.14514.

[5] G. Cavalcanti, *New aspects of dd^c-lemma*, Math.DG/0501406, D. Phil thesis.

[6] C.C. Chevalley, *The algebraic theory of Spinors*, Columbia University Press, New York, 1954. viii+131 pp. MR 0060497, Zbl 0057.25901.

[7] R. Goto, *Moduli spaces of topological calibrations, Calabi-Yau, hyperKähler, G_2 and Spin(7) structures*, Internat. J. Math. **15** (2004), no. 3, 211–257, MR 2060789, Zbl 1046.58002.

[8] R. Goto, *On deformations of generalized Calabi-Yau, hyperKähler, G_2 and Spin(7) structures*, Math.DG/0512211.

[9] R. Goto, *Poisson structures and generalized Kähler submanifolds*, J. Math. Soc. Japan **61** (2009), no. 1, 107–132, MR 2272873, Zbl 1160.53014.

[10] M. Gualtieri, *Generalized complex geometry,* thesis, Math. DG/0401221, refined version, Math.DG/0703298.

[11] M. Gualtieri, *Hodge decomposition for generalized Kähler manifolds*, Math. DG/0409093, Lecture at the String Theory and Geometry workshop, (2007) Oberwolfach.

[12] N. Hitchin, *Generalized Calabi-Yau manifolds*, Math. DG/0401221, Q. J. Math. **54** (2003), no. 3, 281–308, MR 2013140, Zbl 1076.3201.

[13] N.Hitchin, *Instantons, Poisson structures and generalized Kähler geometry*, Comm. Math. Phys. **265** (2006), no. 1, 131–164, MR 2217300, Zbl 1110.53056.

[14] N.Hitchin, *Bihermitian metrics on Del Pezzo surfaces*, Math.DG/0608213, J. Symplectic Geom. **5** (2007), no. 1, 1–8, MR 2371181, Zbl pre05237677.

[15] D. Huybrechts, *Generalized Calabi-Yau structures, K3 surfaces and B-fields*, math.AG/0306132, Internat. J. Math. **16** (2005), no. 1, 13–36, MR 2115675, Zbl 1120.14027.

[16] K. Kodaira *Complex manifolds and deformations of complex structures*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 283. Springer-Verlag, New York, 1986, MR 0815922, Zbl 0581.32012. Grundlehren der Mathematischen Wissenschaften, **283**, Springer-Verlag, (1986).

[17] K. Kodaira & D.C. Spencer, *On deformations of complex, analytic structures I,II*, Ann. of Math. (2) **67** (1958) 328–466, MR 0112154, Zbl 0128.16901.

[18] K. Kodaira & D.C. Spencer, *On deformations of complex analytic structure, III. stability theorems for complex structures*, Ann. of Math. (2) **71** (1960) 43–76, MR 0115189, Zbl 0128.16902.

[19] Y. Kosmann-Schwarzbach, *Derived brackets*, Lett. Math. Phys. **69** (2004), 61–87, MR 2104437, Zbl 1055.17016.

[20] Y. Lin & S. Tolman, *Symmetries in generalized Kähler geometry*, Comm. Math. Phys. **268** (2006), no. 1, 199–222, MR 2249799, Zbl 1120.53049 , Math. DG/0509069.

[21] Z.-J. Liu, A. Weinstein & Ping. Xu, *Manin triples for Lie bialgebroids*, J. Differential Geom. **45** (1997), no. 3, 547–574, MR 1472888, Zbl 0885.58030.

[22] F. Sakai, *Anti-Kodaira dimension of ruled surfaces*, Sci. Rep. Saitama Univ. Ser. A **10** (1982), no. 2, 1–7, MR 0662405, Zbl 0496.14022.

[23] J. P. Serre, *Lie Algebras & Lie groups*, Lecture Notes in Mathematics, 1500. Springer-Verlag, Berlin, 2006, MR 2179691, Zbl 0742.17008.

[24] G. Tian, *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric*, Mathematical aspects of string theory (San Diego, Calif., 1986), 629–646, Adv. Ser. Math. Phys., 1, World Sci. Publishing, Singapore, 1987, MR 0915841, Zbl 0696.53040.

Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka, 560
Japan
E-mail address: goto@math.sci.osaka-u.ac.jp