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THE PLATEAU PROBLEM IN ALEXANDROV SPACES

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Abstract

We study the Plateau Problem of finding an area minimizing disk bounding a given Jordan curve in Alexandrov spaces with curvature $\geq \kappa$. These are complete metric spaces with a lower curvature bound given in terms of triangle comparison. Imposing an additional condition that is satisfied by all Alexandrov spaces according to a conjecture of Perel'man, we develop a harmonic map theory from two dimensional domains into these spaces. In particular, we show that the solution to the Dirichlet problem from a disk is Hölder continuous in the interior and continuous up to the boundary. Using this theory, we solve the Plateau Problem in this setting generalizing classical results in Euclidean space (due to J. Douglas and T. Rado) and in Riemannian manifolds (due to C.B. Morrey).

1. Introduction

The Plateau Problem is the problem of finding a surface minimizing the area amongst all surfaces which are images of a map from a disk and spanning a given Jordan curve Γ in a space X. If X is the Euclidean space $\mathbf{R}^n$, we can formulate this problem more precisely as follows. If D is the unit disk in $\mathbf{R}^2$, the area of a map $u : D \to \mathbf{R}^n$ is

$$A(u) = \int_D \sqrt{\left|\frac{\partial u}{\partial x}\right|^2 \left|\frac{\partial u}{\partial y}\right|^2 - \left(\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y}\right)^2}\, dxdy. \tag{1}$$

The Plateau Problem in $\mathbf{R}^n$. *Given a Jordan curve $\Gamma \subset \mathbf{R}^n$, let*

$$\begin{aligned}\mathcal{F} \;\; = \;\; \big\{ v : \overline{D} \to \mathbf{R}^n : \; & v \in W^{1,2}(D) \cap C^0(\overline{D}) \text{ and} \\ & v\big|_{\partial D} \text{ monotonically parameterizes } \Gamma \big\}.\end{aligned}$$

Find $u \in \mathcal{F}$ so that $A(u) \leq A(v)$ for all $v \in \mathcal{F}$.

The mathematical problem of proving the existence of an area minimizing surface spanning a given contour was raised by J. Lagrange in the mid-eighteenth century, but the problem is named after the Belgian

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physicist J. Plateau who studied soap films. It was not until the 1930's that J. Douglas [**D**] and T. Rado [**R1**] [**R2**] properly formulated and independently solved this problem. In the 1950's, C.B. Morrey [**Mo**] generalized the problem by replacing the ambient Euclidean space by a space belonging to a very general class of Riemannian manifolds (that includes all compact ones). Further generalization is due to I. Nikolaev [**N**] who replaced the Riemannian manifold by a complete metric space with curvature bounded from above in the sense of Alexandrov. Our interest here is to extend the generalization to the case when the ambient space is an Alexandrov space, i.e. when the curvature is bounded from *below*.

An important ingredient for the Plateau Problem (and minimal surface theory in general) is the theory of harmonic maps from a domain of dimension 2. In fact, the solution of the Plateau Problem in Euclidean space and Riemannian manifolds can be given by a map that is harmonic and conformal. With the assumption of non-positive curvature, the harmonic map theory into a singular target space (with the domain assumed to be a Riemannian domain of arbitrary dimension) was first considered in the foundational paper of M. Gromov and R. Schoen [**GS**] and further generalized by N. Korevaar and R. Schoen [**KS1**], [**KS2**]. This theory was also developed independently by J. Jost (see [**Jo**] and references therein). A generalization to the case when the curvature is bounded from above by an arbitrary constant was given by T. Serbinowski [**Se1**]. The aspect that makes the harmonic map theory tractable in this setting is the convexity property of the energy functional under the assumption of an upper curvature bound. The regularity theory for the Dirichlet problem (i.e. the problem of finding a map of least energy amongst all maps with a given boundary condition) states that the Dirichlet solution is Lipschitz continuous in the interior [**KS1**] and Hölder continuous up to the boundary if given a Hölder continuous boundary condition [**Se2**]. Recall also that there often exists a heavy reliance on the upper sectional curvature bound when one studies harmonic maps into Riemannian manifolds (see for example [**ES**]). The harmonic map approach to the Plateau Problem in metric spaces of curvature bounded from above is discussed by the first author in [**Me1**], [**Me2**], [**Me3**]. (This differs substantially from the approach pursued in [**N**].)

To tackle the Plateau Problem when the ambient space has a lower curvature bound, we will develop the relevant harmonic map theory. More specifically, we study the Dirichlet problem for maps into an Alexandrov space X. The difficulty here is that we do not have the nice convexity properties of the energy functional and cannot mimic the techniques developed for maps into non-positively curved spaces. In fact, the solution of the Dirichlet problem is not continuous in general, even for maps into Riemannian manifolds (cf. [**Ha**]). On the other hand, the regularity of harmonic maps into Riemannian manifolds from a two dimensional domain was established by [**Mo**], [**Gu**], [**Sc**] and [**He**]. We generalize this result when the target space is an Alexandrov space X of curvature $\geq \kappa$ assuming an additional condition (that X satisfies Perel'man's conjecture described later in this section).

Regularity Theorem (cf. Theorem 6, Theorem 9)**.** *Let X be a compact Alexandrov space satisfying Perel'man's conjecture. A Dirichlet solution $u : D \to X$ is Hölder continuous in the interior of D and continuous up to ∂D.*

We note that this is the optimal regularity result. A two-dimensional cone C with vertex angle $< \pi$ is positively curved and a harmonic map from a disk into C is Hölder continuous, but not Lipschitz, at a point in the pre-image of the vertex.

Using the theorem above, we solve the Plateau Problem by using the Dirichlet solution as a means to obtain an area minimizing disk. One fundamental point we need to clarify is the notion of area associated to a map into an Alexandrov space. Note that the area given by (1) for maps into Euclidean space is obtained by integrating the area element associated with the pull-back metric. The notion of the pull-back metric for maps into non-positively curved metric spaces was given in [**KS1**] and for metric spaces of general upper curvature bound in [**Me2**]. We prove that this notion also makes sense for maps into Alexandrov spaces (cf. Theorem 11). Using the pull-back metric, we define the area functional for maps into X and formulate the Plateau Problem analogously to the statement of the Plateau problem in Euclidean space. The proof of the existence of the solution of the Plateau Problem parallels a well-known argument for the Euclidean case [**L**]. Combined with the regularity theorem for the Dirichlet problem, this gives us:

Theorem (cf. Theorem 13 and Theorem 17)**.** *Let X be a compact Alexandrov space satisfying Perel'man's conjecture and $\Gamma \subset X$ be a Jordan curve. Suppose there exists a continuous map $u_0 : \overline{D} \to X$ of finite energy whose restriction to ∂D monotonically parameterizes Γ. Then there exists a continuous map $u : \overline{D} \to X$ which minimizes area amongst all other continuous maps whose restriction of ∂D monotonically parameterizes Γ. Furthermore, u is conformal, energy minimizing and Hölder continuous in the interior of D.*

We now discuss the space X in the theorems above in more detail. Recall that an Alexandrov space with curvature bounded below by κ

is one in which geodesic triangles are *thicker* than comparison triangles in the two-dimensional simply connected surface of constant curvature κ. This notion of curvature was developed by a Russian school of mathematicians led by A.D. Alexandrov starting in the 1940's. More recently, Alexandrov spaces re-emerged into prominence as they are the limiting spaces of a sequence of certain Riemannian manifolds under the Gromov-Hausdorff convergence. Perel'man's Stability Theorem [**P**] states that if two Alexandrov spaces of the same dimension are sufficiently close in the Gromov-Hausdorff distance, they are actually homeomorphic. In fact, Perel'man asserts something more: the homeomorphism between the two spaces can be chosen to be bi-Lipschitz. The proof of Perel'man's claim in its full generality is not yet available to our knowledge. For a good discussion on the Stability Theorem and related issues, we refer to Kapovitch [**Ka**]. We note the following two properties of an Alexandrov space: First, the tangent cone $T_P X$ at a point of an n-dimensional Alexandrov space X is a cone $\mathcal{C}(\Pi_P)$ over the space of directions Π_P at P which is itself an $(n-1)$ Alexandrov space. Second, the Hausdorff-Gromov distance between a neighborhood around P in X and a neighborhood around the vertex of $\mathcal{C}(\Pi_P)$ at this point can be made arbitrarily small by taking the neighborhoods sufficiently small. Thus, Perel'man's claim implies that if X is an n-dimensional Alexandrov space, then X satisfies the property that X is locally bi-Lipschitz equivalent to a cone over a $(n-1)$-dimensional Alexandrov space. Furthermore, this $(n-1)$-dimensional Alexandrov space is locally bi-Lipschitz equivalent to a cone over an $(n-2)$-dimensional Alexandrov space and so forth. This motivates us to say that an Alexandrov space X satisfies the *Perel'man conjecture* if it has this property.

The outline of this paper is as follows. In section 2, we give definitions of Alexandrov spaces and other related concepts. We also recall Korevaar and Schoen's Sobolev space theory into metric spaces. Section 3 contains the two dimensional harmonic map theory. In particular, we discuss the existence of the solution to the Dirichlet problem and prove its interior and boundary regularity. In section 4, the solution of the Plateau Problem is shown. This section also contains the proof of the existence of the pull-back inner product that allows us to make sense of the area functional (subsection 4.1).

Because the interior regularity proven in subsection 3.1 is central to this paper and because of the technical nature of its proof, we conclude this section by illustrating the ideas behind this argument. The main step of the proof is to establish that, for any $D_r(x_0) \subset D$, we have a good bound on the energy of a map $u\big|_{D_r(x_0)}$ in terms of the energy of $u\big|_{\partial D_r(x_0)}$. This in turn implies an energy decay estimate which, by Morrey's Energy Decay Lemma, implies the Hölder continuity. If the image $\Gamma_0 \subset X$ of the boundary map $u\big|_{\partial D_r(x_0)}$ is long, then its energy is large and thus we restrict our attention to the case when Γ_0 is short. Hence, we can assume that Γ_0 is contained in a neighborhood that is bi-Lipschitz equivalent to a neighborhood of the vertex of the cone $\mathcal{C}(\Pi_P)$ for some $P \in X$. Since the ratio of the energy of a given map and the energy of this map composed with a bi-Lipschitz map is bounded from above and below by a constant depending on the bi-Lipschitz constant, we can further assume for the sake of simplicity that $u\big|_{D_r(x_0)}$ maps into this cone. We now consider the following two cases: (1) the length of Γ_0 is short relative to its distance from the vertex V of the cone and (2) the length of Γ_0 is long relative to its distance from the vertex. In case (1), we extend the map $u|_{\partial D_r(x_0)}$ to a map φ defined on $D_r(x_0)$ by setting $\varphi(x_0) = V$ and linearly mapping the radial ray from x_0 to a point $\xi \in \partial D_r(x_0)$ to a ray from V to $u(\xi)$. By the construction, the energy of φ is bounded in terms of the energy of $u\big|_{\partial D_r(x_0)}$. The main step follows immediately since $u\big|_{D_r(x_0)}$ is energy minimizing and has the same boundary values as φ. In case (2), Γ_0 is contained in a neighborhood U far away from the vertex and hence U is bi-Lipschitz equivalent to product of $\Pi_P \times I$ for some interval $I \subset \mathbf{R}$. We construct a map φ by separately considering the Dirichlet problem in Π_P and in I. Therefore, if we have a good energy bound for the Dirichlet problem in Π_P, then we are done. Since the dimension of Π_P is one less than that of X, we are able to prove the main step by an inductive argument on the dimension of X.

2. Definitions and Background Material

2.1. Alexandrov Spaces. We begin with a discussion of Alexandrov spaces and refer to [**Sh**], [**BBI**], [**BGP**], [**OS**] for more details.

Definition. We say a complete metric space (X, d) (or more simply X) is an *Alexandrov space of curvature bounded from below by* κ if it satisfies the following conditions:

(1) X is a length space; i.e. for any two points $P, Q \in X$, there exists a curve γ_{PQ} between P and Q with length equal to $d(P, Q)$.

(2) Let S_κ be a simply connected surface of constant curvature κ. Denote the distance function of S_κ by $\bar{d}$ and the geodesic between $\bar{P}, \bar{Q} \in S_\kappa$ by $\bar{P}\bar{Q}$. Given a triple $P, Q, R \in X$ with $d(P,Q), d(Q,R), d(P,R) < \frac{\pi}{\sqrt{\kappa}}$, let $\triangle(PQR)$ be a geodesic triangle. Then there exists a geodesic triangle $\triangle(\bar{P}\bar{Q}\bar{R})$ in S_κ such that $d(P,Q) = \bar{d}(\bar{P},\bar{Q}), d(P,R) = \bar{d}(\bar{P}\bar{R}), d(R,Q) = \bar{d}(\bar{R},\bar{Q})$ and if we take two points $\bar{S} \in \bar{P}\bar{Q}$ and $\bar{T} \in \bar{P}\bar{R}$ with $d(P,S) = \bar{d}(\bar{P},\bar{S})$, $d(P,T) = \bar{d}(\bar{P},\bar{T})$, then $d(S,T) \geq \bar{d}(\bar{T},\bar{S})$. The triangle $\triangle(\bar{P}\bar{Q}\bar{R}) \subset S_\kappa$ will be called a comparison triangle of

$\triangle(PQR) \subset X$.

For simplicity, will say that X is an *Alexandrov space* if there exists some κ so that X is an Alexandrov space of curvature bounded from below by κ. In this paper, it is not important whether κ is positive, zero or negative; we only use the fact that there exists some lower bound on curvature. Hence, we may as well assume $\kappa < 0$.

Let $\alpha(s) : [0, a] \to X$ and $\beta(t) : [0, b] \to X$ be arclength parameterizations of two geodesics emanating from a point $P \in X$ and let $\theta(t, s)$ be the angle at $\bar{P}$ of a comparison triangle $\triangle\overline{\alpha(t)}\bar{P}\overline{\beta(s)}$ in S_κ. In particular, if X is an Alexandrov space of curvature bounded from below by $\kappa = -1$ then $\theta(t, s) \in [0, \pi]$ is given by the equality

$$\cosh \bar{d}(\bar{\alpha}(t), \bar{\beta}(s)) = \cosh t \cosh s - \sinh t \sinh s \cos \theta(t, s).$$

Condition (2) implies that $t \mapsto \theta(t, s)$ and $s \mapsto \theta(t, s)$ are monotone non-increasing. The angle between geodesics α and β is defined to be

$$\angle(\alpha, \beta) = \lim_{t,s \to 0} \theta(t, s).$$

We will need the following geometric fact:

Lemma 1. *Let X be an Alexandrov space. For any $\rho > 0$, there exists $\delta = \delta(\rho) > 0$ sufficiently small so that if*

(i) $P, R, T \in X$ with $P \neq R$, $d_{PR} < \delta$ and

$$\left|\frac{1}{2}d_{PR} - d_{PT}\right| < \delta^2 d_{PR}\ ,\ \left|\frac{1}{2}d_{PR} - d_{RT}\right| < \delta^2 d_{PR}, \tag{2}$$

(ii) γ_{TR} is a geodesic from T to R and $R' \in \gamma_{TR}$ with

$$d_{RR'} = \delta d_{PR} \tag{3}$$

(iii) $\gamma_{PR'}$ is a geodesic from P to R' with T' as its midpoint,

then

$$d_{TT'} < \rho d_{PR}. \tag{4}$$

Remark. The idea behind Lemma 1 is as follows. One of the distinguishing features of a space X with a lower curvature bound is the non-uniqueness of geodesics between two given points. Related to this non-uniqueness statement is the following fact: given two points $P, R \in X$, any point T whose distances to P and to R are both approximately half of d_{PR} as in (i) may be far away from the midpoint of a geodesic γ_{PR}. For example, let P be the north pole and R be the south pole on the standard 2-sphere and T be a point on the equator. There exists a geodesic γ_{PR} from P and R whose midpoint is the antipodal point of T. In a smooth Riemannian manifold, the point T satisfying (i) is close to the midpoint of γ_{PR} if P and R are contained in a sufficiently small neighborhood, but in an Alexandrov space, such a neighborhood does not generally exist. On the other hand, Lemma 1 says that we can choose a point R' close to R as in (ii) so that T is close to a midpoint T' of a geodesic $\gamma_{PR'}$.

Proof. We assume that X is an Alexandrov space of curvature bounded from below by -1. (Given an Alexandrov space of curvature bounded from below by $\kappa < 0$, we can rescale the distance function by a factor of $\frac{1}{|\kappa|}$ to construct an Alexandrov space of curvature bounded from below by -1. Since the assumption and the conclusion of the lemma is scale invariant, the assumption that the curvature is bounded from below by -1 is without a loss of generality.) Fix $\delta > 0$ and let $P, R, T, \gamma_{TR}, R', \gamma_{PR'}, T'$ satisfy (i), (ii) and (iii) above. Since

$$d_{TR'} = d_{TR} - d_{RR'}, \quad d_{T'R'} = \frac{1}{2} d_{PR'}, \quad d_{PR} - d_{RR'} \leq d_{PR'} \leq d_{PR} + d_{RR'},$$

(2) and (3) imply

$$d_{TR'}, d_{T'R'} = \left(\frac{1}{2} + O(\delta) \right) d_{PR}. \tag{5}$$

Define α by setting

$$\cosh d_{TT'} = \cosh d_{TR'} \cosh d_{T'R'} - \sinh d_{TR'} \sinh d_{T'R'} \cos \alpha.$$

Using Taylor expansion, we obtain

$$\begin{aligned} d_{TT'}^2 &= d_{TR'}^2 + d_{T'R'}^2 - 2 d_{TR'} d_{T'R'} \cos \alpha + O(d_{PR}^3) \\ &= (d_{TR'} - d_{T'R'})^2 + 2 d_{TR'} d_{T'R'} (1 - \cos \alpha) + O(d_{PR}^3). \end{aligned}$$

Furthermore, apply (5) to obtain

$$d_{TT'}^2 = O(\delta^2) d_{PR}^2 + 2 \left(\frac{1}{2} + O(\delta) \right)^2 (1 - \cos \alpha) d_{PR}^2 + O(d_{PR}^3).$$

Thus, if we can show that α can be made arbitrarily small by taking δ (and therefore d_{PR}) sufficiently small, then we obtain $O(\delta^2) + 2\left(\frac{1}{2} + O(\delta)\right)^2 (1 - \cos \alpha) < \frac{\rho^2}{2}$ for sufficiently small δ and hence

$$d_{TT'}^2 \leq \frac{\rho^2}{2} d_{PR}^2 + O(d_{PR}^3) < \rho^2 d_{PR}^2$$

for δ sufficiently small. Thus, we are left to show that α is small if δ is chosen to be small. To see this, we let $\gamma_{TR'} \subset \gamma_{TR}$ be a geodesic from T to R' and $\gamma_{RR'} \subset \gamma_{TR}$ be a geodesic from R to R'. Next, let α_0 be the angle between $\gamma_{PR'}$ and $\gamma_{TR'}$ and β_0 the angle between $\gamma_{PR'}$ and $\gamma_{RR'}$. Lastly, let β be the angle defined by

$$\cosh d_{PR} = \cosh d_{PR'} \cosh d_{RR'} - \sinh d_{PR'} \sinh d_{RR'} \cos \beta.$$

By construction, $\alpha_0 + \beta_0 = \pi$, and by the monotonicity property of angles in Alexandrov space, $\alpha_0 \geq \alpha$ and $\beta_0 \geq \beta$. Hence

$$\begin{aligned}
\cosh d_{PR} &\leq \cosh d_{PR'} \cosh d_{RR'} - \sinh d_{PR'} \sinh d_{RR'} \cos \beta_0 \\
&= \cosh d_{PR'} \cosh d_{RR'} + \sinh d_{PR'} \sinh d_{RR'} \cos \alpha_0 \\
&\leq \cosh d_{PR'} \cosh d_{RR'} + \sinh d_{PR'} \sinh d_{RR'} \cos \alpha.
\end{aligned}$$

Expanding by Taylor series, we obtain

$$\begin{aligned}
d_{PR}^2 &\leq d_{PR'}^2 + d_{RR'}^2 + 2 d_{PR'} d_{RR'} \cos \alpha + O(d_{PR}^3) \\
&\leq (d_{PR'} + d_{RR'})^2 + 2 d_{PR'} d_{RR'} (\cos \alpha - 1) + O(d_{PR}^3).
\end{aligned}$$

By the triangle inequality along with (2), we have

$$\begin{aligned}
d_{PR'} &\leq d_{PT} + d_{TR'} \\
&= d_{PT} + d_{TR} - d_{RR'} \\
&\leq d_{PR} + 2\delta^2 d_{PR} - d_{RR'}.
\end{aligned}$$

Furthermore, the triangle inequality and (3) gives

$$d_{PR'} \geq d_{PR} - d_{RR'} = (1-\delta) d_{PR}.$$

Combining the last three inequalities, we obtain

$$d_{PR}^2 \leq d_{PR}^2 (1 + O(\delta^2))^2 + d_{PR}^2 (\delta - \delta^2)(\cos \alpha - 1) + O(d_{PR}^3).$$

Dividing by d_{PR}^2 and δ and rearranging terms, we get

$$(1-\delta)(1-\cos\alpha) \leq 4\delta + 4\delta^2 + O(d_{PQ}).$$

Hence, we see that α is small if δ is sufficiently small. q.e.d.

Definition. The *space of directions* Σ_P at $P \in X$ is the closure of the set of equivalence classes of geodesics emanating from P endowed with the distance function $d_{\Sigma_P}([\alpha],[\beta]) = \angle(\alpha, \beta)$. Here, α is said to be equivalent to β if and only if $\angle(\alpha,\beta) = 0$. (Since X is assumed to be an Alexandrov space, for arclength parameterized geodesics $\alpha : [0,a] \to X$ and $\beta : [0,b] \to X$ with $0 < a \leq b$, we have that α and β are equivalent if and only if $\alpha(s) = \beta(s)$ for all $s \in [0,a]$.)

Definition. The *tangent cone* T_P at $P \in X$ is a *cone* over the space of directions Σ_P. More precisely, T_P is defined to be the set

$$\Sigma_P \times [0, \infty) / \sim$$

where $\sim$ identifies all element of the form $([\alpha], 0)$ along with a distance function d_{T_P} defined by

$$d_{T_P}^2(([\gamma], s), ([\sigma], t)) = s^2 + t^2 - 2st \cos d_{\Sigma_P}([\gamma],[\sigma]).$$

The equivalence class of $([\alpha], 0)$ will be called the vertex of T_P.

Finally, we define the notion of an Alexandrov spaces satisfying the Perel'man conjecture given by the following inductive definition.

Definition. Let X be a compact Alexandrov space. Then the Hausdorff dimension of X is an integer (cf. [**BGP**]). We say that a 1-dimensional compact Alexandrov space is said to satisfy the *Perel'man conjecture* if and only if it is a finite interval of length $\leq \pi$ or a circle of length $\leq 2\pi$. Assuming that we have given the definition of an $(n-1)$-dimensional compact Alexandrov space X satifying the Perel'man conjecture, we say that an n-dimensional compact Alexandrov space satisfies the *Perel'man conjecture* if every point $P \in X$ has a neighborhood U_P (hereafter referred to as a *conic neighborhood*) which is bi-Lipschitz homeomorphic to a neighborhood of the vertex of a cone over an $(n-1)$-dimensional compact Alexandrov space of diameter $\leq \pi$ which satisfies the Perel'man conjecture.

Let X be a n-dimensional compact Alexandrov space satisfying the Perel'man conjecture. For each $P \in X$, let U_P be a conic neighborhood of P. Because of the assumption that X is compact, there exists a finite set of point $F \subset X$ so that $\{U_P\}_{P\in F}$ is a covering of X. We will refer to $\{U_P\}$ as a finite cover of X by conic neighborhoods. A number $\lambda > 0$ is a *Lebesgue number* of a finite cover $\{U_P\}_{P\in F}$ if $A \subset U_P$ for some $P \in F$ whenever the diameter of A is $\leq \lambda$.

Perel'man's Stability Theorem is the following:

Theorem (cf. [**P**], [**Ka**])**.** *Let X be a compact n-dimensional Alexandrov space of curvature bounded from below by κ. There exists $\epsilon = \epsilon(X) > 0$ so that if Y is an n-dimensional Alexandrov space of curvature bounded from below by κ with the Hausdorff-Gromov distance between X and Y less than ϵ, then there exists a homeomorphism between X and Y.*

Perel'man asserts that there actually exists a *bi-Lipschitz* homeomorphism between X and Y above. A consequence of Perel'man's claim is that the condition that an n-dimensional Alexandrov space satisfies Perel'man's conjecture is actually redundant. This follows immediately from the fact that, for any point P in a n-dimensional Alexandrov space X, the pointed Hausdorff limit of the scaling $(\lambda X; P)$ of X is isometric to $(T_P(X); V)$. In other words, a small neighborhood around P is close in Hausdorff-Gromov distance to a small neighborhood around V in T_P which is a cone over a $(n-1)$-dimensional space of directions.

2.2. Sobolev Space $W^{1,2}(\Omega, X)$. We summarize Korevaar and Schoen's Sobolev space theory of [**KS1**] Chapter 1. Let Ω be a compact Riemannian domain and (X, d) a complete metric space. A Borel measurable map $u : \Omega \to X$ is said to be in $L^2(\Omega, X)$ if for $P \in X$,

$$\int_\Omega d^2(u(x), P) d\mu < \infty.$$

This condition is independent of $P \in X$ by the triangle inequality. For $\epsilon > 0$, set $\Omega_\epsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \epsilon\}$ and let $S(x, \epsilon)$ denote the sphere of radius ϵ centered at x in Ω. Construct the ϵ-approximate energy function $e_\epsilon(x) : \Omega_\epsilon \to \mathbf{R}$ by

$$e_\epsilon(x) = \frac{1}{\omega_n} \int_{S(x,\epsilon)} \frac{d^2(u(x), u(y))}{\epsilon^2} \frac{d\sigma}{\epsilon^{n-1}}$$

where ω_n is the volume of the unit sphere in $\mathbf{R}^n$. (This differs from the ϵ-approximate energy function given in [**KS1**] (1.2ii) by a factor of ω_n.) The ϵ-approximate energy function e_ϵ is a L^1-function; more specifically (cf. [**KS1**] (1.2iii)),

$$\int_{\Omega_\epsilon} e_\epsilon(x) d\mu \leq C\epsilon^{-2}. \tag{6}$$

Let ν be any Borel measure on the interval $(0, 2)$ satisfying

$$\nu \geq 0 \ , \ \nu((0,2)) = 1, \ \int_0^2 \lambda^{-2} d\nu(\lambda) < \infty. \tag{7}$$

Consider an averaged approximate energy density function defined by

$${}_\nu e_\epsilon(x) = \begin{cases} \displaystyle\int_0^2 e_{\lambda\epsilon}(x) d\nu(\lambda) & \text{for } x \in \Omega_{2\epsilon} \\ 0 & \text{for } x \in \Omega - \Omega_{2\epsilon}. \end{cases} \tag{8}$$

By (6) and (7), we see that ${}_\nu e_\epsilon \in L^1(\Omega)$. Thus, we can define a functional $E^u_\epsilon : C_c(\Omega) \to \mathbf{R}$ by setting

$$E^u_\epsilon(f) = \int_\Omega f(x) {}_\nu e_\epsilon(x) d\mu.$$

We will say that u is a finite energy map or $u \in W^{1,2}(\Omega, X)$ if

$$E^u = \sup_{f \in C_c(\Omega), 0 \leq f \leq 1} \limsup_{\epsilon \to 0} E^u_\epsilon(f) < \infty.$$

By [**KS1**] Theorem 1.5.1, the above definition of finite energy map is independent of the choice of ν satisfying (7); in other words, if $u \in L^2(\Omega, X)$ has finite energy with respect to some measure ν_1 satisfying (7), then it has finite energy with respect to all such ν. Furthermore, the same theorem says that if $u \in W^{1,2}(\Omega, X)$, then the measures ${}_\nu e_\epsilon d\mu$ converge weakly to the same measure; in other words, there exists a measure de such that

$${}_\nu e_\epsilon(x) d\mu \rightharpoonup de, \ \forall \nu \text{ satisfying (7)}.$$

One of the remarkable results in [**KS1**] is that the notion of the energy density function $|du|^2$ defined in the classical case for maps between Riemannian manifolds has an analogue in the singular setting. More precisely, by [**KS1**] Theorem 1.10, the measure de is absolutely continuous with respect to the Lebesgue measure and thus there exists an L^1-function $|\nabla u|^2$, called the *energy density*, so that

$$de = |\nabla u|^2 d\mu.$$

Moreover, $|\nabla u|^2$ coincides (up to a constant multiple) with the norm squared of the gradient when $X = \mathbf{R}$.

There is also a corresponding generalization for the directional energy density function. Let $\Gamma(T\overline{\Omega})$ be the set of Lipschitz tangent vector fields on $\overline{\Omega}$ and $V \in \Gamma(T\overline{\Omega})$. For simplicity, we denote $x + \epsilon V$ to be the flow along V at time ϵ with initial point x. Define the ϵ-approximate energy density function

$${}^{V}e_{\epsilon}(x) = \frac{d^2(u(x), u(x+\epsilon V))}{\epsilon^2}$$

and an averaged approximate directional energy density function ${}^{V}_{\nu}e_{\epsilon}(x)$ in the same way as (8). By [**KS1**] Theorems 1.8.1 and 1.9.6, if $u \in W^{1,2}(\Omega, X)$, then meaures ${}^{V}e_{\epsilon}d\mu$ and ${}^{V}_{\nu}e_{\epsilon}d\mu$ converge weakly to a measure which is absolutely continuous with respect to the Lebesgue measure. Thus, there exists a function $|u_*(V)|^2$, called the *directional energy density*, so that

$$(9) \qquad {}^{V}e_{\epsilon}d\mu,\ {}^{V}_{\nu}e_{\epsilon}d\mu \rightharpoonup |u_*(V)|^2 d\mu.$$

The following equality is contained in Theorem 1.11 of [**KS1**]; for any $h \in C^{0,1}(\overline{\Omega})$, we have

$$(10) \qquad |u_*(hV)|^2 = |h|^2 |u_*(V)|^2.$$

If Ω is a Lipschitz domain and $u \in W^{1,2}(\Omega, X)$, the restriction of u to the boundary $\partial\Omega$ makes sense; more precisely, there exists a well-defined notion of a trace of u, denoted $Tr(u)$, which is an element of $L^2(\partial\Omega, X)$. Two maps $u, v \in W^{1,2}(\Omega, X)$ have the same trace (i.e. $Tr(u) = Tr(v)$) if and only if $d(u(x), v(x)) \in W^{1,2}_0(\Omega)$ (cf. Theorem 1.12.2 of [**KS1**]).

Many standard statements in elementary calculus can be translated for the metric space setting using the above notions. First, consider a map $\gamma : [a, b] \subset \mathbf{R} \to X \in W^{1,2}([a, b], X)$. From [**KS1**] Lemma 1.9.5, we have

$$\left|\gamma_*\left(\frac{\partial}{\partial t}\right)\right|^2(t) = \lim_{\epsilon\to 0} \frac{d^2(\gamma(t), \gamma(t+\epsilon))}{\epsilon^2}, \quad \text{for a.e. } t \in [a, b].$$

We set the notation

$$\left|\gamma_*\left(\frac{\partial}{\partial t}\right)\right|(t) := \sqrt{\left|\gamma_*\left(\frac{\partial}{\partial t}\right)\right|^2(t)}.$$

Thus,

$$\left|\gamma_*\left(\frac{\partial}{\partial t}\right)\right|(t) = \lim_{\epsilon\to 0}\frac{d(\gamma(t),\gamma(t+\epsilon))}{\epsilon}, \quad \text{for a.e. } t \in [a,b]. \tag{11}$$

Furthermore, (1.9xvi) of [**KS1**] says that

$$\int_a^b \left|\gamma_*\left(\frac{\partial}{\partial t}\right)\right|(t)dt = \lim_{||P||\to 0}\sum_{i=0}^{k-1} d(\gamma(t_{i+1}),\gamma(t_i))$$

for arbitrary partitions

$$P : a = t_0 < t_1 < \cdots < t_k = b$$

of $[a,b]$. This in turn implies, by the triangle inequality and the Cauchy-Schwartz inequality, that

(12)

$$d(\gamma(t_1),\gamma(t_2)) \leq \int_{t_1}^{t_2}\left|\gamma_*\left(\frac{\partial}{\partial t}\right)\right|dt \leq \left(\int_a^b\left|\gamma_*\left(\frac{\partial}{\partial t}\right)\right|^2 dt\right)^{1/2}|t_1-t_2|^{1/2}.$$

Now with D a unit disk, let (x,y) and (r,θ) be its Cartesian and polar coordinates respectively. By Lemma 1.9.5 of [**KS1**], we have for a.e. $x \in (-1,1)$ and a.e. $y \in (-1,1)$,

$$\frac{d^2(u(x,y),u(x+\epsilon,y))}{\epsilon^2} = \left|u_*\left(\frac{\partial}{\partial x}\right)\right|^2(x,y) \tag{13}$$

and

$$\frac{d^2(u(x,y),u(x,y+\epsilon))}{\epsilon^2} = \left|u_*\left(\frac{\partial}{\partial y}\right)\right|^2(x,y). \tag{14}$$

For a point $(r,\theta) \in D$ with $r \neq 0$, let $R = (r_1,r_2)\times(\theta_1,\theta_2) \subset D\backslash\{0\}$ be its neighborhood. By Theorem 1.11 of [**KS1**], if $u \in W^{1,2}(D,X)$ (with respect to the usual Euclidean metric on D), then $u \in W^{1,2}(R,X)$ (with respect to the usual Euclidean metric on the product $(r_1,r_2)\times(\theta_1,\theta_2)$). Furthermore, inequality (1.11iii) in the same theorem implies that $u\big|_{I_r} \in W^{1,2}(I_r,X)$ and $u\big|_{l_\theta} \in W^{1,2}(I_\theta,X)$ for a.e. $r \in (r_1,r_2)$ and a.e. $\theta \in (\theta_1,\theta_2)$. Here, I_r and I_θ are the vertical and horizontal lines of $(r_1,r_2)\times(\theta_1,\theta_2)$ respectively. Therefore, we can apply Lemma 1.9.5 to conclude that for a.e. $(r,\theta)\in D$, we have

$$\frac{d^2(u(r,\theta),u(r+\epsilon,\theta))}{\epsilon^2} = \left|u_*\left(\frac{\partial}{\partial r}\right)\right|^2(r,\theta) \tag{15}$$

and

$$\frac{d^2(u(r,\theta),u(r,\theta+\epsilon))}{\epsilon^2} = \left|u_*\left(\frac{\partial}{\partial \theta}\right)\right|^2(r,\theta). \tag{16}$$

We now assume that X is an Alexandrov space (with or without the assumption that it satisfies Perel'man's Conjecture). For purposes of expositional clarity, we record the statements (17) and (18) below which can be justified using Theorem 11 in Section 4.1. Let (x, y) be the standard Cartesian coordinates on the disk. Then using the fact that the inverse of the standard Euclidean metric on D is given by $g^{xx} = g^{yy} = 1$ and $g^{xy} = 0$ in Cartesian coordinates (x, y) and $g^{rr} = 1$, $g^{\theta\theta} = \frac{1}{r^2}$ and $g^{r\theta} = g^{\theta r} = 0$ in polar coordinates (r, θ), we can write

$$\begin{aligned} |\nabla u|^2(r,\theta) &= \left|u_*\left(\frac{\partial}{\partial x}\right)\right|^2(r,\theta) + \left|u_*\left(\frac{\partial}{\partial y}\right)\right|^2(x,y) \\ &= \left|u_*\left(\frac{\partial}{\partial r}\right)\right|^2(r,\theta) + \frac{1}{r^2}\left|u_*\left(\frac{\partial}{\partial \theta}\right)\right|^2(r,\theta), \end{aligned} \tag{17}$$

for almost every point in D. The validity of (17) follows immediately from an application of equality (49) in Theorem 11 below combined with the fact that in polar coordinates $g^{rr} = 1$, $g^{\theta\theta} = \frac{1}{r^2}$ and $g^{r\theta} = g^{\theta r} = 0$. Furthermore,

$$E^u = E^{u\circ\psi}, \ \forall \text{ conformal reparameterization } \psi : D \to D. \tag{18}$$

The statement (18) follows by a well-known computation in the smooth setting which can be adapted to this setting by the change of variables formula (48) of Theorem 49.

We will now prove some lemmas that we will need later. For notational simplicity, we set

$$D_\epsilon(Z, W) = \frac{d(u(x + \epsilon Z), u(x + \epsilon W))}{\epsilon}, \quad Z, W \in \Gamma(T\overline{\Omega}).$$

Lemma 2. *Let $V \in \Gamma(T\overline{\Omega})$ and $f \in C_c(\Omega)$, $f \geq 0$. Then $D_\epsilon(0, V)$ converges to $|u_*(V)|$ pointwise almost everywhere, in L^2 and in L^2-norm, i.e.*

$$D^2_\epsilon(0, V) \to |u_*(V)|^2 \quad a.e.\ x \in \Omega, \tag{19}$$

$$\int_\Omega f(D_\epsilon(0, V) - |u_*(V)|)^2 \to 0 \tag{20}$$

and

$$\int_\Omega f D^2_\epsilon(0, V) \to \int_\Omega f|u_*(V)|^2. \tag{21}$$

Proof. As

$$D_\epsilon(0, V) = \frac{d(u(x), u(x + \epsilon V))}{\epsilon},$$

the convergence of (21) follows from (9). To see the convergence of (19), first observe that (1.9 xix) of [**KS1**] implies that $|u_*(V)| = 0$ almost everywhere on $\{x : V(x) = 0\}$. Since $D_\epsilon(0, V) = 0$ on this set, we only need to verify the convergence on $\{x : V(x) \neq 0\}$. After applying a $C^{1,1}$ change of coordinates from the initial coordinate chart, we can assume that V is a coordinate direction. Thus Lemma 1.9.5 of

[**KS1**] implies that $D_\epsilon^2(0,V) \to |u_*(V)|^2$ almost everywhere. Now that we have verified (21) and (19), we will show that (20) follows from those two convergence statements. First, we write

$$f(D_\epsilon(0,V) - |u_*(V)|)^2 \leq 2fD_\epsilon^2(0,V) + 2f|u_*(V)|^2.$$

Since

$$\int_\Omega (2fD_\epsilon^2(0,V) + 2f|u_*(V)|^2) \to \int_\Omega 4f|u_*(V)|^2$$

by (21) and $f(D_\epsilon(0,V) - |u_*(V)|^2) \to 0$ a.e. by (19), we can apply the Dominated Convergence Theorem to conclude (20). q.e.d.

Lemma 3. *Let $V, U \in \Gamma(T\overline{\Omega})$ and $f \in C_c(\Omega)$, $f \geq 0$. Then as $\epsilon \to 0$, we have*

$$\int_\Omega f(D_\epsilon(0,V) - D_\epsilon(U,U+V))^2 \to 0, \tag{22}$$

$$\int_\Omega fD_\epsilon^2(U,U+V)d\mu \to \int_\Omega f|u_*(V)|^2 d\mu \tag{23}$$

and

$$D_\epsilon^2(U,U+V) \to |u_*(V)|^2 \quad a.e. \tag{24}$$

Proof. For this proof, we set

$$D_\epsilon = D_\epsilon(0,V), D = |u_*(V)| \;\text{ and }\; T_\epsilon\varphi(x) = \varphi(x + \epsilon U)$$

for any function $\varphi : \Omega \to \mathbf{R}$. To see why (22) is true, first note that $T_\epsilon D_\epsilon = D_\epsilon(U,U+V)$ and

$$\begin{aligned}\sqrt{f}T_\epsilon D_\epsilon &= T_\epsilon(T_{-\epsilon}\sqrt{f})T_\epsilon D_\epsilon \\ &= T_\epsilon(\sqrt{f} - (\sqrt{f} - T_{-\epsilon}\sqrt{f}))T_\epsilon D_\epsilon \\ &= T_\epsilon(\sqrt{f}D_\epsilon) - (T_\epsilon\sqrt{f} - \sqrt{f})T_\epsilon D_\epsilon.\end{aligned}$$

Thus, denoting the L^2 norm by $\|\cdot\|_2$, we obtain

$$\|\sqrt{f}T_\epsilon D_\epsilon - \sqrt{f}D_\epsilon\|_2 \leq \|T_\epsilon(\sqrt{f}D_\epsilon) - \sqrt{f}D_\epsilon\|_2 + \|(T_\epsilon\sqrt{f} - \sqrt{f})T_\epsilon D_\epsilon\|_2.$$

Furthermore, several application of the triangle inequality yields

$$\begin{aligned}&\|\sqrt{f}T_\epsilon D_\epsilon - \sqrt{f}D_\epsilon\|_2 \\ &\quad\leq \|T_\epsilon(\sqrt{f}D_\epsilon) - \sqrt{f}D\|_2 + \|\sqrt{f}D - \sqrt{f}D_\epsilon\|_2 \\ &\quad\quad + \|(T_\epsilon\sqrt{f} - \sqrt{f})T_\epsilon D_\epsilon\|_2 \\ &\quad\leq \|T_\epsilon(\sqrt{f}D_\epsilon) - T_\epsilon(\sqrt{f}D)\|_2 + \|T_\epsilon(\sqrt{f}D) - \sqrt{f}D\|_2 \\ &\quad\quad + \|\sqrt{f}D - \sqrt{f}D_\epsilon\|_2 + \|(T_\epsilon\sqrt{f} - \sqrt{f})T_\epsilon D_\epsilon\|_2 \\ &\quad\leq \|T_\epsilon(\sqrt{f}D) - \sqrt{f}D\|_2 + 2\|\sqrt{f}D - \sqrt{f}D_\epsilon\|_2 \\ &\quad\quad + \|(T_\epsilon\sqrt{f} - \sqrt{f})T_\epsilon D_\epsilon\|_2.\end{aligned}$$

As $\epsilon \to 0$, the first term on the right hand side converges to 0 since $\sqrt{f}D \in L^2(\Omega)$, the second term by Lemma 2 and the third term since $T_\epsilon\sqrt{f} \to \sqrt{f}$ uniformly. Thus, we have established (22).

To see why (23) is true, one can use the change of coordinates method outlined in the proof of Lemma 2.3.1 of [**KS1**]. The convergence of (24) follows immediately from (22) and (23). q.e.d.

Lemma 4. *Let $V, U \in \Gamma(T\overline{\Omega})$ and $f \in C_c(\Omega)$, $f \geq 0$. Then for all $\eta > 0$ there exists $\epsilon_0, \delta > 0$ such that for all $\tilde{\Omega} \subset \Omega$ with $\mu(\tilde{\Omega}) < \delta$ and $\epsilon < \epsilon_0$, we have*

$$\int_{\tilde{\Omega}} f(x) D_\epsilon^2(0, V)dx < \eta \text{ and } \int_{\tilde{\Omega}} f(x) D_\epsilon^2(U, U + V)dx < \eta. \tag{25}$$

Proof. We use the notation in the proof of Lemma 3. Since fD^2 is a non-negative integrable function on Ω, there exists $\delta > 0$ such that if $\mu(\tilde{\Omega}) < \delta$ then

$$2 \int_{\tilde{\Omega}} fD^2 < \frac{\eta}{2}.$$

By Lemma 2, there exists $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$, then

$$2 \int_{\tilde{\Omega}} (\sqrt{f}D_\epsilon - \sqrt{f}D)^2 \leq 2 \int_{\Omega} (\sqrt{f}D_\epsilon - \sqrt{f}D)^2 < \frac{\eta}{2}.$$

Thus, the first inequality of (25) follows by observing that

$$\int_{\tilde{\Omega}} fD_\epsilon^2 = \int_{\tilde{\Omega}} (\sqrt{f}(D + D_\epsilon - D))^2 \leq 2 \int_{\tilde{\Omega}} fD^2 + 2 \int_{\tilde{\Omega}} (\sqrt{f}D_\epsilon - \sqrt{f}D)^2.$$

The second inequality follows from

$$\begin{aligned} &\int_{\tilde{\Omega}} fD_\epsilon T_\epsilon D_\epsilon \\ &= \int_{\tilde{\Omega}} fD_\epsilon^2 + \int_{\tilde{\Omega}} fD_\epsilon(T_\epsilon D_\epsilon - D_\epsilon) \\ &\leq \int_{\tilde{\Omega}} fD_\epsilon^2 + \left(\int_{\tilde{\Omega}} fD_\epsilon^2 \right)^{1/2} \left(\int_{\tilde{\Omega}} f(T_\epsilon D_\epsilon - D_\epsilon)^2 \right)^{1/2} \end{aligned}$$

and the observation that the second term converges to 0 as $\epsilon \to 0$ by Lemma 3. q.e.d.

3. The Dirichlet Problem

We let D be a unit disk in the plane. The Dirichlet Problem for an Alexandrov space X is formulated as follows:

The Dirichlet Problem. *Let $\psi \in W^{1,2}(D, X)$,*

$$W_\psi^{1,2} = \{v \in W^{1,2}(D, X) : Tr(v) = Tr(\psi)\}$$

and $E_\psi = \inf\{E^v : v \in W^{1,2}_\psi\}$. *Find* $u \in W^{1,2}_\psi$ *such that* $E^u = E_\psi$.

If $u \in W^{1,2}(D, X)$ has the property that $E^u \leq E^v$ for any $v \in W^{1,2}(D, X)$ with $Tr(v) = Tr(u)$, then u will be referred to as a Dirichlet solution (for the boundary data $Tr(u)$). We first establish the following existence result:

Theorem 5. *Let X be a compact Alexandrov space. Given any $\psi \in W^{1,2}(D, X)$, there exists a Dirichlet solution $u \in W^{1,2}_\psi(D, X)$.*

Proof. The proof is an easy application of the results of Chapter 1 in [**KS1**]. We take a sequence of maps $\{u_k\} \subset W^{1,2}_\psi(D, X)$ such that E^{u_k} converges to E_ψ. Since X is compact, there exists $C > 0$ so that

$$\int_D d^2(u_k(x), Q) d\mu(x) + E^{u_k} \leq C.$$

By the precompactness theorem (Theorem 1.13 of [**KS1**]), there exists a subsequence $\{u_{k_i}\}$ that converges in $L^2(D, X)$ to $u \in W^{1,2}(D, X)$. By the lower semicontinuity of energy (Theorem 1.6.1 of [**KS1**]) and the trace theory (Theorem 1.12.2 of [**KS1**]), $E^u = E_\psi$ and $Tr(u) = Tr(\psi)$. q.e.d.

The rest of this section is devoted to the regularity issues of the Dirichlet solution.

3.1. The Interior Hölder Continuity. The goal of this subsection is to prove:

Theorem 6. *Let X be a compact Alexandrov space satisfying the Perel'man conjecture. Let $u \in W^{1,2}(D, X)$ be a Dirichlet solution. Then for each $R \in (0, 1)$, there exists C and $\alpha > 0$ dependent only on R, E^u and X so that*

$$d(u(z_1), u(z_2)) \leq C|z_1 - z_2|^\alpha, \ \forall z_1, z_2 \in D_R(0).$$

Here, $D_R(z_0) \subset \mathbf{R}^2$ is the disk of radius R centered at z_0. In particular, $D_1(0) = D$. Before we prove Theorem 6, we will need several preliminary lemmas. In the following, let Π be a compact Alexandrov space with diameter $\leq \pi$. We define two metric spaces $\mathcal{P}(\Pi)$ and $\mathcal{C}(\Pi)$ associated with Π. The first is the product of Π with $\mathbf{R}$; more precisely, $\mathcal{P}(\Pi)$ is the set

$$\Pi \times \mathbf{R} = \{(P, t) : P \in X, t \in \mathbf{R}\}$$

endowed with the distance function $d_{\mathcal{P}}$ defined by

$$d^2_{\mathcal{P}}((P, t), (Q, s)) = d^2(P, Q) + (t - s)^2.$$

For any $r_1, r_2 \in [0, \infty)$ with $r_1 < r_2$, we define the truncated product space as

$$\mathcal{P}(\Pi, r_1, r_2) = \{(P, t) \in \mathcal{P}(\Pi) : r_1 < t < r_2\}.$$

The second space is the cone over Π; more precisely, $\mathcal{C}(\Pi)$ is the set

$$\Pi \times [0,\infty)/\sim \text{ where } (P,0)\sim(Q,0)$$

endowed with the distance function $d_{\mathcal{C}}$ defined by

$$d^2_{\mathcal{C}}((P,t),(Q,s)) = t^2+s^2-2ts\cos d(P,Q).$$

The vertex of $\mathcal{C}(\Pi)$ (i.e. any point of the form $(P,0)$) will be denoted $\mathbf{O}$. For any $r_1, r_2 \in [0,\infty)$ with $r_1 < r_2$, we define the truncated cone as

$$\mathcal{C}(\Pi, r_1, r_2) = \{(P,t)\in\mathcal{C}(\Pi) : r_1 < t < r_2\}.$$

Given a map $u \in W^{1,2}(D,\mathcal{P}(\Pi))$ (resp. $u \in W^{1,2}(D,\mathcal{C}(\Pi))$) we will denote energy, energy density function and directional energy function by $E^u_{\mathcal{P}}$, $|\nabla u|^2_{\mathcal{P}}$ and $|u_*(V)|^2_{\mathcal{P}}$ (resp. $E^u_{\mathcal{C}}$, $|\nabla u|^2_{\mathcal{C}}$ and $|u_*(V)|^2_{\mathcal{C}}$) to avoid confusion.

If $0 < r_1 < r_2 < \infty$, then $u \in W^{1,2}(D,\mathcal{P}(\Pi,r_1,r_2))$ if and only if $u \in W^{1,2}(D,\mathcal{C}(\Pi,r_1,r_2))$. Indeed, a simple computation shows that there exists

$$L = L(r_1, r_2) \tag{26}$$

so that

$$\frac{1}{\sqrt{L}} d_{\mathcal{P}}(P,Q) \le d_{\mathcal{C}}(P,Q) \le \sqrt{L} d_{\mathcal{P}}(P,Q)$$

and hence

$$\frac{1}{L} E^u_{\mathcal{P}} \le E^u_{\mathcal{C}} \le L E^u_{\mathcal{P}}. \tag{27}$$

The key step in proving Theorem 6 is the following lemma.

Lemma 7. *Let X be a compact Alexandrov space satisfying Perel'man's conjecture, $\{U_p\}_{p\in F}$ be a finite cover of X by conic neighborhoods and λ be its Lebesgue number. There exists κ depending only on X so that if $u \in W^{1,2}(D,X)$ is a Dirichlet solution, $Tr(u) = \gamma \in W^{1,2}(\partial D, X)$ and*

$$\int_{\partial D} \left|\gamma_*\left(\frac{\partial}{\partial\theta}\right)\right|^2 d\theta < \frac{\lambda^2}{2\pi}, \tag{28}$$

then

$$E^u \le \kappa \int_{\partial D} \left|\gamma_*\left(\frac{\partial}{\partial\theta}\right)\right|^2 d\theta. \tag{29}$$

Proof. We prove this by an induction on the dimension of X. We first verify the inductive step. Assume Lemma 7 is true whenever the dimension is n and suppose that the dimension of X is $n+1$. By the definition of conic neighborhoods, for each $p \in F$, there exists a bi-Lipshitz map

$$\varphi_p : U_p \to \varphi_p(U_p) \subset \mathcal{C}(\Pi_p)$$

where Π_p is the space of directions at p of X. Let K, η be sufficiently large so that for all $p \in F$ and $P, Q \in U_p$,

$$\frac{1}{\sqrt{K}} d(P,Q) \leq d_C(\varphi_p(P), \varphi_p(Q)) \leq \sqrt{K} d(P,Q) \tag{30}$$

and

$$\frac{\pi}{\eta} < \frac{1}{4}. \tag{31}$$

In view of (12), assumption (28) implies that for any $\theta_1, \theta_2 \in \partial D$,

$$d(\gamma(\theta_1), \gamma(\theta_2)) \leq \int_{\partial D} \left| \gamma_* \left(\frac{\partial}{\partial \theta} \right) \right| d\theta \leq \sqrt{2\pi} \left(\int_{\partial D} \left| \gamma_* \left(\frac{\partial}{\partial \theta} \right) \right|^2 d\theta \right)^{1/2} \leq \lambda.$$

Therefore, the image of γ is contained in U_p for some $p \in F$ and we can let

$$\sigma = \varphi_p \circ \gamma : \partial D \to \varphi_p(U_p) \subset \mathcal{C}(\Pi_p).$$

We will write $\sigma = (\sigma_1, \sigma_2)$ where $\sigma_1 : \partial D \to \Pi_p$ and $\sigma_2 : \partial D \to \mathbf{R}$ are the natural projections to the first and second factors respectively. We consider two cases:

Case 1. $\exists \theta_0 \in \partial D$ such that

$$d^2_{\mathcal{C}}(\sigma(\theta_0), \mathbf{O}) \leq \eta \int_{\partial D} \left| \sigma_* \left(\frac{\partial}{\partial \theta} \right) \right|^2_{\mathcal{C}} d\theta. \tag{32}$$

Let (r, θ) be the polar coordinates of D and define $\psi = (\psi_1, \psi_2) : D \to \mathcal{C}(\Pi_p)$ by setting

$$\psi(r, \theta) := (\sigma_1(\theta), r\sigma_2(\theta)).$$

It is clear by construction that $\psi \in W^{1,2}(D, \mathcal{C}(\Pi_p))$ and $Tr(\psi) = \sigma$. Furthermore, we have

$$d^2_{\mathcal{C}}(\psi(r_1, \theta), \psi(r_2, \theta)) = |r_1 - r_2|^2 d^2_{\mathcal{C}}(\sigma(\theta), \mathbf{O}) \tag{33}$$

and

$$d^2_{\mathcal{C}}(\psi(r, \theta_1), \psi(r, \theta_2)) = r^2 d^2_{\mathcal{C}}(\sigma(\theta_1), \sigma(\theta_2)) \tag{34}$$

by the definition of ψ and the definition of the distance function $d_{\mathcal{C}}$. Dividing (33) by $|r_1 - r_2|^2$ and (34) by $|\theta_1 - \theta_2|^2$, taking the limit as $r_1 \to r_2$ and $\theta_1 \to \theta_2$ respectively and noting (11), (15) and (16), we conclude that

$$\begin{aligned} \left| \psi_* \left(\frac{\partial}{\partial r} \right) \right|^2_{\mathcal{C}} (r, \theta) &= d^2_{\mathcal{C}}(\sigma(\theta), \mathbf{O}) \\ \text{and} \qquad \left| \psi_* \left(\frac{\partial}{\partial \theta} \right) \right|^2_{\mathcal{C}} (r, \theta) &= r^2 \left| \sigma_* \left(\frac{\partial}{\partial \theta} \right) \right|^2_{\mathcal{C}} (\theta) \end{aligned} \tag{35}$$

for a.e. $(r,\theta) \in D$. From the triangle inequality, (32) and (12), we see that

$$\begin{aligned} d^2_{\mathcal{C}}(\sigma(\theta),\mathbf{O}) &\leq \left(d_{\mathcal{C}}(\sigma(\theta_0),\mathbf{O}) + d_{\mathcal{C}}(\sigma(\theta),\sigma(\theta_0))\right)^2 \\ &\leq 2\left(d^2_{\mathcal{C}}(\sigma(\theta_0),\mathbf{O}) + d^2_{\mathcal{C}}(\sigma(\theta),\sigma(\theta_0))\right) \\ &\leq 2\left(\eta\int_{\partial D}\left|\sigma_*\left(\frac{\partial}{\partial\theta}\right)\right|^2_{\mathcal{C}} d\theta + \left(\int_{\theta_0}^{\theta}\left|\sigma_*\left(\frac{\partial}{\partial\theta}\right)\right|_{\mathcal{C}}\right)^2\right) \\ &\leq 2(\eta+2\pi)\int_{\partial D}\left|\sigma_*\left(\frac{\partial}{\partial\theta}\right)\right|^2_{\mathcal{C}} d\theta. \end{aligned} \tag{36}$$

Therefore (17), (35) and (36) imply

$$\begin{aligned} E^{\psi}_{\mathcal{C}} &= \int_0^{2\pi}\int_0^1\left(\left|\psi_*\left(\frac{\partial}{\partial r}\right)\right|^2_{\mathcal{C}} + \frac{1}{r^2}\left|\psi_*\left(\frac{\partial}{\partial\theta}\right)\right|^2_{\mathcal{C}}\right) r dr d\theta \\ &= \int_0^{2\pi}\int_0^1\left(d^2_{\mathcal{C}}(\sigma(\theta),\mathbf{O}) + \left|\sigma_*\left(\frac{\partial}{\partial\theta}\right)\right|^2_{\mathcal{C}}(\theta)\right) r dr d\theta \\ &\leq \Lambda\int_{\partial D}\left|\sigma_*\left(\frac{\partial}{\partial\theta}\right)\right|^2_{\mathcal{C}} \end{aligned} \tag{37}$$

for some constant Λ dependent only on η. Finally, by the fact that u is energy minimizing and by (30), we see that

$$E^u \leq E^{\varphi_p^{-1}\circ\psi} \leq K E^{\psi}_{\mathcal{C}} \leq K\Lambda\int_{\partial D}\left|\sigma_*\left(\frac{\partial}{\partial\theta}\right)\right|^2_{\mathcal{C}} \leq K^2\Lambda\int_{\partial D}\left|\gamma_*\left(\frac{\partial}{\partial\theta}\right)\right|^2.$$

CASE 2. $\forall\theta \in \partial D$,

$$d^2_{\mathcal{C}}(\sigma(\theta),\mathbf{O}) > \eta\int_{\partial D}\left|\sigma_*\left(\frac{\partial}{\partial\theta}\right)\right|^2_{\mathcal{C}} d\theta.$$

Integrating over $\theta \in \partial D$, we obtain

$$\frac{1}{2\pi}\int_{\partial D} d^2_{\mathcal{C}}(\sigma,\mathbf{O})d\theta > \eta\int_{\partial D}\left|\sigma_*\left(\frac{\partial}{\partial\theta}\right)\right|^2_{\mathcal{C}} d\theta$$

or equivalently

$$\frac{1}{\frac{1}{2\pi}\int_{\partial D} d^2_{\mathcal{C}}(\sigma,\mathbf{O})d\theta}\int_{\partial D}\left|\sigma_*\left(\frac{\partial}{\partial\theta}\right)\right|^2_{\mathcal{C}} d\theta < \frac{1}{\eta}.$$

If we define $\tilde{\sigma} = (\tilde{\sigma}_1,\tilde{\sigma}_2) : \partial D \to \mathcal{C}(\Pi_p)$ by

$$(\tilde{\sigma}_1(\theta),\tilde{\sigma}_2(\theta)) = \left(\sigma_1(\theta), \frac{1}{\frac{1}{2\pi}\int_{\partial D} d^2_{\mathcal{C}}(\sigma,\mathbf{O})d\theta}\sigma_2(\theta)\right)$$

then we have

$$\frac{1}{2\pi}\int_{\partial D} d^2(\tilde{\sigma}(\theta),\mathbf{O}) = 1 \tag{38}$$

and

$$\int_{\partial D} \left| \tilde{\sigma}_* \left(\frac{\partial}{\partial \theta} \right) \right|_{\mathcal{C}}^2 d\theta < \frac{1}{\eta}. \tag{39}$$

Now note that $\tilde{\sigma}$ is continuous; indeed, for any $\theta, \theta' \in \partial D$,

$$\begin{aligned} d_{\mathcal{C}}(\tilde{\sigma}(\theta), \tilde{\sigma}(\theta')) &\leq \int_{\theta}^{\theta'} \left| \tilde{\sigma}_* \left(\frac{\partial}{\partial \theta} \right) \right|_{\mathcal{C}} d\theta \\ &\leq \left(\int_{\partial D} \left| \tilde{\sigma}_* \left(\frac{\partial}{\partial \theta} \right) \right|_{\mathcal{C}}^2 d\theta \right)^{1/2} |\theta - \theta'|^{1/2} \\ &\leq \frac{1}{\sqrt{\eta}} |\theta - \theta'|^{1/2} \end{aligned}$$

by (39). Thus, (38) implies there exists $\theta' \in \partial D$ so that $d_{\mathcal{C}}(\tilde{\sigma}(\theta'), \mathbf{O}) = 1$. Furthermore, that fact that $|\theta - \theta'| \leq \pi$ implies that

$$d_{\mathcal{C}}(\tilde{\sigma}(\theta), \tilde{\sigma}(\theta')) < \sqrt{\frac{\pi}{\eta}} < \frac{1}{2}$$

by choice of η in (31). Thus,

$$|1 - d_{\mathcal{C}}(\tilde{\sigma}(\theta), \mathbf{O})| = |d_{\mathcal{C}}(\tilde{\sigma}(\theta'), \mathbf{O}) - d_{\mathcal{C}}(\tilde{\sigma}(\theta), \mathbf{O})| \leq d_{\mathcal{C}}(\tilde{\sigma}(\theta), \tilde{\sigma}(\theta')) < \frac{1}{2}$$

which implies

$$\frac{1}{2} < d_{\mathcal{C}}(\tilde{\sigma}(\theta), \mathbf{O}) \leq \frac{3}{2}.$$

Let $v_1 : D \to \Pi_p$ be the Dirichlet solution with $Tr(v_1) = \tilde{\sigma}_1$ and $v_2 : D \to \mathbf{R}$ be the Dirichlet solution with $Tr(v_2) = \tilde{\sigma}_2$. Since the dimension of Π_p is n, the inductive hypothesis implies that exists constant κ' so that

$$E^{v_1}_{\Pi_p} \leq \kappa' \int_{\partial D} \left| (\tilde{\sigma}_1)_* \left(\frac{\partial}{\partial \theta} \right) \right|_{\Pi_p}^2 d\theta$$

where we have used the subscript to denote quantities associated to the metric space Π_p. If we let $v = (v_1, v_2) \in \mathcal{P}(\Pi_p)$, then the definition of a product space immediately implies that there exists $\kappa'' > 0$ such that

$$\begin{aligned} E^v_{\mathcal{P}} &= E^{v_1}_{\Pi_p} + \int_D |\nabla v_2|^2 \\ &\leq \kappa' \int_{\partial D} \left| (\tilde{\sigma}_1)_* \left(\frac{\partial}{\partial \theta} \right) \right|_{\Pi_p}^2 d\theta + \kappa'' \int_{\partial D} \left| (\tilde{\sigma}_2)_* \left(\frac{\partial}{\partial \theta} \right) \right|_{\mathcal{P}}^2 d\theta \\ &= (\kappa' + \kappa'') \int_{\partial D} \left| \tilde{\sigma}_* \left(\frac{\partial}{\partial \theta} \right) \right|_{\mathcal{P}}^2 d\theta. \end{aligned}$$

Combined with (27), this implies that

$$E^v_{\mathcal{C}} \leq L^2 (\kappa' + \kappa'') \int_{\partial D} \left| \tilde{\sigma}_* \left(\frac{\partial}{\partial \theta} \right) \right|_{\mathcal{C}}^2 d\theta$$

where $L = L(\frac{1}{2}, \frac{3}{2})$ as in (26). If we define

$$w = (w_1, w_2) = (v_1, \left(\frac{1}{2\pi}\int_{\partial D} d^2_{\mathcal{C}}(\sigma, \mathbf{O})d\theta\right) v_2),$$

then $Tr(w) = \sigma$ and

$$E^w_{\mathcal{C}} \leq L^2(\kappa' + \kappa'') \int_{\partial D} \left|\sigma_*\left(\frac{\partial}{\partial\theta}\right)\right|^2_{\mathcal{C}} d\theta.$$

Finally, by the fact that u is energy minimizing and by (30), we see that

$$E^u \leq E^{\varphi_p^{-1}\circ w} \leq KE^w_{\mathcal{C}} \leq K^2L^2(\kappa' + \kappa'') \int_{\partial D} \left|\gamma_*\left(\frac{\partial}{\partial\theta}\right)\right|^2 d\theta. \tag{40}$$

Let $\kappa = \max\{K^2\Lambda, K^2L^2(\kappa' + \kappa'')\}$. In view of inequalities (37) and (40), we obtain (29). This finishes the proof of the inductive step.

Now assume that the dimension of X is 2. Then the space of direction at any point of X is either an interval or a circle and we can follow the proof of the inductive step to prove the base case of the inductive argument. q.e.d.

To summarize, we have demonstrated that if u is an energy minimizing map with Sobolev trace map γ that has small energy, then we have an estimate of the energy of u in terms of its trace. We use this fact along with the Morrey's Energy Decay Lemma for maps into X to prove Hölder continuity. We let $D_R(z_0)$ denote the disk of radius R centered at z_0 and $E^u[D_r(z_0)]$ the energy of u in the disk $D_r(z_0)$.

Lemma 8 (Morrey)**.** *Let $u \in W^{1,2}(D, X)$ satisfy*

$$E^u[D_r(z_0)] \leq C_R^2 r^{2\alpha}, 0 \leq r < 1 - R \tag{41}$$

for $R \in (0, 1)$ and $z_0 \in D_R(0) \subset D$ where C_R is a constant depending on R. Then there exists a constant K so that for every $z_1, z_2 \in D_R(0)$,

$$d(u(z_1), u(z_2)) \leq KC_R|z_1 - z_2|^\alpha.$$

Proof. Using the Sobolev theory of maps into metric space targets developed in Chapter 1 of [**KS1**], the assertion of the lemma follows from Morrey's argument in [**Mo**]. q.e.d.

Proof of Theorem 6. Fix a finite cover of X by conic neighborhood and let λ be its Lebesgue number. Let $R \in (0, 1)$ and let $z_0 \in D_R$. By an argument similar to that for [**KS1**] Lemma 1.9.1, u restricted to $\partial D_r(z_0)$ is absolutely continuous and $W^{1,2}$ for almost every choice of such $r \in (0, 1 - R)$; in other words, $u\big|_{\partial D_r(z_0)} \in W^{1,2}(\partial D_r(z_0), X)$ for a.e. $r \in (0, 1 - R)$. Furthermore, note that since $|\nabla u|^2 \in L^2(D)$, $|\nabla u|^2$ restricted to $\partial D_r(z_0)$ is in $L^2(\partial D_r(z_0))$ for a.e. $r \in (0, 1 - R)$ by the Fubini's Theorem. If s is the artlength parameter of $\partial D_r(z_0)$ and

(r,θ) is the polar coordinates in $D_r(z_0)$ centered at z_0, then we have the equality $s = r\theta$. Therefore, by (10) and (17), we have that

(42)

$$\left|u_*\left(\frac{\partial}{\partial s}\right)\right|^2 = \frac{1}{r^2}\left|u_*\left(\frac{\partial}{\partial \theta}\right)\right|^2 \leq \left|u_*\left(\frac{\partial}{\partial r}\right)\right|^2 + \frac{1}{r^2}\left|u_*\left(\frac{\partial}{\partial \theta}\right)\right|^2 = |\nabla u|^2$$

for a.e. $r \in (0, 1-R)$ and $s \in \partial D_r(z_0)$.

Let $\hat{u}$ be the composition of u with the dilation and translation of the plane which takes D to $D_r(z_0)$. If

$$\int_{\partial D_r(z_0)} \left|u_*\left(\frac{\partial}{\partial s}\right)\right|^2 ds < \frac{\lambda^2}{2\pi r},$$

then change of variables $s = r\theta$ gives

$$\int_{\partial D} \left|\hat{u}_*\left(\frac{\partial}{\partial \theta}\right)\right|^2 d\theta < \frac{\lambda^2}{2\pi}$$

by (10). Therefore, we obtain

$$\begin{aligned}
E^u[D_r(z_0)] &= E^{\hat{u}} \quad \text{(conformal invariance of energy (18))} \\
&\leq \kappa \int_{\partial D} \left|\hat{u}_*\left(\frac{\partial}{\partial \theta}\right)\right|^2 d\theta \quad \text{(Lemma 7)} \\
&= \kappa r \int_{\partial D_r(z_0)} \left|u_*\left(\frac{\partial}{\partial s}\right)\right|^2 ds \ \text{(change of variables } s = r\theta) \\
&\leq \kappa r \int_{\partial D_r(z_0)} |\nabla u|^2 ds \quad \text{(by inequality (42))} \\
&= \kappa r \frac{d}{dr} E^u[D_r(z_0)],
\end{aligned}$$

for a.e. $r \in (0, 1-R)$. If

$$\int_{\partial D_r(z_0)} \left|u_*\left(\frac{\partial}{\partial s}\right)\right|^2 ds \geq \frac{\lambda^2}{2\pi r},$$

then using the inclusion $D_r(z_0) \subset D$, we obtain

$$\begin{aligned}
E^u[D_r(z_0)] &\leq E^u \\
&\leq E^u \frac{2\pi r}{\lambda^2} \int_{\partial D_r(z_0)} \left|u_*\left(\frac{\partial}{\partial s}\right)\right|^2 d\theta \\
&\leq \frac{2\pi E^u}{\lambda^2} r \int_{\partial D_r(z_0)} |\nabla u|^2 d\theta \\
&= \frac{2\pi E^u}{\lambda^2} r \frac{d}{dr} E^u[D_r(z_0)]
\end{aligned}$$

for a.e. $r \in (0, 1-R)$, Thus, for almost every $r \in (0, 1-R)$,

$$E^u[D_r(z_0)] \leq \max\left\{\kappa, \frac{2\pi E^u}{\lambda^2}\right\} r \frac{d}{dr} E^u[D_r(z_0)].$$

Letting $C_0 = \max\{\kappa, \frac{2\pi E^u}{\lambda^2}\}$, we obtain the differential inequality,

$$\frac{1}{C_0 r} \leq \frac{\frac{d}{dr}E^u[D_r(z_0)]}{E^u[D_r(z_0)]}.$$

Integrating this in the interval $[r, 1-R]$ gives us the estimate needed to employ Lemma 8. q.e.d.

3.2. Boundary Regularity. The goal of this section is to prove:

Theorem 9. *Let X be a compact Alexandrov space satisfying Perel'-man's conjecture. If $\gamma \in C^0(\partial D, X)$ is a continuous map and $u \in W^{1,2}(D, X)$ its Dirichlet solution, then u is continuous in $\overline{D}$.*

To prove the boundary regularity, we need the following lemma which gives a lower bound on the energy of a harmonic map if a point is mapped sufficiently away from the boundary values. For any measurable set $A \subset D$ and $v \in W^{1,2}(D, X)$, we let $E^v[A] = \int_A |\nabla v|$.

Lemma 10. *Let $\epsilon, M > 0$. There exists $\eta = \eta(\epsilon, M) > 0$ so that for any $\varphi \in C^0(\partial D, X)$ and its Dirichlet solution $v \in W^{1,2}(D, X)$ with $d(v(0), \varphi(\partial D)) > \epsilon$ and $E^v \leq M$, we have*

$$E^v[v^{-1}(B_\epsilon(v(0)))] \geq \eta.$$

Proof. We prove this theorem by way of contradiction. Suppose that the statement is false. Then there exists a sequence of Dirichlet solutions $v_i \in W^{1,2}(D, X)$ with $\varphi_i = Tr(v_i)$ satisfying $d(v_i(0), \varphi_i(\partial D)) > \epsilon$ and

$$E^{v_i}[v_i^{-1}(B_\epsilon(v_i(0)))] \to 0. \tag{43}$$

Since X is compact, we may assume that $v_i(0) \to p \in X$ by taking a subsequence if necessary. Suppose $x \in D$ has the property that $d(v_i(x), p) < \frac{\epsilon}{2}$. The triangle inequality $d(v_i(x), v_i(0)) \leq d(v_i(x), p) + d(v_i(0), p)$ implies that $d(v_i(x), v_i(0)) < \epsilon$ for sufficietly large i. Therefore, $v_i^{-1}(B_{\frac{\epsilon}{2}}(p)) \subset v_i^{-1}(B_\epsilon(v_i(0)))$ which implies

$$E^{v_i}[v_i^{-1}(B_{\frac{\epsilon}{2}}(p))] \leq E^{v_i}[v_i^{-1}(B_\epsilon(v_i(0)))]. \tag{44}$$

Since $E^{v_i} \leq M$ for all i, we can apply the precompactness theorem and the trace theory (cf. [**KS1**] Theorem 1.13 and Theorem 1.12.2 respectively) to obtain a subsequence (which we denote $\{v_i\}$ by an abuse of notation) so that $v_i \to v$ in $L^2(D, X)$ and $\varphi_i = Tr(v_i) \to \varphi = Tr(v)$ in $L^2(\partial D, X)$. Fix $\delta \in (0,1)$ and let $D_{1-\delta}$ be a disk of radius $1-\delta$ centered at the origin. By Theorem 6, $v_i\big|_{D_{1-\delta}}$ is Hölder continuous; more specifically,

$$d(v_i(z_1), v_i(z_2)) \leq \mathcal{C}(X, \delta) \mid z_1 - z_2 \mid^{\alpha(X,\delta)}, \ \forall z_1, z_2 \in D_{1-\delta}.$$

Note that the modulus of continuity depends only on the geometry of the target and on the arbitrary constant δ. Hence, $\{v_i\big|_{D_{1-\delta}}\}$ form an equicontinuous family and converge uniformly to a Hölder continuous map according to the Arzela-Ascoli Theorem. The limit map must be the restriction of v constructed above to the smaller disk $D_{1-\delta}$. Consequently, $v(0) = p$ and, since δ is arbitrary, v is continuous in D. In particular, this implies $v^{-1}(B_{\frac{\epsilon}{4}}(p))$ is an open set. By the triangle inequality, $d(v_i(z), p) \leq d(v_i(z), v(z)) + d(v(z), p)$, and hence if $z \in D_{1-\delta}$ and $d(v(z), p) < \frac{\epsilon}{4}$ then $d(v_i(z), p) \leq \frac{\epsilon}{2}$ for sufficiently large i depending only on ϵ, X and δ and not on the chosen z since the convergence of v_i to v is uniform in $D_{1-\delta}$. Therefore, $v^{-1}(B_{\frac{\epsilon}{4}}(p)) \cap D_{1-\delta} \subset v_i^{-1}(B_{\frac{\epsilon}{2}}(p)) \cap D_{1-\delta}$ for sufficiently large i and

$$\begin{aligned} \int_{v^{-1}(B_{\frac{\epsilon}{4}}(p)) \cap D_{1-\delta}} \mid \nabla v_i \mid^2 d\mu &\leq \int_{v_i^{-1}(B_{\frac{\epsilon}{2}}(p)) \cap D_{1-\delta}} \mid \nabla v_i \mid^2 d\mu \\ &\leq E^{v_i}[v_i^{-1}(B_{\frac{\epsilon}{2}}(p))]. \end{aligned}$$

By the lower semicontinuity of the energy functional (cf. [**KS1**] Theorem 1.6.1), (43) and (44), we conclude that

$$\int_{v^{-1}(B_{\frac{\epsilon}{4}}(p)) \cap D_{1-\delta}} \mid \nabla v \mid^2 d\mu = 0.$$

Therefore,

$$E^v[v^{-1}(B_{\frac{\epsilon}{4}}(p))] = 0$$

by the Lebesgue Dominated Convergence Theorem which in turn implies that v must be constant on each connected component of $v^{-1}(B_{\frac{\epsilon}{4}}(p))$. In particular, it must be identically equal to p on the component K of $v^{-1}(B_{\frac{\epsilon}{4}}(p))$ containing 0. The continuity of v implies that $v^{-1}(p)$ is closed and hence K is closed. Since K is both open and closed, $K = D$. Therefore, v and hence φ is identically equal to p.

On the other hand, the triangle inequality says

$$d^2(\varphi_i, p) \leq 2d^2(\varphi_i, \varphi) + 2d^2(\varphi, p)$$

and hence

$$2\pi\epsilon \leq 2\int_{\partial D} d^2(\varphi_i, \varphi) d\theta + 2\int_{\partial D} d^2(\varphi, p) d\theta.$$

Letting $i \to 0$, we obtain

$$2\pi\epsilon \leq 2\int_{\partial D} d^2(\varphi, p) d\theta = 0,$$

a contradiction. q.e.d.

The proof of boundary regularity is an easy application of Lemma 10.

Proof of Theorem 9. Suppose a Dirichlet solution $u : D \to X$ with a continuous trace $\gamma : \partial D \to X$ is not continuous at some point $x_0 \in \partial D$. There exists $\epsilon > 0$ and $x_i \to x_0$ with

$$d(u(x_i), \gamma(x_0)) > 2\epsilon. \tag{45}$$

By an easy modification of the Courant-Lebesgue lemma for our setting and the continuity of γ, we may choose $\delta_i \to 0$ such that u restricted to $\partial D_{\delta_i}(x_0) \cap D$ is continuous and the length of the curve

$$\Gamma_i := u(\partial D_{\delta_i}(x_0) \cap D) \cup \gamma(D_{\delta_i}(x_0) \cap \partial D)$$

converges to 0 as $i \to \infty$. This combined with (45) implies that

$$d(\Gamma_i, u(x_i)) > \epsilon \tag{46}$$

for sufficiently large i. By choosing subsequence if necessary, assume that $x_i \in D_{\delta_i}(x_0) \cap D$. By the Riemann Mapping Theorem, there exists a conformal map ψ_i from $D_{\delta_i}(x_0) \cap D$ to D which sends x_i to 0. Let

$$v_i = u \circ \psi_i^{-1} : D \to X \quad \text{and} \quad \varphi_i = Tr(v_i).$$

Note that $v_i(0) = u_i(x_i)$ and the image of φ_i is Γ_i. Furthermore, (46) implies that $d(v_i(0), \varphi_i(\partial D)) > \epsilon$. Thus, Lemma 10 says there exists $\eta > 0$ such that $E^{v_i}[v_i^{-1}(B_\epsilon(v_i(0)))] \geq \eta$ for all i. By conformal invariance of energy (i.e. (18)), $E^u[D_{\delta_i}(x_0) \cap D] \geq \eta$. However, since $u \in W^{1,2}(D, X)$ (and hence $|\nabla u|^2 \in L^1(D)$), we see that $E^u[D_{\delta_i}(x_0) \cap D] \to 0$ as $i \to \infty$ and we arrive at our contradiction. q.e.d.

4. The Plateau Problem

4.1. The pull-back inner product. Before we can properly state the Plateau Problem for an Alexandrov space, we must formulate a notion of area. Our definition is analogous to the usual definition of the area functional for a map from a surface into a Riemannian manifold; in other words, it is obtained by integrating the area element of the pull-back metric. Thus, we first need to generalize the notion of the pull-back metric in this setting. This is accomplished by (47) and Theorem 11 below.

Let Ω be a Riemannian domain, X an Alexandrov space and $u \in W^{1,2}(\Omega, X)$. (Note that we do not need to assume X is finite dimensional or satisfies Perel'man's conjecture in this subsection.) For $Z, W \in \Gamma(T\overline{\Omega})$ (i.e. Z, W are Lipschitz vector fields on $\overline{\Omega}$), we define

$$\pi(Z, W) = \frac{1}{4}|u_*(Z + W)|^2 - \frac{1}{4}|u_*(Z - W)|^2. \tag{47}$$

If (Ω, g) has local coordinates $(x^1, x^2, \ldots, x^n)$ and corresponding tangent basis $\{\partial_1, \partial_2, \ldots, \partial_n\}$, we write

$$\pi_{ij} = \pi(\partial_i, \partial_j).$$

We show in Theorem 11 below that π generalizes the notion of the pull-back metric. The analogous result for the case when X is a NPC (non-positively curved) space is proven in [**KS1**] and the case when the curvature of X is bounded from above is proven in [**Me2**].

Theorem 11. *Let X be an Alexandrov space and $u \in W^{1,2}(\Omega, X)$. The operator*

$$\pi : \Gamma(T\overline{\Omega}) \times \Gamma(T\overline{\Omega}) \to L^1(\Omega, R)$$

defined above (and referred to as the inner product associated with u) is continuous, symmetric, bilinear, non-negative and tensorial; more specifically

$$\begin{aligned} \pi(Z,Z) &= |u_*(Z)|^2 \\ \pi(Z,W) &= \pi(W,Z) \\ \pi(Z, hV+W) &= h\pi(Z,V) + \pi(Z,W) \text{ for any } h \in C^{0,1}(\overline{\Omega}). \end{aligned}$$

For $Z = Z^i\partial_i$ and $W = W^i\partial_j$, we have

$$\pi(Z,W) = \pi_{ij} Z^i W^j.$$

If $\psi : \Omega_1 \to \Omega$ is a $C^{1,1}$ map, then writing $v = u \circ \psi$ and π_v for the inner product associated with v, we have the formula

$$(\pi_v)_{ij} = \pi_{lm} \frac{\partial \psi^l}{\partial x^i} \frac{\partial \psi^m}{\partial x^j}. \tag{48}$$

Hence in local coordinates,

$$|\nabla u|^2 = g^{ij} \pi_{ij} \tag{49}$$

where (g^{ij}) is (as usual) the inverse matrix to the Riemannian metric matrix (g_{ij}).

Proof. Assuming Proposition 12 below, we can follow the proof of Theorem 2.3.2 of [**KS1**] to prove Theorem 11. q.e.d.

Proposition 12. *Let Ω be a Riemannian domain and let X be an Alexandrov space. If $u \in W^{1,2}(\Omega, X)$, then for any $Z, W \in \Gamma(\overline{T\Omega})$ the parallelogram identity*

$$|u_*(Z+W)|^2 + |u_*(Z-W)|^2 = 2|u_*(Z)|^2 + 2|u_*(W)|^2 \tag{50}$$

holds.

Proof of Proposition 12. Recall that for any $Z, W \in \Gamma(\overline{T\Omega})$, we denote by $x + \epsilon Z$ the flow along V with initial point $x \in \Omega$ at time ϵ and

$$D_\epsilon(Z,W) := \frac{d(u(x+\epsilon Z), u(x+\epsilon W))}{\epsilon}.$$

Now fix $f \in C_c(\Omega), f \geq 0$ and $Z, W \in \Gamma(\overline{\Omega})$. Let

$$\begin{aligned}
\Omega^+ &= \big\{x \in \mathrm{spt} f : |u_*(Z)|^2, |u_*(W)|^2, \\
&\qquad\qquad |u_*(Z+W)|^2, |u_*(Z-W)|^2 \neq 0\big\}, \\
\Omega_N &= \left\{x \in \mathrm{spt} f : \frac{1}{2N} < |u_*(Z)|^2, |u_*(W)|^2, \right. \\
&\qquad\qquad \left. |u_*(Z+W)|^2, |u_*(Z-W)|^2 < \frac{N}{2}\right\}, \\
F(x,\epsilon) &= 2D_\epsilon^2\left(Z, \frac{Z+W}{2}\right) + 2D_\epsilon^2\left(W, \frac{Z+W}{2}\right) + D_\epsilon^2(0, Z+W) \\
&\quad - D_\epsilon^2(0,Z) - D_\epsilon^2(Z, Z+W) - D_\epsilon^2(W, Z+W) - D_\epsilon^2(0,W).
\end{aligned}$$

We claim the following:

Claim 1. $\mu(\Omega^+ \backslash \Omega_N) \to 0$ as $N \to \infty$.

Claim 2. Fix N. For any $\rho > 0$, let $\delta(\rho)$ be as in Lemma 1. Then there exists a function $G_\rho(x,\epsilon)$ so that if the following three inequalities:

$$\begin{aligned}
&\frac{1}{N} < D_\epsilon(0, Z+W), D_\epsilon(0,Z), D_\epsilon(0,W), \\
&\qquad D_\epsilon(Z,W), D_\epsilon(Z, Z+W), D_\epsilon(W, Z+W) < N
\end{aligned} \tag{51}$$

$$\left|\frac{1}{2}D_\epsilon(0, Z+W) - D_\epsilon\left(0, \frac{Z+W}{2}\right)\right| < \delta(\rho)^2 D_\epsilon(0, Z+W) \tag{52}$$

$$\left|\frac{1}{2}D_\epsilon(0, Z+W) - D_\epsilon\left(Z+W, \frac{Z+W}{2}\right)\right| < \delta(\rho)^2 D_\epsilon(0, Z+W) \tag{53}$$

are satisfied for $\epsilon > 0$ and $x \in \Omega_N$, then

$$F(x,\epsilon) \geq G_\rho(x,\epsilon). \tag{54}$$

Furthermore, there exists a function $G_\rho(x)$ so that

$$\lim_{\epsilon \to 0} \int_{\Omega_N} f(x)|G_\rho(x,\epsilon)|d\mu = \int_{\Omega_N} f(x)|G_\rho(x)|d\mu + O(\rho^2) \tag{55}$$

and

$$\lim_{\rho \to 0} \int_{\Omega_N} f(x)|G_\rho(x)|d\mu = 0. \tag{56}$$

Claim 3. For $x \in \Omega - \Omega^+$, the parallelogram identity (50) holds.

Assuming the validity of the three claims, we prove the parallelogram identity as follows. Let $f \in C_c(\Omega)$ be a non-negative function and fix

$\eta > 0$. By Lemma 4, there exists $\epsilon_0, \delta > 0$ so that for any $\tilde{\Omega}$ with $\mu(\tilde{\Omega}) < \delta$ and $\epsilon < \epsilon_0$, we have

$$\int_{\tilde{\Omega}} fF(x,\epsilon) > -\eta.$$

By Lemmas 2 and 3,

$$D_\epsilon(0, Z+W) \to |u_*(Z+W)|,\ D_\epsilon(0,Z) \to |u_*(Z)|,$$

$$D_\epsilon(0,W) \to |u_*(W)|, D_\epsilon(Z,W) \to |u_*(Z-W)|,$$

$$D_\epsilon(Z, Z+W) \to |u_*(W)|,\ D_\epsilon(W, Z+W) \to |u_*(Z)|$$

pointwise almost everywhere. By Egoroff's Theorem, there exists set a A so that $\mu(A) < \frac{\delta}{2}$ and these convergences are uniform on $\Omega - A$. By Claim 1, there exists N sufficiently large so that $\mu(\Omega^+ \backslash \Omega_N) < \frac{\delta}{2}$. Hence,

$$\int_{(\Omega^+ \backslash \Omega_N) \cup A} f(x)F(x,\epsilon) > -\eta.$$

For $\rho > 0$, the uniform convergence implies that there exists $\epsilon_0 > 0$ sufficiently small so that (51), (52) and (53) hold for all $\epsilon < \epsilon_0$ and all $x \in \Omega_N \backslash A$. Thus, by Claim 2 (54),

$$\begin{aligned}
&\int_{\Omega^+} f(x)F(x,\epsilon)d\mu \\
&\quad = \int_{(\Omega^+ \backslash \Omega_N) \cup A} f(x)F(x,\epsilon)d\mu + \int_{\Omega_N \backslash A} f(x)F(x,\epsilon)d\mu \\
&\quad \geq -\eta + \int_{\Omega_N \backslash A} f(x)G_\rho(x,\epsilon)d\mu \\
&\quad \geq -\eta - \int_{\Omega_N \backslash A} f(x)|G_\rho(x,\epsilon)|d\mu.
\end{aligned}$$

Take $\epsilon \to 0$ and apply Lemma 2, Lemma 3 and Claim 2 (55) to obtain

$$\begin{aligned}
&\int_{\Omega^+} f\big(|u_*(Z+W)|^2 + |u_*(Z-W)|^2 - 2|u_*(Z)|^2 - 2|u_*(W)|^2\big)d\mu \\
\geq\ & -\eta - \textstyle\int_\Omega f(x)|G_\rho(x)| + O(\rho^2).
\end{aligned}$$

Now by taking $\rho \to 0$, applying Claim 2 (56) and noting that η can be made arbitrarily small, we obtain

$$\int_{\Omega^+} f(|u_*(Z+W)|^2 + |u_*(Z-W)|^2 - 2|u_*(Z)|^2 - 2|u_*(W)|^2)d\mu \geq 0.$$

Combined with Claim 3,

$$\int_{\Omega} f(|u_*(Z+W)|^2 + |u_*(Z-W)|^2 - 2|u_*(Z)|^2 - 2|u_*(W)|^2)d\mu \geq 0.$$

Replacing Z and W by $\frac{Z+W}{2}$ and $\frac{Z-W}{2}$ respectively in the above argument, we obtain

$$\int_\Omega f(2|u_*(Z)|^2 + 2|u_*(W)|^2 - |u_*(Z+W)|^2 - |u_*(Z-W)|^2) d\mu \geq 0.$$

Finally, since the choice of f is arbitrary, we obtain the parallelogram identity. q.e.d.

We are now left to prove the three claims.

Proof of Claim 1. If

$$\begin{aligned}
\Omega^{\leq \frac{1}{N}} &= \big\{x \in \mathrm{spt} f : 0 < \min\{|u_*(Z)|^2, |u_*(W)|^2, \\
&\qquad |u_*(Z+W)|^2, |u_*(Z-W)|^2\} \leq (2N)^{-1}\big\} \\
\Omega^{\geq N} &= \big\{x \in \mathrm{spt} f : \max\{|u_*(Z)|^2, |u_*(W)|^2, \\
&\qquad |u_*(Z+W)|^2, |u_*(Z-W)|^2\} \geq N/2\big\},
\end{aligned}$$

then $\Omega^+ \backslash \Omega_N = \Omega^{\leq \frac{1}{N}} \cup \Omega^{\geq N}$. Since

$$\Omega^{<\frac{1}{N+1}} \subset \Omega^{<\frac{1}{N}} \text{ and } \cap_{N=1}^\infty \Omega^{<\frac{1}{N}} = \emptyset,$$

we have that $\mu(\Omega^{<\frac{1}{N}}) \to 0$ as $N \to 0$. Furthermore,

$$\begin{aligned}
\frac{N}{2}\mu(\Omega^{>N}) \leq \int_{\Omega^{>N}} & |u_*(Z)|^2 + |u_*(W)|^2 \\
& + |u_*(Z+W)|^2 + |u_*(Z-W)|^2 < \infty.
\end{aligned}$$

which implies $\mu(\Omega^{>N}) \to 0$ as $N \to 0$. This ends the proof of Claim 1.

Proof of Claim 2. For $x \in \Omega$ and $\epsilon > 0$, assume (51), (52) and (53) are satisfied and let

(57)
$$\begin{aligned}
P = u(x), \qquad & Q = u(x+\epsilon Z) \qquad R = u(x+\epsilon(Z+W)), \\
S = u(x+\epsilon W), \quad & T = u\left(x + \epsilon\left(\tfrac{Z+W}{2}\right)\right).
\end{aligned}$$

The inequalities (52) and (53) imply

$$\left|\frac{1}{2}d_{PR} - d_{PT}\right| < \delta^2(\rho) d_{PR}\ , \ \left|\frac{1}{2}d_{PR} - d_{RT}\right| < \delta^2(\rho) d_{PR}.$$

Let γ_{RT} be a geodesic from R to T and R' be a point on γ_{RT} so that

$$d_{RR'} = \delta(\rho) d_{PR}. \tag{58}$$

Let $\gamma_{PR'}$ be a geodesic from P to R' and T' be its midpoint. By Lemma 1, we have

$$d_{TT'} < \rho d_{PR}.$$

Define γ to be the curve which is the sum of geodesics from $Q = u(x + \epsilon Z)$ to T' and from T' to $S = u(x+\epsilon W)$. Let $\bar{d}$ be the distance function in the hyperbolic plane $\mathbf{H}^2$ and construct points $\bar{P}, \bar{Q}, \bar{R}', \bar{S} \in \mathbf{H}^2$ with the property that

$$d_{PQ} = \bar{d}_{\bar{P}\bar{Q}},\ d_{QR'} = \bar{d}_{\bar{Q}\bar{R}'},\ d_{R'S} = \bar{d}_{\bar{R}'\bar{S}},\ d_{SP} = \bar{d}_{\bar{S}\bar{P}},\ d_{PR'} = \bar{d}_{\bar{P}\bar{R}'} \tag{59}$$

and so that geodesic triangles $\triangle \bar{P}\bar{Q}\bar{R}'$ and $\triangle \bar{P}\bar{S}\bar{R}'$ intersect only along the geodesic $\bar{\gamma}_{\bar{P}\bar{R}'}$ from $\bar{P}$ to $\bar{R}'$. If $\bar{T}'$ is the midpoint of $\bar{\gamma}_{\bar{P}\bar{R}'}$,

$$\bar{d}_{\bar{Q}\bar{T}'} \leq d_{QT'}\ \bar{d}_{\bar{T}'\bar{S}} \leq d_{T'S} \tag{60}$$

by the property of an Alexandrov space. Hence

$$\bar{d}_{\bar{Q}\bar{S}} \leq \bar{d}_{\bar{Q}\bar{T}'} + \bar{d}_{\bar{T}'\bar{S}} \leq d_{QT'} + d_{T'S}.$$

Therefore, if

$$E(x,\epsilon) := \bar{d}^2_{\bar{Q}\bar{S}} + \bar{d}^2_{\bar{P}\bar{R}'} - \bar{d}^2_{\bar{P}\bar{Q}} - \bar{d}^2_{\bar{Q}\bar{R}'} - \bar{d}^2_{\bar{R}'\bar{S}} - \bar{d}^2_{\bar{P}\bar{S}},$$

then

$$\begin{aligned} E(x,\epsilon) \quad \leq \quad & L^2(\gamma) + d^2(u(x), R^{'}) - d^2(u(x), u(x+\epsilon Z)) \\ & - d^2(R^{'}, u(x+\epsilon Z)) - d^2(R^{'}, u(x+\epsilon W)) \\ & - d^2(u(x), u(x+\epsilon W)). \end{aligned}$$

Dividing by ϵ^2, we obtain

$$\begin{aligned} & \frac{E(x,\epsilon)}{\epsilon^2} \\ & \leq \quad \left(\frac{L^2(\gamma)}{\epsilon^2} - D^2_\epsilon(0,Z) - D^2_\epsilon(0,W)\right) \\ & \qquad + \left(\frac{d^2(u(x),R^{'})}{\epsilon^2} - \frac{d^2(R^{'},u(x+\epsilon W))}{\epsilon^2} - \frac{d^2(R^{'},u(x+\epsilon Z))}{\epsilon^2}\right) \\ & =: \quad (I) + (II). \end{aligned} \tag{61}$$

Hence, by the triangle inequality, we have,

$$\begin{aligned} L(\gamma) \quad & = \quad d_{QT'} + d_{T'S} \\ & \leq \quad d_{QT} + d_{TS} + 2d_{TT'} \\ & \leq \quad d_{QT} + d_{TS} + 2\rho d_{PR} \\ & = \quad d\left(u(x+\epsilon Z), u\left(x + \epsilon\frac{Z+W}{2}\right)\right) \\ & \qquad + d\left(u(x+\epsilon W), u\left(x + \epsilon\frac{Z+W}{2}\right)\right) \\ & \qquad + 2\rho d(u(x), u(x+(\epsilon Z + W)). \end{aligned}$$

If we square this inequality, divide by ϵ^2 and assume that $\rho << 1$, we have

$$\begin{aligned}
\frac{L^2(\gamma)}{\epsilon^2} &\leq D_\epsilon^2\left(Z, \frac{Z+W}{2}\right) + D_\epsilon^2\left(W, \frac{Z+W}{2}\right) \\
&\quad +2D_\epsilon^2\left(Z, \frac{Z+W}{2}\right) D_\epsilon^2\left(W, \frac{Z+W}{2}\right) \\
&\quad +4\rho D_\epsilon(0, Z+W)\left(D_\epsilon\left(Z, \frac{Z+W}{2}\right) + D_\epsilon\left(W, \frac{Z+W}{2}\right)\right) \\
&\quad +4\rho^2 D_\epsilon^2(0, Z+W) \\
&\leq 2D_\epsilon^2\left(Z, \frac{Z+W}{2}\right) + 2D_\epsilon^2\left(W, \frac{Z+W}{2}\right) \\
&\quad +8\rho\Bigg(D_\epsilon^2(0, Z+W) + D_\epsilon^2\left(Z, \frac{Z+W}{2}\right) \\
&\qquad\qquad +D_\epsilon^2\left(W, \frac{Z+W}{2}\right)\Bigg)
\end{aligned}$$

which immediately implies

$$\begin{aligned}
(I) \;&\leq\; 2D_\epsilon^2\left(Z, \frac{Z+W}{2}\right) + 2D_\epsilon^2\left(W, \frac{Z+W}{2}\right) - D_\epsilon^2(0, Z) - D_\epsilon^2(0, W) \\
&\quad +8\rho\left(D_\epsilon^2(0, Z+W) + D_\epsilon^2\left(Z, \frac{Z+W}{2}\right) + D_\epsilon^2\left(W, \frac{Z+W}{2}\right)\right). \qquad (62)
\end{aligned}$$

Furthermore, assuming $\rho << 1$, we also obtain

$$d_{PR'}^2 \leq (d_{PR} + d_{RR'})^2 = (1+\delta(\rho))^2 d_{PR}^2 \leq (1+3\delta(\rho)) d_{PR}^2$$

$$\begin{aligned}
d_{QR'}^2 &\geq (d_{QR} - d_{RR'})^2 = (d_{QR} - \delta(\rho) d_{PR})^2 \\
&\geq d_{QR}^2 - 2\delta(\rho) d_{QR} d_{PR} \geq (1-\delta(\rho)) d_{QR}^2 - \delta(\rho) d_{PR}^2
\end{aligned}$$

$$\begin{aligned}
d_{SR'}^2 &\geq (d_{SR} - d_{RR'})^2 = (d_{SR} - \delta(\rho) d_{PR})^2 \\
&\geq d_{SR}^2 - 2\delta(\rho) d_{SR} d_{PR} \geq (1-\delta(\rho)) d_{SR}^2 - \delta(\rho) d_{PR}^2,
\end{aligned}$$

which immediately implies

$$\begin{aligned}
\frac{d^2(u(x), R')}{\epsilon^2} &\leq (1+3\delta(\rho)) D_\epsilon^2(0, Z+W) \\
-\frac{d^2(u(x+\epsilon Z), R')}{\epsilon^2} &\leq -(1-\delta(\rho)) D_\epsilon^2(Z, Z+W)) + \delta(\rho) D_\epsilon^2(0, Z+W) \\
-\frac{d^2(u(x+\epsilon Z), R')}{\epsilon^2} &\leq -(1-\delta(\rho)) D_\epsilon^2(W, Z+W)) + \delta(\rho) D_\epsilon^2(0, Z+W).
\end{aligned}$$

These combine to give

$$\begin{aligned}
(II) \;&\leq\; D_\epsilon^2(0, Z+W) - D_\epsilon^2(Z, Z+W) - D_\epsilon^2(W, Z+W) \\
&\quad +5\delta(\rho)(D_\epsilon^2(0, Z+W) + D_\epsilon^2(Z, Z+W) + D_\epsilon^2(W, Z+W)). \qquad (63)
\end{aligned}$$

Combining (61), (62) and (63), we obtain

$$\begin{aligned}
&\frac{E(x,\epsilon)}{\epsilon^2}\\
&\leq \; 2D_\epsilon^2\left(Z, \frac{Z+W}{2}\right) + 2D_\epsilon^2\left(W, \frac{Z+W}{2}\right) + D_\epsilon^2(0, Z+W)\\
&\quad -D_\epsilon^2(0,Z) - D_\epsilon^2(0,W) - D_\epsilon^2(Z, Z+W) - D_\epsilon^2(W, Z+W)\\
&\quad +8\rho\left(D_\epsilon^2(0, Z+W) + D_\epsilon^2\left(Z, \frac{Z+W}{2}\right) + D_\epsilon^2\left(W, \frac{Z+W}{2}\right)\right)\\
&\quad +5\delta(\rho)(D_\epsilon^2(0, Z+W) + D_\epsilon^2(Z, Z+W) + D_\epsilon^2(W, Z+W))\\
&\leq \; F(x,\epsilon)\\
&\quad +8\rho\left(D_\epsilon^2(0, Z+W) + D_\epsilon^2\left(Z, \frac{Z+W}{2}\right) + D_\epsilon^2\left(W, \frac{Z+W}{2}\right)\right)\\
&\quad +5\delta(\rho)(D_\epsilon^2(0, Z+W) + D_\epsilon^2(Z, Z+W) + D_\epsilon^2(W, Z+W)).
\end{aligned}$$

Let

$$\begin{aligned}
&G_1(x,\epsilon)\\
&\quad := \; -8\rho(D_\epsilon^2(0, Z+W) + D_\epsilon^2\left(Z, \frac{Z+W}{2}\right) + D_\epsilon^2\left(W, \frac{Z+W}{2}\right)\\
&\qquad\quad -5\delta(\rho)(D_\epsilon^2(0, Z+W) - D_\epsilon^2(Z, Z+W) - D_\epsilon^2(W, Z+W))
\end{aligned}$$

Inequality (51) implies that

$$\frac{\epsilon}{N} < d_{PQ}, d_{QR}, d_{RS}, d_{PS}, d_{PR}, d_{QS} < N\epsilon.$$

By also using the fact that $d_{RR'} = \delta(\rho) d_{PR} \leq \rho N \epsilon$, we can apply Lemma 18 of the Appendix to obtain,

$$\begin{aligned}
&\left|\frac{E(x,\epsilon)}{\epsilon^2}\right|\\
&\leq \; C_N \left(\left|D_\epsilon^2(V, V+W) - D^2(0,W)\right| + \left|D_\epsilon^2(0,V) - D^2(W, V+W)\right|\right.\\
&\quad \left. + |D_\epsilon(V, V+W) - D(0,W)| + |D_\epsilon(V, V+W) - D(0,W)|\right)\\
&\quad +K_1\rho^2 + K_2\epsilon
\end{aligned}$$

for some constants K_1, K_2 sufficiently large. Define $G_2(x,\epsilon)$ to be the right hand side of the inequality above. Thus, (54) holds if we set $G_\rho(x,\epsilon) = G_1(x,\epsilon) - G_2(x,\epsilon)$. Furthermore, set

$$\begin{aligned}
G_\rho(x) \;:=\; & -8\rho\left(|u_*(Z+W)|^2 + \frac{1}{2}|u_*(Z-W)|\right)\\
& -5\delta(\rho)\left(|u_*(Z+W)|^2 + |u_*(W)|^2 + |u_*(Z)|^2\right) + O(\rho^2).
\end{aligned}$$

Then (55) and (56) hold by Lemmas 2 and 3. This ends the proof of Claim 2.

Proof of Claim 3. Let Ω_0 denote the set of all points in Ω so that $|u_*(Z+W)|^2=0$. If P,Q,R,T be as in (57). Then

$$d_{QT}^2 - d_{PQ}^2 = (d_{QT}-d_{PQ})(d_{QT}+d_{PQ}) \le d_{PT}(d_{QT}+d_{PQ}).$$

Thus, for any $f \in C_c(\Omega_0)$, $0 \le f \le 1$,

$$\begin{aligned}
&\int_{\Omega_0} f\left(D_\epsilon^2\left(Z, \frac{Z+W}{2}\right) - D_\epsilon^2(0,Z)\right)d\mu \\
\le\ & \int_{\Omega_0} f D_\epsilon\left(0, \frac{Z+W}{2}\right)\left(D_\epsilon\left(Z, \frac{Z+W}{2}\right) + D_\epsilon(0,Z)\right) \\
\le\ & \left(\int_{\Omega_0} f D_\epsilon^2\left(0, \frac{Z+W}{2}\right)\right)^{1/2} \\
& \times \left(\int_{\Omega} f D_\epsilon^2\left(Z, \frac{Z+W}{2}\right) + \int_{\Omega_0} f D_\epsilon^2(0,Z)\right)^{1/2}.
\end{aligned}$$

We take the limit as ϵ goes to 0 to obtain

$$\begin{aligned}
&\int_{\Omega_0} f\left(\left|u_*\left(\frac{-Z+W}{2}\right)\right|^2 - |u_*(Z)|^2\right) \\
\le\ & \left(\int_{\Omega_0} f\left|u_*\left(\frac{Z+W}{2}\right)\right|^2\right)^{1/2} \\
& \times \left(\int_{\Omega_0} f\left|u_*\left(\frac{-Z+W}{2}\right)\right|^2 + |u_*(Z+W)|^2\right)^{1/2} \\
\le\ & \left(\frac{1}{4}\int_{\Omega_0} f|u_*(Z+W)|^2\right)^{1/2} \\
& \times \left(\int_{\Omega_0} f\left|u_*\left(\frac{-Z+W}{2}\right)\right|^2 + |u_*(Z+W)|^2\right)^{1/2} \\
=\ & 0.
\end{aligned}$$

Thus we arrive at

$$\left|u_*\left(\frac{Z-W}{2}\right)\right|^2 \le |u_*(Z)|^2 \quad \text{a.e. } x \in \Omega_0.$$

Similarly, using

$$d_{PQ}^2 - d_{QT}^2 = (d_{PQ}-d_{QT})(d_{PQ}+d_{QT}) \le d_{PT}(d_{PQ}+d_{QT}),$$

we obtain the opposite inequality. Hence, we conclude

$$\left|u_*\left(\frac{Z-W}{2}\right)\right|^2 = |u_*(Z)|^2 \quad \text{a.e. } x \in \Omega_0.$$

Interchanging Z and W in the argument above, we also obtain

$$\left|u_*\left(\frac{Z-W}{2}\right)\right|^2 = |u_*(W)|^2 \quad \text{a.e. } x \in \Omega_0.$$

Therefore,

$$\begin{aligned} &|u_*(Z+W)|^2 + |u_*(Z-W)|^2 \\ &= 0 + 4\left|u_*\left(\frac{Z-W}{2}\right)\right|^2 = 2|u_*(Z)|^2 + 2|u_*(W)|^2 \end{aligned}$$

for a.e. $x \in \Omega_0$. Similar arguments apply when we examine points of Ω where the other directional energy measures vanish. This ends the proof of Claim 3.

4.2. The Plateau Problem. We can define the area functional for $u \in W^{1,2}(D,X)$ by

$$A(u) = \int_D \sqrt{\det \pi}\; dxdy = \int_D \sqrt{\pi_{11}\pi_{22} - \pi_{12}^2}\; dxdy.$$

where $\pi_{11} = \pi(\frac{\partial}{\partial x}, \frac{\partial}{\partial x})$, $\pi_{12} = \pi(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ and $\pi_{22} = \pi(\frac{\partial}{\partial y}, \frac{\partial}{\partial y})$. The Plateau Problem for an Alexandrov space is formulated as:

The Plateau Problem *Le X be an Alexandrov space, Γ a closed Jordan curve in X and*

$$\mathcal{F}_\Gamma = \{u \in W^{1,2}(D,X) \cap C^0(\overline{D},X) : \\ u|_{\partial D} \textit{ parametrizes } \Gamma \textit{ monotonically}\}.$$

Find $u \in \mathcal{F}_\Gamma$ so that $A(u) = \inf\{A(v) : \; v \in \mathcal{F}_\Gamma\}$.

The main result of this section is that we can solve the Plateau Problem if there exists at least one continuous finite energy map whose trace monotonically parametrizes Γ.

Theorem 13. *Let X be a compact Alexandrov space. If $\mathcal{F}_\Gamma \neq \emptyset$, there exsits $u \in \mathcal{F}_\Gamma$ so that $A(u) = \inf\{A(v) : \; v \in \mathcal{F}_\Gamma\}$.*

We separate the proof of Theorem 13 into two claims. **Claim 1** is that there exists a map which minimizes the energy functional in $\mathcal{F}_\Gamma$. **Claim 2** is that an energy minimizing map is also an area minimizer. These claims are proved by an extending the arguments used for the Euclidean case (cf. [**L**]). In order to prove the first claim, we need the following lemma.

Lemma 14. *Fix $\xi_1, \xi_2, \xi_3 \in \partial D$ and $P_1, P_2, P_3 \in \Gamma$. If*

$$\mathcal{F}_\Gamma' = \{u \in \mathcal{F}_\Gamma : u(\xi_i) = P_i \textit{ for } \; i = 1,2,3 \textit{ and } E^u \leq 2 \inf_{u \in \mathcal{F}_\Gamma} E(u)\},$$

then

$$F = \{u|_{\partial D} : u \in \mathcal{F}_\Gamma' \;,\; E^u \leq 2 \inf_{u \in \mathcal{F}_\Gamma} E(u)\}$$

forms an equicontinuous family of maps.

Proof. This follows from the same argument given in Proposition 6 of [**L**]. q.e.d.

We now prove the first claim:

Claim 1. There exists $u \in \mathcal{F}_\Gamma$ so that $E^u = \inf_{v\in\mathcal{F}_\Gamma} E^v$.

Proof. For any $v \in \mathcal{F}_\Gamma$, there exists a Möbius transformation ψ so that $v \circ \psi(\xi_i) = P_i$ for $i = 1, 2, 3$. Furthermore, $E^{v\circ\psi} = E^v$ by (18). Therefore,

$$\inf_{u\in\mathcal{F}_\Gamma} E^u = \inf_{u\in\mathcal{F}'_\Gamma} E^u.$$

which implies

$$\inf_{u\in\mathcal{F}_\Gamma} E^u = \inf_{\phi\in\mathcal{F}'_\Gamma} E_\phi.$$

where

$$E_\phi = \inf\{E^v : v \in W^{1,2}_\phi(D, X)\}.$$

Let $\{v_m\} \subset \mathcal{F}'_\Gamma$ be a sequence so that $\lim_{m\to\infty} E_{v_m} = \inf_{v\in\mathcal{F}_\Gamma} E^v$. By the equicontinuity of F (i.e. Lemma 14), there exists a subsequence $\{v_{m'}\}$ so that $\{v_{m'}\big|_{\partial D}\}$ converges uniformly to a continuous map $\phi : \partial D \to \Gamma$. By the uniform convergence, we are guaranteed to have $\phi(p_i) = q_i$ for $i = 1, 2, 3$. Let $u_{m'}$ be the solution to the Dirichlet Problem for boundary data $v_{m'}$. By Theorem 9, $u_{m'} \in W^{1,2}(D, X) \cap C^0(D, X)$ and hence $u_{m'} \in \mathcal{F}_\Gamma$. From the precompactness theorem (cf. Theorem 1.13 of [**KS1**]), we may choose a subsequence of $u_{m'}$ (which we still denote $u_{m'}$ by an abuse of notation) which converges in $L^2(D, X)$ to $u \in W^{1,2}(D, X)$. By the lower semicontinuity of the energy functional (cf. Theorem 1.6.1 of [**KS1**]),

$$E^u \leq \liminf_{m'\to\infty} E^{u_{m'}} \leq \liminf_{m'\to\infty} E^{v_{m'}} = \inf_{v\in\mathcal{F}_\Gamma} E^v. \tag{64}$$

Since the trace functions converge in L^2 distance (cf. Theorem 1.12.2 in [**KS1**]), we have $Tr(u) = \phi$. Moreover, (64) implies that u is a Dirichlet solution for boundary data ϕ and hence $u \in W^{1,2}(D, X) \cap C^0(\overline{D}, X)$ by Theorem 9. Therefore $u \in \mathcal{F}'_\Gamma \subset \mathcal{F}_\Gamma$ and this concludes the proof of the claim. q.e.d.

We now claim that u obtained above not only minimizes energy in $\mathcal{F}_\Gamma$, but also minimizes the area functional. We need the following two lemmas.

Lemma 15. *If $u \in \mathcal{F}_\Gamma$ satisfies $E^u = \inf_{v\in\mathcal{F}'_\Gamma} E^v$, then u is weakly conformal; in other words, u satisfies the conformality relation $\pi_{11} = \pi_{22}$ and $\pi_{12} = 0$.*

Proof. This follows by adapting a well-known computation in the smooth setting to the current situation. This can be justified by the change of variables formula (48). q.e.d.

Lemma 16. *For any $v \in \mathcal{F}_\Gamma$ and $\delta > 0$, there exists a continuous map $\psi_0 : D \to D$ monotonically taking ∂D to ∂D such that $\frac{1}{2}E^{v \circ \psi_0} \leq A(v) + \delta$.*

Proof. Let $\mathbf{C}$ be the complex plane and let $X \times \mathbf{C}$ be the metric space equipped with the distance function

$$\overline{d}((P,z),(Q,w)) = \sqrt{d^2(P,Q) + |z-w|^2}$$

for $P, Q \in X$ and $z, w \in \mathbf{C}$. For $v \in \mathcal{F}_\Gamma$, consider $v_\sigma : D \to X \times \mathbf{C}$ defined by

$$v_\sigma(z) = (v(z), \sigma z).$$

By (19), we see that

$$\begin{aligned} |(v_\sigma)_*(V)|^2 &= \lim_{\epsilon \to 0} \frac{\overline{d}^2(v_\sigma(z), v_\sigma(z+\epsilon V))}{\epsilon^2} \\ &= \lim_{\epsilon \to 0} \frac{d^2(v(z), v(z+\epsilon V)) + |\sigma z - \sigma(z+\epsilon V)|^2}{\epsilon^2} \\ &= |v_*(V)|^2 + \sigma^2 |V|^2 \end{aligned}$$

for $V \in \Gamma(T\overline{D})$ and a.e. $z \in D$. Hence,

$$(\pi_{v_\sigma})_{ij} = (\pi_v)_{ij} + \frac{\sigma^2}{4}\delta_{ij} \tag{65}$$

where π_{v_σ} and π_v are the pull-back inner products associated with v_σ and v respectively and $\delta = (\delta_{ij})$ is the standard Euclidean metric on D. In particular, we have

$$(\pi_{v_\sigma})_{11}(\pi_{v_\sigma})_{22} - (\pi_{v_\sigma})_{12}^2 = (\pi_v)_{11}(\pi_v)_{22} - (\pi_v)_{12}^2 + \frac{\sigma^4}{16} \geq \frac{\sigma^4}{16} > 0$$

and $A(v_\sigma) \leq A(v) + \delta$ for $\sigma > 0$ sufficiently small.

We now fix σ_0 such that

$$A(v_{\sigma_0}) \leq A(v) + \delta$$

and let $\pi = (\pi_{ij})$ be the pull-back inner product associated with v_{σ_0}. Define the π-energy ${}^\pi E^\psi$ for a $W^{1,2}$-map $\psi : D \to D$ by setting

$$\int_D \pi_{ij}(\psi(x,y)) \left(\frac{\partial \psi^i}{\partial x}\frac{\partial \psi^j}{\partial x} + \frac{\partial \psi^i}{\partial y}\frac{\partial \psi^j}{\partial y} \right) dxdy.$$

Also let ${}^\delta E^\psi$ be the standard Dirichlet energy of ψ, i.e. the δ-energy defined analogously to the π-energy. Then by (65),

$$\frac{\sigma_0^2}{4}\, {}^\delta E^\psi \leq {}^\pi E^\psi. \tag{66}$$

Let $\mathcal{D}$ be the class of all diffeomorphisms $\psi : D \to D$ satisfying $\psi(\xi_i) = \xi_i$ for $i = 1, 2, 3$ (with ξ_i defined as in Lemma 14) and $\overline{\mathcal{D}}$ be the closure of $\mathcal{D}$ with respect to the uniform convergence. A π-energy minimizing sequence in $\overline{\mathcal{D}}$ has uniformly bounded δ-energy by (66) and hence is an equicontinuous family. Thus there exists a subsequence converging uniformly to a map $\psi_0 : D \to D$. In particular, the uniform convergence implies ψ_0 maps ∂D monotonically onto ∂D and the lower semicontinuity of the π-energy implies ${}^{\pi}E^{\psi_0} = \inf\{{}^{\pi}E^{\psi} : \psi \in \overline{\mathcal{D}}\}$. Let $\eta_t : D \to D$ be a family of diffeomorphisms depending smoothly on t with $\eta_0 =$ identity. The minimizing property of ψ_0 implies that

$$\frac{d}{dt}\,{}^{\pi}E^{\psi_0 \circ \eta_t}\Big|_{t=0} = 0$$

since the three point condition can be achieved by a Möbius transformation without changing energy. This then implies the conformality relations

$$\pi_{ij}\frac{\partial \psi_0^i}{\partial x}\frac{\partial \psi_0^j}{\partial x} = \pi_{ij}\frac{\partial \psi_0^i}{\partial y}\frac{\partial \psi_0^j}{\partial y} \text{ and } \pi_{ij}\frac{\partial \psi_0^i}{\partial x}\frac{\partial \psi_0^j}{\partial y} = 0$$

almost everywhere. Thus

$$\begin{aligned}
{}^{\pi}E^{\psi_0} &= \int_D \pi_{ij}(\psi_0(x,y))\left(\frac{\partial \psi_0^i}{\partial x}\frac{\partial \psi_0^j}{\partial x} + \frac{\partial \psi_0^i}{\partial y}\frac{\partial \psi_0^j}{\partial y}\right) dxdy \\
&= 2\int_D \Bigg(\pi_{ij}(\psi_0(x,y))\frac{\partial \psi_0^i}{\partial x}\frac{\partial \psi_0^j}{\partial x}\pi_{ij}(\psi_0(x,y))\frac{\partial \psi_0^i}{\partial y}\frac{\partial \psi_0^j}{\partial y} \\
&\qquad -\pi_{ij}(\psi_0(x,y))\frac{\partial \psi_0^i}{\partial x}\frac{\partial \psi_0^j}{\partial y}\pi_{ij}(\psi_0(x,y))\frac{\partial \psi_0^i}{\partial x}\frac{\partial \psi_0^j}{\partial y}\Bigg)^{1/2} dxdy \\
&= 2\int_D \sqrt{\pi_{11}\pi_{22} - \pi_{12}^2}\, dudv \\
&= 2A(v_{\sigma_0}).
\end{aligned}$$

For almost every $y \in (-1,1)$, $\psi_0\big|_{I_y}$ is in $W^{1,2} \cap C^{0,\frac{1}{2}}$ where $I_y \subset D$ is the horizontal line at y. Let $X = \frac{\partial}{\partial x}$. Since $\epsilon \mapsto \psi_0(x+\epsilon, y)$ is the flow starting at $\psi_0(x,y)$ along the (almost everywhere defined) vector field $(\psi_0)_*X$ at time ϵ, the ϵ-energy density functions of $v_{\epsilon_0} \circ \psi_0$ and v_{ϵ_0} satisfy the equality

$${}^{X}_{\nu}e_\epsilon(x,y) = {}^{(\psi_0)_*X}_{\nu}e_\epsilon(\psi_0(x,y)).$$

The same equality is true with X replaced by $Y = \frac{\partial}{\partial y}$. Thus, as in the proof of [**KS1**] Theorem 2.3.2 formula (2.3v), we see that

$$(\pi_{v_\sigma \circ \psi_0})_{ij} = \pi_{lm}\frac{\partial \psi_0^l}{\partial x^i}\frac{\partial \psi_0^m}{\partial x^j}$$

where $\pi_{v_\sigma \circ \psi_0}$ is the pull-back inner product associated with $v_\sigma \circ \psi_0$. This immediately implies $E^{v_{\sigma_0} \circ \psi_0} = {}^\pi E^{\psi_0}$. Therefore, we have

$$E^{v \circ \psi_0} \leq E^{v_{\sigma_0} \circ \psi_0} = {}^\pi E^{\psi_0} = 2A(v_{\sigma_0}) \leq 2(A(v) + \delta).$$

q.e.d.

We now prove our second claim.

Claim 2. *If $u \in \mathcal{F}_\Gamma$ satisfies $E^u = \inf_{v \in \mathcal{F}_\Gamma} E^v$, then $A(u) = \inf\{A(v) : v \in \mathcal{F}_\Gamma\}$.*

Proof. By the Cauchy-Schwarz lemma,

$$\sqrt{\pi_{11}\pi_{22} - (\pi_{12})^2} \leq \sqrt{\pi_{11}\pi_{22}} \leq \frac{1}{2}(\pi_{11} + \pi_{22})$$

with

$$\sqrt{\pi_{11}\pi_{22} - (\pi_{12})^2} = \frac{1}{2}(\pi_{11} + \pi_{22}) \iff \pi_{11} = \pi_{22} \text{ and } \pi_{12} = 0.$$

Since u satisfies the conformality equations by Lemma 15, we deduce that $A(u) = \frac{1}{2}E^u$. Given $v \in \mathcal{F}_\Gamma$ and $\delta > 0$, let ψ_0 be as in Lemma 16. Then $v \circ \psi_0 \in \mathcal{F}_\Gamma$ and

$$A(u) = \frac{1}{2}E^u \leq \frac{1}{2}E^{v \circ \psi_0} \leq A(u) + \delta.$$

Since δ can be chosen arbitrarily small, we are done. q.e.d.

In establishing the above claims, we have also shown:

Theorem 17. *Let X be an Alexandrov space. There exists a solution u of the Plateau Problem that is conformal and energy minimizing. Therefore, if X is compact Alexandrov space satisfying Perel'man's conjecture, then there exists a solution u of the Plateau Problem that is Hölder continuous in the interior of D and continuous up to ∂D.*

5. Appendix

We establish the following fact about quadrilaterals in hyperbolic plane. The purpose is to estimate the difference between the sum of the lengths of the diagonals and the sum of the lengths of the sides.

Lemma 18. *If $\bar{P}, \bar{Q}, \bar{R}', \bar{S} \in \mathbf{H}^2$ so that*

$$\frac{\epsilon}{N} \leq \bar{d}_{\bar{P}\bar{Q}}, \bar{d}_{\bar{Q}\bar{R}'}, \bar{d}_{\bar{R}'\bar{S}}, \bar{d}_{\bar{P}\bar{S}}, \bar{d}_{\bar{P}\bar{R}'}, \bar{d}_{\bar{Q}\bar{S}} \leq N\epsilon,$$

then

$$\begin{aligned}
&|\bar{d}^2_{\bar{Q}\bar{S}} + \bar{d}^2_{\bar{P}\bar{R}'} - \bar{d}^2_{\bar{P}\bar{Q}} - \bar{d}^2_{\bar{Q}\bar{R}'} - \bar{d}^2_{\bar{R}'\bar{S}} - \bar{d}^2_{\bar{P}\bar{S}}| \\
&\leq C_N \Big(|\bar{d}^2_{\bar{Q}\bar{R}'} - \bar{d}^2_{\bar{P}\bar{S}}| + |\bar{d}^2_{\bar{P}\bar{Q}} - \bar{d}^2_{\bar{R}'\bar{S}}| + \epsilon(|\bar{d}_{\bar{Q}\bar{R}'} - \bar{d}_{\bar{P}\bar{S}}| + |\bar{d}_{\bar{P}\bar{Q}} - \bar{d}_{\bar{R}'\bar{S}}|)\Big) \\
&\quad + O(\epsilon^3)
\end{aligned}$$

where C_N is a constant dependent on N and $O(\epsilon^3)$ has the property that $\frac{O(\epsilon^3)}{\epsilon^2} \to 0$ as $\epsilon \to 0$.

Proof. Let

$$E = \bar{d}^2_{\bar{Q}\bar{S}} + \bar{d}^2_{\bar{P}\bar{R}'} - \bar{d}^2_{\bar{P}\bar{Q}} - \bar{d}^2_{\bar{Q}\bar{R}'} - \bar{d}^2_{\bar{R}'\bar{S}} - \bar{d}^2_{\bar{P}\bar{S}}.$$

Define $\lambda, \delta \in [0, \pi]$ by

$$\cosh \bar{d}_{\bar{P}\bar{R}'} = \cosh \bar{d}_{\bar{Q}\bar{R}'} \cosh \bar{d}_{\bar{P}\bar{Q}} - \sinh \bar{d}_{\bar{Q}\bar{R}'} \sinh \bar{d}_{\bar{P}\bar{Q}} \cos \lambda$$
$$\cosh \bar{d}_{\bar{Q}\bar{S}} = \cosh \bar{d}_{\bar{Q}\bar{R}'} \cosh \bar{d}_{\bar{R}'\bar{S}} - \sinh \bar{d}_{\bar{Q}\bar{R}'} \sinh \bar{d}_{\bar{R}'\bar{S}} \cos \delta.$$

By Taylor series expansion, we obtain

$$\bar{d}^2_{\bar{P}\bar{R}'} = \bar{d}^2_{\bar{Q}\bar{R}'} + \bar{d}^2_{\bar{P}\bar{Q}} - 2\bar{d}_{\bar{Q}\bar{R}'}\bar{d}_{\bar{P}\bar{Q}} \cos \lambda + O(\epsilon^3)$$
$$\bar{d}^2_{\bar{Q}\bar{S}} = \bar{d}^2_{\bar{Q}\bar{R}'} + \bar{d}^2_{\bar{R}'\bar{S}} - 2\bar{d}_{\bar{Q}\bar{R}'}\bar{d}_{\bar{R}'\bar{S}} \cos \delta + O(\epsilon^3).$$

We have then

$$\begin{aligned} E &= \bar{d}^2_{\bar{Q}\bar{R}'} - \bar{d}^2_{\bar{P}\bar{S}} - 2\bar{d}_{\bar{Q}\bar{R}'}\bar{d}_{\bar{P}\bar{Q}} \cos \lambda - 2\bar{d}_{\bar{Q}\bar{R}'}\bar{d}_{\bar{R}'\bar{S}} \cos \delta + O(\epsilon^3) \\ &= \bar{d}^2_{\bar{Q}\bar{R}'} - \bar{d}^2_{\bar{P}\bar{S}} - 2\bar{d}_{\bar{Q}\bar{R}'} \big((\bar{d}_{\bar{R}'\bar{S}} - \bar{d}_{\bar{P}\bar{Q}}) \cos \delta + \bar{d}_{\bar{P}\bar{Q}}(\cos \delta + \cos \lambda) \big) \\ &\quad + O(\epsilon^3), \end{aligned}$$

and hence

$$\begin{aligned} E &\leq |\bar{d}^2_{\bar{Q}\bar{R}'} - \bar{d}^2_{\bar{P}\bar{S}}| + 2N\epsilon |\bar{d}_{\bar{R}'\bar{S}} - \bar{d}_{\bar{Q}\bar{P}}| + 2N^2\epsilon^2 |\cos \delta + \cos \lambda| + O(\epsilon^3) \\ &\leq C_N(|\bar{d}^2_{\bar{Q}\bar{R}'} - \bar{d}^2_{\bar{P}\bar{S}}| + \epsilon |\bar{d}_{\bar{R}'\bar{S}} - \bar{d}_{\bar{Q}\bar{P}}| + \epsilon^2 |\cos \delta + \cos \lambda|) \\ &\quad + O(\epsilon^3). \end{aligned} \tag{67}$$

We now estimate $|\cos \delta + \cos \lambda|$. Let A be the area of $\triangle \bar{R}'\bar{Q}\bar{P}$. Since the perimeter of $\triangle \bar{R}'\bar{Q}\bar{P}$ is bounded by some constant times $N\epsilon$, we have $A = O(\epsilon^2)$. Define $\alpha, \beta \in [0, \pi]$ by

$$\cosh \bar{d}_{\bar{Q}\bar{R}'} = \cosh \bar{d}_{\bar{P}\bar{R}'} \cosh \bar{d}_{\bar{P}\bar{Q}} - \sinh \bar{d}_{\bar{P}\bar{R}'} \sinh \bar{d}_{\bar{P}\bar{Q}} \cos \alpha$$
$$\cosh \bar{d}_{\bar{P}\bar{S}} = \cosh \bar{d}_{\bar{P}\bar{R}'} \cosh \bar{d}_{\bar{R}'\bar{S}} - \sinh \bar{d}_{\bar{P}\bar{R}'} \sinh \bar{d}_{\bar{R}'\bar{S}} \cos \beta. \tag{68}$$

The interior angles of the triangle $\triangle \bar{R}'\bar{Q}\bar{P}$ are α, λ and $\delta - \beta$. Since $A = \pi - \alpha - \lambda - (\delta - \beta)$, we see that

$$|\cos \delta + \cos \lambda| \leq |\cos \delta - \cos(\delta + (A + \alpha - \beta))| \leq A + |\alpha - \beta| = |\alpha - \beta| + O(\epsilon^2),$$

where we used the Mean Value Theorem in the second inequality. The fact that the ratios of any two pairwise distances of $\bar{P}, \bar{Q}, \bar{R}'$ and $\bar{S}$ are bounded from below by $\frac{1}{N^2}$ and from above by N^2 implies that α and β are bounded away from 0 and π. Thus, $|\alpha - \beta| \leq L|\cos \alpha - \cos \beta|$ for some constant dependent on N. Therefore, we obtain

$$|\cos \delta + \cos \lambda| \leq L|\cos \alpha - \cos \beta| + O(\epsilon^2)$$

which combined with (67) gives

$$E \leq |\bar{d}^2_{\bar{Q}\bar{R}'} - \bar{d}^2_{\bar{P}\bar{S}}| + 2N\epsilon |\bar{d}_{\bar{R}'\bar{S}} - \bar{d}_{\bar{Q}\bar{P}}| + 2LN^2\epsilon^2 |\cos \alpha - \cos \beta| + O(\epsilon^3). \tag{69}$$

By (68), we also have

$$\begin{aligned}
&\sinh(\bar d_{\bar P\bar R'})\sinh(\bar d_{\bar R'\bar S})\sinh(\bar d_{\bar P\bar Q})|\cos\alpha-\cos\beta| \\
&\quad = \Big|\sinh(\bar d_{\bar P\bar Q})(-\cosh(\bar d_{\bar P\bar S})+\cosh(\bar d_{\bar P\bar R'})\cosh(\bar d_{\bar R'\bar S})) \\
&\qquad -\sinh(\bar d_{\bar R'\bar S})(-\cosh(\bar d_{\bar Q\bar R'})+\cosh(\bar d_{\bar P\bar R'})\cosh(\bar d_{\bar P\bar Q}))\Big|.
\end{aligned}$$

The right hand side can be estimated as

$$\begin{aligned}
&\left|\bar d_{\bar R'\bar S}\left(\bar d^2_{\bar P\bar Q}+\bar d^2_{\bar P\bar R'}-\bar d^2_{\bar Q\bar R'}\right)-\bar d_{\bar P\bar Q}\left(\bar d^2_{\bar P\bar R'}+\bar d^2_{\bar R'\bar S}-\bar d^2_{\bar P\bar S}\right)\right|+O(\epsilon^5) \\
&\quad\leq \bar d^2_{\bar P\bar Q}\left|\bar d_{\bar R'\bar S}-\bar d_{\bar P\bar Q}\right|+\bar d_{\bar P\bar Q}\left(\bar d^2_{\bar P\bar Q}-\bar d^2_{\bar R'\bar S}\right)+\bar d^2_{\bar P\bar R'}\left|\bar d_{\bar R'\bar S}-\bar d_{\bar P\bar Q}\right| \\
&\qquad +\bar d_{\bar P\bar Q}\left(\bar d^2_{\bar P\bar S}-\bar d^2_{\bar Q\bar R'}\right)+\bar d^2_{\bar R'\bar S}\left|\bar d_{\bar P\bar Q}-\bar d_{\bar R'\bar S}\right|+O(\epsilon^5).
\end{aligned}$$

Furthermore,

$$\frac{\epsilon^3}{N^3}\leq \bar d_{\bar P\bar R'}\bar d_{\bar R'\bar S}\bar d_{\bar P\bar Q}\leq \sinh(\bar d_{\bar P\bar R'})\sinh(\bar d_{\bar R'\bar S})\sinh(\bar d_{\bar P\bar Q}).$$

Therefore, we obtain

$$|\cos\alpha-\cos\beta|\leq\frac{C_N}{\epsilon^2}\left(\epsilon|\bar d_{\bar R'\bar S}-\bar d_{\bar P\bar Q}|+(\bar d^2_{\bar P\bar Q}-\bar d^2_{\bar R'\bar S})+(\bar d^2_{\bar P\bar S}-\bar d^2_{\bar Q\bar R'})\right).$$

Combining this with (69), we obtain the desired inequality. q.e.d.

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