J. DIFFERENTIAL GEOMETRY
87 (2011) 343-388

CLOSED MAGNETIC GEODESICS ON S^2

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Abstract

We give existence results for simple closed curves with prescribed geodesic curvature on S^2, which correspond to periodic orbits of a charge in a magnetic field.

1. Introduction

The trajectory of a charged particle on an orientable Riemannian surface (N, g) in a magnetic field given by the magnetic field form $\Omega = k\, dA$, where $k : N \to \mathbb{R}$ is the magnitude of the magnetic field and dA is the area form on N, corresponds to a curve γ on N that solves

$$D_{t,g}\dot\gamma = k(\gamma)J_g(\gamma)\dot\gamma \tag{1.1}$$

where $D_{t,g}$ is the covariant derivative with respect to g, and $J_g(x)$ is the rotation by $\pi/2$ in T_xN measured with g and the orientation chosen on N. A curve γ in N that solves (1.1) will be called a *(k-)magnetic geodesic*. We refer to [**4**, **12**, **6**] for the Hamiltonian description of the motion of a charge in a magnetic field. Taking the scalar product of (1.1) with $\dot\gamma$, we see that if γ is a magnetic geodesic, then $(\gamma, \dot\gamma)$ lies on the energy level $E_c := \{(x, V) \in TN : |V|_g = c\}$.

The geodesic curvature $k_g(\gamma, t)$ of an immersed curve γ at t is defined by

$$k_g(\gamma, t) := |\dot\gamma(t)|_g^{-2}\big\langle \big(D_{t,g}\dot\gamma\big)(t), N_g(\gamma(t))\big\rangle_g,$$

where $N_g(\gamma(t))$ denotes the unit normal of γ at t given by

$$N_g(\gamma(t)) := |\dot\gamma(t)|_g^{-1} J_g(\gamma(t))\dot\gamma(t).$$

By (1.1), a nonconstant curve γ on E_c is a k-magnetic geodesic if and only if its geodesic curvature $k_g(\gamma, t)$ is given by $k(\gamma(t))/c$. We will take advantage of the latter description and consider the equation

$$D_{t,g}\dot\gamma = |\dot\gamma|_g k(\gamma)J_g(\gamma)\dot\gamma. \tag{1.2}$$

We call equation (1.2) the *prescribed geodesic curvature equation*, as its solutions γ are constant speed curves with geodesic curvature $k_g(\gamma, t)$ given by $k(\gamma(t))$. For fixed k and $c > 0$ the equations (1.1) and (1.2) are equivalent in the following sense: If γ is a nonconstant solution of (1.2)

Received 5/9/2009.

with k replaced by k/c, then the curve $t \mapsto \gamma(ct/|\dot\gamma|_g)$ is a k-magnetic geodesic on E_c and a k-magnetic geodesic on E_c solves (1.2) with k replaced by k/c. We emphasize that unless $k \equiv 0$, the solutions of (1.1) lying in different E_c are not reparametrizations of each other.

We study the existence of closed curves with prescribed geodesic curvature or equivalently the existence of periodic magnetic geodesics on prescribed energy levels E_c.

There are different approaches to this problem: the Morse-Novikov theory for (possibly multivalued) variational functionals (see [**30**, **24**, **29**]), the theory of dynamical systems using methods from symplectic geometry (see [**15**, **14**, **11**, **4**, **12**, **13**, **26**]), and Aubry-Mather's theory (see [**6**]).

We suggest a new approach: instead of looking for critical points of the (possibly multivalued) action functional, we consider solutions to (1.2) as zeros of the vector field $X_{k,g}$ defined on the Sobolev space $H^{2,2}(S^1,N)$ as follows: For $\gamma \in H^{2,2}(S^1,N)$ we let $X_{k,g}(\gamma)$ be the unique weak solution of

$$\big(-D^2_{t,g}+1\big)X_{k,g}(\gamma) = -D_{t,g}\dot\gamma + |\dot\gamma|_g k(\gamma)J_g(\gamma)\dot\gamma \tag{1.3}$$

in $T_\gamma H^{2,2}(S^1,N)$. The uniqueness implies that any zero of $X_{k,g}$ is a weak solution of (1.2) which is a classical solution in $C^2(S^1,N)$ applying standard regularity theory. The vector field $X_{k,g}$ as well as the set of solutions to (1.2) is invariant under a circle action: For $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$ and $\gamma \in H^{2,2}(S^1,N)$ we define $\theta * \gamma \in H^{2,2}(S^1,N)$ by

$$\theta * \gamma(t) = \gamma(t+\theta).$$

Moreover, for $V \in T_\gamma H^{2,2}(S^1,N)$ we let

$$\theta * V := V(\cdot + \theta) \in T_{\theta*\gamma}H^{2,2}(S^1,N).$$

Then $X_{k,g}(\theta*\gamma) = \theta*X_{k,g}(\gamma)$ for any $\gamma \in H^{2,2}(S^1,N)$ and $\theta \in S^1$. Thus, any zero gives rise to an S^1-orbit of zeros and we say that two solutions γ_1 and γ_2 of (1.2) are (geometrically) distinct, if $S^1 * \gamma_1 \neq S^1 * \gamma_2$.

We will apply this approach to the case $N = S^2$, equipped with a smooth metric g, and k a positive smooth function on S^2. We shall prove

Theorem 1.1. *Let g be a smooth metric and k a positive smooth function on S^2. Suppose that one of the following three assumptions is satisfied:*

$$4\inf(k) \geq \big(inj(g)\big)^{-1}\big(2\pi + (\sup K_g^-)vol(S^2,g)\big), \tag{1.4}$$

$$K_g > 0 \text{ and } 2\inf(k) \geq \sup(K_g)^{\frac{1}{2}}, \tag{1.5}$$

$$\sup(K_g) < 4\inf(K_g), \tag{1.6}$$

where K_g denotes the Gauss curvature, $K_g^- := -\min(K_g, 0)$, and $inj(g)$ denotes the injectivity radius of (S^2, g). Then there are at least two simple solutions of (1.2) *in $C^2(S^1, S^2)$.*

Concerning the existence of closed k-magnetic geodesics for a positive smooth function k on (S^2, g), the following is known (see [**11**, **12**]):

- (i) if $c > 0$ is sufficiently small, then E_c contains two simple closed magnetic geodesics;
- (ii) if g is sufficiently close to the round metric g_0 and k is sufficiently close to a positive constant, then there is a closed magnetic geodesic in every energy level E_c;
- (iii) if $c > 0$ is sufficiently large, then E_c contains a closed magnetic geodesic.

Using the equivalence between (1.1) and (1.2), we obtain from Theorem 1.1

Corollary 1.2. *Let g be a smooth metric, k a positive smooth function on S^2, and $c > 0$. Suppose that one of the following three assumptions is satisfied:*

$$c \le 4\big(\inf(k)\big)\big(inj(g)\big)\big(2\pi + (\sup K_g^-)vol(S^2, g)\big)^{-1}, \tag{1.7}$$

$$K_g > 0 \text{ and } c \le 2\inf(k)\big(\sup(K_g)\big)^{-\frac{1}{2}}, \tag{1.8}$$

$$\sup(K_g) < 4\inf(K_g).$$

Then E_c contains at least two simple closed magnetic geodesics.

Condition (1.7) should be compared to the existence results in (i) and gives bounds on the required smallness of c in terms of geometric quantities. To show that (1.7) is useful despite the implicit definition of $inj(g)$, we apply an estimate of $inj(g)$ in [**18**] and obtain (1.8) as a special case. The pinching condition (1.6) extends the existence result in (ii) and shows, for instance, that on the round sphere there are two simple closed curves of prescribed geodesic curvature k for any positive function k, which gives a partial solution to a problem posed by Arnold in [**5**, 1994-35,1996-18] concerning the existence of closed magnetic geodesics on S^2 on every energy level E_c.

By the famous Lusternik-Schnirelmann theorem, there are at least three simple closed geodesics on every Riemannian two sphere (S^2, g). As a by-product of our analysis we show that, in general, even if k is very close to 0, there are no more than two simple closed magnetic geodesics on S^2 in E_1 (see also [**14**, sec. 7]).

Theorem 1.3. *Let g_0 be the round metric on S^2. For any positive constant $k_0 > 0$ there is a smooth function k on S^2, which can be chosen arbitrarily close to k_0, such that there are exactly two simple solutions of (*1.2*).*

The proof of our existence results is organized as follows. After setting up notation in Section 2 and introducing the classes of maps and spaces needed for our analysis, we define in Section 3 a S^1-equivariant Poincaré-Hopf index or S^1-degree,

$$\chi_{S^1}(X, M) \in \mathbb{Z},$$

where M is an S^1-invariant subset of prime curves in $H^{2,2}(S^1, S^2)$ and X belongs to a class of S^1-invariant vector fields. The index $\chi_{S^1}(X, M)$ is related to the extension of the Leray-Schauder degree to intrinsic nonlinear problems in [**32**, **9**] and is used combined with the a priori estimates in Section 6 to count simple periodic solutions of (1.2). We remark that the standard degree $\chi(X, M)$, which does not take the S^1 invariance into account, vanishes as it detects only fixed points under the S^1-action, i.e. constant solutions. Equivariant degree theories have been defined and applied to differential equations by many authors; we refer to [**16**, **10**, **8**, **17**, **7**] and the references therein. However, we do not see how to apply these results directly to (1.2).

The vector field $X_{k,g}$ corresponding to the prescribed geodesic curvature problem falls into the class of vector fields, where our S^1-degree is defined. In Section 4 we show that the S^1-degree of an isolated zero orbit of $X_{k,g}$ is given by $-i(P, \theta)$, where $i(P, \theta)$ denotes the fixed point index of the Poincaré map of the corresponding magnetic geodesic. Since the Poincaré map is area preserving, we obtain from [**27**, **23**] that the S^1-degree of an isolated zero orbit is bounded below by -1.

Section 5 is devoted to the computation of $\chi_{S^1}(X_{k_0,g_0}, M)$, where k_0 is a positive constant, g_0 is the round metric of S^2, and M is the set of simple regular curves in $H^{2,2}(S^1, S^2)$. We call equation (1.2) with $k \equiv k_0$ and $g = g_0$ the unperturbed problem, which is analyzed in detail. The set of simple solutions to the unperturbed problem is given by circles of latitude of radius $(1 + k_0^2)^{-1/2}$ with respect to an arbitrarily chosen north pole. To compute the S^1-degree we slightly perturb the constant function k_0 and end up with exactly two nondegenerate solutions of degree -1. This implies that $\chi_{S^1}(X_{k_0,g_0}, M) = -2$.

Section 6 contains the a priori estimates showing that the set of simple solutions to (1.2) is compact in M under each of the assumptions (1.4)–(1.6). Together with the perturbative analysis in Section 5, this yields the proof of Theorem 1.3 and allows us to construct an admissible homotopy of vector fields between X_{k_0,g_0} and $X_{k,g}$ whenever k and g satisfy the assumptions of Theorem 1.1. The homotopy invariance of the S^1-equivariant Poincaré-Hopf index then shows

$$\chi_{S^1}(X_{k,g}, M) = \chi_{S^1}(X_{k_0,g_0}, M) = -2.$$

Since the S^1-degree of an isolated zero orbit is always larger than -1, there are at least two simple solutions of (1.2). The existence result is given in Section 7.

Acknowledgements. I would like to thank Professor Friedrich Tomi for valuable discussions and the unknown referees for several suggestions which helped to improve the article, in particular for drawing my attention to [**27**, **23**].

2. Preliminaries

Let $S^2 = \partial B_1(0) \subset \mathbb{R}^3$ be the standard round sphere with induced metric g_0 and orientation such that the rotation $J_{g_0}(y)$ is given for $y \in S^2$ by

$$J_{g_0}(y)(v) := y \times v \text{ for all } v \in T_y S^2,$$

where $\times$ denotes the cross product in $\mathbb{R}^3$. If we equip S^2 with a general Riemannian metric g, then the rotation by $\pi/2$ measured with g is given by

$$J_g(y)v = \big(G(y)\big)^{-1} J_{g_0}(y)\big(G(y)\big)v \quad \forall v \in T_y S^2,$$

where $G(y)$ denotes a positive symmetric map $G(y) \in \mathcal{L}(T_y S^2)$ satisfying

$$\langle v, w\rangle_{T_y S^2, g} = \langle G(y)v, G(y)w\rangle_{T_y S^2, g_0} \quad \forall v, w \in T_y S^2.$$

We consider for $m \in \mathbb{N}_0$ the set of Sobolev functions

$$H^{m,2}(S^1, S^2) := \{\gamma \in H^{m,2}(S^1, \mathbb{R}^3) : \gamma(t) \in \partial B_1(0) \text{ for a.e. } t \in S^1\}.$$

For $m \geq 1$ the set $H^{m,2}(S^1, S^2)$ is a sub-manifold of the Hilbert space $H^{m,2}(S^1, \mathbb{R}^3)$ and is contained in $C^{m-1}(S^1, \mathbb{R}^3)$. Hence, if $m \geq 1$ then $\gamma \in H^{m,2}(S^1, S^2)$ satisfies $\gamma(t) \in \partial B_1(0)$ for all $t \in S^1$. In this case the tangent space $T_\gamma H^{m,2}(S^1, S^2)$ of $H^{m,2}(S^1, S^2)$ at $\gamma \in H^{m,2}(S^1, S^2)$ is given by

$$T_\gamma H^{m,2}(S^1, S^2) := \{V \in H^{m,2}(S^1, \mathbb{R}^3) : V(t) \in T_{\gamma(t)} S^2 \text{ for all } t \in S^1\}.$$

For $m = 0$ the set $H^{0,2}(S^1, S^2) = L^2(S^1, S^2)$ fails to be a manifold. In this case we define for $\gamma \in H^{1,2}(S^1, S^2)$ the space $T_\gamma L^2(S^1, S^2)$ by

$$T_\gamma L^2(S^1, S^2) := \{V \in L^2(S^1, \mathbb{R}^3) : V(t) \in T_{\gamma(t)} S^2 \text{ for a.e. } t \in S^1\}.$$

A metric g on S^2 induces a metric on $H^{m,2}(S^1, S^2)$ for $m \geq 1$ by setting for $\gamma \in H^{m,2}(S^1, S^2)$ and V, $W \in T_\gamma H^{m,2}(S^1, S^2)$

$$\langle W, V\rangle_{T_\gamma H^{m,2}(S^1,S^2),g} := \int_{S^1} \Big\langle \big((-1)^{\llcorner \frac{m}{2} \lrcorner}(D_{t,g})^m + 1\big)V(t), \big((-1)^{\llcorner \frac{m}{2} \lrcorner}(D_{t,g})^m + 1\big)W(t)\Big\rangle_{\gamma(t),g} dt,$$

where $\llcorner m/2 \lrcorner$ denotes the largest integer that does not exceed $m/2$.

Let X be a differentiable vector field on $H^{2,2}(S^1,S^2)$. Then the covariant (Frechet) derivative D_gX,

$$D_gX:\ TH^{2,2}(S^1,S^2))\to TH^{2,2}(S^1,S^2),$$

of the vector field X with respect to the metric induced by g is defined as follows: For $\gamma\in H^{2,2}(S^1,S^2)$ and $V\in T_\gamma H^{2,2}(S^1,S^2)$ we consider a C^1-curve

$$(-\varepsilon,\varepsilon)\ni s\mapsto \gamma_s\in H^{2,2}(S^1,S^2)$$

satisfying

$$\gamma_0=\gamma \text{ and } \frac{d}{ds}\gamma_s|_{s=0}=V,$$

and define

$$D_gX|_\gamma[V](t):=D_{s,g}\Big(X\big(\gamma_s(t)\big)\Big)|_{s=0}.$$

For the vector field theory on infinite dimensional manifolds, it is convenient to work with Rothe maps instead of compact perturbations of the identity, because the class of Rothe maps is open in the space of linear continuous maps. We recall the definition and properties of Rothe maps given in [**32**] for the sake of the reader's convenience. For a Banach space E, we denote by $\mathcal{GL}(E)$ the set of invertible maps in $\mathcal{L}(E)$, and by $\mathcal{S}(E)$ the set

$$\mathcal{S}(E)=\{T\in\mathcal{GL}(E):\ (tT+(1-t)I)\in\mathcal{GL}(E) \text{ for all } t\in[0,1]\}.$$

Then the set of Rothe maps $\mathcal{R}(E)$ is defined by

$$\mathcal{R}(E):=\{A\in\mathcal{L}(E):\ A=T+C,\, T\in\mathcal{S}(E) \text{ and } C \text{ compact}\}.$$

The set $\mathcal{R}(E)$ is open in $\mathcal{L}(E)$ and consists of Fredholm operators of index 0. Moreover, $\mathcal{GR}(E):=\mathcal{R}(E)\cap\mathcal{GL}(E)$ has two components, $\mathcal{GR}^{\pm}(E)$, with $I\in\mathcal{GR}^{+}(E)$. For $A\in\mathcal{GR}(E)$ we let

$$\operatorname{sgn}A=\begin{cases}+1 & \text{if } A\in\mathcal{GR}^{+}(E),\\ -1 & \text{if } A\in\mathcal{GR}^{-}(E).\end{cases}$$

If $A=I+C\in\mathcal{GL}(E)$, where C is compact, then $A\in\mathcal{GR}(E)$ and $\operatorname{sgn}A$ is given by the usual Leray-Schauder degree of A.

Since g and k are smooth, $X_{k,g}$ is a smooth vector field (see [**31**, sec. 6]) on the set $H^{2,2}_{reg}(S^1,S^2)$ of regular curves,

$$H^{2,2}_{reg}(S^1,S^2):=\{\gamma\in H^{2,2}(S^1,S^2):\ \dot\gamma(t)\neq 0 \text{ for all } t\in S^1\}.$$

To compute $D_gX_{k,g}|_\gamma(V)$ we observe

$$\begin{aligned}D_{s,g}\big(-D^2_{t,g}+1)X_{k,g}(\gamma_s)\big)&=D_{s,g}\big(-D_{t,g}\dot\gamma_s+|\dot\gamma_s|_g k(\gamma_s)J_g(\gamma_s)\dot\gamma_s\big)\\ &=-D^2_{t,g}\frac{d\gamma_s}{ds}-R_g\big(\frac{d\gamma_s}{ds},\dot\gamma_s\big)\dot\gamma_s+D_{s,g}\big(|\dot\gamma_s|_g k(\gamma_s)J_g(\gamma_s)\dot\gamma_s\big).\end{aligned}$$

Evaluating at $s=0$, we obtain

$$
\begin{aligned}
&D_{s,g}\big((-D_{t,g}^2+1)X_{k,g}(\gamma_s)\big)|_{s=0}\\
&\quad= -D_{t,g}^2V - R_g\big(V,\dot\gamma\big)\dot\gamma + |\dot\gamma|_g^{-1}\langle D_{t,g}V,\dot\gamma\rangle_g k(\gamma)J_g(\gamma)\dot\gamma\\
&\qquad + |\dot\gamma|_g\big(k'(\gamma)V\big)J_g(\gamma)\dot\gamma + |\dot\gamma|_g k(\gamma)\big(D_gJ_g|_\gamma V\big)\dot\gamma\\
&\qquad + |\dot\gamma|_g k(\gamma)J_g(\gamma)D_{t,g}V. \qquad (2.1)
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&D_{s,g}\big((-D_{t,g}^2+1)X_{k,g}(\gamma_s)\big)|_{s=0}\\
&\quad= -D_{s,g}D_{t,g}^2X_{k,g}(\gamma_s)|_{s=0} + D_{s,g}X_{k,g}(\gamma_s)|_{s=0}\\
&\quad= \big(-D_{t,g}^2+1\big)D_gX_{k,g}|_\gamma(V) - D_{t,g}\Big(R_g\big(V,\dot\gamma\big)X_{k,g}(\gamma)\Big)\\
&\qquad - R_g\big(V,\dot\gamma\big)D_{t,g}X_{k,g}(\gamma). \qquad (2.2)
\end{aligned}
$$

Equating (2.1) and (2.2) at a critical point γ of $X_{k,g}$ leads to

$$
\begin{aligned}
&\big(-D_{t,g}^2+1\big)D_gX_{k,g}|_\gamma(V)\\
&\quad= -D_{t,g}^2V - R_g\big(V,\dot\gamma\big)\dot\gamma + |\dot\gamma|_g^{-1}\langle D_{t,g}V,\dot\gamma\rangle_g k(\gamma)J_g(\gamma)\dot\gamma\\
&\qquad + |\dot\gamma|_g\big(k'(\gamma)V\big)J_g(\gamma)\dot\gamma + |\dot\gamma|_g k(\gamma)\big(D_gJ_g|_\gamma V\big)\dot\gamma\\
&\qquad + |\dot\gamma|_g k(\gamma)J_g(\gamma)D_{t,g}V. \qquad (2.3)
\end{aligned}
$$

We note that (see also [**32**, thm. 6.1])

$$
\big(-D_{t,g}^2+1\big)D_gX_{k,g}|_\gamma(V) = (-D_{t,g}^2+1)V + T(V),
$$

where T is a linear map from $T_\gamma H^{2,2}(S^1,S^2)$ to $T_\gamma L^2(S^1,S^2)$ that depends only on V and its first derivatives and is therefore compact. Taking the inverse $(-D_{t,g}^2+1)^{-1}$, we deduce that $D_gX_{k,g}|_\gamma$ is the form $identity + compact$ and thus a Rothe map.

For $m\geq 1$ the exponential map $Exp_g : TH^{m,2}(S^1,S^2)\to H^{m,2}(S^1,S^2)$ is defined for $\gamma\in H^{m,2}(S^1,S^2)$ and $V\in T_\gamma H^{m,2}(S^1,S^2)$ by

$$
Exp_{\gamma,g}(V)(t) := Exp_{\gamma(t),g}(V(t)),
$$

where $Exp_{z,g}$ denotes the exponential map on (S^2,g) at $z\in S^2$. Due to its pointwise definition,

$$
\theta * Exp_{\gamma,g}(V)(t) = Exp_{\theta*\gamma,g}(\theta * V)(t).
$$

3. The S^1-Poincaré-Hopf index

For $\gamma\in H^{2,2}(S^1,S^2)$ we define the form $\omega_g(\gamma)\in(T_\gamma H^{2,2}(S^1,S^2))^*$ by

$$
\begin{aligned}
\omega_g(\gamma)(V) :&= \int_0^1 \langle\dot\gamma(t),\big(-(D_{t,g})^2+1\big)V(t)\rangle_{\gamma(t),g}\,dt\\
&= \langle\dot\gamma,V\rangle_{T_\gamma H^{1,2}(S^1,S^2),g}.
\end{aligned}
$$

Approximating $\dot\gamma$ by vector fields contained in $T_\gamma H^{2,2}(S^1,S^2)$, it is easy to see that $\omega_g(\gamma)\neq 0$, if $\gamma\neq$ const. If $\gamma\in H^{3,2}(S^1,S^2)$, then $\omega_g(\gamma)$ extends to a linear form in $(T_\gamma L^2(S^1,S^2))^*$ by

$$\omega_g(\gamma)(V):=\langle\big(-(D_{t,g})^2+1\big)\dot\gamma,V\rangle_{T_\gamma L^2(S^1,S^2),g}.$$

From Riesz' representation theorem there is $W_g(\gamma)\in T_\gamma H^{2,2}(S^1,S^2)$ such that

$$\omega_g(\gamma)(V)=\langle V,W_g(\gamma)\rangle_{T_\gamma H^{2,2}(S^1,S^2),g}\ \forall V\in T_\gamma H^{2,2}(S^1,S^2),$$

and

$$\langle W_g(\gamma)\rangle^\perp=\langle\dot\gamma\rangle^{\perp,H^{1,2}}\cap T_\gamma H^{2,2}(S^1,S^2). \tag{3.1}$$

Hence

$$W_g(\gamma)=(-(D_{t,g})^2+1)^{-1}\dot\gamma$$

and W_g is a C^2 vector field on $H^{2,2}(S^1,S^2)$.

The form $\omega_g(\gamma)$ and the vector $W_g(\gamma)$ are equivariant under the S^1-action in the sense that for all $\theta\in S^1$ and $V\in T_\gamma H^{2,2}(S^1,S^2)$ we have

$$w_{\theta*\gamma,g}(\theta*V)=\omega_g(\gamma)(V)\text{ and }W_{\theta*\gamma,g}=\theta*W_g(\gamma).$$

Using the vector field W_g, we define a vector bundle $SH^{2,2}(S^1,S^2)$ by

$$SH^{2,2}(S^1,S^2):=\{(\gamma,V)\in TH^{2,2}(S^1,S^2):\ \gamma\neq\text{const},\ V\in\langle W_g(\gamma)\rangle^\perp\}.$$

Note that $SH^{2,2}(S^1,S^2)$ is S^1-invariant, as

$$(\gamma,V)\in SH^{2,2}(S^1,S^2)\implies(\theta*\gamma,\theta*V)\in SH^{2,2}(S^1,S^2)\ \forall\theta\in S^1.$$

For $\gamma\in H^{2,2}(S^1,S^2)\setminus\{\text{const}\}$ we consider the map

$$\psi_{\gamma,g}:T_\gamma H^{2,2}(S^1,S^2)\times T_\gamma H^{2,2}(S^1,S^2)\to SH^{2,2}(S^1,S^2)$$

defined by

$$\psi_{\gamma,g}(V,U):=\Big(Exp_{\gamma,g}V,Proj_{\langle W_g(Exp_{\gamma,g}V)\rangle^\perp}\big(DExp_{\gamma,g}|_VU\big)\Big). \tag{3.2}$$

The differential of $\psi_{\gamma,g}$ at $(0,0)$ is given by

$$D\psi_{\gamma,g}|_{(0,0)}(V,U)=(V,U-\|W_g(\gamma)\|^{-2}\langle U,W_g(\gamma)\rangle_{T_\gamma H^{2,2}(S^1,S^2),g}W_g(\gamma)).$$

Consequently, there is $\delta=\delta(\gamma,g)>0$ such that $\psi_{\gamma,g}$ restricted to

$$B_\delta(0)\times B_\delta(0)\cap\langle W_g(\gamma)\rangle^\perp\subset T_\gamma H^{2,2}(S^1,S^2)\times T_\gamma H^{2,2}(S^1,S^2)$$

is a chart for the manifold $SH^{2,2}(S^1,S^2)$ at $(\gamma,0)$. The construction is S^1-equivariant, for

$$\psi_{\theta*\gamma,g}(\theta*V,\theta*U)=\theta*\psi_{\gamma,g}(V,U)\ \forall\theta\in S^1$$

and we may choose $\delta(\gamma, g) = \delta(\theta * \gamma, g)$ for all $\theta \in S^1$. Shrinking $\delta(\gamma, g)$ we may assume, as $Exp_{\gamma,g}$ is also a chart for $H^{k,2}(S^1, S^2)$ with $1 \le k \le 4$ and by (3.1),

(3.3)
$$T_{Exp_{\gamma,g}(V)} H^{1,2}(S^1, S^2) = \langle D_t Exp_{\gamma,g}(V)\rangle \oplus DExp_{\gamma,g}|_V(\langle \dot\gamma\rangle^{\perp, H^{1,2}}),$$

(3.4)
$$T_{Exp_{\gamma,g}(V)} H^{2,2}(S^1, S^2) = \langle W_g(Exp_{\gamma,g}(V))\rangle \oplus DExp_{\gamma,g}|_V(\langle W_g(\gamma)\rangle^{\perp}),$$

(3.5)
$$\mathrm{Proj}_{\langle W_g(Exp_{\gamma,g}(V))\rangle^{\perp}} \circ DExp_{\gamma,g}|_V : \langle W_g(\gamma)\rangle^{\perp} \xrightarrow{\cong} \langle W_g(Exp_{\gamma,g}(V)\rangle^{\perp},$$

and the norm of the projections corresponding to the decompositions in (3.3) and (3.4) as well as the norm of the map in (3.5) and its inverse are uniformly bounded with respect to V.

The S^1-action is only continuous but not differentiable on $H^{2,2}(S^1, S^2)$ as, for instance, the candidate for the differential of the map $\theta \to \theta * \gamma$ at $\theta = 0$, $\dot\gamma$, is in general only in $T_\gamma H^{1,2}(S^1, S^2)$. We prove the existence of a slice of the S^1-action (see [**19**, lem. 2.2.8] and the references therein) at a curve γ with higher regularity and obtain additional differentiability of the slice map.

Lemma 3.1 (Slice lemma)**.** *Let $\gamma \in H^{3,2}(S^1, S^2)$ be a prime curve, i.e. a curve with trivial isotropy group $\{\theta \in S^1 : \theta * \gamma = \gamma\}$. Then there is an open neighborhood $\mathcal{U}$ of 0 in $T_\gamma H^{2,2}(S^1, S^2)$, such that the map*

$$\Sigma_{\gamma,g} : S^1 \times \mathcal{U} \cap \langle W_g(\gamma)\rangle^{\perp} \to H^{2,2}(S^1, S^2),$$

defined by

$$\Sigma_{\gamma,g}(\theta, V) := \theta * Exp_{\gamma,g}(V),$$

is a homeomorphism onto its range, which is open in $H^{2,2}(S^1, S^2)$. Moreover, the inverse $(\Sigma_{\gamma,g})^{-1}$ satisfies

$$Proj_{S^1} \circ (\Sigma_{\gamma,g})^{-1} \in C^2\Big(\Sigma_{\gamma,g}\big(S^1 \times \mathcal{U} \cap \langle W_g(\gamma)\rangle^{\perp}\big), S^1\Big).$$

Proof. Fix a prime curve $\gamma \in H^{3,2}(S^1, S^2)$. We consider for $\delta_0 > 0$ the map

$$F_{\gamma,g} : B_{\delta_0}(0) \times B_{\delta_0}(0) \subset \mathbb{R}/\mathbb{Z} \times T_\gamma H^{2,2}(S^1, S^2) \to \mathbb{R}$$

defined by

$$F_{\gamma,g}(\theta, V) := \omega_g(\gamma)\Big(Exp^{-1}_{\gamma,g}\big(\theta * Exp_{\gamma,g}(V)\big)\Big).$$

Note that, as S^1 acts continuously on $H^{2,2}(S^1, S^2)$ and $Exp_{\gamma,g}$ is a local diffeomorphism, after shrinking $\delta_0 > 0$ the map $F_{\gamma,g}$ is well defined. $Exp_{\gamma,g}$ is a smooth map, such that for fixed θ the map $V \mapsto F_{\gamma,g}(\theta, V)$ is also smooth. Moreover, since $Exp_{\gamma,g}(V)$ is in $H^{2,2}(S^1, S^2)$

and $DExp_{\gamma,g}|_V$ maps L^2 vector fields along γ into L^2 vector fields along $Exp_{\gamma,g}(V)$, the map

$$\theta \mapsto Exp^{-1}_{\gamma,g}\big(\theta * Exp_{\gamma,g}(V)\big)$$

is C^2 from $B_{\delta_0}(0) \subset \mathbb{R}/\mathbb{Z}$ to $T_\gamma L^2(S^1, S^2)$, the space of L^2 vector fields along γ. For $\gamma \in H^{3,2}(S^1, S^2)$ the form $\omega_g(\gamma)$ is in $(T_\gamma L^2(S^1, S^2))^*$. Thus, $\theta \mapsto F_{\gamma,g}(\theta, V)$ is C^2 as well as $F_{\gamma,g}$. Fix $V \in T_\gamma H^{2,2}(S^1, S^2)$. Since

$$D_\theta F_{\gamma,g}|_{(0,0)} = \omega_g(\gamma)(\dot\gamma), \neq 0,$$

by the implicit function theorem and after shrinking $\delta_0 > 0$ we get a unique C^2-map

$$\sigma_{\gamma,g} : B_{\delta_0}(0) \subset T_\gamma H^{2,2}(S^1, S^2) \to B_{\delta_0}(0) \subset \mathbb{R}/\mathbb{Z}$$

such that

$$F_{\gamma,g}(\sigma_{\gamma,g}(V), V) \equiv 0 \text{ in } B_{\delta_0}(0) \subset T_\gamma H^{2,2}(S^1, S^2).$$

Hence, we may define locally around γ

$$V_{\gamma,g}(\alpha) := Exp^{-1}_{\gamma,g}(\sigma_{\gamma,g}(Exp^{-1}_{\gamma,g}(\alpha)) * \alpha) \in \langle W_g(\gamma)\rangle^\perp.$$

Using the uniqueness of $\sigma_{\gamma,g}$ and the fact that γ is prime, it is standard to see that $\Sigma_{\gamma,g}$ is injective and that the inverse is given locally around $\theta_0 * \gamma$ for fixed $\theta_0 \in S^1$ by

$$\Sigma^{-1}_{\gamma,g} = (\theta_0, 0) + (-\sigma_{\gamma,g} \circ Exp^{-1}_{\gamma,g} \circ (-\theta_0 *), V_{\gamma,g} \circ (-\theta_0 *)).$$

This finishes the proof. q.e.d.

We will compute the Poincaré-Hopf index for the following class of vector fields.

Definition 3.2. *Let M be an open S^1-invariant subset of prime curves in $H^{2,2}(S^1, S^2)$. A C^2 vector field X on M is called (M, g, S^1)-admissible, if*

1. *X is S^1-equivariant, i.e. $X(\theta * \gamma) = \theta * X(\gamma)$ for all $(\theta, \gamma) \in S^1 \times M$.*
2. *X is proper in M, i.e. the set $\{\gamma \in M : X(\gamma) = 0\}$ is compact.*
3. *X is orthogonal to W_g, i.e. $w_g(\gamma)(X(\gamma)) = 0$ for all $\gamma \in M$.*
4. *X is a Rothe field, i.e. if $X(S^1 * \gamma) = 0$ then*

$$D_g X|_\gamma \in \mathcal{R}(T_\gamma H^{2,2}(S^1, S^2)) \text{ and } Proj_{\langle W_g(\gamma)\rangle^\perp} \circ D_g X|_\gamma \in \mathcal{R}(\langle W_g(\gamma)\rangle^\perp).$$

5. *X is elliptic, i.e. there is $\varepsilon > 0$ such that for all finite sets of charts*

$$\{(Exp_{\gamma_i,g}, B_{2\delta_i}(0)) : \gamma_i \in H^{4,2}(S^1, S^2) \text{ for } 1 \le i \le n\},$$

and finite sets

$$\{W_i \in T_{\gamma_i} H^{4,2}(S^1, S^2) : \|W_i\|_{T_{\gamma_i} H^{4,2}(S^1,S^2)} < \varepsilon \text{ for } 1 \le i \le n\},$$

there holds: If $\alpha \in \bigcap_{i=1}^{n} Exp_{\gamma_i,g}(B_{\delta_i}(0)) \subset H^{2,2}(S^1,S^2)$ *satisfies*

$$X(\alpha) = \sum_{i=1}^{n} Proj_{\langle W_g(\alpha)\rangle^\perp} \circ DExp_{\gamma_i,g}|_{Exp^{-1}_{\gamma_i,g}(\alpha)}(W_i)$$

then α *is in* $H^{4,2}(S^1,S^2)$.

Property (4) does not depend on the particular element γ of the critical orbit $S^1 * \gamma$, because from $\theta * X(\gamma) = X(\theta * \gamma)$ we get

$$D_g X|_\gamma = (-\theta *) \circ D_g X|_{\theta * \gamma} \circ (\theta *), \tag{3.6}$$

and Rothe maps are invariant under conjugacy. Concerning the regularity property (5), taking $W_i = 0$, we deduce that if $X(\gamma) = 0$ then $\gamma \in H^{4,2}(S^1,S^2)$. Furthermore, if $\gamma \in H^{4,2}(S^1,S^2)$ then the map $\theta \mapsto \theta * \gamma$ is C^2 from S^1 to $H^{2,2}(S^1,S^2)$. Hence, if $X(\gamma) = 0$ then

$$0 = D_\theta(X(\theta * \gamma))|_{\theta=0} = D_g X|_\gamma(\dot\gamma), \tag{3.7}$$

such that the kernel of $D_g X|_\gamma$ at a critical orbit $S^1 * \gamma$ is nontrivial. The parameter $\varepsilon > 0$ ensures that (5) remains stable under small perturbations used in the Sard-Smale lemma below. If X is a vector field orthogonal to W_g and $X(\gamma) = 0$, then

$$0 = D\big(\langle X(\alpha), W_g(\alpha)\rangle_{T_\alpha H^{2,2}(S^1,S^2),g}\big)|_\gamma = \langle D_g X|_\gamma, W_g(\gamma)\rangle_{T_\gamma H^{2,2}(S^1,S^2),g}$$

where the various curvature terms and terms containing derivatives of W_g vanish as $X(\gamma) = 0$. Thus, $X(\gamma) = 0$ implies

$$D_g X|_\gamma : T_\gamma H^{2,2}(S^1,S^2) \to \langle W_g(\gamma)\rangle^\perp, \tag{3.8}$$

and the projection $\mathrm{Proj}_{\langle W_g(\gamma)\rangle^\perp}$ in (4) is unnecessary.

Lemma 3.3. *The vector field* $X_{k,g}$ *defined in (*1.3*) is* S^1*-equivariant, orthogonal to* W_g*, elliptic, and a* C^2*-Rothe field with respect to the set* $H^{2,2}_{reg}(S^1,S^2)$ *of regular curves.*

Proof. From Section 1 and Section 2, the vector field $X_{k,g}$ is S^1-equivariant and a C^2-Rothe field. Furthermore, we obtain for $\alpha \in H^{2,2}(S^1,S^2)$

$$\begin{aligned}\langle X_{k,g}(\alpha), W_g(\alpha)\rangle_{T_\alpha H^{2,2}(S^1,S^2),g} &= \int_{S^1} \langle \dot\alpha(t), (-D^2_{t,g}+1)X_{k,g}(\alpha)(t)\rangle_g\, dt \\ &= \int_{S^1} \langle \dot\alpha(t), -D_t\dot\alpha(t) + |\dot\alpha(t)|_g k(\alpha(t)) J_g(\alpha(t))\dot\alpha(t)\rangle_g\, dt \\ &= -\int_{S^1} \langle \dot\alpha(t), D_t\dot\alpha(t)\rangle_g\, dt = -\int_{S^1} \frac{1}{2}\frac{d}{dt}\langle \dot\alpha, \dot\alpha\rangle_g\, dt = 0.\end{aligned}$$

To show that $X_{k,g}$ is elliptic, we fix

$$\{(\gamma_i, W_i) \in TH^{4,2}(S^1,S^2) :\ W_i \in B_{\delta_i}(0),\ 1 \le i \le n\},$$

where $(Exp_{\gamma_i,g}, B_{2\delta_i}(0))$ is a chart around γ_i, and $\alpha \in \bigcap_{i=1}^{n} Exp_{\gamma_i,g}(B_{\delta_i}(0))$ satisfying

$$X_{k,g}(\alpha) = \sum_{i=1}^{n} \text{Proj}_{\langle W_g(\alpha)\rangle^\perp} \circ DExp_{\gamma_i,g}|_{Exp^{-1}_{\gamma_i,g}(\alpha)}(W_i).$$

Then

$$\begin{aligned} &D_{t,g}\dot\alpha - |\dot\alpha|_g k(\alpha) J_g(\alpha)\dot\alpha \\ &\qquad = (-D^2_{t,g} + 1) \sum_{i=1}^{n} \text{Proj}_{\langle W_g(\alpha)\rangle^\perp} \circ DExp_{\gamma_i,g}|_{Exp^{-1}_{\gamma_i,g}(\alpha)}(W_i). \end{aligned}$$

We fix $1 \le i \le n$ and get

$$\begin{aligned} &D^2_{t,g}\text{Proj}_{\langle W_g(\alpha)\rangle^\perp} \circ DExp_{\gamma_i,g}|_{Exp^{-1}_{\gamma_i,g}(\alpha)}(W_i) \\ &\quad = D^2_{t,g}\big(DExp_{\gamma_i,g}|_{Exp^{-1}_{\gamma_i,g}(\alpha)}(W_i)\big) \\ &\qquad - \langle DExp_{\gamma_i,g}|_{Exp^{-1}_{\gamma_i,g}(\alpha)}(W_i), W_g(\alpha)\rangle D^2_{t,g} W_g(\alpha), \end{aligned}$$

as well as

$$\begin{aligned} &D^2_{t,g}\big(DExp_{\gamma_i,g}|_{Exp^{-1}_{\gamma_i,g}(\alpha)}(W_i)\big)(t) \\ &\quad = D^2 Exp_{\gamma_i(t),g}|_{Exp^{-1}_{\gamma_i,g}(\alpha)(t)} D^2_{t,g} Exp^{-1}_{\gamma_i,g}(\alpha)(t)(W_i(t)) + R_{1,i}(t) \\ &\quad = D^2 Exp_{\gamma_i(t),g}|_{Exp^{-1}_{\gamma_i,g}(\alpha)(t)} D(Exp_{\gamma_i(t),g})^{-1}|_{\alpha(t)} D_{t,g}\dot\alpha(t)(W_i(t)) \\ &\qquad + R_{2,i}(t), \end{aligned}$$

where $R_{1,i}$ and $R_{2,i}$ consist of lower order terms containing only derivatives of α up to order 1 and derivatives of γ_i and W_i up to order 2. Thus α is a solution of

$$\begin{aligned} &(1 - A(t))D_{t,g}\dot\alpha = |\dot\alpha|_g k(\alpha) J_g(\alpha)\dot\alpha + R(t) \\ &\qquad - \sum_{i=1}^{n} \langle DExp_{\gamma_i,g}|_{Exp^{-1}_{\gamma_i,g}(\alpha)}(W_i), W_g(\alpha)\rangle (-D^2_{t,g} + 1) W_g(\alpha), \end{aligned} \tag{3.9}$$

where R contains only derivatives of α up to order 1 and derivatives of γ_i and W_i up to order 2, and $A(t) \in \mathcal{L}(T_{\alpha(t)}S^2)$ is given by

$$V \mapsto \sum_{i=1}^{n} D^2 Exp_{\gamma_i(t),g}|_{Exp^{-1}_{\gamma_i,g}(\alpha)(t)} (D(Exp_{\gamma_i(t),g})^{-1}|_{\alpha(t)} V)(W_i(t)).$$

Since $H^{2,2}$-bounds yield L^∞-bounds, choosing $\max \|W_i\|$ small enough independently of $\{\gamma_i\}$ and α, we may assume $\|A(t)\| < \frac{1}{2}$ and A is of class $H^{2,2}$ with respect to t. Since γ_i and W_i are in $H^{4,2}$ and $(-D^2_{t,g} + 1)W_g(\alpha) = \dot\alpha$, the right hand side of (3.9) is in $H^{1,2}$. By standard regularity results, α is in $H^{3,2}$, such that the right hand side of (3.9) is in $H^{2,2}$, which yields $\alpha \in H^{4,2}$. Consequently, $X_{k,g}$ is elliptic. q.e.d.

Definition 3.4. *Let M be an open S^1-invariant subset of prime curves in $H^{2,2}(S^1, S^2)$, $S^1 * \gamma \subset M$, and X an (M, g, S^1)-admissible vector field on M.*

The orbit $S^1 * \gamma$ is called a critical orbit *of X, if $X(\gamma) = 0$.*

The orbit $S^1 * \gamma$ is called a nondegenerate critical orbit *of X, if $X(\gamma) = 0$ and*

$$D_g X|_\gamma : \langle W_g(\gamma)\rangle^\perp \to \langle W_g(\gamma)\rangle^\perp \tag{3.10}$$

is an isomorphism.

If $S^1 * \gamma$ is critical, then using the chart $\psi_{\gamma,g}$ given in (3.2) *we define, after possibly shrinking $\delta > 0$, a map $X^\gamma \in C^2(B_\delta(0) \cap \langle W_g(\gamma)\rangle^\perp, \langle W_g(\gamma)\rangle^\perp)$ by*

$$X^\gamma(V) := Proj_2 \circ \psi^{-1}_{\gamma,g}\big(Exp_{\gamma,g}(V), X(Exp_{\gamma,g}(V))\big), \tag{3.11}$$

where $Proj_2$ denotes the projection on the second component.

The orbit $S^1 * \gamma$ is called an isolated critical orbit *of X, if $X(\gamma) = 0$ and $V = 0$ is an isolated zero of X^γ.*

The nondegeneracy of a critical orbit does not depend on the choice of γ in $S^1 * \gamma$.

Lemma 3.5. *Under the assumptions of Definition* 3.4*, a tangent vector $V \in B_\delta(0) \cap \langle W_g(\gamma)\rangle^\perp$ is a (nondegenerate) zero of X^γ if and only if $S^1 * Exp_{\gamma,g}(V)$ is a (nondegenerate) critical orbit of X.*

Proof. From the fact that $X(Exp_{\gamma,g}(V)) \perp W_g(Exp_{\gamma,g}(V))$, we get

$$X^\gamma(V) = 0 \Longleftrightarrow X(Exp_{\gamma,g}(V)) = 0.$$

Moreover, if $X^\gamma(V) = 0$, then

$$\begin{aligned} DX^\gamma|_V &= \mathrm{Proj}_2 \circ D\psi^{-1}_{\gamma,g}|_{(Exp_{\gamma,g}(V),0)} \\ &\qquad \circ \big(DExp_{\gamma,g}|_V, D_gX|_{Exp_{\gamma,g}(V)} \circ DExp_{\gamma,g}|_V\big) \\ &= A^{-1} \circ D_gX|_{Exp_{\gamma,g}(V)} \circ DExp_{\gamma,g}|_V, \end{aligned}$$

where $A : \langle W_g(\gamma)\rangle^\perp \to \langle W_g(Exp_{\gamma,g}(V)\rangle^\perp$ is given by

$$A := \mathrm{Proj}_{\langle W_g(Exp_{\gamma,g}(V))\rangle^\perp} \circ DExp_{\gamma,g}|_V.$$

By (3.5) the map A is an isomorphism. Consequently, the map $DX^\gamma|_V$ is invertible, if and only if

$$D_gX|_{Exp_{\gamma,g}(V)} \circ DExp_{\gamma,g}|_V : \langle W_g(Exp_{\gamma,g}(V)\rangle^\perp \xrightarrow{\cong} \langle W_g(Exp_{\gamma,g}(V))\rangle^\perp \tag{3.12}$$

is an isomorphism. The injectivity in (3.12), (3.3), and (3.7) implies that the kernel of the map $D_gX|_{Exp_{\gamma,g}(V)}$ is one dimensional and given by $\langle D_t Exp_{\gamma,g}(V)\rangle$. As $D_gX|_{Exp_{\gamma,g}(V)}$ is a Rothe map and thus a Fredholm operator of index 0, we deduce that (3.12) implies the nondegeneracy of $Exp_{\gamma,g}(V)$. If (3.10) holds with γ replaced by $Exp_{\gamma,g}(V)$, then the kernel of $D_gX|_{Exp_{\gamma,g}(V)}$ is one dimensional, and from (3.3) we infer that (3.12) holds, which finishes the proof. q.e.d.

Definition 3.6. *Let g_t for $t \in [0,1]$ be a family of smooth metrics on S^2, which induces a corresponding family of metrics on $H^{2,2}(S^1, S^2)$, still denoted by g_t. Let M be an open S^1-invariant subset of prime curves in $H^{2,2}(S^1, S^2)$ and X_0, X_1 two vector-fields on M such that X_i is (M, g_i, S^1)-admissible for $i = 0, 1$. A C^2 family of vector-fields $X(t, \cdot)$ on M for $t \in [0,1]$ is called an (M, g_t, S^1)-homotopy between X_0 and X_1, if*

- *$X(0, \cdot) = X_0$ and $X(1, \cdot) = X_1$,*
- *$\{(t, \gamma) \in [0,1] \times M : X(t, \gamma) = 0\}$ is compact,*
- *$X_t := X(t, \cdot)$ is (M, g_t, S^1)-admissible for all $t \in [0,1]$.*

We write (M, g, S^1)-homotopy, if the family of metrics g_t is constant.

If X is an (M, g_t, S^1)-homotopy, then differentiating

$$\langle X(t,\gamma), W_{g_t}(\gamma)\rangle_{T_\gamma H^{2,2}(S^1,S^2),g_t} \equiv 0$$

we see as in (3.8) for $(t_0, \gamma_0) \in X^{-1}(0)$,

$$D_gX|_{t_0,\gamma_0} : \mathbb{R} \times T_{\gamma_0}H^{2,2}(S^1, S^2) \to \langle W_{g_{t_0}}(\gamma_0)\rangle^{\perp, g_{t_0}}. \tag{3.13}$$

Moreover, analogous to (3.11), there is $\delta > 0$ such that

$$X^{t_0,\gamma_0} \in C^2(B_\delta(t_0) \times B_\delta(0) \cap \langle W_{g_{t_0}}(\gamma_0)\rangle^{\perp,g_{t_0}}, \langle W_{g_{t_0}}(\gamma_0)\rangle^{\perp,g_{t_0}}),$$

where

$$X^{t_0,\gamma_0}(t, V) := \mathrm{Proj}_3 \circ \psi^{-1}_{\gamma_0,t_0}\big(t, Exp_{\gamma_0,g_{t_0}}(V), X(t, Exp_{\gamma_0,g_{t_0}}(V))\big),$$

and ψ_{γ_0,t_0} is a chart around $(t_0, \gamma_0, 0)$ of the bundle

$$S_{[0,1]}H^{2,2}(S^1, S^2) := \{(t, \gamma, V) \in [0,1] \times TH^{2,2}(S^1, S^2) : \gamma \neq \text{const} \text{ and } V \in \langle W_{g_t}(\gamma)\rangle^{\perp,g_t}\},$$

defined in a neighborhood of $(t_0, 0, 0)$ in

$$[0,1] \times T_{\gamma_0}H^{2,2}(S^1, S^2) \times \langle W_{g_{t_0}}(\gamma_0)\rangle^{\perp,g_{t_0}}$$

by

$$\psi_{\gamma,t_0}(t, V, U) := \big(t, Exp_{\gamma,g_{t_0}}V, Proj_{\langle W_{g_t}(Exp_{\gamma,g_{t_0}}V)\rangle^{\perp,g_t}}(DExp_{\gamma,g_{t_0}}|_V U)\big). \tag{3.14}$$

Definition 3.7. *Let X be an (M, g_t, S^1)-homotopy and $(t_0, S^1 * \gamma_0) \in [0,1] \times M$. The orbit $(t_0, S^1 * \gamma_0)$ is called a* nondegenerate *zero of X, if $X(t_0, \gamma_0) = 0$ and*

$$D_gX|_{(t_0,\gamma_0)} : \mathbb{R} \times \langle W_{g_{t_0}}(\gamma_0)\rangle^{\perp,g_{t_0}} \to \langle W_{g_{t_0}}(\gamma_0)\rangle^{\perp,g_{t_0}} \tag{3.15}$$

is surjective.

Analogously to Lemma 3.5, we obtain for a homotopy X.

Lemma 3.8. *Under the assumptions of Definition* 3.7*, the tuple* (t, V) *in* $B_\delta(t_0) \times B_\delta(0) \cap \langle W_{g_{t_0}}(\gamma)\rangle^{\perp, g_{t_0}}$ *is a (nondegenerate) zero of* X^{t_0,γ_0} *if and only if the orbit* $(t, S^1 * Exp_{\gamma_0, g_{t_0}}(V))$ *is a (nondegenerate) zero of* X.

We give an S^1 equivariant version of the Sard-Smale lemma [**28**, **25**].

Lemma 3.9. *Let* M *be an open* S^1*-invariant subset of prime curves in* $H^{2,2}(S^1, S^2)$ *and* X *an* (M, g, S^1)*-admissible vector field on* M*. Let* $\mathcal{U}$ *be an open neighborhood of the zeros of* X*. Then there exists a* (M, g, S^1)*-admissible vector field* Y *such that* Y *has only finitely many isolated, nondegenerate zeros,* Y *equals* X *outside* $\mathcal{U}$*, and there is an* (M, g, S^1)*-homotopy connecting* X *and* Y*.*

Proof. As X is proper and $X^{-1}(0) \subset H^{4,2}(S^1, S^2)$ using Lemma 3.1 we may cover $X^{-1}(0)$ with finitely many open sets

$$X^{-1}(0) \subset \bigcup_{i=1}^{n} S^1 * Exp_{\gamma_i, g}\big(B_{\delta_i}(0) \cap \langle W_g(\gamma_i)\rangle^{\perp}\big),$$
$$\bigcup_{i=1}^{n} S^1 * Exp_{\gamma_i, g}\big(B_{3\delta_i}(0) \cap \langle W_g(\gamma_i)\rangle^{\perp}\big) \subset \mathcal{U},$$

where $\delta_i > 0$, the slice $\Sigma_{\gamma_i, g}$ is defined in $S^1 \times B_{3\delta_i}(0)$, and X^{γ_i} is defined in $B_{3\delta_i}(0) \cap \langle W_g(\gamma_i)\rangle^{\perp}$ for $i = 1, \ldots, n$.

Then $DX^{\gamma_i}|_0$ is in $\mathcal{R}(\langle W_g(\gamma_i)\rangle^{\perp})$, which is open in $\mathcal{L}(\langle W_g(\gamma_i)\rangle^{\perp})$. Thus $DX^{\gamma_i}|_V$ remains a Rothe map for V close to 0 and consequently a Fredholm operator of index 0. As Fredholm maps are locally proper, we may assume for all $1 \leq i \leq n$ that the map X^{γ_i} is proper and Rothe on $\overline{B_{2\delta_i}(0)} \cap \langle W_g(\gamma_i)\rangle^{\perp}$, i.e.

$$DX^{\gamma_i}|_V \in \mathcal{R}(\langle W_g(\gamma_i)\rangle^{\perp}) \;\forall V \in \overline{B_{2\delta_i}(0)} \cap \langle W_g(\gamma_i)\rangle^{\perp},$$
$$\overline{B_{2\delta_i}(0)} \cap \langle W_g(\gamma_i)\rangle^{\perp} \cap (X^{\gamma_i})^{-1}(K) \text{ is compact}$$

for all compact sets $K \subset \langle W_g(\gamma_i)\rangle^{\perp}$.

To construct Y, we proceed step by step and construct Y_j such that

(i) Y_j equals X outside $\cup_{i=1}^{j-1} S^1 * Exp_{\gamma_i, g}(B_{2\delta_i}(0) \cap \langle W_g(\gamma_i)\rangle^{\perp})$,

(ii) $Y_j^{-1}(0)$ is a subset of

$$\bigcup_{i=1}^{n} S^1 * Exp_{\gamma_i, g}(B_{\delta_i}(0) \cap \langle W_g(\gamma_i)\rangle^{\perp}),$$

(iii) the critical orbits of Y_j in

$$\bigcup_{i=1}^{j} S^1 * Exp_{\gamma_i, g}(\overline{B_{\delta_i}(0)} \cap \langle W_g(\gamma_i)\rangle^{\perp}),$$

are isolated and nondegenerate.

Since each X^{γ_i} is proper, $\|X(\cdot)\|$ is bounded below by a positive constant in

$$\cup_{i=1}^{n} S^1 * Exp_{\gamma_i, g}(B_{2\delta_i}(0) \setminus \bigcup_{i=1}^{n} S^1 * Exp_{\gamma_i, g}(B_{\delta_i}(0).$$

Consequently, (ii) remains valid for all small perturbations of X.

We start with $Y_0 := X$. In the jth step we consider $Y_{j-1}^{\gamma_j}$. By the Sard-Smale lemma there is $V_j \in \langle W_g(\gamma_j)\rangle^\perp \cap T_{\gamma_j} H^{4,2}(S^1, S^2)$ arbitrarily close to zero, such that $Y_{j-1}^{\gamma_j} + V_j$ has only nondegenerate zeros in $\overline{B_{\delta_i}(0)} \cap \langle W_g(\gamma_j)\rangle^\perp$.

Since $\gamma_j \in H^{4,2}(S^1, S^2)$, the map $\theta \mapsto \theta * \gamma_j$ is in $C^2(S^1, H^{2,2}(S^1, S^2))$ and $S^1 * \gamma_j$ is a C^2 sub-manifold of $H^{2,2}(S^1, S^2)$. Shrinking $\delta_j > 0$, we may assume the distance function $d_g(\cdot, S^1 * \gamma_j)$ in the Riemannian manifold $H^{2,2}(S^1, S^2)$ satisfies

$$d_g(\cdot, S^1 * \gamma_j)^2 \in C^2(S^1 * Exp_{\gamma_j, g}(B_{2\delta_j}(0) \cap \langle W_g(\gamma_j)\rangle^\perp), \mathbb{R}),$$

and there are $\varepsilon_{j,1}, \varepsilon_{j,2} > 0$ such that the set

$$\{\gamma \in S^1 * Exp_{\gamma_j, g}(B_{2\delta_j}(0) \cap \langle W_g(\gamma_j)\rangle^\perp) : \varepsilon_{j,1} \leq d_g(\gamma, S^1 * \gamma_j) \leq \varepsilon_{j,2}\}$$

is contained in

$$S^1 * Exp_{\gamma_j, g}\big((B_{2\delta_j}(0) \setminus \overline{B_{\delta_j}(0)}) \cap \langle W_g(\gamma_j)\rangle^\perp\big).$$

We take a cut-off function $\eta \in C_c^\infty(\mathbb{R}, [0, 1])$ such that $\eta \equiv 1$ in $[0, \varepsilon_{j,1}]$ and $\eta(x) = 0$ for $x \geq \varepsilon_{j,2}$. Using Lemma 3.1, we define

$$\theta_j \in C^2(S^1 * Exp_{\gamma_j, g}(B_{2\delta_j}(0) \cap \langle W_g(\gamma_j)\rangle^\perp), S^1)$$

by $\theta_j := \mathrm{Proj}_{S^1} \circ (\Sigma_{\gamma_j, g})^{-1}$ and the vector field Y_j on M by

$$Y_j(\gamma) := Y_{j-1}(\gamma),$$

if $\gamma \notin S^1 * Exp_{\gamma_j, g}(B_{2\delta_j}(0) \cap \langle W_g(\gamma_j)\rangle^\perp)$ and

$$Y_j(\gamma) := Y_{j-1}(\gamma) + \eta\big(d_g(\gamma, S^1 * \gamma_j)\big) \\ \mathrm{Proj}_2 \circ \psi_{\theta_j(\gamma) * \gamma_j, g}\big(Exp^{-1}_{\theta_j(\gamma) * \gamma_j, g}(\gamma), \theta_j(\gamma) * V_j\big),$$

if $\gamma \in S^1 * Exp_{\gamma_j, g}(B_{2\delta_j}(0) \cap \langle W_g(\gamma_j)\rangle^\perp)$.

Note that the map $\theta \mapsto (\theta * \gamma_j, \theta * V_j)$ is in $C^2(S^1, TH^{2,2}(S^1, S^2))$ as $(\gamma_j, V_j) \in TH^{4,2}(S^1, S^2)$. It is easy to see that Y_j is an S^1 equivariant C^2 vector field, which is orthogonal to W_g by construction. If $\|V_j\|$ is small enough, then (i)–(iii) continue to hold for Y_j as well as the Rothe property, because Rothe maps and nondegenerate critical orbits are stable under small perturbations. Moreover, $\cos(t)^2 Y_{j-1} + \sin(t)^2 Y_j$ is proper for any $t \in [0, \pi/2]$, because $\cos(t)^2 Y_{j-1} + \sin(t)^2 Y_j$ equals Y_{j-1} outside $S^1 * Exp_{\gamma_j, g}(B_{2\delta_j}(0) \cap \langle W_g(\gamma_j)\rangle^\perp)$, which is proper, and the zeros of $\cos(t)^2 Y_{j-1} + \sin(t)^2 Y_j$ inside $S^1 * Exp_{\gamma_j, g}(\overline{B_{2\delta_j}(0)} \cap \langle W_g(\gamma_j)\rangle^\perp)$ are contained in the compact set $S^1 * Exp_{\gamma_j, g}((Y_{j-1}^{\gamma_j})^{-1}([0, 1]V_j))$. If Y_{j-1} is elliptic with constant $\varepsilon_{j-1} > 0$, then taking $\|V_j\|$ small enough,

$\cos(t)^2 Y_{j-1} + \sin(t)^2 Y_j$ remains elliptic with constant $\varepsilon_j = \varepsilon_{j-1}/2$, because $Y_j(\gamma)$ and $Y_{j-1}(\gamma)$ differ only by

$$\lambda \mathrm{Proj}_{\langle W_g(\gamma)\rangle^\perp} \circ DExp_{\theta_j(\gamma)*\gamma_j,g}|_{Exp^{-1}_{\theta_j(\gamma)*\gamma_j,g}(\gamma)}(\theta_j(\gamma) * V_j),$$

where $\lambda \in [0,1]$ and $\theta_j(\gamma) * \gamma_j$ and $\theta_j(\gamma) * V_j$ are in $H^{4,2}$.

For $j = n$ we arrive at the desired vector-field Y. q.e.d.

Essentially the same arguments lead to the following lemma.

Lemma 3.10. *Let M be an open S^1-invariant subset of prime curves in $H^{2,2}(S^1, S^2)$, g_t for $t \in [0,1]$ a smooth family of metrics on S^2, and X an (M, g_t, S^1)-homotopy between two vector-fields X_0 and X_1 on M, which have only finitely many critical orbits in M that are all nondegenerate. Let $\mathcal{U}$ be an open neighborhood of the zeros of X. Then there exists an (M, g_t, S^1)-homotopy Y and $\varepsilon > 0$ such that $Y_t(\gamma) = X_t(\gamma)$ for*

$$(t, \gamma) \in ([0, \varepsilon] \cup [1 - \varepsilon, 1]) \times M \cup ([0,1] \times M) \setminus \mathcal{U},$$

and

$$DY|_{t,\gamma} : \mathbb{R} \times \langle W_{g_t}(\gamma)\rangle^{\perp,g_t} \to \langle W_{g_t}(\gamma)\rangle^{\perp,g_t}$$

is surjective for all zeros (t, γ) of Y.

For the rest of this section we let M be an open S^1-invariant subset of prime curves in $H^{2,2}(S^1, S^2)$ and X an (M, g, S^1)-admissible vector field on M. We shall define the S^1-equivariant Poincaré-Hopf index $\chi_{S^1}(X, M)$ of the vector-field X with respect to the set M. We begin with the definition of the local degree of an isolated, nondegenerate critical orbit of X.

We fix a nondegenerate critical orbit $S^1 * \gamma_0$ of X in M. As X is (M, g, S^1)-admissible, $DX|_{\gamma_0} \in \mathcal{GR}(\langle W_g(\gamma_0)\rangle^\perp)$ and we define the local degree of X at $S^1 * \gamma_0$ by

$$\deg_{loc,S^1}(X, S^1 * \gamma_0) := \mathrm{sgn} D_g X|_{\gamma_0}.$$

From (3.6) the local degree does not depend on the choice of γ_0 in $S^1 * \gamma_0$.

Definition 3.11 (S^1-degree)**.** *Let X be (M, g, S^1)-admissible. From Lemma 3.9 there is a vector field Y, which is (M, g, S^1)-homotopic to X, with only finitely many critical orbits that are all nondegenerate. The S^1-equivariant Poincaré-Hopf index (or S^1-degree) of X in M is defined by*

$$\chi_{S^1}(X, M) := \sum_{\{S^1*\gamma\subset M:\, Y(S^1*\gamma)=0\}} \deg_{loc,S^1}(Y, S^1 * \gamma).$$

If $S^1 * \gamma_0$ is an isolated critical orbit of X, we define the local S^1-degree of X in $S^1 * \gamma_0$ by

$$\deg_{loc,S^1}(X, S^1 * \gamma_0) := \chi_{S^1}(X, S^1 * B_\delta(\gamma_0)),$$

where we choose $\delta > 0$ such that $S^1 * \gamma_0$ is the unique critical orbit of X in the geodesic ball $B_\delta(S^1 * \gamma)$.

To show that the definition does not depend on the particular choice of Y or δ, and that the S^1-degree does not change under homotopies in the class of (M, g, S^1)-admissible vector-fields, we prove

Lemma 3.12. *Let g_t for $t \in [0,1]$ be a continuous family of metrics on $H^{2,2}(S^1, S^2)$. Suppose X is an (M, g_t, S^1)-homotopy between X_0 and X_1, such that the zeros of X_0 and X_1 are isolated and nondegenerate. Then*

$$\chi_{S^1}(X_0, M) = \chi_{S^1}(X_1, M).$$

Proof. By Lemma 3.10 we may assume that the homotopy X is nondegenerate, i.e. $DX^{t,\gamma}$ is surjective whenever $X(t, S^1 * \gamma) = 0$.

Fix $(t_0, \gamma_0) \in X^{-1}(0)$. From the implicit function theorem, Lemma 3.1, and Lemma 3.8 there is a regular C^1 curve $c = (c_t, c_\gamma) \in C^1(I, \mathbb{R} \times M)$ with $I = (-1, 1)$ for $t_0 \in (0, 1)$ and $I = [0, 1)$ for $t_0 \in \{0, 1\}$, such that $X(c(s)) \equiv 0$, $c(0) = (t_0, \gamma_0)$, and the map

$$S^1 \times I \ni (\theta, s) \mapsto (c_t(s), \theta * c_\gamma(s)) = \theta * c(s)$$

parametrizes the zero set $X^{-1}(0)$ locally around (t_0, γ_0), where we define the action of S^1 on tuples (t, γ) by $\theta * (t, \gamma) := (t, \theta * \gamma)$.

The ellipticity of X_t shows that $c_\gamma(s) \in H^{4,2}(S^1, S^2)$; thus $\dot{c}_\gamma(s)$ is in $T_{c_\gamma(s)} H^{2,2}(S^1, S^2)$ and from (3.1) we deduce that

$$\dot{c}_\gamma(s) \text{ is transversal to } \langle W_{g_{c_t(s)}}(c_\gamma(s)) \rangle^{\perp, g_{c_t(s)}}.$$

Since $0 \neq c'(0) \in \mathbb{R} \times \langle W_{g_{t_0}}(\gamma_0) \rangle^{\perp, g_{t_0}}$, we see from the construction of c that we may assume for all $s \in I$,

$$c'(s) \text{ is transversal to } \big(0, \dot{c}_\gamma(s)\big). \tag{3.16}$$

By the S^1-equivariance of X, (3.16), and the fact that $D_{g_{c_t(s)}} X|_{c_t(s), c_\gamma(s)}$ is a Fredholm operator of index 1 with image $\langle W_{g_{c_t(s)}}(c_\gamma(s)) \rangle^{\perp, g_{c_t(s)}}$ of codimension 1, we find

$$\text{kernel } D_{g_{c_t(s)}} X|_{c_t(s), c_\gamma(s)} = \langle c'(s), (0, \dot{c}_\gamma(s)) \rangle. \tag{3.17}$$

Fix (c_1, I_1) and (c_2, I_2) such that $S^1 * c_1(s_1) = S^1 * c_2(s_2)$ for some $s_1 \in I_1$ and $s_2 \in I_2$. Then from the uniqueness part in the construction of c_2 we get $\theta_2 \in S^1$ such that $\theta_2 * c_2(s_2) = c_1(s_1)$. From its construction, $\theta_2 * c_2'(s_2)$ is contained in the kernel of $DX|_{c_1(s_1)}$ spanned by $\langle c_1'(s_1), (0, (\dot{c_1})_\gamma(s_1)) \rangle$. Since $c_1'(s_1)$ and $\theta_2 * c_2'(s_2)$ are both transversal to $(0, (\dot{c_1})_\gamma(s_1))$, there is $0 \neq \lambda_1 \in \mathbb{R}$ and $\lambda_2 \in \mathbb{R}$ such that

$$\theta_2 * c_2'(s_2) = \lambda_1 c_1'(s_1) + \lambda_2 (0, (\dot{c_1})_\gamma(s_1)).$$

We choose a function $\theta_2 \in C^1(I, \mathbb{R}/\mathbb{Z})$ satisfying $\theta_2(s_2) = \theta_2$ and $\theta_2'(s_2) = -\lambda_2$, define $\bar{c}_2 \in C^1(I, M)$ by $\bar{c}_2(s) := \theta_2(s) * c_2(s)$, and get

$$\bar{c}_2'(s_2) = \theta_2 * c_2'(s_2) + (0, \theta_2 * (\dot{c}_2)_\gamma(s_2))\theta_2'(s_2) = \lambda_1 c_1'(s_1).$$

With an additional change in the s parameter, we may easily arrive at $\bar{c}_2'(s_2) = c_1'(s_1)$ in such a way that the map $(\theta, s) \mapsto \theta * \bar{c}_2(s)$ still parametrizes $S^1 * c_2(I_2)$. This gives a recipe how to obtain from two overlapping local parametrizations (c_1, I_1) and (c_2, I_2) of $X^{-1}(0)$ a parametrization of the union $S^1 * c_1(I_1) \cup S^1 * c_2(I_2)$. As in the classification of one dimensional manifolds [**21**], we deduce that $X^{-1}(0)$ is a two dimensional manifold with components diffeomorphic to $S^1 \times S^1$ or $S^1 \times [0,1]$.

Let P be a component of $X^{-1}(0)$ with boundary, i.e., of the type $S^1 \times [0,1]$, such that a parametrization of P is given by

$$(\theta, s) \in S^1 \times [0,1] \mapsto \theta * c(s),$$

where $c \in C^1([0,1], [0,1] \times M)$. First we change c to arrive at

$$c'(s) \in \mathbb{R} \times \langle W_{g_{c_t(s)}}(c_\gamma(s))\rangle^{\perp, g_{c_t(s)}} \subset \mathbb{R} \times T_{c_\gamma(s)} H^{2,2}(S^1, S^2). \tag{3.18}$$

To this end we note that from the definition of W_g we have

$$\mathbb{R} \times T_{c_\gamma(s)} H^{2,2}(S^1, S^2) = \mathbb{R} \times \langle W_{g_{c_t(s)}}(c_\gamma(s))\rangle^{\perp, g_{c_t(s)}} \oplus \langle (0, \dot{c}_\gamma(s))\rangle$$

and denote by Proj_1 the projection onto $\mathbb{R} \times \langle W_{g_{c_t(s)}}(c_\gamma(s))\rangle^{\perp, g_{c_t(s)}}$ with respect to this decomposition. There holds

$$c'(s) = \mathrm{Proj}_1(c'(s)) + \lambda(s)(0, \dot{c}_\gamma(s)).$$

We take $\theta \in C^1([0,1], \mathbb{R})$ such that $\theta'(s) = -\lambda(s)$ and define $\bar{c}(s) := \theta(s) * c(s)$. Then

$$\begin{aligned} \bar{c}'(s) &= \Big(c_t'(s), \theta(s) * \big(c_\gamma'(s) - \lambda(s)\dot{c}_\gamma(s)\big)\Big) \\ &\quad \in \mathbb{R} \times \theta(s) * \langle W_{g_{c_t(s)}}(c_\gamma(s))\rangle^{\perp, g_{c_t(s)}} = \mathbb{R} \times \langle W_{g_{c_t(s)}}(\bar{c}_\gamma(s))\rangle^{\perp, g_{c_t(s)}}. \end{aligned}$$

Thus, replacing c with $\bar{c}$, we may assume (3.18) holds.

Consider for $s \in [0,1]$ the family of operators

$$F_s : \mathbb{R} \times \langle W_{g_{c_t(s)}}(c_\gamma(s))\rangle^{\perp, g_{c_t(s)}} \to \mathbb{R} \times \langle W_{g_{c_t(s)}}(c_\gamma(s))\rangle^{\perp, g_{c_t(s)}}$$

defined by

$$F_s(\tau, V) := \big(\langle c'(s), (\tau, V)\rangle_{\mathbb{R} \times T_{c_\gamma(s)} H^{2,2}(S^1, S^2)}, D_{g_{c_t(s)}} X|_{c(s)}(\tau, V)\big).$$

Since

$$\mathrm{kernel}(D_{g_{c_t(s)}} X|_{c(s)}) \cap \mathbb{R} \times \langle W_{g_{c_t(s)}}(c_\gamma(s))\rangle^{\perp, g_{c_t(s)}} = \langle c'(s)\rangle,$$
$$D_g X|_{c(s)}(\mathbb{R} \times \langle W_{g_{c_t(s)}}(c_\gamma(s))\rangle^{\perp, g_{c_t(s)}}) = \langle W_{g_{c_t(s)}}(c_\gamma(s))\rangle^{\perp, g_{c_t(s)}},$$

each F_s is an isomorphism. Moreover, the Rothe property of X implies that each F_s is a Rothe map, because F_s is obtained from $DX|_{c(s)}$

through a change in finite dimensions. Consequently, $\operatorname{sgn}(F_s)$ is well defined and by its homotopy invariance independent of $s \in [0,1]$. If $c_t'(s) \neq 0$, we have again by the homotopy invariance $\operatorname{sgn}(F_s) = \operatorname{sgn}(\tilde{F}_s)$, where

$$\tilde{F}_s(\tau, V) := F_s\big(\tau, V + (c_t'(s))^{-1}\tau c_\gamma'(s)\big).$$

We have

$$\tilde{F}_s = \begin{pmatrix} (c_t'(s))^{-1}\|c'(s)\|^2 & \langle c_\gamma'(s), \cdot\rangle \\ 0 & D_\gamma X|_{c(s)} \end{pmatrix} \sim \begin{pmatrix} (c_t'(s))^{-1}\|c'(s)\|^2 & 0 \\ 0 & D_\gamma X|_{c(s)} \end{pmatrix}.$$

Hence, for all $s \in [0,1]$ such that $c_t'(s) \neq 0$, there holds

$$\operatorname{sgn}(F_s) = \operatorname{sgn}(\tilde{F}_s) = \operatorname{sgn}(c_t'(s))\operatorname{sgn}(D_\gamma X|_{c(s)}). \tag{3.19}$$

Let $S^1 * \alpha_1, \ldots, S^1 * \alpha_{k_0}$ be the critical orbits of X_0 and $S^1 * \beta_1, \ldots, S^1 * \beta_{k_1}$ be the critical orbits of X_1. The critical orbits of X_0 and X_1 are boundary points of $X^{-1}(0)$. From (3.19) we get

- $\operatorname{sgn} DX_0|_{\alpha_i} = -\operatorname{sgn} DX_0|_{\alpha_j}$, if $S^1 * \alpha_i$ and $S^1 * \alpha_j$ are boundary orbits of the same component of $X^{-1}(0)$,
- $\operatorname{sgn} DX_1|_{\beta_i} = -\operatorname{sgn} DX_1|_{\beta_j}$, if $S^1 * \beta_i$ and $S^1 * \beta_j$ are boundary orbits of the same component of $X^{-1}(0)$,
- $\operatorname{sgn} DX_0|_{\alpha_i} = \operatorname{sgn} DX_1|_{\beta_j}$, if $S^1 * \alpha_i$ and $S^1 * \beta_j$ are boundary orbits of the same component of $X^{-1}(0)$.

Putting the above facts together, we see that

$$\chi_{S^1}(X_0, M) = \chi_{S^1}(X_1, M).$$

q.e.d.

4. The Degree of an Isolated Critical Orbit

Let $\gamma \in H^{2,2}(S^1, S^2)$ be a prime, regular curve such that $S^1 * \gamma$ is an isolated critical orbit of $X_{k,g}$. Then the curve

$$\mu(t) := \gamma\big(t|\dot{\gamma}|_g^{-1}\big)$$

is a closed k-magnetic geodesic with minimal period $\omega := |\dot{\gamma}|_g$ such that $t \mapsto (\mu(t), \dot{\mu}(t))$ lies in the bundle

$$E_1 := \{(x, V) \in TS^2 : \ |V|_g = 1\}.$$

We fix a transversal section Σ in E_1 at the point $\theta := (\gamma(0), \omega^{-1}\dot{\gamma}(0))$ and denote by $P : B_1 \cap \Sigma \to B_2 \cap \Sigma$ the corresponding Poincaré map, where B_1, B_2 are open neighborhoods of θ (see [**1**, chap. 7–8]).

In this section we shall show

Lemma 4.1. *Under the above assumptions, θ is an isolated fixed point of P and there holds*

$$\deg_{loc,S^1}(X_{k,g}, S^1 * \gamma) = -i(P, \theta) \geq -1,$$

where $i(P, \theta)$ denotes the index of the isolated fixed point θ.

We consider the linearizations of equation (1.1) and (1.2) given by

$$\begin{aligned} 0 &= -D_{t,g}^2 V - R_g\big(V,\dot\mu\big)\dot\mu + k(\mu)J_g(\mu)D_{t,g}V \\ &\quad + \big(k'(\mu)V\big)J_g(\mu)\dot\mu + k(\mu)\big(D_g J_g|_\mu V\big)\dot\mu \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} 0 &= -D_{t,g}^2 V - R_g\big(V,\dot\gamma\big)\dot\gamma + |\dot\gamma|_g k(\gamma)J_g(\gamma)D_{t,g}V \\ &\quad + |\dot\gamma|_g\big(k'(\gamma)V\big)J_g(\gamma)\dot\gamma + |\dot\gamma|_g k(\gamma)\big(D_g J_g|_\gamma V\big)\dot\gamma \\ &\quad + |\dot\gamma|_g^{-1}\langle D_{t,g}V,\dot\gamma\rangle_g k(\gamma)J_g(\gamma)\dot\gamma \\ &= -D_{t,g}^2 V + T(V, D_{t,g}V), \end{aligned} \tag{4.2}$$

where $T(V, D_{t,g}V)$ abbreviates all terms containing V or $D_{t,g}V$. For $(V_1, V_2) \in T_{\gamma(0)}S^2 \times T_{\gamma(0)}S^2$ we denote by $\Phi(\cdot,(V_1,V_2))$, respectively $U(\cdot,(V_1,V_2))$, the solution to (4.1), respectively (4.2), with initial values

$$V(0) = V_1 \text{ and } D_{t,g}V(0) = V_2.$$

Then

$$dP|_\theta = \mathrm{Proj}_{T_\theta\Sigma} \circ \big(\Phi(\omega,\cdot), D_{t,g}\Phi(\omega,\cdot)\big)|_{T_\theta\Sigma},$$

where $\mathrm{Proj}_{T_\theta\Sigma}$ is the projection onto $T_\theta\Sigma$ with kernel given by

$$\langle (\dot\mu(0), D_{t,g}\dot\mu(0))^T, (0,\dot\mu(0))^T\rangle.$$

Lemma 4.2. *Suppose θ is a nondegenerate fixed point of P, i.e. the linearized Poincaré map $dP|_\theta : T_\theta\Sigma \to T_\theta\Sigma$ has no eigenvalues equal to one. Then $S^1 * \gamma$ is a nondegenerate critical orbit of $X_{k,g}$ and*

$$\deg_{loc,S^1}(X_{k,g}, S^1 * \gamma) = -sgn\big(\det(dP|_\theta - I)\big).$$

Proof. Since index and nondegeneracy do not depend on the transversal section, we may assume $T_\theta\Sigma = T_\omega\Sigma$, where we write for $q \in \mathbb{R}$

$$\begin{aligned} T_q\Sigma := \{&(V_1,V_2)^T \in T_\theta(TS^2) \cong T_{\gamma(0)}S^2 \times T_{\gamma(0)}S^2 : \\ &(V_1,V_2)^T \text{ is orthogonal to } (\dot\gamma(0), qD_{t,g}\dot\gamma(0))^T \text{ and } (0,\dot\gamma(0))^T\} \end{aligned}$$

with respect to the componentwise scalar product.

From (1.2) and the symmetries of the curvature tensor and J_g we obtain, for any solution V to (4.1) or (4.2),

$$\begin{aligned} \frac{d}{dt}\langle D_{t,g}V(t),\dot\gamma(t)\rangle_g &= \langle D_{t,g}^2 V(t),\dot\gamma(t)\rangle_g + \langle D_{t,g}V(t), D_{t,g}\dot\gamma(t)\rangle_g \\ &= \langle |\dot\gamma|_g k(\gamma)(D_g J_g|_{\gamma(t)}V(t))\dot\gamma(t),\dot\gamma(t)\rangle_g \\ &= 0, \end{aligned} \tag{4.3}$$

because for a variation $(s,t) \mapsto \Gamma(s,t)$ of γ with $\partial_s \Gamma(0,\cdot) = V$ we find

$$\begin{aligned} 0 &= \frac{d}{ds} \langle J_g(\Gamma(s,t)) \partial_t \Gamma(s,t), \partial_t \Gamma(s,t) \rangle_g |_{s=0} \\ &= \langle (D_g J_g|_{\gamma(t)} V(t)) \dot\gamma(t), \dot\gamma(t) \rangle_g \\ &\quad + \underbrace{\langle J_g(\gamma(t)) V(t), \dot\gamma(t) \rangle_g + \langle J_g(\gamma(t)) \dot\gamma(t), V(t) \rangle_g}_{=0} \end{aligned}$$

Up to scaling with ω in t, equations (4.1) and (4.2) only differ by

$$|\dot\gamma|_g^{-1} \langle D_{t,g} V, \dot\gamma \rangle_g k(\gamma) J_g(\gamma) \dot\gamma.$$

Since $\langle D_{t,g} U, \dot\gamma \rangle_g$ is constant, we get for V_2 orthogonal to $\dot\gamma(0)$

$$U\big(t\omega^{-1}, (V_1, \omega V_2)\big) = \Phi\big(t, (V_1, V_2)\big).$$

Consequently,

$$dP|_\theta = \mathrm{Proj}_{T_\omega \Sigma} \circ A_\omega^{-1} \circ \big(U(1,\cdot), D_{t,g} U(1,\cdot)\big) \circ A_\omega|_{T_\omega \Sigma}$$

where A_ω, $\mathrm{Proj}_{T_\theta \Sigma} \in \mathcal{L}(T_{\gamma(0)} S^2 \times T_{\gamma(0)} S^2)$ are given by

$$A_\omega(V_1, V_2) := (V_1, \omega V_2),$$

$$\mathrm{Proj}_{T_\theta \Sigma} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} := \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} - \frac{\langle \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \begin{pmatrix} \dot\gamma(0) \\ \omega D_{t,g} \dot\gamma(0) \end{pmatrix} \rangle_g}{\| \begin{pmatrix} \dot\gamma(0) \\ D_{t,g} \dot\gamma(0) \end{pmatrix} \|_g^2} \begin{pmatrix} \dot\gamma(0) \\ \omega^{-1} D_{t,g} \dot\gamma(0) \end{pmatrix}.$$

We note that $A_\omega : T_\theta \Sigma \xrightarrow{\cong} T_1 \Sigma$ and $\mathrm{Proj}_{T_\theta \Sigma} \circ A_\omega^{-1} = A_\omega^{-1} \circ \mathrm{Proj}_{T_1 \Sigma}^{\perp}$. Hence, we may replace in the following $dP|_\theta$ by $dP|_\gamma : T_1 \Sigma \to T_1 \Sigma$

$$dP|_\gamma := \mathrm{Proj}_{T_1 \Sigma}^{\perp} \circ \big(U(1,\cdot), D_{t,g} U(1,\cdot)\big)|_{T_1 \Sigma}.$$

To show that $S^1 * \gamma$ is a nondegenerate critical orbit, we fix $V \in \langle W_g(\gamma) \rangle^\perp$ such that $DX_{k,g}|_\gamma(V) = 0$. There are $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} := \begin{pmatrix} V(0) \\ D_{t,g} V(0) \end{pmatrix} + \lambda_1 \begin{pmatrix} \dot\gamma(0) \\ D_{t,g} \dot\gamma(0) \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ \dot\gamma(0) \end{pmatrix} \in T_1 \Sigma.$$

Using the fact that V, $\dot\gamma$, and $t\dot\gamma$ solve (4.2), we get

$$\mathrm{Proj}_{T_1 \Sigma}^{\perp} \begin{pmatrix} U(1, (W_1, W_2)) \\ D_{t,g} U(1, (W_1, W_2)) \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}.$$

Since $dP|_\gamma$ has no eigenvalues equal to one, (W_1, W_2) equals $(0,0)$ and $V = \lambda_1 \dot\gamma + \lambda_2 t \dot\gamma$. From the periodicity of V we obtain $\lambda_2 = 0$, and the fact that $V \in \langle W_g(\gamma) \rangle^\perp$ gives $\lambda_1 = 0$. Consequently, $S^1 * \gamma$ is a nondegenerate critical orbit of $X_{k,g}$.

We consider $\tilde X \in \mathcal{L}(T_\gamma H^{2,2}(S^1, S^2) \times \mathbb{R})$ defined by

$$\tilde X(V, \delta) := \big(DX_{k,g}|_\gamma(V) + \delta W_g(\gamma), \delta - \varepsilon \langle V, W_g(\gamma) \rangle_g\big),$$

where $\varepsilon > 0$ will be chosen later. Then $\tilde{X}$ is of the form *identity* + *compact*. If $\tilde{X}(V_0, \delta_0) = 0$, then $\delta_0 = 0$, since $DX_{k,g}|_\gamma(V_0)$ is orthogonal to $W_g(\gamma)$. From $\varepsilon > 0$ we get $V_0 \in \langle W_g(\gamma)\rangle^\perp$ and finally $V_0 = 0$. Thus, $\tilde{X}$ is invertible. With respect to the decomposition

$$T_\gamma H^{2,2}(S^1, S^2) \times \mathbb{R} = \langle W_g(\gamma)\rangle^\perp \times \{0\} \oplus \langle W_g(\gamma)\rangle \times \{0\}\{0\} \times \mathbb{R}$$

we have

$$\tilde{X} = \begin{pmatrix} DX_{k,g}|_\gamma & * & 0 \\ 0 & 0 & 1 \\ 0 & -\varepsilon & 1 \end{pmatrix},$$

such that

$$\deg(\tilde{X}, (0,0)) = \deg(DX_{k,g}|_\gamma|_{\langle W_g(\gamma)\rangle^\perp}, 0) = \deg_{loc,S^1}(X_{k,g}, S^1 * \gamma),$$

where deg denotes the usual Leray-Schauder degree.

Using the ideas in [**20**, chap. 3], we define a homotopy

$$\Phi : [0,1] \times T_\gamma H^{2,2}(S^1, S^2) \times \mathbb{R} \to T_\gamma H^{2,2}(S^1, S^2) \times \mathbb{R},$$

by

$$\begin{aligned} &\Phi(s, (V, \delta)) := \\ &\Big(V + s(-D_{t,g}^2 + 1)^{-1}\big(T(V, D_{t,g}V) - V\big) + s\delta W_g(\gamma) \\ &\quad + (1-s)(-D_{t,g}^2 + 1)^{-1} \\ &\quad \Big(D_{t,g}^2 U(\cdot, (V(0), D_{t,g}V(0)), \delta) - U(\cdot, (V(0), D_{t,g}V(0)), \delta)\Big), \\ &\quad \delta - \varepsilon \int_0^1 (-D_{t,g}^2 + 1)\Big((1-s)V + sU(\cdot, (V(0), D_{t,g}V(0)), \delta)\Big)\dot{\gamma}\Big), \end{aligned}$$

where $(-D_{t,g}^2 + 1)^{-1}$ maps $T_\gamma L^2(S^1, S^2)$ to $T_\gamma H^{2,2}(S^1, S^2)$ and the function $U(\cdot, (V_1, V_2), \delta)$ denotes the solution to

$$0 = -D_{t,g}^2 V + T(V, D_{t,g}V) + \delta\dot{\gamma},$$

with initial values $V(0) = V_1$ and $D_{t,g}V(0) = V_2$.

Fix $(s_0, V_0, \delta_0) \in [0,1] \times T_\gamma H^{2,2}(S^1, S^2) \times \mathbb{R}$ such that

$$\Phi(s_0, (V_0, \delta_0)) = (0, 0).$$

Then V_0 is a periodic solution of

$$\begin{aligned} 0 = -D_{t,g}^2 V_0 + V_0 + sT(V_0, D_{t,g}V_0) - sV_0 + s\delta\dot{\gamma} + (1 - s_0) \\ \Big(D_{t,g}^2 U(\cdot, (V(0), D_{t,g}V(0)), \delta) - U(\cdot, (V(0), D_{t,g}V(0)), \delta)\Big). \end{aligned} \tag{4.4}$$

Since $U(\cdot, (V_0(0), D_{t,g}V_0(0)), \delta_0)$ is a solution to (4.4) with the same initial values, we see that

$$V_0 = U(\cdot, (V_0(0), D_{t,g}V_0(0)), \delta_0).$$

In this case $\Phi(s_0, (V_0, \delta_0)) = (0,0)$ is equivalent to

$$\big(DX_{k,g}|_\gamma(V_0) + \delta_0 W_g(\gamma), \delta_0 - \varepsilon\langle V_0, W_g(\gamma)\rangle_g\big) = (0,0),$$

which shows that $V_0 = 0$ and $\delta_0 = 0$. Consequently,

$$\deg(\tilde{X}, (0,0)) = \deg(\Phi(1,\cdot), (0,0)) = \deg(\Phi(0,\cdot), (0,0)).$$

We choose $\tilde{E}_i = (E_i, \delta_i) \in T_\gamma H^{2,2}(S^1, S^2) \times \mathbb{R}$ for $1 \le i \le 5$ such that

$$\{(E_i(0), D_{t,g}E_i(0), \delta_i)^T :\ 1 \le i \le 5\}$$

is an orthonormal basis of $T_{\gamma(0)}S^2 \times T_{\gamma(0)}S^2 \times \mathbb{R}$ with respect to the componentwise scalar product. Since $\Phi(0, (V,\delta)) = (V,\delta)$ for all

$$(V,\delta) \in W_0 := \{(V,0) \in T_\gamma H^{2,2}(S^1,S^2) \times \mathbb{R} :\ V(0) = 0 = D_{t,g}V(0)\},$$

there holds for $W_1 := \langle \tilde{E}_i :\ 1 \le i \le 5\rangle$

$$\deg(\Phi(0,\cdot),(0,0)) = \deg(P_{W_1} \circ \Phi(0,\cdot)|_{W_1}, 0),$$

where $P_{W_1} : T_\gamma H^{2,2}(S^1,S^2) \times \mathbb{R} \to W_1$ is given by

$$P_{W_1}(V,\delta) := \sum_{i=1}^{5} \big(\langle V(0), E_i(0)\rangle + \langle D_{t,g}V(0), D_{t,g}E_i(0)\rangle + \delta\delta_i\big)\tilde{E}_i.$$

Note that P_{W_1} is the projection onto W_1 with kernel W_0.

We define $Ev_0 :\ T_\gamma H^{2,2}(S^1,S^2) \times \mathbb{R} \to T_{\gamma(0)}S^2 \times T_{\gamma(0)}S^2 \times \mathbb{R}$ by

$$Ev_0(V,\delta) := \big(V(0), D_{t,g}V(0), \delta\big).$$

Then $Ev_0|_{W_1}$ is an isomorphism and we have

$$\deg(P_{W_1} \circ \Phi(0,\cdot)|_{W_1}, 0) = \deg(Ev_0 \circ P_{W_1} \circ \Phi(0,\cdot) \circ (Ev_0|_{W_1})^{-1}, 0).$$

We note that for a function U,

$$(-D_{t,g}^2 + 1)^{-1}\big((-D_{t,g}^2+1)U\big) = U + Q,$$

where Q solves $(-D_{t,g}^2 + 1)Q = 0$ with boundary conditions

$$\begin{aligned} Q(0) - Q(1) &= U(1) - U(0),\\ D_{t,g}Q(0) - D_{t,g}Q(1) &= D_{t,g}U(1) - D_{t,g}U(0). \end{aligned}$$

We let B_1 and B_2 be the smooth parallel vector fields along γ such that $\{B_1(0), B_2(0)\}$ is a basis of $T_{\gamma(0)}S^2$. Then the set Λ_0 of functions Q with $(-D_{t,g}^2 + 1)Q = 0$ is given by

$$\Lambda_0 := \{e^t \sum_{i=1}^{2} \lambda_i B_i(t) + e^{-t}\sum_{i=1}^{2} \mu_i B_i(t) :\ \lambda_1,\ \lambda_2,\ \mu_1,\ \mu_2 \in \mathbb{R}\}.$$

We define $L_0,\ L_1 :\ \Lambda_0 \to T_{\gamma(0)}S^2 \times T_{\gamma(0)}S^2$ by

$$\begin{aligned} L_0(Q) &:= \big(Q(0) - Q(1), D_{t,g}Q(0) - D_{t,g}Q(1)\big),\\ L_1(Q) &:= \big(Q(0), D_{t,g}Q(0)\big). \end{aligned}$$

It is easy to see that L_0 and L_1 are isomorphisms. We have

$$\begin{aligned}&Ev_0 \circ P_{W_1} \circ \Phi(0,\cdot) \circ (Ev_0|_{W_1})^{-1}(V_1,V_2,\delta)\\&= \Bigg(L_1 \circ L_0^{-1} \begin{pmatrix} U(1,(V_1,V_2),\delta) - U(0,(V_1,V_2),\delta) \\ D_{t,g}U(1,(V_1,V_2),\delta) - D_{t,g}U(0,(V_1,V_2),\delta) \end{pmatrix},\\&\qquad \delta - \varepsilon \int_0^1 (-D_{t,g}^2 + 1)\Big(U(\cdot,(V_1,V_2),\delta)\Big)\dot\gamma \Bigg).\end{aligned}$$

The map $L_1 \circ L_0^{-1}$ may be computed explicitly solving a system of linear equations. Using the fact that the parallel transport is an isometry, it is easy to see that $\det(L_1 \circ L_0^{-1}) > 0$. Thus we may replace $L_1 \circ L_0^{-1}$ by id without changing the degree. Hence we need to compute the degree of $Y \in \mathcal{L}(T_{\gamma(0)}S^2 \times T_{\gamma(0)}S^2 \times \mathbb{R})$ given by

$$\begin{aligned}Y(V_1,V_2,\delta) := \Bigg(&\begin{pmatrix} U(1,(V_1,V_2),\delta) - U(0,(V_1,V_2),\delta) \\ D_{t,g}U(1,(V_1,V_2),\delta) - D_{t,g}U(0,(V_1,V_2),\delta) \end{pmatrix},\\&\delta - \varepsilon \int_0^1 (-D_{t,g}^2 + 1)\Big(U(\cdot,(V_1,V_2),\delta)\Big)\dot\gamma \Bigg).\end{aligned}$$

To compute $\deg(Y,0)$, we decompose $T_{\gamma(0)}S^2 \times T_{\gamma(0)}S^2 \times \mathbb{R}$ into

$$T_1\Sigma \times \{0\} \oplus \big\langle (\dot\gamma(0), D_{t,g}\dot\gamma(0), 0)\big\rangle \oplus \big\langle (0,\dot\gamma(0),0)\big\rangle \oplus \big\langle (0,0,1)\big\rangle,$$

where the decomposition is orthogonal with respect to the componentwise scalar product. We have

$$\begin{aligned}Y(\dot\gamma(0), D_{t,g}\dot\gamma(0), 0) &= \big(0,0,-\varepsilon\|\dot\gamma\|_{H^{1,2}}^2\big),\\ Y(0,\dot\gamma(0),0) &= \big(\dot\gamma(0), D_{t,g}\dot\gamma(0), -\varepsilon(\|\sqrt{t}\dot\gamma\|_{L^2}^2 + \|\sqrt{t}D_{t,g}\dot\gamma\|_{L^2}^2)\big).\end{aligned}$$

We obtain analogously to (4.3)

$$\begin{aligned}\frac{d}{dt}\langle D_{t,g}U(t,(0,0),1), \dot\gamma(t)\rangle_g &= \langle \dot\gamma(t), \dot\gamma(t)\rangle_g > 0,\\ \langle Y(0,0,1), (0,\dot\gamma(0),0)\rangle_g &= \langle D_{t,g}U(1,(0,0),1), \dot\gamma(0)\rangle_g > 0.\end{aligned}$$

Choosing $\varepsilon > 0$ small enough we find

$$\langle Y(0,0,1),(0,0,1)\rangle_g = 1 - O(\varepsilon) > 0.$$

Moreover, again using (4.3), we get for all $(V_1,V_2) \in T_1\Sigma$

$$\langle Y(V_1,V_2,0), (0,\dot\gamma(0),0)\rangle_g = 0.$$

Consequently, we obtain with respect to the above decomposition

$$\begin{aligned}\deg(Y,0) &= \operatorname{sgn}\det \begin{pmatrix} P|_\gamma - id & 0 & 0 & * \\ * & 0 & 1 & * \\ 0 & 0 & 0 & + \\ * & - & - & + \end{pmatrix}\\ &= -\operatorname{sgn}\det(P|_\gamma - id),\end{aligned}$$

which proves the claim. q.e.d.

Proof of Lemma 4.1. The fact that θ is an isolated fixed point is obvious from the properties of the Poincaré map P (see [**1**, thm. 7.1.2]).

We may choose $\delta > 0$ such that $S^1 * \gamma$ is the unique critical orbit of $X_{k,g}$ in the geodesic ball $B_\delta(S^1 * \gamma)$, θ is the unique fixed point of P in the geodesic ball $B_\delta(\theta)$, $i(P,\theta) = i(P, B_\delta(\theta))$, and

$$\deg_{S^1}(X_{k,g}, B_\delta(S^1 * \gamma)) = \deg_{S^1,loc}(X_{k,g}, S^1 * \gamma).$$

From the homotopy invariance and a Kupka-Smale theorem for magnetic flows in [**22**], we may assume by Lemma 4.2 that the critical orbits of $X_{k,g}$ in $B_\delta(S^1 * \gamma)$ and the fixed points of P in $B_\delta(\theta)$ are nondegenerate. Again using Lemma 4.2, we find

$$\deg_{S^1}(X_{k,g}, B_\delta(S^1 * \gamma)) = -i(P, B_\delta(\theta)).$$

Finally, since we may assume that the Poincaré map is area preserving (see [**1**, thm. 8.1.3]), we obtain from [**27**, **23**]

$$i(P,\theta) \leq 1.$$

This yields the claim. q.e.d.

5. The Unperturbed Problem

Let $S^2 = \partial B_1(0) \subset \mathbb{R}^3$ be the standard round sphere with induced metric g_0. Then the prescribed geodesic curvature equation with $k \equiv k_0$ on (S^2, g_0) is given by

$$Proj_{\gamma^\perp} \ddot{\gamma} = |\dot{\gamma}| k_0 \gamma \times \dot{\gamma}, \tag{5.1}$$

where $\gamma \in H^{2,2}(S^1, S^2)$, $\dot{\gamma}$ and $\ddot{\gamma}$ are the usual derivatives of γ considered as a curve in $\mathbb{R}^3$, and $|\dot{\gamma}|$ is the Euclidean norm of $\dot{\gamma}$ in $\mathbb{R}^3$.

To compute the S^1-degree of the unperturbed equation (5.1), we proceed in three steps.

Step 1: We compute explicitly the set $\mathcal{Z}$ of simple solutions in $H^{2,2}(S^1, S^2)$ to (5.1) and show that $\mathcal{Z}$ is a finite dimensional, nondegenerate manifold, in the sense that we have for all $\alpha \in \mathcal{Z}$

$$T_\alpha \mathcal{Z} = \text{kernel}(D_{g_0} X_{k_0,g_0}|_\alpha),$$
$$T_\alpha H^{2,2}(S^1, S^2) = T_\alpha \mathcal{Z} \oplus R(D_{g_0} X_{k_0,g_0}|_\alpha).$$

Step 2: We perform a finite dimensional reduction of a slightly perturbed problem: We consider for $k_1 \in C^2(S^2, \mathbb{R})$, which will be chosen later, and $\varepsilon \in \mathbb{R}$, which is assumed to be very small, the perturbed vector field $X_{g_0,\varepsilon}$ defined by

$$\begin{aligned} X_{g_0,\varepsilon}(\gamma) &:= (-D^2_{t,g_0} + 1)^{-1}\big(- D_{t,g_0}\dot{\gamma} + |\dot{\gamma}|_{g_0}(k_0 + \varepsilon k_1(\gamma))\gamma \times \dot{\gamma}\big) \\ &= X_{k_0,g_0}(\gamma) + \varepsilon K_1(\gamma), \end{aligned}$$

where the vector field K_1 is given by

$$K_1(\gamma) := (-D^2_{t,g_0} + 1)^{-1} |\dot{\gamma}|_{g_0}\big(k_1(\gamma)\gamma \times \dot{\gamma}\big).$$

We show that if $\alpha_0 \in \mathcal{Z}$ is a nondegenerate zero of the vector field $\alpha \mapsto P_1(\alpha) \circ K_1(\alpha)$ on $\mathcal{Z}$, where $P_1(\alpha)$ is a projection onto $T_\alpha \mathcal{Z}$ defined below, then there is a unique nondegenerate critical orbit $S^1 * \gamma(\varepsilon)$ for any $0 < \varepsilon << 1$ such that $\gamma(\varepsilon)$ converges to α_0 as $\varepsilon \to 0^+$ and

$$\deg_{loc,S^1}(X_{g_0,\varepsilon}, S^1 * \gamma(\varepsilon)) = -\deg_{loc}(P_1(\cdot) \circ K_1(\cdot), \alpha_0).$$

Step 3: We choose

$$k_1(x) := \langle x, e_3 \rangle \text{ for } x \in S^2 = \partial B_1(0) \subset \mathbb{R}^3,$$

where $\{e_1, e_2, e_3\}$ denotes the standard basis of $\mathbb{R}^3$, and show that $P_1(\cdot) \circ K_1(\cdot)$ has exactly two nondegenerate zeros of degree $+1$. This yields the formula $\chi_{S^1}(X_{k_0,g_0}, M) = -2$, where M is the subset of $H^{2,2}(S^1, S^2)$ consisting of simple and regular curves.

5.1. The simple solutions of (5.1)**.** Differentiating twice the identity $|\gamma|^2 = 1$, we find $\langle \ddot{\gamma}, \gamma \rangle + |\dot{\gamma}|^2 \equiv 0$ and (5.1) is equivalent to

$$\ddot{\gamma} = |\dot{\gamma}| k_0 \gamma \times \dot{\gamma} - |\dot{\gamma}|^2 \gamma. \tag{5.2}$$

In order to solve the ordinary differential (5.2), we fix initial conditions

$$\gamma(0) = \gamma_0 \in S^2 \text{ and } \dot{\gamma}(0) = \tilde{v}_0 \in T_{\gamma_0} S^2.$$

If $\tilde{v}_0 = 0$, then γ is given by the constant curve $\gamma \equiv \gamma_0$. We may assume in the sequel

$$\lambda := |\tilde{v}_0| > 0.$$

If $k_0 \neq 0$, then there is a unique $r = r(k_0) \in (-1, 1) \setminus \{0\}$ such that

$$k_0 = \frac{\sqrt{1 - r^2}}{r}.$$

For $k_0 = 0$, the case of geodesics, we may take $r = \pm 1$.

For $\lambda > 0$ and a positive oriented orthonormal system $\{v_0, v_1, w\}$, we define the function $\alpha \in C^\infty(\mathbb{R}, S^2)$ by

$$\alpha(t, \lambda, v_0, v_1, w) := \sqrt{1 - r^2} w + r \cos(\lambda r^{-1} t) v_1 + r \sin(\lambda r^{-1} t) v_0.$$

A direct calculation shows that $\alpha(\cdot, \lambda, v_0, v_1, w)$ solves (5.2). Moreover, if we take for given $(\gamma_0, \tilde{v}_0)$ the positive oriented orthonormal system (v_0, v_1, w) defined by

$$v_0 := \lambda^{-1} \tilde{v}_0,\ v_1 := r\gamma_0 + \sqrt{1 - r^2}(v_0 \times \gamma_0),\ w := (v_1 \times v_0)$$

and $\lambda > 0$ as above, then $\alpha(\cdot, \lambda, v_0, v_1, w)$ satisfies the initial conditions

$$\alpha(0, \lambda, v_0, v_1, w) = \gamma_0,\ \dot{\alpha}(0, \lambda, v_0, v_1, w) = \tilde{v}_0.$$

Since we are only interested in solutions in $H^{2,2}(S^1, S^2)$, we get an extra condition on λ, i.e. the 1-periodicity leads to

$$\lambda \in 2\pi \mathbb{Z} r.$$

Hence the simple solutions in $H^{2,2}(S^1,S^2)$ of (5.1) are given by

$$\mathcal{Z} := \{\alpha(\cdot, 2\pi|r|, v_0, v_1, w) : \{v_0, v_1, w\} \text{ is a positive orthonormal system in } \mathbb{R}^3\}.$$

$S0(3)$ acts on solutions: if γ solves (5.1), so does $A\circ\gamma$ for any $A \in SO(3)$. We have

$$A \circ \alpha(\cdot, 2\pi|r|, v_0, v_1, w) = \alpha(\cdot, 2\pi|r|, A(v_0), A(v_1), A(w)),$$

and the set of solutions is parametrized by $SO(3)$. It is easy to see that

$$\alpha(\cdot, 2\pi|r|, v_0, v_1, w) = \theta * \alpha(\cdot, 2\pi|r|, v_0', v_1', w')$$

for some $\theta \in S^1$ if and only if $w = w'$. Consequently, the set of critical orbits is parametrized by $w \in S^2$. In the sequel we fix $k_0 > 0$ and $r > 0$.

To compute the kernel of $D_{g_0}X_{k_0,g_0}|_\alpha$ at $\alpha = \alpha(\cdot, 2\pi r, v_0, v_1, w)$ for some fixed system (v_0, v_1, w), we note that for $V \in T_\alpha H^{2,2}(S^1, S^2)$

$$R_{g_0}(V, \dot\alpha)\dot\alpha = V|\dot\alpha|^2 - \langle V, \dot\alpha\rangle\dot\alpha$$

and hence by (2.3)

$$\begin{aligned} D_{g_0}X_{k_0,g_0}|_\alpha(V) &= (-D^2_{g_0,t} + 1)^{-1}\big(- D^2_{t,g_0}V - V|\dot\alpha|^2 + \langle V, \dot\alpha\rangle\dot\alpha \\ &\quad + |\dot\alpha|^{-1}\langle D_{t,g_0}V, \dot\alpha\rangle k_0(\alpha \times \dot\alpha) + |\dot\alpha| k_0(\alpha \times D_{t,g_0}V)\big). \end{aligned} \tag{5.3}$$

Due to the geometric origin of equation (5.1) we deduce that

$$\begin{aligned} &W_1(t, v_0, v_1, w) := \dot\alpha = 2\pi r(-\sin(2\pi t)v_1 + \cos(2\pi t)v_0), \\ &W_1(0, v_0, v_1, w) = 2\pi r v_0,\ D_{t,g_0}W_1(0, v_0, v_1, w) = -4\pi^2 r^3 k_0(k_0 v_1 - w), \\ &W_0(t, v_0, v_1, w) := t\dot\alpha, \\ &W_0(0, v_0, v_1, w) = 0,\ D_{t,g_0}W_0(0, v_0, v_1, w) = 2\pi r v_0, \end{aligned}$$

solve the equation

$$\begin{aligned} 0 = &- D^2_{t,g_0}W - W|\dot\alpha|^2 + \langle W, \dot\alpha\rangle\dot\alpha \\ &+ |\dot\alpha|^{-1}\langle D_{t,g_0}W, \dot\alpha\rangle k_0(\alpha \times \dot\alpha) + |\dot\alpha| k_0(\alpha \times D_{t,g_0}W). \end{aligned} \tag{5.4}$$

The vector-field W_1 corresponds to invariance with respect to the S^1-action, $\theta \mapsto \alpha(\cdot + \theta)$, and W_0 stems from the change of parameter $s \mapsto \alpha(\cdot s)$. The $SO(3)$ invariance leads to two additional vector-fields in the kernel of $D_{g_0}X_{k_0,g_0}|_\alpha$, i.e. we let

$$\begin{aligned} w_{1,s} &:= \cos(s)w + sin(s)v_1,\ v_0 = v_0, \\ v_{1,s} &= v_0 \times w_{1,s} = \cos(s)v_1 - \sin(s)w, \\ w_{2,s} &:= \cos(s)w + sin(s)v_0,\ v_1 = v_1, \\ v_{0,s} &= w_{2,s} \times v_1 = \cos(s)v_0 - \sin(s)w \end{aligned}$$

and get

$$W_2(t, v_0, v_1, w) := \frac{d}{ds}\big(\alpha(\cdot, 2\pi r, v_0, v_{1,s}, w_{1,s})\big|_{s=0} = rk_0v_1 - r\cos(2\pi t)w,$$
$$W_2(0, v_0, v_1, w) = \sqrt{1-r^2}v_1 - rw,\ D_{t,g_0}W_2(0, v_0, v_1, w) = 0,$$
$$W_3(t, v_0, v_1, w) := \frac{d}{ds}\big(\alpha(\cdot, 2\pi r, v_{0,s}, v_1, w_{2,s})\big|_{s=0} = rk_0v_0 - r\sin(2\pi t)w,$$
$$W_3(0, v_0, v_1, w) = rk_0v_0,\ D_{t,g_0}W_3(0, v_0, v_1, w) = 2\pi r^3(k_0v_0 - w). \tag{5.5}$$

We will omit the dependence of W_i on (v_0, v_1, w), if there is no possibility of confusion. Since the initial values of $W_0, \dots, W_3$ are linearly independent in $\big(T_{\alpha(0)}S^2\big)^2$, any solution of (5.4) is a linear combination of $W_0, \dots, W_3$. As only $W_1, \dots, W_3$ are periodic, we obtain

$$\operatorname{kernel}(D_{g_0}X_{k_0,g_0}|_\alpha) = \langle W_1,\, W_2,\, W_3\rangle = T_\alpha\mathcal{Z}. \tag{5.6}$$

To find the image of $D_{g_0}X_{k_0,g_0}|_\alpha$ we note that the moving frame $\{\dot\alpha, \alpha\times\dot\alpha\}$ is an orthogonal system in $T_\alpha S^2$ for any $t \in S^1$. Thus any $V \in T_\alpha H^{2,2}(S^1, S^2)$ may be written as

$$V = \lambda_1\dot\alpha + \lambda_2(\alpha\times\dot\alpha)$$

for some functions $\lambda_1,\ \lambda_2 \in H^{2,2}(S^1, \mathbb{R})$. Using the fact that

$$D_{t,g_0}\dot\alpha = |\dot\alpha|k_0(\alpha\times\dot\alpha) \text{ and } D_{t,g_0}(\alpha\times\dot\alpha) = -|\dot\alpha|k_0\dot\alpha,$$

we may express $D_{t,g_0}V$ and $(D_{t,g_0})^2V$ in terms of λ_1 and λ_2. This leads to

$$\begin{aligned} D_{g_0}X_{k_0,g_0}|_\alpha(V) = (-D^2_{t,g_0} + 1)^{-1}\big((-\lambda_1'' + 2\pi\sqrt{1-r^2}\lambda_2')\dot\alpha \\ + (-\lambda_2'' - (2\pi)^2\lambda_2)(\alpha\times\dot\alpha)\big). \end{aligned} \tag{5.7}$$

Concerning $W_1, \dots, W_3$ and W_g we find

$$\begin{aligned} &W_1(t) = \dot\alpha(t),\\ &W_2(t) = -\frac{1}{2\pi r}\big(\sqrt{1-r^2}\sin(2\pi t)\dot\alpha(t) + \cos(2\pi t)(\alpha\times\dot\alpha)\big),\\ &W_3(t) = -\frac{1}{2\pi r}\big(-\sqrt{1-r^2}\cos(2\pi t)\dot\alpha(t) + \sin(2\pi t)(\alpha\times\dot\alpha)\big),\\ &W_{g_0}(\alpha) = (1+|\dot\alpha|^2k_0^2)^{-1}\dot\alpha = (1+|\dot\alpha|^2k_0^2)^{-1}W_1(\alpha). \end{aligned} \tag{5.8}$$

Lemma 5.1. *For any solution α of the unperturbed problem, there holds*

$$\{0\} = \langle W_1(\alpha), W_2(\alpha), W_3(\alpha)\rangle \cap R\big(D_{g_0}X_{k_0,g_0}|_\alpha\big),$$
$$\langle W_1(\alpha)\rangle^\perp = \langle W_2(\alpha), W_3(\alpha)\rangle \oplus R\big(D_{g_0}X_{k_0,g_0}|_\alpha\big).$$

Proof. We omit the dependence of W_i on α. For $\lambda_1, \lambda_2 \in H^{2,2}(S^1, \mathbb{R})$ we have

$$\begin{aligned}(-D^2_{t,g_0} + 1)\big(\lambda_1 \dot\alpha + \lambda_2(\alpha \times \dot\alpha)\big) &= \big(-\lambda_1'' + 4\pi\sqrt{1-r^2}\lambda_2' + (4\pi^2(1-r^2)+1)\lambda_1\big)\dot\alpha \\ &\quad + \big(-\lambda_2'' - 4\pi\sqrt{1-r^2}\lambda_1' + (4\pi^2(1-r^2)+1)\lambda_2\big)\alpha \times \dot\alpha.\end{aligned}$$

Hence we get by direct calculations

$$(-D^2_{t,g_0} + 1)(W_1) = (4\pi^2(1 + r^2 + 1)\dot\alpha,$$

$$\begin{aligned}(-D^2_{t,g_0} + 1)(-2\pi r W_2) &= \sqrt{1-r^2}(-4\pi^2 r^2 + 1)\sin(2\pi t)\dot\alpha \\ &\quad + (4\pi^2 r^2 + 1)\cos(2\pi t)(\alpha \times \dot\alpha),\end{aligned} \tag{5.9}$$

$$\begin{aligned}(-D^2_{t,g_0} + 1)(-2\pi r W_3) &= -\sqrt{1-r^2}(-4\pi^2 r^2 + 1)\cos(2\pi t)\dot\alpha \\ &\quad + (4\pi^2 r^2 + 1)\sin(2\pi t)(\alpha \times \dot\alpha).\end{aligned} \tag{5.10}$$

Consequently, by (3.8) and (5.8) the vector W_1 is orthogonal to $\langle W_2, W_3\rangle$ and to $R\big(D_{g_0} X_{k_0,g_0}|_\alpha\big)$ in $T_\alpha H^{2,2}(S^1, S^2)$. As in $L^2(S^1,\mathbb{R})$

$$\lambda_2'' + (2\pi)^2\lambda_2 \perp_{L^2} \langle \cos(2\pi t), \sin(2\pi t)\rangle, \ \langle \lambda_1'', \lambda_2'\rangle \perp_{L^2} \text{const},$$

we get

$$\begin{aligned}\{0\} = (-D^2_{t,g_0} + 1)\big(\langle W_1, W_2, W_3\rangle\big) \\ \cap (-D^2_{t,g_0} + 1) D_{g_0} X_{k_0,g_0}|_\alpha (T_\alpha H^{2,2}(S^1, S^2))\end{aligned}$$

and the claim follows, for $D_{g_0} X_{k_0,g_0}|_\alpha$ is a Fredholm operator of index 0. q.e.d.

To analyze the image of $D_{g_0} X_{k_0,g_0}$ we see for $\alpha \in \mathcal{Z}$

$$\begin{aligned}R(D_{g_0} X_{k_0,g_0}|_\alpha) &= \big\{(-D^2_{t,g_0} + 1)^{-1}\big((-\lambda_1'' + 2\pi\sqrt{1-r^2}\lambda_2')\dot\alpha \\ &\qquad - (\lambda_2'' + (2\pi)^2\lambda_2)(\alpha \times \dot\alpha)\big) : \ \lambda_1, \, \lambda_2 \in H^{2,2}(S^1,\mathbb{R})\big\} \\ &= \big\{(-D^2_{t,g_0} + 1)^{-1}\big(\lambda_1\dot\alpha + \lambda_2(\alpha \times \dot\alpha)\big) : \ \lambda_{1,2} \text{ in} \\ &\qquad L^2(S^1,\mathbb{R}), \, \lambda_1 \perp_{L^2} 1, \, \lambda_2 \perp_{L^2} \langle \cos(2\pi t), \sin(2\pi t)\rangle\big\} \\ &= \langle(\alpha \times \dot\alpha)\rangle \oplus E_+,\end{aligned} \tag{5.11}$$

where E_+ is given by

$$\begin{aligned}E_+ = \big\{&(-D^2_{t,g_0} + 1)^{-1}\big(\lambda_1\dot\alpha + \lambda_2(\alpha \times \dot\alpha)\big) : \\ &\lambda_1, \, \lambda_2 \in L^2(S^1,\mathbb{R}), \, \lambda_1 \perp_{L^2} 1, \, \lambda_2 \perp_{L^2} \langle 1, \cos(2\pi t), \sin(2\pi t)\rangle\big\}.\end{aligned}$$

We have for $V = \lambda_1\dot\alpha + \lambda_2(\alpha \times \dot\alpha)$ in $T_\alpha H^{2,2}(S^1, S^2)$

$$D_{g_0} X_{k_0,g_0}|_\alpha(V) \in E_+ \iff \lambda_2 \perp_{L^2} 1 \iff V \perp_{L^2} (\alpha \times \dot\alpha).$$

We fix $V = (-D^2_{t,g_0}+1)^{-1}(\lambda_1\dot\alpha + \lambda_2(\alpha\times\dot\alpha)) \in E_+$. Then

$$\begin{aligned}\int_{S^1} V(\alpha\times\dot\alpha) &= \int_{S^1}(-D^2_{t,g_0}+1)^{-1}(\lambda_1\dot\alpha+\lambda_2(\alpha\times\dot\alpha))(\alpha\times\dot\alpha)\\ &= \int_{S^1}(\lambda_1\dot\alpha+\lambda_2(\alpha\times\dot\alpha))(-D^2_{t,g_0}+1)^{-1}(\alpha\times\dot\alpha)\\ &= (4\pi^2(1-r^2)+1)^{-1}\int_{S^1}(\lambda_1\dot\alpha+\lambda_2(\alpha\times\dot\alpha))(\alpha\times\dot\alpha) = 0.\end{aligned}$$

Consequently, $D_{g_0}X_{k_0,g_0}|_\alpha(E_+) = E_+$.

Since E_+ is L^2-orthogonal to $\alpha\times\dot\alpha$ and $\dot\alpha$, we may write

$$V = (\nu_1+f_1)\dot\alpha + (\nu_2+f_2)(\alpha\times\dot\alpha),$$

with $\nu_{1,2}\perp_{L^2}\langle 1,\sin(2\pi\cdot),\cos(2\pi\cdot)\rangle$ and $f_{1,2}\in\langle\sin(2\pi\cdot),\cos(2\pi\cdot)\rangle$. Then

$$\begin{aligned}&\langle(-D^2_{t,g_0}+1)D_{g_0}X_{k_0,g_0}|_\alpha(V),V\rangle_{L^2}\\ &\quad= \int_{S^1}(\nu_1')^2 - 2\pi\sqrt{1-r^2}\,\nu_1'\nu_2 + (\nu_2')^2 - 4\pi^2(\nu_2)^2\\ &\qquad + (f_1')^2 - 2\pi\sqrt{1-r^2}\,f_1'f_2.\end{aligned}\tag{5.12}$$

For $\nu_2\perp\langle 1,\,\cos(2\pi\cdot),\sin(2\pi\cdot)\rangle$ we have

$$\int_{S^1}(\nu_2')^2 - 4\pi^2(\nu_2)^2 \ge \int_{S^1}16\pi^2(\nu_2)^2,$$

hence

$$\int_{S^1}(\nu_1')^2 - 2\pi\sqrt{1-r^2}\,\nu_1'\nu_2 + (\nu_2')^2 - 4\pi^2(\nu_2)^2 \ge \frac{3}{4}(\nu_1')^2 + 12\pi^2(\nu_2)^2.$$

Concerning the remaining term in (5.12), we note that as $(-D^2_{t,g_0}+1)$ maps

$$\big\{\lambda_1\dot\alpha+\lambda_2(\alpha\times\dot\alpha):\ \lambda_1,\lambda_2\in\langle\sin(2\pi\cdot),\cos(2\pi\cdot)\rangle\big\}$$

into itself and $V\in E_+$, there holds

$$f_1\dot\alpha + f_2(\alpha\times\dot\alpha) \in (-D^2_{t,g_0}+1)^{-1}\big\langle\big(\cos(2\pi\cdot)\dot\alpha\big),\big(\sin(2\pi\cdot)\dot\alpha\big)\big\rangle.$$

Hence, by explicit computations there are $x,y\in\mathbb{R}$ satisfying

$$\begin{aligned}f_1(t) &= x\cos(2\pi t) + y\sin(2\pi t),\\ f_2(t) &= \frac{8\pi^2\sqrt{1-r^2}}{4\pi^2(2-r^2)+1}\big(y\cos(2\pi t) - x\sin(2\pi t)\big).\end{aligned}$$

This gives

$$\int_{S^1}(f_1')^2 - 2\pi\sqrt{1+r^2}\,f_1'f_2 = \frac{2\pi^2(1+4\pi^2r^2)}{4\pi^2(2-r^2)+1}(x^2+y^2).$$

This shows that

$$\langle(-D^2_{t,g_0}+1)D_{g_0}X_{k_0,g_0}|_\alpha(V),V\rangle_{L^2} > 0 \text{ for all } V\in E_+\setminus\{0\},$$

and the homotopy

$$[0,1] \ni s \mapsto (1-s)\big(D_{g_0} X_{k_0,g_0}|_\alpha\big)|_{E_+} + s\, id|_{E_+}$$

is admissible. We use the decomposition in (5.11) and

$$D_{g_0} X_{k_0,g_0}|_\alpha(\alpha \times \dot\alpha) = -\frac{4\pi^2}{4\pi^2(1-r^2)+1}(\alpha \times \dot\alpha)$$

to see that

$$\operatorname{sgn}\big(D_{g_0} X_{k_0,g_0}|_\alpha\big)|_{R(D_{g_0} X_{k_0,g_0}|_\alpha)} = -1. \tag{5.13}$$

5.2. The finite dimensional reduction. We fix $\alpha_0 \in \mathcal{Z}$ and a parametrization φ of $\mathcal{Z}$, which maps an open neighborhood of 0 in $T_{\alpha_0}\mathcal{Z}$ into $\mathcal{Z}$, such that

$$\varphi(0) = \alpha_0 \text{ and } D\varphi|_0 = id.$$

As $\mathcal{Z}$ consists of smooth functions, $\mathcal{Z}$ is a sub-manifold of $H^{m,2}(S^1,S^2)$ for $1 \le m < \infty$. We define Φ from an open neighborhood $\mathcal{U}$ of 0 in

$$T_{\alpha_0} H^{2,2}(S^1,S^2) = \langle W_1(\alpha_0), W_2(\alpha_0), W_3(\alpha_0)\rangle \oplus R(DX_{g_0,0}|_{\alpha_0})$$

to $H^{2,2}(S^1,S^2)$ by

$$\Phi(W,U) := Exp_{\alpha_0,g_0}\big(Exp^{-1}_{\alpha_0,g_0}(\varphi(W)) + U\big).$$

Then $(\Phi,\mathcal{U})$ is a chart of $H^{2,2}(S^1,S^2)$ around α_0 such that $\mathcal{U}$ is an open neighborhood of 0 in $T_{\alpha_0}H^{2,2}(S^1,S^2)$, and

$$\Phi(0) = \alpha_0,\ D\Phi|_0 = id,$$
$$\Phi^{-1}\big(\mathcal{Z} \cap \Phi(\mathcal{U})\big) = \mathcal{U} \cap \langle W_1(\alpha_0), W_2(\alpha_0), W_3(\alpha_0)\rangle.$$

From the properties of Exp_{α_0,g_0} the map Φ is a chart of $H^{k,2}(S^1,S^2)$ around α_0 for any $1 \le k \le 4$. Shrinking $\mathcal{U}$ we may assume that (3.3)–(3.5) continue to hold with $Exp_{\gamma,g}$ replaced by Φ, i.e.

$$T_{\Phi(V)}H^{1,2}(S^1,S^2) = \langle \frac{d}{dt}\Phi(V)\rangle \oplus D\Phi|_V(\langle\dot\alpha_0\rangle^{\perp,H^{1,2}}), \tag{5.14}$$

$$T_{\Phi(V)}H^{2,2}(S^1,S^2) = \langle W_{g_0}(\Phi(V))\rangle \oplus D\Phi|_V(\langle W_{g_0}(\alpha_0)\rangle^{\perp}), \tag{5.15}$$

$$\mathrm{Proj}_{\langle W_{g_0}(\Phi(V)\rangle^\perp} \circ D\Phi|_V : \langle W_{g_0}(\alpha_0)\rangle^\perp \xrightarrow{\cong} \langle W_{g_0}(\Phi(V)\rangle^\perp, \tag{5.16}$$

and the norm of the projections in (5.14) and (5.15) as well as the norm of the map in (5.16) and its inverse are uniformly bounded with respect to V. For $\alpha_0 \in \mathcal{Z}$, the vectors $W_1(\alpha_0)$ and $W_{g_0}(\alpha_0)$ are collinear and we use $\langle W_1(\alpha_0)\rangle$ instead of $\langle W_{g_0}(\alpha_0)\rangle$ in the analysis of the unperturbed problem below.

As in (3.2), we get a chart Ψ for the bundle $SH^{2,2}(S^1,S^2)$ around $(\alpha_0, 0)$,

$$\Psi : \mathcal{U} \times \mathcal{U} \cap \langle W_1(\alpha_0)\rangle^\perp \to SH^{2,2}(S^1,S^2),$$
$$\Psi(V,U) := \big(\Phi(V), Proj_{\langle W_{g_0}(\Phi(V))\rangle^\perp} \circ D\Phi|_V(U)\big).$$

Analogous to (3.11), we define

$$X^{\Phi}_{g_0,\varepsilon} : \mathcal{U} \cap \langle W_1(\alpha_0)\rangle^{\perp} \to \langle W_1(\alpha_0)\rangle^{\perp}$$

by

$$X^{\Phi}_{g_0,\varepsilon}(V) := Proj_2 \circ \Psi^{-1}\big(\Phi(V), X_{g_0,\varepsilon}(\Phi(V))\big).$$

Replacing $Exp_{\gamma,g}$ by Φ, it is easy to see that Lemma 3.5 carries over to $X^{\Phi}_{g_0,\varepsilon}$, i.e.

$$V \in \mathcal{U} \cap \langle W_1(\alpha_0)\rangle^{\perp} \text{ is a (nondegenerate) zero of } X^{\Phi}_{g_0,\varepsilon} \text{ if and only if}$$
$$S^1 * \Phi(V) \text{ is a (nondegenerate) critical orbit of } X_{g_0,\varepsilon}, \tag{5.17}$$

and if $X^{\Phi}_{g_0,\varepsilon}(V) = 0$, then after shrinking $\mathcal{U}$,

$$DX^{\Phi}_{g_0,\varepsilon}|_V = A_V^{-1} \circ DX_{g_0,\varepsilon}|_{\Phi(V)} \circ D\Phi|_V, \tag{5.18}$$

where the isomorphism $A_V : \langle W_1(\alpha_0)\rangle^{\perp} \to \langle W_{g_0}(\Phi(V))\rangle^{\perp}$ is given by

$$A_V = Proj_{\langle W_{g_0}(\Phi(V))\rangle^{\perp}} \circ D\Phi|_V.$$

From Lemma 5.1 we may assume

$$\mathcal{U} \cap \langle W_1(\alpha_0)\rangle^{\perp} = \mathcal{U}_1 \times \mathcal{U}_2,$$

where $\mathcal{U}_1$ and $\mathcal{U}_2$ are open neighborhoods of 0 in $\langle W_2(\alpha_0), W_3(\alpha_0)\rangle$ and $R\big(D_{g_0}X_{k_0,g_0}|_{\alpha_0}\big)$. We denote for $\alpha \in \mathcal{Z}$ by $P_2(\alpha)$ the projection onto $R(DX_{g_0,0}|_{\alpha})$ with respect to the decomposition

$$\langle W_1(\alpha)\rangle^{\perp} = \langle W_2(\alpha), W_3(\alpha)\rangle \oplus R\big(D_{g_0}X_{k_0,g_0}|_{\alpha}\big),$$

and by $P_1(\alpha)$ the projection onto $\langle W_2(\alpha), W_3(\alpha)\rangle$. Moreover, for $W \in \mathcal{U}_1$, we define for $i = 1, 2$

$$P_i^{\Phi}(W) := (A_W)^{-1} \circ P_i(\Phi(W)) \circ A_W.$$

The projections $P_1^{\Phi}(W)$ and $P_2^{\Phi}(W)$ correspond to the decomposition

$$\langle W_1(\alpha_0)\rangle^{\perp} = \langle W_2(\alpha_0), W_3(\alpha_0)\rangle \oplus R\big(D_{g_0}X^{\Phi}_{g_0,0}|_W\big), \tag{5.19}$$

as we have for $W \in \mathcal{U}_1$

$$DX^{\Phi}_{g_0,0}|_W = A_W^{-1} \circ DX_{g_0,0}|_{\Phi(W)} \circ A_W.$$

Lemma 5.2. *For $\alpha_0 \in \mathcal{Z}$ after possibly shrinking $\mathcal{U}$, there are $\varepsilon_0 > 0$ and*

$$U \in C^2([-\varepsilon_0, \varepsilon_0] \times \mathcal{U}_1, \langle W_1(\alpha_0)\rangle^{\perp}),$$
$$R \in C^2([-\varepsilon_0, \varepsilon_0] \times \mathcal{U}_1, \langle W_2(\alpha_0), W_2(\alpha_0)\rangle),$$

such that for all $(\varepsilon, W) \in [-\varepsilon_0, \varepsilon_0] \times \mathcal{U}_1$

$$\begin{aligned}
R(\varepsilon, W) &= X^{\Phi}_{g_0,\varepsilon}(W + U(\varepsilon, W)),\\
0 &= P^{\Phi}_1(W) \circ U(\varepsilon, W),\\
O(\varepsilon)_{\varepsilon\to 0} &= \|U(\varepsilon, W)\| + \|D_W U(\varepsilon, W)\| + \|R(\varepsilon, W)\| + \|D_W R(\varepsilon, W)\|,\\
R(\varepsilon, W) &= \varepsilon P^{\Phi}_1(W) \circ K^{\Phi}_1(W) + o(\varepsilon)_{\varepsilon\to 0},\\
U(\varepsilon, W) &= -\varepsilon(DX^{\Phi}_{g_0,0}|_W)^{-1} \circ P^{\Phi}_2(W) \circ K^{\Phi}_1(W) + o(\varepsilon)_{\varepsilon\to 0}.
\end{aligned}$$

Moreover, $U(\varepsilon, W)$ *and* $R(\varepsilon, W)$ *are unique in the following sense: If* (ε, W, U, R) *in* $[-\varepsilon_0, \varepsilon_0] \times \mathcal{U}_1 \times \mathcal{U} \cap \langle W_1(\alpha_0)\rangle^{\perp} \times \mathcal{U}_1$ *satisfies*

$$X^{\Phi}_{g_0,\varepsilon}(W + U) = R \text{ and } P^{\Phi}_1(W)\big(U\big) = 0,$$

then $U = U(\varepsilon, W)$ *and* $R = R(\varepsilon, W)$*.*

Proof. We define a C^2-function H

$$\begin{aligned}
H : \mathbb{R} \times \mathcal{U}_1 \times \mathcal{U} \cap \langle W_1(\alpha_0)\rangle^{\perp} &\times \langle W_2(\alpha_0), W_3(\alpha_0)\rangle\\
&\to \langle W_1(\alpha_0)\rangle^{\perp} \times \langle W_2(\alpha_0), W_3(\alpha_0)\rangle,
\end{aligned}$$

by

$$H(\varepsilon, W, U, R) := \big(X^{\Phi}_{g_0,\varepsilon}(W + U) - R, P^{\Phi}_1(W)(U)\big).$$

We have in $\mathcal{L}(\langle W_1(\alpha_0)\rangle^{\perp} \times \langle W_2(\alpha_0), W_3(\alpha_0)\rangle)$

$$D_{(U,R)}H|_{(0,W,0,0)} = \begin{pmatrix} DX^{\Phi}_{g_0,0}|_W & -id \\ P^{\Phi}_1(W) & 0 \end{pmatrix},$$

where we used the fact that $X^{\Phi}_{g_0,0}(W) = 0$ and (5.18). From (5.6) and Lemma 5.1 we see that $D_{(U,R)}H|_{(0,0,0,0)}$ is an isomorphism. By the implicit function theorem, after possibly shrinking $\mathcal{U}$, we get $\varepsilon_0 > 0$ and unique functions $U = U(\varepsilon, W)$ and $R = R(\varepsilon, W)$ such that $H(\varepsilon, W, U(\varepsilon, W), R(\varepsilon, W)) = 0$ for all $(\varepsilon, W) \in [-\varepsilon_0, \varepsilon_0] \times \mathcal{U}_1$, and $D_{(U,R)}H|_{\varepsilon,W,U,R}$ is uniformly invertible for $(\varepsilon, W, U, R) \in [-\varepsilon_0, \varepsilon_0] \times \mathcal{U}_1 \times \mathcal{U}_2$. This yields the existence and uniqueness part of the claim.

The uniqueness implies $U(0, W) = 0$ and $R(0, W) = 0$ for all $W \in \mathcal{U}_1$. As U and R are differentiable, we find $U(\varepsilon, W) = O(\varepsilon)$ and $R(\varepsilon, W) = O(\varepsilon)$ as $\varepsilon \to 0$. Moreover, taking the derivative with respect to W, we see

$$0 = D_W H|_{(0,W,0,0)} + D_{(U,R)}H|_{(0,W,0,0)}\big(D_W U(0, W), D_W R(0, W)\big)^T.$$

Since $H(0, W, 0, 0) \equiv 0$, we have $D_W H|_{(0,W,0,0)} = 0$, which implies

$$(D_W U(0, W), D_W R(0, W)) = (0, 0),$$

because $D_{(U,R)}H|_{(0,W,0,0)}$ is invertible. This gives the desired estimate for $D_W U$ and $D_W R$.

Moreover, taking the derivative with respect to ε at $(0, W, 0, 0)$, we see as above

$$\begin{aligned} 0 &= D_\varepsilon H|_{(0,W,0,0)} + D_{(U,R)} H|_{(0,W,0,0)} \big(D_\varepsilon U(0,W), D_\varepsilon R(0,W)\big)^T \\ &= (K_1^\Phi(W), 0) + \begin{pmatrix} DX^\Phi_{g_0,0}|_W & -id \\ P_1^\Phi(W) & 0 \end{pmatrix} \begin{pmatrix} D_\varepsilon U(0,W) \\ D_\varepsilon R(0,W) \end{pmatrix}. \end{aligned}$$

Consequently,

$$\begin{aligned} D_\varepsilon R(0,W) &= P_1^\Phi(W) \circ K_1^\Phi(W), \\ D_\varepsilon U(0,W) &= -(DX^\Phi_{g_0,0}|_W)^{-1} \circ P_2^\Phi(W) \circ K_1^\Phi(W). \end{aligned}$$

This yields the claim. q.e.d.

Lemma 5.3. *Under the assumptions of Lemma* 5.2*, we have as* $\varepsilon \to 0$

$$X^\Phi_{g_0,\varepsilon}(W + U(\varepsilon, W)) = \varepsilon P_1^\Phi(W) \circ K_1^\Phi(W) + O(\varepsilon^2)_{\varepsilon\to 0},$$

where K_1^Φ *is the vector-field* K_1 *in the coordinates* Φ*, i.e.*

$$K_1^\Phi = X^\Phi_{g_0,1} - X^\Phi_{g_0,0}.$$

Proof. Since $U(\varepsilon, W) = O(\varepsilon)$, we find

$$\begin{aligned} &X^\Phi_{g_0,\varepsilon}(W + U(\varepsilon, W)) \\ &\quad = P_1^\Phi(W) \circ X^\Phi_{g_0,\varepsilon}(W + U(\varepsilon, W)) \\ &\quad = P_1^\Phi(W) \circ X^\Phi_{g_0,0}(W + U(\varepsilon, W)) + \varepsilon P_1^\Phi(W) \circ K_1^\Phi(W + U(\varepsilon, W)) \\ &\quad = P_1^\Phi(W) \circ DX^\Phi_{g_0,0}|_W U(\varepsilon, W) + \varepsilon P_1^\Phi(W) \circ K_1^\Phi(W) + O(\varepsilon^2) \\ &\quad = \varepsilon P_1^\Phi(W) \circ K_1^\Phi(W) + O(\varepsilon^2)_{\varepsilon\to 0}. \end{aligned}$$

q.e.d.

Lemma 5.4. *Under the assumptions of Lemma* 5.2*, suppose* 0 *is a nondegenerate zero of the vector-field* $P_1^\Phi(\cdot) \circ K_1^\Phi(\cdot)$ *on* $\mathcal{U}_1$*, in the sense that* $P_1^\Phi(0) \circ K_1^\Phi(0) = 0$ *and*

$$D_W(P_1^\Phi(\cdot) \circ K_1^\Phi(\cdot))|_0 \in \mathcal{L}(\langle W_2(\alpha_0), W_3(\alpha_0)\rangle)$$

is an isomorphism. Then, after possibly shrinking ε_0 *and* $\mathcal{U}$*, for any* $0 < |\varepsilon| \le \varepsilon_0$ *there is a unique* $W(\varepsilon) \in \mathcal{U}_1$ *such that*

$$\begin{aligned} X^\Phi_{g_0,\varepsilon}(W(\varepsilon) + U(\varepsilon, W(\varepsilon))) &= 0, \\ W(\varepsilon) \to 0 \ \textit{as}\ \varepsilon &\to 0. \end{aligned}$$

Moreover, $V(\varepsilon) := W(\varepsilon) + U(\varepsilon, W(\varepsilon))$ *is the only zero of* $X^\Phi_{g_0,\varepsilon}$ *in* $\mathcal{U} \cap \langle W_1(\alpha_0)\rangle^\perp$ *and is nondegenerate with*

$$sgn(DX^\Phi_{g_0,\varepsilon}|_{V(\varepsilon)}) = -det(D_W(P_1^\Phi(\cdot) \circ K_1^\Phi(\cdot))|_0).$$

Proof. Using Lemma 5.2 and the estimates for U and $D_W U$, we find

$$\begin{aligned} D_W\Big(X^\Phi_{g_0,\varepsilon}(\cdot+U(\varepsilon,\cdot))\Big)|_W &= D_W\Big(P^\Phi_1(\cdot)\circ X^\Phi_{g_0,\varepsilon}(\cdot+U(\varepsilon,\cdot))\Big)\\ &= (D_W P^\Phi_1|_W)\circ X^\Phi_{g_0,\varepsilon}(W+U(\varepsilon,W))\\ &\quad + P^\Phi_1(W)\circ DX^\Phi_{g_0,\varepsilon}|_{W+U(\varepsilon,W)}\circ(Id+D_W U|_{(\varepsilon,W)})\\ &= (D_W P^\Phi_1|_W)\circ\Big(\varepsilon K^\Phi_1(W)+DX^\Phi_{g_0,\varepsilon}|_W U(\varepsilon,W)+O(\varepsilon^2)\Big)\\ &\quad + P^\Phi_1(W)\circ\big(\varepsilon DK^\Phi_1|_W+D^2X^\Phi_{g_0,0}|_W U(\varepsilon,W)+O(\varepsilon^2)\big). \end{aligned} \tag{5.20}$$

Differentiating the identity for fixed ε

$$P^\Phi_1(W)\circ DX^\Phi_{g_0,0}|_W U(\varepsilon,W)\equiv 0$$

with respect to W we obtain

$$\begin{aligned} 0 &= (D_W P^\Phi_1|_W)\circ DX^\Phi_{g_0,0}|_W U(\varepsilon,W)\\ &\quad + P^\Phi_1(W)\circ\Big(D^2X^\Phi_{g_0,0}|_W U(\varepsilon,W)+DX^\Phi_{g_0,0}|_W\circ D_W U|_{(\varepsilon,W)}\Big). \end{aligned} \tag{5.21}$$

Since $P^\Phi_1(W)\circ DX^\Phi_{g_0,0}|_W\equiv 0$, combining (5.20) and (5.21) leads to

$$D_W\Big(X^\Phi_{g_0,\varepsilon}(\cdot+U(\varepsilon,\cdot))\Big)|_W=\varepsilon D_W\big(P^\Phi_1(\cdot)\circ K^\Phi_1(\cdot)\big)|_W+O(\varepsilon^2). \tag{5.22}$$

We define $F:[-\varepsilon_0,\varepsilon_0]\times\mathcal{U}_1\to\langle W_2(\alpha_0),W_3(\alpha_0)\rangle$ by

$$F(\varepsilon,W):=\varepsilon^{-1}P^\Phi_1(W)\circ X^\Phi_{g_0,\varepsilon}(W+U(\varepsilon,W)).$$

Note that by Lemma 5.3 the function F extends continuously to $\varepsilon=0$. By (5.22) we have

$$D_W F|_{(\varepsilon,W)}=D_W\big(P^\Phi_1(\cdot)\circ K^\Phi_1(\cdot)\big)|_W+O(\varepsilon),$$

and F is in C^1 with $D_W F|_{(0,0)}$ invertible. Consequently, by the implicit function theorem after shrinking ε_0 and $\mathcal{U}$, there is a unique C^1-function $W=W(\varepsilon)$ such that $F(\varepsilon,W(\varepsilon))\equiv 0$ and for $\varepsilon\neq 0$

$$X^\Phi_{g_0,\varepsilon}(W(\varepsilon)+U(\varepsilon,W(\varepsilon)))\equiv 0.$$

Shrinking $\mathcal{U}$, we may assume that any $V\in\mathcal{U}\cap\langle W_1(\alpha_0)\rangle^\perp$ admits a unique decomposition $V=W_V+U_V$, where $U_V=P^\Phi_2(W_V)V$. From the construction in Lemma 5.2 and the analysis above, we see that for $(\varepsilon,V)\in[-\varepsilon_0,\varepsilon_0]\setminus\{0\}\times\mathcal{U}\cap\langle W_1(\alpha_0)\rangle^\perp$

$$\begin{aligned} X^\Phi_{g_0,\varepsilon}(V)=0 &\Longleftrightarrow X^\Phi_{g_0,\varepsilon}(W_V+U_V)=0\\ &\Longleftrightarrow U_V=U(\varepsilon,W_V)\text{ and }X^\Phi_{g_0,\varepsilon}(W_V+U(\varepsilon,W_V))=0\\ &\Longleftrightarrow V=W(\varepsilon)+U(\varepsilon,W(\varepsilon)). \end{aligned}$$

We use the decomposition in (5.19) to compute the local degree of $X^{\Phi}_{g_0,\varepsilon}$ in $V(\varepsilon) := W(\varepsilon) + U(\varepsilon, W(\varepsilon))$ as $\varepsilon \to 0$. As $U(\varepsilon, W) = O(\varepsilon)$ we find

$$\begin{aligned} DX^{\Phi}_{g_0,\varepsilon}|_{V(\varepsilon)} &= DX^{\Phi}_{g_0,0}|_{W(\varepsilon)} + D^2X^{\Phi}_{g_0,0}|_{W(\varepsilon)}U(\varepsilon, W(\varepsilon)) \\ &\quad + \varepsilon DK^{\Phi}_1|_{W(\varepsilon)} + O(\varepsilon^2). \end{aligned} \tag{5.23}$$

Differentiating for fixed $\tilde{W} \in \langle W_2(\alpha_0), W_3(\alpha_0)\rangle$ the identity

$$DX^{\Phi}_{g_0,0}|_W \tilde{W} \equiv 0,$$

we obtain $D^2X^{\Phi}_{g_0,0}|_W \tilde{W} \equiv 0$ and thus by (5.23)

$$DX^{\Phi}_{g_0,\varepsilon}|_{V(\varepsilon)}\tilde{W} = \big(\varepsilon DK^{\Phi}_1|_{W(\varepsilon)} + O(\varepsilon^2)\big)\tilde{W}.$$

For $\tilde{U} \in R\big(D_{g_0}X^{\Phi}_{g_0,0}|_W\big)$ we get from (5.23)

$$DX^{\Phi}_{g_0,\varepsilon}|_{V(\varepsilon)}\tilde{U} = \big(DX^{\Phi}_{g_0,0}|_{W(\varepsilon)} + O(\varepsilon)\big)\tilde{U}.$$

Consequently, with respect to the decomposition in (5.19),

$$\begin{aligned} DX^{\Phi}_{g_0,\varepsilon}|_{V(\varepsilon)} &= \begin{pmatrix} \varepsilon P^{\Phi}_1(W(\varepsilon)) \circ DK^{\Phi}_1|_{W(\varepsilon)} & 0 \\ 0 & DX^{\Phi}_{g_0,0}|_{W(\varepsilon)} \end{pmatrix} \\ &\quad + \begin{pmatrix} O(\varepsilon^2) & O(\varepsilon) \\ O(\varepsilon) & O(\varepsilon) \end{pmatrix}. \end{aligned}$$

This shows that shrinking $\varepsilon_0 > 0$, we may assume that $V(\varepsilon)$ is a nondegenerate zero of $X^{\Phi}_{g_0,\varepsilon}$ for all $0 < |\varepsilon| \le \varepsilon_0$ and by (5.13)

$$\begin{aligned} \operatorname{sgn}(DX^{\Phi}_{g_0,\varepsilon}|_{V(\varepsilon)}) &= \det\big(D(P^{\Phi}_1(\cdot) \circ K^{\Phi}_1(\cdot))|_0\big)\operatorname{sgn}(DX^{\Phi}_{g_0,0}|_{W(\varepsilon)}) \\ &= -\det\big(D(P^{\Phi}_1(\cdot) \circ K^{\Phi}_1(\cdot))|_0\big). \end{aligned}$$

This finishes the proof. q.e.d.

We consider $P_1(\cdot) \circ K_1(\cdot)$ as a vector field on $\mathcal{Z}$. If $\alpha_0 \in \mathcal{Z}$ is a zero of $P_1(\cdot) \circ K_1(\cdot)$, then we obtain due to the S^1 invariance that $S^1 * \alpha_0 \subset \mathcal{Z}$ is a zero orbit and

$$\begin{aligned} W_1(\alpha_0) &\in \operatorname{kernel}(D_{\mathcal{Z}}(P_1(\cdot) \circ K_1(\cdot))|_{\alpha_0}), \\ R(D_{\mathcal{Z}}(P_1(\cdot) \circ K_1(\cdot))|_{\alpha_0}) &\perp W_1(\alpha_0), \end{aligned}$$

where $D_{\mathcal{Z}}$ denotes the covariant derivative on $\mathcal{Z}$. In the sequel we will therefore consider $D_{\mathcal{Z}}(P_1(\cdot) \circ K_1(\cdot))|_{\alpha_0}$ as a map

$$D_{\mathcal{Z}}(P_1(\cdot) \circ K_1(\cdot))|_{\alpha_0} : \langle W_2(\alpha_0), W_3(\alpha_0)\rangle \to \langle W_2(\alpha_0), W_3(\alpha_0)\rangle.$$

Lemma 5.5. *Under the assumptions of Lemma 5.2, suppose α_0 is a nondegenerate zero of the vector field $P_1(\cdot) \circ K_1(\cdot)$ on $\mathcal{Z}$, in the sense that $P_1(\alpha_0) \circ K_1(\alpha_0) = 0$ and*

$$D_{\mathcal{Z}}(P_1(\cdot) \circ K_1(\cdot))|_{\alpha_0} \in \mathcal{L}(\langle W_2(\alpha_0), W_3(\alpha_0)\rangle)$$

is an isomorphism. Then for any $0 < \varepsilon < \varepsilon_0$, there is $\gamma(\varepsilon) \in \Phi(\mathcal{U})$ satisfying

$$X_{g_0,\varepsilon}(\gamma(\varepsilon)) = 0 \text{ and } \gamma(\varepsilon) \to \alpha_0 \text{ as } \varepsilon \to 0.$$

Moreover, $S^1 * \gamma(\varepsilon)$ is the unique critical orbit of $X_{g_0,\varepsilon}$ in $\Phi(\mathcal{U})$ and is nondegenerate with

$$\deg_{loc,S^1}(X_{g_0,\varepsilon}, S^1 * \gamma(\varepsilon)) = -\det(D_{\mathcal{Z}}(P_1(\cdot) \circ K_1(\cdot))|_{\alpha_0}).$$

Proof. We note that as $P_1(\alpha_0) \circ K_1(\alpha_0) = 0$

$$D_{\mathcal{Z}}(P_1(\cdot) \circ K_1(\cdot))|_{\alpha_0} = D(P_1^\Phi(\cdot) \circ K_1^\Phi(\cdot))|_0.$$

Consequently, the assumptions of Lemma 5.4 are satisfied and we may define for $0 < \varepsilon < \varepsilon_0$ the curve $\gamma(\varepsilon)$ by

$$\gamma(\varepsilon) := \Phi(V(\varepsilon)) \in H^{2,2}(S^1, S^2).$$

From (5.17) we infer that $\gamma(\varepsilon)$ is the unique zero of $X_{g_0,\varepsilon}$ in $\Phi(\langle W_1\rangle^\perp \cap \mathcal{U})$ and $S^1 * \gamma(\varepsilon)$ is a nondegenerate critical orbit. It is easy to see that the existence of a slice in Lemma 3.1 remains valid if we replace Exp_{α_0,g_0} by Φ. Consequently, $S^1 * \gamma(\varepsilon)$ is the unique critical orbit of $X_{g_0,\varepsilon}$ in $S^1 * \Phi(\langle W_1\rangle^\perp \cap \mathcal{U})$, which is an open neighborhood of $S^1 * \alpha_0$ in $H^{2,2}(S^1, S^2)$.

We fix $0 < \varepsilon < \varepsilon_0$ and consider for $s \in [0,1]$ the family of maps

$$Y_s := A^{-1}_{V(\varepsilon)} \circ DX_{g_0,\varepsilon}|_{\gamma(\varepsilon)} \circ \big((1-s) + s\mathrm{Proj}_{\langle W_1(\gamma(\varepsilon))\rangle^\perp}\big) D\Phi|_{V(\varepsilon)}.$$

Since $DX_{g_0,\varepsilon}|_{\gamma(\varepsilon)}$ restricted to $\langle W_1(\gamma(\varepsilon))\rangle^\perp$ is of the form $id - compact$, writing

$$DX_{g_0,\varepsilon}|_{\gamma(\varepsilon)} = DX_{g_0,\varepsilon}|_{\gamma(\varepsilon)} \circ \mathrm{Proj}_{\langle W_1(\gamma(\varepsilon))\rangle^\perp} + DX_{g_0,\varepsilon}|_{\gamma(\varepsilon)} \circ \mathrm{Proj}_{\langle W_1(\gamma(\varepsilon))\rangle},$$

we deduce that $Y_s = id - compact$ for all $s \in [0,1]$. From Lemma 5.4 we have that Y_0 is invertible and satisfies

$$Y_0 = DX^\Phi_{g_0,\varepsilon}|_{V(\varepsilon)} \text{ and } \mathrm{sgn}(Y_0) = -\det(D_{\mathcal{Z}}(P_1(\cdot) \circ K_1(\cdot))|_{\alpha_0}).$$

As $DX^\Phi_{g_0,\varepsilon}|_{V(\varepsilon)}$ is invertible, the kernel of $DX_{g_0,\varepsilon}|_{\gamma(\varepsilon)}$ is given by $\langle\dot\gamma(\varepsilon)\rangle$. Since $\gamma(\varepsilon)$ converges to α_0 as $\varepsilon \to 0$ and $\dot\alpha_0 = W_1(\alpha_0)$, we get

$$\dot\gamma(\varepsilon) = W_1(\gamma(\varepsilon)) + o(1)_{\varepsilon\to 0},$$

which implies together with (5.14) that $\langle\dot\gamma(\varepsilon)\rangle$ is transversal to the image of

$$\big((1-s) + s\mathrm{Proj}_{\langle W_1(\gamma(\varepsilon))\rangle^\perp}\big) \circ D\Phi|_{V(\varepsilon)}$$

for all $s \in [0,1]$. Consequently, Y_s remains invertible when s moves from 0 to 1. Due to the homotopy invariance, we finally obtain

$$\begin{aligned}\operatorname{sgn}(Y_0) &= -\operatorname{sgn}\big(\det(D_{\mathcal{Z}}(P_1(\cdot)\circ K_1(\cdot))|_{\alpha_0})\big)\\ &= \operatorname{sgn}(Y_1) = \operatorname{sgn}\Big(A^{-1}_{V(\varepsilon)}\circ DX_{g_0,\varepsilon}|_{\gamma(\varepsilon)}\circ A_{V(\varepsilon)}\Big)\\ &= \operatorname{sgn}(DX_{g_0,\varepsilon}|_{\gamma(\varepsilon)}) = \deg_{loc,S^1}(X_{g_0,\varepsilon}, S^1 * \gamma(\varepsilon)).\end{aligned}$$

This finishes the proof. q.e.d.

5.3. The S^1-degree of (5.1). We define the function k_1 by

$$k_1(x) := \langle x, e_3\rangle \text{ for } x \in S^2 = \partial B_1(0) \subset \mathbb{R}^3. \tag{5.24}$$

The corresponding vector-field K_1 on $H^{2,2}(S^1,S^2)$ is given by

$$K_1(\alpha) = (-D^2_{t,g_0}+1)^{-1}(|\dot\alpha|\langle\alpha,e_3\rangle(\alpha\times\dot\alpha)).$$

We note that for $\alpha = \alpha(\cdot,2\pi r,v_0,v_1,w)\in\mathcal{Z}$ we have

$$\begin{aligned}&(-D^2_{t,g_0}+1)(2\pi r)^{-1}K_1(\alpha)\\ &\quad= \big(\sqrt{1-r^2}\langle w,e_3\rangle + r\cos(2\pi\cdot)\langle v_1,e_3\rangle + r\sin(2\pi\cdot)\langle v_0,e_3\rangle\big)(\alpha\times\dot\alpha)\\ &\quad= \frac{-2\pi r^2}{4\pi^2r^2+1}(-D^2_{t,g_0}+1)\big(\langle v_1,e_3\rangle W_2(\alpha)+\langle v_0,e_3\rangle W_3(\alpha)\big)+h(\alpha),\end{aligned}$$

where $(-D^2_{t,g_0}+1)^{-1}h(\alpha)$ is in the image of $D_{g_0}X_{k_0,g_0}|\alpha$ by (5.7)–(5.10). Hence,

$$P_1(\alpha)\circ K_1(\alpha) = \frac{-4\pi^2r^3}{4\pi^2r^2+1}\langle v_1,e_3\rangle W_2(\alpha) + \frac{-4\pi^2r^3}{4\pi^2r^2+1}\langle v_0,e_3\rangle W_3(\alpha),$$

and there are exactly two critical orbits of $P_1(\alpha)\circ K_1(\alpha)$ on $\mathcal{Z}$ given by

$$\{\alpha=\alpha(\cdot,2\pi r,v_0,v_1,w)\in\mathcal{Z}:\ w=\pm e_3\} = S^1*\alpha_+\cup S^1*\alpha_-,$$

where

$$\alpha_+ = \alpha(\cdot,2\pi r,e_1,e_2,e_3) \text{ and } \alpha_- = \alpha(\cdot,2\pi r,-e_1,e_2,-e_3).$$

The curves $\alpha_\pm$ correspond to two parallels with respect to the north pole e_3 and curvature k_0. Using the formulas for W_2 and W_3 in (5.5), we find with respect to the basis $\{W_2(\alpha_\pm),W_3(\alpha_\pm)\}$

$$D(P_1(\cdot)\circ K_1(\cdot))|_{\alpha_\pm} = \frac{4\pi^2r^3}{4\pi^2r^2+1}\begin{pmatrix}\pm1 & 0\\ 0 & \pm1\end{pmatrix}.$$

Thus, we may apply Lemma 5.5 and get two critical orbits $\alpha_\pm(\varepsilon)$ for $X_{g_0,\varepsilon}$ converging to $\alpha_\pm$ as $\varepsilon\to 0$.

Lemma 5.6. *Let M be the subset of $H^{2,2}(S^1,S^2)$ consisting of simple and regular curves. Then $\chi_{S^1}(X_{k_0,g_0},M) = -2$.*

Proof. We choose $k_1 = \langle \cdot, e_3 \rangle$ as above. From Lemmas 5.2–5.4 there are $\varepsilon_0 > 0$ and an open neighborhood $\mathcal{U}$ of $\mathcal{Z}$ such that for all $0 < \varepsilon < \varepsilon_0$ the critical orbits of $X_{g_0,\varepsilon}$ in $\mathcal{U}$ are given exactly by $S^1 * \alpha_\pm(\varepsilon)$. Indeed, suppose there are $\varepsilon_n \to 0^+$ and a sequence (α_n) of zeros of X_{g_0,ε_n} converging to $\mathcal{Z}$ different from $S^1 * \alpha_\pm(\varepsilon_n)$. Up to a subsequence

$$\alpha_n \to \alpha_0 \in \mathcal{Z}$$

as $n \to \infty$. For large n we use the chart Φ around α_0 as in Lemma 5.2. From the existence of a slice in Lemma 3.1 we get a sequence $\theta_n \in \mathbb{R}/\mathbb{Z}$ converging to 0 such that

$$\theta_n * \alpha_n = \Phi(V_n) \text{ for some } V_n \in \langle W_1(\alpha_0)\rangle^\perp.$$

As in the proof of Lemma 5.4 we may decompose

$$V_n = \Phi^{-1}(\theta_n * \alpha_n) = W_n + U_n,$$

where $W_n \in \langle W_2(\alpha_0), W_3(\alpha_0)\rangle$ and $U_n \in R(DX^\Phi_{k_0,g_0}|_{W_n})$. From the uniqueness part of Lemma 5.2, as $X_{g_0,\varepsilon_n}(W_n + U_n) = 0$, we get $U_n = U(\varepsilon_n, W_n)$. By Lemma 5.3 we see that necessarily $P_1(\alpha_0) \circ K_1(\alpha_0) = 0$, such that $S^1 * \alpha_0 \in \{S^1 * \alpha_\pm\}$. From Lemma 5.5 we finally deduce that $S^1 * \alpha_n \in \{S^1 * \alpha_\pm(\varepsilon_n)\}$, a contradiction.

From the definition of the S^1-equivariant Poincaré-Hopf index and the classification of the simple zeros of X_{k_0,g_0}, there holds for small $\varepsilon > 0$

$$\chi_{S^1}(X_{k_0,g_0}, M) = \chi_{S^1}(X_{k_0,g_0}, \mathcal{U}) = \chi_{S^1}(X_{g_0,\varepsilon}, \mathcal{U}) = -2.$$

q.e.d.

6. A priori estimates

We fix a continuous family of metrics $\{g_t : t \in [0,1]\}$ on S^2 and a continuous family of positive continuous function $\{k_t : t \in [0,1]\}$ on S^2. We let X_t be the vector field on $H^{2,2}(S^1, S^2)$ defined by

$$X_t := X_{k_t,g_t}.$$

We denote by $M \subset H^{2,2}(S^1, S^2)$ the set

$$M := \{\gamma \in H^{2,2}(S^1, S^2) : \gamma \text{ is simple and regular}\}.$$

We shall give sufficient conditions assuring that the set

$$X^{-1}(0) := \{(\gamma, t) \in M \times [0,1] : X_t(\gamma) = 0\}$$

is compact in $M \times [0,1]$. Fix $(\gamma, t) \in X^{-1}(0)$. The Gauss-Bonnet formula yields

$$\int_\gamma k_t \, ds + \int_{\Omega_\gamma} K_{g_t} \, dg_t = 2\pi,$$

where Ω_γ denotes the interior of γ with respect to the normal N_{g_t} and K_{g_t} is the Gauss curvature of (S^2, g_t). To obtain a contradiction, assume that there is (γ_n, t_n) in $X^{-1}(0)$ such that $L(\gamma_n) \to 0$ as $n \to \infty$. Then the left hand side in the Gauss-Bonnet formula, as k_t and K_{g_t} are uniformly bounded, tends to 0, which is impossible. Consequently, the length $L(\gamma)$ of γ satisfies

$$c \leq L(\gamma) \leq \big(\inf\{k_t(x)\}\big)^{-1}\big(2\pi + \sup_{t\in[0,1]}\{(\sup K^-_{g_t})vol(S^2, g_t)\}\big), \tag{6.1}$$

for some positive constant $c = c(\{k_t\}, \{g_t\})$ and $K^-_{g_t} := -\min(K_{g_t}, 0)$.

Suppose (γ_n, t_n) in $X^{-1}(0)$ converges to (γ_0, t_0) in $H^{2,2}(S^1, S^2)$, such that

$$\gamma_0 \notin M.$$

Then by (6.1) the curve γ_0 is non-constant and regular, hence there is $s_1 \neq s_2$ in $\mathbb{R}/\mathbb{Z}$ such that $\gamma_0(s_1) = \gamma_0(s_2)$. As γ_n are simple curves, parametrized proportional to arc-length, we see that $\dot\gamma_0(s_1) = \pm\dot\gamma_0(s_2)$. If $\dot\gamma_0(s_1) = \dot\gamma_0(s_2)$, then by the unique solvability of the initial value problem

$$\gamma_0(\cdot + (s_1 - s_2)) = \gamma_0(\cdot).$$

If $\dot\gamma_0(s_1) = -\dot\gamma_0(s_2)$, then we write γ close to s_1 and s_2 as a graph over the tangent direction $\dot\gamma_0(s_1)$ in normal coordinates $Exp_{\gamma_0(s_1)}$. By the maximum principle we find

$$\begin{aligned}
\gamma_0(s_1 + t) &= Exp_{\gamma_0(s_1),g}\big(t\dot\gamma_0(s_1) + a(t)N_g(\gamma_0(s_1))\big),\\
\gamma_0(s_2 + t) &= Exp_{\gamma_0(s_1),g}\big(-t\dot\gamma_0(s_1) - b(t)N_g(\gamma_0(s_1))\big),
\end{aligned}$$

where $a(t)$ and $b(t)$ are positive for $t \neq 0$. Consequently, if $\dot\gamma_0(s_1) = -\dot\gamma_0(s_2)$, then γ_0 touches itself at $\gamma_0(s_1)$, locally separated by the geodesic through $\gamma_0(s_1)$ with velocity $\dot\gamma_0(s_1)$. Thus, γ_0 is an m-fold covering for some $m \in \mathbb{N}$ of a curve α, which is almost simple in the sense that α can only touch itself as described above. Using stereographic coordinates $\mathcal{S}$ there is a point p_0 close to the curve γ_0, such that the winding number of $\mathcal{S}(\gamma_0)$ around $\mathcal{S}(p_0)$ is $\pm m$. Since γ_0 is a limit of simple curves, by the stability of the winding number, we deduce $m = 1$.

We denote by (Ω_0, g) the interior of γ_0 considered as a Riemannian surface with boundary of positive geodesic curvature. Fix a touching point $\gamma_0(s_1) = \gamma_0(s_2)$. The point $\gamma_0(s_1) = \gamma_0(s_2)$ corresponds to two different boundary points of Ω_0. Denote by β the curve of minimal length in Ω_0 connecting the two boundary points. From a regularity result for variational problems with constraints (see [**2**, **3**]) the minimizer β is a C^1-curve. By the maximum principle, β cannot touch the boundary of Ω_0 and is therefore a C^2 geodesic in the interior of Ω_0. Moreover, as a minimizer, β is stable, and going back to S^2 the curve β is a geodesic loop which is stable with respect to variations with fixed end-points. Thus

$$inj(g_{t_0}) \leq \frac{1}{2}L(\beta) < \frac{1}{4}L(\gamma_0). \tag{6.2}$$

This leads to

Lemma 6.1. *$X^{-1}(0)$ is compact in $M \times [0,1]$ under each of the following assumptions:*

(6.3)
$$\inf_{(t,x)\in[0,1]\times S^2}\{k_t\} \geq \frac{1}{4}\sup_{t\in[0,1]}\Big(\big(inj(g_t)\big)^{-1}\big(2\pi + (\sup K^-_{g_t})vol(S^2,g_t)\big)\Big),$$

(6.4)
$$K_{g_t} > 0 \ \forall t\in[0,1] \ \textit{and} \ \inf_{(t,x)\in[0,1]\times S^2}\{k_t\} \geq \frac{1}{2}\sup_{t\in[0,1]}\Big(\big(\sup K_{g_t}\big)^{\frac{1}{2}}\Big),$$

$$K_{g_t} > 0 \ \forall t\in[0,1] \ \textit{and} \ \big(\sup K_{g_t}\big) < 4\big(\inf K_{g_t}\big) \ \textit{for all} \ t\in[0,1], \tag{6.5}$$

where $inj(g_t)$ denotes the injectivity radius of (S^2,g_t).

Proof. We first show that $X^{-1}(0)$ is closed under each of the above assumptions. Suppose $(\gamma_n,t_n)\in X^{-1}(0)$ converges to some (γ_0,t_0) in $H^{2,2}(S^1,S^2)$. To obtain a contradiction, assume $(\gamma_0,t_0)\notin X^{-1}(0)$, i.e., γ_0 is not simple. Then by the above analysis γ_0 touches itself at some point $\gamma_0(s_1)=\gamma_0(s_2)$ and there is a stable, nontrivial geodesic loop β, which yields a bound from above on the injectivity radius in (6.2) by the length of γ_0. If γ_0 is too short, this is impossible. The estimate on the length of γ_0 in (6.1) leads to the contradiction under the assumption (6.3). If $K_{t_0}>0$, then by [**18**, thm 2.6.9],

$$inj(g_{t_0}) \geq \pi\big(\sup K_{t_0}\big)^{-\frac{1}{2}}, \tag{6.6}$$

and (6.4) is a special case of (6.3).

Moreover, by Bonnet-Meyer's theorem, as β is a stable geodesic loop, its length is bounded by

$$L(\beta) \leq \frac{\pi}{\sqrt{\inf K_{t_0}}},$$

which yields together with (6.6) the contradiction assuming (6.5).

To deduce the compactness of $X^{-1}(0)$, we fix a sequence (γ_n,t_n) in $X^{-1}(0)$. By (6.1) the length $L_{g_{t_n}}(\gamma_n)$ is uniformly bounded. Since each γ_n is parametrized proportional to arc-length, $(|\dot\gamma_n|_{g_{t_n}})$ is uniformly bounded. Using the equation (1.2) and standard elliptic regularity, (γ_n) is bounded in $H^{4,2}(S^1,S^2)$. Hence we may choose a subsequence, which converges in $H^{2,2}(S^1,S^2)$ and by the first part of the proof in $X^{-1}(0)$ under each of the above assumptions. This yields the claim. q.e.d.

Proof of Theorem 1.3. We fix $k_0 > 0$, let $k_1 \in C^\infty(S^2, \mathbb{R})$ be given by (5.24), and consider the metrics $g_t \equiv g_0$, the functions $k_t := k_0 + tk_1$, and the corresponding vector fields $X_t := X_{k_t,g_0}$. The zeros of X_0 in M are given by $\mathcal{Z}$, the manifold of solutions to the unperturbed problem. The compactness of $X^{-1}(0)$ implies that the zeros of X_t in M converge to $\mathcal{Z}$ as $t \to 0$. From the proof of Lemma 5.6 there are exactly two critical orbits $S^1 * \alpha_\pm(t)$ for $|t| > 0$ small enough close to $\mathcal{Z}$ which are nondegenerate and converge to the orbits of the parallels $\alpha(\cdot, 2\pi|r|, \pm e_1, e_2, \pm e_3)$ as $t \to 0$. Consequently, there are exactly two simple solutions of (1.2) with $g = g_0$ and $k = k_0 + tk_1$ if $|t| > 0$ is small enough. q.e.d.

7. Existence results

We give the proof of our main existence result.

Proof of Theorem 1.1. We consider the family of metrics $\{g_t : t \in [0, 1]\}$ defined by

$$g_t := (1 - t)g_0 + tg.$$

Since $\{g_t\}$ is a compact family of metrics, there is a constant $k_0 > 0$ such that

$$k_0 > \frac{1}{4} \sup_{t\in[0,1]} \Big(\big(inj(g_t)\big)^{-1} \big(2\pi + (\sup K^-_{g_t}) vol(S^2, g_t)\big)\Big).$$

We denote by M the set of simple regular curves in $H^{2,2}(S^1, S^2)$. From condition (6.3) in Lemma 6.1 the homotopy

$$[0, 1] \ni t \mapsto X_{k_0,g_t}$$

is (M, g_t, S^1)-admissible, and hence from Lemma 3.12 and Lemma 5.6,

$$-2 = \chi_{S^1}(X_{k_0,g_0}, M) = \chi_{S^1}(X_{k_0,g}, M).$$

We define the family of functions $\{k_t : t \in [0, 1]\}$ by

$$k_t := (1 - t)k_0 + tk$$

and consider the homotopy

$$[0, 1] \ni t \mapsto X_{k_t,g}.$$

Under each of the above assumptions, we may apply Lemma 6.1 to deduce that the homotopy is (M, g, S^1)-admissible, and thus

$$-2 = \chi_{S^1}(X_{k_0,g}, M) = \chi_{S^1}(X_{k,g}, M).$$

Since the local degree of an isolated critical orbit is larger than -1 by Lemma 4.1, there are at least two simple solutions to (1.2). q.e.d.

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