

## AXIAL MINIMAL SURFACES IN $\mathbf{S}^2 \times \mathbf{R}$ ARE HELICOIDAL

DAVID HOFFMAN & BRIAN WHITE

### Abstract

We prove that if a complete, properly embedded, finite-topology minimal surface in  $\mathbf{S}^2 \times \mathbf{R}$  contains a line, then its ends are asymptotic to helicoids, and that if the surface is an annulus, it must be a helicoid.

### 1. Introduction

There is a rich theory of complete properly embedded minimal surfaces of finite topology in  $\mathbf{R}^3$ . In particular, we now have a good understanding of the ends of such surfaces: aside from the plane, every such surface either has one end, in which case it is asymptotic to a helicoid [BB08], or it has more than one end, in which case each end is asymptotic to a plane or to a catenoid [Col97], [HM89], [MR93]. For the rest of this introduction, let us use “minimal surface” to mean “complete, properly embedded minimal surface with finite topology”. (Colding and Minicozzi [CM08] have proved that every complete embedded minimal surface with finite topology in  $\mathbf{R}^3$  is properly embedded, so the assumption of properness is not necessary.)

It is interesting to try to classify the ends of minimal surfaces in homogeneous 3-manifolds other than  $\mathbf{R}^3$ . This paper deals with the ambient manifold  $\mathbf{S}^2 \times \mathbf{R}$ . (The fundamental paper on minimal surfaces in  $\mathbf{S}^2 \times \mathbf{R}$  is Rosenberg [Ros02]. The survey [Ros03] is a good introduction to this paper as well as to the papers of [MR05], [Hau06], and [PR99] mentioned below.) In that case, the only compact minimal surfaces are horizontal 2-spheres. Any noncompact example has exactly two ends, both annular, one going up and one going down. Therefore the genus-zero, noncompact minimal surfaces in  $\mathbf{S}^2 \times \mathbf{R}$  are all annuli. The minimal annuli that are foliated by horizontal circles have been classified by Hauswirth [Hau06]. They form a two-parameter family that contains on its boundary the helicoids (defined in Section 1.2) and the unduloids constructed by Pedrosa and Ritore [PR99]. There are no other known minimal annuli.

These facts suggest the following two questions posed by Rosenberg:

---

Received 10/16/2009.

- 1) Is every minimal annulus in  $\mathbf{S}^2 \times \mathbf{R}$  one of the known examples?  
That is, is every minimal annulus fibered by horizontal circles?
- 2) If so, must each end of any minimal surface in  $\mathbf{S}^2 \times \mathbf{R}$  be asymptotic to one of the known minimal annuli?

In this paper, we show that the answer to both questions is “yes” in case the surface is an *axial surface*, i.e., in case the surface contains a vertical line. In particular, the axial minimal annuli in  $\mathbf{S}^2 \times \mathbf{R}$  are precisely the helicoids.

The assumption that the surface contains a line is a very strong one, but there are many minimal surfaces that have that property. Indeed, in [HW08] we outlined the proof of the existence of axial examples of every genus  $g$  and every vertical flux. (See equation (5) in Remark 3.1 for a definition of vertical flux. The forthcoming [HW11] contains the full existence proof.) By Theorem 1.3 below, those examples are all asymptotic to helicoids, so we call them “genus- $g$  helicoids”.

Combining these results with Theorem 1.3, we have:

**1.1. Theorem.** *For every helicoid  $H$  of finite pitch in  $\mathbf{S}^2 \times \mathbf{R}$  and for every genus  $g > 0$ , there are at least two genus- $g$  properly embedded, axial minimal surfaces whose ends are, after suitable rotations, asymptotic to  $H$ . The two surfaces are not congruent to each other by any orientation-preserving isometry of  $\mathbf{S}^2 \times \mathbf{R}$ . If  $g$  is even, they are not congruent to each other by any isometry of  $\mathbf{S}^2 \times \mathbf{R}$ .*

The totally geodesic cylinder  $\mathbf{S}^1 \times \mathbf{R}$  may be thought of as a helicoid of infinite pitch. In this case, the methods of [HW11, HW08] still produce two examples for each genus, but the proof that the two examples are not congruent breaks down. Earlier, by a different method, Rosenberg [Ros02] explicitly constructed, for each  $g$ , an axial, genus- $g$  minimal surface asymptotic to a cylinder.

**1.2. Helicoids.** Let  $O$  and  $O^*$  be a pair of antipodal points in  $\mathbf{S}^2 \times \{0\}$  and let  $Z$  and  $Z^*$  be vertical lines passing through those points. Let  $\sigma_{\alpha,v}$  denote the screw motion of  $\mathbf{S}^2 \times \mathbf{R}$  consisting of rotation through angle  $\alpha$  about the axes  $Z$  and  $Z^*$  followed by vertical translation by  $v$ . A *helicoid* with axes  $Z$  and  $Z^*$  is a surface of the form

$$(1) \quad \bigcup_{t \in \mathbf{R}} \sigma_{\kappa t, t} X$$

where  $X$  is a horizontal great circle that intersects  $Z$  and  $Z^*$ . The *pitch* of the helicoid is  $2\pi/\kappa$ , whose absolute value equals twice the vertical distance between successive sheets of the surface. Unlike the situation in  $\mathbf{R}^3$ , helicoids of different pitch do not differ by a homothety of  $\mathbf{S}^2 \times \mathbf{R}$ ; there are no such homotheties. Note that a cylinder is a helicoid with infinite pitch ( $\kappa = 0$ ), and that as  $\kappa \rightarrow \infty$  the helicoids associated with  $\kappa$  converge to a minimal lamination of  $\mathbf{S}^2 \times \mathbf{R}$  by level spheres with singular set of convergence equal to the axes  $Z \cup Z^*$ .

The main result of this paper is the following theorem:

**1.3. Theorem.** *Let  $M$  be a properly embedded, axial minimal surface in  $\mathbf{S}^2 \times \mathbf{R}$  with bounded curvature and without boundary.*

- (1) *If  $E$  is an annular end of  $M$ , then  $E$  is asymptotic to a helicoid;*
- (2) *If  $M$  is an annulus, then  $M$  is a helicoid;*
- (3) *If  $M$  has finite topology, then each of its two ends is asymptotic to a helicoid, and the two helicoids are congruent to each other by a rotation.*

**1.4. Remarks.** Meeks and Rosenberg [MR05] proved that a properly embedded minimal surface with finite topology in  $\mathbf{S}^2 \times \mathbf{R}$  has bounded curvature. Thus our assumption that the surfaces we consider have bounded curvature is always satisfied.

In statement (1), it is not necessary that  $E$  be part of a complete surface without boundary. The statement is true (with essentially the same proof) for any properly embedded annulus  $E \subset \mathbf{S}^2 \times [z_0, \infty)$  such that  $\partial E \subset \partial \mathbf{S}^2 \times \{z_0\}$  and such that  $E$  contains a vertical ray.

We do not know whether the two helicoids referred to in statement (3) must be the same. See the discussion in Remark 3.2 below.

**Acknowledgments.** We would like to thank Harold Rosenberg for helpful discussions. The research of the second author was supported by the NSF under grant DMS 0707126.

## 2. A convexity lemma

**2.1. Axial surfaces are symmetric and have two axes.** Suppose that  $M$  is a properly embedded, axial minimal surface in  $\mathbf{S}^2 \times \mathbf{R}$ . Then  $M$  contains a vertical line  $Z$ . We claim that  $M$  must also contain the antipodal line  $Z^*$ , i.e., the line consisting of all points at distance  $\pi r$  from  $Z$ , where  $r$  is the radius of the  $\mathbf{S}^2$ . To see this, let  $\rho_Z : \mathbf{S}^2 \times \mathbf{R} \rightarrow \mathbf{S}^2 \times \mathbf{R}$  denote rotation by  $\pi$  about  $Z$ . By Schwarz reflection,  $\rho_Z$  induces an orientation-reversing isometry of  $M$ . In particular,  $\rho_Z$  interchanges the two components of the complement of  $M$ . Thus no point in the complement of  $M$  is fixed by  $\rho_Z$ , so the fixed points of  $\rho_Z$  must all lie in  $M$ . The fixed point set of  $\rho_Z$  is precisely  $Z \cup Z^*$ , so  $Z^*$  must lie in  $M$ , as claimed.

**2.2. The angle function  $\theta$ .** We will assume from now on that an axial surface in  $\mathbf{S}^2 \times \mathbf{R}$  has axes  $Z$  and  $Z^*$  that pass through a fixed pair  $O$  and  $O^*$  of antipodal points in  $\mathbf{S}^2 = \mathbf{S}^2 \times \{0\}$ . Fix a stereographic projection from  $(\mathbf{S}^2 \times \{0\}) \setminus \{O^*\}$  to  $\mathbf{R}^2$ , and let  $\theta$  be the angle function on  $(\mathbf{S}^2 \times \{0\}) \setminus \{O, O^*\}$  corresponding to the polar coordinate  $\theta$  on  $\mathbf{R}^2$ . Extend  $\theta$  to all of  $(\mathbf{S}^2 \times \mathbf{R}) \setminus (Z \cup Z^*)$  by requiring that it be invariant under vertical translations. Of course,  $\theta$  is only well-defined up to integer multiples of  $2\pi$ .

If  $H$  is a helicoid with axes  $Z$  and  $Z^*$ , we will call the components of  $H \setminus (Z \cup Z^*)$  *half-helicoids*. From (1) it follows that the half-helicoids are precisely the surfaces given by

$$(2) \quad \theta = \kappa z + b.$$

Here  $2\pi/\kappa$  is the pitch of the helicoid. Note that at each level  $z$ , specifying  $\theta = c$  specifies a great semicircle connecting  $Z$  to  $Z^*$ . Thus the half helicoid specified by  $\theta = \kappa z + b$  is foliated by great semicircles turning at a constant rate. Rotating  $H$  by an angle  $\gamma$  changes the corresponding  $b$  to  $b + \gamma$ .

Note that the entire helicoid  $H$  consists of  $Z \cup Z^*$  (where  $\theta$  is not defined) together with all points not in  $Z \cup Z^*$  such that

$$\theta \cong \kappa z + b \pmod{\pi}.$$

**2.2. The restriction of  $\theta$  to an annular slice.** Let  $I \subset \mathbf{R}$  be a closed interval (possibly infinite) and let

$$E = M \cap (\mathbf{S}^2 \times I)$$

be the portion of  $M$  in  $\mathbf{S}^2 \times I$ . Suppose that  $E$  is an annulus. Then  $E \setminus (Z \cup Z^*)$  consists of two simply connected domains that are congruent by the involution  $\rho_Z$ . Denote by  $D$  one of these domains, and consider the restriction of  $\theta$  to  $D$ . Because  $D$  is simply connected, we may choose a single-valued branch of this function, and we will also refer to it as  $\theta$  when there is no ambiguity. Since  $I$  is closed,  $\overline{D} \setminus D \subset Z \cup Z^*$ . Because  $\overline{D}$  has a well-defined tangent halfplane at each point of  $\overline{D} \setminus D$ , the function  $\theta$  extends continuously to all of  $\overline{D}$ .

At each level  $z$ , the angle function  $\theta$  has a maximum and a minimum on the compact set  $\overline{D} \cap (\mathbf{S}^2 \times \{z\})$ . This allows us to make the following definition:

**2.3. Definition.**

$$\begin{aligned} \alpha(z) &= \max\{\theta(p, z) : (p, z) \in \overline{D} \cap (\mathbf{S}^2 \times \{z\})\}, \\ \beta(z) &= \min\{\theta(p, z) : (p, z) \in \overline{D} \cap (\mathbf{S}^2 \times \{z\})\}, \\ \phi(z) &= \alpha(z) - \beta(z). \end{aligned}$$

Note that  $\alpha(z) = \beta(z)$  if and only if  $D \cap (\mathbf{S}^2 \times \{z\})$  is half of a great circle. Note also that  $E$  is a portion of a helicoid if and only if

$$\alpha(z) \equiv \beta(z) \equiv \kappa z + b$$

for some  $\kappa$  and  $b$ .

**2.4. Lemma.** *The functions  $\alpha$ ,  $-\beta$ , and  $\phi = \alpha - \beta$  are convex, and they are strictly convex unless they are linear (in which case  $E$  is contained in a helicoid and  $\phi \equiv 0$ ).*

*Proof.* Suppose that  $\alpha$  is not strictly convex. Then there exists  $z_1 < z_2$  and  $0 < \lambda < 1$  such that

$$(3) \quad \alpha(z_1 + \lambda(z_2 - z_1)) \geq \alpha(z_1) + \kappa(z_1 + \lambda(z_2 - z_1)),$$

where

$$\kappa = \frac{\alpha(z_2) - \alpha(z_1)}{z_2 - z_1}$$

is the slope of the line segment connecting  $(z_1, \alpha(z_1))$  and  $(z_2, \alpha(z_2))$ . It follows that there is a line of the form

$$y = \kappa z + \tilde{b}$$

that lies above the graph of  $\alpha$  between  $z_1$  and  $z_2$  and touches that graph at a point  $(\tilde{z}, \alpha(\tilde{z}))$  with  $z_1 < \tilde{z} < z_2$ . By the definition of  $\alpha$ , there is a point  $(\tilde{p}, \tilde{z})$  in the compact set  $\overline{D} \cap (\mathbf{S}^2 \times \{\tilde{z}\})$  with

$$\theta(\tilde{p}, \tilde{z}) = \alpha(\tilde{z}).$$

Then in a neighborhood of  $(\tilde{p}, \tilde{z})$ , the surface  $D$  lies on one side of the half-helicoid  $H$  given by  $\theta = \kappa z + \tilde{b}$  as in (2), and the two surfaces touch at  $(\tilde{p}, \tilde{z})$ . By the maximum principle (or the boundary maximum principle if  $(\tilde{p}, \tilde{z})$  is a boundary point of  $D$ ), a neighborhood in  $\overline{D}$  of  $(\tilde{p}, \tilde{z})$  lies in  $H$ . By analytic continuation, all of  $D$  lies in  $H$ . Thus  $\theta = \kappa z + \tilde{b}$  on  $D$ , so  $\alpha(z) \equiv \beta(z) \equiv \kappa z + \tilde{b}$  and  $\phi(z) \equiv 0$ .

The statements about convexity and strict convexity of  $-\beta$  (or, equivalently, about concavity and strict concavity of  $\beta$ ) are proved in exactly the same way. The assertions about  $\alpha - \beta$  follow, since the sum of two convex functions is convex, and the sum is strictly convex if either summand is strictly convex. q.e.d.

### 3. The proof of Theorem 1.3

Consider first an annular end  $E$ . We may suppose that  $E$  is properly embedded in  $\mathbf{S}^2 \times [a, \infty)$ . Let

$$c := \limsup_{z \rightarrow \infty} \phi(z)$$

where  $\phi = \alpha - \beta$  is as in Definition 2.3. Choose  $z_n \rightarrow \infty$  such that

$$\lim_{n \rightarrow \infty} \phi(z_n) = c.$$

(A priori,  $c$  might be infinite, but we will show below that it is equal to zero.)

Let  $E_n$  and  $D_n$  be the result of translating  $E$  and  $D$  downward by  $z_n$ . Since we are assuming that the curvature of  $E$  is bounded, we may assume by passing to a subsequence that the  $E_n$  converge smoothly to a properly embedded minimal annulus  $\hat{E}$  (see [MR05]), and that the  $D_n$  converge smoothly to  $\hat{D}$ , one of the connected components of

$\hat{E} \setminus (Z \cup Z^*)$ . The smooth convergence  $D_n \rightarrow \hat{D}$  implies that the angle-difference functions

$$\phi_n(z) := \phi(z + z_n)$$

for the disks  $D_n$  converge smoothly to the angle-difference function  $\hat{\phi}$  for the disk  $\hat{D}$ . Note that because  $\hat{\phi}$  is the angle-difference function for the simply connected surface  $\hat{D}$ , it has a well defined and finite value for all  $z \in \mathbf{R}$ . In particular, its value at  $z = 0$  is finite. Also,

$$\hat{\phi}(z) = \lim_{n \rightarrow \infty} \phi_n(z) = \lim_{n \rightarrow \infty} \phi(z + z_n) \leq c$$

by the definition of  $c$ , with equality if  $z = 0$ . Thus  $\hat{\phi}(z)$  attains its maximum value of  $c$  at  $z = 0$ . Consequently,  $\hat{\phi}$  is not strictly convex, so by Lemma 2.4,  $\hat{D}$  is contained in helicoid, and therefore  $\hat{\phi} \equiv 0$ . In particular,  $c = 0$ .

Returning our attention to the original surface  $D$ , we will now prove statement (1) of the theorem. Since  $\alpha \geq \beta$ , the sets

$$\begin{aligned} \{(z, y) : y > \alpha(z), z > a\}, \\ \{(z, y) : y < \beta(z), z > a\} \end{aligned}$$

are disjoint, convex subsets of  $\mathbf{R}^2$ , so they are separated by a line  $y = az + d$ . Thus

$$\beta(z) \leq az + d \leq \alpha(z).$$

Since  $\alpha(z) - \beta(z) \rightarrow 0$  as  $z \rightarrow \infty$ , the graphs of  $\alpha$  and of  $\beta$  are asymptotic to the line  $y = az + d$ . Thus  $D$  is  $C^0$ -asymptotic to the half-helicoid whose equation is  $\theta = az + d$ . Since the curvature of  $D$  is bounded, the surface  $D$  is smoothly asymptotic to that half-helicoid. It follows immediately that the end

$$E = \overline{D} \cup \rho_Z \overline{D}$$

is asymptotic to the corresponding helicoid. This proves statement (1) of the theorem.

To prove statement (2), suppose that  $M$  is a properly embedded, axial minimal annulus. Let  $D$  be one of the simply connected components of  $M \setminus (Z \cup Z^*)$ . We know from Lemma 2.4 that  $\phi$  is convex on all of  $\mathbf{R}$ , and from the proof above of the first statement of Theorem 1.3 (applied to the ends of  $M$ ) that

$$\lim_{z \rightarrow \pm\infty} \phi(z) = 0.$$

Thus  $\phi(z) \equiv 0$ , so by Lemma 2.4,  $M$  is a helicoid.

Statement (3) of the theorem follows from a standard flux argument as follows. Let  $s < t$  and let

$$\Sigma = \Sigma(s, t) = M \cap (\mathbf{S}^2 \times (s, t)).$$

Let  $\nu(p)$  be the outward unit co-normal at  $p \in \partial\Sigma \subset M$  (a vector field along  $\partial\Sigma$  that is tangent to  $M$ ). Since  $\partial/\partial\theta$  is a Killing field on  $\mathbf{S}^2 \times \mathbf{R}$ ,

$$\int_{\partial\Sigma} \left( \nu \cdot \frac{\partial}{\partial\theta} \right) ds = 0$$

by the first variation formula. Equivalently, if we let  $M_a = M \cap \{z \leq a\}$ , then the flux

$$(4) \quad \int_{\partial M_a} \left( \nu \cdot \frac{\partial}{\partial\theta} \right) ds$$

is independent of  $a$ . We call (4) the *rotational flux* of  $M$  (with respect to the axes  $Z$  and  $Z^*$ ).

If  $M$  is asymptotic (as  $z \rightarrow \infty$  or as  $z \rightarrow -\infty$ ) to a helicoid  $H$ , then  $M$  and  $H$  clearly have the same rotational flux. Thus to prove statement (3) of the theorem, it suffices to check that if two helicoids with axes  $Z \cup Z^*$  have the same rotational flux, then they are congruent by rotation. If we let  $F(\kappa)$  denote the rotational flux of the helicoid  $H(\kappa)$  given by

$$\theta \cong \kappa z \pmod{\pi},$$

then it suffices to show that  $F(\kappa)$  depends strictly monotonically on  $\kappa$ . To see it does, consider expression (4) for  $F(\kappa)$ :

$$F(\kappa) = \int_{\partial(H(\kappa) \cap \{z \leq 0\})} \left( \nu \cdot \frac{\partial}{\partial\theta} \right) ds.$$

Note that the domain  $\partial(H(\kappa) \cap \{z \leq 0\})$  is a fixed great circle  $C$ , and that for each point in  $C \setminus \{O, O^*\}$ , the integrand is a strictly increasing function of  $\kappa$  (because the larger  $\kappa$  is, the smaller the angle between the vectors  $\partial/\partial\theta$  and  $\nu$ ). Thus  $F(\kappa)$  is a strictly increasing function of  $\kappa$ . q.e.d.

**3.1. Remark.** The reader may wonder why we used rotational flux rather than the vertical flux

$$(5) \quad \int_{\partial M_a} \left( \nu \cdot \frac{\partial}{\partial z} \right) ds.$$

The problem with vertical flux is that the helicoid  $H(\kappa)$  and its mirror image  $H(-\kappa)$  have the same vertical flux. Thus vertical flux alone does not rule out the possibility that the two ends of  $M$  might be asymptotic to helicoids that are mirror images of each other.

**3.2. Remark.** We have not proved that the constant terms  $b$  in the equations

$$\theta \cong \kappa z + b \pmod{\pi}$$

for the helicoids asymptotic to the ends of  $M$  are the same. There is some reason to expect that  $b$  can change from end to end.

A change in  $b$  corresponds to a rotation, and when  $\kappa \neq 0$  (i.e., when the helicoid is not a cylinder), a rotation by  $\beta$  is equivalent to a vertical translation by  $\beta/\kappa$ . In the Introduction, we discussed known examples

of properly embedded axial minimal surfaces of finite genus. Those examples may be regarded as desingularizing the intersection of a helicoid  $H$  with the totally geodesic sphere  $\mathbf{S}^2 \times \{0\}$ . Such desingularization might well cause a slight vertical separation of the top and bottom ends of the helicoid, in order to “make room” for the sphere. A similar situation exists in  $\mathbf{R}^3$  when considering the Costa-Hoffman-Meeks surfaces as desingularizations of the intersection of a vertical catenoid with a horizontal plane passing through the waist of the helicoid [HM90], [HK97]. While the top and bottom catenoidal ends have the same logarithmic growth rate, corresponding to the vertical flux, numerical evidence from computer simulation of these surfaces indicates a vertical separation of those ends. (In other words, the top end is asymptotic to the top of a catenoid, the bottom end is asymptotic to the bottom of a catenoid, and numerical evidence indicates that the two catenoids are related by a nonzero vertical translation.)

### References

- [BB08] Jacob, Bernstein & Christine Breiner, *Conformal Structure of Minimal Surfaces with Finite Topology*, 92008), arXiv:0810.4478v1 [math.DG], To appear in Comment. Math. Helv.
- [CM08] Tobias H. Colding & William P. Minicozzi II, *The Calabi-Yau conjectures for embedded surfaces*, Ann. of Math. (2), **167**, (2008), no. 1, 211–243, MR 2373154 (2008k:53014), Zbl 1142.53012.
- [Col97] Pascal Collin, *Topologie et courbure des surfaces minimales proprement plongées de  $\mathbf{R}^3$* , Ann. of Math. (2), **145**, (1997), 1, 1–31, French MR 1432035 (98d:53010), Zbl 0886.53008.
- [Hau06] Laurent Hauswirth, *Minimal surfaces of Riemann type in three-dimensional product manifolds*, Pacific J. Math., **224**, (2006), 1, 91–117, MR 2231653 (2007e:53004), Zbl 1108.49031.
- [HK97] David Hoffman & Karcher Hermann, *Complete embedded minimal surfaces of finite total curvature*, Geometry, V, Encyclopaedia Math. Sci., **90**, Springer, Berlin, (1997), 5–93, 267–272, MR 1490038 (98m:53012), Zbl 0890.53001.
- [HM89] David Hoffman & William H. Meeks III, *The asymptotic behavior of properly embedded minimal surfaces of finite topology*, J. Amer. Math. Soc., **2**, (1989), no. 4, 667–682, MR 1002088 (90f:53010), Zbl 0683.53005.
- [HM90] ———, *Limits of minimal surfaces and Scherk’s fifth surface*, Arch. Rational Mech. Anal., **111**, (1990), no. 2, 181–195, MR 1057654 (92a:53008), Zbl 0709.53006.
- [HW08] David Hoffman & Brian White, *On the number of minimal surfaces with a given boundary*, Astérisque, **322**, (2008), 207–224, MR 1057654 (92a:53008), Zbl 1184.53012.
- [HW11] ———, *Helicoid-like minimal surfaces of arbitrary genus in  $\mathbf{S}^2 \times \mathbf{R}$* , in preparation,
- [MR93] William H. Meeks III & Harold Rosenberg, *The geometry and conformal structure of properly embedded minimal surfaces of finite topology in  $\mathbf{R}^3$* ,



- Invent. Math., **114**, (1993), no. 3, 625–639, MR 1244914 (94i:53003), Zbl 0803.53007.
- [MR05] William H. Meeks & Harold Rosenberg, *The theory of minimal surfaces in  $\mathbb{M} \times \mathbb{R}$* , Comment. Math. Helv., **80**, (2005), no. 4, 811–858, MR 2182702, Zbl 1085.53049.
- [PR99] Renato H.L. Pedrosa & Manuel Ritoré, *Isoperimetric domains in the Riemannian product of a circle with a simply connected space form and applications to free boundary problems*, Indiana Univ. Math. J., **48**, (1999), no. 4, 1357–1394, MR 1757077 (2001k:53120), Zbl 0956.53049.
- [Ros02] Harold Rosenberg, *Minimal surfaces in  $\mathbb{M}^2 \times \mathbb{R}$* , Illinois J. Math., **46**, (2002), no. 4, 1177–1195, MR 1988257, Zbl 1036.53008.
- [Ros03] ———, *Some recent developments in the theory of minimal surfaces in 3-manifolds*, Publicações Matemáticas do IMPA [IMPA Mathematical Publications], (2003), 24<sup>o</sup> Colóquio Brasileiro de Matemática [24th Brazilian Mathematics Colloquium], Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, MR 2028922 (2005b:53015), Zbl 1064.53007.

DEPARTMENT OF MATHEMATICS  
STANFORD UNIVERSITY  
STANFORD, CA 94305

*E-mail address:* hoffman@math.stanford.edu

DEPARTMENT OF MATHEMATICS  
STANFORD UNIVERSITY  
STANFORD, CA 94305

*E-mail address:* white@math.stanford.edu