

WEIGHTED BERGMAN KERNELS ON ORBIFOLDS

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Abstract

We describe a notion of ampleness for line bundles on orbifolds with cyclic quotient singularities that is related to embeddings in weighted projective space, and prove a global asymptotic expansion for a weighted Bergman kernel associated to such a line bundle.

1. Introduction

Let (X, ω) be a compact n -dimensional Kähler manifold and L be a positive line bundle on X equipped with a hermitian metric h whose curvature form is $-2\pi i\omega$. These induce an L^2 -metric on the space of sections $H^0(L^k)$, and given an orthonormal basis $\{s_\alpha\}$ the Bergman kernel is the smooth function

$$B_k(x) = \sum_{\alpha} |s_{\alpha}(x)|^2,$$

where $|s_{\alpha}(x)|$ is the pointwise norm induced by h (more precisely, its k th power considered as a metric on L^k). More invariantly, the L^2 -metric on $H^0(L^k)$ induces a Fubini-Study metric on $\mathbb{P}(H^0(L^k)^*) \supset X$ and $\mathcal{O}(1)$ over it; restricting to X we get a metric h_{FS} on $\mathcal{O}_X(1) \cong L^k$ which may not equal h . Then $B_k = h/h_{FS}$ is their ratio.

Work of Fefferman [Fef74], Yau [Yau86], Tian [Tia90], Zelditch [Zel98], Catlin [Cat99] and Ruan [Rua98] describes the asymptotic behaviour of this function for large k : there exist smooth functions b_1, \dots, b_N and a global expansion

$$(1.1) \quad B_k = k^n + b_1 k^{n-1} + \dots + b_N k^{n-N} + O(k^{n-N-1})$$

for $k \gg 0$, where $O(k^{n-N-1})$ can even be taken in the C^∞ norm. Moreover, the b_j can be expressed in terms of the derivatives of the metric [Lu00]. In particular, b_1 is half the scalar curvature of ω , leading to the importance of this expansion in the theory of constant scalar curvature Kähler metrics [Yau86, Don01].

Throughout this paper, X will instead be an orbifold (usually compact, except when we work locally) with cyclic stabiliser groups and an

orbifold line bundle L over it. We let ord_x denote the order of a point $x \in X$: the size of the cyclic stabiliser group of any lift of this point in an orbifold chart. When X is compact, $\text{ord}(X)$ denotes the least common multiple of this finite collection of integers.

Then the same expansion of B_k holds away from the orbifold locus, [DLM06, Son04]. And while the precise form of the expansion over the orbifold locus is given in [DLM06], the result is not smooth; see Remark 1.12.

We illustrate the situation locally using the simplest example: the orbifold $\mathbb{C}/(\mathbb{Z}/2)$ with local coordinate z on \mathbb{C} acted on by $\mathbb{Z}/2$ via $z \mapsto -z$. Then $x = z^2$ is a local coordinate on the quotient thought of as a manifold. An ordinary (non-orbifold) line bundle is one pulled back from the quotient, i.e., one which has trivial $\mathbb{Z}/2$ -action upstairs when considered as a trivial line bundle there. This has invariant sections $\mathbb{C}[x] = \mathbb{C}[z^2]$. We do not consider such line bundles, as their sections only see the quotient manifold $\text{Spec } \mathbb{C}[x]$ (to which the usual Bergman expansion applies), missing the extra functions of $\sqrt{x} = z$ that the orbifold sees.

Instead we use the nontrivial *orbifold line bundle* given by the trivial line bundle upstairs with nontrivial $\mathbb{Z}/2$ -action (acting as -1 on the trivialisation). This has invariant sections $\sqrt{x}\mathbb{C}[x] = z\mathbb{C}[z^2]$, while its square has trivial $\mathbb{Z}/2$ -action and has sections $\mathbb{C}[x] = \mathbb{C}[z^2]$ as above. Therefore, the sections of its powers generate the entire ring of functions $\mathbb{C}[\sqrt{x}] = \mathbb{C}[z]$ upstairs, and see the full orbifold structure.

Definition 1.2. [RT11] An orbifold line bundle L over a cyclic orbifold X is *locally ample* if in an orbifold chart around $x \in X$, the stabiliser group acts faithfully on the line L_x . We say L is *orbi-ample* if it is both locally ample and globally positive. (That is, $L^{\text{ord}(X)}$ is ample in the usual sense when thought of as a line bundle on the underlying space of X ; by the Kodaira-Baily embedding theorem [Bai57] one can equivalently ask that L admits a hermitian metric with positive curvature.)

In [RT11, sections 2.5–2.6] this notion of ampleness is shown to be equivalent to the existence of an orbifold embedding

$$(X, L) \hookrightarrow (\mathbb{W}\mathbb{P}, \mathcal{O}(1))$$

of X into a weighted projective space $\mathbb{W}\mathbb{P}$. That is, the orbifold structure of X is pulled back from that of $\mathbb{W}\mathbb{P}$, and L is the pullback of the orbifold hyperplane bundle $\mathcal{O}_{\mathbb{W}\mathbb{P}}(1)$.

Just as in our example above, for any orbifold line bundle L which has nontrivial stabiliser action on the line L_x over an orbifold point x , the sections of L (i.e., the invariant sections upstairs) all vanish at x . Therefore, for most powers L^k of an orbi-ample line bundle, the Bergman kernel B_k will vanish at orbifold points. Conversely, for powers

divisible by ord_x , $B_k(x)$ will be nonzero, so in general $B_k(x)$ will have some periodic behaviour in k at orbifold points. The purpose of this paper is to get a smooth global expansion by taking weighted sums of Bergman kernels associated to various powers of L , flattening out the periodicity. In the companion paper [RT11] we apply this to orbifold Kähler metrics of constant scalar curvature and their relationship to stability.

The Bergman kernel measures the local density of holomorphic sections, so to get an expansion on an orbifold, we first discuss the general local situation—the quotient of an open set in \mathbb{C}^n by a linear action of the cyclic group $\mathbb{Z}/m := \mathbb{Z}/m\mathbb{Z}$. The reader will not lose much by considering only the $\mathbb{C}/(\mathbb{Z}/2)$ example above, with the orbifold line bundle with trivialisation which has weight ± 1 under the $\mathbb{Z}/2$ -action.

By the local ampleness condition, there exists an identification between \mathbb{Z}/m and the group of m th roots of unity with respect to which there exists a local equivariant (not invariant!) trivialisation of L of weight -1 . Then, with respect to this trivialisation, sections of L^k downstairs, i.e., invariant sections upstairs on \mathbb{C}^n , are the same thing as functions upstairs of \mathbb{Z}/m -weight $k \bmod m$. (In particular, they vanish at the origin unless $k \equiv 0 \bmod m$.)

We consider the weighted Bergman kernel

$$(1.3) \quad B_k^{\text{orb}} := \sum_i c_i B_{k+i},$$

where c_i are positive constants and i runs over some fixed finite index set of nonnegative integers. This is the global expression downstairs on the quotient; upstairs locally we are taking a similar expression, but only include the sections of L^{k+i} that have \mathbb{Z}/m -weight $0 \bmod m$ (or, using the trivialisation, the functions of weight $k+i \bmod m$).

Since functions of nonzero weight vanish at the orbifold point, the ordinary Bergman kernel at the origin upstairs is equal to the sum of the $|s_\alpha|^2$ over an orthonormal basis of *only the invariant sections*; i.e., it equals the “downstairs” Bergman kernel B_{k+i} at the origin. (Using the trivialisation, this corresponds to summing over only the functions of weight $k+i$.) Therefore, by (1.1) applied to the ordinary upstairs Bergman kernel at the origin, the downstairs kernel $B_{k+i}(0)$ has the asymptotic expansion for fixed i and as k tends to infinity

$$(k+i)^n + \frac{1}{2} \text{Scal}(\omega)(k+i)^{n-1} + \dots$$

Summing over i , we find that in (1.3) the functions upstairs of \mathbb{Z}/m -weight u contribute, to leading order in k ,

$$(1.4) \quad \sum_{i \equiv u-k} c_i k^n$$

to $B_k^{\text{orb}}(0)$. (Here and throughout the paper, the subscript means that the sum is taken over all i equal to $u - k \pmod{m}$.)

For instance, in our $\mathbb{C}/(\mathbb{Z}/2)$ example above with the nontrivial orbifold line bundle, the functions of $\mathbb{Z}/2$ -weight 0 (respectively 1) mod 2 contribute only to the terms $B_{k+i}(0)$ with $k+i$ even (respectively odd). To even things out and ensure that B_k^{orb} has a smooth global expansion, we want to use *all* local functions of all \mathbb{Z}/m -weights equally (thus returning us to something close to the original Bergman kernel *upstairs*). So we choose the c_i so that the functions of each \mathbb{Z}/m -weight contribute to the sum (1.3) with the same coefficient.

We therefore want for each k for (1.4) to be equal for all u , i.e.,

$$\sum_{i \equiv u} c_i \text{ is independent of } u.$$

More generally, if $N \leq n$ and all of the coefficients of k^n, \dots, k^{n-N} in

$$(1.5) \quad \sum_i c_i (k+i)^n$$

have equal contributions in each weight (mod m), i.e.,

$$(1.6) \quad \sum_{i \equiv u} i^p c_i \text{ is independent of } u \text{ for } p = 0, \dots, N,$$

then the same will be true of each $\sum_i c_i (k+i)^{n-j}$ for $j = 0, \dots, N$, and B_k^{orb} will admit an asymptotic expansion in k of order N . In fact, we impose a slightly stronger condition than (1.6) in order to get the expansion at the C^r level.

Theorem 1.7. *Let (X, ω) be a compact n -dimensional Kähler orbifold with cyclic quotient singularities, and L be an orbifold ample line bundle on X equipped with a hermitian metric whose curvature form is $-2\pi i \omega$. Fix $N, r \geq 0$ and suppose c_i are a finite number of positive constants chosen so that if X has an orbifold point of order m , then*

$$(1.8) \quad \frac{1}{m} \sum_i i^p c_i = \sum_{i \equiv u \pmod{m}} i^p c_i \quad \text{for all } u \text{ and } p = 0, \dots, N+r.$$

Then the function

$$B_k^{\text{orb}} := \sum_i c_i B_{k+i}$$

admits a global C^r -expansion of order N . That is, there exist smooth functions b_0, \dots, b_N on X such that

$$B_k^{\text{orb}} = b_0 k^n + b_1 k^{n-1} + \dots + b_N k^{n-N} + O(k^{n-N-1}),$$

where the $O(k^{n-N-1})$ term is to be taken in the C^r -norm. Furthermore, the b_j are universal polynomials in the constants c_i and the derivatives

of the Kähler metric ω ; in particular,

$$b_0 = \sum_i c_i \quad \text{and} \quad b_1 = \sum_i c_i \left(ni + \frac{1}{2} \text{Scal}(\omega) \right),$$

where $\text{Scal}(\omega)$ is the scalar curvature of ω .

Remark 1.9. If condition (1.8) holds for some m , then it also holds if m is replaced by any factor; hence for the theorem it is sufficient to assume it holds when $m = \text{ord}(X)$ is the lowest common multiple of the orders of the stabiliser groups of all points of X . So in particular, the $N = r = 0$ case yields a top-order C^0 -expansion for B_k^{orb} when $c_i = 1$ for $i = 1, \dots, \text{ord}(X)$ and $c_i = 0$ otherwise. It turns out (Lemma 3.5) that (1.8) is equivalent to asking that the function $\sum_i c_i z^i$ has a root of order $N + r + 1$ at each nontrivial m -th root of unity. Thus there is little loss in assuming the c_i are defined by

$$\sum_i c_i z^i := (z^{m-1} + z^{m-2} + \dots + 1)^{N+r+1}.$$

In fact, the condition (1.8) is natural in the sense that it is also necessary for the existence of an expansion; see Remark 3.7. Also, if X is a manifold, then there is no condition on the c_i , and the theorem is equivalent to the expansion (1.1) for manifolds stated above.

Remark 1.10. By a C^r -expansion of order N we mean there is a constant $C_{r,N}$ such that

$$\left\| B_k^{\text{orb}} - (b_0 k^n + b_1 k^{n-1} + \dots + b_N k^{n-N}) \right\|_{C^r} \leq C_{r,N} k^{n-N-1},$$

where the norm $\|f\|_{C^r}$ is the sum of the supremum norms of f and its first r derivatives over all of X in the orbifold sense (i.e., we measure the norm of derivatives using the metric upstairs on orbifold charts). The theorem generalises to hermitian metrics h on L with curvature form $-2\pi i \omega_h$ not necessarily equal to ω , the only change being that the coefficients in the expansion become [MM07, Theorem 4.1.3, Remark 5.4.13]

$$(1.11) \quad \begin{aligned} b_0 &= \frac{\omega_h^n}{\omega^n} \sum_i c_i, \\ b_1 &= \frac{\omega_h^n}{\omega^n} \sum_i c_i \left[ni + \text{tr}_{\omega_h}(\text{Ric}(\omega)) - \frac{1}{2} \text{Scal}(\omega_h) \right]. \end{aligned}$$

Moreover, the constants $C_{r,N}$ can be taken to be uniform as (h, ω) runs over a compact set.

The strategy of our proof is to use known results on manifolds. For global quotients one can obtain what we want rather easily by averaging the expansion upstairs. To apply this to general orbifolds, we need to be

able to work locally, which is what the approach of Berman-Berndtsson-Sjöstrand [BBS08] to the expansion (1.1) allows us to do. This starts by proving a local expansion for the Bergman kernel, and then uses the Hörmander estimate to pass to a global expansion. In Section 3 we average these local expansions to get a local orbifold expansion, and use it in Section 4 to get a global expansion in much the same way as for manifolds.

Remark 1.12. There are several approaches to the analysis that one may take to prove the above result. For example, once one arranges the Bergman kernels in the combination of Theorem 1.7 the main result follows quickly from the precise control of the singularity of the Bergman kernel expansion over the orbifold locus proved in [DLM06, (5.25)]; see [DLM11] for details.

One consequence is a Riemann-Roch expression for orbi-ample line bundles obtained by integrating the expansion over X . The difference between this and the general Kawasaki-Riemann-Roch formula for orbifolds [Kaw79] is that the periodic term coming from the orbifold singularities vanishes due to the weighted sum over i . We denote the orbifold canonical bundle by K_{orb} (defined, for instance, in [RT11, section 2.4]).

Corollary 1.13. *Let X be an n -dimensional orbifold with cyclic quotient singularities and L be an orbi-ample line bundle. Let $m = \text{ord}(X)$ and suppose*

$$\frac{1}{m} \sum_i c_i = \sum_{i \equiv u \pmod m} c_i \quad \text{and} \quad \frac{1}{m} \sum_i i c_i = \sum_{i \equiv u \pmod m} i c_i \quad \text{for all } u.$$

Then

$$\sum_i c_i h^0(L^{k+i}) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$$

with

$$\begin{aligned} a_0 &= \frac{\sum_i c_i}{n!} \int_X c_1(L)^n, \\ a_1 &= \frac{\sum_i i c_i}{(n-1)!} \int_X c_1(L)^n - \frac{\sum_i c_i}{2(n-1)!} \int_X c_1(K_{\text{orb}}) c_1(L)^{n-1}. \end{aligned}$$

Proof. This follows from $\int_X B_k \frac{\omega^n}{n!} = h^0(L^k)$ and the fact that the integral of the scalar curvature of any orbifold Kähler metric is the topological quantity $-\frac{1}{(n-1)!} \int_X c_1(K_{\text{orb}}) c_1(L)^{n-1}$. q.e.d.

As an indication of the connection with constant scalar curvature metrics, we give an analogue of Donaldson's Theorem [Don01, theorem 2] for orbifolds (for a more applicable generalisation, see [RT11, theorem 5.1]).

Corollary 1.14. *Suppose that (1.8) is satisfied for $N = 1$ and $r = 2$. For each $k \gg 0$, let h_k be a hermitian metric on L with curvature $-2\pi i\omega_k$. Consider the diagonal sequence of Bergman kernels B_k^{orb} associated to the metrics (h_k, ω_k) and the power L^k of L , and suppose that for each k this function is constant over X . If also ω_k converges in C^2 to a Kähler metric ω_∞ , then $\text{Scal}(\omega_\infty)$ is constant.*

Proof. Let $\text{vol} := \int_X \frac{c_1(L)^n}{n!}$ be the volume of L . Notice that since the Bergman kernel is constant, by the previous corollary and integrating over X , we have $\text{vol } B_k^{\text{orb}} = a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$ with a_i as above. Hence, applying Theorem 1.7 to the pair (h_k, ω_k) ,

$$\frac{1}{\text{vol}}(a_0 k^n + a_1 k^{n-1} + O(k^{n-2})) = B_k^{\text{orb}} = b_0 k^n + b_1 k^{n-1} + O(k^{n-2}),$$

where the $O(k^{n-2})$ term is independent of (h_k, ω_k) since they lie in a compact set. Furthermore, $b_0 = \sum_i c_i = \frac{a_0}{\text{vol}}$ and $b_1 = \sum_i c_i (ni + \frac{1}{2} \text{Scal}(\omega_k))$. Thus the k^n terms cancel, so b_1 tends to $a_1 \text{vol}^{-1}$ uniformly over X as k tends to infinity. Since ω_k tends to ω_∞ in C^2 , $\text{Scal}(\omega_k)$ tends to $\text{Scal}(\omega_\infty)$ and thus $\text{Scal}(\omega_\infty)$ is constant. q.e.d.

Interpretation using peaked sections. Intuitively, one can understand what is going on as follows. On a manifold $B_k(x)$ is the pointwise norm square of a *peaked section* s_x at x of unit L^2 -norm. This is glued from zero outside a small ball of radius $O(k^{-1/2})$ centred on x and the standard local model (with Gaussian norm) inside the ball. (If s_x is constructed to have zero L^2 -inner product with any section vanishing at x , then the Bergman kernel at x is *precisely* $|s_x(x)|^2$; if not, then this is still a good enough approximation for the discussion here.)

To describe the orbifold case, consider the simple model consisting of a single orbifold chart $\mathbb{C} \rightarrow \mathbb{C}/(\mathbb{Z}/m)$. Suppose that x is a smooth point, which in this model case is any point other than the origin. Then for large k we can pick a ball of radius $O(k^{-1/2})$ centred at x and disjoint from the origin. It therefore admits a copy upstairs in \mathbb{C} , i.e., a ball around one of its preimages. Then just as in the smooth case, there is a peaked section supported in this ball, and moving it around \mathbb{C} by the group action gives an invariant section with m peaks. This can then be equivariantly glued to zero, giving an invariant section that has L^2 -norm approximately m , but considered as a section downstairs it has norm 1 (since the volume downstairs is defined to be the volume upstairs divided by the order of the chart). Therefore its pointwise norm squared at x is approximately the value of the Bergman kernel.

Now instead consider an orbifold point, which here is the origin. A section upstairs on \mathbb{C} , peaked about the origin, can be equivariant *only* when $k \equiv 0 \pmod{m}$. Therefore for most k , the Bergman kernel will have value zero at the origin. But when k is a multiple of m , we get an invariant section with L^2 -norm 1 upstairs, and therefore L^2 -norm

only $\frac{1}{m}$ when considered as an orbifold section downstairs. Therefore we multiply the section by \sqrt{m} to get one of unit L^2 -norm, and find that the Bergman kernel of the orbifold at the origin is approximately m times as big as at manifold points.

Thus when we sum the Bergman kernels over a period L^k, L^{k+1}, \dots of length at least m we stand a chance of them averaging out to something which is again approximately constant to top order in k . The condition that they do is that the average of the coefficients c_i with which we make each L^{k+i} contribute is equal to the sum of the coefficients c_i which contribute at the origin (i.e., those for which $k+i \equiv 0 \pmod{m}$). But this is precisely the $p=0$ condition in (1.8). A similar analysis for $p=1$ means that we can ensure that we smooth out all of the i -dependence of the orbifold Bergman kernel except the part coming from scalar curvature.

Conventions: We refer to [RT11] for detailed definitions and conventions. For this paper we need only that an orbifold with cyclic stabiliser groups is an analytic space covered by charts of the form $U \rightarrow U/G \subset X$, where U is an open set in \mathbb{C}^n and G is a finite cyclic group acting effectively and *linearly* on U ; see for example [RT11, section 2.1]. It is important for our applications that we allow orbifold structure in codimension one. Thus if $m \geq 2$, then $\mathbb{C}/(\mathbb{Z}/m)$ is considered a distinct orbifold from \mathbb{C} even though they have the same underlying space. Quantities on X (hermitian metrics, Kähler metrics, sections, supremum norms, etc.) are always taken in the orbifold sense, as an invariant quantity upstairs in orbifold charts. An orbifold line bundle is an equivariant line bundle on orbifold charts; the gluing condition is described in [RT11].

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2. Bergman Kernels

From now on, let (X, ω) be a compact n -dimensional Kähler orbifold and L be an orbifold line bundle with a hermitian metric h whose curvature form is $-2\pi i\omega$. By abuse of notation, we denote the induced metric on L^k also by h , which along with the volume form defined by ω yields an L^2 -inner product on $H^0(L^k)$ given by

$$(s, t)_{L^2} = \int_X (s, t)_h \frac{\omega^n}{n!} \quad \text{for } s, t \in H^0(L^k).$$

The Bergman kernel of L^k is defined to be $B_k(x) = \sum_{\alpha} |s_{\alpha}(x)|_h^2$, where $\{s_{\alpha}\}$ is any L^2 -orthonormal basis for $H^0(L^k)$. Recall that the *reproducing kernel* of L^k is the section $K = K_k$ of $L^k \boxtimes \bar{L}^k$ on $X \times X$ given by

$$K(y, x) = \sum_{\alpha} s_{\alpha}(y) \boxtimes \bar{s}_{\alpha}(x),$$

and therefore

$$B_k(x) = |K(x, x)|_h.$$

More invariantly, for $x \in X$ let K_x be the section of $L^k \otimes \bar{L}_x^k$ on X given by $K_x(y) := K(y, x)$, so that $(s, K_x)_{L^2}$ is an element of the line L_x^k for any section s of L^k . Then the defining property of the reproducing kernel is that this should equal $s(x) \in L_x^k$:

$$s(x) = (s, K_x)_{L^2} \quad \text{for all } s \in H^0(L^k).$$

We base our discussion of the functions B_k on the work of [BBS08] that starts by constructing a local expansion, as we describe next. For other approaches to Bergman kernels, see for example the book [MM07].

Let U be an open ball in \mathbb{C}^n centred at the origin, and $\phi: U \rightarrow \mathbb{R}$ be smooth and strictly plurisubharmonic. Setting $\omega = \frac{i}{2\pi} \partial \bar{\partial} \phi$, define an inner product on the space of smooth functions by

$$(u, v)_{\phi}^2 = \int_U u \bar{v} e^{-\phi} \frac{\omega^n}{n!},$$

and let $H(U)_{\phi}$ be the space of holomorphic functions of finite norm. (The reader should of course think of this situation as arising when U is a chart of a manifold over which the hermitian line bundle $L|_U$ has curvature $-2\pi i \omega$ and a holomorphic trivialisation of norm $e^{-\phi/2}$.) Let χ be a smooth cutoff function supported on U that takes the value 1 on $\frac{1}{2}U$.

Definition 2.1. We say that a sequence \mathcal{K}_k , $k \gg 0$, of smooth functions on $U \times U$ are *local reproducing kernels* mod $O(k^{-N-1})$ for $H(U)_{k\phi}$ if there is a neighbourhood $U_0 \subset U$ of the origin such that for any $x \in U_0$ and any $u \in H(U)_{k\phi}$,

$$u(x) = (\chi u, \mathcal{K}_{k,x})_{k\phi} + O(k^{-N-1} e^{k\phi(x)/2}) \|u\|_{k\phi},$$

where $\mathcal{K}_{k,x}(y) := \mathcal{K}_k(y, x)$, and the $O(k^{-N-1} e^{k\phi(x)/2})$ term is uniform on U_0 .

So whereas the reproducing kernels K_k are globally and uniquely defined, the local kernels \mathcal{K}_k are far from unique. The next theorem exhibits preferred ones, which will be shown later to be local approximations to the global kernels.

We need the notion of a symmetric almost sesqui-holomorphic extension $\psi: U \times U \rightarrow \mathbb{C}$ of ϕ . That is, $\psi(x, y) = \overline{\psi(y, x)}$, $\psi(x, x) = \phi(x)$, and $\bar{\partial}(\psi(x, \bar{y}))$ vanishes to all orders on $\{x = y\}$, i.e., $D^\alpha(\bar{\partial}(\psi(x, \bar{y})))|_{\{x=y\}} = 0$ for all α . (This notion is discussed in [BBS08, section 2.6] and [Rub08, section 3.3.3.1]; if ϕ is analytic, then one can take ψ to be holomorphic in the first variable and anti-holomorphic in the second.) By shrinking U if necessary we can ensure there is a $\delta > 0$ such that

$$(2.2) \quad \operatorname{Re}(\psi(x, y)) \leq \phi(x)/2 + \phi(y)/2 - \delta\|x - y\|^2$$

for $x, y \in U$ [BBS08, equation 2.7], [Rub08, lemma 3.8]. We write $f = O(k^{-\infty})$ to mean $f = O(k^{-M})$ for any M .

Theorem 2.3 (Berman-Berndtsson-Sjöstrand). *For fixed $N, r \geq 0$ there exist smooth functions \tilde{b}_j defined on $U \times U$ such that*

$$\mathcal{K}_k(y, x) = \left(k^n + \tilde{b}_1(y, x)k^{n-1} + \cdots + \tilde{b}_{N+r}(y, x)k^{n-N-r} \right) e^{k\psi(y, x)}$$

is a local reproducing kernel mod $O(k^{-N-r-1})$ for $H(U)_{k\phi}$. Each \tilde{b}_j can be written as a polynomial in the derivatives $\partial_x^\alpha \bar{\partial}_y^\beta \psi(x, y)$, and in particular $\tilde{b}_1(x, x) = \frac{1}{2} \operatorname{Scal}(\omega)$.

Moreover, if $D_{x,y}$ is a differential operator of any order in x and y , then $e^{-k(\phi(x)/2 + \phi(y)/2)} D_{x,y} \bar{\partial}_x \mathcal{K}_k$ and $e^{-k(\phi(x)/2 + \phi(y)/2)} D_{x,y} \partial_y \mathcal{K}_k$ are both $O(k^{-\infty})$.

Proof. This is [BBS08, proposition 2.5 and section 2.4] when ϕ is analytic, and generalised to the smooth case in *loc. cit.* proposition 2.7. (The cited work is stated in the case that U is a ball of radius 1, but the case for general radius follows immediately on rescaling; there is also an extra factor of π^{-n} due to differences in conventions for the curvature form.) q.e.d.

Observe that if a finite group G acts on U preserving ϕ , then we can take ψ and χ to be G -invariant, in which case $\mathcal{K}_k(\zeta x, \zeta x) = \mathcal{K}_k(x, x)$ for all $\zeta \in G$.

3. Local expansion

We next extend the local expansion of the previous section to orbifold charts, which begins by averaging the local reproducing kernels \mathcal{K}_k . Building on the notation of the previous section, suppose that a cyclic group G of order m acts linearly and faithfully on U and that our plurisubharmonic function ϕ and cutoff function χ are invariant. The inner product on the space of functions is replaced with

$$(u, v)_{\phi, m} = \frac{1}{m} \int_U u \bar{v} e^{-\phi} \frac{\omega^n}{n!},$$

and $H(U)_{\phi,m}$ is the space of holomorphic functions of finite norm. Fix a generator ζ of G and suppose we are given a character of G which maps ζ to a primitive m -th root of unity λ . We say a function u on U has weight j if $u(\zeta x) = \lambda^j u(x)$ for all $x \in U$. (The reader should now think of U as arising from an orbifold chart $U \rightarrow U/G$, and the character as describing the action of $G = \mathbb{Z}/m$ on $L|_U$ under some (noninvariant) trivialisation; the condition that λ is primitive comes from the local condition for orbi-ample line bundles, and the sections of $L^k|_U$ now arise from functions on U of weight $k \bmod m$.)

Let $\mathcal{K}_k = (k^n + \tilde{b}_1 k^{n-1} + \cdots + \tilde{b}_{N+r} k^{n-N-r}) e^{k\psi}$ be the preferred local reproducing kernel for $H(U)_{k\phi}$ from Theorem 2.3 and define

$$(3.1) \quad \mathcal{K}_k^{\text{av}}(y, x) := \frac{1}{m} \sum_{u,v=0}^{m-1} \lambda^{k(v-u)} \mathcal{K}_k(\zeta^u y, \zeta^v x).$$

One sees that $\mathcal{K}_k^{\text{av}}$ has weight k in the first variable and weight $-k$ in the second variable (as indeed it must if it is to be a section of $L^k|_U \boxtimes \bar{L}^k|_U$).

Lemma 3.2. $\mathcal{K}_k^{\text{av}}$ is a local reproducing kernel mod $O(k^{-N-r-1})$ for the subspace of $H(U)_{k\phi,m}$ consisting of functions that have weight k .

Proof. Let u have weight k . Then by the change of variables $y' = \zeta^u y$ and the invariance of ϕ and χ we have, up to terms of order $O(e^{k\phi(x)/2} k^{-N-r-1}) \|u\|_{k\phi}$,

$$\begin{aligned} \int_U \chi(y) u(y) \overline{\lambda^{k(v-u)} \mathcal{K}_{k,\zeta^v x}(\zeta^u y)} e^{-k\phi(y)} \frac{\omega^n}{n!} &= \lambda^{-kv} u(\zeta^v x) \\ &= u(x), \end{aligned}$$

so

$$\begin{aligned} (\chi u, \mathcal{K}_{k,x}^{\text{av}})_{k\phi,m} &= \frac{1}{m} \int_U \chi(y) u(y) \overline{\mathcal{K}_{k,x}^{\text{av}}(y)} e^{-k\phi(y)} \frac{\omega^n}{n!} \\ &= \frac{1}{m^2} \sum_{u,v=0}^{m-1} u(x) = u(x). \end{aligned}$$

q.e.d.

We show in the next section that downstairs on the orbifold, the Bergman kernel B_k is locally approximated by $\mathcal{K}_k^{\text{av}}(x, x) e^{-k\phi(x)}$, and thus the weighted Bergman kernel $B_k^{\text{orb}} = \sum_i c_i B_{k+i}$ is approximated by

$$\mathcal{B}_k^{\text{orb}}(x) := \sum_i c_i \mathcal{K}_{k+i}^{\text{av}}(x, x) e^{-(k+i)\phi(x)},$$

where, we recall, the sum is over a fixed finite index of nonnegative integers i .

Theorem 3.3. *Suppose the c_i satisfy*

$$(3.4) \quad \frac{1}{m} \sum_i i^p c_i = \sum_{i \equiv u} i^p c_i \quad \text{for } 0 \leq u \leq m-1, 0 \leq p \leq N+r.$$

Then there is a local C^0 -expansion of $\mathcal{B}_k^{\text{orb}}$ of order $N+r$

$$\mathcal{B}_k^{\text{orb}}(x) = b_0(x)k^n + b_1(x)k^{n-1} + \cdots + b_{N+r}(x)k^{n-N-r} + O(k^{n-N-r-1})$$

on U . Moreover, the b_j depend only on c_i and the derivatives of the metric; in particular, $b_0 = \sum_i c_i$ and $b_1 = \sum_i c_i(ni + \frac{1}{2} \text{Scal}(\omega))$.

Our proof requires a reformulation of the condition made on the c_i .

Lemma 3.5. *The constants c_i satisfy (3.4) if and only if the function $\sum_i c_i z^i$ has a root of order $N+r+1$ at every m -th root of unity other than 1.*

Proof. If (3.4) holds, then for each $0 \leq p \leq N+r$ the quantity $c = \sum_{i \equiv u} i^p c_i$ is independent of u . Hence if $\sigma^m = 1$ with $\sigma \neq 1$, then

$$\sum_i i^p c_i \sigma^i = \sum_{u=0}^{m-1} \sum_{i \equiv u} i^p c_i \sigma^i = \sum_{u=0}^{m-1} \sigma^u \sum_{i \equiv u} i^p c_i = c \sum_{u=0}^{m-1} \sigma^u = 0,$$

proving that $\sum_i c_i z^i$ has a root of order $N+r+1$ at σ . For the converse, let σ be a primitive root of unity, so the hypothesis is that $\sum_i i^p c_i \sigma^{si} = 0$ for $1 \leq s \leq m-1$ and $0 \leq p \leq N+r$, so for any given u ,

$$\frac{1}{m} \sum_i i^p c_i = \frac{1}{m} \sum_{s=0}^{m-1} \sigma^{-su} \sum_i i^p c_i \sigma^{si} = \frac{1}{m} \sum_i i^p c_i \sum_{s=0}^{m-1} \sigma^{(i-u)s} = \sum_{i \equiv u} i^p c_i.$$

q.e.d.

Proof of Theorem 3.3. Write

$$\begin{aligned} \mathcal{B}_k^{\text{orb}}(x) &= \frac{1}{m} \sum_i c_i \sum_{u,v=0}^{m-1} \lambda^{(k+i)(v-u)} \mathcal{K}_{k+i}(\zeta^u x, \zeta^v x) e^{-(k+i)\phi(x)} \\ &= S_1 + S_2, \end{aligned}$$

where S_1 consists of the terms with $u = v$ and S_2 consists of the terms with $u \neq v$. We show below that S_2 is $O(k^{n-N-r-1})$. Observe that

$$\begin{aligned} S_1 &= \frac{1}{m} \sum_i c_i \sum_{u=0}^{m-1} \mathcal{K}_{k+i}(\zeta^u x, \zeta^u x) e^{-(k+i)\phi(x)} \\ &= \sum_i c_i \mathcal{K}_{k+i}(x, x) e^{-(k+i)\phi(x)}, \end{aligned}$$

by the invariance of \mathcal{K}_{k+i} , and since each $\mathcal{K}_{k+i} e^{-(k+i)\phi}$ can be expanded in k , the same is true of S_1 . In fact, by the binomial expansion

$$\begin{aligned}
(k+i)^{n-j} &= \sum_{u=0}^{N+r} \binom{n-j}{u} k^{n-j-u} i^u + O(k^{n-N-r-1}), \\
S_1 &= \sum_i c_i \sum_{j=0}^{N+r} (k+i)^{n-j} \tilde{b}_j(x, x) \\
&= \sum_{j=0}^{N+r} b_j(x) k^{n-j} + O(k^{n-N-r-1}),
\end{aligned}$$

where

$$b_j(x) = \sum_{q=0}^j \left(\tilde{b}_q(x, x) \binom{n-q}{j-q} \sum_i c_i i^{j-q} \right).$$

In particular, b_0 and b_1 are as claimed in the statement of the theorem.

Now to bound S_2 , write $S_2 = \frac{1}{m} \sum_{u \neq v} S_{u,v}$, where

$$S_{u,v} = \sum_i c_i \mathcal{K}_{k+i}(\zeta^u x, \zeta^v x) \lambda^{(k+i)(v-u)} e^{-(k+i)\phi(x)}.$$

Fixing $u \neq v$, let $\sigma = \lambda^{v-u}$ and note that since λ is primitive, $\sigma \neq 1$. Furthermore, set

$$\eta = \eta(x) = e^{\psi(\zeta^u x, \zeta^v x) - \phi(x)},$$

so

$$\begin{aligned}
S_{u,v} &= \sum_i c_i \sum_{j=0}^{N+r} (k+i)^{n-j} \tilde{b}_j(\zeta^u x, \zeta^v x) \sigma^{k+i} \eta^{k+i} \\
&= \sigma^k \sum_{j=0}^{N+r} k^{n-j} \sum_{q=0}^j \tilde{b}_q(\zeta^u x, \zeta^v x) \binom{n-q}{j-q} \sum_i c_i i^{j-q} \sigma^i \eta^{k+i} \\
&\quad + O(k^{n-N-r-1}).
\end{aligned}$$

Therefore, to prove the theorem it is sufficient to show that for $0 \leq l \leq N+r$,

$$(3.6) \quad u_{l,k} := \sum_i c_i i^l \sigma^i \eta^{k+i} = O(k^{l-N-r-1}).$$

To this end, write

$$u_{l,k} = \left[\frac{\sum_i c_i i^l \sigma^i \eta^i}{(\eta-1)^{N+r-l+1}} \right] (\eta-1)^{N+r-l+1} \eta^k.$$

From Lemma 3.5, the function $\eta \mapsto \sum_i c_i i^l \sigma^i \eta^i$ has a root of order $N+r-l+1$ at $\eta=1$, and so the term in square brackets is bounded. So it is sufficient to prove the following:

Claim: Let $s \geq 1$ and $u \neq v$. Then

$$(\eta-1)^s \eta^k = O(k^{-s}).$$

Proof: Since G acts linearly on U , we can write $U = U_1 \times U_2$, where the action is faithful on U_1 and trivial on U_2 . For $x \in U$, write $x = (x_1, x_2)$ under this decomposition. Then there exist positive constants c, c' such that $c\|x_1\|^2 \leq \|\zeta^u x - \zeta^v x\|^2 \leq c'\|x_1\|^2$ for all $x \in U$. Hence from (2.2), there is a $\delta' > 0$ such that

$$\operatorname{Re}(\psi(\zeta^u x, \zeta^v x) - \phi(x)) \leq -\delta'\|x_1\|^2$$

for all $x \in U$. If $x_1 = 0$, then $\eta = 1$ and we are done. Assuming $x_1 \neq 0$, recall that if z is a complex number with $\operatorname{Re} z < 0$, then $|e^{kz}| \leq \frac{s!}{(-k \operatorname{Re} z)^s}$ for all k . Thus there is a constant C such that

$$\left| (\eta(x) - 1)^s \eta(x)^k \right| \leq \frac{C}{k^s} \left| \frac{\eta(x) - 1}{\|x_1\|^2} \right|^s.$$

But $\psi(\zeta^u x, \zeta^v x) - \phi(x) = O(\|x_1\|^2)$, so $\frac{\eta(x)-1}{\|x_1\|^2}$ is bounded and the claim follows. q.e.d.

Remark 3.7. Conversely, suppose that $\mathcal{B}_k^{\text{orb}}$ admits an asymptotic expansion in C^0 of order N at the point $x = 0$ in U which is fixed by the group action. We will show that the c_i must satisfy the conditions (3.4) for $r = 0$.

Since λ is primitive and sections of orbi-ample line bundles vanish at orbifold points,

$$\mathcal{K}_{k+i}^{\text{av}}(0, 0) = \begin{cases} m\mathcal{K}_{k+i}(0, 0) & \text{if } k+i \equiv 0 \pmod{m}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned} \mathcal{B}_k^{\text{orb}}(0) &= m \sum_{i \equiv -k} c_i \mathcal{K}_{k+i}^{\text{av}}(0, 0) = m \sum_{i \equiv -k} c_i \sum_{p=0}^N \tilde{b}_p(0, 0) (k+i)^{n-p} \\ &= \sum_{p=0}^N b_p k^{n-p} + O(k^{n-N-1}), \end{aligned}$$

where

$$(3.8) \quad b_p = \sum_{q=0}^p \left(\tilde{b}_q(0, 0) \binom{n-q}{p-q} \sum_{i \equiv -k} c_i i^{p-q} \right).$$

Each b_p is periodic in k ; the existence of an asymptotic expansion of $\mathcal{B}_k^{\text{orb}}(0)$ implies that in fact each b_p is independent of k .

We now use induction on p to show the conditions (3.4), i.e., that $\sum_{i \equiv -k} c_i i^p$ is also independent of k . For $p = 0$, the sum (3.8) reduces to $b_0 = \sum_{i \equiv -k} c_i$ since $\tilde{b}_0(0, 0) = 1$. Therefore, b_0 's independence of k implies the $p = 0$ case of (3.4). For general p , the sum (3.8) is $\binom{n}{p} \sum_{i \equiv -k} c_i i^p$ plus terms shown inductively to be independent of k . Therefore, we recover the conditions (3.4) for $r = 0$.

4. Global Expansion

We will require the following standard estimate, which we prove for completeness. At first we do not require X to be compact.

Lemma 4.1. *Let X be a Kähler manifold of dimension n . There exists a constant C such that for any $p \in X$ we have $|f(p)| \leq C(a\|\bar{\partial}f\|_{C^0} + a^{-n}\|f\|_{L^2})$ for all smooth functions f and all sufficiently small $a > 0$. Moreover, if X is compact then we can choose $a > 0$ uniformly over all $p \in X$.*

Proof. We first prove the one dimensional case. Let $\epsilon = \sup |\bar{\partial}f|$, pick a local coordinate $z = re^{i\theta}$ about the point $p = 0$, and let D_a denote the ball $|z| = r \leq a$ in this coordinate. By the fundamental theorem of calculus,

$$(4.2) \quad \frac{1}{2\pi a} \int_{\partial D_a} \frac{\partial f}{\partial \theta} d\theta = 0.$$

We know that

$$\bar{\partial}f = \frac{1}{2} \left(\frac{\partial f}{\partial r} + \frac{i}{r} \frac{\partial f}{\partial \theta} \right) (dr - ir d\theta)$$

has a pointwise bound on its norm of $O(\epsilon)$. Here we measure norms of 1-forms in the standard metric, which by the compactness of X is boundedly close to the Kähler metric in D_a .

Then $\frac{1}{r} \frac{\partial f}{\partial \theta}$ is within ϵ of $i \frac{\partial f}{\partial r}$, and (4.2) gives

$$\frac{1}{2\pi} \left| \int_{\partial D_a} \frac{\partial f}{\partial r} d\theta \right| = \left| \frac{1}{2\pi} \frac{d}{da} \int_{\partial D_a} f d\theta \right| \leq \epsilon.$$

Integrating with respect to a gives the estimate

$$|f(0)| \leq \frac{1}{2\pi} \left| \int_{\partial D_a} f d\theta \right| + a\epsilon.$$

Multiplying by a and integrating again yields

$$\frac{a^2}{2} |f(0)| \leq \frac{1}{2\pi} \int_{D_a} |f| r dr d\theta + \frac{a^3}{3} \epsilon.$$

By the Cauchy-Schwarz inequality, this gives the desired bound on $|f(0)|$ in terms of the L^2 -norm of f and the pointwise supremum of $|\bar{\partial}f|$.

Now we pass to the general case. Pick local coordinates z_i in which p is the origin, and apply the above argument over one dimensional discs in the z_n direction to bound

$$(4.3) \quad |f(z_1, \dots, z_{n-1}, 0)| \leq \frac{1}{\pi a^2} \int_{|z_n| \leq a} |f(z_1, \dots, z_{n-1}, z_n)| + \frac{2a}{3} \epsilon.$$

Similarly,

$$|f(z_1, \dots, z_{n-2}, 0, 0)| \leq \frac{1}{\pi a^2} \int_{|z_{n-1}| \leq a} |f(z_1, \dots, z_{n-2}, z_{n-1}, 0)| + \frac{2a}{3} \epsilon,$$

which by (4.3) can be bounded by

$$|f(z_1, \dots, z_{n-2}, 0, 0)| \leq \frac{1}{\pi^2 a^4} \int_{|z_n|, |z_{n-1}| \leq a} |f(z_1, \dots, z_n)| + \frac{2a}{3}\epsilon + \frac{2a}{3}\epsilon.$$

Inductively, we find that

$$|f(0)| \leq \frac{1}{\pi^n a^{2n}} \int_{|z_i| \leq a} |f(z_1, \dots, z_n)| + \frac{2na}{3}\epsilon.$$

Again an application of Cauchy-Schwartz gives the result. q.e.d.

Corollary 4.4. *Let X be a compact Kähler orbifold and L be an orbifold line bundle with hermitian metric. There exists a constant C such that for any section $s \in \Gamma(L^k)$ and $x \in X$ we have $|s(x)| \leq C(k^{-\frac{1}{2}}\|\bar{\partial}s\|_{C^0} + k^{\frac{n}{2}}\|s\|_{L^2})$.*

Proof. This follows from the previous result, since we can work locally in an orbifold chart where upstairs we have a smooth Kähler metric. We trivialise L (and so each L^k) upstairs with a holomorphic section of norm 1 at x which is possibly not invariant under G . In a ball of radius $a = k^{-1/2}$ about x , the hermitian metric is then boundedly close to taking absolute values. Picking the coordinates in the above proof to be invariant under the finite group gives, for $k \gg 0$, a bound in terms of an integral upstairs over a G -invariant ball. The actual integral on the orbifold differs from this by dividing by the order of the group; since this is finite we get the same order bound. q.e.d.

We now apply the results from the previous section to orbifold charts. As usual, X is an orbifold with cyclic quotient singularities, and L is an orbi-ample line bundle with hermitian metric with curvature $-2\pi i\omega$. Suppose that $U \rightarrow U/G$ is a small orbifold chart in X , where U is a small ball centred at the origin in \mathbb{C}^n and G is cyclic of order m . By the orbi-ampleness condition, L has a trivialisation with weight -1 under the action of G via an identification between G and the m th roots of unity. Then sections of $L^k|_U$ are given locally by holomorphic functions f of weight k . The pointwise norm of such a section is $|f(z)|e^{-k\phi(z)/2}$, where the plurisubharmonic function ϕ is the norm squared of the trivialising section.

As previously mentioned, the local reproducing kernel $\mathcal{K}_k^{\text{av}}(y, x)$ on $U \times U$ defined in (3.1) has weight k in the first variable, and weight $-k$ in the second and χ is an invariant cutoff function supported on U and identically 1 on $\frac{1}{2}U$. Thus multiplying by the local trivialisation, we think of $\chi\mathcal{K}_{k,x}^{\text{av}}$ as a smooth section of $L^k \otimes \bar{L}_x^k$. The local reproducing property for $\mathcal{K}_k^{\text{av}}$ proved in Lemma 3.2 says there is a neighbourhood $U_0 \subset U$ of the origin such that if $x \in U_0$ and $t \in H^0(L^k) \otimes \bar{L}_x^k$, then

$$t(y) = (t, \chi\mathcal{K}_{k,y}^{\text{av}})_{L^2} + O(k^{-N-r-1})\|t\|_{L^2} \quad \text{for } y \in U_0,$$

where the error term is measured using the hermitian metric on $L_y^k \otimes \bar{L}_x^k$. In what follows, U will be fixed, but U_0 will be allowed to shrink as required. If $s \in H^0(L \boxtimes \bar{L})$, then applying this to $t = s_x = s(\cdot, x)$ gives

$$(4.5) \quad s(y, x) = (s_x, \chi \mathcal{K}_{k,y}^{\text{av}})_{L^2} + O(k^{-N-r-1}) \|s_x\|_{L^2}.$$

Now recall that $K_k = \sum_{\alpha} s_{\alpha} \boxtimes \bar{s}_{\alpha}$ denotes the global reproducing kernel discussed at the start of Section 2. The next proposition shows how this is approximated by the local reproducing kernels $\mathcal{K}_k^{\text{av}}$.

Proposition 4.6. *There is a neighbourhood $U_0 \subset \frac{1}{4}U$ of the origin such that*

$$K_k|_{U_0 \times U_0} = \mathcal{K}_k^{\text{av}} + O(k^{n-N-r-1}).$$

Proof. This is identical to the manifold case [BBS08, theorem 3.1], and for convenience we sketch the details. The Bergman kernel has the extremal characterisation $B_k(x) = \sup_{\|s\|=1} |s(x)|^2$ where the supremum is over all sections of L^k of unit L^2 -norm. Using this, it is shown in [Ber04, theorem 1.1] that there is a constant C such that $B_k(x) \leq Ck^n$ uniformly on X (the cited work is for noncompact manifolds; what is important is that it is local, so the inequality only improves if we apply it upstairs on an orbifold chart and restrict to invariant sections).

By the Cauchy-Schwarz inequality applied to $K_k = \sum_{\alpha} s_{\alpha} \boxtimes \bar{s}_{\alpha}$, we have $|K_k(y, x)|_h \leq \sqrt{B_k(x)B_k(y)} = O(k^n)$, so in particular $\|K_{k,x}\|_{L^2} = O(k^n)$ uniformly in x . Putting $s := K_k$ (which is holomorphic) into the local reproducing property, (4.5) gives

$$K_k(y, x) = (K_{k,x}, \chi \mathcal{K}_{k,y}^{\text{av}})_{L^2} + O(k^{n-N-r-1})$$

on $U_0 \times U_0$. Then for $x, y \in U_0$ we can write

$$(4.7) \quad K_k(y, x) = \mathcal{K}_k^{\text{av}}(y, x) - \overline{w(y, x)} + O(k^{n-N-r-1}),$$

where

$$\begin{aligned} w(y, x) &:= \overline{\mathcal{K}_k^{\text{av}}(y, x)} - (\chi \mathcal{K}_{k,y}^{\text{av}}, K_{k,x})_{L^2} \\ &= \chi(x) \mathcal{K}_{k,y}^{\text{av}}(x) - (\chi \mathcal{K}_{k,y}^{\text{av}}, K_{k,x})_{L^2}, \end{aligned}$$

as $\mathcal{K}_k^{\text{av}}$ is hermitian. With the aim of bounding w , define $u_y(\cdot) := w(y, \cdot)$ so

$$u_y(\cdot) = \chi \mathcal{K}_{k,y}^{\text{av}} - (\chi \mathcal{K}_{k,y}^{\text{av}}, K_{k,\cdot})_{L^2}.$$

Notice that the inner product in this expression is precisely the projection of $\chi \mathcal{K}_{k,y}^{\text{av}}$ onto the space of holomorphic sections of L^k . So, said another way, u_y is the L^2 -minimal solution of the equation

$$(4.8) \quad \bar{\partial} u_y = \bar{\partial}(\chi \mathcal{K}_{k,y}^{\text{av}}).$$

Now

$$\bar{\partial}(\chi \mathcal{K}_{k,y}^{\text{av}})(t) = \mathcal{K}_{k,y}^{\text{av}}(t) \bar{\partial} \chi(t) + \chi(t) \bar{\partial} \mathcal{K}_{k,y}^{\text{av}}(t),$$

which we claim is $O(k^{-\infty})$ in C^0 . Since we are assuming that $U_0 \subset \frac{1}{4}U$, χ is identically 1 on $2U_0$ and supported on U so $\bar{\partial}\chi(t)$ vanishes for $t \in 2U_0$ and for $t \notin U$. On the other hand, by (2.2) there is a $\delta_1 > 0$ such that

$$\operatorname{Re}[2\psi(t, y) - \phi(t) - \phi(y)] \leq -\delta_1 \|t - y\|^2$$

on $U \times U$. If $t \in U \setminus 2U_0$, so that t is a bounded distance away from y , then by Theorem 2.3,

$$\begin{aligned} |\mathcal{K}_{k,y}(t)|_h^2 &= O(k^{2n})e^{2k\psi(t,y)}e^{-\phi(t)}e^{-\phi(y)} \\ &= O(k^{2n})e^{k[2\psi(t,y)-\phi(t)-\phi(y)]} \\ &\leq O(k^{2n})e^{-\delta_2 k} = O(k^{-\infty}). \end{aligned}$$

Thus $\mathcal{K}_{k,y}\bar{\partial}\chi = O(k^{-\infty})$ in C^0 on all of X , and furthermore the $O(k^{-\infty})$ term is independent of $y \in U_0$. Applying this to $\xi^v y$ as v ranges over a period, we get $\mathcal{K}_{k,y}^{\text{av}}\bar{\partial}\chi = O(k^{-\infty})$ as well. Now from the second statement in Theorem 2.3, $\chi\bar{\partial}\mathcal{K}_{k,y}^{\text{av}} = O(k^{-\infty})$, hence $\bar{\partial}(\chi\mathcal{K}_{k,y}^{\text{av}}) = O(k^{-\infty})$ in C^0 , and therefore in L^2 as well.

Thus applying the Hörmander estimate to sections of $L \otimes \bar{L}_y$, we conclude that $\|u_y\|_{L^2} = O(k^{-\infty})$. Using this and that the bound on $\bar{\partial}u_y$ is uniform in x and y , Corollary 4.4 gives the pointwise estimate

$$|w(y, x)| = |u_y(x)| = O(k^{-\infty}) \quad \text{on } U_0 \times U_0.$$

Therefore (4.7) becomes $K_k = \mathcal{K}_k^{\text{av}} + O(k^{n-N-r-1})$ on $U_0 \times U_0$, as required. q.e.d.

So $\mathcal{K}_k^{\text{av}}$ approximates K_k to order $O(k^{n-N-r-1})$ in the C^0 norm. We next use this to get a C^r expansion at the expense of a factor of $O(k^r)$.

Lemma 4.9. *Suppose $f_k(x, y)$ is a sequence of functions on $U_0 \times U_0$ such that $(\bar{\partial}_x f_k)e^{-k(\phi(x)/2+\phi(y)/2)}$ and $(\partial_y f_k)e^{-k(\phi(x)/2+\phi(y)/2)}$ are both of order $O(k^{-\infty})$. Suppose also that $f_k(x, y)e^{-k(\phi(x)/2+\phi(y)/2)} = O(k^q)$ uniformly on $U_0 \times U_0$. Then for any differential operator in x and y of total order p ,*

$$(Df_k)e^{-k(\phi(x)/2+\phi(y)/2)} = O(k^{q+p}) \text{ on } \frac{1}{2}U_0 \times \frac{1}{2}U_0.$$

Proof. Fix $(x, y) \in \frac{1}{2}U_0 \times \frac{1}{2}U_0$. Consider first the case that $f(x, y) = f_k(x, y)$ is holomorphic in the first variable and anti-holomorphic in the second. Suppose $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ and let $F(t) = f(x_1, \dots, t, \dots, x_n, y_1, \dots, y_n)$ and similarly $\Phi(t) = \phi(x_1, \dots, t, \dots, x_n)$, where the t lies in the i -th coordinate. If B denotes the ball of radius k^{-1} around x_i , then for k sufficiently large $(x_1, \dots, t, \dots, x_n) \in U$ for

all $t \in B$. So applying the Cauchy formula to B gives

$$\begin{aligned} \frac{\partial f}{\partial x_j} \Big|_{(x,y)} e^{-k\left(\frac{\phi(x)}{2} + \frac{\phi(y)}{2}\right)} &= \int_{\partial B} \frac{F(t) e^{-k(\Phi(t)/2 + \phi(y)/2)}}{2\pi i (t - x_j)^2} e^{-k(\phi(x)/2 - \Phi(t)/2)} dt \\ &= \int_{\partial B} \frac{O(k^q)}{(t - x_j)^2} dt \\ &= O(k^{q+1}), \end{aligned}$$

where the second inequality uses $\phi(x) - \Phi(t) = O(|x_j - t|) = O(k^{-1})$ on ∂B , so the exponential term is $O(1)$. The other derivatives are treated similarly.

More generally, if we assume only $\bar{\partial}_x f = O(k^{-\infty})$, then the calculation above holds up to a term $e^{-k(\phi(x)/2 + \phi(y)/2)} \int_B \partial F / \partial \bar{t} dt d\bar{t}$ which is of order $O(k^{-\infty})$, so the conclusion still holds. q.e.d.

Recall that the Bergman kernel is $B_k(x) = |K_k(x, x)|_h$.

Corollary 4.10. *For k sufficiently large, $B_k(x) = \mathcal{K}_k^{\text{av}}(x, x) e^{-k\phi(x)} + O(k^{n-N-1})$ in C^r on $\frac{1}{2}U_0$. Moreover, if D is a differential operator of order p , then*

$$(4.11) \quad DB_k = O(k^{n+p}) \text{ in } C^0$$

uniformly on X .

Proof. Let g_k be the local expression of K_k in our trivialisation of $L|_U$, so $B_k(x) = |K_k(x, x)|_h = g_k(x, x) e^{-k\phi(x)}$. Now Proposition 4.6 restricted to the diagonal implies $(g_k - \mathcal{K}_k^{\text{av}}) e^{-k\phi(x)} = O(k^{n-N-r-1})$ in C^0 on U_0 . So we can apply Lemma 4.9 to $f_k := g_k - \mathcal{K}_k^{\text{av}}$ to deduce that if D is a differential operator of order q in x and y then

$$D(g_k - \mathcal{K}_k^{\text{av}}) e^{-k(\phi(x)/2 + \phi(y)/2)} = O(k^{n-N-r-1+q})$$

on $\frac{1}{2}U_0$. Therefore, taking the first r derivatives of

$$B_k(x) - \mathcal{K}_k^{\text{av}}(x, x) e^{-k\phi(x)} = (g_k(x, x) - \mathcal{K}_k^{\text{av}}(x, x)) e^{-k\phi(x)}$$

with respect to x using the Leibnitz rule proves the first statement.

For the second statement, Proposition 4.6 implies $g_k e^{-k(\phi(x)/2 + \phi(y)/2)}$ is $O(k^n)$ on $U_0 \times U_0$. Thus from Lemma 4.9 we get that if D has order at most q then $(Dg_k) e^{-k(\phi(x)/2 + \phi(y)/2)} = O(k^{n+q})$ on $\frac{1}{2}U_0$. Hence taking the derivatives of $B_k(x) = g_k(x, x) e^{-k\phi(x)}$ with the Leibnitz rule gives $DB_k = O(k^{n+p})$ on $\frac{1}{2}U_0$ for any differential operator of order p . So by compactness the same holds uniformly on all of X . q.e.d.

The proof of our main theorem is now immediate. Each x is contained in some open set U_0 for which Theorem 3.3 and Corollary 4.10 hold, and

by compactness there is a finite cover by such sets. Thus

$$\begin{aligned} B_k^{\text{orb}}(x) &= \sum_i c_i B_{k+i}(x) = \sum_i c_i \mathcal{K}_{k+i}^{\text{av}}(x, x) e^{-(k+i)\phi(x)} + O(k^{n-N-1}) \\ &= \sum_{j=0}^N b_j(x) k^{n-j} + O(k^{n-N-1}) \end{aligned}$$

in C^r , as required.

In [RT11] we use a variant of our main theorem, which for convenience we record here.

Corollary 4.12. *Let γ be a homogeneous polynomial in two variables of degree $d \leq N$, and suppose that as usual the c_i are chosen to satisfy (1.8). Then there is a C^r -expansion*

$$\sum_i c_i \gamma(k, i) B_{k+i} = b_0 k^{n+d} + b_1 k^{n+d-1} + \dots + O(k^{n-N+d-1}).$$

Moreover, if $\gamma(i, k) = Ak^d + Bk^{d-1}i + \dots$ then

$$b_0 = A \sum_i c_i \quad \text{and} \quad b_1 = \sum_i c_i (A[ni + \text{Scal}(\omega)/2] + iB).$$

Proof. By linearity we may suppose $\gamma(i, k) = i^a k^{d-a}$ for some $a \leq d$. Since c_i satisfy (1.8) for $p = 0, \dots, N+r$, we get that $c'_i := i^a c_i$ satisfy this condition for $p = 0, \dots, N+r-a$. Applying our main theorem with the constants c'_i proves the corollary. q.e.d.

Remark 4.13. For example,

$$\sum_i (k+i) c_i B_{k+i} = b_0 k^{n+1} + b_1 k^n + \dots$$

where $b_0 = \sum_i c_i$ and $b_1 = \sum_i c_i [(n+1)i + \text{Scal}(\omega)/2]$.

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