My Personal Experience on the Subject of Geometric Analysis

Shing-Tung Yau
Harvard University

Dedicated to the 20th Anniversary of NCTS Math
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I am giving a lecture for the 20th anniversary of the National Center for Theoretical Sciences in Taiwan. I recalled my discussion with Professor Liu 22 years ago. At that time, he was Director of National Science Foundation of Taiwan. Prof. Liu has been a good friend since 1991 when he was President of Tsinghua University in Hsinchu.

In that year, I took my family to spend a sabbatical year in Taiwan. Originally my major goal was to teach my two young boys Chinese and to know more about the Chinese people and the land. However, during the course of my lectures in that year, I had made a lot of good friends.

Taiwan 1991

I also admired the tremendous administrative skill of Prof. Liu as President of Tsinghua. After he moved from the position of President of Tsinghua University, he became Minister of Transportation and Director of National Science Foundation.

When he was Director of National Science Foundation, he asked me whether Taiwan should fund half a million US dollars annually to the center in Korea that was newly established under the leadership of Prof. C.N. Yang. I told him that as far as mathematics is concerned, my colleagues in taiwan is at least as good as those colleagues in Korea and half a million US dollars per year is a huge sum (at least at that time).

I proposed that he should consider to set up a center in Taiwan instead of contributing money to Korea. Prof. Liu agreed immediately. On the other hand, Prof. Liu felt that in order not to upset Prof. Yang, there should be one component of the center for theoretical physics. Hence the NCTS will have one part on mathematics and the other part on physics.

The NCTS was launched 20 years ago. It was not an easy task to launch this center.

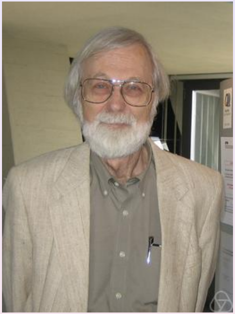
Over the years, many fields in mathematics were developed in the center. Two notable subjects were geometric analysis and number theory. The former was led by C.-S. Lin, and the later by the leadership of Jing Yu and Winnie Li. Since Ken Ribet will talk about number theory in this conference, I will spend my lecture on the subject of geometric analysis. Lin has done many creative works in this subject. But I shall talk about my own experiences in the past fifty years.

The first serious paper I wrote was during my first year in Berkeley after I arrived from Hong Kong. In those days, Evans Hall was not built and the math department was in Campbell Hall. There were more than one hundred faculty. Assistant professors and visitors were put in the temporary wood buildings right in front of the site of Evans Hall. Graduate students had no offices. But the research atmosphere was excellent.

Berkeley 1969

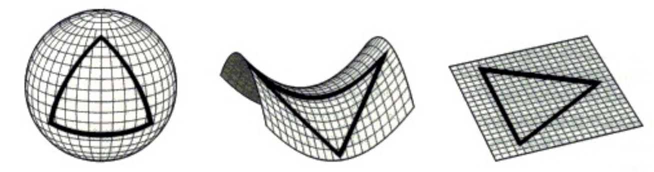
I took a lot of classes. But I regularly put my books on a desk in the library which is located on the first floor of Campbell Hall. I wandered around the library constantly and glanced at all the books and journals that were available. In those days , there were not many journals and it is feasible to read most journals, although I could not understand the details of those papers.

However one day right before Christmas when virtually nobody was around, I found the paper of Milnor appeared in Journal of Differential Geometry that related local geometry (described by curvature) with global geometry (described by fundamental group). I was fascinated by this well-written paper. I read the details of the paper and started to think what would happen if one relaxes some of the curvature conditions.



John Milnor

By reading the references recommended by Milnor, I succeeded to prove something interesting. It used some theory of infinite groups. The paper appeared in Annals of Mathematics. Starting from this first paper, I was motivated strongly to understand how global geometry is influenced by local geometry.



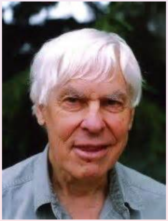
As a first year graduate student, I was taking classes from Charles Morrey on his newly finished book:

Multiple integrals
in the calculus of variation.

All my classmates complained on the style of Morrey's teaching.

Charles Morrey

They were much more excited by the seminar given by Smale and Palais on what they called global analysis. Their class was packed with people. It looked interesting to me also. But soon I realized that by and large, they try to avoid the key problem of doing estimate by forming some hypothesis called condition C.



Smale

Palais

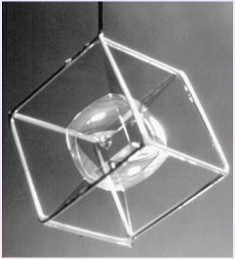
I learnt from Morrey's class how important is to establish estimates in nonlinear partial differential equations. Good estimates in nonlinear equations depend on skillful manipulation of arguments in analysis. They often look like simple once the final argument is discovered. Most students thought that it was just calculus and not exciting at all. They did not know how much effort that the author needed to spend on figuring out those estimates that often give penetrating insight of the phenomenon governed by the nonlinear equation.

Soon I realized that those estimates give a bridge to go from local information to global information. They can be found by maximum principle or by integration by parts. So by the end of my three quarters of study with Morrey, I got some feeling about nonlinear partial differential equations.

During the spring quarter, the Cambodian bombing gave rise to huge demonstrations all over United States. Most classes were canceled. I was the only student in the class of Morrey. The class was moved to his office and we enjoyed many interesting conversations. Morrey liked me and suggested me to study with him for my PhD. I believed he had only one PhD student in his whole life. Perhaps that may change my research life if I worked with Morrey. But I had written two papers in geometry already and was reluctant to change from geometry to nonlinear partial differential equations.

But the influence of Morrey on my career is deep. I decided that my future direction in research would be an attempt to merge the field of nonlinear partial differential equations with geometry. That was easy to say than actually carry it out. Most geometers at that time are very content to carry out local algebraic computation and I was not satisfied with that.

I was rather fascinated by the work of Morrey on his solution of the classical Plateau problem for rather general Riemannian manifolds and his proof of the 2-dimensional uniformization theorem for Riemann surfaces. This is one of the most influential work on nonlinear elliptic equations of two variables.



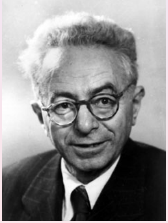
Minimal surface

Ten years afterwards, when I was faculty in IAS in Princeton, I met Whitney who is one of the founders of modern differential topology. He told me he started the theory of classification of immersions of closed curves into the plane in late 1940s in answering a challenge of his classmate Morrey in relation to the Plateau problem. Whitney's theory was generalized by Stephen Smale and Morris Hirsch later to higher-dimensional manifolds.

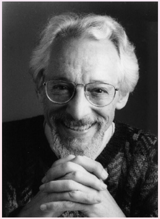
Hassler Whitney

The famous work of Smale on turning the sphere inside out is a special case of this immersion theory. Soon afterwards, Gromov studied the submersion problem and generalized it much further and coined the name of h-principle. Although Plateau problem was solved by Douglas and Rado for Euclidean space and Morrey for general Riemannian manifold, it left many questions to be answered.

An important question is the possible nature of singularity that the minimal surface solution for the Plateau problem may acquire. Courant thought that there may be branched points until Osserman showed that the surface is in fact an immersion. This shows the relevance of Plateau problem to the immersion theory.



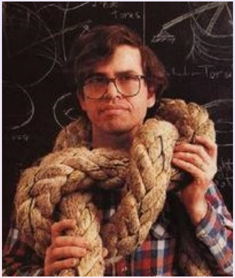
Courant



Osserman

It was an outstanding question whether the Douglas-Morrey solution is embedded if the boundary curve lies on the boundary of a convex body. Meeks and I settled this question affirmatively in 1977 using topological tools coming from solution of Dehn's Lemma in 3-dimensional topology.

We were able to turn the argument around by using minimal surfaces to solve important problems in 3-manifold topology. Together with the work of Thurston, it solved the famous Smith problem for finite group actions on the 3-dimensional sphere (that they are conjugate to linear actions).



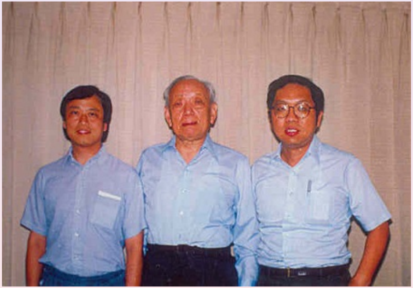
William Thurston

Although topologists used methods related to minimal surfaces for quite sometime, they forgot that it was initiated by us. In any case, my interest in uniformization theorem has been lasting all the way up to now. The first important question was when will a complete manifold parabolic?

Hence I spent a lot of time to try to prove the Liouville theorem: that under the condition that the Ricci curvature is nonnegative, the manifold cannot support nontrivial positive harmonic functions. It took me about two years to find a gradient estimate for positive harmonic function to achieve this.

At Stony Brook

This gradient estimate played an important role for many of my works in geometry and analysis. It leads to my work with S.-Y. Cheng on eigenfunctions and my work with Peter Li on the Li-Yau inequality for parabolic equations.



With Chern and Cheng

The higher-dimensional generalization of uniformization theorem has been my focus of research since my time as graduate student. When I was graduate student, there was already a famous conjecture due to Ted Frankel which asserted that compact Kähler manifolds with positive bisectional curvature is biholomorphic to complex projective space.

Ted Frankel

This beautiful conjecture was discussed pretty intensively by Kobayashi and Ochiai in their seminars.

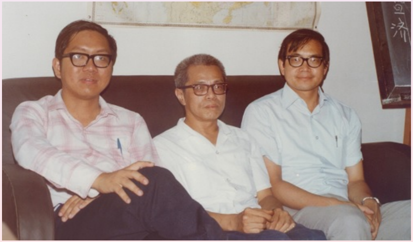
With Chern and Kobayashi in Japan

But I felt that the complete picture should include the following two conjectures:

1. Noncompact complete Kähler manifolds with positive bisectional curvature is biholomorphic to $\mathbb{C}^n$.
2. Compact Kähler manifolds with negative sectional curvature is covered by a bounded domain.

A complete simply connected Kähler manifold with strongly negative curvature should have open immersion onto a bounded domain. This program was laid out by me when I was second year student in Berkeley. Some progress was made by me by proving that complete Kähler manifolds with nonnegative Ricci curvature cannot admit nontrivial bounded holomorphic functions. I gave a talk on this program in 1975 in the Williamstown conference on several complex variables.

This was first time I met Siu who showed interest in this program immediately. We started to work on some simple case of the problem assuming decay of the curvature. Although the result is not as strong as I wished to see, it gave some method to construct peak functions. We did give the proof of the Frenkel conjecture using minimal surface technique that I had been fond to use in those days.



With Yum-Tong Siu and Qikeng Lu

But here we need to use second variation formula for high-codimensional surfaces. The treatment depends on the fact that any complex vector bundle over a Riemann surface can be turned into holomorphic bundle. This argument was picked up by Mario Micallef and John Moore.

My conjecture about manifolds biholomorphic to complex Euclidean space is close to be completely solved. Many Geometers such as Bando, Cao, Shi, Albert Chau, Tam, Zhu, Chen and Gang Liu made important contributions. Under the assumptions the volume growth of the manifolds reaches its maximum, Gang Liu was able to settle the conjecture based on previous results developed by many people. For negatively curved manifolds, virtually no progress was made because we have no means to construct bounded holomorphic functions.

The reason I was interested in uniformization problem is based on the basic philosophy on geometric analysis:

> *A geometric structure is determined by a class of solutions to some equations constructed from the geometric structure.*

The existence and parametrization of such solutions therefore plays important role to understand the underlying geometric structure.

Similar to the situation of Riemann, it is important to construct global solutions by piecing local solutions together. The possibility of doing such constructions often are described by some obstruction groups. The vanishing of obstructions group using the curvature of the underlying geometric structure is usually linked to the concept of vanishing theorem, which was initiated by Bochner and Kodaira.

Kunihiko Kodaira

But the vanishing theorems of Bochner-Kodaira are much more related to Hilbert space of square integrable solutions. The problem of understanding Kähler manifolds with negative curvature is that it is difficult to go from such solutions to bounded solutions. Much of the current works on higher dimensional uniformization depend on Hamilton's Kähler-Ricci flow. Unfortunately, except for Riemann surfaces, Kähler-Ricci flow cannot handle metrics with negative curvature well.

In the above proposal of uniformization program, I used the concept of sectional curvature. And in many cases, there is a parallel concept of positivity or negativity appeared in algebraic geometry. For example, Mori was able to prove the Frenkel conjecture in the category of algebraic geometry, where he replaced positivity of curvature by the concept of positivity of tangent bundle in the sense of algebraic geometry. (It is called Hartshore conjecture.)

Shigefumi Mori

Negativity of bisectional curvature should be replaced by negativity of tangent bundle. But the negativity of bisectional curvature seems to be stronger than negativity of tangent bundle. There was example produced by Bun Wong where there are plenty examples of simply connected algebraic manifolds with negative tangent bundle. But up to now, we have no examples of simply connected manifolds with negative bisectional curvature.

It is easy to generalize such problems to replace positivity by nonnegativity and replace manifolds by Hermitian symmetric space. One approach was to use the work of Hamilton on Ricci flows.

With Richard Hamilton

At the time in 1971, when I was thinking about this version of uniformization of complex manifolds using sectional curvature, I was also interested to replace sectional curvature by Ricci curvature. This was fascinating because the class of algebraic manifolds is much richer and potentially most interesting algebraic manifolds can be built out of them.

I learnt general relativity at that time, and I was very impressed by the significance of Ricci curvature in general relativity. Based on Einstein equation, it can be used to produce the matter tensor of the universe that it is modeled after. But differential geometers have studied the Riemannian setting of Einstein equations for a long time.

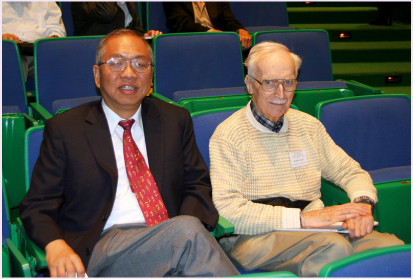
For the same mysterious reason, it appears that the Einstein equation in Riemannian setting gives more smooth solutions than in the Lorentzian setting. Naturally we have also three important cases depending on the sign of scalar curvature.

I am especially fascinated by finding constructions of compact manifolds with zero Ricci curvature. At the time when I was graduate student, the only known examples are those with zero curvature tensor. I was determined to find nontrivial examples which is not covered by Euclidean space. The most attractive space we try to construct such examples was K3 surfaces.

For a long time, we were very much interested to determine whether K3 surface admit any Riemannian metrics with zero Ricci curvature or not. But this is only a special question of which Kähler manifold admits a Kähler-Einstein metric or not. K3 surface is rather special because Hitchin observed that the only metric with nonnegative scalar curvature over it must be Ricci-flat and Kähler.

With Atiyah, Hitchin at Durham 1982

K3 surface is a good testing manifold to answer the conjecture of Calabi: whether any volume form on a Kähler manifold can be realized, up to a constant, as volume form of a Kähler metric in any Kähler class.



With Calabi at Harvard

This can be reduced to a problem of solving a complex Monge-Ampère equation which went back to Kähler. The uniformization theory of algebraic manifolds in terms of Ricci curvature is fascinating because they are reduced to solving certain nonlinear Monge-Ampère equation.

Erich Kähler

Some part of the observation of Calabi was actually made by Kähler in the first paper (1932) on the subject of Kähler geometry, where Kähler described how the potential of Kähler-Einstein metric satisfies the complex Monge-Ampère equation. In fact, Kähler noticed that the Ricci tensor defines a closed $(1,1)$-form which is an invariant depending only on the complex structure of the manifold.

This is the first Chern class of the manifold. Existence of Kähler-Einstein metric means that the first Chern class must be propositional to the Kähler class. This gives an important integrability condition for the existence of Kähler-Einstein metric. The important question is whether this is the only obstruction for existence of Kähler-Einstein metric.

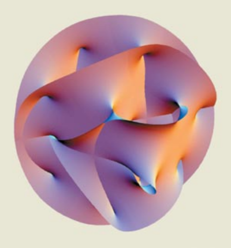
The above mentioned Calabi conjecture implies that for compact Kähler manifolds with zero first Chern class, it has a unique Kähler metric with zero Ricci curvature in each Kähler class.

At the time when I looked at this problem in the winter of 1970, nobody believed that is possible and few geometers knew well how to prove existence theorems on a manifold. Most geometers were trying to give a counterexample to the conjecture of Calabi as it was considered to be too good to be true.

I was very much excited by this problem of Calabi. Not only it is consistent with my strong desire to generalize the theory of uniformization to higher dimensional Kähler manifolds, but also by my great interest in a phenomena in the Einstein equation of general relativity:

> *Gravity may be created by the topology of the manifold and not necessary by matter distribution.*

Kähler manifolds with zero Ricci curvature was later coined by Candelas, Horowitz, Strominger and Witten to be Calabi-Yau manifold.



Calabi-Yau manifold

The simplest examples of Calabi-Yau manifolds are elliptic curves, Abelian varieties and K3 surfaces. These manifolds have been studied by number theorists and algebraic geometers for a long time. Calabi-Yau manifolds are natural generalization of such manifolds. The later developments by string theorists have greatly enriched the whole subject of Calabi-Yau manifolds that were unexpected by classical means.

But even before the Calabi-Yau caught the attention of physicists, I was able to exploit the properties of Kähler-Einstein manifolds to solve several important questions in algebraic geometry. A very important consequence is the proof of the conjecture of Severi that there is only one complex structure on the projective plane.

Francesco Severi

It took me almost six years to solve the Calabi conjecture and only after I convinced that it is correct. It took me three years to prepare the technical estimates to solve the problem. S.-Y. Cheng and I developed nonlinear analysis on manifolds. And we put in a lot of focus on the subject of real Monge-Ampère equations which is related to the Minkowski problem and the subject of affine geometry.

I believe that this is the first turning point for the math community's attention to the subject of geometric analysis. In a very brief time, several important questions in geometry were solved by nonlinear partial differential equations. They should all be looked as major developments of geometric analysis.

The notable advances were the work of Sacks-Uhlenbeck on existence of minimal two-spheres, the proof of the positive mass conjecture by Schoen-Yau, the proof of geometric version of Dehn's Lemma by Meeks- Yau, the proof of Frenkel conjecture due to Siu-Yau and the spectacular work of Richard Hamilton on Ricci flow. All these works are basically accomplished in the 1970s.

When it was turning to the 1980s, fundamental properties of gauge theory on general manifolds is getting attention due to the works of Uhlenbeck, Taubes and Donaldson.

Uhlenbeck

Taubes

Donaldson

Donaldson-Uhlenbeck-Yau established the fundamental work on the proof of existence of Hermitian Yang-Mills connection on slope stable holomorphic bundles. Donaldson was able to build new topological invariants for 4-manifolds using moduli space of anti-self-dual connections.

There were of course many more activities in the subject of geometric analysis in the 1980s and 1990s, especially those works inspired by ideas from physics and applied mathematics. A notable development is the work of Witten on Morse theory which has inspired the work of Floer, Fukaya and others. The work of Seiberg-Witten has become a very powerful tool in geometry and topology, partially based on the work of Cliff Taubes on the existence of pseudoholomorphic curves.

The theory of uniformization for manifolds are as exciting as before. If we drop the assumption that the metric is Kähler, existence theory become difficult as we are dealing with nonlinear system of partial differential equations. Most of the existence theorems for Einstein metrics are either reduced to Kähler geometry, or using symmetry reduction, or combining both ideas. Singular perturbation was used by Cliff Taubes and Joyce to construct interesting metrics.

But most of the Einstein metrics constructed in this way are Einstein metrics with nonnegative scalar curvature. For a long time, it was not clear whether there is any topological obstruction for existence of metrics with negative Ricci curvature until Zhiyong Gao and I constructed metrics with negative Ricci curvature on the 3-sphere. After that, it was clear that any manifolds with dimension greater than three should admit a metric with negative Ricci curvature. This was carried out in detail by Lochkamp.

While there are many different curvatures weaker than Ricci curvature for manifolds, scalar curvature stood out as the most important one. Two most important directions stood out:

1. The Yamabe problem which was initiated by Yamabe and Trudinger, and was completed by Aubin and Schoen.
2. The classification of manifolds with positive scalar curvature is important, but not complete.

This is related to the positive mass conjecture. It depends on two different tools:

1. The spinor argument due to Lichnerowicz which is continued by Gromov-Lawson.
2. The other argument was based on stable minimal surfaces due to Schoen-Yau in relation to the proof of positive mass conjecture.

Construction of metrics with positive scalar curvature based on $SU(2)$ group was due to Lawson-Yau.

In 1978, in the course of settling positive mass conjecture, Schoen-Yau observed that connected sum of metrics with positive scalar curvature still admit metrics with positive scalar curvature. Immediately afterwards, Schoen-Yau found that surgery on codimension three submanifolds in a manifold with positive scalar curvature still admits such a metric.

Subsequently, Gromov-Lawson claimed the same result and Stolz observed that topological results based on the surgery result can go long way to give classification of manifolds with positive scalar curvature in many cases. Metrics with positive scalar curvature is very much related to the initial data set appeared in general relativity.

This relationship was worked out by Schoen-Yau by solving the Jang equation. A complete parametrization of the initial data set will be very important for understanding the phase space for the Einstein equations.

With Singer and Schoen

A very important step was made by Robert Bartnik. But much is still needed to be done. Metrics with positive scalar curvature also appeared in the theory of AdS/CFT. It was explained in my paper with Witten on the importance of such metrics which is the conformal boundary of an Einstein manifold which is asymptotically hyperbolic. Much of the theory in this direction is needed to worked out.

Conclusion

I have given a very brief outline how a portion of geometric analysis was carried out in relation to geometry, nonlinear partial differential equations and mathematical physics. My personal experience is that this is a beautiful subject with great depth and it touches almost all branch of mathematics.

Thank you!