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STABILITY OF HODGE BUNDLES AND A NUMERICAL CHARACTERIZATION OF SHIMURA VARIETIES

Martin Möller, Eckart Viehweg & Kang Zuo

Abstract

Let U be a connected non-singular quasi-projective variety and $f : A \to U$ a family of abelian varieties of dimension g. Suppose that the induced map $U \to \mathcal{A}_g$ is generically finite and there is a compactification Y with complement $S = Y \setminus U$ a normal crossing divisor such that $\Omega^1_Y(\log S)$ is nef and $\omega_Y(S)$ is ample with respect to U.

We characterize whether U is a Shimura variety by numerical data attached to the variation of Hodge structures, rather than by properties of the map $U \to \mathcal{A}_g$ or by the existence of CM points.

More precisely, we show that $f : A \to U$ is a Kuga fibre space, if and only if two conditions hold. First, each irreducible local subsystem $\mathbb{V}$ of $R^1 f_* \mathbb{C}_A$ is either unitary or satisfies the Arakelov equality. Second, for each factor M in the universal cover of U whose tangent bundle behaves like that of a complex ball, an iterated Kodaira-Spencer map associated with $\mathbb{V}$ has minimal possible length in the direction of M. If in addition $f : A \to U$ is rigid, it is a connected Shimura subvariety of $\mathcal{A}_g$ of Hodge type.

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Introduction

Let Y be a non-singular complex projective variety of dimension n, and let U be the complement of a normal crossing divisor S. We are interested in families $f : A \to U$ of polarized abelian varieties, up to isogeny, and we are looking for numerical invariants which take the minimal possible value if and only if U is a Shimura variety of certain type, or to be more precise, if $f : A \to U$ is a Kuga fibre space as recalled in Section 2.1. Those invariants will be attached to $\mathbb{C}$-subvariations of Hodge structures $\mathbb{V}$ of $R^1 f_* \mathbb{C}_A$. We will always assume that the family has semistable reduction in codimension one, hence that the local system $R^1 f_* \mathbb{C}_A$ has unipotent monodromy around the components of S.

In [**VZ04**] we restricted ourselves to curves Y, and we gave a characterization of Shimura curves in terms of the degree of $\Omega^1_Y(\log S)$ and the degree of the Hodge bundle $f_*\Omega^1_{X/Y}(\log f^{-1}(S))$ for a semistable model $f : X \to Y$ of $A \to U$. For infinitesimally rigid families this description was an easy consequence of Simpson's correspondence, whereas in the non-rigid case we had to use the classification of certain discrete subgroups of $\mathbb{P}\mathrm{Sl}_2(\mathbb{R})$. In [**VZ07**] we started to study families over a higher dimensional base U, restricting ourselves to the rigid case. There it became evident that one has to consider numerical invariants of all the irreducible $\mathbb{C}$-subvariations of Hodge structures $\mathbb{V}$ of $R^1 f_* \mathbb{C}_A$, and that for ball quotients one needed some condition on the second Chern classes, or equivalently on the length of the Higgs field of certain wedge products of $\mathbb{V}$. In [**VZ07**] we have chosen the condition that the discriminant of one of the Hodge bundles is zero. This was needed to obtain the purity of the Higgs bundles (see Definition 1.4) for the special variations of Hodge structures considered there, but it excluded several standard representations.

In this article we give a numerical characterization of a Shimura variety of Hodge type, or of a Kuga fibre space in full generality, including rigid and non-rigid ones. We refer to Section 1 for definitions and background theorems.

Consider a complex polarized variation of Hodge structures $\mathbb{V}$ on U of weight one. The most important numerical invariant of $\mathbb{V}$ or of the associated Higgs bundle $(E = E^{1,0} \oplus E^{0,1}, \theta)$ is the *slope* $\mu(\mathbb{V})$. Recall that the slope $\mu(\mathcal{F})$ of a torsion free coherent sheaf $\mathcal{F}$ on Y, is defined by the rational number

$$\mu(\mathcal{F}) := \frac{\mathrm{c}_1(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}.\mathrm{c}_1(\omega_Y(S))^{\dim(Y)-1}. \tag{0.1}$$

Correspondingly we define $\mu(\mathbb{V}) := \mu(E^{1,0}) - \mu(E^{0,1})$. As we will see, $\mu(\mathbb{V})$ is related to μ-stability, a concept which will be stated in Definition 4.2.

For any polarized family $f : A \to U$ of abelian varieties and for any irreducible $\mathbb{C}$-subvariation of Hodge structures $\mathbb{V}$ on U in $R^1 f_*\mathbb{C}_A$ the unipotence of the local monodromies at infinity implies by [**VZ07**, Theorem 1] the Arakelov type inequality

$$\mu(\mathbb{V}) = \mu(E^{1,0}) - \mu(E^{0,1}) \le \mu(\Omega^1_Y(\log S)). \tag{0.2}$$

We say that $\mathbb{V}$ satisfies the *Arakelov equality*, if equality holds in (0.2).

A second numerical property will be important to us, the length of the iterated Kodaira Spencer map $\varsigma(\mathbb{V})$ as defined in Section 1.2.

In these terms, we can state numerically how the variation of Hodge structures over a Kuga fiber space U looks like. We refer to Section 1.1 for the decomposition of $\Omega^1_Y(\log S)$ into summands Ω_i according to the splitting of its universal covering.

Proposition 0.1. *Let $f : A \to U$ be a Kuga fibre space, such that the induced polarized variation of Hodge structures $\mathbb{W} = R^1 f_*\mathbb{C}_A$ has unipotent local monodromies at infinity. Then, replacing U by a finite étale covering if necessary, there exists a compactification Y satisfying the Assumption* 1.1 *and the Condition* 1.2*, such that for all irreducible non-unitary $\mathbb{C}$ subvariations of Hodge structures $\mathbb{V}$ of $\mathbb{W}$ with Higgs bundle (E, θ) one has:*

i. *There exists some $i = i(\mathbb{V})$ such that the Higgs field θ factorizes through*

$$\theta : E^{1,0} \longrightarrow E^{0,1} \otimes \Omega_i \xrightarrow{\subset} E^{0,1} \otimes \Omega^1_Y(\log S).$$

ii. *The Arakelov equality* $\mu(\mathbb{V}) = \mu(\Omega^1_Y(\log S))$ *holds.*
iii. *The sheaves* $E^{1,0}$ *and* $E^{0,1}$ *are* μ*-stable.*
iv. *The sheaf* $E^{1,0} \otimes E^{0,1\vee}$ *is* μ*-polystable.*
v. *Assume for* $i = i(\mathbb{V})$ *that* M_i *is a complex ball of dimension* $n_i \geq 1$*. Then*

$$\varsigma(\mathbb{V}) = \frac{\mathrm{rk}(E^{1,0}) \cdot \mathrm{rk}(E^{0,1}) \cdot (n_i + 1)}{\mathrm{rk}(E) \cdot n_i}.$$

The first consequence will be referred to as $\mathbb{V}$ being *pure of type* $i = i(\mathbb{V})$.

We will verify the first four of those properties, presumably well known to experts, at the end of Section 3. The fifth one will be proved in Section 7.

Our main interest is the question, which of the conditions stated in Proposition 0.1 will force an arbitrary family $f : A \to U$ of abelian varieties to be a Kuga fibre space. We will need the existence of a projective compactification Y of U satisfying the Assumption 1.1. Remark however, that this condition automatically holds true for compact nonsingular subvarieties $U = Y$ of the fine moduli scheme $\mathcal{A}_g^{[N]}$ of polarized abelian varieties of dimension g with a level N structure for $N \geq 3$.

The main result of this article characterizes a Kuga fibre space as a family of abelian varieties $f : A \to U$ for which the slopes $\mu(\mathbb{V})$ are maximal and the complexity $\varsigma(\mathbb{V})$ is minimal for all $\mathbb{C}$-subvariations of Hodge structures $\mathbb{V} \subset R^1 f_* \mathbb{C}_A$. In Section 1.1 we will recall Yau's uniformization theorem and the corresponding decomposition of $\Omega^1_Y(\log S)$. Type B factors are those whose corresponding factor in the uniformization is a complex ball of dimension greater than one.

Theorem 0.2. *Let* $f : A \to U$ *be a family of polarized abelian varieties such that* $R^1 f_* \mathbb{C}_A$ *has unipotent local monodromies at infinity, and such that the induced morphism* $U \to \mathcal{A}_g$ *is generically finite. Assume that* U *has a projective compactification* Y *satisfying the Assumptions* 1.1*. Then the following two conditions are equivalent:*

a. *There exists an étale covering* $\tau : U' \to U$ *such that* $f' : A' = A \times_U U' \to U'$ *is a Kuga fibre space.*
b. *For each irreducible subvariation of Hodge structures* $\mathbb{V}$ *of* $R^1 f_* \mathbb{C}_A$ *with Higgs bundle* (E, θ) *one has:*
   1. *Either* $\mathbb{V}$ *is unitary or the Arakelov equality* $\mu(\mathbb{V}) = \mu(\Omega^1_Y(\log S))$ *holds.*
   2. *If for a* μ*-stable direct factor* Ω_i *of* $\Omega^1_Y(\log S)$ *of type* B *the composition*

$$\theta_i : E^{1,0} \xrightarrow{\theta} E^{0,1} \otimes \Omega^1_Y(\log S) \xrightarrow{\mathrm{pr}} E^{0,1} \otimes \Omega_i \xrightarrow{\subset} E^{0,1} \otimes \Omega^1_Y(\log S)$$

is non-zero, then $\varsigma((E, \theta_i)) = \dfrac{\mathrm{rk}(E^{1,0}) \cdot \mathrm{rk}(E^{0,1}) \cdot (n_i + 1)}{\mathrm{rk}(E) \cdot n_i}$.

If under the assumption a) or b) $f : A \to U$ is infinitesimally rigid, then U' is a Shimura variety of Hodge type.

For the notions in the last sentence, see Section 2. We add that a family $f : A \to U$ meeting the hypothesis of this theorem also meets the hypotheses of Proposition 0.1 and thus the properties iii) and iv) on stability follow as a corollary.

The proof of Theorem 0.2 will be given in Section 7. As indicated, the subvariations of Hodge structures which are pure of type B will play a special role. In Section 7 we will obtain a slightly more precise information, stated as Addendum 7.20.

We do not know whether the condition 2) in Theorem 0.2, b) is really needed. As we will show in Section 9 for $\mathrm{rk}(\mathbb{V}) \leq 7$ this condition is not necessary, provided that $\omega_Y(S)$ is ample or more generally if the Condition 9.2 holds true. However, the necessity of the equality (1.3) in the characterization of ball quotients might indicate that a condition on the first Chern class, as given by the Arakelov equality, can not be sufficient to characterize complex balls.

The main technical step towards this theorem is a purity theorem for variation of Hodge structures. Proposition 0.1, i) states that for Kuga fibre spaces the variation of Hodge structures decomposes as a direct sum of pure and of unitary subvariations. This statement has a converse.

Theorem 0.3. *Under the Assumptions* 1.1 *consider an irreducible non-unitary polarized* $\mathbb{C}$*-variation of Hodge structures* $\mathbb{V}$ *of weight* 1 *with unipotent monodromy at infinity. If* $\mathbb{V}$ *satisfies the Arakelov equality, then* $\mathbb{V}$ *is pure for some* $i = i(\mathbb{V})$.

The proof of Theorem 0.3 will cover most of the Sections 4, 5 and 6. We will have to consider small twists of the slopes $\mu(\mathcal{F})$.

In [**VZ07**] we had to exclude direct factors of $\Omega^1_Y(\log S)$ of type C, and we used a different numerical condition for $\mathbb{V}$ of type B. Recall that the discriminant of a torsion free coherent sheaf $\mathcal{F}$ on Y is given by

$$\delta(\mathcal{F}) = \big[2 \cdot \mathrm{rk}(\mathcal{F}) \cdot \mathrm{c}_2(\mathcal{F}) - (\mathrm{rk}(\mathcal{F}) - 1) \cdot \mathrm{c}_1(\mathcal{F})^2\big].\mathrm{c}_1(\omega_Y(S))^{\dim(Y)-2},$$

and that the μ-semistability of $E^{1-q,q}$ implies that $\delta(E^{1-q,q}) \geq 0$. So the Arakelov equality implies that

$$\delta(\mathbb{V}) := \mathrm{Min}\{\delta(E^{1,0}), \delta(E^{0,1})\} \geq 0.$$

In [**VZ07**] we gave two criteria forcing $f : A \to U$ to be a Kuga fiber space. The first one, saying that all the direct factors of $\Omega^1_Y(\log S)$ are of type A, is now a special case of Theorem 0.2. In the second criterion we allowed the direct factors of $\Omega^1_Y(\log S)$ to be of type A and B, but excluded factors of type C. There, for all irreducible subvariations $\mathbb{V}$ of Hodge structures we required $\delta(\mathbb{V}) = 0$.

The bridge between the criterion [**VZ07**] and Theorem 0.2 is already contained in [**VZ07**, Proposition 3.4]. If $f : A \to U$ satisfies the hypothesis of Theorem 0.2, then for every irreducible subvariation of Hodge structures $\mathbb{V}$ of $R^1 f_* \mathbb{C}_A$ pure of type $i = i(\mathbb{V})$, with Ω_i of type B satisfying Arakelov equality and $\delta(\mathbb{V}) = 0$, we have either

$$\text{(0.3)} \qquad \mathrm{rk}(E^{1,0}) = \mathrm{rk}(E^{0,1}) \cdot n_i \quad \text{or} \quad \mathrm{rk}(E^{1,0}) \cdot n_i = \mathrm{rk}(E^{0,1}).$$

In particular, the condition in Corollary 1.3, equivalent to Theorem 0.2 b.2), holds.

Up to now we did not mention any condition guaranteeing the existence of fibres with complex multiplication or the equality between the monodromy group and the derived Mumford-Tate group $\mathrm{MT}(f)^{\mathrm{der}}$ (see Section 2.3), usually needed in the construction of Shimura varieties of Hodge type. In fact, as in [**Mo98**], we will rather concentrate on the condition that $U \to \mathcal{A}_g$ is totally geodesic. This will allow in the proof of Theorem 0.2 to identify $f : A \to U$ with a Kuga fibre space $\mathcal{X}(G, \tau, \varphi_0)$. Next, for rigid families we will refer to [**Abd94**] and [**Mo98**] for the proof that they are Shimura varieties of Hodge type (see Section 2 for more details), hence that there are fibres with complex multiplication.

This implies that for a rigid family $f : A \to U$ the group $\mathrm{MT}(f)^{\mathrm{der}}$ is the smallest the $\mathbb{Q}$-algebraic subgroup containing the monodromy group and that U is up to étale coverings equal to $\mathcal{X}(\mathrm{MT}(f)^{\mathrm{der}}, \mathrm{id}, \varphi_0)$.

In [**VZ07**] we used for the last step an explicit identification of possible Hodge cycles. Although not really needed, we will sketch a similar calculation in Section 8. There it will be sufficient to assume that the non-unitary irreducible direct factors of $R^1 f_* \mathbb{C}_A$ satisfy the Arakelov equality, and we will explicitly construct a subgroup $\mathrm{MT}^{\mathrm{mov}}(f)^{\mathrm{der}}$, isomorphic to the monodromy group $\mathrm{Mon}^0(f)$, which up to constant factors coincides with the Mumford-Tate group $\mathrm{MT}(f)^{\mathrm{der}}$. Hence, Using the notations of Section 2.1, $\mathcal{X}(\mathrm{Mon}^0(f), \mathrm{id}, \varphi_0) \cong \mathcal{X}(\mathrm{MT}^{\mathrm{mov}}(f)^{\mathrm{der}}, \mathrm{id}, \varphi_0)$.

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1. Notation

1.1. Positivity and uniformization. Our main numerical criterion is the Arakelov (in)equality, that involves slopes of Higgs bundles. Consequently, we require throughout some positivity properties of the sheaf of differential forms on the compactification Y of U.

Assumptions 1.1. We suppose that the compactification Y of U is a non-singular projective algebraic variety, such that $S = Y \setminus U$ is a normal crossing divisor, and such that

- $\Omega^1_Y(\log S)$ is nef and $\omega_Y(S) = \Omega^n_Y(\log S)$ is ample with respect to U.

By definition a locally free sheaf $\mathcal{F}$ is *numerically effective (nef)* if for all morphisms $\tau : C \to Y$, with C an irreducible curve, and for all invertible quotients $\mathcal{N}$ of $\tau^*\mathcal{F}$ one has $\deg(\mathcal{N}) \geq 0$. An invertible sheaf $\mathcal{L}$ is *ample with respect to* U if for some $\nu \geq 1$ the sections in $H^0(Y, \mathcal{L}^\nu)$ generate the sheaf $\mathcal{L}^\nu$ over U and if the induced morphism $U \to \mathbb{P}(H^0(Y, \mathcal{L}^\nu))$ is an embedding.

At some places we need the following property.

($\star$) If $\mathcal{F}$ and $\mathcal{G}$ are two μ-stable locally free sheaves, then $\mathcal{F} \otimes \mathcal{G}$ is μ-polystable.

This is certainly implied by the ampleness of $\omega_Y(S)$. S.T. Yau conjectures, that Property ($\star$) remains true if $\omega_Y(S)$ is only nef and big. Hopefully there will soon be a proof in a forthcoming article by Sun and Yau. While this is not available, we state minimal alternative statements, that give the desired conclusion.

If U is the base of a Kuga fibre space or more generally if the universal covering $\pi : \tilde{U} \to U$ is a bounded symmetric domain, we will need a second type of condition to hold true for the compactification Y of U.

Condition 1.2. Assume that the universal covering $\tilde{U}$ of U decomposes as the product $M_1 \times \cdots \times M_s$ of irreducible bounded symmetric domains.

- Then the sheaf $\Omega^1_Y(\log S)$ is μ-polystable. If $\Omega^1_Y(\log S) = \Omega_1 \oplus \cdots \oplus \Omega_{s'}$ is the decomposition as a direct sum of μ-stable sheaves, then $s = s'$ and for a suitable choice of the indices $\pi^*\Omega_i|_U = \mathrm{pr}_i^* \Omega^1_{M_i}$.

Mumford studied in [**Mu77**, Section 4] non-singular toroidal compactifications Y of some finite étale covering of the base U of a Kuga fibre space. As we will recall in Section 3, they satisfy the Assumption 1.1 and we will verify in Corollary 3.2, that such *Mumford compactifications* satisfy Condition 1.2.

We need Yau's Uniformization Theorem ([**Ya93**], recalled in [**VZ07**, Theorem 1.4]), saying in particular that the Assumption 1.1 forces the sheaf $\Omega^1_Y(\log S)$ to be μ-polystable. So one has a direct sum decomposition

$$\Omega^1_Y(\log S) = \Omega_1 \oplus \cdots \oplus \Omega_s \tag{1.1}$$

in μ-stable sheaves of rank $n_i = \mathrm{rk}(\Omega_i)$. We say that Ω_i *is of type A*, if it is invertible, and *of type B*, if $n_i > 1$ and if for all $m > 0$ the sheaf $S^m(\Omega_i)$ is μ-stable. Finally, it is *of type C* in the remaining cases, i.e. if for some $m > 1$ the sheaf $S^m(\Omega_i)$ is not μ-stable, hence a direct sum of two or more μ-stable subsheaves.

Let again $\pi : \tilde{U} \to U$ denote the universal covering with covering group Γ. As in the Condition 1.2, the decomposition (1.1) of $\Omega^1_Y(\log S)$ corresponds to a product structure

$$\tilde{U} = M_1 \times \cdots \times M_s, \tag{1.2}$$

where $n_i = \dim(M_i)$. If $\tilde{U}$ is a bounded symmetric domain, the M_i in (1.2) are irreducible bounded symmetric domains. If the image of the fundamental group is an arithmetic group, there exists a Mumford compactification and the decomposition (1.1) coincides with the one in Condition 1.2.

Yau's Uniformization Theorem gives in addition a criterion for each M_i to be a bounded symmetric domain. In fact, if Ω_i is of type A, then M_i is a one-dimensional complex ball. It is a bounded symmetric domain of rank > 1, if Ω_i is of type C.

If Ω_i is of type B, then M_i is an n_i-dimensional complex ball if and only if

$$\left[2 \cdot (n_i + 1) \cdot \mathrm{c}_2(\Omega_i) - n_i \cdot \mathrm{c}_1(\Omega_i)^2\right].\mathrm{c}(\omega_Y(S))^{\dim(Y)-2} = 0. \tag{1.3}$$

By Yau's Uniformization Theorem the Assumption 1.1 implies the Condition 1.2, generalizing the result in the case of Mumford compactifications.

1.2. Variations of Hodge structure and Higgs bundles. In this section we recall basic properties about Higgs bundles and define the invariant $\varsigma(\mathbb{V})$. We show that condition 2) in Theorem 0.2 is a condition on an upper bound of $\varsigma(\mathbb{V})$ and we give a reformulation of that condition.

Consider a complex polarized variation of Hodge structures $\mathbb{V}$ on U of weight k, as defined in [**De87**, page 4] (see also [**Si88**, page 898]), and with unipotent local monodromy around the components of S. The $\mathcal{F}$-filtration on $\mathcal{V}_0 = \mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_U$ extends to a filtration of the Deligne extension $\mathcal{V}$ of $\mathcal{V}_0$ to Y, again denoted by $\mathcal{F}$ (see [**Sch73**]). By Griffiths' Transversality Theorem (see [**Gr70**], for example) the Gauss-Manin connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_Y(\log S)$ induces an $\mathcal{O}_Y$-linear map

$$\mathfrak{gr}_{\mathcal{F}}(\mathcal{V}) = \bigoplus_{p+q=k} E^{p,q} \xrightarrow{\bigoplus \theta_{p,q}} \bigoplus_{p+q=k} E^{p,q} \otimes \Omega^1_Y(\log S) = \mathfrak{gr}_{\mathcal{F}}(\mathcal{V}) \otimes \Omega^1_Y(\log S),$$

with $\theta_{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_Y(\log S)$. So by [**Si92**] $\big(E = \mathfrak{gr}_{\mathcal{F}}(\mathcal{V}),\ \theta = \bigoplus \theta_{p,q}\big)$ is the (logarithmic) Higgs bundle induced by $\mathbb{V}$. We will write $\theta^{(m)}$ for the iterated Higgs field

$$E^{k,0} \xrightarrow{\theta_{k,0}} E^{k-1,1} \otimes \Omega^1_Y(\log S) \xrightarrow{\theta_{k-1,1}} E^{k-2,2} \otimes S^2(\Omega^1_Y(\log S)) \xrightarrow{\theta_{k-2,2}} \cdots \xrightarrow{\theta_{k-m+1,m-1}} E^{k-m,m} \otimes S^m(\Omega^1_Y(\log S)). \tag{1.4}$$

For families of polarized abelian varieties we are considering subvariations $\mathbb{V}$ of the complex polarized variation of Hodge structures $R^1 f_* \mathbb{C}_A$. Of course, $\mathbb{V}$ is polarized by restricting the polarization of $R^1 f_* \mathbb{C}_A$, and $\mathbb{V}$ has weight 1. Then its Higgs field is of the form

$$(E = E^{1,0} \oplus E^{0,1}, \theta) \quad \text{with} \quad \theta : E^{1,0} \to E^{0,1} \otimes \Omega^1_Y(\log S).$$

Variations of Hodge structures of weight $k > 1$ will only occur as tensor representations of $\mathbb{W}_{\mathbb{Q}} = R^1 f_* \mathbb{Q}_A$ or of irreducible direct factors $\mathbb{V}$ of $R^1 f_* \mathbb{C}_A$, in particular in the definition of the invariant $\varsigma(\mathbb{V})$.

Given a Higgs bundle

$$\big(E = E^{1,0} \oplus E^{0,1},\ \theta : E^{1,0} \to E^{0,1} \otimes \Omega^1_Y(\log S)\big)$$

and some $\ell > 0$ one has the induced Higgs bundle

$$\bigwedge^{\ell}(E, \theta) = \Big(\bigoplus_{i=0}^{\ell} E^{\ell-i,i},\ \bigoplus_{i=0}^{\ell-1} \theta_{\ell-i,i} \Big) \quad \text{with}$$

$$E^{\ell-m,m} = \bigwedge^{\ell-m}(E^{1,0}) \otimes \bigwedge^{m}(E^{0,1}) \quad \text{and with} \tag{1.5}$$

$$\theta_{\ell-m,m} : \bigwedge^{\ell-m}(E^{1,0}) \otimes \bigwedge^{m}(E^{0,1}) \longrightarrow \bigwedge^{\ell-m-1}(E^{1,0}) \otimes \bigwedge^{m+1}(E^{0,1}) \otimes \Omega^1_Y(\log S)$$

induced by θ.

If $\ell = \mathrm{rk}(E^{1,0})$, then $E^{\ell,0} = \det(E^{1,0})$. In this case $\langle \det(E^{1,0}) \rangle$ denotes the Higgs subbundle of $\bigwedge^{\ell}(E, \theta)$ generated by $\det(E^{1,0})$. Writing as in (1.4)

$$\theta^{(m)} = \theta_{\ell-m+1,m-1} \circ \cdots \circ \theta_{\ell,0},$$

we define as a measure for the complexity of the Higgs field

$$\varsigma((E,\theta)) := \mathrm{Max}\{\ m \in \mathbb{N};\ \theta^{(m)}(\det(E^{1,0})) \neq 0\} = \mathrm{Max}\{\ m \in \mathbb{N};\ \langle \det(E^{1,0}) \rangle^{\ell-m,m} \neq 0\}.$$

If (E, θ) is the Higgs bundle of a variation of Hodge structures $\mathbb{V}$ we will usually write $\varsigma(\mathbb{V}) = \varsigma((E,\theta))$.

Applying Simpson's correspondence [**Si92**] to the Higgs subbundle $\langle \det(E^{1,0}) \rangle$ of $\bigwedge^{\mathrm{rk}(E^{1,0})}(E, \theta)$ we will obtain in Lemma 7.2 and Lemma 7.3 as a consequence of Theorem 0.3 the following result.

Corollary 1.3. *Assume in Theorem* 0.3 *that for* $i = i(\mathbb{V})$ *the sheaf* Ω_i *is of type A or B. Then*

$$\varsigma(\mathbb{V}) \geq \frac{\mathrm{rk}(E^{1,0}) \cdot \mathrm{rk}(E^{0,1}) \cdot (n_i + 1)}{\mathrm{rk}(E) \cdot n_i}. \tag{1.6}$$

If Ω_i is invertible, hence of type A, we will see in Lemma 7.3 that both the left hand side and the right hand side of (1.6) are equal to $\mathrm{rk}(E^{1,0})$.

Remark that the condition 1) in Theorem 0.2, b) allows to apply Theorem 0.3. Since the condition 2) automatically holds true if $\mathbb{V}$ is unitary, or if it is pure of type A or C, we can as well restate the condition 2) as

2'. *If* $\mathbb{V}$ *is non unitary and pure of type* $i = i(\mathbb{V})$ *with* Ω_i *of type B, then*
$$\varsigma(\mathbb{V}) = \frac{\mathrm{rk}(E^{1,0}) \cdot \mathrm{rk}(E^{0,1}) \cdot (n_i + 1)}{\mathrm{rk}(E) \cdot n_i}.$$

1.3. Purity of variation of Hodge structures. With the types of the factors of $\Omega^1_Y(\log S)$ defined, we restate the concept of purity given in the introduction and summarize when this is automatic by superrigidity.

Definition 1.4.

1. A subsheaf $\mathcal{F} \subset E^{1,0}$ is *pure of type* i if the composition
$$\mathcal{F} \xrightarrow{\subset} E^{1,0} \xrightarrow{\theta} E^{0,1} \otimes \Omega^1_Y(\log S) \xrightarrow{\mathrm{pr}} E^{0,1} \otimes \Omega_j$$
is zero for $j \neq i$ and non-zero for $j = i$.
2. A variation of Hodge structures $\mathbb{V}$ (or the corresponding Higgs bundle $(E^{1,0} \oplus E^{0,1}, \theta)$) is *pure of type* i, if $E^{1,0}$ is pure of type i.
3. If $\mathbb{V}$ (or (E, θ)) is pure of type i and if Ω_i is of type A, B, or C, we sometimes just say that $\mathbb{V}$ (or (E, θ)) *is pure of type A, B, or C.*

Consider the Higgs bundles (E, θ_j) with the pure Higgs field θ_j, given by the composition

$$\theta_j : E^{1,0} \xrightarrow{\theta} E^{0,1} \otimes \Omega^1_Y(\log S) \xrightarrow{\mathrm{pr}_j} E^{0,1} \otimes \Omega_j \xrightarrow{\subset} E^{0,1} \otimes \Omega^1_Y(\log S).$$

In general (E, θ_j) will not correspond to a variation of Hodge structures.

However if (E, θ) is the Higgs bundle of a non-unitary variation of Hodge structures, it is pure of type i if and only if θ_j is zero for $j \neq i$. Moreover one has $\theta_i = \theta$ in this case.

If in the decomposition (1.1) all the μ-stable direct factors Ω_i are of type C, hence if $\tilde{U}$ is the product of bounded symmetric domains $M_i = G_i/K_i$ of rank > 1, the Margulis Superrigidity Theorem and a simple induction argument (see the proof of Proposition 6.9) imply that up to tensor products with unitary representations each representation ρ of the fundamental group Γ is coming from a representation of the group $G = G_1 \times \cdots \times G_s$. Then by Schur's lemma the irreducibility of ρ implies that it is the tensor product of representations ρ_j of the G_j. Correspondingly an irreducible variation of Hodge structures $\mathbb{V}$ is the tensor product of a unitary bundle and of polarized $\mathbb{C}$ variations of Hodge structures $\mathbb{V}_j$ given by ρ_j. Since the weight of $\mathbb{V}$ is one, all the $\mathbb{V}_j$, except for one, have to be variations of Hodge structures of weight zero, hence they are also unitary and the induced Higgs field is zero. So $\mathbb{V}$ is pure. As we will see in Proposition 6.9 one can extend this result to all U with $\tilde{U}$ a bounded symmetric domain.

Consequently, the main point of Theorem 0.3 is about type C factors.

2. Kuga fibre spaces and Shimura varieties of Hodge type

2.1. Kuga fibre spaces and totally geodesic subvarieties. The data to construct a *Kuga fibre space* (see [**Mu69**] and the references therein) are

i. a rational vector space V of dimension $2g$ with a lattice L,
ii. a non-degenerate skew-symmetric bilinear form $Q : V \times V \to \mathbb{Q}$, integral on $L \times L$,
iii. a $\mathbb{Q}$-algebraic group G and an injective map $\tau : G \to \mathrm{Sp}(V, Q)$,
iv. an arithmetic subgroup $\Gamma \subset G$ such that $\tau(\Gamma)$ preserves L,
v. a complex structure

$$\varphi_0 : S^1 = \{z \in \mathbb{C}^* \,;\, |z| = 1\} \to \mathrm{Sp}(V, Q)$$

such that $\tau(G)$ is normalized by $\varphi_0(S^1)$ and such that for all $v \in V \setminus \{0\}$ we have $Q(v, \varphi_0(\sqrt{-1})v) > 0$.

We will allow ourselves to replace the arithmetic subgroup in iv) by a subgroup of finite index, whenever it is convenient. In particular, we will assume that Γ is neat, as defined in [**Mu77**, page 599].

For Γ sufficiently small, $(L, Q, G, \tau, \varphi_0, \Gamma)$ defines a Kuga fibre space, i.e. a family of abelian varieties, by the following procedure. Let $K^0_{\mathbb{R}}$ be the connected component of the centralizer of $\varphi_0(S^1)$ in $G_{\mathbb{R}}$. Then there is a map

$$M := G^0_{\mathbb{R}}/K^0_{\mathbb{R}} \longrightarrow \mathrm{Sp}(V, Q)_{\mathbb{R}}/(\text{centralizer of}\,\varphi_0) \cong \mathbb{H}_g$$

and the pullback of the universal family over $\mathbb{H}_g$ descends to the desired family over

$$\mathcal{X} := \mathcal{X}(G, \tau, \varphi_0) := \Gamma\backslash G^0_{\mathbb{R}}/K^0_{\mathbb{R}}.$$

In the sequel we will usually suppress V and Q from the notation and write just $\mathrm{Sp}(Q)$ or Sp, if no ambiguity arises.

Two different sets of data $(L, Q, G, \tau, \varphi_0, \Gamma)$ and $(L', Q', G', \tau', \varphi'_0, \Gamma')$ may define isomorphic Kuga fibre spaces over $\mathcal{X}(G, \tau, \varphi_0) \cong \mathcal{X}(G', \tau', \varphi'_0)$. Note that different groups G and G' might lead to the same Kuga fibre space and that $K^0_{\mathbb{R}}$ is not necessarily compact but the extension of a central torus in $G_{\mathbb{R}}$ by a compact group. Note moreover that replacing

φ_0 by $\tau(g)\varphi_0\tau(g)^{-1}$ for any $g \in G$ gives an isomorphic Kuga fibre space - this just changes the reference point.

Kuga fibre spaces are the objects that naturally arise when studying polarized variations of Hodge structures satisfying the Arakelov equality. We restrict the translation procedure into the language of Shimura varieties to the case of 'Hodge type', see Section 2.4.

We provide symmetric domains throughout with the Bergman metric (e.g. [**Sa80**, §II.6]). By condition v) in Mumford's definition of a Kuga fibre space, $M \to \mathbb{H}_g$ is a strongly equivariant map in the sense of [**Sa80**]. By [**Sa80**, Theorem II.2.4], it is a *totally geodesic embedding*, i.e. each geodesic curve in $\mathbb{H}_g$ which is tangent to M at some point of M is a curve in M. The converse is dealt with in Section 2.4.

2.2. Étale coverings. Replacing the group Γ by a subgroup of finite index corresponds to replacing U by an étale covering, and by definition one obtains again a Kuga fibre space. So we will consider Kuga fibre spaces and Shimura varieties (see Section 2.4) as equivalence classes up to étale coverings. The way we stated Theorem 0.2 or the Corollary 7.22 we are allowed to replace U by an étale covering, whenever it is convenient.

Since $U \to \mathcal{A}_g$ is induced by a genuine family of polarized abelian varieties $f : A \to U$ and since the subgroup of N-division points is étale over U, an étale covering U' of U maps to the moduli scheme $\mathcal{A}_g^{(N)}$ of abelian varieties with a level N structure, say for $N = 3$. We will drop the $'$ as well as the $^{(N)}$, and we will assume in the sequel:

Assumptions 2.1. $\mathcal{A}_g$ is a fine moduli scheme, $\varphi : U \to \mathcal{A}_g$ is generically finite, and $f : A \to U$ is the pullback of the universal family.

As we will see in the beginning of the Section 7, for φ finite and $\varphi(U)$ non-singular the Arakelov equality will force φ to be étale. At other places, for example if we talk about geodesics, we will have to assume that $\varphi(U)$ is non-singular, and that φ is étale. Then however, since $\mathcal{A}_g$ is supposed to be a fine moduli scheme, we can as well assume that φ is an embedding.

2.3. The Hodge group, the Mumford-Tate group and the monodromy group. We start be recalling the definitions of the Hodge and Mumford-Tate group. Let A_0 be an abelian variety and $W_{\mathbb{Q}} = H^1(A_0, \mathbb{Q})$, equipped with the polarization Q. The *Hodge group* $\mathrm{Hg}(A_0) = \mathrm{Hg}(W_{\mathbb{Q}})$ is defined in [**Mu66**] (see also [**Mu69**]) as the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{Sp}(W_{\mathbb{Q}}, Q)$, whose extension to $\mathbb{R}$ contains the complex structure

$$\varphi_0 : S^1 \longrightarrow \mathrm{Sp}(W_{\mathbb{Q}}, Q),$$

where z acts on (p, q) cycles by multiplication with $z^p \cdot \bar{z}^q$.

In a similar way, one defines the *Mumford-Tate group* $\mathrm{MT}(W_{\mathbb{Q}}) = \mathrm{MT}(A_0)$. The complex structure φ_0 extends to a morphism of real algebraic groups

$$h^{W_{\mathbb{Q}}} : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \longrightarrow \mathrm{Gl}(W_{\mathbb{Q}} \otimes \mathbb{R}),$$

and $\mathrm{MT}(W_{\mathbb{Q}})$ is the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{Gl}(W_{\mathbb{Q}})$, whose extension to $\mathbb{R}$ contains the image of $h^{W_{\mathbb{Q}}}$.

By [**De82**] the group $\mathrm{MT}(W_{\mathbb{Q}})$ is reductive, and it coincides with the largest $\mathbb{Q}$-algebraic subgroup of the linear group $\mathrm{Gl}(W_{\mathbb{Q}})$, which leaves all $\mathbb{Q}$-Hodge tensors invariant, hence all elements

$$\eta \in \left[W_{\mathbb{Q}}^{\otimes m} \otimes W_{\mathbb{Q}}^{\vee \otimes m'}\right]^{0,0}.$$

Here $W_{\mathbb{Q}}^{\vee}$ is regarded as a Hodge structure concentrated in the bidegrees $(0,-1)$ and $(-1,0)$, and hence $W_{\mathbb{Q}}^{\otimes m} \otimes W_{\mathbb{Q}}^{\vee \otimes m'}$ is of weight $m - m'$. So the existence of some η forces m and m' to be equal.

Let $f : A \to U$ be a family of polarized abelian varieties and $\mathbb{W}_{\mathbb{Q}} = R^1 f_* \mathbb{Q}_A$ the induced polarized $\mathbb{Q}$-variation of Hodge structures on U. By [**De82**], [**An92**] or [**Sc96**] there exist a union Σ of countably many proper closed subvarieties of U such that for $y \in U \setminus \Sigma$ the group $\mathrm{MT}(\mathbb{W}_{\mathbb{Q}}|_y)$ is independent of y. We will fix such a 'very general' point y, write $W_{\mathbb{Q}}$ instead of $\mathbb{W}_{\mathbb{Q}}|_y$. We define $\mathrm{MT}(\mathbb{W}_{\mathbb{Q}})$ or $\mathrm{MT}(f)$ to be $\mathrm{MT}(W_{\mathbb{Q}})$.

The monodromy group $\mathrm{Mon}(\mathbb{W}_{\mathbb{Q}})$ is defined as the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{Gl}(W_{\mathbb{Q}})$ which contains the image of the monodromy representation of $\pi_1(U, y)$, and $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}})$ denotes its connected component containing the identity. We will often write Mon^0 or $\mathrm{Mon}^0(f)$ instead of $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}})$.

By [**De82**] $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}})$ is a normal subgroup of the derived subgroup $\mathrm{MT}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}}$. Note that the derived subgroup of the Hodge group $\mathrm{Hg}(A_0)$ coincides with the derived Mumford-Tate group $\mathrm{MT}(R^1 f_* \mathbb{Q}_A)^{\mathrm{der}}$.

2.4. Shimura varieties of Hodge type and totally geodesic subvarieties. A Kuga fibre space $\mathcal{X}(G, \tau, \varphi_0)$ is *of Hodge type*, if it is isomorphic to a Kuga fibre space $\mathcal{X}(G', \tau', \varphi_0')$ such that G' is the Hodge group of the abelian variety defined by φ_0'. Let us next compare this notion with the one of Shimura varieties of Hodge type.

In [**De79**], the notion of a *connected Shimura datum* (G, M) consists of a reductive $\mathbb{Q}$-algebraic group G and a $G(\mathbb{R})^+$-conjugacy class M of homomorphisms $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$ with the following properties:

(SV1) for $h \in M$, only the characters $z/\overline{z}$, 1, $\overline{z}/z$ occur in the representation of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ on $\mathrm{Lie}(G)$.

(SV2) $\mathrm{ad}(h(i))$ is a Cartan involution of G^{ad}.

(SV3) G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

A *connected Shimura variety* is defined to be the pro-system $(\Gamma\backslash M)_\Gamma$, with Γ running over all arithmetic subgroups Γ of $G(\mathbb{Q})$ whose image in G^{ad} is Zariski-dense. Since we do not bother about canonical models and since we allow to replace the base U by an étale cover any time, we say that *U is a Shimura variety of Hodge type*, if U is equal to $\Gamma\backslash M$ for some Γ. Usually Γ is required moreover to be a congruence subgroup, but we drop this condition to simplify matters of passing to étale covers at some places.

We let $\mathrm{CSp}(Q)$ (or CSp) be the group of symplectic similitudes with respect to a symplectic form Q. The Shimura datum $(\mathrm{CSp}(Q), M(Q))$ attached to the symplectic group consists of all maps $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathrm{CSp}(Q)_\mathbb{R}$ defined on $\mathbb{R}$-points by the block diagonal matrix

$$h(x+iy) = \mathrm{diag}\left(\begin{pmatrix} x & -y \\ y & x\end{pmatrix}, \ldots, \begin{pmatrix} x & -y \\ y & x\end{pmatrix}\right) \tag{2.1}$$

with respect to a symplectic basis $\{a_i, b_i\}$, $i = 1, \ldots, g$ of the underlying vector space V.

A Shimura datum (G, M) is *of Hodge type*, if there is a map $\tau : G \to \mathrm{CSp}(Q)$ such that composition of $h \in M$ with τ maps M to $M(Q)$.

There is a bijection between isomorphism classes of Kuga fibre spaces of Hodge type and the universal families of Shimura varieties of Hodge type:

Given $(L, Q, G, \tau, \varphi_0, \Gamma)$, let $Z \cong \mathbb{G}_m$ be the center of CSp, define $G' := G \cdot Z \subset \mathrm{CSp}$ and define $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G'_\mathbb{R}$ by on $\mathbb{C}$-points by $h(z) = \varphi_0(z/\overline{z})|z|$. Finally, let M' be the $G'_\mathbb{R}$ conjugacy class of h. One checks that (G', M') is a Shimura datum of Hodge type. Conversely given (G', M') of Hodge type, let $G := G' \cap \mathrm{Sp}$ and let φ_0 be the restriction of a generic $h \in M'$ to $S^1 \subset \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m(\mathbb{C})$. Together with τ being the inclusion map, this defines a Kuga fibre space of Hodge type.

We keep the Assumptions 2.1. We will not assume at the moment that U is a Shimura variety or that any numerical condition holds on the variation of Hodge structure. We follow Moonen ([**Mo98**]) and recall the construction of the smallest Shimura subvariety $\mathcal{X}^{\mathrm{MT}}$ of Hodge type in $\mathcal{A}_g$ that contains the image of U.

Theorem 2.2 ([**Mo98**]). *Given a generically finite map $\varphi : U \to \mathcal{A}_g$, there exists a Shimura datum (G, M) such that a Shimura variety $\mathcal{X}^{\mathrm{MT}} \cong \Gamma\backslash M$ attached to this Shimura datum is the unique smallest Shimura subvariety of Hodge type in $\mathcal{A}_g$ that contains the image of U.*

Further, G may be chosen to be the Mumford-Tate group at a very general point y of U.

Although the Shimura variety $\mathcal{X}^{\mathrm{MT}}$ is unique, the Shimura datum is unique only up to the centralizer of G in CSp, see [**Mo98**, Remark 2.9].

Proof. Let G be the Mumford-Tate group at a very general point y of U. In the topological space of all maps $h \in M(Q)$ that factor through $G_{\mathbb{R}}$, choose M to be the connected component containing the complex structure at y. By definition of the Mumford-Tate group, M is not empty and by the argument of [**De79**, Lemma 1.2.4], M is an $G(\mathbb{R})^{+}$-conjugacy class. Hence (G, M) is a Shimura datum of Hodge type. Since y was very general, $\varphi : U \to \mathcal{A}_g$ factors through $\mathcal{X}^{\mathrm{MT}}$. The minimality of $\mathcal{X}^{\mathrm{MT}}$ follows from the minimality condition in the definition of the Mumford-Tate group. q.e.d.

We now suppose that U is a totally geodesic non-singular subvariety of the Shimura variety $\mathcal{X}^{\mathrm{MT}} \subset \mathcal{A}_g$. As in Section 2.2 we can also allow a morphism $\varphi : U \to \mathcal{A}_G$ as long as $\varphi(U) \subset \mathcal{A}_g$ is a non-singular totally geodesic subvariety and $U \to \varphi(U)$ étale.

Theorem 2.3 ([**Mo98**] Corollary 4.4)**.** *If $U \subset \mathcal{X}^{\mathrm{MT}}$ is totally geodesic, then U is the base of a Kuga fibre space. It is a Shimura variety of Hodge type up to some translation in the following sense:*

After replacing U by a finite étale cover, there are Kuga fibre spaces over $\mathcal{X}_1$ and $\mathcal{X}_2$ and an isomorphism $\mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}^{\mathrm{MT}}$, such that U is the image of $\mathcal{X}_1 \times \{b\}$ for some point $b \in \mathcal{X}_2(\mathbb{C})$.

For some $a \in \mathcal{X}_2(\mathbb{C})$, the subvariety $\mathcal{X}_1 \times \{a\}$ in $\mathcal{X}^{\mathrm{MT}}$ is a Shimura variety of Hodge type.

Proof. In loc. cit. the author deals with Shimura subvarieties of arbitrary period domains and shows that there exist totally geodesic subvarieties $\mathcal{X}_i$ such that U is the image of $\mathcal{X}_1 \times \{b\}$.

We repeat part of his arguments to justify that $\mathcal{X}_1$ is the base of a Kuga fibre space.

More precisely, let (G, M) be the Shimura datum underlying $\mathcal{X}^{\mathrm{MT}}$. We have a decomposition of the adjoint Shimura datum

$$(G^{\mathrm{ad}}, M) \cong ((\mathrm{Mon}^0)^{\mathrm{ad}}, M_1) \times (G_2^{\mathrm{ad}}, M_2)$$

into connected Shimura data given as follows. Since G is reductive, there is a complement G_2 of Mon^0, i.e. such that $\mathrm{Mon}^0 \times G_2 \to G$ is surjective with finite kernel. Write $G_1 := \mathrm{Mon}^0$ and let M_i be the set of maps

$$\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \longrightarrow G \longrightarrow G^{\mathrm{ad}} \longrightarrow (G_i)^{\mathrm{ad}}.$$

For suitable arithmetic subgroups Γ_i a component of the quotients $\mathcal{X}_i := \Gamma_i \backslash M_i$ have the claimed property by [**Mo98**] Corollary 4.4.

It suffices to take $\tau : \mathrm{Mon}^0 \to \mathrm{Sp}$ the natural inclusion and φ_0 the restriction of any $h \in M$ to $S^1 \subset \mathbb{C}^*$. Then φ_0 normalizes Mon^0 and for a suitable choice of Γ, U is the base of the Kuga fibre space given by $(L, Q, \mathrm{Mon}^0, \tau, \varphi_0, \Gamma)$. q.e.d.

We call $f : A \to U$ rigid, if the induced morphism $U \to \mathcal{A}_g$ to the moduli stack has no non-trivial deformations, hence if there is no smooth projective morphism $\hat{f} : \hat{A} \to U \times T$ with $\dim(T) > 0$ and extending f, such that the induced morphism $U \times T \to \mathcal{A}_g$ is generically finite. In a similar way, $f : A \to U$ is called infinitesimally rigid, if the morphism from U to the moduli stack has no infinitesimal deformations. Using Faltings' description of the infinitesimal deformations (see [**Fa83**]) this holds if and only if there are no antisymmetric endomorphisms of the variation of Hodge structures pure of type $(-1, 1)$. In particular, if $\mathrm{End}(R^1 f_* \mathbb{Q}_X)^{-1,1} = 0$ the family is infinitesimally rigid.

Corollary 2.4 (See also [**Abd94**]). *If in Theorem* 2.3 *the subvariety* U *is rigid, then* U *is a Shimura variety of Hodge type.*

3. Stability for homogeneous bundles and the Arakelov equality for Shimura varieties

To prove a first part of the properties of Kuga fibre space stated in Proposition 0.1 we recall from [**Mu77**] and [**Mk89**] some facts on homogeneous vector bundles on Hermitian symmetric domains and deduce stability results.

Let M be a Hermitian symmetric domain and let $G = \mathrm{Aut}(M)$ be the holomorphic isometries of M. $\mathrm{Aut}(M)$ is the identity component of the isometry group of M and $M \cong G/K$ for a maximal compact subgroup $K \subset G$. Let V_0 be a vector space with a representation $\rho : K \to \mathrm{Gl}(V_0)$ and any ρ-invariant metric h_0. Then

$$V = G \times_K V_0 := G \times V_0 / \sim, \quad \text{where} \quad (g, v) \sim (gk, \rho(k^{-1})v) \quad \text{for } k \in K$$

with the metric h inherited from h_0 is a vector bundle on G/K, homogeneous under the action of G, or as we will say, a *homogeneous* bundle.

Let U be non-singular algebraic variety. In this section we suppose that the universal covering of U is a symmetric domain $M = G/K$ and that the image of the fundamental group of U in G is a neat arithmetic subgroup. We call a bundle E_U on U *homogeneous*, if its pullback to M is homogeneous. We call E_U *irreducible*, if the pullback is given by an irreducible representation ρ.

For the rest of this section, we work over a smooth toroidal compactification Y of U with $S = Y \setminus U$ a normal crossing divisor, as studied in [**Mu77**]. If Y^* denotes the Baily-Borel compactification of U, there exists a morphism $\delta : Y \to Y^*$ whose restriction to U is the identity.

Obviously, the cotangent bundle of a symmetric domain $M = G/K$ is the homogeneous bundle associated with the adjoint representation on $(\mathrm{Lie}(G)/\mathrm{Lie}(K))^\vee$.

We will not need the exact definition of a singular Hermitian metric, 'good on Y' in the sequel. Let us just recall that this implies that the curvature of the Chern connection ∇_h of h represents the first chern class of E.

Theorem 3.1 ([**Mu77**] Theorem 3.1 and Proposition 3.4)**.**

a. *Suppose that E_U is a homogeneous bundle with Hermitian metric h induced by h_0 as above. Then there exists a unique locally free sheaf E on Y with $E|_U = E_U$, such that h is a singular Hermitian metric good on Y.*
b. *For $E_U = \Omega^1_U$ one obtains the extension $E = \Omega^1_Y(\log S)$.*
c. *For $E_U = \omega_U$ one obtains the extension $E = \omega_Y(S)$ and this sheaf is the pullback of an invertible ample sheaf on Y^*.*

Corollary 3.2. *Assume that U maps to the moduli stack $\mathcal{A}_g$ of polarized abelian varieties, and that this morphism is induced from a homomorphism $G \to \mathrm{Sp}$ by taking the double quotient with respect to the maximal compact subgroup and a lattice as in Section* 2.

Then the Mumford compactification Y satisfies the Assumptions 1.1 *and Condition* 1.2.

Proof. If the bounded symmetric domain M decomposes as $M_1 \times \cdots \times M_s$, hence if $\mathrm{Aut}(M) =: G = G_1 \times \cdots \times G_s$, the sheaves $\Omega^1_{M_i}$ are homogeneous bundles associated with $(\mathrm{Lie}(G_i)/\mathrm{Lie}(K_i))^\vee$. They descend to sheaves $\Omega_{i\,U}$ on U which extend to Ω_i on Y. The uniqueness of the extensions implies that $\Omega^1_Y(\log S) = \Omega_1 \oplus \cdots \oplus \Omega_s$.

Let $f : A \to U$ denote the universal family over U, and let $F^{1,0}_U = f_*\Omega^1_{A/U}$ denote the Hodge bundle. Since $U \to \mathcal{A}_g$ is induced by a homomorphism $G \to \mathrm{Sp}$, and since the bundle $\Omega^1_{\mathcal{A}_g}$ is homogeneous on $\mathcal{A}_g$, its pullback to U is homogeneous under G. The latter is isomorphic to $S^2(F^{1,0}_U)$.

The sheaf Ω^1_U is a homogeneous direct factor, hence the uniqueness of the extension in Theorem 3.1 implies that $\Omega^1_Y(\log S)$ is a direct factor of the extension of $S^2(F^{1,0}_U)$ to Y. We may assume that the local monodromies of $R^1 f_*\mathbb{C}_A$ around the components of $S = Y \setminus U$ are unipotent. Then the Mumford extension is $S^2(F^{1,0})$, where $F = F^{1,0} \oplus F^{0,1}$ is the logarithmic Higgs bundle of $R^1 f_*\mathbb{C}_A$. Moreover, as shown by Kawamata (e.g. [**Vi95**, Theorem 6.12]), the sheaf $F^{1,0}$ is nef. So $S^2(F^{1,0})$ and the direct factor $\Omega^1_Y(\log S)$ are both nef.

The nefness of $\omega_Y(S)$ and ampleness with respect to $U = Y \setminus S$ follows directly from the second part of [**Mu77**, Proposition 3.4]. In fact, as remarked in the proof of [**Mu77**, Proposition 4.2], this sheaf is just the pullback of the ample sheaf on the Baily-Borel compactification of U.

It remains to verify that $\Omega^1_Y(\log S)$ is μ-polystable and that for all i Ω_i is μ-stable.

Using standard calculation of Chern characters on products, as in Section 5, it is easy to show that the slopes $\mu(\Omega_i)$ coincide with $\mu(\Omega^1_Y(\log S))$. The μ-stability of Ω_i follows from Lemma 3.5 by a case by case verification that for M_i irreducible the representation attached to the homogeneous bundle Ω_{M_i} is irreducible.

Alternatively, since we have verified the Assumptions 1.1, we can use Yau's Uniformization Theorem, stated in [**VZ07**, Theorem 1.4]. It implies that $\Omega^1_Y(\log S)$ is μ-polystable. Then the sheaves Ω_i, constructed above, are μ-polystable as well. Moreover, if Ω_i decomposes as a direct sum of two μ-polystable subsheaves the corresponding M_i is the product of two subspaces. So if we choose the decomposition $M = M_1 \times \cdots \times M_s$ with M_i irreducible, the sheaves Ω_i are μ-stable. q.e.d.

Example 3.3. Let $E^{p,q}_U$ be a Hodge bundle of a uniformizing $\mathbb{C}$-variation of Hodge structures $\mathbb{V}$ over U. Then $E^{p,q}_U$ is a homogeneous vector bundle and the corresponding invariant metric h is the Hodge metric, induced by the variation of Hodge structures. Let Y be a Mumford compactification of U. By Theorem 3.1 there exists a good extension of $E^{p,q}_U$ to Y.

On the other hand, as described in the introduction, one has the canonical Deligne extension of $\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_U$ to Y. The compatibility of this extension with the $\mathcal{F}$-filtration (see [**Sch73**]) gives another extension $E^{p,q}$ of $E^{p,q}_U$ to Y.

Lemma 3.4. *In the Example* 3.3 *the canonical Deligne extension* $E^{p,q}$ *of* $E^{p,q}_U$ *to* Y *coincides with the Mumford extension of* $E^{p,q}_U$ *in Theorem* 3.1*, a).*

Proof. Let $e_1, \ldots, e_n$ be a local basis for the canonical extension $E^{p,q}$. Building up on [**Sch73**], [**CKS86**, Theorem 5.21] describes the growth of the Hodge metric near S. In particular $||e_i||$ is bounded from above by the logarithm of the coordinate functions $z_1, \ldots, z_k$. The Deligne extension is uniquely determined by the condition of logarithmic growth for the Hodge metric near S.

Since the metric h coincides with the Hodge metric and since Mumford's notion 'good' implies that $h(e_i)$ is bounded from above by the logarithm of the coordinate functions $z_1, \ldots, z_k$, uniqueness implies that the Deligne extension and the Mumford extension coincide. q.e.d.

Lemma 3.5. *Suppose that the vector bundle* E *on* Y *is Mumford's extension of an irreducible homogeneous vector bundle* $E|_U$*. Then* E *is stable with respect to the polarization* $\omega_Y(S)$*.*

Proof. By definition of Mumford's extension ([**Mu77**, Theorem 3.1]), E carries a metric h coming from the G-invariant metric, again denoted by h, on the pull back $\tilde{E}$ of E to M. As mentioned already, for a singular metric, good in the sense of Mumford, the curvature of the Chern connection ∇_h of h represents the first chern class of E.

We claim that the restriction of ∇_h to U is a Hermitian Yang-Mills connection with respect to the Kähler-Einstein metric g on Ω^1_U. In fact, the pull back vector bundle $\tilde{E}$ on M is an irreducible homogeneous vector bundle.

So our claim says that this G-invariant metric h on $\tilde{E}$ is Hermitian-Yang-Mills with respect to the G-invariant (Kähler-Einstein) metric g on Ω^1_M with the argument adapted from the proof of [**Ko86**, Theorem 3.3 (1)]. The $g-$trace of the curvature $\wedge_g(\Theta_h)$ of h is a $G-$invariant endomorphism on the vector bundle $\tilde{E}$, and

$$\wedge_g(\Theta_h)_0 := \wedge_g(\Theta_h)|_{\tilde{E}_0}$$

is an $K-$invariant endomorphism on the vector space $\tilde{E}_0$. Since the maximal compact subgroup K acts on $\tilde{E}_0$ irreducibly, $\wedge_g(\Theta_h)_0$ must be a scalar multiple of the identity on $\tilde{E}_0$. The facts that G operates on M transitively and that the induced action of G on $\tilde{E}$ commutes with $\wedge_g(\Theta_h)$ imply that $\wedge_g(\Theta_h)$ is a constant scalar multiple of the identity endomorphism. So, h is a Hermitian-Yang-Mills metric with respect to the G-invariant (Kähler-Einstein) metric g on Ω^1_M. Here we regard Ω^1_M as an irreducible homogeneous vector bundle. On the quotient U we obtain the Hermitian-Yang-Mills metric h on $E|_U$ with respect to the Kähler-Einstein metric g on Ω^1_U.

Suppose that $F \subset E$ is a subbundle and let s_U be the C^∞ orthogonal splitting over U. By Theorem 5.20 in [**Kol85**] the curvature of the Chern connection to $h|_F$ represents the $c_1(F)$. The Chern-Weil formula implies

$$R(\nabla_{(h|_F)}) = R(\nabla_h)|_F + s_u \wedge s_u^*.$$

The Hermitian Yang-Mills property of h yields $\mu(F) \leq \mu(E)$ and equality holds if and only if s_U is holomorphic.

If the equality holds, the pullback of s_U to M gives an orthogonal splitting of Hermitian vector bundles

$$\pi^* E|_U \cong \pi^* F|_U \oplus \pi^* F^\perp|_U.$$

By Proposition 2 on p. 198 of [**Mk89**] this contradicts the irreducibility of $E|_U$. Thus E is μ-stable. q.e.d.

Lemma 3.6. *Suppose that E_i are vector bundles on Y, that are Mumford's extensions of irreducible homogeneous vector bundles $E_i|_U$. Then $E_1 \otimes E_2$ is μ-polystable.*

Proof. Let ρ_i be the representation corresponding to E_i. Since the E_i are μ-stable, $E_1 \otimes E_2$ is μ-semistable. Repeating the calculation of the curvature of the Chern connection from the previous Lemma, the existence of a subbundle of $E_1 \otimes E_2$ of the same slope as $E_1 \otimes E_2$ implies that the representation $\rho_1 \otimes \rho_2$ corresponding to $E_1 \otimes E_2$ is not irreducible. Since K is reductive, $\rho_1 \otimes \rho_2$ decomposes as a direct sum of irreducible representations. Each of them defines a μ-stable bundle, again by the previous Lemma, and equality of slopes follows from semistability. q.e.d.

Before proving the first part of Proposition 0.1 for the Mumford compactification Y, let us show that the Arakelov equality is independent of the compactification Y and compatible with replacing U by an étale covering U'.

Lemma 3.7. *Let $\delta : U' \to U$ be a finite étale morphism and let Y, S and Y', S' be two compactifications of U and U', both satisfying the Assumptions* 1.1. *Let μ denote the slope on Y with respect to $\omega_Y(S)$ and μ' the one on Y' with respect to $\omega_{Y'}(S')$. Given a complex polarized variation of Hodge structures $\mathbb{V}$ on U with unipotent monodromy at infinity, let (E, θ) and (E', θ') be the logarithmic Higgs bundles of $\mathbb{V}$ and $\mathbb{V}' = \delta^* \mathbb{V}$. Then*

i. $\deg(\delta) \cdot \mu(E^{1-q,q}) = \mu'(E'^{1-q,q})$, *for* $q = 0, 1$.
ii. $\deg(\delta) \cdot \mu(\Omega^1_Y(\log S)) = \mu'(\Omega^1_{Y'}(\log S'))$.
iii. *In particular the Arakelov equality on Y implies the one on Y'.*

Proof. Choose a compactification $\bar{Y}$ of U', with $\bar{S} = \bar{Y} \setminus U$ a normal crossing divisor, such that the inclusion $U' \to Y'$ extends to a birational morphism $\bar{\sigma} : \bar{Y} \to Y'$ and such that the finite morphism $\delta : U' \to U$ extends to a generically finite morphism $\bar{\delta} : \bar{Y} \to Y$. This implies for the sheaves $\mathcal{L} = \bar{\delta}^* \omega_Y(S)$ and $\mathcal{L}' = \bar{\sigma}^* \omega_{Y'}(S')$ and for all $\nu \geq 0$ the equality

$$H^0(\bar{Y}, \mathcal{L}^\nu) = H^0(\bar{Y}, \omega_{\bar{Y}}(\bar{S})^\nu) = H^0(\bar{Y}, (\mathcal{L}')^\nu).$$

Since by Assumption 1.1 both $\mathcal{L}$ and $\mathcal{L}'$ are nef and big, there are effective exceptional divisors E and E' such that

$$\omega_{\bar{Y}}(\bar{S}) = \mathcal{L} \otimes \mathcal{O}_{\bar{Y}}(E) = \mathcal{L}' \otimes \mathcal{O}_{\bar{Y}}(E').$$

We claim that $E = E'$ and thus $\mathcal{L} = \mathcal{L}'$. We suppose that E and E' are irreducible, the general case follows by applying the following argument to each irreducible component. Assume that $0 \neq E \neq E'$. Then, on the one hand the above equality of global sections can be reinterpreted by saying that the multiplication map by the section $s_{\nu E'} \in H^0(\bar{Y}, \mathcal{O}_{\bar{Y}}(\nu E'))$ vanishing precisely along E' (to the order ν) gives an isomorphism

$$H^0(\bar{Y}, (\mathcal{L}')^\nu) \hookrightarrow H^0(\bar{Y}, (\mathcal{L}')^\nu(\nu E' - \nu E)),$$

or, that for any section $t \in H^0(\bar{Y}, (\mathcal{L}')^\nu)$ the section

$$t s_{\nu E'} \in H^0(\bar{Y}, (\mathcal{L}')^\nu(\nu E'))$$

vanishes to the order at least ν along E.

On the other hand, since $\mathcal{L}'$ is big, one can find an effective divisor F on $\bar{Y}$ and some β_0 sufficiently large, such that the sheaf $(\mathcal{L}')^{\beta_0} \otimes \mathcal{O}_{\bar{Y}}(-F)$ is ample. We write $F = aE + G$ with E not a component of G. Tensoring by a power of $\mathcal{L}'$ we may arrange that $(\mathcal{L}')^\beta \otimes \mathcal{O}_{\bar{Y}}(-aE - G)$ is ample and $\beta > a$. For m sufficiently large, the sheaf $(\mathcal{L}')^{m\beta} \otimes \mathcal{O}_{\bar{Y}}(-maE - mG)$ is very ample. Consequently, there exists a section in $H^0(\bar{Y}, (\mathcal{L}')^{m\beta} \otimes$

$\mathcal{O}_{\bar{Y}}(-maE - mG))$ that does not vanish along E. We consider it as section $t \in H^0(\bar{Y}, (\mathcal{L}')^{m\beta})$ that vanishes along E to the order ma. Hence $ts_{\nu E'}$ vanishes along E to the same order ma, contradicting the vanishing order at least $m\beta$ obtained from the first argument. The case $E' \neq 0$ can be ruled out interchanging the roles of $\mathcal{L}$ and $\mathcal{L}'$.

Let us write $\bar{\mu}$ for the slope with respect to the invertible sheaf $\mathcal{L} = \mathcal{L}'$ on $\bar{Y}$. The Deligne extension of $\mathbb{V} \otimes_C \mathcal{O}_U$ is compatible with pullbacks. This implies that $\bar{\delta}^* E^{1-q,q} = \bar{\sigma}^* E'^{1-q,q}$, and by the projection formula

$$\deg(\delta) \cdot \mu(E^{1-q,q}) = \bar{\mu}(\bar{\delta}^* E^{1-q,q}) = \bar{\mu}(\bar{\sigma}^* E'^{1-q,q}) = \mu'(E'^{1-q,q})$$

$$\text{and} \quad \dim(U) \cdot \deg(\delta) \cdot \mu(\Omega^1_Y(\log S)) = \deg(\delta) \cdot \mu(\omega_Y(S)) = \bar{\mu}(\mathcal{L}) = \bar{\mu}(\mathcal{L}') = \mu'(\omega_{Y'}(S')) = \dim(U) \cdot \mu'(\Omega^1_{Y'}(\log S')).$$

Of course, iii) follows from i) and ii). q.e.d.

We now prove Proposition 0.1 except for the statement v). The latter will be shown at the end of Section 7, by applying Addendum 7.20, III.

Proof of Proposition 0.1*, part i)–iv) for Mumford's compactification.* Those properties can be verified over some étale covering of U. So one may assume that $U \to \mathcal{A}_g$ factors through a fine moduli scheme, hence by Theorem 2.3 through $\mathcal{X}^{\rm MT} = \mathcal{X}_1 \times \mathcal{X}_2$ with image of the form $\mathcal{X}_1 \times \{b\}$. Let $\mathbb{T}$ denote the irreducible direct factor of the uniformizing $\mathbb{C}$-variation of Hodge structures on the Shimura variety $\mathcal{X}_1 \times \mathcal{X}_2$, with $\mathbb{V} \subset \mathbb{T}|_{\mathcal{X}_1 \times \{b\}}$.

By Schur's Lemma and [**De87**, Prop. 1.13] a polarized variation of Hodge structures on $\mathcal{X}_1 \times \mathcal{X}_2$ is a direct sum of exterior products of complex polarized variations of Hodge structures (see [**VZ05**, Prop. 3.3]). The irreducibility of $\mathbb{T}$ implies that $\mathbb{T} = {\rm pr}_1^*\mathbb{V}_1 \otimes {\rm pr}_2^*\mathbb{V}_2$ for suitable irreducible $\mathbb{C}$-variations of Hodge structures $\mathbb{V}_i$ on $\mathcal{X}_i$. Remark that $\mathbb{V}$, $\mathbb{T}$ and $\mathbb{V}_1$ are concentrated in bidegrees $(1,0)$ and $(0,1)$. Hence $\mathbb{V}_2$ has weight zero and is concentrated in bidegree $(0,0)$. Since ${\rm pr}_2^*\mathbb{V}_2|_{\mathcal{X}_1 \times \{b\}}$ is a trivial Hodge structure, independent of the point b, the local system $\mathbb{T}|_{\mathcal{X}_1 \times \{b\}}$ is just a direct sum of several copies of $\mathbb{V}_1$. This remains true if one replaces b by a different point $a \in \mathcal{X}_2$. The irreducibility of $\mathbb{V}$ implies that $\mathbb{V} \cong \mathbb{V}_1$, so passing from b to a one does not change the irreducible components of the complex variation of Hodge structures.

So we may suppose without loss of generality that U is a Shimura variety of Hodge type given by the datum (G, M).

Our first aim is to exhibit $E^{1,0}$ and $E^{0,1}$ as homogeneous vector bundles. Let $\tau : G \to {\rm CSp}$ be the map given by the property 'of Hodge type'. Choose a base point on the symmetric domain M and its image on $M' := M(Q)$. There are maximal compact subgroups K of $G^{\rm der}$ and $K' \cong U(g)$ of Sp such that $U \to \mathcal{A}_g$ is uniformized by the map $M = G^{\rm der}/K \to {\rm Sp}/K' =: M'$. Let $\pi_U : D \to U$ and $\pi_{\mathcal{A}_g} : D' \to \mathcal{A}_g$ be the natural quotients modulo arithmetic subgroups. The choice of the base point in M' is equivalent to the choice of a Q-symplectic basis $\{a_i, b_i\}$ of V such that we have $h(i)(a_i) = b_i$ and $h(b_i) = -a_i$ by 2.1.

Since the $(1,0)$- and $(0,1)$-parts of $\pi^*_{\mathcal{A}_g}(R^1 f_* \mathbb{C}_A)$ are the i resp. $-i$-eigenspaces of $h(i)$, they are homogeneous bundles. Moreover, they are given by the representations ρ_{can} and $\overline{\rho_{\text{can}}}$, where $\rho_{\text{can}} : U(g) \to \text{GL}(g)$ is the standard representation. The $(1,0)$- and $(0,1)$-parts of $\pi^*_U(R^1 f_* \mathbb{C}_A)$ are consequently homogeneous bundles too, given by the representation $\rho_{\text{can}} \circ \tau|_K$ and $\overline{\rho_{\text{can}}} \circ \tau|_K$.

Next, we link two notions of irreducibility. Since π_U is the quotient map by an arithmetic group $\Gamma \subset G(\mathbb{Q})$, whose image in G^{ad} is Zariski-dense, $\mathbb{C}$-irreducible summands of $R^1 f_* \mathbb{C}_A$ are in bijection with $\mathbb{C}$ irreducible summands of the representation

$$\widetilde{\tau} : \widetilde{G^{\text{ad}}} \longrightarrow G \longrightarrow \text{CSp}.$$

Here $\widetilde{G^{\text{ad}}} \to G^{\text{ad}}$ is the universal covering and the map to $\widetilde{G^{\text{ad}}} \to G$ is induced by the canonical splitting of $\text{Lie}(G)$ into its abelian and its semisimple part. We determine these $\mathbb{C}$ irreducible summands, following [**De79**, §2.3.7 (a)], see also [**Sa65**] or [**Sa80**].

By [**De79**, §2.3.4] the simple components of $G_{\mathbb{R}}$ are absolutely simple. Write

$$G^{\text{ad}}_{\mathbb{R}} = \bigtimes_{i \in I} G_i$$

and partition the index set $I = I_c \cup I_{\text{nc}}$ according to whether G_i is compact or not. By [**De79**, §1.3.8 (a) and §2.3.7] the irreducible direct factors of $V_{\mathbb{C}}$ are of the form $\otimes_{t \in T} W_t$ for some $T \subset I$, where W_t is an irreducible representation of $\widetilde{G_{i,\mathbb{R}}}$. Moreover, the condition (SV1) forces $T \cap I_{\text{nc}}$ to contain at most one element, see [**De79**, Lemma 1.3.7] This shows i).

If $T \cap I_{\text{nc}} = \emptyset$, then $\mathbb{V}$ is unitary. We thus restrict to the other case from now on. Then the condition 'Shimura variety' imposes the restrictions to the representation of the non-compact group as in the hypothesis of Lemma 3.8, stated below. From this lemma we deduce that in each case the representation of $K \subset G^{\text{ad}}$ is irreducible.

Now we know by Lemma 3.5 that for each irreducible summand $\mathbb{V}$ of $R^1 f_* \mathbb{C}_A$, both $E^{1,0}$ and $E^{0,1}$ are μ-stable. By Lemma 3.6, the bundle $\text{Hom}(E^{1,0}, E^{0,1})$ is μ-polystable with the μ-stable summands given as homogeneous bundles by the irreducible summands of the representation $\rho \otimes \rho^\vee$, where $\rho = \rho_{\text{can}} \circ \tau$. This proves iii) and iv). Since $M \to M'$ is induced by a group homomorphism and hence totally geodesic, the tangent map

$$T_M \longrightarrow T'_M|_M = \text{Hom}(E^{1,0}, E^{0,1})$$

is onto a direct summand. Since it is a map between homogeneous bundles, the direct summand corresponds to an irreducible summand of the representation $\rho \otimes \rho^\vee$. Consequently, the map

$$\overline{(T_U)} \longrightarrow \overline{\mathrm{Hom}(E^{1,0}, E^{0,1})|_U}$$

between the Mumford extensions is an injection onto a μ-stable summand. Since the Mumford extension of T_U is $T_Y(-\log S)$ and the Mumford extension of $E^{p,q}$ is the Deligne extension, we obtain

$$\mu(T_Y(-\log S)) = \mu(E^{1,0}) - \mu(E^{0,1}),$$

i.e. the Arakelov equality, stated as ii). q.e.d.

We keep the notations of the preceding proof, that will be completed with the following lemma. We follow [**De79**] and define a cocharacter $\chi : \mathbb{G}_m \to (G_i)_{\mathbb{C}}$ induced by $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$ in the following way. Fix an isomorphism

$$(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)_{\mathbb{C}} \cong \mathbb{G}_m \times \mathbb{G}_m$$

such that the inclusion

$$(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)(\mathbb{R}) \to (\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)(\mathbb{C})$$

is given by $z \mapsto (z, \overline{z})$. Let $i : \mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m$ be the inclusion given by the identity in the second argument. Then $\chi := h_{\mathbb{C}} \circ i$.

Given χ, we let $\tilde{\chi}$ be the inductive system of fractional lifts of χ to $\widetilde{G_i}$ ([**De79**, §1.3.4]).

Lemma 3.8. *Let $\tau_{i,t} : \widetilde{G_i} \to \mathrm{GL}(W_t)$ be an irreducible representation whose highest weight α is a fundamental weight and such that*

$$\langle \widetilde{\chi}, \alpha + \iota(\alpha) \rangle = 1, \tag{3.1}$$

where ι is the opposition involution. Then W_t is the sum of two non-empty weight spaces, denoted by $W_t^{1,0}$ and $W_t^{0,1}$. Both weight spaces are irreducible representations of the maximal compact subgroup K_i of G_i.

Proof. The equivalence of the condition (3.1) and the decomposition into two weight spaces is in ([**De79**, §1.3.8]). The possible solutions to 3.1 are listed on [**Sa65**, p. 461]. We distinguish the cases according to the Dynkin diagram of G_i. We use that the cocharacter $\widetilde{\chi}$ satisfying (3.1) determines a special node in the Dynkin diagram ([**De79**, §1.2.5]).

Type a_n: In this case $G_i = \mathrm{SU}(p,q)$ with $p + q = n - 1$, depending on the signature of the bilinear form induced by the Cartan involution $\mathrm{ad}(h(i))$. We may assume $p \geq q$. The maximal compact subgroup is

$$K_i = S(U(p) \times U(q)).$$

If $q > 1$ only the standard representation satisfies 3.1. The weight spaces $W_t^{1,0}$ and $W_t^{0,1}$ carry the standard representation of $\mathrm{SU}(p)$ and $\mathrm{SU}(q)$ respectively and are hence irreducible.

If $q = 1$ all j-th wedge product representations for $j = 1, \dots, n-1$ satisfy 3.1. The weight spaces $W_t^{1,0}$ (resp. $W_t^{0,1}$) carry the j-th (resp. $j-1$-st) exterior power representation of $\mathrm{SU}(p)$, which is also irreducible.

Type b_n: In this case is $G_i = \mathrm{SO}(2, 2n-1)$ (type IV_{2n-1} in [**Sa80**]) and the only representation that satisfies 3.1 is the spin representation of the double cover $\mathrm{Spin}(2, 2n-1) \to G_i$. The maximal compact subgroup is

$$K_i \cong \mathrm{SO}(2n-1, \mathbb{R}) \times \mathrm{SO}(2, \mathbb{R}).$$

We claim that one weight space carries the tensor product of the spin representation of $\mathrm{SO}(2n-1)$ and one of the natural representations $\mathrm{SO}(2, \mathbb{R}) \to U(1)$ while the other weight space carries the tensor product of the spin representation and the complex conjugate representation of $\mathrm{SO}(2, \mathbb{R})$. In both cases the representations are well known to be irreducible.

In order to prove the claim we write down the spin representation explicitly and exhibit its weight spaces. We follow the notations of [**Sa65**, §3.5]. Let G_i be the group of transformations of $V_\mathbb{R}$ preserving a bilinear form S of signature $(2n-1, 2)$. Let $\{e_1, \dots, e_{2n-1}\}$ (resp. $\{e_{2n}, e_{2n+1}\}$) be an orthonormal bases of V^+ (resp. V^-), the subspaces where the form is positive (resp. negative) definite. We let $f_j = (e_{2j-1} + ie_{2j})/2$ for $j = 1, \dots, n-1$ and $f_n = (e_{2n} + ie_{2n+1})$. Denote by W the complex vector space generated by the f_j. The exterior algebra $E = \Lambda(W)$ embeds into the Clifford algebra of $C(V, S)$. For an ordered subset $\mathcal{J} = \{i_1, \dots, i_a\} \subset N := \{1, \dots, n\}$ we consider the elements $f_\mathcal{J} = f_{i_1} \cdots f_{i_a}$ and their complex conjugates in the Clifford algebra. We identify E with the left ideal $E \cdot \overline{f_N}$ and obtain a representation of $\mathrm{Spin}(2, 2n-1)$ on E.

We may choose in

$$\mathrm{Lie}(G_i) = \left\{ \begin{pmatrix} X_1 & X_{12} \\ X_{12}^T & X_2 \end{pmatrix} ; \, X_1, X_{12}, X_2 \,\text{real}, X_1, X_2 \,\text{skew symmetric} \right\}.$$

a maximal abelian subalgebra,

$$\mathfrak{h} = \left\{ \mathrm{diag} \left(\begin{pmatrix} 0 & -\xi_1 \\ \xi_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & -\xi_{n-1} \\ \xi_{n-1} & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 & -\xi_n \\ \xi_n & 0 \end{pmatrix}, \xi_i \in \mathbb{R} \right) \right\}.$$

Then by the calculation in [**Sa65**, p. 455]) the $f_\mathcal{J}$ are eigenvectors with corresponding weight $\frac{i}{2}(\sum_{i \notin \mathcal{J}} \xi_i - \sum_{i \in \mathcal{J}} \xi_i)$. The map χ corresponding to the special node is generated by the element $H_0 \in \mathrm{Lie}(G_i)$ with $X_1 = 0$, $X_{12} = 0$ and $X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We deduce that the weight spaces $W_i^{1,0}$ (resp. $W_i^{0,1}$) are generated by the $f_\mathcal{J}$ with $n \notin \mathcal{J}$ (resp. by the $f_\mathcal{J}$ with $n \in \mathcal{J}$).

From this we first read off that $\mathrm{SO}(2,\mathbb{R})$ acts on the weight spaces as claimed. Fix the root system

$$\{i(\xi_1-\xi_2),\ldots,i(\xi_{2n-2}-\xi_{2n-1}),i\xi_{2n-1}\}$$

of $\mathrm{so}(2n-1)$. Consider $W_i^{1,0}$ as a representation of $\widetilde{\mathrm{SO}(2n-1)}$ of dimension 2^{n-1}. A vector of highest weight is $f_{N\setminus\{n\}}$ with weight $i/2\sum_{i=1}^{n-1}\xi_i$. Consequently, the representation contains a spin representation of $\mathrm{Spin}(2n-1)\to\mathrm{SO}(2n-1)$. For dimension reasons the representation is irreducible. The same argument applies to $W_i^{0,1}$.

Type c_n: In this case $G_i=\mathrm{Sp}(n)$, and as in the beginning of the proof of Proposition 0.1 above, the weight spaces carry the standard representation of $U(n)$ and its complex conjugate. Thus, they are irreducible.

Type d_n: This case splits into two sub-cases according to the χ or equivalently according to the position of the corresponding special node in the Dynkin diagram.

Special node at the 'fork' end. In this case

$$G_i=\mathrm{SU}^-(n,\mathbb{H})\cong\mathrm{SU}(n,n)\cap\mathrm{SO}(2n,\mathbb{C})\subset\mathrm{Sl}(2n,\mathbb{C})$$

where $\mathbb{H}$ denotes the Hamiltonians. In this matrix representation the weight spaces are given by the n first (resp. last) column vectors. The maximal compact subgroup $K_i\cong U(n)$ sits in G_i via

$$A+iB\mapsto\begin{pmatrix}A & B\\ -B & A\end{pmatrix}.$$

Consequently, both weight spaces are n-dimensional and carry the irreducible standard representation of $U(n)$.

Special node at the opposite end. This is completely similar to the case b_n replacing 'spin' by 'half spin' representations throughout.

Exceptional Lie algebras do not admit any solution to 3.1. q.e.d.

4. Slopes and filtrations of coherent sheaves

We will need small twists of the slope $\mu(\mathcal{F})$ defined with respect to the nef and big invertible sheaf $\omega_Y(S)$ in 0.1. So we will decompose the slope in a linear combination of different slopes and we will deform the coefficients a little bit. In particular, as in [**La04**], we will compare the Harder-Narasimhan filtrations for small twists of slopes.

On the non-singular projective variety Y of dimension n consider $n-1$-tuples of $\mathbb{R}$-divisors

$$\underline{D}^{(\iota)}=(D_1^{(\iota)},\ldots,D_{n-1}^{(\iota)}),$$

for $\iota=1,\ldots,m$. The collection of those divisors will be denoted by $\underline{D}^{(\bullet)}$. Given two such tuples $\underline{D}^{(\bullet)}$ and $\underline{D}'^{(\bullet)}$ we define the sum componentwise, hence

$$\underline{D}^{(\bullet)}+\underline{D}'^{(\bullet)}=\big[(D_1^{(\iota)}+D_1'^{(\iota)},\ldots,D_{n-1}^{(\iota)}+D_{n-1}'^{(\iota)});\ \iota=1,\ldots,m\big].$$

Definition 4.1. We call $\underline{D}^{(\bullet)}$ a *semi-polarization* if the $\mathbb{R}$-divisors $D_j^{(\iota)}$ are nef for $\iota = 1, \ldots, m$ and for $j = 1, \ldots, n-1$ and if the intersection cycle
$$(\underline{D}^{(\iota)})^{n-1} := D_1^{(\iota)} . \cdots . D_{n-1}^{(\iota)}$$
is not numerically trivial for $\iota = 1, \ldots, m$.

For a coherent torsion free sheaf $\mathcal{F}$ on Y and for each $\iota \in \{1, \ldots, m\}$ one defines the slope
$$\mu_{\underline{D}^{(\iota)}}(\mathcal{F}) = \frac{\mathrm{c}_1(\mathcal{F}).(\underline{D}^{(\iota)})^{n-1}}{\mathrm{rk}(\mathcal{F})},$$
and adding up
(4.1)
$$\mu_{\underline{D}^{(\bullet)}}(\mathcal{F}) = \mu_{[\underline{D}^{(1)}, \ldots, \underline{D}^{(m)}]}(\mathcal{F}) = \sum_{\iota=1}^{m} \mu_{\underline{D}^{(\iota)}}(\mathcal{F}) = \sum_{\iota=1}^{m} \frac{\mathrm{c}_1(\mathcal{F}).(\underline{D}^{(\iota)})^{n-1}}{\mathrm{rk}(\mathcal{F})}.$$
In the sequel we will assume that $\underline{D}^{(\bullet)}$ is a semi-polarization, and we fix a torsion free coherent sheaf $\mathcal{F}$ on Y. If there is no ambiguity, we write μ' in this Section instead of $\mu_{\underline{D}^{(1)}, \ldots, \underline{D}^{(m)}}$, and we reserve the notion μ for the special case where the slope is taken with respect to $\omega_Y(S)$.

Given an exact sequence of torsion free coherent sheaves
$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0,$$
an easy calculation shows that
$$\mu'(\mathcal{F}) = \frac{\mathrm{rk}(\mathcal{F}')}{\mathrm{rk}(\mathcal{F})}\mu'(\mathcal{F}') + \frac{\mathrm{rk}(\mathcal{F}'')}{\mathrm{rk}(\mathcal{F})}\mu'(\mathcal{F}''). \tag{4.2}$$
In order to define 'stability' for locally free or torsion free coherent sheaves one has to take care of boundary divisors of slope zero, i.e. of prime divisors D with $\mu'(\mathcal{O}_Y(D)) = 0$. Since the divisors $D_j^{(\iota)}$ are nef, this is equivalent to the condition $D.(\underline{D}^{(\iota)})^{n-1} = 0$, for $\iota = 1, \ldots, m$.

Definition 4.2. Keeping the notations introduced above, let $\mathcal{F}$ and $\mathcal{G}$ be two coherent torsion free sheaves on Y.

a. A subsheaf $\mathcal{G}$ of $\mathcal{F}$ is *μ'-equivalent* to $\mathcal{F}$, if $\mathcal{F}/\mathcal{G}$ is a torsion sheaf and if $\mathrm{c}_1(\mathcal{F}) - \mathrm{c}_1(\mathcal{G})$ is the class of an effective divisor D with $\mu'(\mathcal{O}_Y(D)) = 0$, or equivalently with $D.(\underline{D}^{(\iota)})^{n-1} = 0$, for $\iota = 1, \ldots, m$. We call *μ'-equivalence* the equivalence relation on coherent sheaves generated by μ-equivalent inclusions.
b. A morphism $\mathcal{G} \to \mathcal{F}$ is *surjective up to μ'-equivalence*, if its image is μ'-equivalent to $\mathcal{F}$.
c. $\mathcal{G} \subset \mathcal{F}$ is *saturated*, if $\mathcal{F}/\mathcal{G}$ is torsion free.
d. $\mathcal{F}$ is *μ'-stable*, if $\mu'(\mathcal{G}) < \mu'(\mathcal{F})$ for all non-trivial subsheaves $\mathcal{G}$ of $\mathcal{F}$ with $\mathrm{rk}(\mathcal{G}) < \mathrm{rk}(\mathcal{F})$.

e. $\mathcal{F}$ is *μ'-semistable*, if $\mu'(\mathcal{G}) \leq \mu'(\mathcal{F})$ for all non-trivial subsheaves $\mathcal{G}$ of $\mathcal{F}$.
f. $\mathcal{F}$ is *μ'-polystable* if it is the direct sum of μ'-stable sheaves of the same slope.
g. A saturated subsheaf $\mathcal{G}$ of $\mathcal{F}$ is called a *maximal destabilizing subsheaf*, if for all subsheaves $\mathcal{E}$ of $\mathcal{F}$ one has $\mu'(\mathcal{E}) \leq \mu'(\mathcal{G})$ and if the equality implies that $\mathcal{E} \subset \mathcal{G}$.

We will give a nicer description of the relation 'μ-equivalence' in a special case at the beginning of Section 5.

Lemma 4.3.
1. *If $\mathcal{F}$ is μ'-stable and if $\mathcal{G} \subset \mathcal{F}$ is a subsheaf with $\mu'(\mathcal{G}) = \mu'(\mathcal{F})$ then $\mathcal{F}$ and $\mathcal{G}$ are μ'-equivalent.*
2. *A μ'-polystable sheaf $\mathcal{F}$ is μ'-semistable.*
3. *In particular, if $\mathcal{H}$ is invertible, then $\bigoplus \mathcal{H}$ is μ'-semistable.*

Proof. If $\mathcal{G}$ is a subsheaf of $\mathcal{F}$ with $\mathrm{rk}(\mathcal{G}) = \mathrm{rk}(\mathcal{F})$ then $\mathrm{c}_1(\mathcal{F}) - \mathrm{c}_1(\mathcal{G})$ is an effective divisor D. Since all the $D_j^{(\iota)}$ are nef, one finds $D.(\underline{D}^{(\iota)})^{n-1} \geq 0$ and hence $\mu'(\mathcal{G}) \leq \mu'(\mathcal{F})$. This implies 2) in case that $\mathcal{F}$ is μ'-stable.

For μ'-polystable sheaves 2) follows by induction on the number of direct factors, and 3) is an example for the statement in 2).

If $\mathcal{F}$ is μ'-stable and $\mu'(\mathcal{G}) = \mu'(\mathcal{F})$, then by definition $\mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{G})$, hence $D.(\underline{D}^{(\iota)})^{n-1} = 0$ as claimed in 1). q.e.d.

Later the divisors $D_i^{(\iota)}$ will correspond to the determinant of the μ-polystable direct factors Ω_j of $\Omega^1_Y(\log S)$ in the decomposition 1.1, each one occurring as often as the rank of Ω_j, except the one corresponding to the upper index ι. For one ι we will multiply in 4.1 $\mu_{\underline{D}^{(\iota)}}$ by a factor $1+\epsilon$.

We consider in this section a more general and more flexible set-up than needed in the sequel, hoping that it might be of use in a different context. We choose a second tuple

$$\underline{H}^{(\iota)} = (H_1^{(\iota)}, \ldots, H_{n-1}^{(\iota)})$$

of nef $\mathbb{R}$-divisors, for $\iota = 1, \ldots, m$, and the polynomial

$$\mu'_t(\mathcal{F}) = \mu_{\underline{D}^{(\bullet)}+t\cdot\underline{H}^{(\bullet)}}(\mathcal{F}) = \sum_{\iota=1}^{m} \frac{\mathrm{c}_1(\mathcal{F}).(\underline{D}^{(\iota)} + t\cdot \underline{H}^{(\iota)})^{n-1}}{\mathrm{rk}(\mathcal{F})}.$$

Of course one has $\mu'_0(\mathcal{F}) = \mu'(\mathcal{F})$. The cycle $(\underline{D}^{(\iota)} + t\cdot\underline{H}^{(\iota)})^{n-1}$ can be written as

$$D_1^{(\iota)}.\cdots.D_{n-1}^{(\iota)} + \sum_{I\in\mathcal{I}} t^{n-|I|-1}\cdot D_{i_1}^{(\iota)}.\cdots.D_{i_{|I|}}^{(\iota)}.H_{j_1}^{(\iota)}.\cdots.H_{j_{n-1-|I|}}^{(\iota)}$$

where the sum is taken over the set $\mathcal{I}$ of ordered subsets

$$I = \{i_1, \ldots, i_{|I|}\} \quad \text{of} \quad \{1, \ldots, n-1\}$$

of cardinality $|I| < n-1$, and where $\{j_1, \ldots, j_{j_{n-1-|I|}}\}$ is the complement of I in $\{1, \ldots, n-1\}$, again as an ordered set. For a coherent sheaf $\mathcal{G}$ one has

(4.3)

$$\mu'_t(\mathcal{F}) - \mu'_t(\mathcal{G}) = \mu'(\mathcal{F}) - \mu'(\mathcal{G}) + \sum_{\mathcal{I}} t^{n-|I|-1} \cdot (\mu'^I(\mathcal{F}) - \mu'^I(\mathcal{G})),$$

$$\text{(4.4)} \qquad \text{with} \quad \mu'^I(\mathcal{G}) = \sum_{\iota=1}^{m} \frac{\mathrm{c}_1(\mathcal{G}).D^{(\iota)}_{i_1}.\cdots.D^{(\iota)}_{i_{|I|}}.H^{(\iota)}_{j_1}.\cdots.H^{(\iota)}_{j_{n-1-|I|}}}{\mathrm{rk}(\mathcal{G})}.$$

Lemma 4.4. *For a coherent sheaf $\mathcal{F}$ of rank r consider the sets*

$$\mathrm{S} = \{\mu'(\mathcal{G});\ \mathcal{G} \subset \mathcal{F}\} \subset \mathbb{R} \quad \textit{and}$$

$$\mathcal{S} = \left\{\mu'_t(\mathcal{G}) = \sum_{\nu=0}^{n-1} a_\nu \cdot t^\nu;\ \mathcal{G} \subset \mathcal{F}\right\} \subset \mathbb{R}[t].$$

Then

i. *the set* S *is discrete and bounded from above.*
ii. *There exists some $\epsilon_0 > 0$ and some 'maximal' element $G(t) \in \mathcal{S}$, such that for all $F(t) \in \mathcal{S}$ with $F(t) \neq G(t)$ one has $G(\epsilon) > F(\epsilon)$ for $0 < \epsilon \leq \epsilon_0$.*

Proof. Let S′ be the set of all coefficients occurring in $F(t) \in \mathcal{S}$. We will first show, that the set S′ is discrete and bounded from above. Since $\mathrm{S} \subset \mathrm{S}'$, this implies i).

For $\mathcal{H}$ invertible and sufficiently ample $\mathcal{F}^\vee \otimes \mathcal{H}$ is generated by global sections. Hence $\mathcal{F}$ is embedded in $\bigoplus \mathcal{H}$. Then under the projection to suitable factors, any subsheaf $\mathcal{G} \subset \mathcal{F}$ of rank r' is isomorphic to a subsheaf of $\bigoplus^{r'} \mathcal{H}$ and $\mathrm{c}_1(\mathcal{G}) = r \cdot \mathrm{c}_1(\mathcal{H}) - D$ for some effective divisor D.

Since the divisors $D^{(\iota)}_j$ and $H^{(\iota)}_j$ are all nef, the intersection of the 1-dimensional cycles

$$D^{(\iota)}_{i_1}.\cdots.D^{(\iota)}_{i_{|I|}}.H^{(\iota)}_{j_1}.\cdots.H^{(\iota)}_{j_{n-1-|I|}}$$

in 4.4 with any divisor is a non-negative multiple of a fixed real number, So one may write

$$\sum_{\iota=1}^{m} (\underline{D}^{(\iota)} + t \cdot \underline{H}^{(\iota)})^{n-1} = \sum_{\nu=0}^{n-1} \left(\sum_{\mu} \alpha_{\nu,\mu} C_{\mu,\nu}\right) t^\nu$$

for $\alpha_{\mu,\nu} \in \mathbb{R}$ and for linear combinations $C_{\mu,\nu}$ of curves with $D.C_{\mu,\nu} \geq 0$ for all effective divisors D. Then $-\mathrm{S}'$ is discrete, as a subset of the union of translates of finite many copies of

$$\bigcup_{\nu} \sum_{\mu} \alpha_{\mu,\nu} \cdot \mathbb{N}.$$

Moreover S′ it is bounded above by the maximal coefficient c of $\mu'_t(\mathcal{H})$.

On the set $\mathcal{S}$ consider the lexicographical order. So $\sum_{\nu=0}^{n-1} a_\nu \cdot t^\nu < \sum_{\nu=0}^{n-1} b_\nu \cdot t^\nu$ if $a_\nu = b_\nu$ for $\nu < j$ and if $a_j < b_j$. Obviously $\mathcal{S}$ contains a maximal element $G(t) = \sum_{\nu=0}^{n-1} b_\nu \cdot t^\nu$ for this order.

Choose $\epsilon_0 \in (0,1)$ to be a real number with

$$\frac{1}{\sqrt{\epsilon_0}} \geq \sup_{c \in \mathrm{S}} \left\{ \sum_{\nu=j+1}^{n-1} (c - b_\nu) t^{\nu-j-1};\ t \in [0,1],\ j = 1, \dots, r-1 \right\},$$

and such that for $\nu = 0, \dots, r$ one has $[b_\nu - \sqrt{\epsilon_0}, b_\nu + \sqrt{\epsilon_0}] \cap \mathrm{S}' = \{b_\nu\}$.

Since $G(t) > F(t)$, for some j and for $0 < \epsilon \leq \epsilon_0$ one finds

$$G(\epsilon) - F(\epsilon) = \sum_{\nu=j}^{n-1} (b_\nu - a_\nu) \cdot \epsilon^\nu \geq$$

$$\epsilon^j \cdot \left((b_j - a_j) + \epsilon \cdot \sum_{\nu=j+1}^{n-1} (b_\nu - c) \cdot \epsilon^{\nu-j-1} \right) > \epsilon^j \cdot \left(\sqrt{\epsilon_0} - \epsilon \cdot \frac{1}{\sqrt{\epsilon_0}} \right) \geq 0.$$

q.e.d.

We will consider next values of the polynomials $F(t) \in \mathcal{S}$ for small $\epsilon \in \mathbb{R}_{\geq 0}$.

Definition 4.5. For $\epsilon \in \mathbb{R}_{\geq 0}$ consider a filtration $0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_\ell = \mathcal{F}$ with $\mathcal{G}_\alpha / \mathcal{G}_{\alpha-1}$ torsion free and μ'_ϵ-semistable, for $\alpha = 1, \dots, \ell$, and with

(4.5)
$$\mu'_{\epsilon,\max}(\mathcal{F}) = \mu'_\epsilon(\mathcal{G}_1) \geq \mu'_\epsilon(\mathcal{G}_2/\mathcal{G}_1) \geq \cdots \geq \mu'_\epsilon(\mathcal{G}_\ell/\mathcal{G}_{\ell-1}) = \mu'_{\epsilon,\min}(\mathcal{F}).$$

The filtration is called a *μ'_ϵ-Harder-Narasimhan filtration* if the inequalities in (4.5) are all strict, and it is called a *weak μ'_ϵ-Jordan-Hölder filtration* if $\mu'_{\epsilon,\max}(\mathcal{F}) = \mu'_{\epsilon,\min}(\mathcal{F})$.

Lemma 4.6. *Let $\mathcal{F}$ be a coherent torsion free sheaf on Y.*

a. *For all $\epsilon \geq 0$ there exists a Harder-Narasimhan filtration*

$$\mathcal{G}_0 = 0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_\ell = \mathcal{F}$$

of $\mathcal{F}$ with respect to μ'_ϵ and this filtration is unique.

b. *There exists some $\epsilon_0 > 0$ such that the filtration in a) is independent of ϵ for $\epsilon_0 \geq \epsilon > 0$.*

c. *If $\mathcal{F}$ is μ'-stable, then for some $\epsilon_0 > 0$ and for all $\epsilon_0 \geq \epsilon \geq 0$ the sheaf $\mathcal{F}$ is μ'_ϵ-semistable.*

Proof. For $\epsilon > 0$ we apply Lemma 4.4, ii). For the polynomial $G(t)$, given there, choose a subsheaf $\mathcal{G} \subset \mathcal{F}$ with $G(t) = \mu'_t(\mathcal{G})$, for all $t \in \mathbb{R}$. Moreover for $0 < \epsilon \leq \epsilon_0$ the slope $\mu'_\epsilon(\mathcal{G}) = G(\epsilon)$ is maximal among the possible slopes of subsheaves of $\mathcal{F}$. This allows to assume that $\mathcal{G}$ is saturated. If there are several subsheaves of $\mathcal{F}$ with the same slope, we choose a saturated one of maximal rank.

If for $\mathcal{E} \subset \mathcal{F}$ one has $\mu'_\epsilon(\mathcal{E}) = \mu'_\epsilon(\mathcal{G})$, then by 4.2 the slope of $\mathcal{E} \oplus \mathcal{G}$ is $\mu'_\epsilon(\mathcal{G})$. The maximality of the slope of $\mathcal{G}$ implies $\mu'_\epsilon(\mathcal{E} \cap \mathcal{G}) \leq \mu'_\epsilon(\mathcal{G})$ and $\mu'_\epsilon(\mathcal{E} + \mathcal{G}) \leq \mu'_\epsilon(\mathcal{G})$. By 4.2 this is only possible if $\mu'_\epsilon(\mathcal{E} + \mathcal{G}) = \mu'_\epsilon(\mathcal{G})$. Then the maximality of the rank of $\mathcal{G}$ implies that $\mathrm{rk}(\mathcal{E} + \mathcal{G}) = \mathrm{rk}(\mathcal{G})$, and $\mathcal{E} \subset \mathcal{G}$.

So $\mathcal{G}$ is a maximal destabilizing subsheaf of $\mathcal{F}$, and it is independent of $\epsilon \in (0, \epsilon_0]$. The existence and uniqueness of a μ'_ϵ-Harder-Narasimhan filtration follows by induction on the rank. Here of course we have to lower ϵ_0 in each step.

For $\epsilon = 0$ the existence and uniqueness of the Harder-Narasimhan filtration follows by the same argument, replacing the reference to part ii) of Lemma 4.4 by the one to part i).

Assume now that $\mathcal{F}$ is μ'-stable and consider the Harder-Narasimhan filtration in a). Then

$$\mu'(\mathcal{G}_1) = \lim_{\epsilon \to 0} \mu'_\epsilon(\mathcal{G}_1) \geq \lim_{\epsilon \to 0} \mu'_\epsilon(\mathcal{F}) = \mu'(\mathcal{F}).$$

By assumption, $\mathcal{F}$ is stable, with respect to μ', hence $\mathcal{G}_1 = \mathcal{F}$, and $\ell = 1$. q.e.d.

Although this will not be used in the sequel, let us state a strengthening of the last part of Lemma 4.6.

Addendum 4.7. *For ϵ_0 sufficiently small, the sheaf $\mathcal{F}$ in part c) is μ'_ϵ-stable for all $\epsilon_0 \geq \epsilon \geq 0$.*

Proof. Part i) of Lemma 4.4 and the μ'-stability of $\mathcal{F}$ imply that

$$\gamma = \mathrm{Inf}\{\mu'(\mathcal{F}) - \mu'(\mathcal{G});\ \mathrm{rk}(\mathcal{G}) < \mathrm{rk}(\mathcal{F})\} > 0.$$

Let us return to the slopes μ'^I introduced in 4.3 and 4.4. By part a) of Lemma 4.6 there exists a Harder-Narasimhan filtration

$$\mathcal{G}_0^I = 0 \subset \mathcal{G}_1^I \subset \cdots \subset \mathcal{G}_{\ell_I}^I = \mathcal{F}$$

with respect to μ'^I. In particular for $\mathcal{G} \subset \mathcal{F}$ one has $\mu'^I(\mathcal{G}) \leq \mu'^I(\mathcal{G}_1^I)$.

Choose $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, and for all $I \in \mathcal{I}$ with $|I| < n - 1$ one has

$$\frac{1}{|\mathcal{I}| + 1} \cdot \gamma \geq \epsilon^{n-|I|-1} \cdot \left(\mu'^I(\mathcal{G}_1^I) - \mu'^I(\mathcal{F})\right)$$

For a subsheaf $\mathcal{G} \subset \mathcal{F}$ of strictly smaller rank one finds

$$\mu'^I(\mathcal{F}) - \mu'^I(\mathcal{G}) \geq \mu'^I(\mathcal{F}) - \mu'^I(\mathcal{G}_1^I),$$

and thereby

$$\mu'_\epsilon(\mathcal{F}) - \mu'_\epsilon(\mathcal{G}) \geq \gamma + \sum_{I\in\mathcal{I}} \epsilon^{n-1-|I|} \cdot (\mu'^I(\mathcal{F}) - \mu'^I(\mathcal{G})) \geq$$

$$\gamma + \sum_{I\in\mathcal{I}} \epsilon^{n-1-|I|} \cdot (\mu'^I(\mathcal{F}) - \mu'^I(\mathcal{G}_1^I)) \geq \gamma - \frac{|\mathcal{I}|}{|\mathcal{I}|+1} \cdot \gamma > 0.$$

q.e.d.

Corollary 4.8. *Assume in Lemma* 4.6 *that $\mathcal{F}$ is μ'-semistable. Then there exists a weak μ'-Jordan-Hölder filtration*

$$\mathcal{G}_0 = 0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_\ell = \mathcal{F}$$

and some $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$ the filtration $\mathcal{G}_\bullet$ is a μ'_ϵ-Harder-Narasimhan-filtration.

Proof. The filtration $\mathcal{G}_\bullet$, constructed Lemma 4.6, b), is a μ'_ϵ-Harder-Narasimhan filtration for all $0 < \epsilon \leq \epsilon_0$. Taking the limit of the slopes for μ'_ϵ one obtains

$$\mu'_{\max}(\mathcal{F}) = \mu'(\mathcal{G}_1) \geq \mu'(\mathcal{G}_2/\mathcal{G}_1) \geq \cdots \geq \mu'(\mathcal{G}_\ell/\mathcal{G}_{\ell-1}) = \mu'_{\min}(\mathcal{F}),$$

and since $\mathcal{F}$ is μ'-semistable, those are all equalities. q.e.d.

5. Splittings of Higgs bundles

The negativity of kernels of Higgs bundles provide a well-known criterion for the orthogonal complement of a subbundle $\mathcal{K}$ of $E^{1,0}$ to be a holomorphic subbundle: It suffices to show that the slope of the cokernel $\mathcal{Q}$ with respect to the canonical polarization is zero. In this section we extend this to a criterion that zero slope with respect to canonical *semi*-polarizations implies – best that one can expect – vanishing of $\partial/\partial\bar{z}$-derivatives of the orthogonal splitting map $\mathcal{Q} \to E^{1,0}$ in the corresponding directions.

Assume again that Y is non-singular, that $U \subset Y$ the complement of a normal crossing divisor S, and that the positivity conditions stated as Assumptions 1.1 hold true. Then one has the decomposition (see 1.1)

$$\Omega^1_Y(\log S) = \Omega_1 \oplus \cdots \oplus \Omega_s$$

as a direct sum of μ-stable subsheaves Ω_i of rank n_i.

Lemma and Definition 5.1.

i. *The μ-stable direct factors Ω_i and their determinants* $\det(\Omega_i)$ *are nef. The cycles* $\mathrm{c}_1(\Omega_i)^{n_i+1}$ *are numerically trivial.*
ii. *For $\nu_1, \ldots, \nu_s$ with $\nu_1 + \cdots + \nu_s = n$ the product* $\mathrm{c}_1(\Omega_1)^{\nu_1}.\cdots.\mathrm{c}_1(\Omega_s)^{\nu_s}$ *is a positive multiple of* $\mathrm{c}_1(\omega_Y(S))^n$*, if $\nu_\iota = n_\iota$ for $\iota = 1, \ldots, s$. Otherwise it is zero.*
iii. $\mathrm{c}_1(\Omega_1)^{n_1}.\cdots.\mathrm{c}_1(\Omega_s)^{n_s} > 0$.

iv. *Let D be an effective $\mathbb{Q}$ divisor. Then $D.\mathrm{c}_1(\omega_Y(S))^{n-1} = 0$ if and only if*

$$D.\mathrm{c}_1(\Omega_1)^{\nu_1}.\cdots.\mathrm{c}_1(\Omega_s)^{\nu_s} = 0$$

for all $\nu_1, \ldots, \nu_s$ with $\nu_1 + \cdots + \nu_s = n-1$.

v. *Let NS_0 denote the subspace of the Neron-Severi group $\mathrm{NS}(Y)_{\mathbb{Q}}$ of Y which is generated by all prime divisors D satisfying the equivalent conditions in iv). Then all effective divisors B with class in NS_0 is supported in S.*

vi. *If for some $\alpha \in \mathbb{Q}$ one has $\mathrm{c}_1(\Omega_i) - \alpha \cdot \mathrm{c}_1(\Omega_j) \in \mathrm{NS}_0$ then $i = j$.*

Proof. Parts i), ii), iii) and vi) have been shown in [**VZ07**, Lemmata 1.6 and 1.9]. Part iv) follows from the nefness of $\det(\Omega_i)$. For v) consider a prime divisor D whose support meets U. Since $\omega_Y(S)$ is nef and ample with respect to U, the restriction $\omega_Y(S)|_D$ is nef and big, and hence $D.\mathrm{c}_1(\omega_Y(S))^{n-1} > 0$. So the nefness of $\omega_Y(S)$ implies that none of the components of B in v) can meet U. q.e.d.

Using the notations from Section 4 consider $m = 1$ and the tuple $\underline{D}^{(1)}$ where all divisors are $D_j^{(1)} = K_Y + S$ for some canonical divisor K_Y. Then the slope $\mu_{\underline{D}^{(1)}}(\mathcal{F})$, considered there, is equal to $\mu(\mathcal{F})$. Using Lemma 5.1 the μ-equivalence, as given by Definition 4.2, can be made more precise. Recall that we define two torsion free coherent sheaves $\mathcal{G}$ and $\mathcal{F}$ to be μ-equivalent, if there is a chain of μ-equivalent inclusions

$$\mathcal{G} = \mathcal{G}_1 \hookrightarrow \mathcal{F}_1 \hookleftarrow \mathcal{G}_2 \hookrightarrow \mathcal{F}_2 \hookleftarrow \cdots\cdots \hookrightarrow \mathcal{F}_{\ell-1} \hookleftarrow \mathcal{G}_\ell \hookrightarrow \mathcal{F}_\ell = \mathcal{F}.$$

Addendum 5.2. *Let $\tau : U' \to Y$ be the complement of all prime divisors $D \leq S$ with $D \in \mathrm{NS}_0$. Let $\mathcal{F}$ and $\mathcal{G}$ be torsion free coherent sheaves on Y.*

vii. *Assume that $\mathcal{G}$ is a subsheaf of $\mathcal{F}$ which is μ-equivalent to $\mathcal{F}$. Then $\mathrm{c}_1(\mathcal{F}) - \mathrm{c}_1(\mathcal{G})$ lies in the subspace NS_0, defined in Lemma 5.1 v), and $\mathcal{G}|_{U'} \to \mathcal{F}|_{U'}$ is an isomorphism. In particular this holds if $\mathcal{G} \hookrightarrow \mathcal{F}$ is an inclusion of μ-semistable sheaves of the same slope and rank.*

viii. *The following conditions are equivalent:*
   a. *$\mathcal{G}$ and $\mathcal{F}$ are μ-equivalent.*
   b. *There exists an isomorphism $\tau^*\mathcal{G} \to \tau^*\mathcal{F}$.*
   c. *There exists an effective divisor $B \in \mathrm{NS}_0$ with $\mathcal{G} \subset \mathcal{F} \otimes \mathcal{O}_Y(B)$.*

ix. *Let $\theta : \mathcal{G} \to \mathcal{F}$ be a morphism of μ-semistable sheaves of the same slope, and let $\mathrm{Im}'(\theta)$ denote the saturated image, i.e. the kernel of*

$$\mathcal{F} \longrightarrow (\mathcal{F}/\mathrm{Im}(\theta))/_{\mathrm{torsion}}.$$

Then $\mathrm{Im}'(\theta)$ is a μ-semistable subsheaf of $\mathcal{F}$ of slope $\mu(\mathcal{F})$, and the inclusion $\mathrm{Im}(\theta) \hookrightarrow \mathrm{Im}'(\theta)$ is an isomorphism over U'.

Proof. Part vii) follows directly from the definition of μ-equivalence in 4.2 and from the definition of NS_0 in Lemma and Definition 5.1. As a consequence, in viii) the condition a) implies b).

On the other hand, given an isomorphism $\tau^*\mathcal{G} \cong \tau^*\mathcal{F}$, hence $\tau_*\tau^*\mathcal{G} \cong \tau_*\tau^*\mathcal{F}$, one finds effective divisors B and B', both supported in $Y \setminus U'$, with

$$\mathcal{G} \hookrightarrow \mathcal{G} \otimes \mathcal{O}_Y(B') = \mathcal{F} \otimes \mathcal{O}_Y(B) \hookleftarrow \mathcal{F}.$$

In particular b) implies c). Finally, since $B \in \mathrm{NS}_0$ one finds that c) implies a).

For part ix) one just has to remark that the nefness of $\omega_Y(S)$ implies that

$$\mu(\mathcal{G}) \leq \mu(\mathrm{Im}(\theta)) \leq \mu(\mathrm{Im}'(\theta)) \leq \mu(\mathcal{F}).$$

q.e.d.

Example and Definition 5.3. Let $\mathcal{F}$ be a μ-semistable torsion free coherent sheaf. As for slopes defined by polarizations (e.g. [**HL97**, page 23]) one finds for semi-polarizations a maximal μ-polystable subsheaf $\mathrm{Soc}(\mathcal{F}) = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_\ell$ of slope $\mu(\mathcal{F})$. Remark that in general the saturated hull of $\mathrm{Soc}(\mathcal{F})$ is no longer μ-polystable, but for some effective divisor $B \in \mathrm{NS}_0$ it will be contained in the μ-polystable sheaf $(\mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_\ell) \otimes \mathcal{O}_Y(B)$, and both are μ-equivalent.

In Section 7 we will need the *cosocle* $\mathrm{Cosoc}(\mathcal{F})$ of $\mathcal{F}$, defined as the dual of the socle of $\mathcal{F}^\vee$. In down to earth terms this is the largest μ-polystable sheaf of slope $\mu(\mathcal{F})$ for which there exists a morphism $\theta : \mathcal{F} \to \mathrm{Cosoc}(\mathcal{F})$, surjective over some open set.

In the sequel we consider again an irreducible polarized complex variation of Hodge structures $\mathbb{V}$ of weight 1 with unipotent monodromy at infinity and with Higgs bundle

$$\big(E = E^{1,0} \oplus E^{0,1},\ \theta : E^{1,0} \to E^{0,1} \otimes \Omega^1_Y(\log S)\big).$$

We assume that $\mathbb{V}$ is non-unitary, hence that $\theta \neq 0$.

Recall that a Higgs subsheaf $(\mathcal{G}, \theta|_{\mathcal{G}})$ of a Higgs bundle (E, θ) is a subsheaf with $\theta(\mathcal{G}) \subset \mathcal{G} \otimes \Omega^1_Y(\log S)$. Correspondingly a torsion free Higgs quotient sheaf is of the form $\mathcal{Q} = E/\mathcal{G}$, where $\mathcal{G}$ is saturated and a Higgs subsheaf. By [**VZ07**, Proposition 2.4] one obtains as a corollary of Simpson's correspondence:

Lemma 5.4. *Let $\underline{D}^{(\iota)}$ be a finite system of $n-1$-tuples of nef $\mathbb{R}$-divisors. Let (E, θ) be the Higgs bundle of a complex polarized variation of Hodge structures with unipotent monodromy at infinity. Then:*

i. $\mu_{\underline{D}^{(\bullet)}}(\mathcal{G}) \leq 0$ *for all Higgs subsheaves* $\mathcal{G}$.
ii. $\mu_{\underline{D}^{(\bullet)}}(\mathcal{Q}) \geq 0$ *for all torsion free Higgs quotient sheaves* $\mathcal{Q}$.

iii. *If for one ι and for all j the divisors $\underline{D}_j^{(\iota)}$ are ample with respect to U, then the following conditions are equivalent for a saturated Higgs subsheaf $\mathcal{G}$ of E and for $\mathcal{Q} = E/\mathcal{G}$:*
1. $\mu_{\underline{D}^{(\bullet)}}(\mathcal{G}) = 0$.
2. $\mu_{\underline{D}^{(\bullet)}}(\mathcal{Q}) = 0$.
3. *$\mathcal{G}$ is a direct factor of the Higgs bundle E.*

Let us write D_i for a divisor with $\mathcal{O}_Y(D_i) = \det(\Omega_i)$ and consider for $\iota = 1, \ldots, s$ the tuple $\underline{\widetilde{D}}^{(\iota)}$

$$(5.1) \qquad (\overbrace{D_\iota, \ldots, D_\iota}^{n_\iota - 1}, \overbrace{D_1, \ldots, D_1}^{n_1}, \ldots \overbrace{D_{\iota-1}, \ldots, D_{\iota-1}}^{n_{\iota-1}}, \overbrace{D_{\iota+1}, \ldots, D_{\iota+1}}^{n_{\iota+1}}, \ldots, \overbrace{D_s, \ldots, D_s}^{n_s}).$$

For some binomial coefficients one can write

$$\mu(\mathcal{F}) = \sum_{\iota=1}^{s} \alpha_\iota \cdot \mu_{\underline{\widetilde{D}}^{(\iota)}}(\mathcal{F}).$$

To get rid of the α_ι we replace $\underline{\widetilde{D}}^{(\iota)}$ by the tuple $\underline{D}^{(\iota)}$ obtained by multiplying each of the divisors in $\underline{\widetilde{D}}^{(\iota)}$ by $\sqrt[n-1]{\alpha_\iota}$. So for the intersection cycle one gets

$$(\underline{D}^{(\iota)})^{n-1} = \alpha_\iota \cdot (\underline{\widetilde{D}}^{(\iota)})^{n-1}$$

and one finds

$$(5.2) \quad \mu_{\underline{D}^{(\iota)}}(\mathcal{F}) = \alpha_\iota \cdot \mu_{\underline{\widetilde{D}}^{(\iota)}}(\mathcal{F}) \quad \text{and} \quad \mu(\mathcal{F}) = \sum_{\iota=1}^{s} \mu_{\underline{D}^{(\iota)}}(\mathcal{F}) = \mu_{\underline{D}^{\bullet}}(\mathcal{F}).$$

Remark that $\mu_{\underline{D}^{(\iota)}}(\Omega_i) \neq 0$ if and only if $\iota = i$.

Properties 5.5.
1. If the local system $\mathbb{V}$ is irreducible and non-unitary there exists some ι with $\mu_{\underline{D}^{(\iota)}}(E^{1,0}) > 0$.
2. If $\mathcal{L}$ is an invertible sheaf, nef and big, then for all j one has $\mu_{\underline{D}^{(j)}}(\mathcal{L}) > 0$.

Proof. For part 1) remark that Lemma 5.4, ii) and iii) imply that $\mu(E^{1,0}) > 0$. For 2) recall that for $\nu \gg 1$ the sheaf $\mathcal{L}^\nu \otimes \Omega_j^{-1}$ has a section with divisor Γ. Since the D_j are all nef, $\mu_{\underline{D}^{(j)}}(\mathcal{O}_Y(\Gamma)) \geq 0$ and hence

$$\nu \cdot \mu_{\underline{D}^{(j)}}(\mathcal{L}) \geq \alpha_j \mathrm{c}_1(\Omega_1)^{n_1} . \cdots . \mathrm{c}_1(\Omega_s)^{n_s} > 0.$$

q.e.d.

Next we consider a small twist of μ by choosing for $\epsilon \geq 0$

$$\mu_\epsilon^{\{\iota\}}(\mathcal{F}) = \epsilon \cdot \mu_{\underline{D}^{(\iota)}}(\mathcal{F}) + \mu(\mathcal{F}).$$

For $s > 1$ none of the divisors $\underline{D}_j^{(\iota)}$ is ample. So we are not allowed to apply part iii) of Lemma 5.4 to the slope $\mu_{\underline{D}^{(\iota)}}$.

For $\mu_\epsilon^{\{\iota\}}$ things are better. For a for a Higgs subbundle $\mathcal{G}$ of E the first part of Lemma 5.4 only implies that $\mu_{\underline{D}^{(\iota)}}(\mathcal{G}) \leq 0$. Since $\mu(\mathcal{G}) \leq 0$ the equality $\mu_\epsilon^{\{\iota\}}(\mathcal{G}) = 0$ can only hold for $\epsilon > 0$ if $\mu(\mathcal{G}) = \mu_{\underline{D}^{(\iota)}}(\mathcal{G}) = 0$. This implies that the saturated hull of $\mathcal{G}$ in E is a direct factor, contradicting the irreducibility of $\mathbb{V}$. So $\mathrm{rk}(\mathcal{G}) < \mathrm{rk}(E)$ implies that $\mu_\epsilon^{\{\iota\}}(\mathcal{G}) < 0$.

As we will show in Section 6.1 the same holds for the slopes $\mu_{\underline{D}^{(\iota)}}$ if the universal covering $\tilde{U}$ is a bounded symmetric domain. Without this information, one just has the following criterion.

Proposition 5.6. *Let*

$$0 \longrightarrow \mathcal{K} \longrightarrow E^{1,0} \longrightarrow \mathcal{Q} \longrightarrow 0$$

be an exact sequence, and let $s : \mathcal{Q} \to E^{1,0}$ be the orthogonal complement of $\mathcal{K}$. Assume that for some ι the slope $\mu_{\underline{D}^{(\iota)}}(\mathcal{Q}) = 0$. Then

a. *The composition*

$$\mathcal{Q} \xrightarrow{\ s\ } E^{1,0} \xrightarrow{\ \theta\ } E^{0,1} \otimes \Omega^1_Y(\log S) \xrightarrow{\ \mathrm{pr}_\iota\ } E^{0,1} \otimes \Omega_\iota$$

is zero.

b. *$s : \mathcal{Q} \to E^{1,0}$ is holomorphic in the direction Ω_ι.*

Remark that a priori s is a C^∞ map. So part b) of the Proposition needs some explanation. Recall that we have the decomposition $\tilde{U} = M_1 \times \cdots \times M_s$, corresponding to the decomposition of $\Omega^1_Y(\log S)$ in μ-stable direct factors. Write $n_0 = 0$, again $n_i = \mathrm{rk}(\Omega_i) = \dim(M_i)$ and $m_i = \sum_{j=0}^{i} n_j$.

Given a point $y \in U$ let us choose a local coordinate system $z_1, \ldots, z_n$ in a neighborhood of y such that $\pi^*(z_{m_{i-1}+1}), \ldots, \pi^*(z_{m_i})$ are coordinates on M_i.

Definition 5.7. The inclusion $s : \mathcal{Q} \to E^{1,0}$ is holomorphic in the direction Ω_ι if its image is invariant under the action of $\partial/\partial\bar{z}_k$ on $E^{1,0}$ for $k = m_{\iota-1} + 1, \ldots, m_\iota$.

Proof of Proposition 5.6. We assume $\iota = 1$. Locally, in some open set $W \subset U$ choose complex coordinates $z_1, \ldots, z_n$ as above and unitary frames of $E^{1,0}$ and $E^{0,1}$. That is, choose C^∞-sections $e_1, \ldots, e_\ell$ of $E^{1,0}$ and $f_1, \ldots, f_{\ell'}$ of $E^{0,1}$ orthogonal with respect to the scalar product $h(\cdot,\cdot)$ coming from the Hodge metric, and such that $e_1, \ldots, e_k$ generate

$\mathcal{K}$ while $e_{k+1}, \ldots, e_\ell$ generate $s(Q)$. Write the Higgs field θ in these coordinates as

$$\theta(e_\alpha) = \sum_{i=1}^{n} \sum_{\beta=1}^{\ell'} \theta^i_{\alpha,\beta} f_\beta dz_i.$$

By [**Gr70**, Theorem 5.2] the curvature R of the metric connection ∇_h on $E^{1,0}$ is given by

$$R_{E^{1,0}} = \theta \wedge \theta^* = \sum_{i,j=1}^{n} (R_{E^{1,0}})^{i,j} dz_i \wedge \bar{z}_j, \tag{5.3}$$

where

$$(R_{E^{1,0}})^{i,j}_{\alpha,\beta} = \sum_{\gamma=1}^{\ell'} \theta^i_{\alpha,\gamma} \overline{\theta^i_{\beta,\gamma}}.$$

For the subbundle $\mathcal{K} \subset E^{1,0}$ the composition

$$b : \mathcal{K} \longrightarrow E^{1,0} \xrightarrow{\nabla_h} E^{1,0} \otimes \Omega^1_U \longrightarrow \mathcal{Q} \otimes \Omega^1_U$$

of the metric connection and the quotient is called second fundamental form. Taking complex conjugates we obtain a map

$$c : \bar{\mathcal{K}} \cong \mathcal{K}^\vee \longrightarrow \bar{\mathcal{Q}} \otimes \Omega^{0,1}_U \cong \mathcal{Q}^\vee \otimes \Omega^{0,1}_U.$$

Both maps are only C^∞. We write the map c in coordinates

$$c(e_\alpha) = \sum_{i=1}^{n} \sum_{\beta=1}^{k} c^i_{\alpha,\beta} e_\beta dz_i.$$

By [**Gr70**, Theorem 5.2] the curvature of the metric connection on $\mathcal{Q}$ is given by

$$R_{\mathcal{Q}} = (\theta s) \wedge (\theta s)^* + c \wedge c^* = \sum_{i,j=1}^{n} (R_{\mathcal{Q}})^{i,j} dz_i \wedge \bar{z}_j, \quad \text{where}$$

(5.4)

$$(R_{\mathcal{Q}})^{i,j}_{\alpha,\beta} = \sum_{\gamma=1}^{\ell'} \theta^i_{\alpha,\gamma} \overline{\theta^i_{\beta,\gamma}} + \sum_{\gamma=1}^{k} c^i_{\alpha,\gamma} \overline{c^i_{\beta,\gamma}}, \quad \text{for} \quad \alpha, \beta \in \{k+1, \ldots, \ell\}.$$

We conclude that for all i, the matrices $(R_{\mathcal{Q}})^{i,i}$ are positive semi-definite. Moreover their traces are zero if and only if $\theta^i_{\alpha,\beta} = 0$ and $c^i_{\alpha,\beta} = 0$ for all $\alpha, \beta \in \{k+1, \ldots, \ell\}$.

We write $R(\Omega_i)$ for the curvature of $\det(\Omega_i)$. By Lemma 5.1 ii) and after rescaling z_i by suitable constants we may assume that over W

$$R(\Omega_i) = dz_{m_{j-1}+1} \wedge d\bar{z}_{m_{j-1}+1} + \cdots + dz_{m_j} \wedge d\bar{z}_{m_j},$$

keeping the convention $m_0 = n_0 = 0$. Then

$$R(\Omega_1)^{n_1-1} \wedge R(\Omega_2)^{n_2} \wedge \cdots \wedge R(\Omega_s)^{n_s} = \sum_{i=1}^{n_1} C_i \cdot \bigwedge_{j\neq i} dz_j \wedge d\bar{z}_j,$$

for some binomial coefficients $C_i > 0$. The hypothesis $\mu_{\underline{D}^{(1)}}(\mathcal{Q}) = 0$ is equivalent to

$$0 = \left(\frac{\sqrt{-1}}{2\pi}\right) \cdot \int_U \operatorname{tr}(R_{\mathcal{Q}}) \wedge R(\Omega_1)^{n_1-1} \wedge R(\Omega_2)^{n_2} \wedge \cdots \wedge R(\Omega_s)^{n_s}.$$

Since $\operatorname{tr}(R_{\mathcal{Q}})$ and all the $R(\Omega_i)$ are positive semidefinite, the integral has to be zero on all open sets, in particular on W. We deduce

$$\begin{aligned} 0 &= \int_W \left(\sum_{i,j=1}^{n} \operatorname{tr}(R_Q)^{i,j} dz_i \wedge d\bar{z}_j \right) \wedge \left(\sum_{i=1}^{n_1} C_i \cdot \bigwedge_{j\neq i} dz_j \wedge d\bar{z}_j \right) \\ &= \int_W C_i \operatorname{tr}(R_{\mathcal{Q}})^{i,i} \bigwedge_{j=1}^{n} dz_j \wedge d\bar{z}_j. \end{aligned} \tag{5.5}$$

Hence $\operatorname{tr}(R_{\mathcal{Q}})^{i,i} = 0$ for all i and we obtain the vanishing on U of the composition $\operatorname{pr}_\iota \circ \theta \circ s$ as claimed in a) and of all $c^i_{\alpha,\beta}$. Since the $(0,1)$-part of the metric connection ∇_h is $\bar{\partial}$, the vanishing of $c^i_{\alpha,\beta}$ is what is claimed in b). Since the sheaves Ω_ι, $E^{1,0}$ and $E^{0,1}$ are locally free, both vanishing statements extend to the whole of Y. q.e.d.

6. Purity of Higgs bundles with Arakelov equality

In this section we will prove Theorem 0.3. So keeping the assumptions from Section 5 we will assume in addition that $\mathbb{V}$ is non-unitary and that it satisfies the Arakelov equality

$$\mu(\mathbb{V}) = \mu(E^{1,0}) - \mu(E^{0,1}) = \mu(\Omega^1_Y(\log S)).$$

By [**VZ07**, Theorem 1] we know that $E^{1,0}$ and $E^{0,1}$ are both μ-semistable. We keep the notations from the last section. In particular as in 5.1 and 5.2 we define tuples $\underline{D}^{(\iota)}$ of divisors for $\iota = 1, \ldots, s$ with $\mu = \mu_{\underline{D}^{(\bullet)}}$. Moreover

$$\mu_\epsilon^{\{\iota\}} = \mu + \epsilon \cdot \mu_{\underline{D}^{(\iota)}}$$

denotes a small perturbation of the slope μ. First we show that this is the slope associated with a small perturbation of the original collection of divisors by a suitable collection of nef divisors, as studied in Section 4.

Lemma 6.1. *For some tuples of nef $\mathbb{R}$-divisors $\underline{H}^{(i)}$ one has*

$$\mu_\epsilon^{\{\iota\}} = \mu_{\underline{D}^{(\bullet)}+\epsilon\cdot\underline{H}^{(\bullet)}}.$$

Proof. There are several choices for the $\underline{H}^{(i)}$. In the description of the tuple of divisors $\tilde{\underline{D}}^{(\iota)}$ in 5.1 denote the first entry by D_ℓ. Then the first entry in $\underline{D}^{(\iota)}$ is $\sqrt[n-1]{\alpha_\iota} \cdot D_\ell$. Here $\ell = \iota$, if $n_\iota > 1$, or some other index in case that $n_\iota = 1$.

Then choose the tuples of $\mathbb{R}$-divisors $\underline{H}^{(\bullet)}$ with $H_j^{(i)} = 0$ for $i = 1, \ldots, s$ and for $j = 1, \ldots, n-1$, except for $H_1^{(\iota)}$ which is chosen to be $\sqrt[n-1]{\alpha_\iota} \cdot D_\ell$. This implies that $\underline{D}^{(i)} + \epsilon \cdot \underline{H}^{(i)} = \underline{D}^{(i)}$ for $i \neq \iota$, whereas

$$(\underline{D}^{(\iota)} + \epsilon \underline{H}^{(\iota)})^{n-1} = (1+\epsilon) \cdot (\underline{D}^{(\iota)})^{n-1}.$$

So for a sheaf $\mathcal{F}$ one finds

$$\mu_{\underline{D}^{(\bullet)}+\epsilon\cdot\underline{H}^{(\bullet)}}(\mathcal{F}) = \sum_{i=1}^{s} \mu_{\underline{D}^{(i)}+\epsilon\cdot\underline{H}^{(i)}}(\mathcal{F}) = (1+\epsilon)\cdot\mu_{\underline{D}^{(\iota)}}(\mathcal{F}) + \sum_{i\neq\iota} \mu_{\underline{D}^{(i)}}(\mathcal{F})$$

$$= \epsilon \cdot \mu_{\underline{D}^{(\iota)}}(\mathcal{F}) + \sum_{i=1}^{s} \mu_{\underline{D}^{(i)}}(\mathcal{F}) = \epsilon \cdot \mu_{\underline{D}^{(\iota)}}(\mathcal{F}) + \mu_{\underline{D}^{(\bullet)}}(\mathcal{F}) = \mu_\epsilon^{\{\iota\}}(\mathcal{F}).$$

q.e.d.

By Corollary 4.8 one finds a filtration $\mathcal{G}_\bullet^{(\iota)}$ of $E^{1,0}$ and some $\epsilon_0 > 0$ such that $\mathcal{G}_\bullet^{(\iota)}$ is a $\mu_\epsilon^{\{\iota\}}$-Harder-Narasimhan filtration of $E^{1,0}$, for all $\epsilon \in (0, \epsilon_0]$, and a weak μ-Jordan-Hölder filtration. Of course we may choose ϵ_0 to be independent of ι.

So the quotient sheaves $\mathcal{G}_i^{(\iota)}/\mathcal{G}_{i-1}^{(\iota)}$ are $\mu_\epsilon^{\{\iota\}}$-semistable for all $\epsilon \in [0, \epsilon_0]$, however not necessarily $\mu_{\underline{D}^{(\iota)}}$-semistable.

Lemma 6.2. *Let $\mathcal{F}$ be a μ-stable subsheaf of $E^{1,0}$ with $\mu(\mathcal{F}) = \mu(E^{1,0})$. Then $\mathcal{F}$ is pure of type ι for some $\iota \in \{1, \ldots, s\}$. Moreover, each subsheaf $\mathcal{F}'$ of $E^{1,0}$ which is isomorphic to $\mathcal{F}$ is pure of the same type ι.*

Recall from Definition 1.4 that $\mathcal{F}$ is pure of type ι if the restriction $\theta|_{\mathcal{F}}$ of the Higgs field factors like

$$\mathcal{F} \xrightarrow{\theta_\iota} E^{0,1} \otimes \Omega_\iota \xrightarrow{\subset} E^{0,1} \otimes \Omega^1_Y(\log S).$$

Equivalently, writing T_i for the dual of Ω_i and $\theta_i^\vee$ for the composite

$$E^{1,0} \otimes T_i \xrightarrow{\theta\otimes\mathrm{id}_{T_i}} E^{0,1} \otimes \Omega_i \otimes T_i \xrightarrow{\text{contraction}} E^{0,1},$$

one requires $\theta_i^\vee(\mathcal{F} \otimes T_i)$ to be zero for $i \neq \iota$. Since $\mathbb{V}$ is non-unitary this is only possible if $\theta_\iota^\vee(\mathcal{F} \otimes T_\iota) \neq 0$.

Proof of Lemma 6.2. Assume that $\mathcal{F}' \cong \mathcal{F}$ and that for some $i \neq i'$ one has

$$\theta_i^\vee(\mathcal{F} \otimes T_i) \neq 0 \quad \text{and} \quad \theta_{i'}^\vee(\mathcal{F}' \otimes T_{i'}) \neq 0.$$

We will write $\mathcal{B}_i$ and $\mathcal{B}_{i'}$ for the saturated hull of those images. The Arakelov equality implies that $\theta_i^\vee$ and $\theta_{i'}^\vee$ are morphisms between μ-semistable sheaves of the same slope, hence $\mu(\mathcal{B}_\iota) = \mu(\mathcal{F}) + \mu(T_\iota)$ for $\iota = i,\ i'$.

The sheaves $\mathcal{F}$ and T_ι are μ-stable. By Lemma 4.6 for $\epsilon > 0$, sufficiently small, and for all j the sheaves $\mathcal{F}$ and T_ι are $\mu_\epsilon^{\{j\}}$-semistable. Hence $\mathcal{F} \otimes T_\iota$ is $\mu_\epsilon^{\{j\}}$-semistable, and consequently,

$$\mu_\epsilon^{\{j\}}(\mathcal{B}_\iota) \geq \mu_\epsilon^{\{j\}}(\mathcal{F}) + \mu_\epsilon^{\{j\}}(T_\iota) \quad \text{and} \quad \mu_{\underline{D}^{(j)}}(\mathcal{B}_\iota) \geq \mu_{\underline{D}^{(j)}}(\mathcal{F}) + \mu_{\underline{D}^{(j)}}(T_\iota).$$

For $\iota = i$ and $j \neq i$ one obtains

$$0 \geq \mu_{\underline{D}^{(j)}}(\mathcal{B}_i) \geq \mu_{\underline{D}^{(j)}}(\mathcal{F}) + \mu_{\underline{D}^{(j)}}(T_i) = \mu_{\underline{D}^{(j)}}(\mathcal{F}),$$

and for $\iota = i' \neq k$

$$0 \geq \mu_{\underline{D}^{(k)}}(\mathcal{B}_{i'}) \geq \mu_{\underline{D}^{(k)}}(\mathcal{F}) + \mu_{\underline{D}^{(k)}}(T_{i'}) = \mu_{\underline{D}^{(k)}}(\mathcal{F}') = \mu_{\underline{D}^{(k)}}(\mathcal{F}).$$

Then $i \neq i'$ implies that $\mu_{\underline{D}^{(j)}}(\mathcal{F}) \leq 0$ for all j, hence $\mu(\mathcal{F}) \leq 0$. Since $\mathbb{V}$ is non-unitary and since $\mu(\mathcal{F}) = \mu(E^{1,0})$ this contradicts part iii) of Lemma 5.4. q.e.d.

Let us define

$$\mathcal{K}^{(\iota)} = \operatorname{Ker}\bigl(E^{1,0} \longrightarrow E^{0,1} \otimes \Omega_Y^1(\log S) \longrightarrow E^{0,1} \otimes \bigoplus_{j \neq \iota} \Omega_j\bigr). \tag{6.1}$$

Corollary 6.3. *There exists some ι with $\mathcal{K}^{(\iota)} \neq 0$.*

Proof. Choose a direct factor $\mathcal{F}$ of the socle of $E^{1,0}$, hence a μ-stable subsheaf $\mathcal{F} \subset E^{1,0}$ with $\mu(\mathcal{F}) = \mu(E^{1,0})$. Then by Lemma 6.2 the bundle $\mathcal{F}$ is contained in $\mathcal{K}^{(\iota)}$ for some ι. q.e.d.

Lemma 6.4. *Assume that $\mathbb{V}$ is pure of type ι, hence that $E^{1,0} = \mathcal{K}^{(\iota)}$. Then for all $j \neq \iota$ one has $\mu_{\underline{D}^{(j)}}(E^{1,0}) = 0$.*

Proof. If $E^{1,0} = \mathcal{K}^{(\iota)}$, the saturated image $\mathcal{B}_\iota$ of

$$\theta_\iota^\vee : E^{1,0} \otimes T_\iota \longrightarrow E^{0,1}$$

has to be non-zero. $\theta_\iota^\vee$ is a map of μ-semistable sheaves of the same slope, hence $\mu(\mathcal{B}_\iota) = \mu(E^{1,0}) - \mu(\Omega_\iota)$. For ϵ sufficiently small $E^{1,0} \otimes T_\iota$ is $\mu_\epsilon^{\{j\}}$-semistable, and

$$\mu_\epsilon^{\{j\}}(\mathcal{B}_\iota) \geq \mu_\epsilon^{\{j\}}(E^{1,0}) - \mu_\epsilon^{\{j\}}(\Omega_\iota).$$

Then for $j \neq \iota$ one finds

$$\mu_{\underline{D}^{(j)}}(\mathcal{B}_\iota) \geq \mu_{\underline{D}^{(j)}}(E^{1,0}) - \mu_{\underline{D}^{(j)}}(\Omega_\iota) = \mu_{\underline{D}^{(j)}}(E^{1,0}),$$

which by Lemma 5.4 can neither be positive, nor negative, hence it must be zero. q.e.d.

A similar argument will be used to obtain a stronger statement, which finally will lead to a contradiction, except if $E^{1,0} = \mathcal{K}^{(\iota)}$ for some ι.

Lemma 6.5. *Let ℓ be the length of the filtration $\mathcal{G}_\bullet^{(\iota)}$.*

a. *Then $\mathcal{G}_{\ell-1}^{(\iota)} \subset \mathcal{K}^{(\iota)}$.*

b. *If $\mathcal{K}^{(\iota)} \neq E^{1,0}$ then $\mu_{\underline{D}^{(\iota)}}(\mathcal{G}_\ell^{(\iota)}/\mathcal{G}_{\ell-1}^{(\iota)}) = 0$.*

Proof. Let $\nu \in \{1, \ldots, \ell+1\}$ be the largest number with $\mathcal{G}_{\nu-1}^{(\iota)} \subset \mathcal{K}^{(\iota)}$. If $\nu = \ell + 1$ then $E^{1,0} = \mathcal{G}_\ell^{(\iota)} = \mathcal{K}^{(\iota)}$ and there is nothing to show. So let us assume that $\nu \leq \ell$, or equivalently that $\mathcal{K}^{(\iota)} \neq E^{1,0}$.

For all $j \neq \iota$ the restriction of $\theta_j^\vee$ to $\mathcal{G}_\nu^{(\iota)}$ induces a morphism

$$\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)} \otimes T_j \longrightarrow E^{0,1}, \tag{6.2}$$

and by assumption for at least one $j \neq \iota$ this morphism is non-zero. So we will fix such an index j and assume in the sequel that the saturated image $\mathcal{B}_j$ of $\theta_j^\vee|_{\mathcal{G}_\nu^{(\iota)}}$ is non-zero.

Since $\mathcal{G}_\bullet^{(\iota)}$ is a weak μ-Jordan Hölder filtration, the morphism in (6.2) is a morphism between μ-semistable sheaves of the same slope, non-zero by assumption. Then

$$\mu(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)} \otimes T_j) = \mu(E^{1,0}) + \mu(T_j) = \mu(E^{0,1}) = \mu(\mathcal{B}_j).$$

Since $j \neq \iota$

$$\mu_{\underline{D}^{(\iota)}}(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)} \otimes T_j) = \mu_{\underline{D}^{(\iota)}}(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)}), \quad \text{and hence}$$
$$\mu_\epsilon^{\{\iota\}}(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)} \otimes T_j) = \epsilon \cdot \mu_{\underline{D}^{(\iota)}}(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)}) + \mu(\mathcal{B}_j).$$

For $0 < \epsilon \leq \epsilon_0$ the sheaf $\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)} \otimes T_j$ is $\mu_\epsilon^{\{\iota\}}$-semistable, which implies that

$$\epsilon \cdot \mu_{\underline{D}^{(\iota)}}(\mathcal{B}_j) + \mu(\mathcal{B}_j) = \mu_\epsilon^{\{\iota\}}(\mathcal{B}_j) \geq \epsilon \cdot \mu_{\underline{D}^{(\iota)}}(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)}) + \mu(\mathcal{B}_j). \tag{6.3}$$

By the choice of $\mathcal{G}_\bullet^{(\iota)}$ as a $\mu_\epsilon^{\{\iota\}}$-Harder-Narasimhan filtration one has an inequality

$$\mu_\epsilon^{\{\iota\}}(\mathcal{G}_\ell^{(\iota)}/\mathcal{G}_{\ell-1}^{(\iota)}) \leq \mu_\epsilon^{\{\iota\}}(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)}), \tag{6.4}$$

with equality if and only if $\nu = \ell$. Since $\mathcal{G}_\bullet^{(\iota)}$ is a weak μ-Jordan-Hölder filtration the slope $\mu(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)})$ is independent of ν. So the inequality 6.4 carries over to one for the slope $\mu_{\underline{D}^{(\iota)}}$. As we have seen in Lemma 5.4, i) one has $\mu_{\underline{D}^{(\iota)}}(\mathcal{B}_j) \leq 0$, so rewriting the inequalities 6.3 and 6.4 one gets

$$\mu_{\underline{D}^{(\iota)}}(\mathcal{G}_\ell^{(\iota)}/\mathcal{G}_{\ell-1}^{(\iota)}) \leq \mu_{\underline{D}^{(\iota)}}(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)}) \leq 0. \tag{6.5}$$

Lemma 5.4, ii) implies however that $\mu_{\underline{D}^{(\iota)}}(\mathcal{G}^{(\iota)}_\ell/\mathcal{G}^{(\iota)}_{\ell-1}) \geq 0$. So both inequalities in 6.5 are equalities and b) holds true. Moreover 6.4 is an equality, hence $\nu = \ell$, as claimed in a). q.e.d.

Corollary 6.6. *If in Lemma* 6.5 *the sheaf* $\mathcal{Q} = E^{1,0}/\mathcal{K}^{(\iota)}$ *is non-zero, it is* μ *and* $\mu^{\{\iota\}}_\epsilon$*-semistable. One has*

$$\mu^{\{\iota\}}_\epsilon(\mathcal{Q}) = \mu(\mathcal{Q}) = \mu(E^{1,0}) = \mu(E^{0,1}) + \mu(\Omega^1_Y(\log S)),$$

and hence $\mu_{\underline{D}^{(\iota)}}(\mathcal{Q}) = 0$.

Proof. Since $\mathcal{K}^{(\iota)}$ as the kernel of a morphism between μ-semistable sheaves of the same slope is μ-semistable, $\mathcal{Q}$ has the same property.

By Lemma 6.5, b) the slope $\mu_{\underline{D}^{(\iota)}}(\mathcal{G}^{(\iota)}_\ell/\mathcal{G}^{(\iota)}_{\ell-1}) = 0$. Since $\mathcal{G}^{(\iota)}_\bullet$ is a weak μ-Jordan-Hölder filtration and a $\mu^{\{\iota\}}_\epsilon$-Harder-Narasimhan filtration, for all $0 \leq \epsilon \leq \epsilon_0$ the quotient $\mathcal{G}^{(\iota)}_\ell/\mathcal{G}^{(\iota)}_{\ell-1}$ is $\mu^{\{\iota\}}_\epsilon$-semistable and has slope

$$\mu^{\{\iota\}}_\epsilon(\mathcal{G}^{(\iota)}_\ell/\mathcal{G}^{(\iota)}_{\ell-1}) = \mu(\mathcal{G}^{(\iota)}_\ell/\mathcal{G}^{(\iota)}_{\ell-1}) = \mu(E^{1,0}).$$

For $j \neq \iota$ the sheaf Ω_j is $\mu^{\{\iota\}}_\epsilon$-stable of slope $\mu^{\{\iota\}}_\epsilon(\Omega_j) = \mu(\Omega_j) = \mu(\Omega^1_Y(\log S))$, hence

$$\mathcal{G}^{(\iota)}_\ell/\mathcal{G}^{(\iota)}_{\ell-1} \otimes \bigoplus_{j\neq\iota} T_j$$

is again $\mu^{\{\iota\}}_\epsilon$-semistable of slope $\mu(E^{0,1})$.

Let $\mathcal{B}$ be the saturated image of $\mathcal{G}^{(\iota)}_\ell/\mathcal{G}^{(\iota)}_{\ell-1} \otimes \bigoplus_{j\neq\iota} T_j$ in $E^{0,1}$. Then $\mu(\mathcal{B}) = \mu(E^{0,1})$ and $\mu^{\{\iota\}}_\epsilon(\mathcal{B}) \geq \mu(E^{0,1})$.

On the other hand Lemma 5.4 implies that $\mu_{\underline{D}^{(\iota)}}(\mathcal{B}) \leq 0$, hence $\mu^{\{\iota\}}_\epsilon(\mathcal{B}) = \mu(E^{0,1})$. So $\mathcal{B}$ as a quotient of a $\mu^{\{\iota\}}_\epsilon$-semistable sheaf of the same slope has to be $\mu^{\{\iota\}}_\epsilon$-semistable of slope

$$\mu(E^{0,1}) = \mu^{\{\iota\}}_\epsilon\big(\mathcal{G}^{(\iota)}_\ell/\mathcal{G}^{(\iota)}_{\ell-1} \otimes \bigoplus_{j\neq\iota} T_j\big).$$

Since $\mathcal{Q} = E^{1,0}/\mathcal{K}^{(\iota)}$ is a subsheaf of $\mathcal{B} \otimes \bigoplus_{j\neq\iota}\Omega_j$ one finds

$$\mu^{\{\iota\}}_\epsilon(\mathcal{Q}) \leq \mu^{\{\iota\}}_\epsilon\big(\mathcal{B} \otimes \bigoplus_{j\neq\iota}\Omega_j\big) = \mu(E^{0,1}),$$

and since it is a quotient of $\mathcal{G}^{(\iota)}_\ell/\mathcal{G}^{(\iota)}_{\ell-1}$ one has $\mu^{\{\iota\}}_\epsilon(\mathcal{Q}) \geq \mu(E^{0,1})$. One obtains the equality of slopes in Corollary 6.6. Finally $\mathcal{Q}$ as a subsheaf of a $\mu^{\{\iota\}}_\epsilon$-semistable sheaf of the same slope is itself $\mu^{\{\iota\}}_\epsilon$-semistable. q.e.d.

Proof of Theorem 0.3. Renumbering the factors we will assume that $\mathcal{K}^{(1)} \neq 0$, and we will write

$$\Omega = \bigoplus_{j=2}^{s} \Omega_j \quad \text{and} \quad T = \Omega^\vee.$$

So $\mathcal{K}^{(1)}$ is the kernel of the composition

$$E^{1,0} \longrightarrow E^{0,1} \otimes \Omega^1_Y(\log S) \xrightarrow{\text{pr}} E^{0,1} \otimes \Omega.$$

Let $\mathcal{K}_1$ be the kernel of

$$E^{1,0} \longrightarrow E^{0,1} \otimes \Omega^1_Y(\log S) \xrightarrow{\text{pr}_1} E^{0,1} \otimes \Omega_1.$$

Claim 6.7. $E^{1,0}$ is the direct sum $\mathcal{K}^{(1)} \oplus \mathcal{K}_1$.

Proof. As a consequence of Corollary 6.6 the sheaf $\mathcal{Q} = E^{1,0}/\mathcal{K}^{(1)}$ satisfies $\mu_{\underline{D}^{(1)}}(\mathcal{Q}) = 0$. So Proposition 5.6, a), tells us that the orthogonal complement $s(\mathcal{Q})$ is contained in $\mathcal{K}_1$.

The intersection of $\mathcal{K}^{(1)}$ and $\mathcal{K}_1$ lies in the kernel of θ. Hence it is zero and the induced map $\mathcal{K}_1 \to \mathcal{Q}$ is injective. On the other hand

$$\mathcal{Q} \xrightarrow{s} E^{1,0} \longrightarrow \mathcal{Q}$$

factors through $\mathcal{K}_1 \to \mathcal{Q}$, and the latter must be surjective. This implies that

$$E^{1,0} = \mathcal{K}^{(1)} \oplus \mathcal{K}_1.$$

q.e.d.

Let $\mathcal{B}^{(1)}$ and $\mathcal{B}_1$ be the saturated images of

$$E^{1,0} \otimes T \longrightarrow E^{0,1} \quad \text{and} \quad E^{1,0} \otimes T_1 \longrightarrow E^{0,1},$$

respectively.

Claim 6.8. $\mathcal{B}^{(1)} \cap \mathcal{B}_1 = 0$.

Proof. Dualizing the exact sequences

$$0 \longrightarrow \mathcal{B}^{(1)} \longrightarrow E^{0,1} \longrightarrow \mathcal{C}^{(1)} = E^{0,1}/\mathcal{B}^{(1)} \longrightarrow 0$$
$$\text{and} \quad 0 \longrightarrow \mathcal{B}_1 \longrightarrow E^{0,1} \longrightarrow \mathcal{C}_1 = E^{0,1}/\mathcal{B}_1 \longrightarrow 0$$

one obtains that

$$\mathcal{C}^{(1)\vee} = \text{Ker}(E^{0,1\vee} \xrightarrow{\tau} \mathcal{B}^{(1)\vee}) \quad \text{and} \quad \mathcal{C}_1^\vee = \text{Ker}(E^{0,1\vee} \xrightarrow{\tau_1} \mathcal{B}_1^\vee).$$

The dual Higgs bundle $E^\vee$ has $E^{0,1\vee}$ as subsheaf of bidegree $(1,0)$ and $E^{0,1\vee}$ is of bidegree $(0,1)$. The composite

$$E^{0,1\vee} \xrightarrow{\tau} \mathcal{B}^{(1)\vee} \xrightarrow{\subset} E^{1,0\vee} \otimes \Omega \quad \text{and} \quad E^{0,1\vee} \xrightarrow{\tau_1} \mathcal{B}_1^\vee \xrightarrow{\subset} E^{1,0\vee} \otimes \Omega_1$$

are the components of the dual Higgs field. Applying Claim 6.7 to $E^\vee$ one obtains a decomposition $E^{0,1\vee} = \mathcal{C}^{(1)\vee} \oplus \mathcal{C}_1^\vee$, hence $E^{0,1} \cong \mathcal{C}^{(1)} \oplus \mathcal{C}_1$ and $\mathcal{B}^{(1)} \cap \mathcal{B}_1 = 0$. q.e.d.

So one obtains a direct sum decomposition of Higgs bundles

$$(E,\theta) = \big(\mathcal{K}^{(1)} \oplus \mathcal{B}^{(1)}, \theta^{(1)} = \theta|_{\mathcal{K}^{(1)}}\big) \oplus \big(\mathcal{K}_1 \oplus \mathcal{B}_1, \theta_1 = \theta|_{\mathcal{K}_1}\big)$$

corresponding to a decomposition $\mathbb{V} = \mathbb{V}^{(1)} \oplus \mathbb{V}_1$. The irreducibility of $\mathbb{V}$ and the assumption $\mathcal{K}^{(1)} \neq 0$ imply $\mathbb{V}_1 = 0$, hence $\mathcal{K}_1 = 0$. q.e.d.

6.1. Using superrigidity. As mentioned in the introduction, the purity of the Higgs fields in Theorem 0.3 follows from the Margulis Superrigidity Theorem, without using the Arakelov equality, provided all the direct μ-stable factors of $\Omega^1_Y(\log S)$ are of type C. We will show below, that for variations of Hodge structures of weight 1 it is sufficient to assume that the universal covering $\tilde{U}$ of U is a bounded symmetric domain. In different terms, if Ω_i is of type B we suppose that it satisfies the Yau-equality

$$2(n_i+1)\cdot \mathrm{c}_2(\Omega_i).\mathrm{c}_1(\Omega_i)^{n_i-2}.\mathrm{c}_1(\omega_Y(S))^{n-n_i} = n_i \cdot \mathrm{c}_1(\Omega_i)^{n_i}.\mathrm{c}_1(\omega_Y(S))^{n-n_i}$$

([**Ya93**], see also [**VZ07**, Theorem 1.4]).

Proposition 6.9. *Suppose that $\tilde{U}$ is a bounded symmetric domain and that $\mathbb{V}$ is an irreducible complex polarized variation of Hodge structures of weight 1 with unipotent monodromy at infinity. Then the associated Higgs bundle $(E^{1,0}\oplus E^{0,1},\theta)$ is pure of type ι for some $\iota \in \{1,\dots,s\}$.*

Sketch of the proof. By assumption $U = \Gamma\backslash\tilde{U}$ is the quotient of a bounded symmetric domain $\tilde{U} = M_1 \times \cdots \times M_s$ by a lattice Γ. We can write $M_i = G_i/K_i$ as quotient of a real, non-compact, simple Lie group by a maximal compact subgroup.

Assume first that $U = U_1 \times U_2$. By [**VZ05**, Proposition 3.3] an irreducible local system on $\mathbb{V}$ is of the form $\mathrm{pr}_1^*\mathbb{V}_1 \otimes \mathrm{pr}_2^*\mathbb{V}_2$, for irreducible local systems $\mathbb{V}_i$ on U_i with Higgs bundles (E_i,θ_i). Since $\mathbb{V}$ is a variation of Hodge structures of weight 1, one of those, say $\mathbb{V}_2$ has to have weight zero, hence it must be unitary.

Then the Higgs field on U factors through $E^{0,1}\otimes\Omega^1_{U_1}$. By induction on the dimension we may assume that $\mathbb{V}_1$ is pure of type ι for some ι with M_ι a factor of $\tilde{U}_1$. So the same holds true for $\mathbb{V}$.

Hence we may assume that U is irreducible, or even that

(6.6) no finite étale covering of U is a product of proper subvarieties.

By [**Zi84**] § 2.2, replacing Γ by a subgroup of finite index, hence replacing U by a finite unramified cover, there is a partition of $\{1,\dots,s\}$ into subsets I_k such that $\Gamma = \prod_k \Gamma_k$ and Γ_k is an irreducible lattice in $\prod_{i\in I_k} G_i$. Here irreducible means that for any $N \subset \prod_{i\in I_k} G_i$ a normal subgroup, Γ_k is dense in $\prod_{i\in I_k} G_i/N$. The condition (6.6) is equivalent to the irreducibility of Γ, so $I_1 = \{1,\dots,s\}$.

If $s=1$ or if $\mathbb{V}$ is unitary, the statement of the proposition is trivial. Otherwise, $G := \prod_{i=1}^s G_i$ is of real rank ≥ 2 and the conditions of Margulis' superrigidity theorem (e.g. [**Zi84**, Theorem 5.1.2 ii)]) are met. As consequence, the homomorphism $\Gamma \to \mathrm{Sp}(V,Q)$, where V is a fibre of $\mathbb{V}$ and where Q is the symplectic form on V, factors through a representation $\rho : G \to \mathrm{Sp}(V,Q)$. Since the G_i are simple, we can repeat the argument used in the proof of [**VZ05**, Proposition 3.3] in the product case: ρ is a tensor product of representations, all of which but one have weight 0. q.e.d.

Corollary 6.10. *Under the assumptions made in Proposition* 6.9 *let* $\mathcal{Q} \neq 0$ *be a quotient of* $E^{1,0}$ *with* $\mu_{\underline{D}^{(i)}}(\mathcal{Q}) = 0$*, for some* $i \in \{1, \ldots, s\}$*. Then* $\mathcal{Q} = E^{1,0}$*.*

Proof. By Proposition 6.9 $\mathbb{V}$ is pure of type ι for some ι. On the other hand Proposition 5.6 implies that the orthogonal complement of $\mathcal{Q}$ lies in the kernel of the composite

$$E^{1,0} \xrightarrow{\theta} E^{0,1} \otimes \Omega^1_Y(\log S) \xrightarrow{\mathrm{pr}_i} E^{0,1} \otimes \Omega_i.$$

Since θ is injective and factors through $E^{0,1} \otimes \Omega_\iota$ this implies that $i = \iota$ and we assume that both are 1.

Now one argues as in the proof of Proposition 5.6. The metric connection ∇_h is zero in directions not contained in M_1, hence in the equation (5.3) one finds that $(R_{E^{1,0}})^{i,j} = 0$ as soon as $i > n_1$ or $j > n_1$. Similarly $c^i_{\alpha,\beta} = 0$ for $i > n_1$, and hence the equation (5.4) implies that $(R_{\mathcal{Q}})^{i,j}_{\alpha,\beta} = 0$ for $i > n_1$ or $j > n_1$. One has again

$$\mu_{\underline{D}^{(j)}}(\mathcal{Q}) = \left(\frac{\sqrt{-1}}{2\pi}\right) \cdot \int_U \mathrm{tr}(R_{\mathcal{Q}}) \wedge R(\Omega_1)^{n_j - 1} \wedge R(\Omega_1)^{n_1} \wedge \cdots \\ \wedge R(\Omega_{j-1})^{n_{j-1}} \wedge R(\Omega_{j+1})^{n_{j+1}} \wedge \cdots \wedge R(\Omega_s)^{n_s}.$$

As in equation (5.5) this is the same as

$$\int_W \left(\sum_{i,j=1}^{n} \mathrm{tr}(R_Q)^{i,j} dz_i \wedge d\bar{z}_j \right) \wedge \left(\sum_{i=n_{j-1}+1}^{n_j} C_i \cdot \bigwedge_{j \neq i} dz_j \wedge d\bar{z}_j \right).$$

For $j > 1$ this is zero, hence $\mu(\mathcal{Q}) = 0$ and one can apply Lemma 5.4. q.e.d.

7. Stability of Higgs bundles, lengths of iterated Higgs fields and splitting of the tangent map

In this section we prove Theorem 0.2, the numerical characterization of Shimura varieties, the equivalent numerical and geometrical characterizations of ball quotients stated as Addendum 7.20, the Corollary 7.22 and we finish the proof of Proposition 0.1. Moreover we recall the proof of Corollary 1.3, essentially contained in [**VZ07**, Section 5].

As usual we assume that U has a non-singular projective compactification Y with boundary $S = Y \setminus U$ a normal crossing divisor, satisfying the Assumptions 1.1. In addition, replacing U by an étale covering, we will assume as in Section 2.2 that the family $f : A \to U$ is induced by a generically finite morphism $\varphi : U \to \mathcal{A}_g$ to a fine moduli scheme $\mathcal{A}_g$ of polarized abelian varieties with a suitable level structure.

So we consider again an irreducible non-unitary complex polarized variation of Hodge structures $\mathbb{V}$ on U, satisfying the Arakelov equality, and with unipotent local monodromy operators at infinity.

By Theorem 0.3, the logarithmic Higgs bundle $(E = E^{1,0} \oplus E^{0,1}, \theta)$ of $\mathbb{V}$ is pure of type ι, i.e. the Higgs field factors through $E^{0,1} \otimes \Omega_\iota$. We write $\ell = \mathrm{rk}(E^{1,0})$ and $\ell' = \mathrm{rk}(E^{0,1})$, and n_ι denotes $\mathrm{rk}(\Omega_\iota) = \dim(M_\iota)$. The Arakelov equality says that

$$\mu(E^{1,0}) - \mu(E^{0,1}) = \mu(\Omega^1_Y(\log S)) = \mu(\Omega_\iota).$$

Since $\mathrm{c}_1(E^{1,0}) + \mathrm{c}_1(E^{0,1}) = 0$ and hence $\ell \cdot \mu(E^{1,0}) + \ell' \cdot \mu(E^{0,1}) = 0$, one can restate the Arakelov equality as

$$\frac{\ell + \ell'}{\ell'} \cdot \mu(E^{1,0}) = \mu(\Omega_\iota). \tag{7.1}$$

Let us formulate two easy consequences of the Arakelov equality.

Lemma 7.1. *Assume that each irreducible non-unitary $\mathbb{C}$-subvariation of Hodge structures of $R^1 f_* \mathbb{C}_A$ satisfies the Arakelov equality. Then:*

1) *If φ is generically finite, then for each direct factor Ω_ι of $\Omega^1_Y(\log S)$ there exists at least one non-unitary local subsystem $\mathbb{V}$ which is pure of type ι.*
2) *If $\varphi(U)$ is non-singular, then $\varphi : U \to \varphi(U)$ is étale.*

Proof. Let $F^{1,0}$ be the $(1,0)$-part in the Hodge filtration of $R^1 f_* \mathbb{C}_A$. Since U is generically finite over $\mathcal{A}_g$ the sheaf $\det(f_* \Omega^1_{X/Y}) = f_* \omega_{X/Y}$ is big. Since it is nef, using the slopes introduced in Section 6, one finds by the Property 5.5 2) that

$$\mu_{\underline{D}^{(\iota)}}(f_* \omega_{X/Y}) = g \cdot \mu_{\underline{D}^{(\iota)}}(f_* \Omega^1_{X/Y}) = g \cdot \mu_{\underline{D}^{(\iota)}}(F^{1,0}) > 0$$

for all ι. Consider an irreducible complex polarized subvariation of Hodge structures $\mathbb{V}$ with Higgs bundle $(E^{1,0} \oplus E^{0,1}, \theta)$. If $\mathbb{V}$ is unitary $\mu_{\underline{D}^{(j)}}(E^{1,0}) = 0$ for all j. Otherwise by Theorem 0.3 $\mathbb{V}$ is pure of type $i = i(\mathbb{V})$. Lemma 6.4 implies that $\mu_{\underline{D}^{(j)}}(E^{1,0}) = 0$ for $j \neq i(\mathbb{V})$.

Given ι, the inequality $\mu_{\underline{D}^{(\iota)}}(F^{1,0}) > 0$ implies that there exist direct factors $E^{1,0}$ with $\mu_{\underline{D}^{(\iota)}}(E^{1,0}) > 0$. For the corresponding irreducible subvariations $\mathbb{V}$ of $R^1 f_* \mathbb{C}_A$ one finds $\iota = i(\mathbb{V})$.

For the second statement we choose a nonsingular compactification Z and a normal crossing divisor $\Sigma \subset Z$ with $\varphi(U) = Z \setminus \Sigma$. Let us choose a blowing up $\delta : Y' \to Y$ with centers in S such that $S' = \delta^*(S)$ is again a normal crossing divisor, and such that φ extends to $\varphi : Y' \to Z$. By the Arakelov equality the image $\mathcal{I}$ of $\hat{\tau} : F^{1,0} \otimes F^{0,1^\vee} \to \Omega^1_Y(\log S)$ has the same slope as $\Omega^1_Y(\log S)$. Since the second sheaf is μ-polystable, $\mathcal{I}$ is a subsheaf of a direct sum of certain direct factors of $\Omega^1_Y(\log S)$ and both are μ-equivalent. The first part of Lemma 7.1 implies that all direct factors occur, hence $\mathcal{I} \hookrightarrow \Omega^1_Y(\log S)$ is an isomorphism over some open set U'. The part viii) of Addendum 5.2 allows to choose $U' = U$.

Since $\mathcal{A}_g$ is a fine moduli scheme, the Higgs bundle is the pullback of the Higgs bundle on $\varphi(U)$. Hence $\delta^*(\hat{\tau})$ factors through

$$\varphi^*\Omega^1_Z(\log \Sigma) \longrightarrow \Omega^1_{Y'}(\log S')$$

with image in $\delta^*\Omega^1_Y(\log S) \subset \Omega^1_{Y'}(\log S')$. Since the last inclusion is an isomorphism over U, the surjectivity of the Higgs field on U implies the morphism φ is unramified on U. q.e.d.

Let us return to the Higgs bundle $\bigwedge^\ell(E,\theta)$ introduced in (1.5) and to the Higgs subbundle $\langle \det(E^{1,0})\rangle$ generated by $\det(E^{1,0})$. From now on we will write $\langle \det(E^{1,0})\rangle$ for the saturated Higgs subbundle of $\bigwedge^\ell(E,\theta)$. So $\langle \det(E^{1,0})\rangle^{\ell-m,m}$ denotes the saturated hull of the image of the induced map

$$\theta^{(m)^\vee} : \det(E^{1,0}) \otimes S^m(T) \longrightarrow E^{\ell-m,m} = \bigwedge^{\ell-m}(E^{1,0}) \otimes \bigwedge^{m} E^{0,1}.$$

Note that by this change of notation we neither change the slopes, nor the length

$$\varsigma(E) = \varsigma((E,\theta)) = \mathrm{Max}\{\ m \in \mathbb{N};\ \langle \det(E^{1,0})\rangle^{\ell-m,m} \neq 0\}.$$

So the next Lemma implies Corollary 1.3.

Lemma 7.2. *Assume that Ω_ι is of type A or B, hence that $S^m(\Omega_\iota)$ is μ-stable for all m. Then the Arakelov equality implies that*

$$\mathrm{Min}\{\ell,\ell'\} \geq \varsigma(E) \geq \frac{\ell\cdot\ell'\cdot(n_\iota+1)}{(\ell+\ell')\cdot n_\iota}. \tag{7.2}$$

The right hand side of 7.2 is an equality if and only if $\langle \det(E^{1,0})\rangle$ is a direct factor of $\bigwedge^\ell(E,\theta)$.

Proof. For $0 \leq m \leq \varsigma = \varsigma(E)$ the sheaf $\langle \det(E^{1,0})\rangle^{\ell-m,m}$ is a μ-stable sheaf of slope

$$(\ell-m)\cdot\mu(E^{1,0}) + m\cdot\mu(E^{0,1}) = \ell\cdot\mu(E^{1,0}) - m\cdot\mu(\Omega^1_Y(\log S)) = \left(\frac{\ell\cdot\ell'}{\ell+\ell'} - m\right)\cdot\mu(\Omega^1_Y(\log S)),$$

and of rank $\binom{n_\iota+m-1}{m}$. The degree of this sheaf with respect to the polarization $\omega_Y(S)$ is non-positive, hence

$$(7.3)\quad 0 \geq \frac{\mu(\langle\det(E^{1,0})\rangle)}{\mu(\Omega^1_Y(\log S))} = \sum_{m=0}^{\varsigma} \binom{n_\iota+m-1}{m}\cdot\left(\frac{\ell\cdot\ell'}{\ell+\ell'}-m\right) = \left(\frac{\ell\cdot\ell'}{n_\iota\cdot(\ell+\ell')}-\frac{\varsigma}{n_\iota+1}\right)\cdot(\varsigma+1)\cdot\binom{\varsigma+n_\iota}{\varsigma+1},$$

and one obtains the second inequality stated in 7.2. This is an equality if and only if (7.3) is an equality. By Simpson's correspondence for polystable Higgs bundles the latter holds if and only if $\langle\det(E^{1,0})\rangle$ is a Higgs direct factor of $\bigwedge^\ell(E,\theta)$. The first inequality in 7.2 is obvious, since $E^{\ell-m,m}$ is zero for $m \geq \mathrm{Min}\{\ell,\ell'\}$. q.e.d.

We now distinguish three cases, according to the type of the bounded symmetric domain attached to Ω_ι.

7.1. Type A: Ω_ι is invertible. This case is easy to understand. Let us recall the arguments used already in [**VZ07**]. The Arakelov equality and Lemma 5.4 imply that

$$(7.4)\qquad E^{1,0} \longrightarrow E^{0,1}\otimes\Omega_\iota,$$

is injective and surjective on some open dense subscheme. So $\ell=\ell'$ and the inequality (7.2) implies that $\varsigma((E,\theta))=\ell$.

Both sides in (7.4) are μ-semistable of the same slope, and they are μ-equivalent. A μ-stable subsheaf $\mathcal{F}$ of $E^{1,0}$ of slope $\mu(E^{1,0})$ generates a Higgs subbundle $\mathcal{F}\oplus\mathcal{F}\otimes T_\iota$, whose first Chern class is zero. So the irreducibility implies that $\mathcal{F}=E^{1,0}$ and we can state:

Proposition 7.3. *If Ω_ι is invertible, then the Arakelov equality (7.1) implies that $E^{1,0}$ and $E^{0,1}$ are both μ-stable of the same rank, that $\varsigma((E,\theta))=\ell$ and that $\langle\det(E^{1,0})\rangle$ is a direct factor of $\bigwedge^\ell(E,\theta)$.*

7.2. Type B: $S^m(\Omega_\iota)$ stable for all m and not invertible. In this case we do not know whether the factor M_ι of the universal covering $\tilde{U}$ corresponding to Ω_ι is a bounded domain, and the Arakelov equality just implies that certain numerical and stability conditions are equivalent.

Proposition 7.4. *Let $\mathbb{V}$ be an irreducible non-unitary complex polarized variation of Hodge structures of weight 1, pure of type A or B, with unipotent local monodromy at infinity, and with Higgs bundle (E,θ). Assume that $\mathbb{V}$ satisfies the Arakelov equality. Consider the following conditions:*

a. *$E^{1,0}$ and $E^{0,1}$ are μ-stable.*
b. *${E^{1,0}}^\vee\otimes E^{0,1}$ is μ-polystable.*
c. *The saturated image of $T_\iota \to \mathcal{H}om(E^{1,0},E^{0,1})$ is a direct factor of the sheaf $\mathcal{H}om(E^{1,0},E^{0,1})$.*

d. *The Higgs bundle* $\langle \det(E^{1,0})\rangle$ *is a direct factor of the Higgs bundle* $\bigwedge^\ell(E,\theta)$.
e. $\mu(\langle \det(E^{1,0})\rangle) = 0$.
f. $\varsigma((E,\theta)) = \frac{\ell\cdot\ell'\cdot(n_\iota+1)}{(\ell+\ell')\cdot n_\iota}$.

Then:

i. *The conditions c), d), e), and f) are equivalent and they imply that* M_ι *is a complex ball of dimension* n_ι.
ii. *The condition b) implies c).*
iii. *Whenever the condition* $(\star)$ *is satisfied, for example if* U *is projective or of dimension one, a) implies b).*

If $\mathbb{V}$ is pure of type A, we know that the conditions a), d), and f) automatically hold true. Nevertheless we included this case in the statement , since we will later refer to the equivalence between c) and f).

Proof of Proposition 7.4. The stability of $E^{1,0}$ implies the one for $E^{1,0^\vee}$, and hence b) follows from a) and from $(\star)$.

For part ii) remark that the Arakelov equality says that

$$\mu(T_\iota) = \mu(\mathcal{H}om(E^{1,0}, E^{0,1})).$$

So c) is a consequence of b).

By Simpson's correspondence d) and e) are equivalent, and the numerical condition in f) is equivalent to d) by Lemma 7.2. So for i) it remains to verify the equivalence of c) and d).

Claim 7.5. The condition d) implies c).

Proof. The inclusion $\langle \det(E^{1,0})\rangle^{\ell-1,1} \to E^{\ell-1,1}$ is given by $T_\iota \to \mathcal{H}om(E^{1,0}, E^{0,1})$, tensorized with $\det(E^{1,0})$. So Condition d) implies that the saturated image of $\langle \det(E^{1,0})\rangle^{\ell-1,1}$ is a direct factor of $E^{\ell-1,1} = E^{1,0^\vee} \otimes E^{0,1} \otimes \det(E^{1,0})$, hence that c) holds. q.e.d.

Remark 7.6. The implication 'c) implies d)' has been claimed in [**VZ07**, page 327] in a more special situation. There however, as pointed out by the referee of the present article, the argument is not complete. We did not verify that the image of $\Phi_{m+1} \circ \theta^{\ell-m,m}$ really lies in $\langle \det(E^{1,0})\rangle^{\ell-m,m}$. This can easily be done, using Claim 7.7 below and the property $(**)$ on page 294 of [**VZ07**]. Here, without using the last condition, we will work out the argument in details without further reference to [**VZ07**].

Let us write

$$E^{\ell-m,m} = \bigwedge^{\ell-m}(E^{1,0}) \otimes \bigwedge^{m}(E^{0,1}) \cong \bigwedge^{m}(E^{1,0^\vee}) \otimes \bigwedge^{m}(E^{0,1}) \otimes \det(E^{1,0}).$$

Using the right hand isomorphism we will regard $E^{\ell-m,m}$ as a subsheaf of

$$S^m(E^{1,0^\vee}\otimes E^{0,1})\otimes \det(E^{1,0}).$$

Then the dual Higgs field $\theta^\vee_{\ell-m,m}: E^{\ell-m,m}\otimes T_\iota \to E^{\ell-m-1,m+1}$ is given by a quotient of the multiplication map

$$S^m(E^{1,0^\vee}\otimes E^{0,1})\otimes (E^{1,0^\vee}\otimes E^{0,1}) \longrightarrow S^{m+1}(E^{1,0^\vee}\otimes E^{0,1}).$$

tensored with $\det(E^{1,0})$ and restricted to $E^{\ell-m,m}\otimes T_\iota$. Since the slope is additive for tensor products $\mu(E^{\ell-m,m})$ is equal to $(\ell-m)\cdot\mu(E^{1,0})+m\cdot\mu(E^{0,1})$. The Arakelov equality implies that

$$\mu(E^{\ell-m,m}) = m\cdot\mu(T_\iota)+\ell\cdot\mu(E^{1,0}) = m\cdot\mu(T_\iota)+\mu(\det(E^{1,0})). \tag{7.5}$$

Claim 7.7. Let V be a μ-semistable subsheaf of $E^{\ell-m,m}$ of slope of $\mu(E^{\ell-m,m})$. Assume that for some $b>0$ there exists a morphism

$$E^{\ell-m-1,m+1}\longrightarrow S^{m+1}(T_\iota)^{\oplus b}\otimes\det(E^{1,0})$$

such that the composition

$$\gamma'_m : V\otimes T_\iota \xrightarrow{\subset} E^{\ell-m,m}\otimes T_\iota \longrightarrow E^{\ell-m-1,m+1}\longrightarrow S^{m+1}(T_\iota)^{\oplus b}\otimes \det(E^{1,0})$$

is surjective up to μ-equivalence, as defined in Definition 4.2. Then there exists a morphism

$$E^{\ell-m,m}\to S^m(T_\iota)^{\oplus b}\otimes\det(E^{1,0}),$$

whose restriction $\gamma_m : V\to S^m(T_\iota)^{\oplus b}\otimes\det(E^{1,0})$ induces γ'_m, in the sense that γ'_m is the composite of $\gamma_m\otimes \mathrm{id}_{T_\iota}$ with the multiplication map

$$S^m(T_\iota)^{\oplus b}\otimes T_\iota\otimes\det(E^{1,0})\longrightarrow S^{m+1}(T_\iota)^{\oplus b}\otimes\det(E^{1,0}).$$

In particular γ_m is again surjective up to μ-equivalence.

Proof. The morphism γ'_m is generically surjective, hence one has a generically surjective morphism

$$\gamma'_m\otimes\mathrm{id}_{\Omega_\iota} : V\otimes T_\iota\otimes\Omega_\iota \longrightarrow S^{m+1}(T_\iota)^{\oplus b}\otimes\Omega_\iota\otimes\det(E^{1,0}),$$

factoring through $E^{\ell-m,m}\otimes T_\iota\otimes\Omega_\iota$. Restricting to $V\subset V\otimes T_\iota\otimes\Omega_\iota$ and composing with the natural contraction map

$$\alpha_m : S^{m+1}(T_\iota)^{\oplus b}\otimes\Omega_\iota\otimes\det(E^{1,0})\longrightarrow S^m(T_\iota)^{\oplus b}\otimes\det(E^{1,0})$$

one gets $\gamma_m : V\xrightarrow{\subset} E^{\ell-m,m}\longrightarrow S^m(T_\iota)^{\oplus b}\otimes\det(E^{1,0})$. By construction γ'_m is the restriction of the composition

$$V\otimes T_\iota \xrightarrow{\subset} V\otimes T_\iota\otimes\Omega_\iota\otimes T_\iota \xrightarrow{\gamma'_m\otimes\mathrm{id}_{\Omega_\iota\otimes T_\iota}}$$
$$S^{m+1}(T_\iota)^{\oplus b}\otimes\Omega_\iota\otimes T_\iota\otimes\det(E^{1,0})\longrightarrow S^{m+1}(T_\iota)^{\oplus b}\otimes\det(E^{1,0}),$$

where the last arrow is the map $\mathrm{id}_{S^{m+1}(T_\iota)^{\oplus b}}\otimes\alpha\otimes\mathrm{id}_{\det(E^{1,0})}$ and where $\alpha : T_\iota\otimes\Omega_\iota\to\mathcal{O}_Y$ denotes again the contraction map. The last of the morphisms is up to the tensor product with the identity on $\det(E^{1,0})$ a direct sum of morphisms factoring like

$$S^{m+1}(T_\iota)\otimes\Omega_\iota\otimes T_\iota \xrightarrow{\alpha_m\otimes \mathrm{id}_{T_\iota}} S^m(T_\iota)\otimes T_\iota \xrightarrow{\mathrm{mult}} S^{m+1}(T_\iota).$$

So one obtains γ'_m as the composite of γ_m with the multiplication map. In particular γ_m is the direct sum of non-zero morphisms and the stability of $S^m(T_\iota)$ implies that the image of γ_m is μ-equivalent to $S^m(T_\iota)^{\oplus b}\otimes\det(E^{1,0})$. q.e.d.

Let us return to the notations introduced in the first part of Section 5. In particular NS_0 denotes the subgroup of the Neron-Severi group $NS(Y)_{\mathbb{Q}}$ generated by prime-divisors D with $\mu(\mathcal{O}_Y(D))=0$, and U' is the complement of those prime-divisors.

Let us write $\mathcal{S}'^{\ell-m,m}$ for the cosocle of $E^{\ell-m,m}$. As remarked in the Example and Definition 5.3 it is a μ-polystable sheaf of slope $\mu(E^{\ell-m,m})$ of maximal rank, for which there exists a morphism $\theta: E^{\ell-m,m}\to \mathcal{S}'^{\ell-m,m}$, which is surjective over some open set. Using parts vii) and ix) of the Addendum 5.2 one finds that θ is surjective over U'.

Let $\mathcal{S}^{\ell-m,m}$ be the direct sum of all direct factors of $\mathcal{S}'^{\ell-m,m}$, which are μ-equivalent to the μ-stable sheaf $S^m(T_\iota)\otimes\det(E^{1,0})$. Remark that $\mathcal{S}^{\ell-m,m}$ is not unique. By Addendum 5.2 vii) we may choose an effective divisor $B_m\in\mathrm{NS}_0$ such that for some b_m

$$S^m(T_\iota)^{\oplus b_m}\otimes\det(E^{1,0})\hookrightarrow \mathcal{S}^{\ell-m,m}=S^m(T_\iota)^{\oplus b_m}\otimes\det(E^{1,0})\otimes\mathcal{O}_Y(B_m).$$

In particular both sheaves are μ-equivalent. Let us denote the induced morphism by $\beta_m: E^{\ell-m,m}\to\mathcal{S}^{\ell-m,m}$.

As a next step, we will show by induction on m, that for a suitable choice of the divisors B_m the dual Higgs field $\theta^\vee_{\ell-m,m}$ defines a morphism $\mathcal{S}^{\ell-m,m}\to\mathcal{S}^{\ell-m-1,m+1}$. The induction step is given by:

Claim 7.8. We assume that c) holds (and of course the Arakelov equality). Then choosing the effective divisor $B_{m+1}\in\mathrm{NS}_0$ and hence $\mathcal{S}^{\ell-m-1,m+1}$ large enough, there exists a commutative diagram

$$\begin{array}{ccc} E^{\ell-m,m}\otimes T_\iota & \xrightarrow{\beta_m\otimes\mathrm{id}_{T_\iota}} & \mathcal{S}^{\ell-m,m}\otimes T_\iota \\ {\scriptstyle\theta^\vee_{\ell-m,m}}\big\downarrow & & \big\downarrow{\scriptstyle\tau^\vee} \\ E^{\ell-m-1,m+1} & \xrightarrow{\beta_{m+1}} & \mathcal{S}^{\ell-m-1,m+1}. \end{array} \tag{7.6}$$

The morphism $\tau^\vee$ has an explicit description. For simplicity we just formulate this on the open subscheme U'. Part vii) of Addendum 5.2 allows to extend this description to the boundary, perhaps after replacing B_{m+1} by a larger divisor in NS_0.

Claim 7.9. For some morphism $\tau'_m: S^m(T_\iota)^{\oplus b_m}\to S^m(T_\iota)^{\oplus b_{m+1}}$ the morphism $\tau^\vee|_{U'}$ is the composite of $\tau'_m\otimes\mathrm{id}_{T_\iota}|_{U'}$ and the direct product of b_{m+1} copies of the multiplication map $S^m(T_\iota)\otimes T_\iota|_{U'}\to S^{m+1}(T_\iota)|_{U'}$.

Proof of the Claims 7.8 *and* 7.9. By the Arakelov equality, as restated in 7.5 and by the choice of the sheaves $\mathcal{S}^{\ell-\bullet,\bullet}$ the four sheaves in 7.6 all have the same slope and they are all μ-semistable. By Addendum 5.2 for each of the morphisms the image coincides with the saturated image over the open set U'. In particular the restriction of β_m and β_{m+1} to U' is surjective.

Writing V_m for the kernel of β_m, hence $V_m \otimes T_\iota$ for the one of $\beta_m \otimes \mathrm{id}_{T_\iota}$, consider the image $\mathcal{I}$ of $V_m \otimes T_\iota$ under $\theta^\vee_{\ell-m,m}$. We claim that $\mathcal{I}$ is contained in V_{m+1}.

If not $\beta_{m+1} \circ \theta^\vee_{\ell-m,m}(V_m \otimes T_\iota)$ is a non-zero subsheaf of $\mathcal{S}^{\ell-m-1,m+1}$. By Addendum 5.2, ix) its saturated hull is a μ-semistable subsheaf of $\mathcal{S}^{\ell-m-1,m+1}$. Since both are of the same slope, and since the second one is μ-polystable, the saturated image has to be a direct factor, hence isomorphic to $S^{m+1}(T_\iota)^b \otimes \det(E^{1,0}) \otimes \mathcal{O}_Y(B_{m+1})$ for some $b > 0$.

By Claim 7.7 one obtains a morphism $E^{\ell-m,m} \to S^m(T_\iota)^b \otimes \det(E^{1,0}) \otimes \mathcal{O}_Y(B_{m+1})$ whose restriction to V_m is non-zero. Obviously this contradicts the definition of $\mathcal{S}^{\ell-m-1,m+1}$ as a maximal μ-polystable quotient and of V_m as the kernel of β_m.

The restriction of $\beta_m \otimes \mathrm{id}_{T_\iota}$ to U' is surjective. Since $\theta^\vee_{\ell-m,m}(\mathcal{I}) \subset V_{m+1}$, the morphism $\tau^\vee$ exists on U', and enlarging B_{m+1} it extends to Y.

In order to get the explicit description stated in Claim 7.9, we apply Claim 7.7 to $V = E^{\ell-m,m}$. The image of $\gamma'_m = \beta_{m+1} \circ \theta^\vee_{\ell-m,m}$ is a μ-semistable subsheaf of the μ-polystable sheaf $\mathcal{S}^{\ell-m-1,m+1}$, hence μ-equivalent to a direct factor of the form $S^{m+1}(T_\iota)^{\oplus b} \otimes \det(E^{1,0})$. Claim 7.7 implies that for some

$$\gamma_m : E^{\ell-m,m} \to S^m(T_\iota)^{\oplus b} \otimes \det(E^{1,0})$$

the morphism $\beta_{m+1} \circ \theta^\vee_{\ell-m,m}$ is the composite of $\gamma_m \otimes \mathrm{id}_{T_\iota}$ with the multiplication map. Since γ_m factors through the cosocle $\mathcal{S}'^{\ell-m,m}$ and hence through $\mathcal{S}^{\ell-m,m}$, one finds the morphism τ'_m. q.e.d.

Recall that $\langle\det(E^{1,0})\rangle$ is the saturated subsheaf of $E^{\ell-m,m}$ which is generated by $\det(E^{1,0})$. If non-zero $\langle\det(E^{1,0})\rangle^{\ell-m,m}$ contains $S^m(T_\iota) \otimes \det(E^{1,0})$ and both are μ-equivalent. As a next step we will show that $\langle\det(E^{1,0})\rangle^{\ell-m,m}|_{U'}$ is a direct factor of $E^{\ell-m,m}|_{U'}$.

Claim 7.10. Assume c). Then the composite

$$\langle\det(E^{1,0})\rangle^{\ell-m,m} \xrightarrow{\subset} E^{\ell-m,m} \xrightarrow{\beta_m} \mathcal{S}^{\ell-m,m}$$

is injective and defines a splitting of the inclusion $\langle\det(E^{1,0})\rangle^{\ell-m,m}|_{U'} \subset E^{\ell-m,m}|_{U'}$.

Proof. If $\langle\det(E^{1,0})\rangle^{\ell-m,m} = 0$ there is nothing to show. Otherwise by the equality 7.5 $\mu(\langle\det(E^{1,0})\rangle^{\ell-m,m}) = \mu(E^{\ell-m,m})$ and by part ix) of the Addendum 5.2 $\langle\det(E^{1,0})\rangle^{\ell-m,m}$ is a μ-semistable subsheaf of $E^{\ell-m,m}$, containing $S^m(T_\iota)\otimes\det(E^{1,0})$ as a μ-equivalent subsheaf.

Recall that T_ι is a direct factor of $E^{1,0^\vee}\otimes E^{0,1}$, and hence $S^m(T_\iota)$ a direct factor of $S^m(E^{1,0^\vee}\otimes E^{0,1})$. This sheaf also contains $\bigwedge^m(E^{1,0^\vee})\otimes\bigwedge^m(E^{0,1})$ as a direct factor. Writing

$$S^m(E^{1,0^\vee}\otimes E^{0,1})=\bigwedge^m(E^{1,0^\vee})\otimes\bigwedge^m(E^{0,1})\oplus R_m,$$

consider the image of $S^m(T_\iota)$ under the projection $S^m(E^{1,0^\vee}\otimes E^{0,1})\to R_m$. If this is zero we are done. If not one has an injection

$$\alpha' : S^m(T_\iota)\oplus S^m(T_\iota)\longrightarrow\bigwedge^m(E^{1,0^\vee})\otimes\bigwedge^m(E^{0,1})\oplus R_m,$$

where the first factor maps to $\bigwedge^m(E^{1,0^\vee})\otimes\bigwedge^m(E^{0,1})$ and the second one to R_m. For both factors the composite with the projection

$$\bigwedge^m(E^{1,0^\vee})\otimes\bigwedge^m(E^{0,1})\oplus R_m\longrightarrow S^m(T_\iota)$$

is non-zero, hence by μ-semistability it is surjective up to μ-equivalence. So α' splits, and $S^m(T_\iota)$ as the image of the composite of α' with the projection to $\bigwedge^m(E^{1,0^\vee})\otimes\bigwedge^m(E^{0,1})$, splits as well.

Since $\langle\det(E^{1,0})\rangle^{\ell-m,m}|_{U'}$ is by definition the image of the bundle $S^m(T_\iota)\otimes\det(E^{1,0})|_{U'}$ in

$$E^{\ell-m,m}|_{U'}=\bigwedge^m(E^{1,0^\vee})\otimes\bigwedge^m(E^{0,1})\otimes\det(E^{1,0})|_{U'},$$

it is a direct factor, hence its image in the cosocle is non-zero. By the choice of $\mathcal{S}^{\ell-m,m}$ we are done. q.e.d.

Claim 7.11. The condition c) implies d).

Proof. Writing τ for the composite of $\tau^\vee\otimes\Omega_\iota$ with the contraction to $\mathcal{S}^{\ell-m-1,m+1}$ one obtains by Claim 7.8 a Higgs bundle

$$(\mathcal{S},\tau)=\Big(\bigoplus_{m=0}^{\ell}\mathcal{S}^{\ell-m,m},\bigoplus_{m=0}^{\ell-1}\big(\mathcal{S}^{\ell-m,m}\xrightarrow{\ \tau\ }\mathcal{S}^{\ell-m,m}\otimes\Omega_\iota\ \big)\Big)$$

together with a map of Higgs bundles

$$\bigwedge^\ell(E,\theta)=\big(\bigoplus_{m=0}^{\ell}E^{\ell-m,m},\theta\big)\xrightarrow{\ \beta\ }(\mathcal{S},\tau). \tag{7.7}$$

For $\varsigma=\varsigma((E,\theta))$ the sheaf

$$\langle\det(E^{1,0})\rangle=\bigoplus_{m=0}^{\varsigma}\langle\det(E^{1,0})\rangle^{\ell-m,m}=\bigoplus_{m=0}^{\ell}\langle\det(E^{1,0})\rangle^{\ell-m,m}$$

is a Higgs subbundle of the left hand side of 7.7, hence its saturated image

$$(\langle \widetilde{\det(E^{1,0})}\rangle, \tau|_{\langle \widetilde{\det(E^{1,0})}\rangle})$$

in the right hand side is a Higgs subbundle of $(\mathcal{S}, \tau)$. By Claim 7.10 the induced map

$$\langle \det(E^{1,0})\rangle^{\ell-m,m} \longrightarrow \langle \widetilde{\det(E^{1,0})}\rangle^{\ell-m,m}$$

is injective and both sheaves are μ-equivalent. Since $\langle \det(E^{1,0})\rangle^{\ell-m,m}$ is μ-stable, $\langle \widetilde{\det(E^{1,0})}\rangle^{\ell-m,m}$ is μ-equivalent to a direct factor of $\mathcal{S}^{\ell-m,m}$ that we denote by $\langle \widehat{\det(E^{1,0})}\rangle^{\ell-m,m}$. By the explicit description of $\tau^\vee$ in Claim 7.9 the Higgs field τ respects the splitting, and one obtains a quotient Higgs bundle $\langle \widehat{\det(E^{1,0})}\rangle$ of $(\mathcal{S}, \tau)$, hence of $\bigwedge^\ell(E, \theta)$. Since $\langle \det(E^{1,0})\rangle$ is a sub-Higgs bundle Lemma 5.4 implies that

$$\mu(\langle \det(E^{1,0})\rangle) \leq 0,$$

and since $\langle \widehat{\det(E^{1,0})}\rangle$ is a quotient-Higgs bundle, $\mu(\langle \widehat{\det(E^{1,0})}\rangle) \geq 0$. So the μ-equivalence of all direct factors implies that

$$\mu(\langle \det(E^{1,0})\rangle) = \mu(\langle \widehat{\det(E^{1,0})}\rangle) = 0.$$

Using Lemma 5.4 again one finds that $\langle \det(E^{1,0})\rangle$ splits as a Higgs subbundle of $\bigwedge^\ell(E, \theta)$. q.e.d.

To finish the proof of Proposition 7.4 it remains to verify:

Claim 7.12. The splitting in d) implies that M_ι is an n_ι-dimensional complex ball.

Proof. (See also [**VZ07**, Section 5]) The Higgs bundle $\langle \det(E^{1,0})\rangle$ splits as a sub-Higgs bundle of $\bigwedge^\ell E$, hence it is itself a Higgs bundle arising from a local system. In particular the Chern classes $\mathrm{c}_1(\langle \det(E^{1,0})\rangle)$ and $\mathrm{c}_2(\langle \det(E^{1,0})\rangle)$ are zero.

Assume for a moment that there exists an invertible sheaf $\mathcal{L}$ with $\det(E^{1,0}) = \mathcal{L}^\ell$, and consider the Higgs bundle

$$\big(F = F^{1,0} \oplus F^{0,1} = \mathcal{L} \oplus \mathcal{L} \otimes T_\iota, \mathcal{L} \to \mathcal{L} \otimes T_\iota \otimes \Omega_\iota\big).$$

Then $S^\ell(F)$ is a Higgs bundle with $\mathcal{L}^\ell \otimes S^m(T_\iota)$ in bidegree $(\ell - m, m)$, hence isomorphic to $\langle \det(E^{1,0})\rangle$. The first Chern class of $\langle \det(E^{1,0})\rangle$ is zero, hence $\mathrm{c}_1(F)$ as well. On the other hand,

$$\mathrm{c}_1(F) = \mathrm{c}_1(\mathcal{L}) + n_\iota \cdot \mathrm{c}_1(\mathcal{L}) - \mathrm{c}_1(\Omega_\iota) = \frac{n_\iota + 1}{\ell}\mathrm{c}_1(E^{1,0}) - \mathrm{c}_1(\Omega_\iota),$$

and $\mathrm{c}_1(\mathcal{L}) = \dfrac{1}{n_\iota + 1}\mathrm{c}_1(\Omega_\iota)$. For the second Chern class it is easier to calculate the discriminant

$$\Delta(\mathcal{F}) = 2 \cdot \mathrm{rk}(\mathcal{F}) \cdot \mathrm{c}_2(\mathcal{F}) - (\mathrm{rk}(\mathcal{F}) - 1) \cdot \mathrm{c}_1(\mathcal{F})^2.$$

By [**VZ07**, Lemma 3.3], a), the discriminant is invariant under tensor products with invertible sheaves, hence $\Delta(\mathcal{L}\oplus\mathcal{L}\otimes T_\iota)=\Delta(\mathcal{O}_Y\oplus T_\iota)$.

Since $\mathrm{c}_1(\langle\det(E^{1,0})\rangle)^2=\mathrm{c}_2(\langle\det(E^{1,0})\rangle)=0$, as a consequence one finds $\Delta(\langle\det(E^{1,0})\rangle)=0$, and [**VZ07**, Lemma 3.3] implies that $\Delta(F)=0$, hence

$$0=\Delta(\mathcal{O}_Y\oplus T_\iota)=2\cdot(n_\iota+1)\cdot\mathrm{c}_2(T_\iota)-n_\iota\cdot\mathrm{c}_1(T_\iota)^2. \tag{7.8}$$

In general on may choose a finite covering $\sigma: Y'\to Y$ such that $\sigma^*(\det(E^{1,0})=\mathcal{L}'^{\ell}$ for some invertible sheaf $\mathcal{L}'$. Repeating the calculations of Chern classes with T_ι replaced by $T'_\iota=\sigma^*(T_\iota)$ one obtains that $2\cdot(n_\iota+1)\cdot\mathrm{c}_2(T'_\iota)-n_\iota\cdot\mathrm{c}_1(T'_\iota)^2=0$ and again (7.8) holds true.

By Yau's Uniformization Theorem, recalled in [**VZ07**, Theorem 1.4], (7.8) implies that M_ι is a complex ball. q.e.d.

The Proposition 7.4 gives a numerical condition on the length of the wedge product of the Higgs field which, together with the Arakelov equality, implies that M_ι is a complex ball. A similar condition holds automatically for local systems which are pure of type A. This is not surprising, since in this case the corresponding factor M_ι automatically is a 1-dimensional ball.

In slight abuse of notation we say that a local system $\mathbb{V}$ is given by a wedge product of the standard representation of $\mathrm{SU}(1,n)$, if the representation defining $\mathbb{V}$ factors through one of the standard wedge product representations (e.g. [**Sa80**], p. 461)

$$\bigwedge^k:\ \mathrm{SU}(1,n)\to SU\left(\binom{n}{k-1},\binom{n}{k}\right)$$

and if, moreover, the period map for $\mathbb{V}$ factors through the (totally geodesic) embedding of symmetric spaces attached to $\bigwedge^k$. In different terms, for $k=1$ the corresponding Higgs field is given by

$$E^{1,0}=\omega_i^{-\frac{1}{n_i+1}}\otimes\Omega_i,\quad E^{0,1}=\omega_i^{-\frac{1}{n_i+1}}$$

and

$$\theta=\mathrm{id}:\omega_i^{-\frac{1}{n_i+1}}\otimes\Omega_i\longrightarrow\omega_i^{-\frac{1}{n_i+1}}\otimes\Omega_i,$$

where $\omega_i^{-\frac{1}{n_i+1}}$ stands for an invertible sheaf, whose (n_i+1)-st power is $\det(\Omega_i)$.

Proposition 7.13. *Let $\mathbb{V}$ be an irreducible complex polarized variation of Hodge structures of weight* 1*, pure of type ι, with unipotent local monodromy at infinity, and with Higgs bundle (E,θ). Assume that Ω_ι is of type A or B, and that the saturated image of $T_\iota\to\mathcal{H}om(E^{1,0},E^{0,1})$ splits.*

Then $\mathbb{V}$ is the tensor product of a unitary representation with a wedge product of the standard representation of $\mathrm{SU}(1,n)$. *In particular the* *period map $\tau : \tilde{U} \to M'$ factors as the projection $\tilde{U} \to M_\iota$ and a totally geodesic embedding $M_\iota \to M'$.*

Proof. Proposition 7.4, i) implies that M_ι is a complex ball.

Before we proceed, we fix some notation. For a simply connected complex space we denote by $\mathrm{Aut}(M)$ the group of biholomorphic self-mappings of M. This coincides with the definition in Section 3 if M is a hermitian symmetric domain. We write, as usual, $\tilde{U} = \prod_k M_k$ and fix origins o_k in all M_k. Let $G_k := \mathrm{Aut}(M_k)$, $K_k := \mathrm{Stab}(o_k)$. Thus for hermitian symmetric domains M_k we have hence $M_k = G_k/K_k$.

Let $\tau : \tilde{U} \to M'$ be the period map for the bundle $\mathbb{V}$. In M' fix an origin o', let $G' := \mathrm{Aut}(M') \cong \mathrm{SU}(\ell, \ell')$, and let $K' := \mathrm{Stab}(o')$. By the purity of the Higgs bundle, τ factors as the projection $\tilde{U} \to M_\iota$ composed with a map $\tau_1 : M_\iota \to M'$.

The next claim derives the second statement from the main hypothesis. Remember that, since the splitting $T_\iota \to \mathcal{H}om(E^{1,0}, E^{0,1})$ comes from a splitting of Higgs bundles, it is orthogonal for the Hodge metric, hence for the Kähler metric.

Claim 7.14. Let $\tau_1 : M_\iota \to M' = G'/K'$ be a holomorphic map to a hermitian symmetric domains. Assume that $\tau_1^* T_{M'} = T_{M_\iota} \oplus R$ is a holomorphic splitting, orthogonal with respect to the Kähler metric on M'. Then $M_\iota \to M'$ is a totally geodesic embedding and M_ι a symmetric domain.

Proof. (From a letter by N. Mok.) First, the splitting condition on $\tau_1^* T_{M'}$ implies that τ_1 is locally an embedding. Second, we check that the image $\tau_1(M_\iota)$ is totally geodesic in M'. This is again a local condition. By [**He62**, Theorem I.14.5] it suffices to check that the splitting $T_{M'}|_{\tau_1(M_\iota)} = T_M \oplus R$ is preserved under parallel transport.

Take any local holomorphic sections s of T_{M_ι} and t of R. Then $\langle s, t \rangle = 0$ with respect to the Hermitian inner product. The derivative of t with respect to a $(1, 0)$ vector is orthogonal to s because s is holomorphic and because $\langle \cdot, \cdot \rangle$ is Hermitian bilinear. Since s and t are arbitrary, it follows that R is invariant under differentiation in the $(1, 0)$ direction. But since R is a holomorphic subbundle, it is invariant under differentiation in the $(0, 1)$-direction. As a consequence R is parallel, and hence its orthogonal complement T_{M_ι} is parallel, too.

Finally, since M' is a global symmetric domain, it has geodesic symmetries at each point of $\tau_1(M_\iota)$. Since $\tau_1(M_\iota)$ is totally geodesic in M', these are geodesic symmetries of $\tau_1(M_\iota)$. Consequently, $\tau_1(M_\iota)$ is a global symmetric domain and τ_1 is (globally) an embedding. q.e.d.

We continue with the proof of Proposition 7.13 and let

$$B := \{\varphi \in \mathrm{Aut}(M') : \varphi(\tau_1(M_\iota)) = \tau_1(M_\iota)\} \subset G'.$$

In the next step we deduce from Claim 7.14 that $\tau_1(M_\iota) = B/K_B$, where K_B is a maximal compact subgroup. The first observation is:

Claim 7.15. The embedding $\tau_1 : M_\iota \to M'$ is induced by a homomorphism from $\tilde{\tau}_1 : G_\iota \to G'$ that factors through B.

Proof. As explained in [**Sa65**, §1.1] or [**Sa80**, II §2], the geodesic holomorphic embedding $M_\iota \to M'$ is induced by a local isomorphism $G_\iota \to G'$ and hence a homomorphism of Lie algebras $\mathrm{Lie}(G_\iota) \to \mathrm{Lie}(G')$. This induces a homomorphism $\hat{\tau}_1 : \tilde{G}_\iota \to G'$ from the universal covering $\tilde{G}_\iota$ of G_ι. It remains to show that $\hat{\tau}_1$ factors through G_ι, then the factorization through B is obvious from the definition.

It suffices to exhibit a factorization of $\hat{\tau}_1$ on the $\mathbb{R}$-valued points. Since

$$G_\iota(\mathbb{R}) \subset G_\iota(\mathbb{C}) \cong \mathrm{Sl}(1+n)(\mathbb{C}),$$

and since $\mathrm{Sl}(1+n)(\mathbb{C})$ is simply connected, this factorization is obvious. q.e.d.

By this claim, the natural map $\mathrm{res} : B \to \mathrm{Aut}(M_\iota) \cong G$ induces a surjection $B/K_B \to M_\iota$. This map is also injective, since elements in B preserve M_ι. Consequently, the kernel Υ of res is a compact subgroup. By Claim 7.15 again, this kernel is a direct factor. In fact, the kernel is a maximal direct factor, since $G_\iota = \mathrm{Aut}(M_\iota)$ does not contain direct compact factors. We deduce that given the choice of origins, the product decomposition $B \cong G_\iota \times \Upsilon$ is canonical.

By definition of a period map, τ is equivariant with respect to the action of $\pi_1(U)$ via

$$\rho_1 : \pi_1(U) \longrightarrow \mathrm{Aut}(\tilde{U}) \quad \text{and} \quad \rho_2 : \pi_1(U) \longrightarrow \mathrm{Aut}(M') \cong G'$$

on domain and range.

The image of ρ_2 lies in B by definition. The usual argument with Schur's Lemma (e.g. Proposition 5.9 or [**VZ05**] Proposition 3.3) implies that ρ_2 is a tensor product of a unitary representation and of a representation that factors through $\tilde{\tau}_\iota : G_\iota \to B \to G'$.

The last step in the proof of Proposition 7.13 is to exploit that there are not many possibilities for $\tilde{\tau}_1$ that give rise to a holomorphic totally geodesic embedding of hermitian symmetric domains.

Claim 7.16. The representation $G_\iota \to B \to G'$ is a wedge product of the standard representation.

Proof. In order to match the hypothesis of [**Sa80**] precisely, we should postcompose the map $\tilde{\tau}_1 : G_\iota \to G'$ by a natural inclusion of G' into the symplectic group. By the table p. 461 and Proposition 1 in [**Sa80**], $\mathrm{incl}\circ\tilde{\tau}_1$ is a direct sum of wedge products of the standard representations. This direct sum has only one summand, since $\mathbb{V}$ was reducible otherwise. q.e.d.

In order to prove the missing part v) of Proposition 0.1 we will use:

Proposition 7.17. *Assume that U is the quotient of a bounded symmetric domain by an arithmetic group. Assume that Ω_ι is of type A or B, that M_ι is the complex ball $\mathrm{SU}(1,n)/K$ and that $\mathbb{V}$ is the tensor product of a unitary representation with a wedge product of the standard representation of $\mathrm{SU}(1,n)$.*

1. *Then $\mathbb{V}$ satisfies the Arakelov equality.*
2. *Let Y' be a Mumford compactification. Writing (E',θ') for the Higgs bundle of $\mathbb{V}$ on Y', the sheaves $E'^{1,0}$ and $E'^{0,1}$ are μ-stable and $E'^{1,0\vee} \otimes E'^{0,1}$ is μ-polystable.*

Proof. Let Y', S' be a Mumford compactification (see Section 3). The bundles $E'^{1,0}$ and $E'^{0,1}$ are irreducible homogeneous bundles as in Lemma 3.8, case a_n and $q = 1$, given by the wedge products of the standard representation of $U(n)$. The same arguments as in the proof of the first parts of Proposition 0.1 now imply 1) and 2).

By Lemma 3.7 the Arakelov equality on a Mumford compactification implies the one on any compactification, satisfying the positivity statement in Assumption 1.1. q.e.d.

7.3. Type C: $S^m(\Omega_\iota)$ is μ-unstable for some $m > 1$. Yau's Uniformization Theorem, recalled in [**VZ07**, Theorem 1.4], implies that M_ι is a bounded symmetric domain of rank greater than one. Using the characteristic subvarieties, introduced by Mok presumably one can write down an explicit formula for $\varsigma(\mathbb{V})$. However we do not need this, since in this case the superrigidity theorems apply. Recall the notations introduced at the beginning of the proof of Proposition 7.13.

Proposition 7.18. *If $\mathbb{V}$ is pure of type ι, The period map factors as the projection $\tilde{U} \to M_\iota$ and a totally geodesic embedding $M_\iota \to M'$.*

Proof. Purity of $\mathbb{V}$ implies that the period map factors through the projection to M_ι. In the case we treat, M_ι has rank greater than one, hence the metric rigidity theorems of Mok and their generalizations due to To apply. More precisely, let h be the pullback the restriction of the Bergman-metric on M' to M_ι. By purity and since M' is a bounded symmetric domain of non-compact type, h descends to a metric of seminegative curvature on the bundle $(\Omega_\iota)^\vee$ on U. Thus the hypothesis of [**Mk89**, Theorem 4] are met, if one takes into account the arguments of To ([**To89**, Corollary 2] and the subsequent remark) to extend from U compact to U of finite volume. We conclude that up, to a constant multiple, h is the Bergman-metric on M_ι and $M_\iota \to M'$ a totally geodesic embedding. q.e.d.

Lemma 7.19. *Let $U \to \mathcal{A}_g$ be a generically finite map with $\tilde{U} = \prod M_i$. Suppose that all for all irreducible summands $\mathbb{V}$ the period map*

$\tau(\mathbb{V})$ is either constant or the composition $\tau(\mathbb{V}) = \tau_j \circ p_{i(\mathbb{V})}$ of a projection and a totally geodesic embedding of $M_{i(\mathbb{V})}$ to the period domain of $\mathbb{V}$. Then the universal covering map $\tau : \tilde{U} \to \tilde{\mathcal{A}}_g$ is a totally geodesic embedding.

Proof. By Lemma 7.1 the hypothesis 'generically finite' implies that for each i there is at least one non-unitary summand $\mathbb{V}$ with $i = i(\mathbb{V})$. The universal covering map is, by definition, the product of the $\tau(\mathbb{V})$ composed with a block diagonal embedding $\prod_j M'_j \to \tilde{\mathcal{A}}_g$. Since the latter is totally geodesic for the Bergman metric, the claim follows from the hypothesis on the $\tau(\mathbb{V})$. q.e.d.

We can now start the proof of the main theorems and its refinements.

Proof of Proposition 0.1. Parts i)–iv) have been verified at the end of Section 3. By assumption $f : A \to U$ is a Kuga fibre space, $\mathbb{V}$ is pure of type $i = i(\mathbb{V})$, and Ω_i is of type B. In particular the assumption made in Proposition 7.17 hold, and on a suitable compactification the sheaf $E^{1,0^\vee} \otimes E^{0,1}$ is μ-polystable. So Proposition 7.4 implies that

$$\varsigma((E,\theta)) = \frac{\ell \cdot \ell' \cdot (n_i + 1)}{(\ell + \ell') \cdot n_i}.$$

Of course this equality is independent of the compactification. q.e.d.

Proof of Theorem 0.2. If for some étale covering $\tau : U' \to U$ the pullback family $f' : A' \to U'$ is a Kuga fibre space, then the two conditions 1) and 2) on U are equivalent to the same conditions on U'. So we may as well assume that $f : A \to U$ is itself a Kuga fibre space. If $\mathbb{V}$ is a non-unitary irreducible subvariation of Hodge structures in $R^1 f_* \mathbb{C}_A$, then part ii) of Proposition 0.1 gives the Arakelov equality, and part i) implies that $\mathbb{V}$ is pure of type $i = i(\mathbb{V})$.

If Ω_i is of type A or C, there is nothing to verify in 2). If Ω_i is of type B, Part v) of Proposition 0.1 shows that

$$\varsigma(\mathbb{V}) = \varsigma((E,\theta)) = \varsigma((E,\theta_i)) = \frac{\mathrm{rk}(E^{1,0}) \cdot \mathrm{rk}(E^{0,1}) \cdot (n_i + 1)}{\mathrm{rk}(E) \cdot n_i}.$$

Assume now that the conditions 1) and 2) in Theorem 0.2, b) hold. Since φ is generically finite, by the first part of Lemma 7.1 one finds for each direct factor Ω^1_ι of $\Omega_Y(\log S)$ some non-unitary subvariation of Hodge structures $\mathbb{V}$, which is pure of type ι.

If Ω_ι is of type C, we find that the map $\tilde{U} \to M'$ to the period domain M' of $\mathbb{V}$ factors as the projection $\tilde{U} \to M_\iota$ and a totally geodesic embedding $M_\iota \to M'$.

By Proposition 7.13 the same holds if Ω_ι is of type A, or if it is of type B and if the condition 2) Theorem 0.2 holds.

So all the hypothesis of the Lemma 7.19 are met and $\tilde{U} \to \tilde{\mathcal{A}}_g$ is a totally geodesic embedding, hence by Theorem 2.3 there exists a Kuga fibre space $f' : A' \to U'$ such that the image of U' in $\mathcal{A}_g$ coincides with the image $\varphi(U)$. In particular this image is non singular. By the second part of Lemma 7.1 $\varphi : U \to \varphi(U)$ is étale, and replacing U' by an étale covering we may assume that U' dominates U.

Finally the last statement in Theorem 0.2 follows from Corollary 2.4. q.e.d.

Addendum 7.20. *Consider as in Theorem* 0.2 *an irreducible complex polarized subvariation of Hodge structures* $\mathbb{V}$ *of* $R^1 f_* \mathbb{C}_A$ *with Higgs bundle* (E, θ). *Assume that* $\mathbb{V}$ *is non-unitary and satisfies the Arakelov equality. Consider the following conditions for* $i = i(\mathbb{V})$*:*

α. $E^{1,0}$ *and* $E^{0,1}$ *are* μ*-stable.*

β. *The kernel of the natural map* $\mathcal{H}om(E^{0,1}, E^{1,0}) \to \Omega_i$ *is a direct factor of* $\mathcal{H}om(E^{0,1}, E^{1,0})$.

γ.
$$\varsigma(\mathbb{V}) = \frac{\mathrm{rk}(E^{1,0}) \cdot \mathrm{rk}(E^{0,1}) \cdot (n_i + 1)}{\mathrm{rk}(E) \cdot n_i}.$$

δ. M_i *is the complex ball* $\mathrm{SU}(1, n_i)/K$, *and* $\mathbb{V}$ *is the tensor product of a unitary representation with a wedge product of the standard representation of* $\mathrm{SU}(1, n_i)$.

η. *Let* M' *denote the period domain for* $\mathbb{V}$. *Then the period map factors as the projection* $\tilde{U} \to M_i$ *and a totally geodesic embedding* $M_i \to M'$.

Then, depending on the type of Ω_i, *the following holds:*

I. *If* Ω_i *is of type A, then* α), β), γ), δ) *and* η) *hold true.*
II. *If* Ω_i *is of type C, then* η) *holds true.*
III. *If* Ω_i *is of type B, then the conditions* β) *and* γ) *are equivalent. They imply the conditions* δ) *and* η).

Proof. Part I) is Proposition 7.3, if one uses in addition the equivalence between f) and c) in Proposition 7.4 and the Proposition 7.13. Part II) is just repeating the conclusion of Proposition 7.18.

For Part III) remark first that the equivalence of the conditions β) and γ) is part of Proposition 7.4. By Proposition 7.13 β) implies δ) and η). q.e.d.

Since the Arakelov equality says that the slopes of $\mathcal{H}om(E^{0,1}, E^{1,0})$ and of Ω_i coincide, the ampleness of $\omega_Y(S)$ implies ($\star$) and thus shows that α) implies β). Since δ) implies α) we can state the following corollary.

Corollary 7.21. *In the Addendum* 7.20 *one has:*

IV. *If* $\omega_Y(S)$ *is ample, for example if* U *is projective or if* $\dim(U) = 1$, *and if* Ω_i *is of type B, then the conditions* α), β), γ) *and* δ) *are equivalent and imply* η).

Without referring to the ampleness or to Yau's conjecture one still has the following.

Corollary 7.22. *Assume in Theorem* 0.2 *and in Addendum* 7.20 *that the Arakelov equality and the condition* η) *hold for all non-unitary subvariations* $\mathbb{V}$ *of* $R^1 f_* \mathbb{C}_A$. *Then there exists an étale covering* $\tau : U' \to U$ *with* U' *a quotient of a bounded symmetric domain by an arithmetic group. Moreover on a Mumford compactification of* U' *the conditions* α), β), δ), *and* γ) *are equivalent for all irreducible non-unitary subvariations* $\mathbb{V}'$ *of* $\tau^* R^1 f_* \mathbb{C}_A$ *which are of type B.*

Proof of Corollary 7.22. Since we assumed that $U \to \mathcal{A}_g$ is generically finite and that the condition η) in Addendum 7.20 holds for all irreducible non-unitary subvariations of Hodge structures, the argument used at the end of the proof of Theorem 0.2 shows that the pullback $f' : A' \to U'$ of $f : A \to U$ to some étale covering U' of U is a Kuga fibre space. Hence there exists a Mumford compactification Y'. The condition δ) allows to apply Proposition 7.17, 2) to obtain the conditions α) and β) q.e.d.

8. The Arakelov equality and the Mumford-Tate group

We keep the assumption that U is the complement of a normal crossing divisor S in a non-singular projective variety Y, that $\Omega^1_Y(\log S)$ is nef, and that $\omega_Y(S)$ is ample with respect to U. Let $f : A \to U$ be a family of polarized abelian varieties such that $R^1 f_* \mathbb{C}_A$ has unipotent local monodromies at infinity, and such that the induced morphism $U \to \mathcal{A}_g$ is generically finite.

If each irreducible subvariation of Hodge structures of $R^1 f_* \mathbb{C}_A$ is either unitary or it satisfies the Arakelov equality and if in addition the second condition in Theorem 0.2 holds, we have shown in the last section that the induced morphism $U \to \mathcal{A}_g$ is totally geodesic. By Moonen's Theorem 2.3 we know that U is the base of a Kuga fibre space, and that it is the translate of a Shimura variety of Hodge type. In particular this implies that the monodromy group Mon^0 of $R^1 f_* \mathbb{C}_A$ is normalized by the complex structure, hence by the derived Hodge group $\mathrm{MT}(R^1 f_* \mathbb{C}_A)^{\mathrm{der}}$. In this section we will verify the last property as a direct consequence of the Arakelov equality, without using the second condition in Theorem 0.2, and we will determine the invariant cycles under Mon^0 explicitly. The final statement is given in Corollary 8.15.

In the first part of this section we will consider arbitrary complex polarized variations of Hodge structures $\mathbb{V}$ of weight k on U, with unipotent local monodromy around the components of S, and we will write $(E = \bigoplus_{m=0}^{k} E^{k-m,m}, \theta)$ for the Higgs bundle. For $k > 1$ and $\dim(U) > 1$ there is not yet any concept of Arakelov inequality where maximality has as nice consequences as in weight one. We thus start with an ad hoc definition of what should be the maximal case and show that this condition is satisfied for some variations of Hodge structures derived from variations of Hodge structures of weight one with Arakelov equality.

Definition 8.1. The Higgs bundle (E, θ) (or the variation of Hodge structures $\mathbb{V}$) satisfies the *Arakelov condition* if there exist integers $m_{\min} \leq m_{\max}$ with

i. $E^{k-m,m} \neq 0$ if and only if $m_{\min} \leq m \leq m_{\max}$.
ii. For $m_{\min} \leq m < m_{\max}$ the morphism $\theta_{k-m,m} = \theta|_{E^{k-m,m}}$ is non-zero.
iii. For $m_{\min} \leq m \leq m_{\max}$ the sheaves $E^{k-m,m}$ are μ-semistable of slope

$$\mu(E^{k-m,m}) = \mu(E^{k-m_{\min},m_{\min}}) - (m - m_{\min}) \cdot \mu(\Omega^1_Y(\log S)). \tag{8.1}$$

Lemma 8.2.

1. *If $\mathbb{V}$ is unitary and irreducible, it satisfies the Arakelov condition.*
2. *If $k = 1$, if $\mathbb{V}$ is irreducible and if both, $E^{1,0}$ and $E^{0,1}$ are non-zero, then $\mathbb{V}$ satisfies the Arakelov condition if and only if the Arakelov equality holds.*
3. *If $\mathbb{V}$ satisfies the Arakelov condition, then the same holds true for its complex conjugate $\mathbb{V}^\vee$.*

Proof. Simpson correspondence implies in 1) that E is concentrated in one bidegree, whereas in 2) it implies that the Higgs field is non-zero. Then 1) and 2) are just reformulations of the definition. 3) is obvious, since the polarization (as indicated by the notation) allows to identify $\mathbb{V}^\vee$ with the dual local system. q.e.d.

Lemma 8.3. *Consider for $i = 1, \ldots, s$ polarized $\mathbb{C}$-variations of Hodge structures $\mathbb{V}_i$ with unipotent local monodromy at infinity and with Higgs bundles*

$$\Big(E_i = \bigoplus_{m=0}^{k_i} E_i^{k_i-m,m}, \theta_i\Big).$$

If the Arakelov condition holds for all the $\mathbb{V}_i$, it also holds for the local system $\mathbb{V} = \mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_s$ and for each irreducible direct factor $\mathbb{V}'$ of $\mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_s$.

Proof. Let (E, θ) denote again the Higgs bundle of $\mathbb{V}$. In order to show that $\mathbb{V}$ satisfies the Arakelov condition we may assume by induction that $s = 2$. Write $m^{(i)}_{\min}$ and $m^{(i)}_{\max}$ for the integers with $E_i^{k_i-\ell_i,\ell_i} \neq 0$ for $m^{(i)}_{\min} \leq \ell_i \leq m^{(i)}_{\max}$. Then for $k = k_1 + k_2$

$$E^{k-r,r} = \bigoplus_{\ell_1+\ell_2=r} E_1^{k_1-\ell_1,\ell_1} \otimes E_2^{k_2-\ell_2,\ell_2} \neq 0,$$

if and only if $m_{\min} = m^{(1)}_{\min} + m^{(2)}_{\min} \leq r \leq m_{\max} = m^{(1)}_{\max} + m^{(2)}_{\max}$. In addition, if $m_{\min} \leq m_{\max} - 1$ then $r = \ell_1 + \ell_2$ for some ℓ_i with either $\ell_1 < m^{(1)}_{\max}$ or $\ell_2 < m^{(2)}_{\max}$. In the first case, for example, part of the Higgs field is given by the tensor product of the Higgs field $\theta_i|_{E_1^{k_1-\ell_1,\ell_1}}$ with the identity on $E_2^{k_2-\ell_2,\ell_2}$, hence non-zero.

The equation 8.1 tells us that as the tensor product of μ-semistable sheaves $E_1^{k_1-\ell_1,\ell_1} \otimes E_2^{k_2-\ell_2,\ell_2}$ is μ-semistable of slope

$$\mu(E_1^{k_1-m^{(1)}_{\min},m^{(1)}_{\min}}) + \mu(E_2^{k_2-m^{(2)}_{\min},m^{(2)}_{\min}}) \\ -(\ell_1 + \ell_2 - m^{(1)}_{\min} - m^{(2)}_{\min}) \cdot \mu(\Omega^1_Y(\log S)). \tag{8.2}$$

So $E^{k-m,m}$ is μ-semistable of slope $\mu(E^{k-m_{\min},m_{\min}}) - (m - m_{\min}) \cdot \mu(\Omega^1_Y(\log S))$, if non-zero.

For the last part, let (E', θ') denote the Higgs bundle of the irreducible subvariation of Hodge structures $\mathbb{V}'$. We choose $m'_{\min}$ and $m'_{\max}$ to be the smallest and largest integer with $E'^{k-m'_{\min},m'_{\min}}$ and $E'^{k-m'_{\max},m'_{\max}}$ non-zero. By Simpson's correspondence (E', θ') can not be a direct sum of two Higgs bundles, hence $\theta'_{E'^{k-m,m}} \neq 0$ for $m'_{\min} \leq m \leq m'_{\max} - 1$. Finally, the μ-semistability as well as the equation 8.2 carry over to direct factors of (E, θ). q.e.d.

Lemma 8.4. *Let $\mathbb{V}$ be a complex polarized variation of Hodge structures $\mathbb{V}$ of weight k, with unipotent local monodromy around the components of S, and satisfying the Arakelov condition.*

a. *There is a unique m such that each unitary local subsystem $\mathbb{U}$ of $\mathbb{V}$ is concentrated in bidegree $(k-m, m)$. In particular all global sections $s \in H^0(U, \mathbb{V})$ are of bidegree $(k-m, m)$.*
b. *If $\mathbb{V}$ is defined over $\mathbb{R}$, hence of the form $\mathbb{V}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, then k is even and $m = \frac{k}{2}$.*

Proof. Obviously b) follows from a) using Lemma 8.2, 3).

By [**De87**] in a) the local system $\mathbb{U}$ is a subvariation of Hodge structures, in particular the corresponding Higgs bundle $(F, 0)$ is a direct factor of (E, θ). So $F^{p,q}$ has to be a direct factor of $E^{p,q}$, in particular it is again μ-semistable of slope $\mu(E^{p,q})$. Since $\mathbb{U}$ is unitary $\mu(F^{p,q}) = \mu(E^{p,q}) = 0$ and since $\mu(\Omega^1_Y(\log S)) > 0$ the equation 8.1 implies that this is only possible for one tuple (p, q).

For the last part of a) one takes for $\mathbb{U}$ the trivial sub-local system generated by $H^0(U, \mathbb{V})$. q.e.d.

In the sequel we consider a $\mathbb{Q}$-variation of Hodge structures $\mathbb{W}_{\mathbb{Q}} = R^1 f_* \mathbb{Q}_A$ with unipotent monodromy at infinity, induced by a family of polarized abelian varieties $f : A \to U$. So $\mathbb{W}_{\mathbb{Q}}$ is polarized of weight 1 and concentrated in bidegrees $(1, 0)$ and $(0, 1)$. For $\mathbb{Q} \subset K$ we will write $\mathbb{W}_K = \mathbb{W}_{\mathbb{Q}} \otimes K$ and $\mathbb{W} = \mathbb{W}_{\mathbb{C}}$.

Lemma 8.5. *There exists a totally real number field K such that:*

1. *One has a decomposition of variations of Hodge structures*
$$\mathbb{W}_K = \mathbb{W}_{1\,K} \oplus \cdots \oplus \mathbb{W}_{\ell\,K} \quad \text{with} \quad \mathbb{W}_{i\,K} = \mathbb{V}'_{i\,K} \otimes_K H_{i\,K},$$
orthogonal with respect to the polarization.
2. $\mathbb{V}'_{i\,\mathbb{R}} = \mathbb{V}'_{i\,K} \otimes_K \mathbb{R}$ *is irreducible for* $i = 1, \ldots, \ell$.
3. $\mathrm{Hom}(\mathbb{V}'_{i\,\mathbb{R}}, \mathbb{V}'_{j\,\mathbb{R}})$ *is a skew field for $i = j$ and is zero otherwise.*
4. *For each i the decomposition in 1) satisfies one of the following conditions:*
   a. $\mathbb{V}'_{i\,K}$ *is a polarized K-variation of Hodge structures of weight* 1 *and $H_{i\,K}$ a trivial K-Hodge structure, i.e. a K-vector space regarded as a Hodge structure concentrated in bidegree* $(0,0)$.
   b. $H_{i\,K}$ *a K-Hodge structure of weight* 1 *and $\mathbb{V}'_{i\,K}$ is a polarizable variation of Hodge structures concentrated in bidegree* $(0,0)$ *and unitary.*

For subsequent use we label the direct factors in Lemma 8.5 such that for some ℓ_2 and for $1 \leq i \leq \ell_2$ the condition a) holds true, whereas for $\ell_2 < i \leq \ell$ one has the condition b).

Proof of Lemma 8.5. By [**De87**, Proposition 1.12] $\mathbb{W}$ decomposes as a direct sum of irreducible $\mathbb{C}$-subvariations of Hodge structures. Replacing the direct factors $\mathbb{V}$ which are not invariant under complex conjugation by $\mathbb{V} \oplus \mathbb{V}^\vee$, one obtains a decomposition of $\mathbb{V}_\mathbb{R}$ as a direct sum of irreducible polarized $\mathbb{R}$-subvariations of Hodge structures. As shown in [**VZ07**, Lemma 9.4], for example, such a decomposition is induced by one which is defined over some totally real number field K, and it can be chosen to be orthogonal with respect to the polarization. The irreducibility implies that $\mathrm{Hom}(\mathbb{V}'_{i\,\mathbb{R}}, \mathbb{V}'_{j\,\mathbb{R}})$ is a skew field if and only if $\mathbb{V}'_{i\,\mathbb{R}} \cong \mathbb{V}'_{j\,\mathbb{R}}$.

Of course we may write the direct sum of all direct factors, isomorphic to some $\mathbb{V}'_{i\,K}$ in the form $\mathbb{V}'_{i\,K} \otimes_K H_{i\,K}$, for some K vector space $H_{i\,K}$. As in [**De87**, Proposition 1.13] or in [**De71**, Theorem 4.4.8] one defines a Hodge structure on $H_{i\,K}$.

In 4) the bidegrees of $\mathbb{V}'_{i\,K}$ and $H_{i\,K}$ have to add up to $(1,0)$ and $(0,1)$. If $H_{i\,K}$ is concentrated in bidegree $(0,0)$ the variation of Hodge structures $\mathbb{W}_{i\,K}$ is just a direct sum of the $\mathbb{V}'_{i\,K}$, again orthogonal with respect to the polarization, and one obtains case a). Otherwise $\mathbb{V}'_{i\,K}$ has to be concentrated in bidegree $(0,0)$. Since it is polarizable, it has to be unitary. q.e.d.

Beside of the totally real number field K in Lemma 8.5 we fix as in Subsection 2.3 a very general point $y \in U$. If a variation of Hodge structures is denoted by a boldface letter, the restriction to the base point $y \in U$ will be denoted by the same letter, not in boldface, so $W_{i\,K}$ and $V'_{i\,K}$ will denote the fibres at y of $\mathbb{W}_{i\,K}$ and $\mathbb{V}'_{i\,K}$, respectively.

As in [**An92**], one can extend the definition of the Hodge and Mumford-Tate group to an arbitrary polarized K-Hodge structure W_K. Since the decomposition in Lemma 8.5 is defined over a real number field and orthogonal, the complex structure factors through

$$\varphi_0 : S^1 \longrightarrow \bigtimes_{i=1}^{\ell} \mathrm{Sp}(W_{i\,K} \otimes_K \mathbb{R}, Q|_{W_{i\,K}}) \subset \mathrm{Sp}(W_K \otimes_K \mathbb{R}, Q).$$

In a similar way the morphism $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathrm{Gl}(W_K \otimes_K \mathbb{R})$ factors through

$$h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \longrightarrow \bigtimes_{i=1}^{\ell} \mathrm{Gl}(W_{i\,K} \otimes_K \mathbb{R}) \subset \mathrm{Gl}(W_K \otimes_K \mathbb{R}).$$

Hence for the *Mumford-Tate group* $\mathrm{MT}(W_K)$, defined as the smallest K-algebraic subgroup of $\mathrm{Gl}(W_K)$ whose extension to $\mathbb{R}$ contains the image of h, one has an inclusion

$$\mathrm{MT}(W_K) \subset \bigtimes_{i=1}^{\ell} \mathrm{MT}(W_{i\,K}). \tag{8.3}$$

By [**An92**] and [**De82**] the group $\mathrm{MT}(W_K)$ is reductive, and by [**An92**, Lemma 2, a)] it can again be defined as the largest K-algebraic subgroup of the linear group $\mathrm{Gl}(W_K)$, which leaves all K-Hodge cycles invariant, hence all elements

$$\eta \in \left[W_K^{\otimes m} \otimes W_K^{\vee \otimes m'}\right]^{0,0}.$$

The decomposition $\mathrm{Gl}(V'_{i\,K}) \times \mathrm{Gl}(H_{i\,K}) \subset \mathrm{Gl}(W_{i\,K})$ allows to define

$$G_K^{\mathrm{mov}} = \bigtimes_{i=1}^{\ell} \mathrm{Gl}(V'_{i\,K}) \times \{\mathrm{id}_{H_{i\,K}}\} \subset \bigtimes_{i=1}^{\ell} \mathrm{Gl}(W_{i\,K}) \subset \mathrm{Gl}(W_K).$$

Addendum 8.6. *Keeping the notations introduced in Lemma* 8.5 *one has*

5. *There exists a $\mathbb{Q}$-algebraic subgroup $G_{\mathbb{Q}}^{\mathrm{mov}}$ of $\mathrm{Gl}(W_{\mathbb{Q}})$ with $G_{\mathbb{Q}}^{\mathrm{mov}} \otimes K = G_K^{\mathrm{mov}}$. Moreover $G_{\mathbb{Q}}^{\mathrm{mov}}$ is independent of K.*

Proof. On may assume that $\mathbb{W}_{\mathbb{Q}}$ is irreducible over $\mathbb{Q}$. Obviously, if K' is a totally real extension of K, then $G_{K'}^{\mathrm{mov}} = G_K^{\mathrm{mov}} \otimes K'$. So one may also assume that K is a Galois extension of $\mathbb{Q}$ with Galois group Γ.

Let $\Gamma' \subset \Gamma$ be the subgroup consisting of all γ for which $\mathbb{W}_{i\,K}$ is isomorphic to the conjugate $\mathbb{W}_{i\,K}^{\gamma}$ under γ. In particular, $\mathbb{V}'_{i\,K} \cong \mathbb{V}'^{\gamma}_{i\,K}$. So the action of γ on $\mathbb{W}_{i\,K}$ is trivial on the first factor $\mathbb{V}'_{i\,K}$, and it leaves $\mathrm{Gl}(V'_{i\,K}) \times \{\mathrm{id}_{H_{i\,K}}\}$ invariant. Since $\mathbb{V}'_{j\,K} = \mathbb{V}'^{\delta}_{i\,K}$ for some $\delta \in \Gamma$, unique up to multiplication with Γ', the group G_K^{mov} is invariant under conjugation by Γ, hence it is defined over $\mathbb{Q}$ and, as said already, it is independent of K. q.e.d.

As in [**De71**, Lemma 4.4.9] the polarization $Q|_{W_{i\,K}}$ is the tensor product of two forms Q'_i and Q_i on $\mathbb{V}'_{i\,K}$ and $H_{i\,K}$, respectively, one being antisymmetric, the other symmetric. This allows to distinguish in Lemma 8.5, 4,b) two sub-cases:

We say that $\mathbb{W}_{i\,K}$ is *of type b1* if Q'_i is antisymmetric and *of type b2* if Q_i is antisymmetric. In the second case $H_{i\,K}$ is a polarized Hodge structure, and we can talk about its Mumford-Tate group.

Lemma 8.7.

a. *In case a) of Lemma* 8.5*, 4), i.e. for* $i = 1, \ldots, \ell_2$*, one has*

$$\mathrm{MT}(W_{i\,K}) = \mathrm{MT}(V_{i\,K}) \times \{\mathrm{id}_{H_{i\,K}}\}.$$

b. *In Lemma* 8.5*, 4.b) one finds:*

1. In case b1, i.e. for symmetric Q_i*, one has an inclusion*

$$\mathrm{MT}(W_{i\,K}) \subset \{\mathrm{id}_{V_{i\,K}}\} \times \mathrm{SO}(H_{i\,K}).$$

In particular, for $\dim(H_{i\,K}) = 2$*, the group* $\mathrm{MT}(W_{i\,K})$ *is commutative.*

2. In case b2, i.e. if Q_i *is antisymmetric, one has*

$$\mathrm{MT}(W_{i\,K}) = \{\mathrm{id}_{V_{i\,K}}\} \times \mathrm{MT}(H_{i\,K}).$$

3. If Q_i *is antisymmetric or if* Q_i *is symmetric and* $\dim(H_{i\,K}) > 2$*, there exists a non-zero antisymmetric endomorphism of* $\mathbb{W}_{i\,K}$ *of bidegree* $(-1, 1)$.

Proof. Consider a non-trivial element

$$\eta \in \big[W_{i\,K}^{\otimes m} \otimes_K W_{i\,K}^{\vee\otimes m'}\big]^{0,0} = \big[V_{i\,K}'^{\otimes m} \otimes_K V_{i\,K}'^{\vee\otimes m'} \otimes_K H_{i\,K}^{\otimes m} \otimes_K H_{i\,K}^{\vee\otimes m'}\big]^{0,0}.$$

So $m = m'$ and η can be written as

$$\eta = \sum_{\iota} \gamma_\iota \otimes h_\iota, \quad \text{with} \quad \gamma_\iota \in V_{i\,K}'^{\otimes m} \otimes \mathbb{V}_{i\,K}'^{\vee\otimes m} \quad \text{and} \quad h_\iota \in H_{i\,K}^{\otimes m} \otimes H_{i\,K}^{\vee\otimes m}.$$

For $i \leq \ell_2$ all the h_ι are of bidegree $(0,0)$. Then η is pure of bidegree $(0,0)$ if and only if this holds for $\gamma_\iota \otimes h_\iota$ for each ι, or equivalently, if γ_ι is a Hodge cycle. Altogether $\mathrm{MT}(W_{i\,K})$ and $\mathrm{MT}(V'_{i\,K}) \times \{\mathrm{id}_{H_{i\,K}}\}$ are two reductive groups leaving the same cycles invariant. By [**De82**, Proposition 3.1 (c)] they coincide.

If $i > \ell_2$, the sections γ_ι are all of bidegree $(0,0)$. So again η is of bidegree $(0,0)$ if and only if the same holds for the elements h_ι. Let Γ be the largest subgroup of $\mathrm{Sp}(H_{i\,K}, Q_i)$ which leaves all tensors h of bidegree $(0,0)$ invariant. Then $\gamma_\iota \otimes h_\iota$ is invariant under $\{\mathrm{id}_{V_{i\,K}}\} \times \Gamma$ if and only if it is invariant under $\mathrm{MT}(W_{i\,K})$. Again by [**De82**, Proposition 3.1 (c)] both groups coincide.

In case b2 the vector space $H_{i\,K}$, together with Q_i, is a polarized variation of Hodge structures, and Γ is the Mumford-Tate group of $H_{i\,K}$.

In case b1 one has $\Gamma \subset \mathrm{SO}(H_{i\,K})$. Since $\mathrm{O}(2,K)$ is commutative one obtains the second part of b.1).

For the third part of b) assume first that Q_i is symmetric. Then $\dim(H_{i\,K}) = \mu$ is even and for $\mu \geq 4$ the elements of $\mathrm{SO}(\mu, K)$ generate the matrix algebra $\mathrm{M}(\mu, K)$. So one finds an antisymmetric endomorphism of $V'_{i\,K} \otimes H_{i\,K}$ of bidegree $(-1, 1)$.

For Q_i antisymmetric there are obviously antisymmetric endomorphisms of $H_{i\,K}$ of bidegree $(-1, 1)$. The product with the identity of $\mathbb{V}'_{i\,K}$ gives the endomorphism we are looking for. q.e.d.

To compare the Mumford-Tate group with the monodromy group in case a) of Lemma 8.5 one needs some additional hypothesis on the variation of Hodge structures, in our case the Arakelov equality. By [**De87**, Proposition 1.12] the variations of Hodge structures $\mathbb{V}'_i = \mathbb{V}'_{i\,K} \otimes_K \mathbb{C}$ can be written as a direct sum of irreducible polarized $\mathbb{C}$-variations of Hodge structures. We distinguish two sub-cases.

Type a1. $\mathbb{V}'_i$ is an irreducible $\mathbb{C}$-variation of Hodge structures. This implies in particular that $\mathbb{V}'_i$ is isomorphic to its complex conjugate $\mathbb{V}'^{\vee}_i$, and that $\mathbb{V}'_i$ is not unitary. In fact, if $\mathbb{V}'_i$ were unitary, it would decompose in two non-trivial subsystems, one of bidegree $(1, 0)$ and the other of bidegree $(0, 1)$, contradicting the irreducibility.

Claim 8.8. Assume that $\mathbb{W}_{i\,K}$ is of type a1, and that it satisfies the Arakelov equality. Then all global sections

$$\eta \in H^0\big(Y, \mathbb{W}_{i\,K}^{\otimes m} \otimes_K \mathbb{W}_{i\,K}^{\vee \otimes m'}\big)$$

are of bidegree $(m - m', m - m')$.

Proof. The Arakelov equality implies that $\mathbb{V}'_i$ satisfies the Arakelov condition. Since H_i is a K-vector space concentrated in bidegree $(0, 0)$, the same holds true for $\mathbb{W}_i = \mathbb{V}'_i \otimes H_i$. So the Claim follows from Lemma 8.4. q.e.d.

Type a2. $\mathbb{V}'_i$ is the direct sum of two irreducible factors $\mathbb{V}_i$ and $\mathbb{V}^{\vee}_i$, dual to each other and interchanged by complex conjugation. Remark that we allow $\mathbb{V}_i$ and $\mathbb{V}^{\vee}_i$ to be unitary. If not, they satisfy the Arakelov equality. Hence by Lemma 8.3 the two variations of Hodge structures $\mathbb{V}_i$, $\mathbb{V}^{\vee}_i$ as well as their tensor product with H_i will satisfy the Arakelov condition and Lemma 8.4 implies:

Claim 8.9. Assume that $\mathbb{W}_{i\,K}$ is of type a2, and either unitary or with Arakelov equality. Then there exist p and q such that all global sections

$$\eta \in H^0\big(Y, (\mathbb{V}_{i\,K} \otimes_K H_{i\,K})^{\otimes m} \otimes_K (\mathbb{V}_{i\,K} \otimes_K H_{i\,K})^{\vee \otimes m'}\big)$$

are of bidegree (p, q), and all global sections

$$\eta \in H^0\big(Y, (\mathbb{V}^{\vee}_{i\,K} \otimes_K H_{i\,K})^{\otimes m} \otimes_K (\mathbb{V}^{\vee}_{i\,K} \otimes_K H_{i\,K})^{\vee \otimes m'}\big)$$

are of bidegree (q,p). Moreover one has $p+q=m-m'$.

Claim 8.10. For $\mathbb{W}_{i\,K}$ of type a2 the Mumford-Tate group respects the decomposition of $\mathbb{V}'_i$, i.e. up to conjugation

$$\mathrm{MT}(W_{i\,K})\otimes_K\mathbb{C}\subset \mathrm{Gl}(V_i\otimes H_i)\times\mathrm{Gl}(V_i^\vee\otimes H_i).$$

Proof. The decomposition in a direct sum can be defined over an imaginary quadratic extension $K(\sqrt{b})$ of K, say with ι as a generator of the Galois group. So the Mumford-Tate group acts trivially on ι-invariant global sections of $\mathrm{End}(\mathbb{W}_i)$. Applying this to $\mathrm{id}_{V_i\otimes H_i}+\mathrm{id}_{V_i^\vee\otimes H_i}$ and to $\sqrt{b}\cdot(\mathrm{id}_{V_i\otimes H_i}-\mathrm{id}_{V_i^\vee\otimes H_i})$ one obtains the claim. q.e.d.

Definition 8.11. Let $G^{\mathrm{mov}}_{\mathbb{Q}}$ be the group defined in Addendum 8.6. Then we define the *moving part of the Mumford-Tate group* as

$$\mathrm{MT}^{\mathrm{mov}}(W_{\mathbb{Q}})=\mathrm{MT}(W_{\mathbb{Q}})\cap G^{\mathrm{mov}}_{\mathbb{Q}}\quad\text{and}\quad \mathrm{MT}^{\mathrm{mov}}(W_K)=\mathrm{MT}(W_K)\cap G^{\mathrm{mov}}_K.$$

Correspondingly we write for any of the components $\mathbb{W}_{i\,K}$ in Lemma 8.5

$$\mathrm{MT}^{\mathrm{mov}}(W_{i\,K})=\mathrm{MT}(W_{i\,K})\cap\big(\mathrm{Gl}(V_{i\,K})\times\{\mathrm{id}_{H_{i\,K}}\}\big).$$

Lemma 8.7 allows to evaluate the moving part of the Mumford-Tate group. In case a), i.e. for $i=1,\ldots,\ell_2$ one finds

$$\mathrm{MT}^{\mathrm{mov}}(W_{i\,K})=\mathrm{MT}(W_{i\,K})=\mathrm{MT}(V'_{i\,K})\times\{\mathrm{id}_{H_{i\,K}}\},$$

whereas in case b) $\mathrm{MT}^{\mathrm{mov}}(W_{i\,K})$ is trivial. By (8.3)

$$\begin{aligned}\mathrm{MT}^{\mathrm{mov}}(W_K)&=\mathrm{MT}(W_K)\cap\Big(\bigtimes_{i=1}^{\ell_2}\mathrm{Gl}(V'_{i\,K})\times\{\mathrm{id}_{H_{i\,K}}\}\Big)\\ &\subset\bigtimes_{i=1}^{\ell_2}\mathrm{MT}^{\mathrm{mov}}(W_{i\,K}).\end{aligned}\tag{8.4}$$

To give a definition of $\mathrm{MT}^{\mathrm{mov}}(W_{\mathbb{Q}})$ in terms of complex structures we define

(8.5)

$$h^{\mathrm{mov}}:\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m\longrightarrow\bigtimes_{i=1}^{\ell}\mathrm{Gl}(W_{i\,K}\otimes_K\mathbb{R})\xrightarrow{\mathrm{proj}}\bigtimes_{i=1}^{\ell_2}\mathrm{Gl}(W_{i\,K}\otimes_K\mathbb{R}).$$

Lemma 8.12.

i. $\mathrm{MT}^{\mathrm{mov}}(W_{\mathbb{Q}})$ *is a normal subgroup of* $\mathrm{MT}(W_{\mathbb{Q}})$.
ii. $\mathrm{MT}^{\mathrm{mov}}(W_K)$ *is the smallest K-algebraic subgroup H_K of* $\mathrm{Gl}(W_K)$, *for which $H_K\otimes_K\mathbb{R}$ contains the image of h^{mov}.*
iii. $\mathrm{MT}^{\mathrm{mov}}(W_{\mathbb{Q}})$ *is the smallest $\mathbb{Q}$-algebraic subgroup $H_{\mathbb{Q}}$ of* $\mathrm{Gl}(W_{\mathbb{Q}})$ *with*
$$\mathrm{MT}^{\mathrm{mov}}(W_K)\subset H_{\mathbb{Q}}\otimes K.$$
iv. $\mathrm{MT}^{\mathrm{mov}}(W_{\mathbb{Q}})$ *is the smallest $\mathbb{Q}$-algebraic subgroup $H_{\mathbb{Q}}$ of* $\mathrm{Gl}(W_{\mathbb{Q}})$, *for which $H_{\mathbb{Q}}\otimes\mathbb{R}$ contains the image of h^{mov}.*

Proof. We may assume again that $\mathbb{W}_{\mathbb{Q}}$ is irreducible an that K is Galois over $\mathbb{Q}$ with Galois group Γ.

Part ii) follows from 8.4 and from the definition of $\mathrm{MT}(W_{i\,K})$, and part iv) follows from ii) and iii).

To verify part iii) remark that $\mathrm{MT}(W_{\mathbb{Q}})$ is the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{Gl}(W_{\mathbb{Q}})$ whose extension to K contains $\mathrm{MT}(W_K)$. By 8.4

$$\mathrm{MT}^{\mathrm{mov}}(W_K) = \mathrm{MT}(W_K) \cap G_K = \mathrm{MT}(W_K) \cap (G_{\mathbb{Q}} \otimes K).$$

Taking conjugates with $\sigma \in \Gamma$ one finds that

$$\mathrm{MT}^{\mathrm{mov}}(W_K)^{\sigma} = \mathrm{MT}(W_K)^{\sigma} \cap (G_{\mathbb{Q}} \otimes K).$$

For the smallest $\mathbb{Q}$ algebraic subgroup $H_{\mathbb{Q}}$ of $\mathrm{Gl}(W_{\mathbb{Q}})$ with the property $\mathrm{MT}^{\mathrm{mov}}(W_K) \subset H_{\mathbb{Q}} \otimes K$ the extension $H_{\mathbb{Q}} \otimes K$ of scalars is the product over all conjugates of $\mathrm{MT}^{\mathrm{mov}}(W_K)$, hence it is equal to $(\mathrm{MT}(W_{\mathbb{Q}}) \otimes K) \cap (G_{\mathbb{Q}} \otimes K)$ and one obtains iii).

Obviously G_K^{mov} is normal in $\bigtimes_{i=1}^{\ell} \mathrm{Gl}(W_{i\,K})$. The latter contains the group $\mathrm{MT}(W_K)$ and all its conjugates under Γ. So $\mathrm{MT}^{\mathrm{mov}}(W_{\mathbb{Q}}) \otimes K$ is a normal subgroup of $\mathrm{MT}(W_{\mathbb{Q}}) \otimes K$ and i) holds true. q.e.d.

Lemma 8.7 implies that $\mathrm{MT}^{\mathrm{mov}}(W_{i\,K})^{\mathrm{der}} = \mathrm{MT}(W_{i\,K})^{\mathrm{der}}$ in case a) and in case b1), provided $\dim(H_{i\,K}) = 2$. In the remaining cases by Lemma 8.7, b.3) there exists a non-zero antisymmetric endomorphism of $\mathbb{W}_{i\,K}$, which by [**Fa83**] implies non-rigidity. So we can state:

Lemma 8.13. *Assume that $\mathbb{W}_{\mathbb{Q}}$ is a rigid polarized variation of Hodge structures of weight* 1. *Then for all i one has* $\mathrm{MT}^{\mathrm{mov}}(W_{i\,K})^{\mathrm{der}} = \mathrm{MT}(W_{i\,K})^{\mathrm{der}}$ *and hence* $\mathrm{MT}^{\mathrm{mov}}(W_{\mathbb{Q}})^{\mathrm{der}} = \mathrm{MT}(W_{\mathbb{Q}})^{\mathrm{der}}$.

Recall that $\mathbb{W}_{\mathbb{Q}}$ is the variation of Hodge structures of a polarized family of abelian varieties $f : A \to U$, and that W_K and $W_{\mathbb{Q}}$ are the restrictions of $\mathbb{W}_K$ and $\mathbb{W}_{\mathbb{Q}}$ to a very general point $y \in U$. So $\mathrm{MT}(W_{\mathbb{Q}})^{\mathrm{der}}$ is compatible with parallel transport and, following the usual convention, we write $\mathrm{MT}(\mathbb{W}_{\mathbb{Q}})$ instead of $\mathrm{MT}(W_{\mathbb{Q}})$ and $\mathrm{MT}^{\mathrm{mov}}(\mathbb{W}_{\mathbb{Q}})$ instead of $\mathrm{MT}^{\mathrm{mov}}(W_{\mathbb{Q}})$ For $L = \mathbb{Q}$ or $L = K$ we consider the monodromy group $\mathrm{Mon}(\mathbb{W}_L)$, defined as the smallest L-algebraic subgroup of $\mathrm{Gl}(W_L)$ which contains the image of the monodromy representation. As usual the upper Index 0 refers to the connected component of the identity. By [**De82**] (see also [**An92**] or [**Mo98**]) we know that the connected component $\mathrm{Mon}^0(f) = \mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}})$ is a normal subgroup of the derived subgroup $\mathrm{MT}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}}$.

Proposition 8.14. *Keeping the notations introduced in Lemma* 8.5*, assume that each irreducible direct factor of $\mathbb{W} = \mathbb{W}_{\mathbb{Q}} \otimes \mathbb{C}$ is either unitary or satisfies the Arakelov equality. Then*

$$\mathrm{MT}^{\mathrm{mov}}(W_K)^{\mathrm{der}} \subset \mathrm{Mon}^0(\mathbb{W}_K).$$

Before proving Proposition 8.14 let us state and prove the corollary we are heading for.

Corollary 8.15. *Let Y be a non-singular projective variety, and let $U \subset Y$ be the complement of a normal crossing divisor S. Assume that $\Omega^1_Y(\log S)$ is nef and that $\omega_Y(S)$ is ample with respect to U. Let $f : A \to U$ be a family of polarized abelian varieties with unipotent local monodromy at infinity and such that for $\mathbb{W}_{\mathbb{Q}} = R^1 f_* \mathbb{Q}_A$ each non-unitary irreducible subvariation of Hodge structures of $\mathbb{W} = \mathbb{W}_{\mathbb{Q}} \otimes \mathbb{C}$ satisfies the Arakelov equality. Then*

$$\mathrm{MT}^{\mathrm{mov}}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}} = \mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) = \mathrm{MT}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}} \cap G^{\mathrm{mov}}_{\mathbb{Q}}. \tag{8.6}$$

In particular $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}})$ *is normalized by* $\mathrm{MT}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}}$.

If $f : A \to U$ is rigid one finds that $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) = \mathrm{MT}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}}$.

Proof. Choose the totally real number field K according to Lemma 8.5. Obviously $\mathrm{Mon}^0(\mathbb{W}_K)$ is contained in $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) \otimes K$, hence by Proposition 8.14 one has an inclusion

$$\mathrm{MT}^{\mathrm{mov}}(W_K)^{\mathrm{der}} \subset \mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) \otimes K.$$

Extending the coefficients to $\mathbb{R}$ one finds by Lemma 8.12, ii) that the group $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) \otimes \mathbb{R}$ contains the image of the moving part of the complex structure h^{mov}, as defined in 8.5. By part iv) of Lemma 8.12 one gets $\mathrm{MT}^{\mathrm{mov}}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}} \subset \mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}})$. By [**De82**] one knows that $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) \subset \mathrm{MT}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}}$. Since obviously $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) \subset G^{\mathrm{mov}}_{\mathbb{Q}}$, one obtains 8.6. The normality of $\mathrm{MT}^{\mathrm{mov}}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}} \subset \mathrm{MT}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}}$ follows from Lemma 8.12, i). Finally the last part of Corollary 8.15 is a consequence of 8.6, using Lemma 8.13. q.e.d.

Using the notations from Section 2.1, we choose $V = H^1(f^{-1}(y), \mathbb{Q})$ for the very general point $y \in U$ and the induced symmetric bilinear form Q.

Since $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) = \mathrm{MT}^{\mathrm{mov}}(R^1 f_* \mathbb{Q}_A)^{\mathrm{der}}$ is normalized by the group $\mathrm{MT}(R^1 f_* \mathbb{Q}_A)^{\mathrm{der}}$, hence by the complex structure φ_0 as well, one obtains Kuga fibre spaces over

$$\mathcal{X}^{\mathrm{mov}} = \mathcal{X}(\mathrm{MT}^{\mathrm{mov}}(\mathbb{W}_{\mathbb{W}})^{\mathrm{der}}, \mathrm{id}, \varphi_0) \subset \mathcal{X} = \mathcal{X}(\mathrm{MT}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}}, \mathrm{id}, \varphi_0).$$

By [**Mu66**] and [**Mu69**] $\mathcal{X}$ is the moduli space of abelian varieties whose Mumford-Tate group is contained in $\mathrm{MT}(R^1 f_* \mathbb{Q}_A)$. So the family $f : A \to U$ induces a morphism $U \to \mathcal{X}$, perhaps after replacing U by an étale covering. Since $\varphi : U \to \mathcal{A}_g$ is generically finite over its image, the morphism $U \to \mathcal{X}$ has the same property.

Assume in Corollary 8.15 that $f : A \to U$ is rigid, and that $\dim(U) \geq \dim \mathcal{X}$. The rigidity implies by Corollary 8.15 that $\mathrm{MT}^{\mathrm{mov}}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}} = \mathrm{MT}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}}$, and hence that $\mathcal{X}^{\mathrm{mov}} = \mathcal{X}$ is a Shimura variety of Hodge type. Since φ is generically finite over its image, $\varphi : U \to \mathcal{X}$ is dominant, hence $\mathcal{X} = \varphi(U)$. By Lemma 7.1, (2) $\varphi : U \to \mathcal{X}$ is étale.

The same argument applies for non-rigid families if one knows that φ factors through $\mathcal{X}^{\mathrm{mov}}$ and if $\dim(U) \geq \dim \mathcal{X}^{\mathrm{mov}}$. So we can state:

Lemma 8.16. *Assume in Corollary* 8.15 *that the induced morphism* $\varphi : U \to \mathcal{A}_g$ *factors through* $\mathcal{X}^{\mathrm{mov}}$ *and that* $\dim(U) \geq \dim(\mathcal{X}^{\mathrm{mov}})$*. Then (replacing* U *by an étale covering, if necessary)* $\varphi : U \to \mathcal{X}^{\mathrm{mov}}$ *is finite, étale, and surjective.*

In particular this holds true if $f : A \to U$ *is rigid, hence* $\mathcal{X}^{\mathrm{mov}} = \mathcal{X}$ *and if* $\dim(U) \geq \dim(\mathcal{X})$*.*

Example 8.17. Assume in Corollary 8.15 that the universal covering $\tilde{U}$ is a bounded symmetric domain, and that $\mathbb{W}_{\mathbb{Q}}$ is the uniformizing local system. So $\tilde{U}$ is isomorphic to $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) \otimes \mathbb{R}$, divided by a maximal compact subgroup.

Assume either that $f : A \to U$ is rigid, or that the morphism $\tilde{\varphi}$ from $\tilde{U}$ to the Siegel upper halfspace $\tilde{\mathcal{A}}_g$ is induced by a homomorphism

$$\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) \otimes \mathbb{R} \to \mathrm{Sp}(2g, \mathbb{R}).$$

Then the assumptions in Lemma 8.16 hold true.

In fact, in both cases we know that $\tilde{\varphi} : \tilde{U} \to \tilde{\mathcal{A}}_g$ factors through $\mathcal{X}^{\mathrm{mov}}$. Moreover the real dimension of $\tilde{U}$ is equal to the dimension of the quotient of $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) \otimes \mathbb{R}$ by a maximal compact subgroup, hence equal to $2 \cdot \dim(\mathcal{X}^{\mathrm{mov}})$.

Remark 8.18. Without any assumption on rigidity Theorem 2.3 gives the existence of a Shimura variety of Hodge type $\mathcal{X}_1 \times \mathcal{X}_2$ such that $U = \mathcal{X}_1 \times \{b\}$. Using the notations introduced above, $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ and $\mathcal{X}^{\mathrm{mov}} = \mathcal{X}_1 \times \{b\}$. By deforming b to a point a with complex multiplication one gets a Shimura variety of Hodge type $\mathcal{X}_1 \times \{a\}$.

As we have seen the non-rigidity comes from the existence of direct factors of type b1 with $\dim(H_{i\,K}) \geq 4$ or of type b2. Passing from b to a corresponds to a modification of the Hodge structure $H_{i\,K}$ in such a way, that $\mathrm{MT}(W_{i\,K})/_{\mathrm{MT}^{\mathrm{mov}}(W_{i\,K})}$ becomes commutative.

Proof of Proposition 8.14. We will apply arguments, similar to the ones used in the proof of [**VZ07**, Proposition 10.3]. By [**Si92**, Lemma 4.4] $\mathrm{Mon}^0(\mathbb{W}_K)$ is reductive, hence by [**De82**, Proposition 3.1 (c)] there is no larger subgroup of $\mathrm{Gl}(W_K)$ which leaves all elements $\eta_y \in W_K^{\otimes m} \otimes_K W_K^{\vee \otimes m'}$ invariant, which are invariant under $\mathrm{Mon}^0(\mathbb{W}_K)$. If we verify that all elements $\eta_y \in W_K^{\otimes m} \otimes_K W_K^{\vee \otimes m'}$ which are invariant under $\mathrm{Mon}^0(\mathbb{W}_K)$ are invariant under $\mathrm{MT}^{\mathrm{mov}}(W_K)^{\mathrm{der}}$, we get the inclusion

$$\mathrm{MT}^{\mathrm{mov}}(W_K)^{\mathrm{der}} \subset \mathrm{Mon}^0(\mathbb{W}_K).$$

If η_y is invariant under $\mathrm{Mon}^0(\mathbb{W}_K)$, one may replace U by an étale cover and assume that η_y is invariant under the monodromy representation, hence it is the restriction of a global section

$$\eta \in H^0\big(Y, \mathbb{W}_K^{\otimes m} \otimes_K \mathbb{W}_K^{\vee \otimes m'}\big).$$

Since K is a totally real number field, $\mathbb{W}_K^\vee$ is isomorphic to $\mathbb{W}_K$, hence $\det(\mathbb{W}_K)^2$ is trivial. Up to a shift of the bigrading, $\mathbb{W}_K^\vee$ can be identified with

$$\bigwedge^{\mathrm{rk}(W_K)-1} \mathbb{W}_K \otimes_K \det(\mathbb{W}_K)^{-1} = \bigwedge^{\mathrm{rk}(W_K)-1} \mathbb{W}_K \otimes_K \det(\mathbb{W}_K),$$

so we may as well consider sections of

$$\begin{aligned} \eta \in H^0\big(Y, \mathbb{W}_K^{\otimes k}\big) &= \bigoplus_{\mathcal{I}'} H^0\Big(Y, \bigotimes_{i=1}^{\ell} \mathbb{W}_{i\,K}^{\otimes \kappa_i}\Big) \\ &= \bigoplus_{\mathcal{I}'} H^0\Big(Y, \bigotimes_{i=1}^{\ell} \mathbb{V}'^{\otimes \kappa_i}_{i\,K}\Big) \otimes_K \bigotimes_{i=1}^{\ell} H_{i\,K}^{\otimes \kappa_i}, \end{aligned} \tag{8.7}$$

where $\mathcal{I}'$ is the set of tuples $\underline{\kappa} = (\kappa_1, \ldots, \kappa_\ell)$ with $\sum_{i=1}^{\ell} \kappa_i = k$, so $\eta = \sum_{\mathcal{I}'} \eta_{\underline{\kappa}}$. Each component of η in this direct sum decomposition is again invariant under $\mathrm{Mon}^0(\mathbb{W}_K)$. So we may as well assume that $\eta = \eta_{\underline{\kappa}^0}$ for a fixed tuple $\underline{\kappa}^0 = (\kappa_1^0, \ldots, \kappa_\ell^0)$ and that

$$\eta_{\underline{\kappa}^0} = \gamma_{\underline{\kappa}^0} \otimes h_{\underline{\kappa}^0} \quad \text{with} \quad \gamma_{\underline{\kappa}^0} \in H^0\Big(Y, \bigotimes_{i=1}^{\ell} \mathbb{V}'^{\otimes \kappa_i^0}_{i\,K}\Big) \quad \text{and} \quad h_{\underline{\kappa}^0} \in \bigotimes_{i=1}^{\ell} H_{i\,K}^{\otimes \kappa_i^0}.$$

Recall that by our choice of the indices we are in case a) of Lemma 8.5, 4) for $i = 1, \ldots, \ell_2$. Let us rearrange the indices in such a way, that $i = 1, \ldots, \ell_1$ the local system $\mathbb{V}'_i = \mathbb{V}'_{i\,K} \otimes_K \mathbb{C}$ remains irreducible (type a1), whereas for $i = \ell_1 + 1, \ldots, \ell_2$ it decomposes (type a2).

Choose a Galois extension L of K with Galois group Γ, such that the local systems $\mathbb{V}'_{i\,L}$ decompose as a direct sum of two subsystems $\mathbb{V}_{i\,L}$ and $\mathbb{V}_{i\,L}^\vee$ for $i = \ell_1 + 1, \ldots, \ell_2$. By abuse of notation we will drop the L, hence $_i$ stands for $_{i\,L}$.

Consider the set $\mathcal{I}$ of tuples of natural numbers

$$\begin{gathered} \underline{k} = (k_1, \ldots, k_{\ell_1}, k_{\ell_1+1}, k'_{\ell_1+1}, \ldots, k_{\ell_2}, k'_{\ell_2}, k_{\ell_2+1}, \ldots, k_\ell), \quad \text{with} \\ k_i = \kappa_i^0 \quad \text{for} \quad i \in \{1, \ldots, \ell_1\} \cup \{\ell_2 + 1, \ldots, \ell\} \quad \text{and} \\ k_i + k'_i = \kappa_i^0 \quad \text{for} \quad i \in \{\ell_1 + 1, \ldots, \ell_2\}. \end{gathered}$$

Then $H^0\Big(Y, \bigotimes_{i=1}^{\ell} \mathbb{V}'^{\otimes \kappa_i^0}_{i\,K}\Big) \otimes_K L$ decomposes as

$$\bigoplus_{\mathcal{I}} H^0\Big(Y, \bigotimes_{i=1}^{\ell_1} \mathbb{V}'^{\otimes k_i}_i \otimes \bigotimes_{i=\ell_1+1}^{\ell_2} \big(\mathbb{V}_i^{\otimes k_i} \otimes \mathbb{V}_i^{\vee \otimes k'_i}\big) \bigotimes_{i=\ell_2+1}^{\ell} \mathbb{V}'^{\otimes k_i}_i\Big).$$

Remark that the local systems $\mathbb{V}'_i$ and $\mathbb{V}_i$ occurring in this decomposition all satisfy the Arakelov condition. Hence $\gamma = \gamma_{\underline{\kappa}^0}$ and $\eta = \eta_{\underline{\kappa}^0}$ decompose as

$$\gamma = \sum_{\mathcal{I}} \gamma_{\underline{k}} \quad \text{and} \quad \eta = \sum_{\mathcal{I}} \gamma_{\underline{k}} \otimes h_{\underline{\kappa}^0}$$

where by Lemma 8.4

$$\gamma_{\underline{k}} \in \bigoplus_{\mathcal{I}} H^0\Big(Y, \bigotimes_{i=1}^{\ell_1} \mathbb{V}_i'^{\otimes k_i} \otimes \bigotimes_{i=\ell_1+1}^{\ell_2} \big(\mathbb{V}_i^{\otimes k_i} \otimes \mathbb{V}_i^{\vee \otimes k'_i}\big) \otimes \bigotimes_{i=\ell_2+1}^{\ell} \mathbb{V}_i'^{\otimes k_i}\Big)$$

is pure of some bidegree $(p_{\underline{k}}, q_{\underline{k}})$.

The Galois group Γ acts on the decomposition, and since η and $h = h_{\underline{\kappa}^0}$ are defined over K the group Γ permutes the components $\gamma_{\underline{k}}$. The sum over the conjugates of a fixed $\gamma_{\underline{k}}$ will again be defined over K, and by abuse of notations, replacing $\mathcal{I}$ by a subset, we can assume that $\mathcal{I}$ consists of one Γ-orbit.

If for some $\underline{k} \in \mathcal{I}$ one has $p_{\underline{k}} \neq q_{\underline{k}}$ then $\gamma_{\underline{k}}$ is not defined over $\mathbb{R}$, and its complex conjugate is of the form $\gamma_{\underline{k'}}$ for some $\underline{k'} \in \mathcal{I}$. In particular $p = \sum_{\mathcal{I}} p_{\underline{k}} = \sum_{\mathcal{I}} q_{\underline{k}}$, and hence the wedge product $\rho = \bigwedge_{\mathcal{I}} \gamma_{\underline{k}}$ is pure of bidegree (p,p) and defined over L. Since wedge products are direct factor of some tensor product, ρ is a section in

$$H^0\Big(Y, \bigotimes^{\nu} \Big(\bigotimes_{i=1}^{\ell_1} \mathbb{V}_i'^{\otimes k_i} \otimes \bigotimes_{i=\ell_1+1}^{\ell_2} \big(\mathbb{V}_i^{\otimes k_i} \otimes \mathbb{V}_i^{\vee \otimes k'_i}\big) \otimes \bigotimes_{i=\ell_2+1}^{\ell} \mathbb{V}_i'^{\otimes k_i}\Big)\Big).$$

The Galois group Γ of L over K permutes the different components $\gamma_{\underline{k}}$, hence it acts on ρ by a character $\chi : \Gamma \to \{\pm 1\}$. So for some $\beta \in L$ the cycle $\beta \cdot \rho$ is invariant under Γ. Choosing

$$h' \in \bigotimes^{\nu} \bigotimes_{i=1}^{\ell} H_{i\,K}^{\otimes \kappa_i}$$

of bidegree (p', p') one obtains a Hodge cycle

$$\beta \cdot \rho \otimes h' \in H^0\big(Y, \mathbb{W}_K^{\otimes k \cdot \nu}\big).$$

So $\beta \cdot \rho \otimes h'$ is invariant under $\mathrm{MT}(W_K)^{\mathrm{der}}$ hence under the subgroup $\mathrm{MT}^{\mathrm{mov}}(W_K)^{\mathrm{der}}$ as well. This group acts trivially on h', hence $\beta \cdot \rho$ has to be invariant under $\mathrm{MT}^{\mathrm{mov}}(W_K)^{\mathrm{der}}$, where we consider the identification

$$\mathbb{G}_K^{\mathrm{mov}} = \bigtimes_{i=0}^{\ell} \mathrm{Gl}(V'_{i\,K}) \times \{\mathrm{id}_{H_{i\,K}}\} \cong \bigtimes_{i=0}^{\ell} \mathrm{Gl}(V'_{i\,K}).$$

This implies that the subspace $J = \langle \gamma_{\underline{k}};\ \underline{k} \in \mathcal{I} \rangle_L \subset$ of

$$\bigoplus_{\mathcal{I}} H^0\Big(Y, \bigotimes_{i=1}^{\ell_1} \mathbb{V}_i'^{\otimes k_i} \otimes \bigotimes_{i=\ell_1+1}^{\ell_2} \big(\mathbb{V}_i^{\otimes k_i} \otimes \mathbb{V}_i^{\vee \otimes k_i'}\big) \otimes \bigotimes_{i=\ell_2+1}^{\ell} \mathbb{V}_i'^{\otimes k_i}\Big)$$

is invariant under the action of $\mathrm{MT}^{\mathrm{mov}}(W_K)^{\mathrm{der}} \otimes L$. Since

$$\mathrm{MT}^{\mathrm{mov}}(W_K)^{\mathrm{der}} \subset \big(\bigtimes_{i=0}^{\ell} \mathrm{Gl}(V'_{i\,K}) \times \{\mathrm{id}_{H_{i\,K}}\}\big)$$

and since we have seen in Claim 8.10 that $\mathrm{MT}(W_{i\,K})^{\mathrm{der}} \otimes_K \mathbb{C}$ respects the decomposition $\mathbb{V}'_{i\,K} \otimes_K \mathbb{C} = \mathbb{V}_i \oplus \mathbb{V}_i^{\vee}$, the action of $\mathrm{MT}^{\mathrm{mov}}(W_K)^{\mathrm{der}} \otimes_K L$ leaves for each $\underline{k} \in \mathcal{I}$ the subspaces

$$\langle \gamma_{\underline{k}} \rangle_L = J \cap H^0\Big(Y, \bigotimes_{i=1}^{\ell_1} \mathbb{V}_i'^{\otimes k_i} \otimes \bigotimes_{i=\ell_1+1}^{\ell_2} \big(\mathbb{V}_i^{\otimes k_i} \otimes \mathbb{V}_i^{\vee \otimes k_i'}\big) \otimes \bigotimes_{i=\ell_2+1}^{\ell} \mathbb{V}_i'^{\otimes k_i}\Big)$$

invariant. So one obtains a homomorphism

$$\mathrm{MT}^{\mathrm{mov}}(W_K)^{\mathrm{der}} \otimes_K L \longrightarrow \mathrm{Gl}(\langle \gamma_{\underline{k}} \rangle_L) = L^*,$$

necessarily trivial. In particular, the class $\gamma_{\underline{k}}$ is invariant under the group $\mathrm{MT}^{\mathrm{mov}}(W_K)^{\mathrm{der}} \otimes_K L$.

Since both $\sum_{\underline{k}} \gamma_{\underline{k}}$ and $\eta = \sum_{\mathcal{I}} \gamma_{\underline{k}} \otimes h_{\underline{\kappa}^0}$ are defined over K, they are invariant under $\mathrm{MT}^{\mathrm{mov}}(W_K)^{\mathrm{der}}$, as claimed. q.e.d.

9. Variations of Hodge structures of low rank

In this section we will discuss the 'complexity condition' 2) in Theorem 0.2, b) for $\mathbb{C}$-variations of Hodge structures of low rank.

Assumptions 9.1. The $\mathbb{C}$-variation of Hodge structures $\mathbb{V}$ is non unitary, irreducible with unipotent monodromy at infinity and it satisfies the Arakelov equality. By Theorem 0.3 $\mathbb{V}$ is pure for some i, and we assume that Ω_i is of type A or B. We write Ω, T, and n for Ω_i, its dual, and its rank and M for the corresponding factor of the universal covering $\tilde{U}$. As usual $(E = E^{1,0} \oplus E^{0,1}, \theta)$ denotes the Higgs bundle of $\mathbb{V}$, the Hodge numbers are $\ell = \mathrm{rk}(E^{1,0})$ and $\ell' = \mathrm{rk}(E^{0,1})$, hence the period map is given by a morphism $M \to \mathrm{SU}(\ell, \ell')$.

We will assume moreover, that $\omega_Y(S)$ is ample or that the following strengthening of the condition $(\star)$ holds.

Condition 9.2.

i. If $\mathcal{F}$ and $\mathcal{G}$ are two μ-stable torsion free coherent sheaves, then $\mathcal{F} \otimes \mathcal{G}$ is μ-polystable.
ii. If $\mathcal{F}$ is a μ-stable torsion free coherent sheaf, then $\mathcal{F}$ admits an admissible Hermite-Einstein metric, as defined in [**BS94**].

The Condition 9.2 will allow to apply [**VZ07**, Lemma 2.7], saying that the Higgs field θ respects the socle filtration. In particular, the μ-polystability of $E^{1,0}$ will imply the μ-polystability of $E^{1,0} \otimes T$, hence the μ-polystability of $E^{0,1}$.

Lemma 9.3. *If $\omega_Y(S)$ is ample, then the Condition 9.2 hold true.*

Proof. In [**BS94**] it is shown, that a reflexive sheaf on a compact Kähler manifold admits an admissible Hermite-Einstein metric if and only if it is μ-polystable. Part i) follows from the fact, that a tensor product of two admissible Hermite-Einstein metrics is again admissible Hermite-Einstein. In fact, in [**BS94**] admissibility of metrics h_i on bundles $\mathcal{V}_i$ asks for two conditions. First, the curvatures F_i should be square integrable and second their traces ΛF_i should be uniformly bounded. The curvature of $h_1 \otimes h_2$ is $F_1 \otimes \mathrm{Id}_2 + \mathrm{Id}_1 \otimes F_2$. Thus, if h_i are admissible, so is $h_1 \otimes h_2$, and the claim follows. q.e.d.

Recall that by 7.2 the length $\varsigma(\mathbb{V}) = \varsigma((E,\theta))$ of the Higgs subbundle $\bigwedge^\ell(E,\theta)$ satisfies

$$\mathrm{Min}\{\ell,\ell'\} \geq \varsigma(\mathbb{V}) \geq \frac{\ell \cdot \ell' \cdot (n+1)}{(\ell+\ell')\cdot n}. \tag{9.1}$$

Since $\mathbb{V}$ irreducible, by Addendum 7.20, III) the bundle $E^{1,0}$ is μ-stable if and only if the right hand side of 9.1 is an equality. Since (9.1) is symmetric in ℓ and ℓ', in order to verify the equality in certain cases, we are allowed to replace $\mathbb{V}$ by $\mathbb{V}^\vee$ and assume that $\ell \leq \ell'$. One obtains:

Property 9.4. The irreducibility of $\mathbb{V}$ implies that $n \cdot \ell \geq \ell' \geq \ell$. If $\ell' = n \cdot \ell$ the numerical condition 2) in Theorem 0.2 holds, hence the right hand side of 9.1 is an equality. In particular this is the case for $n = 1$, as said already in Lemma 7.3.

Example 9.5. Assume $\ell = 1$. Since $E^{1,0}$ is invertible, $E^{0,1}$ is the saturated hull of the μ-stable sheaf $E^{1,0} \otimes T$, hence of rank $\ell' = n$, and (9.1) is an equality.

Lemma 9.6. *The Hodge bundle $E^{1,0}$ can not have a torsion free μ-stable quotient sheaf $\mathcal{V}$ with $\mu(\mathcal{V}) = \mu(E^{1,0})$, such that $\mathcal{V} \otimes T$ is μ-stable.*

Proof. Obviously ii) is a special case of i). Assume there exists a torsion free μ-stable quotient sheaf $\mathcal{V}$ with $\mu(\mathcal{V}) = \mu(E^{1,0})$, such that $\mathcal{V} \otimes T$ is μ-stable. To be allowed to replace $\mathcal{V}$ by its reflexive hull, we only assume that there is a morphism $E^{1,0} \to \mathcal{V}$ which is surjective on some open dense subscheme and that $\mu(\mathcal{V}) = \mu(E^{1,0})$.

In order to keep notations consistent with [**VZ07**, Section 2], we will first study the dual situation, hence a subbundle $\mathcal{V}'$ of $E^{0,1}$. Recall that the socle $\mathcal{S}_1(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ is the smallest saturated subsheaf containing all μ-polystable subsheaves of $\mathcal{F}$ of slope $\mu(\mathcal{F})$. By [**VZ07**, Lemma 2.7] the Property 9.2, i) implies that the Higgs field θ respects the socle, in particular for $\mathcal{V}' \subset \mathcal{S}_1(E^{0,1})$ the preimage $\theta^{-1}(\mathcal{V}' \otimes \Omega)$ is contained in $\mathcal{S}(E^{1,0})$. Since (E, θ) is the Higgs bundle of an irreducible variation of Hodge structures, $\theta^{-1}(\mathcal{V}' \otimes \Omega) \neq 0$. In fact, $\theta^\vee : E^{1,0} \otimes T \to E^{0,1}$ is surjective, since the cokernel would be a Higgs subbundle of (E, θ) of degree zero.

So $\theta^{-1}(\mathcal{V}' \otimes \Omega)$ is a non-trivial subsheaf of the socle, hence μ-polystable. The μ-stability of $\mathcal{V}' \otimes \Omega$ implies that $\theta^{-1}(\mathcal{V}' \otimes \Omega)$ contains a direct factor which is μ-equivalent to $\mathcal{V}' \otimes \Omega$.

Applying this to the cosocle $\mathcal{S}'(E^{1,0})$, i.e. to the dual of $\mathcal{S}(E^{1,0\vee})$ one finds a quotient sheaf of $E^{0,1}$ which is μ-equivalent to $\mathcal{V} \otimes T$. So (E, θ) has a quotient Higgs bundle whose reflexive hull is isomorphic to $\mathcal{Q} = \mathcal{V} \oplus \mathcal{V} \otimes T$. Lemma 5.4, ii), applied to $\mathcal{Q} = \mathcal{V} \oplus \mathcal{V} \otimes T$, and the Arakelov equality imply that

$$0 \leq \mu(\mathcal{Q})\mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\mathcal{V}) \cdot \mu(\mathcal{V}) + \mathrm{rk}(\mathcal{V}) \cdot n \cdot (\mu(\mathcal{V}) - \mu(\Omega)) =$$
$$\mathrm{rk}(\mathcal{V}) \cdot (\mu(E^{1,0}) + n \cdot (\mu(E^{1,0}) - \mu(\Omega))) = \mathrm{rk}(\mathcal{V}) \cdot (\mu(E^{1,0}) + n \cdot \mu(E^{0,1})).$$

On the other hand, the property 9.4 implies that

$$0 = \ell \cdot \mu(E^{1,0}) + \ell' \cdot \mu(E^{1,0}) \geq \ell \cdot (\mu(E^{1,0}) + n \cdot \mu(E^{1,0})),$$

hence that $\mu(\mathcal{Q}) = 0$. Since $\mathbb{V}$ is irreducible, (E, θ) can not have a Higgs subbundle of degree zero, a contradiction. q.e.d.

Example 9.7. If $\ell = 2$ and if the μ-semistable sheaf $E^{1,0}$ was not μ-stable, one would find an invertible quotient, contradicting Lemma 9.6, ii).

Hence $E^{1,0}$ is μ-stable, and the right hand side of (9.1) is an equality. Since $\mathrm{Min}\{\ell, \ell'\} = 2$ the only solution is $\ell' = 2 \cdot n$ and $\varsigma(\mathbb{V}) = 2$.

Next we will consider the case of a rank two quotient of $E^{1,0}$. To this aim, we have to analyze the holonomy group:

Lemma 9.8. *Let $\mathcal{V}$ be a μ-stable torsion free quotient sheaf of $E^{1,0}$ of rank two with $\mu(\mathcal{V}) = \mu(E^{1,0})$. Then $n = 2$ and for some invertible sheaf $\mathcal{N}$ one has an isomorphism $\mathcal{V}^{\vee\vee} \cong T \otimes \mathcal{N}$.*

Proof. By Lemma 9.6, ii) $\mathcal{V}$ has to be μ-stable. Moreover, since the assumptions are compatible with replacing U by an étale covering, $\mathcal{V}$ remains μ-stable under pullback to such a covering. By Lemma 9.6, i) the sheaf $\mathcal{V} \otimes T$ can not be μ-stable. So in order to finish the proof of the Lemma 9.8 it just remains to verify:

Claim 9.9. Let $\mathcal{V}$ be a rank 2 torsion free sheaf on Y, whose pullback to any integral proper étale covering remains μ-stable. If $\mathcal{V} \otimes T$ is not μ-stable, then $n = 2$ and $\mathcal{V}^{\vee\vee} \cong T \otimes \mathcal{N}$.

Proof. For a sheaf $\mathcal{V}$ of rank two, the only irreducible Schur functors are of the form $\{k-a, a\}$, for $a \leq \frac{k}{2}$. By [**FH91**], 6.9 on p. 79, one has

$$\mathbb{S}_{\{k-a,a\}}(\mathcal{V}) = \begin{cases} \mathbb{S}_{\{k-2a\}}(\mathcal{V}) = S^{k-2a}(\mathcal{V}) \otimes \det(\mathcal{V})^a & \text{if} \quad 2a < k \\ \mathbb{S}_{\{a,a\}}(\mathcal{V}) = \det(\mathcal{V})^a & \text{if} \quad 2a = k \end{cases}.$$

Claim 9.10. The sheaves $S^m(\mathcal{V})$ (and $S^m(T)$) are μ-stable, for all m. Moreover, the holonomy group of $S^m(T)$ with respect to the Hermite-Einstein metric is the full group $\mathrm{U}(n)$.

Proof. Otherwise, the holonomy group with respect to the Hermitian-Einstein metric on $S^m(\mathcal{V})$ (or on $S^m(T)$) is not irreducible. Note that the holonomy group of the tensor product of Hermitian vector bundles is just the tensor product of the holonomy groups of the different factors.

Consequently, a non-trivial splitting of $S^m(\mathcal{V})$ (resp. of $S^m(T)$) forces the holonomy group of $\mathcal{V}$ (resp. of T) with respect to the Hermite-Einstein metric to be strictly smaller than $\mathrm{U}(2)$ (resp. smaller than $\mathrm{U}(n)$).

It is known that a proper subgroup of $\mathrm{U}(2)$ is a semi-product of the torus with $\mathbb{Z}_2$. So one obtains a splitting of $\mathcal{V}$ on some étale double cover.

For T we use instead [**Ya93**] (see also [**VZ07**, Section 1]), saying that the holonomy group of T is $\mathrm{U}(n)$. q.e.d.

Let us continue the proof of Claim 9.9. Assume that $\mathcal{V} \otimes T$ contains a subsheaf $\mathcal{N}$ of the same slope and of rank $r < 2 \cdot \mathrm{rk}(T) = 2 \cdot n$. Since $\mathcal{V} \otimes T$ is μ-polystable, $\mathcal{N}$ is a direct factor. Replacing $\mathcal{N}$ by its complement in $\mathcal{V} \otimes T$, if necessary, we may assume that $r \leq n$.

By taking the r-th wedge product one obtains an inclusion of $\mathcal{L} = \bigwedge^r \mathcal{N}$ into $\bigwedge^r (\mathcal{V} \otimes T)$, and both sheaves have the same slope. Here and later on, the wedge products of a torsion free sheaf is the reflexive hull of the corresponding wedge product on the open set, where the sheaf is locally free.

By [**FH91**, p. 80], for example, one has a decomposition

$$\bigwedge^r (\mathcal{V} \otimes T) = \bigoplus \mathbb{S}_\lambda(\mathcal{V}) \otimes \mathbb{S}_{\lambda'}(T)$$

where the sum is taken over all partitions λ of r with at most 2 rows and n columns and where λ' is the partition complementary to λ. The rank one subsheaf $\mathcal{L}$ of $\bigwedge^r (\mathcal{V} \otimes T)$ must inject to $\mathbb{S}_\lambda(\mathcal{V}) \otimes \mathbb{S}_{\lambda'}(T)$ for some λ. Again both sheaves are μ-semistable of slope $\mu(\mathcal{L})$. Moreover, for $\lambda = \{a, a\}$ the rank of $\mathbb{S}_{\lambda'}(T)$ is strictly larger than one, and the Claim 9.10 implies that neither $\mathbb{S}_{\lambda}(\mathcal{V})$ nor $\mathbb{S}_{\lambda'}(T)$ can be invertible.

Let us assume that $n = 2$. If $r = 2$, the only possibilities for λ are $\{2, 0\}$ or $\{1, 1\}$. In the first case $\mathbb{S}_{\lambda}(\mathcal{V}) = \det(\mathcal{V})$, and in the second case $\mathbb{S}_{\lambda'}(T) = \det(T)$. So both are excluded.

If $\mathcal{N}$ is a subbundle of rank one, we obtain a non-trivial map $\mathcal{N} \otimes \Omega \to \mathcal{V}$. Since both sheaves are μ-stable of the same slope this must be an isomorphism on some dense open subset, and since $\Omega = T \otimes \det(\Omega)$ we are done.

So assume from now on that $n \geq 3$. A non-zero projection of $\mathcal{L}$ to some Schur functor $\mathcal{L} \to \mathbb{S}_{\lambda}(\mathcal{V}) \otimes \mathbb{S}_{\lambda'}(T)$ gives again rise to a non-zero map

$$\mathbb{S}_{\lambda}(\mathcal{V})^{\vee} \otimes \mathcal{L} \longrightarrow \mathbb{S}_{\lambda'}(T)$$

between μ-polystable bundles of rank strictly larger than 1 and of the same slope. Claim 9.10 implies that this is an isomorphism.

Hence the holonomy group of $\mathbb{S}_{\lambda'}(T)$ with respect to the Hermitian-Yang-Mills connection is isomorphic to the holonomy group of $\mathbb{S}_{\lambda}(\mathcal{V})^{\vee}$, up to twisting by scalars. Holonomy groups are compatible with Schur functors, so the $\mathbb{S}_{\lambda}$-representation of the holonomy group of $\mathcal{V}$ is isomorphic to $\mathbb{S}_{\lambda'}$ applied to the holonomy group of T_Y, which by Claim 9.10 is $\mathrm{U}(n)$.

Since $\mathbb{S}'_{\lambda}$ is not the determinant representation, this representation is almost faithful (with the kernel contained in the subgroup of scalar matrices). Since the holonomy group of $\mathcal{V}$ is $\mathrm{U}(2)$, it is too small to contain an almost faithful representation of $\mathrm{U}(n)$ for $n \geq 3$ one obtains a contradiction. So n must be two, and we handled this case already.

q.e.d.

Example 9.11. If $\ell = 3$ and if $n \geq 3$, then the right hand side of (9.1) is an equality, hence

$$3 \geq \varsigma(\mathbb{V}) = \frac{3 \cdot \ell' \cdot (n+1)}{(3+\ell') \cdot n} > 1.$$

For $\varsigma(\mathbb{V}) = 3$ one finds $\ell' = n \cdot \ell$. For $\varsigma(\mathbb{V}) = 2$ the only possibility is $n = \ell' = 3$.

Proof. If $E^{1,0}$ is not μ-stable, it has a torsion free quotient sheaf $\mathcal{V}$ of slope $\mu(E^{1,0})$, either of rank one or of rank two. Both cases have been excluded, by the Lemmata 9.6 and 9.8.

For $\varsigma = \varsigma(\mathbb{V})$ the equality implies that $\ell' = \frac{\varsigma \cdot 3 \cdot n}{(3-\varsigma)\cdot n + 3}$. For $\varsigma = 1$ there is no solution in $\mathbb{Z}_{\geq 3}$, and for $\varsigma = 2$ the only solutions are $(\ell', n) = (3,3)$, $(4,6)$ or $(5,15)$. To exclude the last two cases, consider the non-trivial map

$$S^2(T) \otimes \det(E^{1,0}) \xrightarrow{\tau^{(2)}} E^{1,0} \otimes \bigwedge^2 (E^{0,1}).$$

Since both sides have the same slope, $\tau^{(2)}$ must be injective. However the inequality

$$\frac{(n+1)\cdot n}{2} \leq \ell \cdot \frac{\ell' \cdot (\ell' - 1)}{2}.$$

is violated for $(\ell', n) = (4,6)$ or $(5,15)$. q.e.d.

Example 9.12. For $n = 2$ the right hand side of (9.1) is an equality, except possibly for $\ell' = 5$.

Proof. The inequality (9.1) says that

$$3 \geq \varsigma(\mathbb{V}) \geq \frac{3 \cdot \ell' \cdot 3}{(3 + \ell') \cdot 2}.$$

Since $\ell' \geq 3$ the right hand side is strictly larger than 2, hence $\varsigma(\mathbb{V}) = 3$, and the morphism

$$\det(E^{1,0}) \otimes S^3(T) \xrightarrow{\tau^{(3)}} \bigwedge^3 (E^{0,1})$$

is non-zero. Since both sides have the same slope, for $\ell' = 3$ this contradicts the stability of $S^3(T)$. For $\ell' = 4$ the saturated image of $\tau^{(3)}$ is $\bigwedge^3(E^{0,1})$. Hence the latter and $E^{0,1}$ are both μ-stable. The compatibility of the Higgs field with the socle filtration implies that $E^{1,0}$ is μ-stable, and hence the right hand side of (9.1) must be an equality. Obviously this is a contradiction. q.e.d.

Altogether we verified:

Proposition 9.13. *Under the Assumptions* 9.1 *the numerical condition 2) in Theorem* 0.2*, b) holds in the following cases:*

1. $n = 1$.
2. $n = 2$, $\ell \leq 3$, $\ell \leq \ell'$ *and* $\ell' \neq 5$.
3. $n \geq 3$, $\ell \leq 3$, *and* $\ell \leq \ell'$.

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Goethe Universität Frankfurt
FB Mathematik
60328 Frankfurt am Main, Germany
E-mail address: moeller@math.uni-frankfurt.de

Universität Duisburg-Essen
Mathematik
45117 Essen, Germany
E-mail address: viehweg@uni-due.de

Universität Mainz
Fachbereich 17, Mathematik
55099 Mainz, Germany
E-mail address: kzuo@mathematik.uni-mainz.de