

CORK TWISTING EXOTIC STEIN 4-MANIFOLDS

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Abstract

From any 4-dimensional oriented handlebody X without 3- and 4-handles and with $b_2 \geq 1$, we construct arbitrary many compact Stein 4-manifolds that are mutually homeomorphic but not diffeomorphic to each other, so that their topological invariants (their fundamental groups, homology groups, boundary homology groups, and intersection forms) coincide with those of X . We also discuss the induced contact structures on their boundaries. Furthermore, for any smooth 4-manifold pair (Z, Y) such that the complement $Z - \text{int } Y$ is a handlebody without 3- and 4-handles and with $b_2 \geq 1$, we construct arbitrary many exotic embeddings of a compact 4-manifold Y' into Z , such that Y' has the same topological invariants as Y .

1. Introduction

A basic problem of 4-manifold topology is to find all exotic copies of smooth 4-manifolds, in particular to find various methods of constructing different smooth structures on 4-manifolds (e.g., logarithmic transform [12], Fintushel and Stern's rational blowdown [19] and knot surgery [20]). The purpose of this paper is to approach this problem by corks and give applications. Since different smooth structures on a 4-manifold can be explained by existence corks that divide the manifold into two Stein pieces [4], cork twisting Stein manifolds is a central theme of this paper.

The first cork was introduced in [1], and was used in [2] to construct a pair of two simply connected compact 4-manifolds with boundary and second betti number $b_2 = 1$ that are homeomorphic but non-diffeomorphic. Later it turned out that cork twists easily give many such pairs (Akbulut and Matveyev [3]; the authors [6]), where each pair consists of a Stein 4-manifold and a non-Stein 4-manifold, and hence they are not diffeomorphic.

It is thus interesting to find exotic Stein 4-manifold pairs. Uniqueness of diffeomorphism types of Stein 4-manifolds bounding certain 3-manifolds are known (e.g., $\#_n S^1 \times S^2$ ($n \geq 0$); for more examples,

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see [26] and the references mentioned therein). By contrast, Akhmedov-Etnyre, Mark, and Smith [10] constructed infinitely many simply connected compact Stein 4-manifolds which are mutually homeomorphic but non-diffeomorphic, using knot surgery. Moreover, the induced contact structures on their boundary are mutually isomorphic. Though these 4-manifolds have large second betti number, later in [8] for each $b_2 \geq 1$, by using corks, the authors constructed pairs of simply connected compact Stein 4-manifolds which are homeomorphic but non-diffeomorphic.

In this paper, by using properties of Stein 4-manifolds we extend the previous simple cork constructions to an explicit algorithm. Here (4-dimensional oriented) *2-handlebody* means a compact, connected, oriented smooth 4-manifold obtained from the 4-ball by attaching 1- and 2-handles. The algorithm goes roughly as follows: Take any 2-handlebody with $b_2 \geq 1$, and change the handle diagram into a certain form and add appropriate corks to produce compact Stein 4-manifolds; then, by twisting these corks, detect the change of smooth structures by the adjunction inequalities. This construction generalizes the carving technique of [4].

This process gives arbitrary many mutually homeomorphic but not diffeomorphic compact Stein 4-manifolds that have the same topological invariants as the given 2-handlebody (see Theorems 5.16 and 6.3, for details). We obtain the following theorem:

Theorem 1.1. *Let X be any 4-dimensional 2-handlebody with the second betti number $b_2(X) \geq 1$. Then, for each $n \geq 1$, there exist 2-handlebodies X_i ($0 \leq i \leq n$) with the following properties:*

- (1) *The fundamental group, the integral homology groups, the integral homology groups of the boundary, and the intersection form of each X_i ($0 \leq i \leq n$) are isomorphic to those of X .*
- (2) *X_i ($0 \leq i \leq n$) are mutually homeomorphic but non-diffeomorphic.*
- (3) *Each X_i ($1 \leq i \leq n$) has a Stein structure.*
- (4) *X can be embedded into X_0 . Hence, X_0 does not admit any Stein structure if X cannot be embedded into any simply connected minimal symplectic 4-manifold with $b_2^+ > 1$. (For more non-existence conditions, see Theorems 5.16 and 6.3.)*
- (5) *Each X_i ($0 \leq i \leq n$) can be embedded into X .*

As far as the authors know, this result is new even when we ignore Stein structures. Actually, this theorem gives exotic smooth structures for a large class of compact 4-manifolds with boundary (see also Corollary 10.8). In Section 10, we also construct arbitrary many exotic non-Stein 4-manifolds.

For a given embedding of a 4-manifold, applying the algorithm to its complement, we obtain arbitrary many exotic embeddings of a 4-manifold which has the same the topological invariants as the given manifold (see Theorems 5.17 and 6.4).

Theorem 1.2. *Let Z and Y be compact connected oriented smooth 4-manifolds (possibly with boundary). Suppose that Y is embedded into Z and that its complement $X := Z - \text{int } Y$ is a 2-handlebody with $b_2(X) \geq 1$. Then, for each $n \geq 1$, there exist mutually diffeomorphic compact connected oriented smooth 4-manifolds Y_i ($0 \leq i \leq n$) embedded into Z with the following properties.*

- (1) *The pairs (Z, Y_i) ($0 \leq i \leq n$) are mutually homeomorphic but non-diffeomorphic.*
- (2) *The fundamental group, the integral homology groups, the integral homology groups of the boundary, and the intersection form of Y_i 's ($0 \leq i \leq n$) are isomorphic to those of Y .*
- (3) *The each complement $X_i := Z - \text{int } Y_i$ ($0 \leq i \leq n$) has the properties of the X_i in Theorem 1.1 above (corresponding to X).*

Note that any compact connected oriented smooth 4-manifold Z (possibly with boundary) has such a submanifold Y , because the 4-ball contains a 2-handlebody $S^2 \times D^2$, for example. Hence this theorem shows that every compact connected oriented smooth 4-manifold has arbitrary many exotic embeddings into it, and has arbitrary many compact sub 4-manifolds that are mutually homeomorphic but not diffeomorphic.

We further state the properties of X_i 's in Theorem 1.1. At least in the case of $b_2(X) = 1$, the induced contact structures on the boundary ∂X_i 's have the property below. This also shows that X_i 's are mutually non-diffeomorphic.

Corollary 1.3. *Let X be any 2-handlebody with $b_2(X) = 1$. Suppose that the intersection form of X is non-zero. Fix $n \geq 1$, and denote by X_i ($1 \leq i \leq n$) the corresponding compact Stein 4-manifold in Theorem 1.1. Let ξ_i ($1 \leq i \leq n$) be the contact structure on the boundary ∂X_i ($\cong \partial X_1$) induced by the Stein structure on X_i . Then the each smooth 4-manifold X_i ($1 \leq i \leq n - 1$) admits no Stein structure compatible with ξ_j for any $j > i$.*

It is interesting to discuss cork structures of 4-manifolds (see [6, 7, 9]). In [9], the authors constructed the following example: For each $n \geq 2$, there are n mutually disjoint embeddings of the same cork into a simply connected compact 4-manifold Z_n with boundary, so that twisting Z_n along each copy of the cork produces mutually distinct n smooth structures on Z_n . However, $b_2(Z_n)$ increases when n increases. The above X_i 's have the structures below. We also discuss infinitely many disjoint embeddings in Section 8.

Corollary 1.4. *Let X be any 2-handlebody with $b_2 \geq 1$. For each $n \geq 1$, there exist 2-handlebodies X_i ($0 \leq i \leq n$), a cork (C, τ) , and disjointly embedded copies C_i ($1 \leq i \leq n$) of C into X_0 with the following properties:*

- (1) X_i ($0 \leq i \leq n$) are mutually homeomorphic but non-diffeomorphic.
- (2) Each X_i ($1 \leq i \leq n$) is the cork twist of X_0 along (C_i, τ) .
- (3) Each X_i ($1 \leq i \leq n$) is the manifold of Theorem 1.1, corresponding to this X .

In a forthcoming paper, we will discuss Theorems 1.1 and 1.2 in the case of $b_2(X) = 0$, under some conditions.

This paper is organized as follows. In Sections 2 and 3, we briefly discuss basics of corks, Stein 4-manifolds, and contact 3-manifolds. In Section 4, we study effects of certain operations related to corks. In Section 5, we give the algorithm and prove Theorems 1.1 and 1.2 and Corollary 1.4. In Section 6, we strengthen the algorithm. In Section 7, we prove Corollary 1.3. In Section 8, we construct infinitely many disjoint embeddings of a fixed cork into a noncompact 4-manifold. In Section 9, we apply Theorems 1.1 and 1.2 to some examples X and $(Y, Z) = (S^4, \Sigma_g \times D^2)$ ($g \geq 1$), where Σ_g denotes the closed surface of genus g . In Section 10, we construct arbitrary many compact Stein 4-manifolds and arbitrary many non-Stein 4-manifolds which are mutually homeomorphic but non-diffeomorphic.

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2. Corks

In this section, we recall corks. For more details, the reader can consult [6].

Definition 2.1. Let C be a compact contractible Stein 4-manifold with boundary and $\tau : \partial C \rightarrow \partial C$ an involution on the boundary. We call (C, τ) a *cork* if τ extends to a self-homeomorphism of C but cannot extend to any self-diffeomorphism of C . For a cork (C, τ) and a smooth 4-manifold X that contains C , a *cork twist of X along (C, τ)* is defined to be the smooth 4-manifold obtained from X by removing the submanifold C and regluing it via the involution τ . Note that, any cork twist does not change the homeomorphism type of X (see the remark below). A

cork (C, τ) is called a *cork of X* if the cork twist of X along (C, τ) is not diffeomorphic to X .

Remark 2.2. In this paper, we always assume that corks are contractible. (We did not assume this in the more general definition of [6].) Freedman’s theorem (cf. [11]) implies that every self-diffeomorphism of the boundary ∂C extends to a self-homeomorphism of C , when C is a compact contractible smooth 4-manifold.

Definition 2.3. Let W_n be the contractible smooth 4-manifold shown in Figure 1. Let $f_n : \partial W_n \rightarrow \partial W_n$ be the obvious involution obtained by first surgering $S^1 \times D^3$ to $D^2 \times S^2$ in the interior of W_n , then surgering the other imbedded $D^2 \times S^2$ back to $S^1 \times D^3$ (i.e., replacing the dot and “0” in Figure 1). Note that the diagram of W_n is induced from a symmetric link.

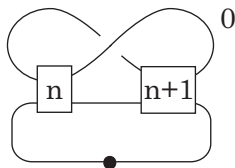


Figure 1. W_n

Theorem 2.4 ([6, Theorem 2.5]). *For $n \geq 1$, the pair (W_n, f_n) is a cork.*

3. Stein 4-manifolds and contact 3-manifolds

In this section, we briefly recall the basics of Stein 4-manifolds and contact 3-manifolds. For the definition of basic terms and more details, the reader can consult Gompf and Stipsicz [23] and Ozbagci and Stipsicz [25]. In this paper, we use Seifert framings and abbreviate them to framings. (When a knot goes over 4-dimensional 1-handles, then convert the diagram into the dotted circle notation and calculate its Seifert framing; cf. [23]). We use the following terminologies throughout this paper.

Definition 3.1. (1) For a Legendrian knot K in $\#n(S^1 \times S^2)$ ($n \geq 0$), we denote by $tb(K)$ and $r(K)$ the Thurston–Bennequin number and the rotation number of K , respectively.

(2) We call a compact connected oriented 4-dimensional handlebody a *2-handlebody* if it consists of one 0-handle and 1- and 2-handles. We call a subhandlebody a *sub 1-handlebody* if it consists of 0- and 1-handles of the whole handlebody.

(3) We call a 2-handlebody a *Legendrian handlebody* if its 2-handles are

attached to an oriented framed Legendrian link in $\partial(D^4 \cup 1\text{-handles}) = \#n(S^1 \times S^2)$ ($n \geq 0$). It is known that every 2-handlebody can be changed into a Legendrian handlebody by an isotopy of the attaching link of 2-handles and orienting its components.

(4) We call a Legendrian handlebody a *Stein handlebody* if the framing of its each 2-handle K is $tb(K) - 1$.

Next, we recall the following useful theorem.

Theorem 3.2 (Eliashberg [14]; cf. [22, 23]). *A compact, connected, oriented, smooth 4-manifold admits a Stein structure if and only if it can be represented as a Stein handlebody.*

We call a compact smooth 4-manifold with a Stein structure a *compact Stein 4-manifold*. Recall that a Stein structure induces an almost complex structure. Thus, the first Chern class c_1 of a compact Stein 4-manifold is defined. The following useful theorems are known and play important roles in this paper.

Theorem 3.3 (Gompf [22]; cf. [23]). *Let X be a Stein handlebody. The first Chern class $c_1(X) \in H^2(X; \mathbf{Z})$ is represented by a cocycle whose value on each 2-handle h attached along a Legendrian knot K is $r(K)$. Here each 2-handle is oriented according to the orientation of the corresponding Legendrian knot.*

Note that the theorem below contains the case where the genus and the self-intersection number are zero, unlike the usual adjunction inequality for closed 4-manifolds.

Theorem 3.4 (Akbulut and Matveyev [3]; cf. [25]). *Let X be a compact Stein 4-manifold and Σ a smoothly embedded genus $g \geq 0$ closed surface in X . Denote by $[\Sigma]$ the second homology class of X represented by Σ . If $[\Sigma] \neq 0$, then the following adjunction inequality holds:*

$$[\Sigma]^2 + |\langle c_1(X), [\Sigma] \rangle| \leq 2g - 2.$$

Proof. For the completeness, we give a minor correction to the proof of [25, Theorem 13.3.8]. In the $g = 0$ case, apply the same argument as the $g \geq 1$ case. (Since [25, Theorem 13.3.6] also holds in the $g = 0$ case ([18]), one can apply it.) q.e.d.

We also use the following lemma, which is easily checked by Figure 2.

Lemma 3.5. *Let K be a Legendrian knot in $\#n(S^1 \times S^2)$ ($n \geq 0$). For any integer pair (t, d) with $t \geq 1$ and $0 \leq d \leq t$, by locally adding zig-zags to K upward or downward, K can be changed so that the following are satisfied.*

- (i) *The Thurston–Bennequin number of K decreases by t .*
- (ii) *The rotation number of K increases by $2d - t$.*

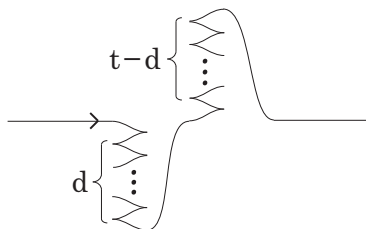


Figure 2. Adding zig-zags

Compact Stein 4-manifolds are known to admit useful embeddings.

Theorem 3.6 (Lisca and Matić [24]; Akbulut and Ozbagci [5]). *Every compact Stein 4-manifold can be embedded into a minimal closed complex surface of general type with $b_2^+ > 1$, and can be embedded into a simply connected, minimal, closed, symplectic 4-manifold with $b_2^+ > 1$. Here minimal means that there is no smoothly embedded 2-sphere with the self-intersection number -1 .*

Proof. (Simply connectedness) Since “simply connected” is not claimed in [24] and [5], we explain this part for completeness. We follow the proof in [5]. We first attach 2-handles to a given Stein 4-manifold to make it a simply connected Stein 4-manifold, then apply the procedure prescribed in [5]. Since this results attaching 2-, 3-, and 4-handles to the boundary, the simply connectedness is preserved. q.e.d.

A compact Stein 4-manifold X induces a contact structure ξ on its boundary ∂X . If its Chern class $c_1(\xi) \in H^2(\partial X; \mathbf{Z})$ is a torsion, then the contact invariant $d_3(\xi) \in \mathbf{Q}$ (called the 3-dimensional invariant) is defined by

$$d_3(\xi) = \frac{1}{4}(c_1(X)^2 - 2e(X) - 3\sigma(X)),$$

where $e(X)$ and $\sigma(X)$ denote the Euler characteristic and the signature of X , respectively. For a computation of $c_1(X)^2$, see [23, 25]. The lemma below is easily verified.

Lemma 3.7 (cf. [23]). *Let X be a compact Stein 4-manifold with $b_2(X) = 1$. Denote the generator of the second homology group of X by v . Suppose $v^2 \neq 0$; then*

$$c_1(X)^2 = \frac{\langle c_1(X), v \rangle^2}{v^2}.$$

4. $W^+(p)$ - and $W^-(p)$ -modifications

In this section, we study the effects of the operations below. We first define them for smooth 2-handlebodies and later redefine them for Legendrian handlebodies. In this paper, the words the “attaching circle of

a “2-handle” and a “smoothly embedded surface” are often abbreviated to a “2-handle” and a “surface,” if they are clear from the context.

Definition 4.1. Assume $p \geq 1$. Let K be a 2-handle of a (smooth) 2-handlebody. Take a small segment of the attaching circle of K as in the first row of Figure 3.

We call the local operations shown in the left and the right side of Figure 3 a $W_1^+(p)$ -modification to K and a $W_1^-(p)$ -modification to K , respectively. Here we do not change the framing of K (ignore the orientations shown in the figure). They are clearly related by a cork twist along (W_1, f_1) as shown in the figure.

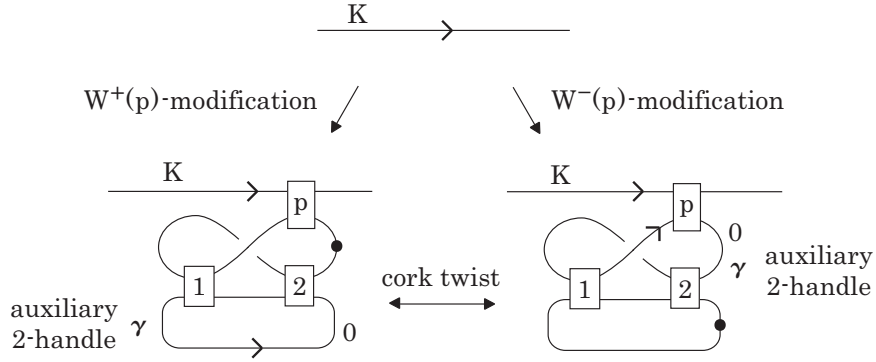


Figure 3. $W_1^\pm(p)$ -modifications ($p \geq 1$) (the framing of K is unchanged)

We will call the 0-framed 2-handle γ on the left (or right) side of the Figure 3 the *auxiliary 2-handle* of the $W_1^\pm(p)$ -modification of K . We will use the same symbol K for the new 2-handle obtained from the original K of X by the modification.

For convenience, we refer to the $W_1^+(0)$ - and $W_1^-(0)$ -modifications as undone operations. For brevity, sometimes we will call these operations W_1^+ - and W_1^- -modifications when we do not need to specify the coefficients, or will call them W_1 -modifications when we do not need to specify both the coefficient and \pm . Clearly the name of this operation comes from the W_1 cork of [6]. Similarly, we can talk about $W^\pm(p)$ -modification for any cork (W, f) coming from a symmetric link.

For the rest of this paper, we will discuss the effects of W -modification where $(W, f) = (W_1, f_1)$. In the rest of this section, we assume $p \geq 1$.

Proposition 4.2. *Let K be a 2-handle of a 2-handlebody X . Any W -modification to K does not change the isomorphism classes of the fundamental group, the integral homology groups, the integral homology groups of the boundary ∂X , and the intersection form of X .*

Proof. Since the 0-framed auxiliary 2-handle links with the 1-handle algebraically once, each operation does not change the fundamental group, the integral homology groups, and the intersection form. We next check the boundary. Recall that the integral homology groups of the boundary of any simply connected 2-handlebody are determined by its intersection form (cf. [23]). So we first replace the dots of the dotted circles of X with 0's, that is, surgery $S^1 \times D^3$'s to $D^2 \times S^2$'s. We now have a simply connected 2-handlebody. Next, apply the W -modification to K . This modification keeps the intersection form and the simply connectedness. Moreover, the boundary of this result is diffeomorphic to the boundary of the result of the W -modification to K of X . Therefore, any W -modification does not affect the homology groups of the boundary ∂X . q.e.d.

Proposition 4.3. *Apply a $W^+(p)$ -modification to a 2-handle K of a 2-handlebody X . Let X^+ and γ denote the result of X and the auxiliary 2-handle, respectively. Suppose that the attaching circle of the original K of X spans a smoothly embedded genus g surface in a sub 1-handlebody $\mathfrak{h}_n(S^1 \times D^3)$ ($n \geq 0$) of X . Then the new K of X^+ spans a smoothly embedded genus $g + p$ surface in a sub 1-handlebody of X^+ after sliding over the 2-handle γ p -times (homologically, this changes K to $K - p\gamma$).*

Proof. The new K is obtained by a band summing the original K and the knot U in the first picture of Figure 4. Hence, it suffices to check that U spans a smoothly embedded surface of genus p after sliding over the 2-handle γ p -times. Introduce a canceling 1- and 2-handle pair and slide γ (geometrically) twice; then we get the second picture. Isotopy gives the third picture. We then slide the knot U over the 0-framed unknot p -times so that U does not link with the lower dotted circle. We get the fourth picture, by ignoring two 2-handles, and isotopy. We can now easily see that U bounds a surface of genus p by the standard argument (cf. [23, Exercise.4.5.12.(b)]). One can check that U is the boundary of D^2 with $2p$ bands attached. Note that in the beginning, we slid γ over the -1 framed 2-handle, which does not affect the result because the sliding was over the canceling 2-handle algebraically zero times. q.e.d.

Lemma 4.4. (1) *The 4-manifolds S_1 and S_2 in Figure 5 are diffeomorphic to $S^2 \times D^2$.*

(2) *Let $\hat{f} : \partial S_1 \rightarrow \partial S_2$ be the diffeomorphism induced by the obvious cork twist of S_1 (i.e., exchanging the dot and 0). Note that the cork twist does not change the boundary of S_1 . Then \hat{f} extends to a diffeomorphism between S_1 and S_2 .*

Proof. (1). The left side of the figure is checked by canceling the 1- and 2-handle pair. The right side is as follows. Slide the middle 2-handle over its meridian as in the second picture of Figure 6. Note that

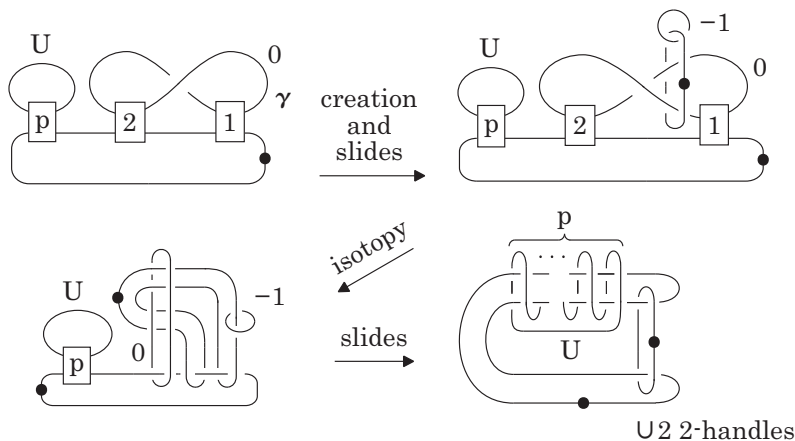


Figure 4

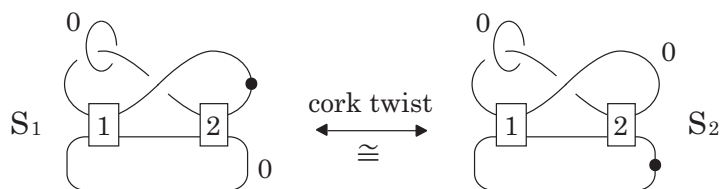


Figure 5. Two diagrams of $S^2 \times D^2$

the middle 2-handle now links with the dotted circle geometrically once. Canceling this 1- and 2-handle pair gives the last picture of the figure.

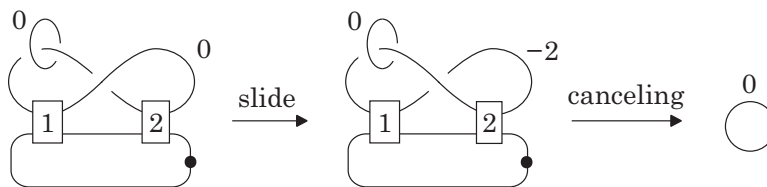


Figure 6

(2). Since f extends to a self-homeomorphism of W , \hat{f} extends to a homeomorphism between S_1 and S_2 . Thus by (1) and Gluck's theorem (Sections 5 and 15 of [21]; cf. [13]), \hat{f} extends to a diffeomorphism between S_1 and S_2 . q.e.d.

Proposition 4.5. *Let K be a 2-handle of a 2-handlebody X . Let Z be any compact connected oriented smooth 4-manifold that contains X as a smooth submanifold.*

(1) Let X^+ be the result of X by a $W^+(p)$ -modification to K . Then the following properties hold.

- (i) X^+ becomes diffeomorphic to X after attaching a 2- and a 3-handle to ∂X^+ , as in Figure 7. Hence, X^+ can be embedded into X and also Z .
- (ii) The fundamental group, the integral homology groups, the integral homology groups of the boundary, and the intersection form of $Z - \text{int } X^+$ are isomorphic to those of $Z - \text{int } X$. Here we see X^+ as a submanifold of Z , through the embedding in (i).

(2) Let X^- be the result of X^+ by replacing the above $W^+(p)$ -modification with the $W^-(p)$ -modification, as in the second row of Figure 8. Then the following properties hold.

- (i) The cork twist of Z along (W, f) is diffeomorphic to Z (see Figure 8). Here this W is the cork in $X^+(\subset Z)$ created from the $W^+(p)$ -modification, and we view X^+ as a submanifold of Z coming from the embedding in (1).(i) above. Hence, X^- can be embedded into X , and also into Z .
- (ii) $Z - \text{int } X^-$ is diffeomorphic to $Z - \text{int } X^+$. Here we see X^+ and X^- as submanifolds of Z , via the embeddings in (1).(i) and (2).(i), respectively.
- (iii) X can be embedded into X^- so that the induced homomorphism $H_*(X; \mathbf{Z}) \rightarrow H_*(X^-; \mathbf{Z})$ is an isomorphism.

(3) There exist homeomorphisms between the pairs (Z, X^+) and (Z, X^-) , and also between the pairs $(Z, Z - \text{int } X^+)$ and $(Z, Z - \text{int } X^-)$.

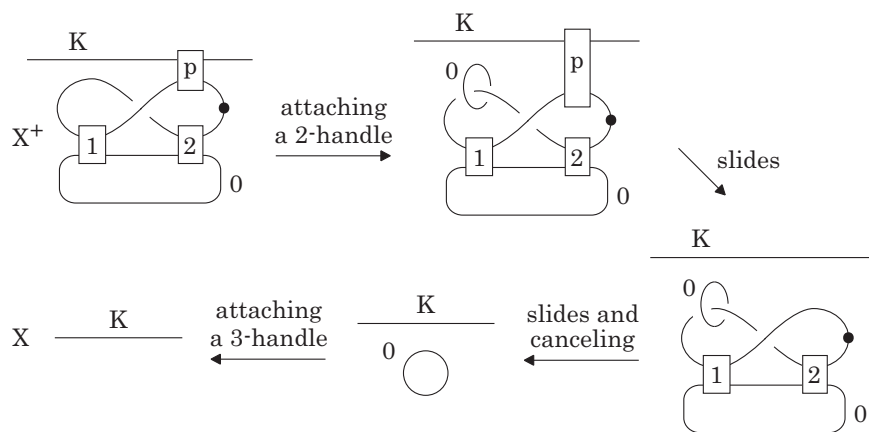


Figure 7. Attaching a 2- and a 3-handle to X^+

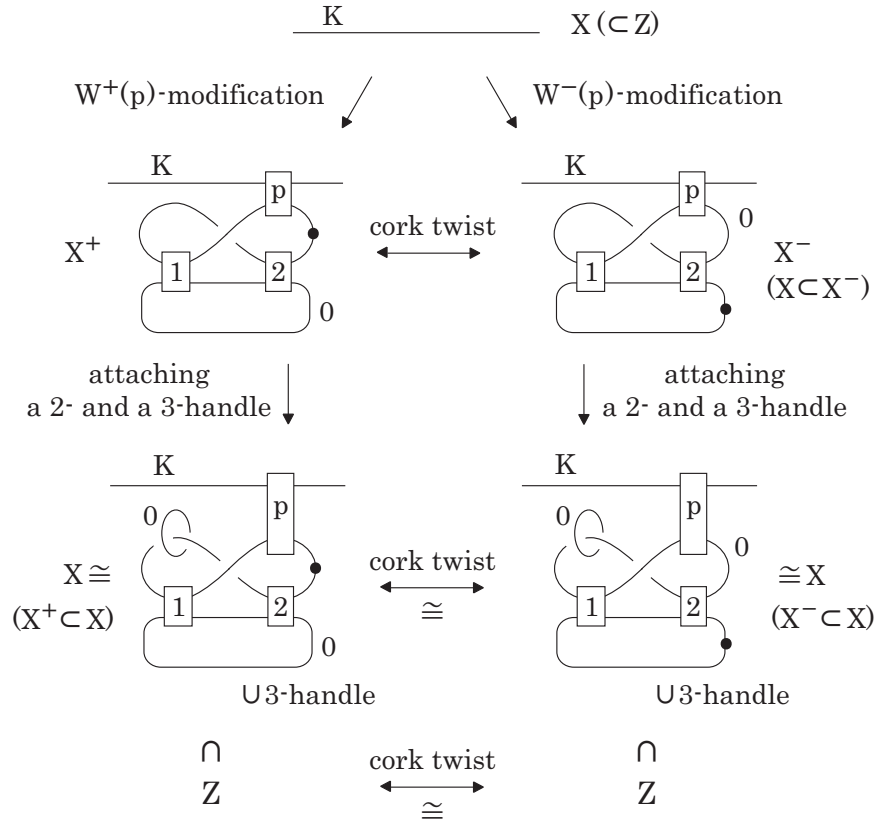


Figure 8. Relations

Proof. (1).(i). The first picture of Figure 7 is a local diagram of X^+ . Following the procedure in the figure, we recover a diagram of X . Hence the claim follows.

(1).(ii). Reverse the procedure in Figure 7 until the second picture, keeping track of the 3-handle introduced in the fifth picture. Then we see, in the second picture of this figure, that the attaching sphere of the 3-handle intersects with the belt circle of the lower 0-framed 2-handle geometrically once. In the second picture, the 3-handle algebraically also cancels the upper 0-framed 2-handle. This is because the meridian of the upper 0-framed 2-handle in the second picture becomes homotopic to a curve linking the 0-framed unknot in the fourth picture geometrically once, although it also tangles around K . Thus $Z - \text{int } X^+$ is obtained from $Z - \text{int } X$ by attaching a dual of this algebraically canceling 2- and 3-handle pair (which is an algebraically canceling 1- and 2-handle pairs).

This fact immediately gives the claim about the fundamental group, the homology groups, and the intersection form. For the claim about the homology groups of the boundary, we first cap off the boundary of Z to form a closed 4-manifold. Now the claim follows from the fact above and the argument in the proof of Proposition 4.2.

(2). Lemma 4.4 gives (i). Note that the $Z = X$ case shows that X^- (i.e., the cork twist of X^+) is embedded into X . Thus, the complements of X^+ and X^- in Z are the same, and hence (ii) follows. Since X^- is obtained from X by attaching an algebraically canceling 1- and 2-handle pair, (iii) follows.

(3). By (2), the cork twist along (W, f) changes (Z, X^+) and $(Z, Z - \text{int}X^+)$ into (Z, X^-) and $(Z, Z - \text{int}X^-)$, respectively. Since f extends to a self-homeomorphism of W , the claim follows. q.e.d.

Next, we define Legendrian versions of W^{+-} - and W^- -modifications for Legendrian handlebodies (recall Definition 3.1).

Let K be a 2-handle of a Legendrian handlebody. Take a small segment of the attaching circle of K as in the first row of Figure 10. Without loss of generality, we can assume that the orientation of the segment of K is from the left to the right (Otherwise, locally apply the Legendrian isotopy in Figure 9. Note that this isotopy does not change the Thurston-Bennequin number and the rotation number).



Figure 9. Legendrian isotopy

Definition 4.6. Let $p \geq 1$. We call the local operations shown in the left and the right side of Figure 10 a $W^+(p)$ -modification to K and a $W^-(p)$ -modification to K , respectively. Here we orient the 2-handles as in the figure. Hence, each operation produces a new Legendrian handlebody from a given Legendrian handlebody. When we see Legendrian handlebodies as smooth handlebodies, these definitions and the orientations are consistent with those in Definition 4.6 and Figure 3. (We can check this just by converting the 1-handle notation.) Note that the auxiliary 2-handle γ to any $W^+(p)$ - (resp. $W^-(p)$ -) modification satisfies the following: its framing is 0 (resp. 0); $tb(\gamma) = 2$ (resp. $tb(\gamma) = 1$); $r(\gamma) = 0$ (resp. $r(\gamma) = 1$).

The above definition clearly shows the following.

Proposition 4.7. *Let K be a 2-handle of a Legendrian handlebody.*

- (1) *Every $W^+(p)$ -modification to K has the following effect.*

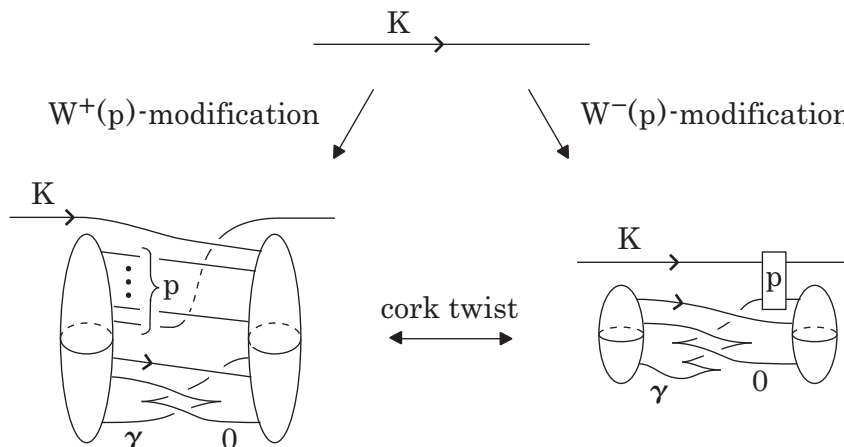


Figure 10. $W^+(p)$ - and $W^-(p)$ -modification ($p \geq 1$). Every framing is Seifert framing. The framing of K is unchanged.

- $tb(K)$ is increased by p , and $r(K)$ is unchanged.
- (2) Every $W^-(p)$ -modification to K has the following effect.
- $tb(K)$ and $r(K)$ are unchanged.

Remark 4.8. For simplicity we used only (W_1, f_1) for W -modifications. Many other corks, including (W_n, f_n) , also work similarly. For example, the operation of “creating a positron” (together with its cork twist) introduced by Akbulut and Matveyev [4] has similar effects. An important effect of W -modifications is to increase the “minimal genera” of second homology classes (under some conditions). This is implied in the next section, through the proof of Theorem 1.1. Essentially different operations (e.g., band sum with a knot with a sufficiently large Thurston-Bennequin number) also have this effect, though they do not share some other effects.

5. Exotic Stein 4-manifolds and exotic embeddings

5.1. Construction. Here we give an algorithm that provides Theorems 1.1 and 1.2. Later, in subsection 9.1, we demonstrate this algorithm on a simple example.

Definition 5.1. Let X be a compact oriented 4-dimensional 2-handlebody with $b_2(X) \geq 1$. Throughout this section, we fix this X . Let $k := b_2(X) - 1$.

Now we begin with the construction. Recall the definitions of Legendrian and Stein handlebodies in Definition 3.1. Apply the following Step 1 to X .

Step 1. Slide and isotope the handles of X so that X is a Legendrian handlebody and that its 2-handles satisfy the following condition.

- 2-handles K_j ($0 \leq j \leq k$) of X do not algebraically go over any 1-handle. So the second homology classes of X given by the 2-handles K_j ($0 \leq j \leq k$) span a basis of $H_2(X; \mathbf{Z})$. Here K_j ($0 \leq j \leq l$) denote all the 2-handles of X ($l \geq k$).

We use the following terminology.

Definition 5.2. We call a Stein handlebody a *good Stein handlebody* if it satisfies the condition described in Step 1.

Remark 5.3 (The outline of the algorithm). Here we briefly summarize the algorithm. (However, beware that the actual construction is rather different.)

- (1) The $b_2(X) = 1$ case: Apply $W^-(p_1)$ -, $W^-(p_2)$ -, \dots , $W^-(p_n)$ -modifications to K_0 of X and call the result $X_0^{(n)}$ (Figure 11). Then replace the above $W^-(p_i)$ -modification with the corresponding $W^+(p_i)$ -modification, and denote the resulting manifold by $X_i^{(n)}$. If we choose $p_1 \ll p_2 \ll \dots \ll p_n$ as sufficiently large integers, then the minimal genera of the second homology classes given by K_0 (after sliding over the auxiliary 2-handle p_i times) in $X_i^{(n)}$ ($0 \leq i \leq n$) become mutually different. We check this using the adjunction inequalities in the Stein manifolds $X_i^{(n)}$ ($1 \leq i \leq n$). We also apply $W^+(q_j)$ -modifications to other 2-handles K_j ($1 \leq j \leq l$) of X at the beginning (Figure 12). In short, the minimal genera detect the smooth structures of $X_i^{(n)}$ ($0 \leq i \leq n$).
- (2) The $b_2(X) \geq 2$ case: This case is a generalization of the $b_2(X) = 1$ case and it is more technical. In this case, we further adjust the rotation number of each K_j ($1 \leq j \leq k$) by the above $W^+(q_j)$ -modification to prevent these handles affecting adjunction inequality arguments. To detect smooth structures, we discuss the minimal genera of bases of $H_2(X_i^{(n)}; \mathbf{Z})$, using adjunction inequalities.

To proceed with the construction, we need the following basic data for X .

Definition 5.4. Denote by m_j, r_j, t_j ($0 \leq j \leq l$), the framing, the rotation number, and the Thurston–Bennequin number of K_j of X , respectively. Let g_j ($0 \leq j \leq k$) be the genus of a smoothly embedded surface in the sub 1-handlebody of X spanned by K_j . Note that the attaching circle of every K_j ($0 \leq j \leq k$) spans a surface because algebraically it does not go over any of the 1-handles (cf. [23]).

Using this data, we here define integers for the construction. Roughly speaking, the following conditions require that each integer is sufficiently large.

Definition 5.5. Put $q_0 = 0$. In the $l \geq 1$ case, define non-negative integers q_j ($1 \leq j \leq l$) so that they satisfy the following conditions.

- (i) $q_j + (t_j - 1) - m_j \geq 0$, for each $1 \leq j \leq l$.
- (ii) $q_j + (t_j - 1) - m_j \geq |r_j|$, for each $1 \leq j \leq k$ (in the $k \geq 1$ case).

Definition 5.6. Put $p_{-1} = p_0 = 0$. Define an increasing integer sequence p_i ($i \geq 1$) so that it satisfies the following conditions.

- (i) $p_i > p_{i-1}$, for each $i \geq 1$.
- (ii) $p_1 + (t_0 - 1) - m_0 \geq 0$.
- (iii) $2p_1 + (t_0 - 1) - m_0 + |r_0| + m_j > 2(g_j + q_j) - 2$, for each $0 \leq j \leq k$.
- (iv) $2p_i + (t_0 - 1) + |r_0| > 2(g_0 + p_{i-1}) - 2$, for each $i \geq 1$.

Remark 5.7. (1) In the case where $t_0 - 1 + |r_0| = 2g_0 - 2$, condition (iv) in Definition 5.6 reduces to (i).

(2) In Definitions 5.4, 5.5 and 5.6; we do not require either the maximalities or the minimalities of those numbers; therefore we can easily define those numbers.

(3) We don't need to calculate g_j and r_j for $k + 1 \leq j \leq l$, we do not use them.

We next adjust the Thurston–Bennequin numbers (and the rotation numbers) of 2-handles except K_0 . Figures 11–13 describe the local operations applied to 2-handles K_j ($0 \leq j \leq l$) of X , through the following Steps 2–5 (without specifying Legendrian diagrams).

Definition 5.8. Let \widehat{X} be the Legendrian handlebody obtained from X by applying the above Step 1 and the Step 2 below. (Skip Step 2 when $l = 0$.)

Step 2. Apply a $W^+(q_j)$ -modification and add zig-zags to each 2-handle K_j ($1 \leq j \leq l$) of X so that the following conditions are satisfied (recall Proposition 4.7, Lemma 3.5 and the conditions of q_j). Let δ_j ($1 \leq j \leq l$, $q_j \neq 0$) be the auxiliary 2-handle to the above $W^+(q_j)$ -modification. In the $l > k$ case, also add a zig-zag to each δ_j ($k + 1 \leq j \leq l$) as follows (ignore (iii) when $k = l$).

- (i) $tb(K_j) = m_j + 1$ ($1 \leq j \leq l$).
- (ii) $|r(K_j)| \leq 1$ ($1 \leq j \leq k$).
- (iii) $tb(\delta_j) = 1$ ($k + 1 \leq j \leq l$, $q_j \neq 0$)

Remark 5.9. Note that the Thurston–Bennequin number of every 2-handle of \widehat{X} except K_0 and all of δ_j ($1 \leq j \leq k$, $q_j \neq 0$) is one more than its framing.

In the rest of this section, fix a positive integer n .

Definition 5.10. Apply Steps 3, 4, and 5, and define $X_0^{(n)}$ then $X_i^{(n)}$ and γ_i as follows.

Step 3. Define $X_0^{(n)}$ as the Legendrian handlebody obtained from \widehat{X} by applying $W^-(p_1)$ -, $W^-(p_2)$ -, \dots , $W^-(p_n)$ -modifications to the 2-handle K_0 .

Step 4. In the $l \geq 1$ case, define $X_{-1}^{(n)}$ as the Legendrian handlebody obtained from $X_0^{(n)}$ by replacing every $W^+(q_j)$ -modification ($1 \leq j \leq l$, $q_j \neq 0$) applied in Step 2 with the corresponding $W^-(q_j)$ -modification. In this case, we also skip the zig-zag operations in Step 2. In the $l = 0$ case, put $X_{-1}^{(n)} := X_0^{(n)}$.

Step 5. Define $X_i^{(n)}$ ($1 \leq i \leq n$) as the Legendrian handlebody obtained from $X_0^{(n)}$ by replacing the $W^-(p_i)$ -modification applied in Step 3 with the corresponding $W^+(p_i)$ -modification. Let γ_i ($1 \leq i \leq n$) denote the auxiliary 2-handle of $X_i^{(n)}$ to the above $W^+(p_i)$ -modification. By adding zig-zags to K_0 and γ_i , we can assume that K_0 and γ_i of each Legendrian handlebody $X_i^{(n)}$ ($1 \leq i \leq n$) satisfy the following conditions (i)–(v) (recall Proposition 4.7 and Lemma 3.5). (Namely, we add zig-zags so that the value $|\langle c_1(X_i^{(n)}), [K_0 - p_i \gamma_i] \rangle|$ becomes as large as possible, see Lemma 5.14.)

- (i) $tb(K_0) = m_0 + 1$.
- (ii) $|r(K_0)| = p_i + (t_0 - 1) - m_0 + |r_0|$.
- (iii) $tb(\gamma_i) = 1$.
- (iv) $|r(\gamma_i)| = 1$.
- (v) In the $r(K_0) \neq 0$ case, the sign of $r(\gamma_i)$ is opposite to the sign of $r(K_0)$.

We have now finished the construction and here discuss Stein structures on $X_i^{(n)}$.

Remark 5.11. (1) If $b_2(X) = 1$ or $(q_1, q_2, \dots, q_k) = 0$, then $X_i^{(n)}$ ($1 \leq i \leq n$) is now a Stein handlebody.

(2) In the case where $b_2(X) \geq 2$ and $(q_1, q_2, \dots, q_k) \neq 0$, $X_i^{(n)}$ ($1 \leq i \leq n$) is not a Stein handlebody yet, because the Thurston–Bennequin number of each δ_j ($1 \leq j \leq k$, $q_j \neq 0$) is still two more than its framing. We can make each $tb(\delta_j)$ one more than its framing by adding a zig-zag either upward or downward. Correspondingly, $r(\delta_j)$ becomes -1 or 1 . This process gives various Stein structures on each $X_i^{(n)}$ ($1 \leq i \leq n$). We later use this flexibility of Stein structures to simplify adjunction inequality arguments.

(3) By adding a zig-zag to each δ_j , $X_0^{(n)}$ becomes a Stein handlebody when the original X is a good Stein handlebody.

(4) $X_{-1}^{(n)}$ is a good Stein handlebody when the original X is a good Stein

handlebody.

(5) As a smooth handlebody, $X_{-1}^{(n)}$ is obtained from X only by W^- -modifications. Proposition 4.5 thus shows that X can be embedded into $X_{-1}^{(n)}$ so that the induced homomorphism $H_*(X; \mathbf{Z}) \rightarrow H_*(X_{-1}^{(n)}; \mathbf{Z})$ is an isomorphism.

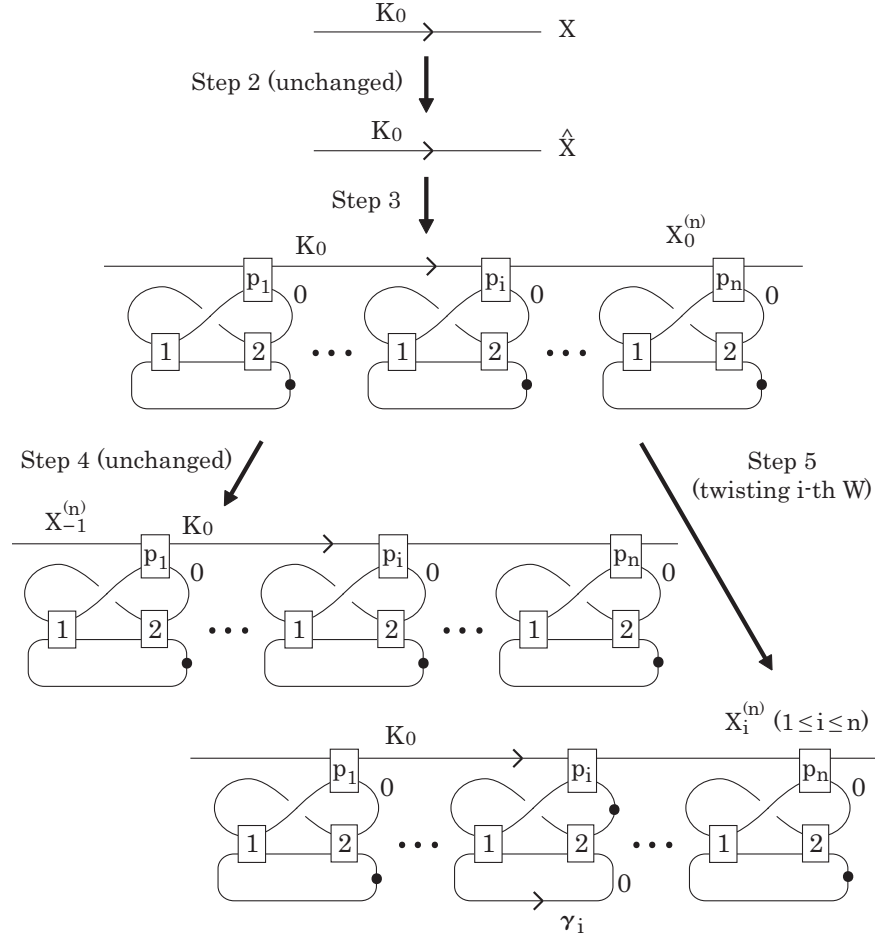


Figure 11. Operations to K_0 (ignoring Legendrian diagrams)

5.2. Detecting smooth structures. We next detect the smooth structures of $X_i^{(n)}$'s, examining the genera of bases of their second homology groups.

Definition 5.12. Ignore (2) when $b_2(X) = 1$.

(1) Define $v_0^{(0)}$ as the element of $H_2(X_0^{(n)}; \mathbf{Z})$ given by the 2-handle K_0 .

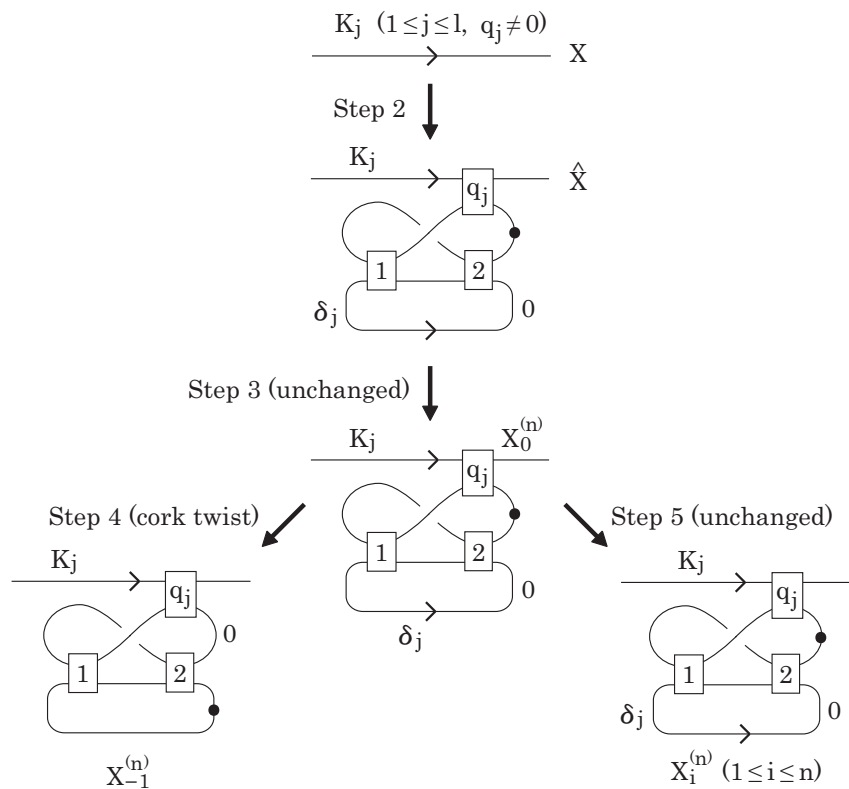


Figure 12. Operations to K_j ($1 \leq j \leq l$, $q_j \neq 0$) (ignoring Legendrian diagrams)

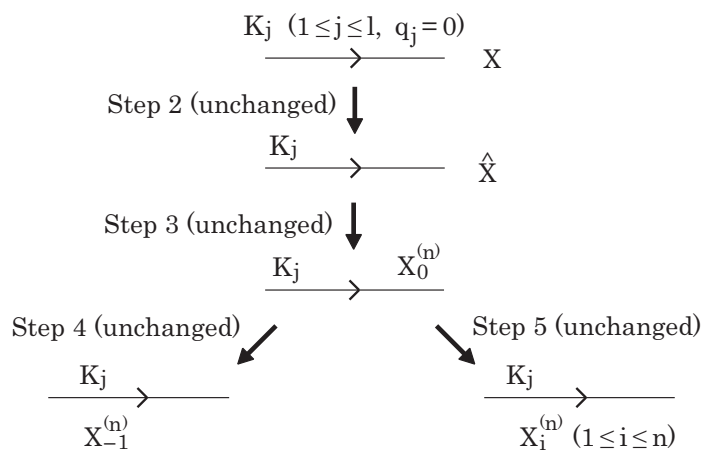


Figure 13. Operations to K_j ($1 \leq j \leq l$, $q_j = 0$) (ignoring Legendrian diagrams)

Define $v_0^{(i)}$ ($1 \leq i \leq n$) as the element of $H_2(X_i^{(n)}; \mathbf{Z})$ given by the 2-handle $K_0 - p_i \gamma_i$, which denotes the result of K_0 by sliding over γ_i p_i -times so that it does not algebraically go over any 1-handle.

(2) For each $0 \leq i \leq n$, define $v_j^{(i)}$ ($1 \leq j \leq k$) as the elements of $H_2(X_i^{(n)}; \mathbf{Z})$ given by the 2-handles $K_j - q_j \delta_j$. Here $K_j - q_j \delta_j$ denotes the result of K_j by sliding over δ_j q_j -times so that it does not algebraically go over any 1-handle.

(3) Let $v_j^{(-1)}$ ($0 \leq j \leq k$) as the element of $H_2(X_{-1}^{(n)}; \mathbf{Z})$ given by the 2-handle K_j .

The Lemma below clearly follows from Proposition 4.3.

Lemma 5.13. (1) For each $0 \leq i \leq n$, the elements $v_j^{(i)}$ ($0 \leq j \leq k$) span a basis of $H_2(X_i^{(n)}; \mathbf{Z})$ and satisfy the following conditions (ignore (ii) when $k = 0$).

(i) $v_0^{(i)}$ is represented by a smoothly embedded genus $g_0 + p_i$ surface, satisfying $v_0^{(i)} \cdot v_0^{(i)} = m_0$.

(ii) Each $v_j^{(i)}$ ($1 \leq j \leq k$) is represented by a smoothly embedded genus $g_j + q_j$ surface, satisfying $v_j^{(i)} \cdot v_j^{(i)} = m_j$.

(2) $v_j^{(-1)}$ ($0 \leq j \leq k$) span a basis of $H_2(X_{-1}^{(n)}; \mathbf{Z})$, and each $v_j^{(-1)}$ ($0 \leq j \leq k$) is represented by a smoothly embedded genus g_j surface satisfying $v_j^{(-1)} \cdot v_j^{(-1)} = m_j$.

We here use the flexibility of Stein structures on $X_i^{(n)}$ ($1 \leq i \leq n$).

Lemma 5.14. For each integer a_0, a_1, \dots, a_k , there exists a Stein structure J on the each smooth 4-manifold $X_i^{(n)}$ ($1 \leq i \leq n$) such that

$$\begin{aligned} & | \langle c_1(X_i^{(n)}, J), a_0 v_0^{(i)} + a_1 v_1^{(i)} + \dots + a_k v_k^{(i)} \rangle | \\ & \geq |a_0| (2p_i + (t_0 - 1) - m_0 + |r_0|) + |a_1 \widehat{q}_1| + |a_2 \widehat{q}_2| + \dots + |a_k \widehat{q}_k|, \end{aligned}$$

where $\widehat{q}_j = q_j - 1$ (if $q_j \neq 0$) and $\widehat{q}_j = 0$ (if $q_j = 0$). Furthermore, the equality holds in the $k = 0$ case (ignoring the last k terms).

Proof. Recall Steps 2 and 5 and Remark 5.11.(2). By appropriately adding a zig-zag to each δ_j ($1 \leq j \leq k$) of $X_i^{(n)}$, Theorem 3.3 easily gives the required claim. q.e.d.

Proposition 5.15. For any $1 \leq i \leq n$, there exists no basis u_0, u_1, \dots, u_k of $H_2(X_i^{(n)}; \mathbf{Z})$ that satisfies the following conditions (ignore (ii) when $b_2(X) = 1$).

(i) u_0 is represented by a smoothly embedded surface with its genus equal to or less than $g_0 + p_{i-1}$ and satisfies $u_0^2 = m_0$.

- (ii) Each u_j ($1 \leq j \leq k$) is represented by a smoothly embedded surface of genus $g_j + q_j$ and satisfies $u_j^2 = m_j$.

Proof. Fix i with $1 \leq i \leq n$. Suppose that a basis u_0, u_1, \dots, u_k of $H_2(X_i^{(n)}; \mathbf{Z})$ satisfies the above conditions (i) and (ii). We can assume that the genus of u_0 is $g_0 + p_{i-1}$, by taking a connected sum with a null-homologous surface in $X_i^{(n)}$. For each $0 \leq j \leq k$, put $u_j := a_0^{(j)} v_0^{(i)} + a_1^{(j)} v_1^{(i)} + \dots + a_k^{(j)} v_k^{(i)}$. Lemma 5.14 and the adjunction inequality for u_0 give the inequality

$$2(g_0 + p_{i-1}) - 2 \geq |a_0^{(0)}| (2p_i + (t_0 - 1) - m_0 + |r_0|) \\ + |a_1^{(0)} \widehat{q}_1| + |a_2^{(0)} \widehat{q}_2| + \dots + |a_k^{(0)} \widehat{q}_k| + m_0.$$

This inequality and condition (iv) of p_i 's in Definition 5.6 easily show

$$0 > (|a_0^{(0)}| - 1)(2p_i + (t_0 - 1) - m_0 + |r_0|).$$

Conditions (i) and (ii) of p_i 's in Definition 5.6 thus give $a_0^{(0)} = 0$. When $k = 0$, this fact contradicts the assumption, hence the required claim follows. We thus assume $k \geq 1$.

Lemma 5.14 and the adjunction inequality for u_j ($1 \leq j \leq k$) give

$$2(g_j + q_j) - 2 \geq |a_0^{(j)}| (2p_i + (t_0 - 1) - m_0 + |r_0|) \\ + |a_1^{(j)} \widehat{q}_1| + |a_2^{(j)} \widehat{q}_2| + \dots + |a_k^{(j)} \widehat{q}_k| + m_j.$$

This inequality and conditions (i) and (iii) of p_i 's in Definition 5.6 easily give the following.

$$0 > (|a_0^{(j)}| - 1)(2p_i + (t_0 - 1) - m_0 + |r_0|).$$

Conditions (i) and (ii) of p_i 's in Definition 5.6 thus give $a_0^{(j)} = 0$. We thus have $a_0^{(0)} = a_0^{(1)} = \dots = a_0^{(k)} = 0$. This fact contradicts the assumption that u_0, u_1, \dots, u_k is a basis. Hence, the required claim follows. \square e.d.

To summarize, here we list up properties of $X_i^{(n)}$. Beware that X_0 in Theorem 1.1 corresponds to $X_{-1}^{(n)}$ in this section and that X_i ($1 \leq i \leq n$) corresponds to $X_i^{(n)}$.

Theorem 5.16. *Let X be any 2-handlebody with $b_2(X) \geq 1$. Fix $n \geq 1$. Let $X_i^{(n)}$ ($-1 \leq i \leq n$) denote the corresponding Legendrian handlebodies in Definition 5.10. Then the following properties hold.*

(1) *The fundamental group, the integral homology groups, the integral homology groups of the boundary, and the intersection form of each $X_i^{(n)}$ ($-1 \leq i \leq n$) are isomorphic to those of X .*

(2) *$X_i^{(n)}$ ($0 \leq i \leq n$) are mutually homeomorphic but not diffeomorphic with respect to the given orientations. When either the following (i) or*

(ii) holds, they are mutually non-diffeomorphic with any orientations. The same properties also hold for $X_i^{(n)}$ ($-1 \leq i \leq n$, $i \neq 0$).

(i) $b_2(X) = 1$.

(ii) The intersection form of X is represented by the zero matrix.

(3) Each $X_i^{(n)}$ ($1 \leq i \leq n$) admits a Stein structure. $X_{-1}^{(n)}$ and $X_0^{(n)}$ admit Stein structures when X is a good Stein handlebody.

(4) X can be embedded into $X_{-1}^{(n)}$ so that the induced homomorphism is an isomorphism between the integral homology groups of X and $X_{-1}^{(n)}$. Therefore $X_{-1}^{(n)}$ does not admit any Stein structure when X cannot be embedded into any simply connected minimal symplectic 4-manifold with $b_2^+ > 1$ (or any minimal complex surface of general type with $b_2^+ > 1$) so that the induced homomorphism between the second homology groups is injective.

(5) Each $X_i^{(n)}$ ($-1 \leq i \leq n$) can be embedded into X .

(6) There exist disjoint copies C_i ($1 \leq i \leq n$) of W_1 in $X_0^{(n)}$ such that $X_i^{(n)}$ is the cork twist of $X_0^{(n)}$ along (C_i, f_1) .

Proof. Proposition 4.2 gives (1). For (3), see Remark 5.11. Proposition 4.5 gives (5). Construction shows (6).

We next check (2). $X_i^{(n)}$ ($-1 \leq i \leq n$) are mutually homeomorphic because they are related to each other by combinations of cork twists. Since p_i ($i \geq 0$) is a strictly increasing sequence, Lemma 5.13 and Proposition 5.15 show the first claim. The second claim in the (ii) case also follows from Lemma 5.13 and Proposition 5.15. In the case $b_2(X) = 1$ and $m_0 \neq 0$, there are no orientation-reversing homeomorphisms between them. Hence they cannot be orientation-reversing diffeomorphic.

Last, we show (4). Remark 5.11.(5) gives the first claim of (4). Suppose that $X_{-1}^{(n)}$ admits a Stein structure. Then $X_{-1}^{(n)}$ admits a Stein handlebody presentation. For every 2-handle of this Stein handlebody, attach a 2-handle along its -2 -framed meridian so that the result is also a Stein handlebody. This new Stein handlebody can be embedded into a simply connected minimal symplectic 4-manifold and a minimal complex surface of general type (see Theorem 3.6). Note that, in each of these closed 4-manifolds, the image of every non-zero second homology class of $X_{-1}^{(n)}$ algebraically intersects with a sphere with its self-intersection number -2 . This fact implies the injectivity of the induced homomorphism between the second homology groups of $X_{-1}^{(n)}$ and the closed manifold. The second claim of (4) thus easily follows. q.e.d.

For a given embedding of a 4-manifold, applying the algorithm to its complement, we get arbitrary many exotic embeddings. In the following,

beware that Y_0 in Theorem 1.2 corresponds to $Y_{-1}^{(n)}$ and that Y_i ($1 \leq i \leq n$) corresponds to $Y_i^{(n)}$.

Theorem 5.17. *Let Z and Y be compact connected oriented smooth 4-manifolds (possibly with boundary). Suppose that Y is embedded into Z and that its complement $X := Z - \text{int } Y$ is a 2-handlebody with $b_2(X) \geq 1$. Fix $n \geq 1$. Then there exist mutually diffeomorphic compact connected oriented smooth 4-manifolds $Y_i^{(n)}$ ($-1 \leq i \leq n$) embedded into Z with the following properties.*

(1) *The pairs $(Z, Y_i^{(n)})$ ($0 \leq i \leq n$) are mutually homeomorphic but non-diffeomorphic with respect to the given orientations. When either the following (i) or (ii) holds, they are mutually non-diffeomorphic for any orientations. The same properties also hold for the pairs $(Z, Y_i^{(n)})$ ($-1 \leq i \leq n, i \neq 0$).*

(i) $b_2(X) = 1$.

(ii) *The intersection form of X is represented by the zero matrix.*

(2) *The fundamental group, the integral homology groups, the integral homology groups of the boundary, and the intersection form of each $Y_i^{(n)}$ ($-1 \leq i \leq n$) are isomorphic to those of Y .*

(3) *Each complement $X_i^{(n)} := Z - \text{int } Y_i^{(n)}$ ($-1 \leq i \leq n$) is the one in Theorem 5.16, corresponding to the above X .*

Proof. Let $X_i^{(n)}$ ($-1 \leq i \leq n$) denote the manifold in Theorem 5.16, corresponding to the above X . Replace every W^- -modification of $X_0^{(n)}$ applied in Step 3 with the corresponding W^+ -modification. As a smooth handlebody, the result of $X_0^{(n)}$ is thus obtained from X only by W^+ -modifications. Applying Proposition 4.5, embed this manifold into X . Proposition 4.5 thus gives an embedding of each $X_i^{(n)}$ ($-1 \leq i \leq n$) into X (and hence Z) by twisting W 's of the above manifold. Put $Y_i^{(n)} := Z - \text{int } X_i^{(n)}$ ($-1 \leq i \leq n$). Now the required claims easily follow from Proposition 4.5 and Theorem 5.16. q.e.d.

We now have Theorems 1.1 and 1.2 and Corollary 1.4.

Proofs of Theorems 1.1 and 1.2 and Corollary 1.4. These clearly follow from Theorems 5.16 and 5.17. q.e.d.

Remark 5.18. (1) Suppose that ∂X is diffeomorphic to $\#_m S^1 \times S^2$ ($m \geq 0$). Since $\#_m S^1 \times S^2$ has a unique Stein filling, each $\partial X_i^{(n)}$ is not diffeomorphic to ∂X , so in particular, each $X_i^{(n)}$ is not homeomorphic to the original X , in this case.

(2) By more restricting the conditions of p_i ($i \geq 1$) in Definition 5.6,

we can easily show the following: $X_0^{(n)}$ produces $2^n - 1$ mutually homeomorphic but non-diffeomorphic compact Stein 4-manifolds by natural combinations of cork twists.

Remark 5.19 (Variants of the construction). There are many variants of the construction; here we remark just a few of them.

(1) We can cut condition (ii) of q_j in Definition 5.5, by choosing each p_i ($i \geq 1$) larger. In this case, we use flexibility of zig-zag operations of K_0 and γ_i , instead of δ_j . Namely, we equip each $X_i^{(n)}$ ($1 \leq i \leq n$) with two Stein structures so that $\langle c_1(X_i^{(n)}), v_0^{(i)} \rangle$ takes two different values, namely, a large positive number and a large negative number. This makes us possible to apply similar adjunction inequality arguments.

(2) Though we used only W_1 for the construction, many other corks (e.g., W_n of [6]) also work. Taking band sums with knots with sufficiently large Thurston–Bennequin numbers are also helpful. We can use band sum operations in Step 2, instead of W^+ -modifications, though Theorem 5.16.(4) and Theorem 5.17 are not guaranteed in this case.

Remark 5.20. Once we apply Step 1 to any given X and calculate the data of X in Definition 5.4, we immediately get a (usually large) smooth handle diagram of each $X_i^{(n)}$, as shown in Figures 11–13. See Subsection 9.1 for the simplest case. Though we can also immediately get a Legendrian (Stein) handlebody diagram of each $X_i^{(n)}$, it usually a very large diagram.

6. Strengthening the construction

In Section 5, we did not completely exclude the possibility that some of $X_i^{(n)}$'s are orientation-reversing diffeomorphic, because the argument was simplified and the conditions of p_i 's in Definition 5.6 were relaxed. In this section, we exclude this possibility by restricting the conditions of p_i 's. We use the same symbols as in Section 5.

Definition 6.1. Let X be any 2-handlebody with $b_2 \geq 1$. Fix $n \geq 1$.

(1) In the $b_2(X) = 1$ case, put $\widehat{X}_i^{(n)} := X_i^{(n)}$ ($-1 \leq i \leq n$), where $X_i^{(n)}$ are the manifolds as in Theorem 5.16.

(2) In the $b_2(X) \geq 2$ case, assume that p_i 's ($i \geq -1$) in Definition 5.6 further satisfy the following conditions (v) and (vi). Then put $\widehat{X}_i^{(n)} := X_i^{(n)}$ ($-1 \leq i \leq n$).

(v) $2p_1 + (t_0 - 1) - m_0 + |r_0| - m_j > 2(g_j + q_j) - 2$, for each $0 \leq j \leq k$.

(vi) $2p_i + (t_0 - 1) - 2m_0 + |r_0| > 2(g_0 + p_{i-1}) - 2$, for each $i \geq 1$.

Let $v_j^{(i)}$ ($1 \leq j \leq k$) denote the basis of $H_2(\widehat{X}_i^{(n)}; \mathbf{Z})$ in Definition 5.12. Since we defined $\widehat{X}_i^{(n)}$ as a special case of $X_i^{(n)}$, the same properties as in Lemma 5.13 hold. For Proposition 5.15, we can easily get the following stronger claim.

Proposition 6.2. *Fix $n \geq 1$. For any $1 \leq i \leq n$, there exists no basis u_0, u_1, \dots, u_k of $H_2(\widehat{X}_i^{(n)}; \mathbf{Z})$ that satisfies the following conditions (ignore (ii) when $b_2(X) = 1$).*

- (i) u_0 is represented by a smoothly embedded surface with its genus equal to or less than $g_0 + p_{i-1}$ and satisfies $|u_0^2| = |m_0|$.
- (ii) Each u_j ($1 \leq j \leq k$) is represented by a smoothly embedded surface of genus $g_j + q_j$ and satisfies $|u_j^2| = |m_j|$.

Using this proposition, we easily get the following strengthened theorems.

Theorem 6.3. *Let X be any 2-handlebody with $b_2(X) \geq 1$. Fix $n \geq 1$. Let $\widehat{X}_i^{(n)}$ ($-1 \leq i \leq n$) denote the corresponding Legendrian handlebody in Definition 6.1. Then the following properties hold.*

- (1) $\widehat{X}_i^{(n)}$ ($0 \leq i \leq n$) are mutually homeomorphic, but mutually non-diffeomorphic for any orientations. The same property also holds for $\widehat{X}_i^{(n)}$ ($-1 \leq i \leq n, i \neq 0$).
- (2) $\widehat{X}_i^{(n)}$ ($-1 \leq i \leq n$) has the same properties as those of $X_i^{(n)}$ ($-1 \leq i \leq n$) in Theorem 5.16.

Theorem 6.4. *Let Z and Y be compact connected oriented smooth 4-manifolds (possibly with boundary). Suppose that Y is embedded into Z and that its complement $X := Z - \text{int } Y$ is a 2-handlebody with $b_2(X) \geq 1$. Fix $n \geq 1$. Then there exist mutually diffeomorphic compact connected oriented smooth 4-manifolds $\widehat{Y}_i^{(n)}$ ($-1 \leq i \leq n$) embedded into Z with the following properties.*

- (1) The pairs $(Z, \widehat{Y}_i^{(n)})$ ($0 \leq i \leq n$) are mutually homeomorphic, but mutually non-diffeomorphic with any orientations. The same property also holds for the pairs $(Z, \widehat{Y}_i^{(n)})$ ($-1 \leq i \leq n, i \neq 0$).
- (2) The fundamental group, the integral homology groups, the integral homology groups of the boundary, and the intersection form of every $\widehat{Y}_i^{(n)}$ ($-1 \leq i \leq n$) are isomorphic to those of Y .
- (3) Each complement $\widehat{X}_i^{(n)} := Z - \text{int } \widehat{Y}_i^{(n)}$ ($-1 \leq i \leq n$) is as in the Theorem 6.3, corresponding to the above X .

7. The contact structures on the boundary

In this section, we discuss the induced contact structures on the boundary $\partial X_i^{(n)}$ in the $b_2(X) = 1$ case. We use the same symbols as in Section 5.

Definition 7.1. Let X be any 2-handlebody with $b_2(X) = 1$. Assume that the intersection form of X is non-zero (i.e., $m_0 \neq 0$). Fix $n \geq 1$. Let $X_i^{(n)}$ and $v_0^{(i)}$ ($1 \leq i \leq n$) denote the corresponding Stein handlebody in Theorem 5.16 and the generator of $H_2(X_i^{(n)}; \mathbf{Z})$ in Definition 5.12, respectively. Let $\xi_i^{(n)}$ ($1 \leq i \leq n$) be the contact structure on $\partial X_i^{(n)}$ induced by the Stein structure on $X_i^{(n)}$.

Lemma 7.2. $d_3(\xi_i^{(n)})$ ($1 \leq i \leq n$) are mutually different.

Proof. Lemma 5.14 shows $|\langle c_1(X_i^{(n)}), v_0^{(i)} \rangle| = 2p_i + t_0 - 1 - m_0 + |r_0|$. The value $|\langle c_1(X_i^{(n)}), v_0^{(i)} \rangle|$ hence strictly increases when i increases. Lemma 3.7 thus shows the claim. q.e.d.

The following proposition gives Corollary 1.3.

Proposition 7.3. Fix $n \geq 1$. Each smooth 4-manifold $X_i^{(n)}$ ($1 \leq i \leq n$) admits no Stein structure compatible with $\xi_j^{(n)}$ for any j with $i < j \leq n$.

Proof. Suppose that some $X_i^{(n)}$ admits a Stein structure J compatible with $\xi_j^{(n)}$ for some j with $i < j \leq n$. Then the corresponding first Chern class $c_1(X_i^{(n)}, J)$ satisfies $|\langle c_1(X_i^{(n)}, J), v_0^{(i)} \rangle| = |\langle c_1(X_j^{(n)}), v_0^{(j)} \rangle|$, because $d_3(\xi_j^{(n)})$ is determined by this value (see Lemma 3.7). The adjunction inequality for the Stein 4-manifold $(X_i^{(n)}, J)$ thus easily shows that $v_0^{(i)}$ cannot be represented by any smoothly embedded surface with genus less than or equal to $g_0 + p_{j-1}$ (see the proof of Proposition 5.15). Since $p_{j-1} \geq p_i$, this fact contradicts the fact that $v_0^{(i)}$ is represented by a surface of genus $g_0 + p_i$ (see Lemma 5.13). q.e.d.

8. Infinitely many disjoint corks in noncompact 4-manifolds

In [9], we constructed infinitely many disjoint embeddings of the cork W_1 into a simply connected noncompact smooth 4-manifold so that this noncompact 4-manifold produces infinitely many different exotic smooth structures by twisting each copy of W_1 . The second betti number of the noncompact 4-manifold is infinite. In this section, we construct such noncompact 4-manifolds for any finite second betti number larger than zero. Namely, we prove the following.

Theorem 8.1. *Let X be any compact 2-handlebody with $b_2(X) \geq 1$. Then there exist infinitely many noncompact 4-manifolds X_i ($i \geq 0$) and infinitely many disjointly embedded copies C_i ($i \geq 1$) of W_1 into X_0 with the following properties.*

- (1) *Each X_i ($i \geq 1$) is the cork twist of X_0 along (C_i, f_1) .*
- (2) *X_i ($i \geq 0$) are mutually homeomorphic but not diffeomorphic.*
- (3) *The fundamental group, the integral homology groups, and the intersection form of every X_i ($i \geq 0$) are isomorphic to those of X .*
- (4) *Each X_i ($i \geq 0$) can be embedded into X .*

In this section, we use the same symbols as in Section 6 (and 5). Let X be any compact 2-handlebody with $b_2(X) \geq 1$. Let \hat{X} and $\hat{X}_i^{(n)}$ ($0 \leq i \leq n$) denote the compact 2-handlebodies in Definitions 5.8 and 6.1, respectively, corresponding to this X . Recall that, for each $n \geq 1$, the smooth 4-manifold $\hat{X}_0^{(n)}$ is obtained from \hat{X} by attaching n pairs of 1- and 2-handles to the boundary (i.e., by $W^-(p_1)$ -, $W^-(p_2)$ -, \dots , $W^-(p_n)$ -modifications to K_0). We thus have an infinite sequence $\hat{X}_0^{(1)} \subset \hat{X}_0^{(2)} \subset \dots \subset \hat{X}_0^{(n)} \subset \hat{X}_0^{(n+1)} \subset \dots$. We are now ready to define noncompact 4-manifolds. See also Figure 14.

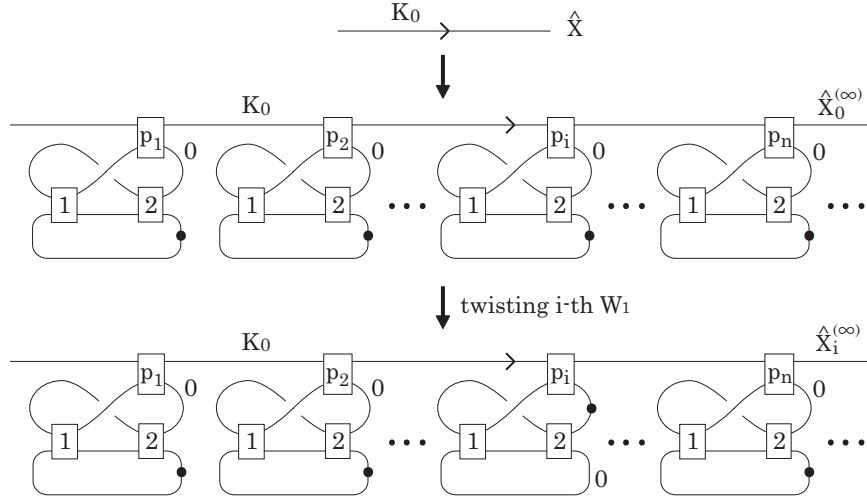


Figure 14. $\hat{X}_i^{(\infty)}$

Definition 8.2. Let $\hat{X}_0^{(\infty)}$ be the noncompact smooth 4-manifold obtained by the inductive limit of the above sequence. Let $\hat{X}_i^{(\infty)}$ ($i \geq 1$) be the noncompact smooth 4-manifold obtained from $\hat{X}_0^{(\infty)}$ by the cork twist along (W, f) where this W is the one given by the above $W^-(p_i)$ -modification to K_0 .

We can now easily prove the above theorem.

Proof of Theorem 8.1. Put $X_i := \widehat{X}_i^{(\infty)}$. (1) is obvious from the definition of $\widehat{X}_i^{(\infty)}$.

(2). Since each $\widehat{X}_i^{(\infty)}$ is obtained from $\widehat{X}_i^{(i+1)}$ by applying infinitely many W^- -modifications, Proposition 4.5 implies that each $\widehat{X}_i^{(\infty)}$ ($i \geq 0$) can be embedded into $\widehat{X}_i^{(i+1)}$ so that the induced homomorphism between the second homology groups is an isomorphism. (Thus this fact shows (4).) It is thus easy to check that the claims similar to Lemma 5.13 and Proposition 6.2 hold for $\widehat{X}_i^{(\infty)}$ ($i \geq 0$). This shows that $\widehat{X}_i^{(\infty)}$ ($i \geq 0$) are mutually non-diffeomorphic for any orientations. On the other hand, $\widehat{X}_i^{(\infty)}$ ($i \geq 0$) are mutually homeomorphic because they are related by cork twists.

(3). The well-known properties of the inductive limit operation and Theorem 6.3.(1) show the $i = 0$ case. Since $\widehat{X}_i^{(\infty)}$ ($i \geq 1$) is a cork twist of $\widehat{X}_0^{(\infty)}$, the $i \geq 1$ case also follows. q.e.d.

Remark 8.3. The corresponding result holds when we use $X_i^{(n)}$ instead of $\widehat{X}_i^{(n)}$, though the orientation problem as in Theorem 5.16 remains, in this case.

9. Examples

9.1. The simplest example. In this subsection, we apply the algorithm in Section 5 for the simplest example. Actually, our algorithm is a generalization of this example. We also demonstrate how to show the (non-) existence of Stein structures on $X_{-1}^{(n)}$ for the example. We use the same symbols as in Section 5.

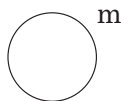


Figure 15. $U(m)$

Let $U(m)$ be the 2-handlebody in Figure 15. We here apply Step 1 in Section 5 to this $U(m)$. There are infinitely many Legendrian handlebody presentations of $U(m)$, and we can use any one. We here adopt the one in Figure 16, where $d := -m - 1$ ($m \leq -2$) and $d := 1$ ($m \geq -1$). In this case, the symbols in Section 5 correspond to the following: $X = U(m)$, and K_0 is the m -framed unknot in the figure, $k = 0$, $m_0 = m$, $t_0 = -d$, $r_0 = d - 1$, and $g_0 = 0$. Put $p_{-1} = p_0 = 0$,

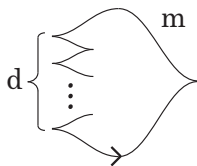


Figure 16. A Legendrian handlebody presentation of $U(m)$

$p_1 = 1$ ($m \leq -2$), $p_1 = m + 2$ ($m \geq -1$), and $p_i = p_1 + i - 1$ ($i \geq 2$). This sequence p_i ($i \geq -1$) satisfies the conditions in Definition 5.6.

We next apply Steps 3–5 in Section 5 (Step 2 is skipped in this case). Denote by $U(m)_i^{(n)}$ the Legendrian handlebody corresponding to $X_i^{(n)}$ for $U(m)$. A smooth handlebody diagram of $U(m)_i^{(n)}$ is given in Figures 17 and 18. Namely, $U(m)_i^{(n)}$ ($1 \leq i \leq n$) is obtained from $U(m)_0^{(n)} (= U(m)_{-1}^{(n)})$ by exchanging the dot and 0 of the i th W_1 component. Theorem 5.16 clearly holds for these $U(m)_i^{(n)}$ ($-1 \leq i \leq n$). In particular, $U(m)_i^{(n)}$ ($0 \leq i \leq n$) are mutually homeomorphic but not diffeomorphic, where we fix $n \geq 1$.

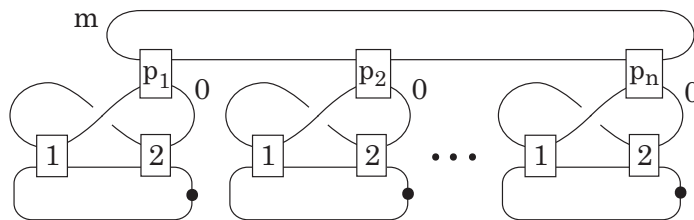


Figure 17. A smooth handle diagram of $U(m)_0^{(n)} (= U(m)_{-1}^{(n)})$

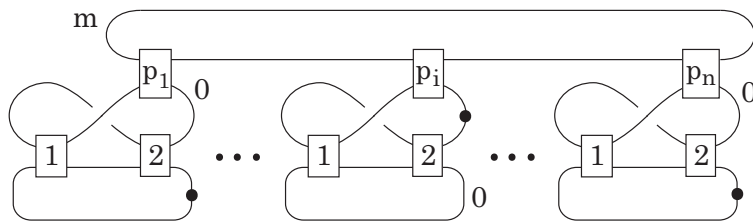


Figure 18. A smooth handle diagram of $U(m)_i^{(n)}$ ($1 \leq i \leq n$)

Finally, we discuss the (non-) existence of Stein structures on $U(m)_{-1}^{(n)} (= U(m)_0^{(n)})$.

(1) The $m \leq -2$ case. In this case, $U(m)$ is a good Stein handlebody; hence, Theorem 5.16 shows that $U(m)_{-1}^{(n)}$ admits a Stein structure.

(2) The $m \geq -1$ case. In this case, $U(m)$ (and thus $U(m)_{-1}^{(n)}$) contains a homologically non-vanishing smoothly embedded sphere with its self-intersection number m . Adjunction inequality shows that $U(m)_{-1}^{(n)}$ does not admit any Stein structure.

9.2. Exotic complements in the 4-sphere. In this subsection, we demonstrate Theorem 5.17 in the $(Z, Y) = (S^4, \Sigma_g \times D^2)$ ($g \geq 1$) case, where Σ_g denotes the closed surface of genus g . Figure 19 is a handlebody diagram of $\Sigma_g \times D^2$.

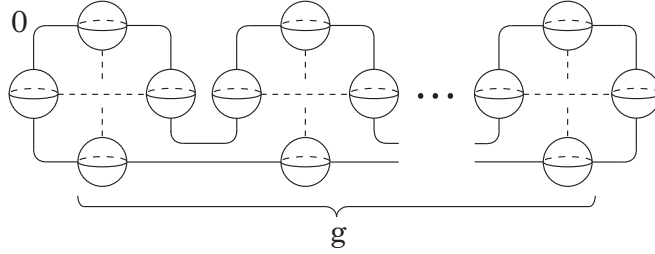


Figure 19. $\Sigma_g \times D^2$ ($g \geq 1$)

We can embed $\Sigma_g \times D^2$ into S^4 as follows: Converting the picture into the dotted circle notation, we get the diagram of $\Sigma_g \times D^2$ in Figure 20. Taking a double, we get the diagram of $\Sigma_g \times S^2$ in Figure 21. By surgering $\Sigma_g \times D^2 \subset \Sigma_g \times S^2$, we get the closed 4-manifold in Figure 22, where the 0-framed meridian of the dotted circle, $2g$ 3-handles, and the 4-handle constitute $\Sigma_g \times D^2$. It is easy to see that this closed 4-manifold is S^4 (cancel 1/2-handle pair, then cancel $2g$ 2/3-handle pairs).

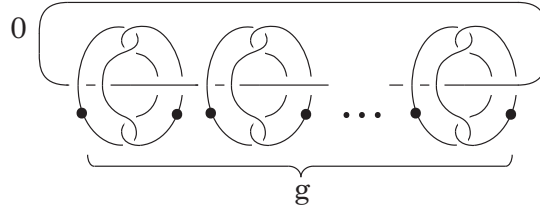


Figure 20. $\Sigma_g \times D^2$ ($g \geq 1$)

We thus have an embedding of $\Sigma_g \times D^2$ into S^4 such that its complement is the 2-handlebody (call X_g) with $b_2 \geq 1$ in Figure 23.

Applying Theorem 5.17 to the above embedding, we get the following proposition. Note that the intersection form of X_g is represented by the zero matrix.

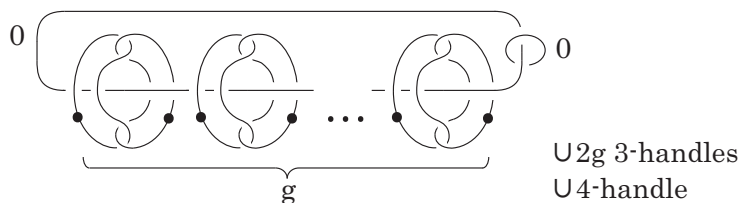


Figure 21. $\Sigma_g \times S^2$ ($g \geq 1$)

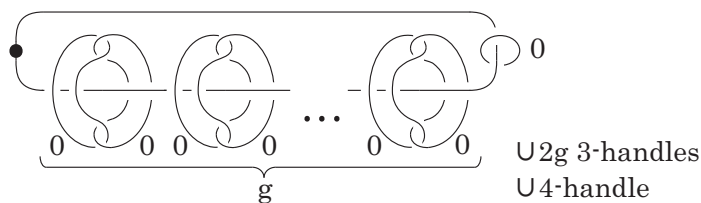


Figure 22. (surgered $\Sigma_g \times S^2$) $\cong S^4$ ($g \geq 1$)

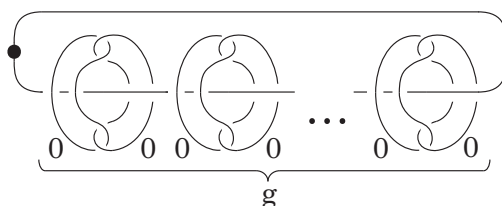


Figure 23. $X_g := S^4 - \text{int}(\Sigma_g \times D^2)$ ($g \geq 1$)

Proposition 9.1. *Fix $g \geq 1$. For each $n \geq 1$, there exist mutually diffeomorphic compact connected oriented smooth 4-manifolds Y_i ($0 \leq i \leq n$) embedded into S^4 with the following properties.*

- (1) *The pairs (S^4, Y_i) ($0 \leq i \leq n$) are mutually homeomorphic but non-diffeomorphic.*
- (2) *The fundamental group, the integral homology groups, the integral homology groups of the boundary, and the intersection form of every Y_i ($0 \leq i \leq n$) are isomorphic to those of $\Sigma_g \times D^2$.*

Remark 9.2. Similarly to the above, we can embed D^2 -bundle R_k ($k \geq 1$) over $\#_k \mathbf{RP}^2$ with Euler number $-2k$ into S^4 and apply our algorithm to the pair (S^4, R_k) . While this procedure seems not to give exotic embeddings of $\#_k \mathbf{RP}^2$, Finashin and Kreck, and Viro [16, 17] constructed infinitely many exotic embeddings of $\#_{10} \mathbf{RP}^2$ into S^4 , by different methods. See also Finashin [15]. However, it is unclear whether the complements of (the neighborhoods of) their examples are mutually homeomorphic but not diffeomorphic.

10. Exotic non-Stein 4-manifolds and exotic Stein 4-manifolds

In this section, we construct arbitrary many non-Stein 4-manifolds and arbitrary many Stein 4-manifolds that are mutually homeomorphic but not diffeomorphic. Namely, we prove the following theorem.

Theorem 10.1. *Let X be a 2-handlebody with $b_2(X) \geq 1$, For each $n \geq 1$, there exist 2-handlebodies X_i^S and X_i^N ($1 \leq i \leq n$) with the following properties.*

- (1) $X_1^S, X_2^S, \dots, X_n^S$ and $X_1^N, X_2^N, \dots, X_n^N$ are mutually homeomorphic but non-diffeomorphic.
- (2) Every X_i^S ($1 \leq i \leq n$) admits a Stein structure, and any X_i^N ($1 \leq i \leq n$) admits no Stein structure.
- (3) The fundamental groups, the integral homology groups, the integral homology groups of the boundary, and the intersection forms of every X_i^S and every X_i^N ($0 \leq i \leq n$) are isomorphic to those of the boundary sum $X \natural U(0)$. Here $U(0)$ denotes the one in Subsection 9.1.

We prove this theorem, using the examples $U(0)_0^{(1)}$ and $U(0)_1^{(1)}$ in subsection 9.1. Figure 24 shows smooth handlebody diagrams of $U(0)_0^{(1)}$ and $U(0)_1^{(1)}$.

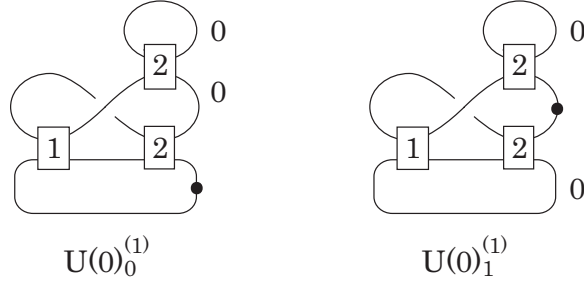


Figure 24. $U(0)_0^{(1)}$ and $U(0)_1^{(1)}$

Definition 10.2. Let X be a 2-handlebody with $b_2(X) \geq 1$. Fix $n \geq 1$. Let $\widehat{X}_i^{(n)}$ ($1 \leq i \leq n$) be the one in Definition 6.1, corresponding to this X . We assume that p_i 's in Definition 6.1 further satisfy the following condition.

- (iii)' $2p_1 + (t_0 - 1) - m_0 + |r_0| > 2$.

For each $1 \leq i \leq n$, put $X_i^S := \widehat{X}_i^{(n)} \natural U(0)_1^{(1)}$ and $X_i^N := \widehat{X}_i^{(n)} \natural U(0)_0^{(1)}$ where \natural denotes the boundary sum.

- Lemma 10.3.** (1) *Every X_i^S ($1 \leq i \leq n$) admits a Stein structure.*
 (2) *Any X_i^N ($1 \leq i \leq n$) admits no Stein structure for any orientations.*
 (3) *$X_1^S, X_2^S, \dots, X_n^S$ and $X_1^N, X_2^N, \dots, X_n^N$ are mutually homeomorphic.*
 (4) *The fundamental groups, the integral homology groups, the integral homology groups of the boundary, and the intersection forms of every X_i^S and every X_i^N ($0 \leq i \leq n$) are isomorphic to those of $X \natural U(0)$.*

Proof. Every X_i^S ($1 \leq i \leq n$) is clearly a Stein handlebody, hence admits a Stein structure. Since $U(0)_0^{(1)}$ (hence X_i^N) contains a homologically non-vanishing smoothly embedded sphere with its self-intersection number 0, the adjunction inequality shows that X_j^N ($1 \leq j \leq n$) does not admit any Stein structure for any orientations. Since $X_1^S, X_2^S, \dots, X_n^S$ and $X_1^N, X_2^N, \dots, X_n^N$ are mutually related by cork twists, the claim (3) follows. The definitions of $U(0)_j^{(1)}$, X_i^S and X_i^N and Theorem 6.3 show claim (4). q.e.d.

Lemma 10.4. *X_i^S ($1 \leq i \leq n$) are mutually non-diffeomorphic for any orientations.*

Proof. The constructions in Sections 5 and 6 and the above condition (iii)' show that each X_i^S can be diffeomorphic to $\widehat{X \natural U(0)}_i^{(n)}$, which denotes the Legendrian handlebody in Theorem 6.3 corresponding to $X \natural U(0)$. Theorem 6.3 thus shows that X_i^S ($1 \leq i \leq n$) are mutually non-diffeomorphic for any orientations. q.e.d.

Let $v_j^{(i)}$ ($0 \leq j \leq k$) denote the elements of $H_2(\widehat{X}_i^{(n)}; \mathbf{Z})$ in Definition 5.12, corresponding to $\widehat{X}_i^{(n)}$ in Definition 10.2 (recall that $\widehat{X}_i^{(n)}$ is defined as a special case of $X_i^{(n)}$). Let w be the generator of $H_2(U(0)_0^{(1)}; \mathbf{Z})$. Let g_j ($0 \leq j \leq k$) and m_j, q_j ($0 \leq j \leq l$) denote the integers in Section 5, corresponding to the above $\widehat{X}_i^{(n)}$. Then the following lemma holds similarly to Lemma 5.13.

Lemma 10.5. *For each $0 \leq i \leq n$, the elements $v_0^{(i)}, v_1^{(i)}, \dots, v_k^{(i)}, w$ span a basis of $H_2(X_i^N; \mathbf{Z})$ and satisfy the following conditions (ignore (ii) when $k = 0$).*

- (i) $v_0^{(i)}$ is represented by a smoothly embedded surface of genus $g_0 + p_i$ and satisfies $v_0^{(i)} \cdot v_0^{(i)} = m_0$.
- (ii) $v_j^{(i)}$ ($1 \leq j \leq k$) is represented by a smoothly embedded surface of genus $g_j + q_j$ and satisfies $v_j^{(i)} \cdot v_j^{(i)} = m_j$.
- (iii) w is represented by a smoothly embedded sphere and satisfies $w^2 = 0$.

We can easily check the following lemma similarly to Propositions 5.15 and 6.2.

Lemma 10.6. *For any $1 \leq i \leq n$, there exists no basis u_0, u_1, \dots, u_{k+1} of $H_2(X_i^N; \mathbf{Z})$ that satisfies the following conditions (ignore (ii) when $b_2(X) = 1$).*

- (i) u_0 is represented by a smoothly embedded surface with its genus equal to or less than $g_0 + p_{i-1}$ and satisfies $u_0^2 = |m_0|$.
- (ii) Each u_j ($1 \leq j \leq k$) is represented by a smoothly embedded surface of genus $g_j + q_j$ and satisfies $u_j^2 = |m_j|$.
- (iii) u_{k+1} is represented by a smoothly embedded sphere and satisfies $u_{k+1}^2 = 0$.

Proof. Suppose that such a basis u_0, u_1, \dots, u_{k+1} exists. Then each u_j ($0 \leq j \leq k+1$) is a linear combination of $v_0^{(i)}, v_1^{(i)}, \dots, v_k^{(i)}, w$ by Lemma 10.5. Since $U(0)$ can be embedded into the 4-ball, Proposition 4.5 shows that $U(0)_0^{(1)}$ can be embedded into the 4-ball. Thus, each X_i^N can be embedded into $\widehat{X}_i^{(n)}$ so that w is sent to 0 and that $v_0^{(i)}, v_1^{(i)}, \dots, v_k^{(i)}$ are sent identically. Therefore we can apply the same argument as Propositions 5.15 and 6.2, and easily get the required claim. q.e.d.

We can now easily prove Theorem 10.1.

Proof of Theorem 10.1. Lemma 10.3 gives the claims (2) and (3). Lemma 10.3 also shows that, for any i, j , two 4-manifolds X_i^S and X_j^N are not diffeomorphic for any orientations. Since the sequence p_i ($i \geq 0$) is strictly increasing, Lemmas 10.5 and 10.6 show that X_i^N ($1 \leq i \leq n$) are mutually non-diffeomorphic for any orientations. Lemmas 10.4 and 10.3.(3) thus give claim (1). q.e.d.

Remark 10.7. Though we used $U(0)_j^{(1)}$ to define X_i^S and X_i^N , we can similarly define X_i^S and X_i^N , using $U(-1)_j^{(1)}$. Put $X_i^S := \widehat{X}_i^{(n)} \natural U(-1)_1^{(1)}$ and $X_i^N := \widehat{X}_i^{(n)} \natural U(-1)_0^{(1)}$, where we assume that p_i 's satisfy

$$(iii)'' \quad 2p_1 + (t_0 - 1) - m_0 + |r_0| - 1 > 0,$$

instead of (iii)' in Definition 10.2. In this case, Theorem 10.1 also holds, where we replace $X \natural U(0)$ with $X \natural U(-1)$ in claim (3). Moreover, each X_i^N does admit a Stein structure after blowing down.

We here give an outline of this proof. The claim corresponding to Lemmas 10.3, 10.4, and 10.5 clearly holds. However, the claim corresponding to Lemma 10.6 is not clear, because $U(-1)$ cannot be embedded into the 4-ball. Here notice that $U(-1)_0^{(1)}$ contains a 2-sphere with the self-intersection number -1 , and that the blowdown of $U(-1)_0^{(1)}$ still

has a Stein handlebody presentation. Using this fact, we can prove the claim corresponding to Lemma 10.6 as follows. Since the blowdown of X_i^N is a Stein handlebody, we can embed it into a minimal complex surface of general type (for this, the property similar to Theorem 3.3 holds; cf. [23]). Then use the blowup formula and the adjunction inequality and apply the argument in the proof of Lemma 10.6.

2-handlebodies give a large class of 4-manifolds with boundary. Actually, we easily get the following.

Corollary 10.8. (1) *For any finitely presented group G , there exist arbitrary many compact Stein 4-manifolds and arbitrary many non-Stein 4-manifolds such that they are mutually homeomorphic but not diffeomorphic and that their fundamental groups are isomorphic to G .*

(2) *For any integral symmetric bilinear form Q over any integral free module, there exist arbitrary many simply connected compact Stein 4-manifolds such that they are mutually homeomorphic but not diffeomorphic and that their intersection forms are isomorphic to Q .*

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