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RESONANCE FOR LOOP HOMOLOGY OF SPHERES

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Abstract

A Riemannian or Finsler metric on a compact manifold M gives rise to a length function on the free loop space ΛM, whose critical points are the closed geodesics in the given metric. If X is a homology class on ΛM, the "minimax" critical level $\mathsf{cr}(X)$ is a critical value. Let M be a sphere of dimension > 2, and fix a metric g and a coefficient field G. We prove that the limit as $\deg(X)$ goes to infinity of $\mathsf{cr}(X)/\deg(X)$ exists. We call this limit $\overline{\alpha} = \overline{\alpha}(M, g, G)$ the *global mean frequency* of M. As a consequence we derive resonance statements for closed geodesics on spheres; in particular either all homology on Λ of sufficiently high degreee lies hanging on closed geodesics of mean frequency (length/average index) $\overline{\alpha}$, or there is a sequence of infinitely many closed geodesics whose mean frequencies converge to $\overline{\alpha}$. The proof uses the Chas-Sullivan product and results of Goresky-Hingston [**7**].

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1. Introduction and Statement of Results

Fix a Riemannian metric g or a Finsler metric f on a compact manifold M. Hence the corresponding norm is given by $\|v\|^2 = g(v,v)$ resp. $\|v\| = f(v)$. As common notation we use the letter g also for a Finsler metric. Let Λ be the space of H^1maps $\gamma : S^1 = \mathbb{R}/\mathbb{Z} \to M$, and let $F : \Lambda \to \mathbb{R}$ be the square root of the energy function:

$$F(\gamma) = \left(\int_0^1 \|\gamma'(t)\|^2 \, dt \right)^{1/2}$$

Then F and the length function $l(\gamma) = \int_0^1 \|\gamma'(t)\| \, dt$ agree on loops that are parameterized proportional to arclength. Fix a coefficient group G. The *critical level* of a homology class $X \in H_*(\Lambda; G)$ is defined to be

$$\begin{aligned} \mathsf{cr}(X) &= \mathsf{cr}_{g,G}(X) = \inf\left\{ a \in \mathbb{R} : \ X \text{ is supported on } \Lambda^{\le a} \right\} \\ &= \inf\left\{ a \in \mathbb{R} : \ X \text{ is in the image of } H_*(\ \Lambda^{\le a}) \right\}, \end{aligned}$$

where $\Lambda^{\le a} \subset \Lambda$ is the space of loops γ with $F(\gamma) \le a$. The critical level of a cohomology class $x \in H^*(\Lambda)$ is defined to be

$$\begin{aligned} \mathsf{cr}(x) &= \mathsf{cr}_{g,G}(x) = \sup\left\{ a \in \mathbb{R} : \ x \text{ is supported on } \Lambda^{\ge a} \right\} \\ &= \sup\left\{ a \in \mathbb{R} : \ x \text{ is in the image of } H^*(\Lambda, \ \Lambda^{< a}) \right\}. \end{aligned}$$

These are critical values of the function F (unless $x = 0 \in H^*(\Lambda)$, in which case $\mathsf{cr}(x) = \infty$). Since the critical points of F are closed geodesics on M, it follows that for each nontrivial homology or cohomology class X, there is a closed geodesic of length $\mathsf{cr}(X)$.

Theorem 1.1. (Resonance Theorem) *A Riemannian or Finsler metric g on $S^n, n > 2$, and a field G determine a* global mean frequency $\overline{\alpha} =: \overline{\alpha}_{g,G} > 0$ *with the property that*

$$\deg(X) - \overline{\alpha}\, \mathsf{cr}(X)$$

is bounded as X ranges over all nontrivial homology and cohomology classes on Λ with coefficients in G.

Thus the countably infinite set of points $(\mathsf{cr}(X), \deg(X))$ in the $(\ell, d)-$plane lies in bounded distance from the line $d = \overline{\alpha}\ell$.

We will derive an explicit bound for $|\deg(X) - \overline{\alpha}\, \mathsf{cr}(X)|$ in Appendix A. If the metric g carries only finitely many closed geodesics, then for $\deg(X)$ sufficiently large, each homology class X satisfies:

$$|\deg(X) - \overline{\alpha}\, \mathsf{cr}(X)| \le n$$

by Lemma 10.2. If γ is a closed geodesic, then so is each of its *iterates* γ^m: $\gamma^m(t) = \gamma(mt)$. The *index* of γ is the index of the hessian of the functional F on the free loop space, and the *average index* is $\alpha_\gamma = \lim_{m\to\infty} \operatorname{ind}(\gamma^m)/m$.

Theorem 1.2. (Density Theorem) *Let $\overline{\alpha}_{g,\mathbb{Q}}$ be the rational global mean frequency of a Riemannian or Finsler metric g on S^n, $n > 2$. For any $\varepsilon > 0$ we have the following estimate (compare Rademacher* [**18**, Thm.1.2]*) for the sum of inverted average indices α_γ of geodesics on (S^n, g) :*

$$\sum_\gamma \frac{1}{\alpha_\gamma} \geq \begin{cases} \frac{1}{n-1} & ; \ n \text{ odd} \\ \frac{1}{2(n-1)} & ; \ n \text{ even} \end{cases}$$

where we sum over a maximal set of prime, geometrically distinct closed geodesics γ whose mean frequency $\overline{\alpha}_\gamma =: \alpha_\gamma/\ell(\gamma)$ satisfies:
$\overline{\alpha}_\gamma \in (\overline{\alpha}_{g,\mathbb{Q}} - \varepsilon, \overline{\alpha}_{g,\mathbb{Q}} + \varepsilon)$.

For each closed geodesic γ,

$$\lim_{d\to\infty} \frac{\#\{m \geq 1\,;\, \mathrm{ind}(\gamma^m) \leq d\}}{d} = \frac{1}{\alpha_\gamma}.$$

Thus $1/\alpha_\gamma$ represents the density, in units of geometrically distinct closed geodesics per degree, of all the iterates of γ. Since γ and all its iterates have the same mean frequency, the sum in the Density Theorem 1.2 represents the total density, in these units, of all the iterates of all the closed geodesics whose mean frequency lies in the given interval.

Corollary 1.3. *Let n be odd. Let g be a Riemannian metric on the n-sphere S^n whose sectional curvature K satisfies*

$$\frac{1}{4} < K \leq 1$$

or let g be a non-reversible Finsler metric with reversibility $\lambda > 1$ (cf. [**19**]*) whose flag curvature K satisfies:*

$$\left(1 - \frac{1}{\lambda + 1}\right)^2 < K \leq 1.$$

Then at least one of the following holds:

(i) *There are at least two closed geodesics γ with mean frequency $\overline{\alpha}_\gamma$ equal to $\overline{\alpha}_{g,\mathbb{Q}}$.*

(ii) *There is a sequence of prime closed geodesics $\{\gamma_j\}$, with mean frequencies $\overline{\alpha}_j \neq \overline{\alpha}_{g,\mathbb{Q}}$ satisfying*

$$\lim_{j\to\infty} \overline{\alpha}_j = \overline{\alpha}_{g,\mathbb{Q}}.$$

Using a Killing field V on the standard sphere one can define an one parameter family f_ε, $\varepsilon \in [0, 1)$ of Finsler metrics of constant flag curvature on the sphere S^n with the following properties: For $\varepsilon = 0$ the metric is the standard Riemannian metric. For $\varepsilon \in (0, 1)$ the metric is a non-reversible Finsler metric of reversibility $\lambda = (1 + \varepsilon)/(1 - \varepsilon)$. The geodesic flow is the composition of the geodesic flow of the standard metric and the flow of the Killing field εV. For ε irrational there are only finitely many closed geodesics, all of which are nondegenerate. These metrics were first introduced by Katok, cf. [**24**], [**18**, thm.5]. The number of geometrically distinct distinct closed geodesics is n if n is even and $n+1$ if n is odd. In the following Corollary we present properties of metrics nearby a Katok metric on S^n :

Corollary 1.4. *Let n be odd. Let S^n carry a Katok metric g_0 of constant flag curvature* 1 *and let $U \subset \Lambda$ be a neighborhood of the set of closed geodesics on (S^n, g_0). Let $N \in \mathbb{Z}$. There is a neighborhood $\mathcal{W}$ of g_0 in the space of metrics so that for every $g \in \mathcal{W}$, at least one of the following is true:*

(i) *There are at least two closed geodesics $\gamma \subset U$ with mean frequency $\overline{\alpha}_\gamma$ equal to $\overline{\alpha}_{g,\mathbb{Q}}$.*

(ii) *There is a sequence of prime closed geodesics $\{\gamma_j\}$, with mean frequencies $\overline{\alpha}_j \neq \overline{\alpha}_{g,\mathbb{Q}}$ satisfying*

$$\lim_{j\to\infty} \overline{\alpha}_j = \overline{\alpha}_{g,\mathbb{Q}}.$$

(iii) *There are at least N closed geodesics with mean frequency equal to $\overline{\alpha}_{g,\mathbb{Q}}$.*

In the Katok metrics the set of prime closed geodesics is compact, and all closed geodesics γ have the same mean frequency $\overline{\alpha}_\gamma = \overline{\alpha}_{g_0,\mathbb{Q}}$. Thus the Katok metrics are *perfectly resonant metrics*. Corollary 1.4 shows that it is difficult to pry the mean frequencies apart! But we will also prove the following:

Theorem 1.5. (Open mapping theorem) *Let M be a compact manifold. Let $\mathcal{G} = \mathcal{G}^r(M)$ be the set of C^r Riemannian or Finsler metrics on M, with $r \geq 2$ for a Riemannian metric and $r \geq 4$ for a Finsler metric. Let*

$$\mathcal{G}_j = \{(g, \gamma_1, \gamma_2, .., \gamma_j) : g \in \mathcal{G} \text{ and each } \gamma_i \text{ is a geodesic in the metric } g\} \subset \mathcal{G} \times \Lambda^j$$

The map

$$\Phi_j : \mathcal{G}_j \longrightarrow \mathbb{R}^j$$

by $\Phi_j(g, \gamma_1, \gamma_2, .., \gamma_j) = (\overline{\alpha}_1, \overline{\alpha}_2, ..., \overline{\alpha}_j)$, where $\overline{\alpha}_i$ is the mean frequency of the geodesic γ_i in the metric g, is an open mapping at each point $(g, \gamma_1, \gamma_2, .., \gamma_j)$ where the γ_i are geometrically distinct and of positive length. That is, at each such point the image of any open set containing the point $(g, \gamma_1, \gamma_2, .., \gamma_j)$ contains a neighborhood of $\Phi_j(g, \gamma_1, \gamma_2, .., \gamma_j)$.

This theorem supports our intuition that the mean frequencies of geometrically distinct closed geodesics can be perturbed independently. The argument of Brian White in [**23**] shows that the space $\mathcal{G}_j$ is a Banach manifold, and that the projection $\mathcal{G}_j \to \mathcal{G}$ is a smooth map of Fredholm index 0.

Lemma 1.6. (Continuity) *Fix a field G. Let $M = S^n$ and $r \geq 2$. The map $\mathcal{G} \to \mathbb{R}$ given by $g \mapsto \overline{\alpha}_{g,G}$ is continuous.*

If a metric g on a manifold M has global mean frequency $\overline{\alpha}$, the scaled metric $s^2 g$ (where all lengths are scaled by a factor s) will have mean frequency $\overline{\alpha}/s$. But we do not know how to perturb $\overline{\alpha}$ keeping the volume fixed, or keeping the mean frequency of a closed geodesic fixed.

In contrast to the Katok metrics, the standard ellipsoid metrics are highly nonresonant. The standard n-dimensional ellipsoid depends on $n+1$ parameters, and has $(n^2+n)/2$ "short" closed geodesics, namely the intersection of the ellipsoid with any of the coordinate planes. We prove the following:

Theorem 1.7. (Ellipsoid Theorem)

(a) *There is a nonempty open set of parameters (a_0, a_1, a_2) for which the mean frequencies of the three short closed geodesics on the 2-dimensional ellipsoid $M(a_0, a_1, a_2)$ are distinct.*

(b) *For $n \geq 3$ there is a nonempty open set of parameters $(a_0, a_{1,} ... a_n)$ for which there are at least $(n+1)/2$ different mean frequencies among the short closed geodesics on the n-dimensional ellipsoid $M(a_0, a_1, \ldots, a_n)$.*

The Density Theorem may not be optimal. If we could prove a resonance theorem for equivariant homology, we could improve the minimum density in Theorem 1.2 to

$$\sum_{\gamma} \frac{1}{\alpha_\gamma} \geq \frac{1}{2}$$

or, in the nondegenerate case, to

$$\sum_{\gamma} \frac{1}{\alpha_\gamma} \geq \begin{cases} \frac{n}{2(n-1)} & ; \qquad n \text{ even} \\ \frac{n+1}{2(n-1)} & ; \qquad n \text{ odd.} \end{cases}$$

where in each case the sum is over closed geodesics whose mean frequency lies in a given ε-neighborhood of $\overline{\alpha}$. The minimum density $\frac{1}{2}$ was obtained by Rademacher, see [**18**], but the sum there was taken in general over closed geodesics whose mean frequency lies in a given ε-neighborhood of an interval. With these improved densities, the number of resonant closed geodesics (2) in Corollary 1.3 (i) and Corollary 1.4 (i) would be replaced by $(n-1)/2$ and, in the nondegenerate case, by $(n+1)/2$. This last estimate is within a factor of 2 of the optimal number, as the odd-dimensional Katok metrics have as few as $n+1$ closed geodesics. Using the *Common Index Jump Theorem* of Long and Zhu [**15**] and recent results of Wang [**22**] one might do even better. However products play a crucial role in our proof; at present there is not a theory incorporating equivariant homology/cohomology and the products $\circledast$ and $\bullet$ (cf. Equation 4) we have used to prove the Resonance Theorem 1.1. There are related products on equivariant homology and cohomology (see [**4**, p.19], and[**7**, 17.4, p.156]), but they vanish for spheres, see Proposition 11.1.

Using a version of the Fadell-Rabinowitz index [**6**] and following similar results by Ekeland-Hofer [**5**] for periodic orbits of of convex Hamiltonian energy hypersurfaces in $\mathbb{R}^{2n}$ Rademacher studied in [**18**] limits closely related to

$$\lim_{m\to\infty} \frac{\mathsf{cr}(z \cup \eta^{\cup m})}{m}$$

for a sequence of equivariant cohomology classes $z \cup \eta^{\cup m} \in H^*_{S\mathbb{O}(2)}\left(\Lambda, \Lambda^0; \mathbb{Q}\right)$ where η is a nonnilpotent element for the cup product $\cup$ in equivariant cohomology. It is not known if there is a metric for which these limits do not exist. However if

$$\limsup_{m\to\infty} \frac{\mathsf{cr}\left(z \cup \eta^{\cup m}\right)}{m} \neq \liminf_{n\to\infty} \frac{\mathsf{cr}\left(z \cup \eta^{\cup m}\right)}{m} \tag{1}$$

for a fixed metric g on M, there would be infinitely many closed geodesics on M for any metric in a neighborhood of g, cf. [**18**, Cor.6.5]. In Section 3 we will look at analogous limits for the loop products, and prove an analogous theorem. Proposition 4.3, Corollary 4.4, and the Theorem 5.1 (*Interval Theorem*) should be compared to Ekeland-Hofer [**5**] and Rademacher [**18**]. If one could extend the resonance theorem to equivariant homology/cohomology, this would imply that the index interval discussed in [**18**] is always a point, and that Inequality 1 never holds.

Organization of the paper

After discussing the Resonance Theorem in Section 2 we present general facts about critical levels, duality, and coefficients in Section 3. Section 4 is concerned with limits of the type given in Equation 1 for the loop homology and loop cohomology products. In Section 5 we prove the *Resonance Theorem* 1.1.

The proof depends on the results of Section 3 and Section 4, and upon three Lemmas 5.3, 5.4, 5.5 regarding the loop homology and cohomology of ΛS^n. In Section 6 we explain how Lemma 5.3 and Lemma 5.4 follow from the results of [**7**] and prove Lemma 5.5. We also compute the Chas-Sullivan product on $H_*\left(\Lambda(S^n); \mathbb{Z}_2\right)$ for $n > 2$ even.

In Section 7 we prove the Continuity and Density Theorems, Corollary 1.3 and Corollary 1.4. Section 8 contains the proof of the Open Mapping Theorem 1.5. The main step is the Perturbation Theorem 8.1; Lemma 8.8 on +-curves in the symplectic group may be of independent interest. In Section 9 we prove the Ellipsoid Theorem. Appendix A (Section 10) contains proofs of several results on closed geodesics that should be familiar to experts. In Appendix B (Section 11) we show that the string brackets on the equivariant homology and cohomology of spheres are trivial.

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2. Discussion of the Resonance Theorem 1.1

Average index and mean frequency of a closed geodesic:

Fix a metric g on M, with M simply connected, and let G be a field. According to a theorem of Gromov [**9**, thm.7.3], [**16**, thm.5.10], the points $(\mathsf{cr}(X), \deg(X))$, $X \in H_*(\Lambda M)$, lie between two lines: There are positive constants C_1 and C_2 such that, for any $X \in H_*(\Lambda M)$,

$$C_1 \,\mathsf{cr}(X) < \deg(X) < C_2 \,\mathsf{cr}(X).$$

The Resonance Theorem 1.1 states if M is a sphere, there is a positive real number $\overline{\alpha} = \overline{\alpha}_{g,G}$ and a constant h so that

$$\overline{\alpha}\,\mathsf{cr}(X) - h < \deg(X) < \overline{\alpha}\,\mathsf{cr}(X) + h.$$

Let M be a compact Riemannian or Finsler manifold of dimension n. If $\gamma \in \Lambda M$, we denote by γ^m the m^{th} iterate of $\gamma : \gamma^m(t) = \gamma(mt)$. A closed geodesic that is not an m^{th} iterate for any $m > 1$ is called *prime.* Closed geodesics are considered *geometrically distinct* if they have different images in M. (For a non-reversible Finsler metric we also consider closed geodesics distinct if they have opposite orientations.) If γ is a closed geodesic on M with length $L > 0$, then γ^m is a closed geodesic of length mL. Bott [**2**] proved that the *average index*

$$\alpha_\gamma = \lim_{m\to\infty} \frac{\operatorname{ind}(\gamma^m)}{m}$$

exists. Therefore so does the *mean frequency* $\overline{\alpha}_\gamma$:

$$\overline{\alpha}_\gamma = \lim_{m\to\infty} \frac{\operatorname{ind}(\gamma^m)}{l(\gamma^m)} = \frac{\alpha_\gamma}{L}\,.$$

The mean frequency can be thought of as the number of conjugate points per unit length. It can be estimated using the sectional or flag curvature, cf. Lemma 9.1. If

$$\delta^2 \le K \le \Delta^2$$

then for every closed geodesic γ on M,

$$\frac{\delta(n-1)}{\pi} \leq \overline{\alpha}_\gamma \leq \frac{\Delta(n-1)}{\pi}.$$

Note also that $\overline{\alpha}_\gamma$ does not depend on the field, and that

$$\overline{\alpha}_\gamma = \overline{\alpha}_{\gamma^m}$$

for $m > 1$.

Statement of the Resonance Theorem 1.1 in terms of the spectral sequence:

Fix a metric g on M, and let G be a field. Let $0 = \ell_0 < \ell_1 < \dots$ be a sequence of real numbers each of which is a regular value, or a nondegenerate critical value, of F. If the critical values are isolated it will make sense to assume that there is a unique critical value in each $(\ell_i, \ell_{i+1}]$. The filtration $\{\Lambda^{\leq \ell_i}\}$ induces a spectral sequence converging to $H_*(\Lambda)$. Each page of the homology spectral sequence is bigraded by the index set $\{i\}$ of the sequence $\{\ell_i\}$, and the whole numbers d . The units of ℓ_i are *length* (the same units as critical values), and the units of d are *degree* (the same units as index) . It is convenient to think of this as a first quadrant spectral sequence in the (ℓ, d)-plane indexed by $\{(\ell_i, d)\}$. Each page of the spectral sequence is the direct sum of its (ℓ_i, d) terms. The (ℓ_i, d) term of the $\mathcal{E}^1$ page is given by

$$H_d(\Lambda^{\leq \ell_i}, \Lambda^{\leq \ell_{i-1}}; G).$$

The k^{th} page is obtained from the previous page by "cancelling" terms by the differential D_k. The differential D_k, $k \geq 1$ has degree $(-k, -1)$ (so that, roughly speaking,
$D_k : H_d(\Lambda^{\leq \ell_i}, \Lambda^{\leq \ell_{i-1}}; G) \to H_{d-1}(\Lambda^{\leq \ell_{i-k}}, \Lambda^{\leq \ell_{i-(k+1)}}; G))$. The homology version of the Resonance Theorem 1.1 states that if M is a sphere, in the $\mathcal{E}^\infty$ page of the spectral sequence all nontrivial entries lie within a bounded distance of a line

$$d = \overline{\alpha}\ell.$$

There is also a spectral sequence converging to $H^*(\Lambda)$, and a similar statement for cohomology: The (ℓ_i, d) term of the $\mathcal{E}^1$ page is given by

$$H^d(\Lambda^{\leq \ell_i}, \Lambda^{\leq \ell_{i-1}}; G)$$

and the differential D_k, $k \geq 1$ has degree $(k, 1)$. If M is a sphere, in the $\mathcal{E}^\infty$ page of the spectral sequence all nontrivial entries lie within a bounded distance of the line

$$d = \overline{\alpha}\ell.$$

If all closed geodesics are isolated, and if each $(\ell_{i-1}, \ell_i]$ contains exactly one critical value L_i, the $\mathcal{E}^1$ page of the spectral sequence is naturally a direct sum of the "contributions" of all the closed geodesics γ. A closed geodesic γ of length L_i contributes its *local homology*

$$H_*(\Lambda^{<L_i} \cup S \cdot \gamma, \Lambda^{<L_i}) \tag{2}$$

to $H_*(\Lambda^{\leq \ell_i}, \Lambda^{\leq \ell_{i-1}})$, where $S \cdot \gamma$ is the orbit of γ under the group S, with $S = \mathbb{O}(2)$ for a Riemannian metric and $S = S\mathbb{O}(2) = S^1$ for an non-reversible Finsler metric. If $X \in H_*(\Lambda^{<L_i} \cup S \cdot \gamma, \Lambda^{<L_i})$ is nontrivial, then we conclude from Lemma 10.1 on the *resonant iterates:*

$$|\deg X - \overline{\alpha}_\gamma \mathsf{cr}(X)| \leq n\,.$$

Thus the contributions to the $\mathcal{E}^1$ page of the homology spectral sequence (or, by the same argument, the cohomology spectral sequence) from the iterates of a single prime closed geodesic γ of mean frequency $\overline{\alpha}_\gamma$ lie at most a vertical distance n from the line

$$d = \overline{\alpha}_\gamma \ell.$$

Examples:

The resonance theorem is obvious in the following three cases:

1) A hypothetical metric with only one closed geodesic. In this case it follows from the above that in the $\mathcal{E}^1$ page of the homology spectral sequence, all nontrivial terms lie at a bounded distance from the line $d = \overline{\alpha}\ell$. Because (over a field) the $\mathcal{E}^\infty$ page is obtained by "cancelling" terms x and y if $Dx = y$, on the $\mathcal{E}^\infty$ page all nontrivial terms again lie at a bounded distance from the same line.
2) The round metric of constant curvature K. In this case all geodesics are closed with (prime) length $2\pi/\sqrt{K}$. The index of the m^{th} iterate of every closed geodesic is $(2m-1)(n-1)$ and therefore as in case (1), in the $\mathcal{E}^1$ page of the homology spectral sequence all nontrivial terms lie at a bounded distance from the line
$$d = \frac{\sqrt{K}(n-1)}{\pi}\ell\,,$$
and this property persists to $\mathcal{E}^\infty$.
3) The Katok metrics on S^n, compare Corollary 1.4. In these examples the closed geodesics have different lengths and different average indices but they all have the same mean frequency since the flag curvatures are constant! Thus in this case also the theorem is clear from the $\mathcal{E}^1$ page.

In all other cases we find the Resonance Theorem surprising.

Consider for example the n-dimensional ellipsoid

$$M = M(a_0, a_1, \dots a_n) = \left\{ (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} \frac{x_i^2}{a_i^2} = 1 \right\}$$

where $a_0 < a_1 < ... < a_n$. The ellipsoid carries $\binom{n+1}{2} = (n^2+n)/2$ *short* closed geodesics, namely the intersection of the ellipsoid with any of the coordinate planes. However Morse proved that in the limit as all $a_i \to 1$, the length of the next shortest prime closed geodesic goes to infinity, cf. [**13**, Lem.3.4.7]. The geodesic flow on the ellipsoid is integrable and there are invariant 2-dimensional tori in the unit tangent bundle. Periodic flow lines are dense, i.e. the unit tangent vectors of closed geodesics form a dense subset of the unit tangent bundle. In particular there are infinitely many closed geodesics, cf. [**13**, Sec.3.5]. It would be very interesting to know a formula for the mean frequencies of the $(n^2+n)/2$ short closed geodesics in terms of the $(n+1)$ parameters a_j! Our intuition is that for a generic ellipsoid, the mean frequencies of the *short* closed geodesics are all different. This would mean that the $\mathcal{E}^1$ page of the spectral sequence has nontrivial entries roughly evenly spaced along $\binom{n+1}{2}$ different lines through the origin in the (ℓ, d) plane. But in the $\mathcal{E}^\infty$ page at most one of these lines remains; all high iterates of all but at most one of the $\binom{n+1}{2}$ short geodesics must be *killed off* before the $\mathcal{E}^\infty$ page. This would mean that the geodesics that dominate at low energy are irrelevant at high energy. While we have not had much success computing the mean frequencies on an ellipsoid, the *Ellipsoid Theorem* 1.7 shows that for an open set of metrics, many of the mean frequencies of the short closed geodesics on an ellipsoid are distinct.

3. Critical levels, duality, and coefficients

In this section let X be a Hilbert manifold and let $F : X \to \mathbb{R}$ be a smooth function satisfying condition C. For $a \in \mathbb{R}$ let $X^a = \{x \in X; F(x) \le a\}$, and $X^{a-} = \{x \in X; F(x) < a\}$. The critical levels $\mathsf{cr}(X) = \mathsf{cr}_G(X)$ or $\mathsf{cr}(x) = \mathsf{cr}_G(x)$ of a relative homology or cohomology class $X \in H_m(X^c, X^a; G)$ or $x \in H^m(X^c, X^a; G)$ are defined by

$$\begin{aligned} \mathsf{cr}_G(X) &= \inf\left\{ b \in [a,c] : x \text{ is in the image of } H_m\left(X^b, X^a; G\right)\right\}; \\ \mathsf{cr}_G(x) &= \sup\left\{ b \in [a,c] : x \text{ is in the image of } H^m\left(X^c, X^{b-}; G\right)\right\} \\ &= \sup\left\{ b \in [a,c] : x \to 0 \text{ in } H^m\left(X^{b-}, X^a; G\right)\right\}. \end{aligned}$$

The critical level of a homology or cohomology class is a critical value of F.

Lemma 3.1. (Duality Lemma) *Let G be a field. Suppose $H_m(X^c, X^a; G) = G$, and that $X \in H_m(X^c, X^a; G)$ and*

$x \in H^m(X^c, X^a; G)$ *have nontrivial Kronecker product* $x(X)$. *Then*

$$\mathsf{cr}(X) = \mathsf{cr}(x)\,.$$

Proof. Clearly

$$\mathsf{cr}(x) \le \mathsf{cr}(X). \tag{3}$$

Suppose $a < b < c$, and let $j : (X^b, X^a) \to (X^c, X^a)$. If $\mathsf{cr}(x) < b$, then $j^*(x) \in H^m(X^b, X^a; G)$ is nontrivial. It follows that there is a class $Y \in H_m(X^b, X^a; G)$ with $j^*(x)(Y) \neq 0$. But then $x(j_*(Y)) = j^*(x)(Y) \neq 0$, from which it follows that $j_*(Y)$ is a multiple of X, and thus $\mathsf{cr}(X) \le b$. q.e.d.

Remark 3.2. Things could be more complicated if the groups have rank > 1. Assume that $H_m(X^c, X^a; G) = G^k$. For simplicity assume that a and c are not critical values. There will be critical values $b_1, \ldots, b_j$ with $a < b_1 < ... < b_j < c$ and subspaces

$$A_1 \subset A_2 \subset ... \subset A_j = H_m(X; G)$$

where A_i is the image of $H_m(X^{b_i}; G)$ in $H_m(X; G)$, and where A_i/A_{i-1} has dimension k_i, with $\Sigma_{i=1}^{j} k_i = k$, and so that if $Y \in (A_i - A_{i-1})$, then $\mathsf{cr}(Y) = b_i$.

Similarly there are subspaces

$$B_j \subset B_{j-1} \subset ... \subset B_1 = H^m(X; G)$$

where B_i is the kernel of the map $H^m(X; G) \to H^m(X^{<b_i}; G)$, so that B_i/B_{i+1} has dimension k_i, and $B_i - B_{i+1}$ is the set of cohomology classes with critical value b_i.

Note also that with respect to the Kronecker product,

$$B_i(A_{i-1}) = 0,$$

and this equation could be used to define B_i once the A_i are defined. For a generic basis for $H_m(X^c, X^a; G)$, the critical value of each generator would be high ($= b_k$), and for the dual basis (or a generic basis) for $H^m(X^c, X^a; G)$, the critical value of each generator would be low ($= b_1$).

Lemma 3.3. (Lemma on the effect of Coefficients) *Suppose* $H^m(X^c, X^a) =: H^m(X^c, X^a; \mathbb{Z}) = \mathbb{Z}$ *and* $H_m(X^c, X^a) =: H_m(X^c, X^a; \mathbb{Z}) = \mathbb{Z}$. *Let* z, Z *be generators of these groups, and let* $w \in H^m(X^c, X^a; G)$ *and* $W \in H_m(X^c, X^a; G)$ *be nontrivial, with* $G = \mathbb{Z}$ *or a field. Then*

$$\mathsf{cr}_{\mathbb{Z}}(z) \le \mathsf{cr}_G(w) \le \mathsf{cr}_{\mathbb{Z}}(Z)$$
$$\mathsf{cr}_{\mathbb{Z}}(z) \le \mathsf{cr}_G(W) \le \mathsf{cr}_{\mathbb{Z}}(Z).$$

Proof. First $H^m(X^c, X^a) = \mathbb{Z}$ implies [**1**, Cor.7.3] that $H_{m-1}(X^c, X^a)$ is torsion free. From this it follows [**1**, p.278] that $H_{m-1}(X^c, X^a) * G = 0$, and thus [**1**, 7.5] that

$$H_m(X^c, X^a) \otimes G \overset{\approx}{\to} H_m(X^c, X^a; G) = G.$$

So $W = Z \otimes g$ for some (nontrivial) $g \in G$. It follows that $\mathsf{cr}_G(W) \leq \mathsf{cr}_{\mathbb{Z}}(Z)$ (since for every representative R for the homology class Z, we have the representative $R \otimes g$ for W , and Supp(R ⊗ g) ⊂ Supp(R).) Moreover [**1**, p.278] Ext $(H_{m-1}(X^c, X^a), G) = 0$, so ([**1**, 7.2]).

$$H^m(X^c, X^a; G) \overset{\approx}{\to} \mathrm{Hom}(H_m(X^c, X^a), G) = G,$$

and when $G = \mathbb{Z}$ the latter isomorphism $\overset{\approx}{\to}$ is induced by the Kronecker product. Thus $z(Z) = 1$. This isomorphism is also natural in the coefficients, so the map $\mathbb{Z} \to G$ taking 1 to the identity of G induces the map $H^m(X^c, X^a) \to H^m(X^c, X^a; G)$ taking 1 to the identity of G. Thus $w = hz$ (as maps $H_m(X^c, X^a) \to G$) for some (nontrivial) $h \in G$. It follows that $\mathsf{cr}_{\mathbb{Z}}(z) \leq \mathsf{cr}_G(w)$. If $G = \mathbb{Z}$, we use the fact that the Kronecker products $w(Z) = hz(Z)$ and $z(W) = gz(Z)$ are nontrivial to conclude that $\mathsf{cr}_{\mathbb{Z}}(w) \leq \mathsf{cr}_{\mathbb{Z}}(Z)$ and $\mathsf{cr}_{\mathbb{Z}}(z) \leq \mathsf{cr}_{\mathbb{Z}}(W)$. When G is a field we have from *Duality Lemma* 3.1 that $\mathsf{cr}_G(w) = \mathsf{cr}_G(W)$, and the Lemma follows. q.e.d.

Remark 3.4. It is not difficult to imagine a scenario in which the critical level of a homology class depends on the coefficients; for example it could be that a homology class X has critical level a, but that some multiple of X is homologous to a class at a lower level. We do not know an example of a loop space where this happens. The simplest case is an interesting question of basic geometry: Let $X \in H_{n-1}(\Lambda S^n)$ be a generator. Is there a metric g on S^n and an integer k so that

$$\mathsf{cr}_{g,\mathbb{Z}}(kX) < \mathsf{cr}_{g,\mathbb{Z}}(X)\,?$$

4. Mean critical levels for homology and cohomology

Fix a Riemannian or Finsler metric g on a compact Riemannian manifold M of dimension n. Fix a coefficient ring G, either a field or $\mathbb{Z}$. Let Λ be the free loop space and $\Lambda^0 \subset \Lambda$ the constant loops. The loop products $\bullet$ and $\circledast$ of Chas-Sullivan [**4**] and Sullivan [**21**]

$$\begin{array}{rrcl} \bullet: & H_j(\Lambda) \otimes H_k(\Lambda) & \longrightarrow & H_{j+k-n}(\Lambda) \\ \circledast: & H^j(\Lambda, \Lambda^0) \otimes H^k(\Lambda, \Lambda^0) & \longrightarrow & H^{j+k+n-1}(\Lambda, \Lambda^0) \end{array} \tag{4}$$

are commutative up to sign. It is shown in [**7**, Prop.5.3, Cor. 10.1] that they satisfy the following basic inequalities:

$$\begin{array}{rcll} \mathsf{cr}(X \bullet Y) & \leq & \mathsf{cr}(X) + \mathsf{cr}(Y) & \text{for all } X, Y \in H_*(\Lambda) \\ \mathsf{cr}(x \circledast y) & \geq & \mathsf{cr}(x) + \mathsf{cr}(y) & \text{for all } x, y \in H^*(\Lambda, \Lambda^0). \end{array} \tag{5}$$

The proofs given there are also valid in the Finsler case. The sequences $\mathsf{cr}(\tau^{\circledast m})$ and $\mathsf{cr}(x \circledast \tau^{\circledast m})$ for *cohomology* classes τ, x are *always* increasing (if finite) by Equation 5. As far as we know the sequences $\mathsf{cr}(U^{\bullet m})$ and $\mathsf{cr}(X \bullet U^{\bullet m})$ for $U, X \in H_*(\Lambda)$ are not necessarily increasing.

We consider the limit as $m \to \infty$ of

$$\frac{\mathsf{cr}(\eta^{\circledast m})}{m} \text{ and } \frac{\mathsf{cr}\,(z \circledast \eta^{\circledast m})}{m}$$

where η and z are both homology or cohomology classes with η non-nilpotent, and where $*$ is the appropriate loop product. Since $\deg(\eta^{\circledast m})$ and $\deg(z \circledast \eta^{\circledast m})$ are approximately linear, the above limits exist if and only if the limits

$$\frac{\mathsf{cr}(\eta^{\circledast m})}{\deg(\eta^{\circledast m})} \text{ and } \frac{\mathsf{cr}\,(z \circledast \eta^{\circledast m})}{\deg(z \circledast \eta^{\circledast m})}$$

exist. (If G is a field, these limits are determined by the Resonance Theorem.)

Let $U, Z \in H_*(\Lambda)$ with $\deg U > n$. We define the *mean levels*

$$\overline{\mu_U}(Z) = \overline{\mu_{g,G,U}}(Z) = \limsup_{m\to\infty} \mathsf{cr}_{g,G}(U^{\bullet m} \bullet Z)/m$$
$$\underline{\mu_U}(Z) = \underline{\mu_{g,G,U}}(Z) = \liminf_{m\to\infty} \mathsf{cr}_{g,G}(U^{\bullet m} \bullet Z)/m.$$

When the two limits coincide we denote the common limit $\mu_U(Z)$.

Lemma 4.1. *If* $j \geq 0$, *and* $U, X, Y \in H_*(\Lambda)$, *then*

$$\overline{\mu_U}(X) = \overline{\mu_U}(U^{\bullet j} \bullet X) \tag{6}$$
$$\overline{\mu_U}(X \bullet Y) \leq \min\{\overline{\mu_U}(X), \overline{\mu_U}(Y)\} \tag{7}$$
$$\overline{\mu_U}(X) \leq \overline{\mu_U}(U). \tag{8}$$

Similar inequalities also hold for $\underline{\mu_U}$ *and* μ_U.

Proof. Equation 6 follows easily from the definitions. Equation 8 follows from the first two equations. We obtain Equation 7 as follows:

$$\begin{aligned}\overline{\mu_U}(X \bullet Y) &= \limsup_{m\to\infty} \mathsf{cr}\left(U^{\bullet m} \bullet X \bullet Y\right)/m \\ &\leq \limsup_{m\to\infty} \{\mathsf{cr}(U^{\bullet m} \bullet X)/m + \mathsf{cr}(Y)/m\} \\ &= \overline{\mu_U}(X).\end{aligned}$$

q.e.d.

We also have mean levels for cohomology;

$$\overline{\mu_\tau}(z) = \limsup_{m\to\infty} \mathsf{cr}\left(\tau^{\circledast m} \circledast z\right)/m \;\; ; \;\; \underline{\mu_\tau}(z) = \liminf_{m\to\infty} \mathsf{cr}(\tau^{\circledast m} \circledast z)/m.$$

If the limits coincide, we denote the common limit $\mu_\tau(z)$. If $j \geq 0$, and $\tau, x, y \in H^*(\Lambda, \Lambda^0)$, then similarly,

$$\begin{aligned} \overline{\mu_\tau}(x) &= \overline{\mu_\tau}(\tau^{\circledast j} \circledast x) \\ \overline{\mu_\tau}(x \circledast y) &\geq \min\{\overline{\mu_\tau}(x), \overline{\mu_\tau}(y)\} \\ \overline{\mu_\tau}(x) &\geq \overline{\mu_\tau}(\tau)\,. \end{aligned} \tag{9}$$

Similar inequalities also hold for $\underline{\mu_\tau}$ and μ_τ.

Lemma 4.2. (Powers Lemma) *If $U \in H_*(\Lambda)$, then $\overline{\mu_U}(U) = \underline{\mu_U}(U)$ and $\mu_U(U)$ exists. Moreover,*

$$\mu_U(U) \leq \mathsf{cr}(U^{\bullet m})/m$$

for all $m \geq 1$.

If $\tau \in H^*(\Lambda, \Lambda^0)$, then $\overline{\mu_\tau}(\omega) = \underline{\mu_\tau}(\tau)$ and $\mu_\omega(\omega)$ exists. Moreover,

$$\mu_\tau(\tau) \geq \mathsf{cr}(\tau^{\circledast m})/m$$

for all $m \geq 1$.

Proof. The lemma follows from Fekete's Lemma [**17**] and the fact that the sequences and $\mathsf{cr}(U^{\bullet m})$ and $\mathsf{cr}(\tau^{\circledast m})$ are subadditive and superadditive. q.e.d.

Index interval

Fix a metric on M and fix G, either a field or $\mathbb{Z}$. The same arguments used in [**18**, Thm.6.2] show the following:

Proposition 4.3. *Let $U, X \in H_*(\Lambda)$, with $\deg U > n$. If $t \in [\underline{\mu_U}(X), \overline{\mu_U}(X)]$ is nonzero (which implies that U is nonnilpotent), then there is a sequence γ_m of closed geodesics on M whose mean frequencies $\overline{\alpha}_m$ converge to $(\deg U - n)/t$.*

Let $\tau, x \in H^*(\Lambda, \Lambda^0)$. If $t \in [\underline{\mu_\tau}(x), \overline{\mu_\tau}(x)]$ is finite (which implies that τ is nonnilpotent), then there is a sequence γ_i of closed geodsics on M whose mean frequencies $\overline{\alpha}_i$ converge to $(\deg \tau + 1 - n)/t$.

Corollary 4.4. *If $[\underline{\mu_U}(X), \overline{\mu_U}(X)]$ or $[\underline{\mu_\tau}(x), \overline{\mu_\tau}(x)]$ is not a point, then M has infinitely many closed geodesics.*

Proof of Proposition 4.3. We have

$$\deg(X \bullet U^{\bullet m}) = \deg X + m(\deg U - n).$$

Thus

$$\begin{aligned} \limsup_{m\to\infty} \mathsf{cr}(X \bullet U^{\bullet m})/\deg(X \bullet U^{\bullet m}) &= \overline{\mu_U}(X)/(\deg U - n) \\ \liminf_{m\to\infty} \mathsf{cr}(X \bullet U^{\bullet m})/\deg(X \bullet U^{\bullet m}) &= \underline{\mu_U}(X)/(\deg U - n). \end{aligned}$$

The basic inequalities 5 imply that

$$\begin{aligned}\frac{\mathsf{cr}(X\bullet U^{\bullet(m+1)})}{\deg(X\bullet U^{\bullet(m+1)})}-\frac{\mathsf{cr}(X\bullet U^{\bullet m})}{\deg(X\bullet U^{\bullet m})}&\le\frac{\mathsf{cr}(U)}{\deg(X\bullet U^{\bullet m})}\\&\le\frac{1}{m}\frac{\mathsf{cr}(U)}{(\deg U-n)}.\end{aligned}$$

The sequence

$$a_m=\frac{\mathsf{cr}(X\bullet U^{\bullet m})}{\deg(X\bullet U^{\bullet m})}$$

has the property that $a_{m+1}-a_m\le\varepsilon_m$, with $\varepsilon_m\to 0$. It follows that the set of limit points of this sequence is an interval; thus every point in

$$\left[\frac{\underline{\mu_U}(X)}{\deg U-n},\frac{\overline{\mu_U}(X)}{\deg U-n}\right]$$

is a limit point of the sequence. For each m there is a closed geodesic γ_m of length $\mathsf{cr}(X\bullet U^{\bullet m})$, whose index λ_m satisfies

$$\deg(X\bullet U^{\bullet m})-2n+1\le\lambda_m\le\deg(X\bullet U^{\bullet m}).$$

Thus (inverting, and using the fact that $l(\gamma_m)$ and $\deg(X\bullet U^{\bullet m})\to\infty$), the sequence $\{\lambda_m/\ell(\gamma_m)\}$ has the same limit points as $\{\deg(X\bullet U^{\bullet m})/cr(X\bullet U^{\bullet m})\}$, i.e. the interval

$$\left[\frac{\deg U-n}{\overline{\mu_U}(X)},\frac{\deg U-n}{\underline{\mu_U}(X)}\right].$$

Using the *Resonant Iterates Lemma* 10.1 we find

$$\left|\frac{\mathrm{ind}\gamma^m}{l(\gamma_m)}-\overline{\alpha}_m\right|\le\frac{n}{l(\gamma_m)}$$

and thus the sequence $\{\overline{\alpha}_m\}$ has the same interval of limit points, and the lemma is proved for homology.

The argument for cohomology is similar; in particular $\deg(x\circledast\tau^{\circledast m})=\deg x+m(\deg\tau+n-1)$ and, using the *Powers Lemma* 4.2:

$$\begin{aligned}\frac{\mathsf{cr}(x\circledast\tau^{\circledast(m+1)})}{\deg(x\circledast\tau^{\circledast(m+1)})}-\frac{\mathsf{cr}(x\circledast\tau^{\circledast m})}{\deg(x\circledast\tau^{\circledast m})}&\ge-\frac{\mathsf{cr}(x\circledast\tau^{\circledast m})}{m^2(\deg\tau+n-1)}\\&\ge-\frac{\overline{\xi_\tau}(x)}{m(\deg\tau+n-1)}\end{aligned}$$

so that if $\overline{\xi_\tau}(x)<\infty$, the sequence $a_m=\mathsf{cr}(x\circledast\tau^{\circledast m})/\deg(x\circledast\tau^{\circledast m})$ has the property that $a_m-a_{m+1}\le\varepsilon_m$, with $\varepsilon_m\to 0$. q.e.d.

5. Proof of the Interval Theorem and Resonance Theorem

Our first task in this section is to prove:

Theorem 5.1. (Interval Theorem) *Fix a metric on $S^n, n > 2$, and let G be a field or $\mathbb{Z}$. Let*

$$\mu^- =: \lim_{m\to\infty} \frac{\mathsf{cr}_{\mathbb{Z}}(\omega^{\circledast m})}{m} \;\; ; \;\; \mu^+ = \lim_{m\to\infty} \frac{\mathsf{cr}_{\mathbb{Z}}(\Theta^{\bullet m})}{m}$$

where Θ and ω are the fundamental nonnilpotent elements in loop homology and cohomology, cf. Lemma 5.4 below. Any sequence of homology or cohomology classes $\{X_k\}$ with $X_k \in H_*\left(\Lambda, \Lambda^0; G\right)$ or $X_k \in H^*\left(\Lambda, \Lambda^0; G\right)$, with $\lim_{k\to\infty} \deg X_k = \infty$, has

$$\left[\liminf_{k\to\infty} \frac{\mathsf{cr}_G(X_k)}{\deg X_k}\, , \, \limsup_{k\to\infty} \frac{\mathsf{cr}_G(X_k)}{\deg X_k}\right] \subseteq \left[\frac{\mu^-}{2(n-1)}\, , \, \frac{\mu^+}{2(n-1)}\right].$$

Then we will use the interval theorem to prove the following theorem already stated as Theorem 1.1 in the Introduction:

Theorem 5.2. (Resonance Theorem) *A Riemannian or Finsler metric g on $S^n, n > 2$, and a field G determine a* global mean frequency *$\overline{\alpha} =: \overline{\alpha}_{g,G} > 0$ with the property that*

$$\deg(X) - \overline{\alpha}\, \mathsf{cr}(X)$$

is bounded as X ranges over all nontrivial homology and cohomology classes on Λ with coefficients in G.

Fix a metric g on $M = S^n$, $n > 2$. The interval theorem will follow from basic properties of loop products, the results of Sections 1 and 2, and two lemmas. The lemmas, Lemma 5.3 and Lemma 5.4, are proved in [**7**], and will be discussed in Section 6. For the Resonance Theorem 5.2 we need a third lemma; Lemma 5.5 will be proved in Section 6.

Lemma 5.3. *For each j, each of $H^j(\Lambda, \Lambda^0; \mathbb{Z})$, $H^j(\Lambda; \mathbb{Z})$, and $H_j(\Lambda; \mathbb{Z})$ is one of the following: 0, $\mathbb{Z}$, or $\mathbb{Z}_2 =: \mathbb{Z}/2\mathbb{Z}$. If G is a field,*

$$\operatorname{rank} H^j(\Lambda, \Lambda^0; G) \leq \operatorname{rank} H^j(\Lambda; G) = \operatorname{rank} H_j(\Lambda; G) \leq 1.$$

This lemma allows us to use the *Duality Lemma* 3.1, and *Lemma* 3.3 *on the effect of coefficients.*

Lemma 5.4. *Fix G, a field or $\mathbb{Z}$. There is a homology class $\Theta \in H_{3n-2}(\Lambda; G)$ and a cohomology class $\omega \in H^{n-1}(\Lambda, \Lambda^0; G)$, with the property that $\Theta^{\bullet m}$ is a generator of $H_{(2m+1)(n-1)+1}(\Lambda; G) = G$, and $\omega^{\circledast m}$ is a generator of $H^{(2m-1)(n-1)}(\Lambda, \Lambda^0; G) = G$. Multiplication with Θ and multiplication with ω are dual:*

$$x \circledast \omega^{\circledast m}(Y \bullet \Theta^{\bullet m}) = x(Y) \tag{10}$$

for every $x \in H^*(\Lambda; G)$, $Y \in H_*(\Lambda, \Lambda^0; G)$ and $m \geq 0$. There is a finite set Q of classes of degree $\leq 3n-2$ in $H_*(\Lambda; G)$, and a finite set q of classes in $H^*(\Lambda, \Lambda^0; G)$ so that every class $X \in H_j(\Lambda; G)$ with degree $j > n$ can be written uniquely in the form $X = aY \bullet \Theta^{\bullet m}$ for some $a \in G$ and $Y \in Q$, and every class $x \in H^j(\Lambda, \Lambda^0; G)$ can be written uniquely in the form $x = ay \circledast \omega^{\circledast m}$ for some $a \in G$ and $y \in q$. If G is a field, these can be chosen so that for every $Y \in Q$ (resp. $y \in q$) there is a unique $y \in q$ (resp. $Y \in Q$) with $y(Y) = 1$.

Note that $\bullet\Theta$ and $\circledast\omega$ both increase degree by $2(n-1)$. Let W be the generator of $H_{n-1}(\Lambda; G)$, so that $\omega(W) = 1$ then $W \bullet \Theta^{\bullet m}$ is a generator of $H_{(2m+1)(n-1)}(\Lambda; G) = G$, and

$$\omega^{m+1}(W \bullet \Theta^{\bullet m}) = 1\,. \tag{11}$$

Similarly, if θ is the generator of $H^{3n-2}(\Lambda, \Lambda^0)$, so that $\theta(\Theta) = 1$ then

$$\theta \circledast \omega^{m-1}(\Theta^{\bullet m}) = 1\,. \tag{12}$$

Fix an integral domain G. Let

$$\mu_G^+ = \lim_{m\to\infty} \frac{\mathsf{cr}_G(\Theta^{\bullet m})}{m}\ ;\ \mu_G^- = \lim_{m\to\infty} \frac{\mathsf{cr}_G(\omega^{\circledast m})}{m}.$$

(These limits exist by the *Power Lemma* 4.2.) We will first establish that

$$\mu^- =: \mu_{\mathbb{Z}}^- \leq \mu_G^- \leq \mu_G^+ \leq \mu_{\mathbb{Z}}^+ =: \mu^+\,. \tag{13}$$

The first and third inequalities follow from the *Lemma* 3.3 *on effect of Coefficients*, and Lemma 5.4. From now on in this section

$$\mathsf{cr} =: \mathsf{cr}_G.$$

We have

$$\begin{aligned}
\lim_{m\to\infty} \frac{\mathsf{cr}(\Theta^{\bullet m})}{m} &=: \mu_\Theta(\Theta) && \text{(by definition, Equation 8)}\\
&\geq \overline{\mu_\Theta}(W) && \text{(Equation 6)}\\
&= \limsup_{m\to\infty} \frac{\mathsf{cr}(W \bullet \Theta^{\bullet m})}{m} && \text{(by definition)}\\
&\geq \limsup_{m\to\infty} \frac{\mathsf{cr}(\omega^{\circledast m+1})}{m} && \text{(Equation 3, Equation 11)}\\
&= \limsup_{m\to\infty} \frac{\mathsf{cr}(\omega^{\circledast m})}{m} && \text{(Equation 9)}\\
&= \lim_{m\to\infty} \frac{\mathsf{cr}(\omega^{\circledast m})}{m} && \text{(Power Lemma 4.2)}
\end{aligned}$$

which proves Equation 13. Let Q, q be as in Lemma 5.4. For every $Y \in Q$ and $y \in q$, by the basic inequalities 5,

$$\mathsf{cr}(\omega^{\circledast m}) \leq \mathsf{cr}(y \circledast \omega^{\circledast m}) \tag{14}$$

$$\mathsf{cr}(Y \bullet \Theta^{\bullet m}) \leq \mathsf{cr}(Y) + \mathsf{cr}(\Theta^{\bullet m}). \tag{15}$$

We will show that the Interval Theorem 5.1 follows, considering seperately the cases when G is a field and $G = \mathbb{Z}$.

If G is a field, given $X \in H_j(\Lambda; G)$ nontrivial with $j > n$, write $X = Y \bullet \Theta^{\bullet m}$ for $Y \in Q$; suppose $y(Y) = 1$. By the Duality Lemma 3.1, and Equation 10,

$$\mathsf{cr}_G(y \circledast \omega^{\circledast m}) = \mathsf{cr}(X) = \mathsf{cr}_G(Y \bullet \Theta^{\bullet m})$$

and thus with Equation 14 and Equation 15, we have

$$\mathsf{cr}(\omega^{\circledast m}) \leq \mathsf{cr}(X) \leq \mathsf{cr}(Y) + \mathsf{cr}(\Theta^{\bullet m})\,. \tag{16}$$

Since

$$\lim_{m \to \infty} \frac{\deg(y \circledast \omega^{\circledast m})}{m} = \lim_{m \to \infty} \frac{\deg(Y \bullet \Theta^{\bullet m})}{m} = 2(n-1)$$

for all y, Y, the Interval Theorem 5.1 is a consequence.

If $G = \mathbb{Z}$, the Interval Theorem for homology/cohomology in dimensions in which the homology and cohomology are $\mathbb{Z}$ follows from Lemma 5.3, Lemma 5.4, Equation 14, and Equation 15 by Lemma 3.3 on effect of Coefficients. Note the theorem applies since $\Theta^{\bullet m}$ and $\omega^{\circledast m}$ are generators in their dimensions. If $X \in H_k(\Lambda; \mathbb{Z}) = \mathbb{Z}_2$, note that the reduction $\overline{X} \in H_k(\Lambda; \mathbb{Z}_2)$ of X mod 2 satisfies $\mathsf{cr}_{\mathbb{Z}_2}(\overline{X}) \leq \mathsf{cr}_{\mathbb{Z}} X$. We leave the remaining details of the case $G = \mathbb{Z}$ to the reader.

For the proof of the resonance theorem we will need also the following: Let

$$\Delta : H_j(\Lambda; G) \to H_{j+1}(\Lambda; G)$$

be the map induced by the S^1-action and the Künneth map ([**4**, p.16], [**7**, p.58]). If a homology class X is supported on a closed set $A \subset \Lambda$, then ΔX is supported on $S^1 \cdot A$; thus

$$\mathrm{cr}(\Delta X) \leq \mathrm{cr}(X). \tag{17}$$

Lemma 5.5. *If n is odd,*

$$\Delta(W \bullet \Theta^{\bullet m}) = (2m+1)\, \Theta^m\,.$$

If n is even, then there is an integer $k = k(n)$ so that

$$\Delta(W \bullet \Theta^{\bullet m}) = (mk+1)\, \Theta^m\,.$$

Proof of the Resonance Theorem 1.1. Now let G be a field. Let

$$\overline{\mu} =: \mu_G = \lim_{m \to \infty} \frac{\mathsf{cr}_G(\omega^{\circledast m})}{m}.$$

By Equation 5, $\mathsf{cr}(\omega^{\circledast m})$ is increasing. By Equation 5, Equation 12 and the duality Lemma 3.1 it follows that $\mathsf{cr}(\Theta^{\bullet m})$ is also increasing. Suppose G has characteristic p. The coefficient $2m-1$ or $(m-1)k+1$ of Θ^{m-1} in Lemma 5.5 cannot be congruent to 0 mod p for two consecutive values of m. Using Equation 17, for each m at least one of the following is true:

$$\mathsf{cr}\left(\Theta^{m-1}\right) \leq \mathsf{cr}(\omega^{\circledast m})\ ;\ \mathsf{cr}\left(\Theta^{m}\right) \leq \mathsf{cr}\left(\omega^{\circledast(m+1)}\right),$$

and thus for each m

$$\mathsf{cr}\left(\Theta^{m-1}\right) \leq \mathsf{cr}\left(\omega^{\circledast(m+1)}\right). \tag{18}$$

Let $X \in H_j(\Lambda; G)$ be nontrivial with $j > n$; say $X = Y \bullet \Theta^{\bullet m}$, with $\deg Y \leq 3n-2$ and $y(Y) \neq 0$. By the Duality Lemma 3.1,

$$\mathsf{cr}(X) = \mathsf{cr}\left(y \circledast \omega^{\circledast m}\right).$$

Using Equation 18 and Equation 16 together with the Powers Lemma 4.2 we have

$$(m-2)\,\overline{\mu} \leq \mathsf{cr}(X) \leq (m+2)\,\overline{\mu} + \mathsf{cr}(Y)\,. \tag{19}$$

Moreover

$$\deg X = 2m(n-1) + \deg Y$$

so

$$2m(n-1) \leq \deg X \leq 2m(n-1) + 3n - 2\,. \tag{20}$$

Putting together Equation 19 and Equation 20 we have

$$\begin{aligned}\frac{\overline{\mu}}{2(n-1)}\deg X \;-\; &\frac{\overline{\mu}\,(7n-6)}{2(n-1)} \leq \mathsf{cr}(X)\\ &\leq \frac{\overline{\mu}}{2(n-1)}\deg X + (\mathsf{cr}(Y) + 2\,\overline{\mu}) + \frac{(3n-2)[\mathsf{cr}(Y) + 2\,\overline{\mu}]}{\deg X + 2 - 3n}.\end{aligned}$$

This is of the form

$$\frac{1}{\overline{\alpha}}\deg X - A \leq \mathsf{cr}(X) \leq \frac{1}{\overline{\alpha}}\deg X + B + \frac{C}{\deg X - D}$$

with positive constants A, B, C, D, and

$$\overline{\alpha} = \frac{2(n-1)}{\overline{\mu}}.$$

The resonance theorem follows. q.e.d.

The following is a consequence of Theorem 5.1 (interval theorem):

Corollary 5.6. *Fix a metric g on S^n with $n > 2$. For any field G, the global mean frequency $\overline{\alpha}_{g,G}$ satisfies:*

$$\frac{1}{\overline{\alpha}_{g,G}} \in \left[\frac{\mu^-}{2(n-1)}, \frac{\mu^+}{2(n-1)}\right].$$

It is not known whether there is a metric on S^n with $\mu^- \neq \mu^+$.

6. Proof of Lemma 5.5; computation of the mod 2 Chas-Sullivan product

In this section we present the proof of Lemma 5.5 and sketch the proofs of Lemma 5.3 and Lemma 5.4.

We first sum up some facts about the loop products for spheres. The Chas-Sullivan product over the integers was computed by Cohen, Jones, and Yan in [**3**]. We include an argument below that also works over the field $\mathbb{Z}_2$.

The identity element of the homology ring is a generator E of $H_n(S^n) \subset H_n(\Lambda S^n)$, represented by the space Λ^0 of trivial loops.

For $n > 1$ odd,

$$(H_*(\Lambda S^n;\mathbb{Z}), \bullet) = \wedge(A) \otimes \mathbb{Z}[U] \tag{21}$$

where $A \in H_0(\Lambda S^n)$, and $U \in H_{2n-1}(\Lambda S^n)$.

The case $n > 2$ even with $\mathbb{Z}_2$ coefficients is formally the same:

$$(H_*(\Lambda S^n;\mathbb{Z}_2), \bullet) = \wedge(A) \otimes \mathbb{Z}_2[U] \tag{22}$$

where $A \in H_0(\Lambda S^n)$ and $U \in H_{2n-1}(\Lambda S^n)$.

For the convenience of the reader, here is the $\mathcal{E}^1$ page of the spectral sequence converging to the homology of ΛS^n for n odd, with integer coefficients, or (respectively) for n even with, mod 2 coefficients, that come from the filtration determined by the function F when $M = S^n$ carries the round metric where the prime closed geodesics have length L. An element in a nonempty box is a generator of homology. The number at the left side of each column is the degreee d.

The column of a class is determined by the filtration level: For example the term $A \bullet U^{\bullet 6}$ in the right-most column represents a generator of $H_{6n-6}(\Lambda^{\leq 3L}, \Lambda^{\leq 2L})$ (with $\mathbb{Z}$ or, resp., $\mathbb{Z}_2$ coefficients).

The column containing A and E is $H_*(\Lambda^0)$. The homology is $\mathbb{Z}$ (resp. $\mathbb{Z}_2$) in every spot where there is a generator, and is 0 otherwise. For the round metric (any n, any coefficents), F is a perfect Morse function, which means the spectral sequence degenerates at the $\mathcal{E}^1$ page. The rank of $H_j(\Lambda S^n)$ is thus the number of generators in the j^{th} row, and the column in which a generator appears tells the critical level of that homology class in the round metric.

$H_*(\Lambda S^n)$; n ODD OR $\mathbb{Z}_2$ COEFFICIENTS:

$\uparrow d$				
$6n-5$				$U^{\bullet 5}$
$6n-6$				$A \bullet U^{\bullet 6}$
$5n-4$			$U^{\bullet 4}$	
$5n-5$				$A \bullet U^{\bullet 5}$
$4n-3$			$U^{\bullet 3}$	
$4n-4$			$A \bullet U^{\bullet 4}$	
$3n-2$		$U^{\bullet 2}$		
$3n-3$			$A \bullet U^{\bullet 3}$	
$2n-1$		U		
$2n-2$		$A \bullet U^{\bullet 2}$		
n	E			
$n-1$		$A \bullet U$		
0	A			
	0	L	$2L$	$3L \ \to \ell$

The relationship between this diagram and the notation of Section 5 is (when n is odd, or mod 2)(see [**7**]):

$$\Theta = U^{\bullet 2} \ ; \ W = A \bullet U.$$

The first equation may be taken for our purposes as a definition. The second equation follows from the fact that both homology classes are generators in their degree, and thus unique up to a sign. The sign can be computed geometrically using the definitions; in any case the proof of the theorem is not sensitive to a change in the sign of W.

For $n > 2$ even with integer coefficients,

$$H_*(\Lambda S^n; \bullet) = \wedge(W) \otimes \mathbb{Z}[A, \Theta]/A^2, AW, 2A\Theta$$

where $W \in H_{n-1}(\Lambda S^n)$, $\Theta \in H_{3n-2}(\Lambda S^n)$ as in Section 5 and $A \in H_0(\Lambda S^n)$. Here is the corresponding diagram of the spectral sequence converging to the homology of the free loop space of an even sphere, with integer coefficients. The classes marked $(*)$ are 2-torsion. The homology in dimensions with generator A, E or $W\Theta^m$, or Θ^m, ω^m or $d\omega^m$ is $\mathbb{Z}$.

$H_*(\Lambda S^n)$; n EVEN AND $\mathbb{Z}$ COEFFICIENTS:

$\uparrow d$				
$6n-5$				0
$6n-6$				$A \bullet \Theta^{\bullet 3}(*)$
$5n-4$			$\Theta^{\bullet 2}$	
$5n-5$				$W \bullet \Theta^{\bullet 2}$
$4n-3$			0	
$4n-4$			$A \bullet \Theta^{\bullet 2}(*)$	
$3n-2$		Θ		
$3n-3$			$W \bullet \Theta$	
$2n-1$		0		
$2n-2$		$A \bullet \Theta(*)$		
n	E			
$n-1$		W		
0	A			
	0	L	$2L$	$3L \ \rightarrow \ell$

Remark 6.1. The truth of Lemma 5.3 and Lemma 5.4 is evident from the above, and the (similar) $\mathcal{E}_1$ page for the cohomology spectral sequence; see [**7**].

Lemma 5.5 will clearly follow from the following

Lemma 6.2. *If n is odd, or with* mod 2 *coefficients,*

$$\Delta(A \bullet U^{\bullet m}) = mU^{\bullet m-1} \tag{23}$$

If n is even, there is an integer $k = k(n)$ so that

$$\Delta(W \bullet \Theta^{\bullet m}) = (mk+1)\Theta^{\bullet m}. \tag{24}$$

Proof. When $m = 1$ and $m = 0$, Equation 23 and Equation 24 are (resp.) the statements

$$\Delta(A \bullet U) = E \ ; \ \Delta(W) = E.$$

These are well-known facts about the classes $A \bullet U$ and W, which are the images of the generator of $H_{n-1}(\Omega S^n)$. Also note

$$\Delta A = 0$$

for dimension reasons, and

$$\Delta(U^{\bullet m}) = 0 \ ; \ \Delta(\Theta^{\bullet m}) = 0$$

for all $m \geq 1$: because the homology is trivial in the degree of $\Delta(U^{\bullet m})$ (or $\Delta(\Theta^{\bullet m})$ if n is even) if $n > 3$, and, in the case $n = 3$ because in the standard metric $\Delta(U^{\bullet m})$ has a representative in level $\leq \frac{m}{2}$ if m is even , and level $\leq \frac{m+1}{2}$ if m is odd. But the homology gropups in the degree of $\Delta(U^{\bullet m})$ at that level are trivial. We will prove Equation 23 and Equation 24 using induction on m and Chas and Sullivan's equations [**4**, Cor.5.3, p.19] or [**7**, 17.1.1, p.58].

$$\{X, Y \bullet Z\} = \{X, Y\} \bullet Z + (-1)^{|Y|(|X|+1)} Y \bullet \{X, Z\}. \tag{25}$$

$$\Delta(X \bullet Y) = \Delta X \bullet Y + (-1)^{|X|} X \bullet \Delta Y + (-1)^{|X|}\{X, Y\} \tag{26}$$

where $|X| = \deg X - n$.

The first step is to compute some brackets. Using Equation 26 we have

$$\{A, U\} = (-1)^n \Delta\,(A \bullet U) = (-1)^n E = -E$$

if n is odd or $G = \mathbb{Z}_2$. When n is even we conclude for dimensional reasons that

$$\{W, \Theta\} = -k\Theta$$

for some $k = k(n)$.

Using induction and Equation 25 we have

$$\{A, U^{\bullet m}\} = -mU^{\bullet (m-1)}$$
$$\{W, \Theta^{\bullet m}\} = -mk\Theta^{\bullet m}$$

for all $m \geq 1$.

The lemma follows using induction and Equation 26. q.e.d.

Chas-Sullivan product on $H_*(\Lambda(S^n); \mathbb{Z}_2)$ $n > 2$ even

The inclusion

$$i : \Omega M \to \Lambda M$$

induces maps

$$i_* : H_*(\Omega M) \to H_*(\Lambda M)$$
$$i^! : H_*(\Lambda M) \to H_{*-n}(\Omega M).$$

Let $\times$ denote the Pontrjagin product on $H_*(\Omega M)$. We compute the homology loop product for even spheres and $\mathbb{Z}_2$ coefficients, Equation 22, using the Pontrjagin product and the Gysin formulas [**4**, Prop.3.4]; [**7**, 9.3].

$$i^!(X \bullet Y) = i^!(X) \times i^!(Y) \tag{27}$$

$$(i_*(Z)) \bullet X = i_*(Z \times i^!(X)) \tag{28}$$

$$i_* i^! X = A \bullet X \tag{29}$$

where A is a generator of $H_0(\Lambda M)$. The Pontrjagin ring $(\Omega S^n, \mathbb{Z}_2; \times)$ is the polynomial ring $\mathbb{Z}_2[W]$ on a generator W in dimension $n-1$. It is well known (probably since Morse, though he was more interested in the quotient $\Lambda/\mathbb{O}(2)$ than in Λ; see [**7**] for a recent treatment using Morse theory) that $H_*(\Lambda(S^n); \mathbb{Z}_2)$ has the same rank in each dimension as $\mathbb{Z}_2[W] \otimes H_*(S^n)$. It follows that the spectral sequence for the fibration

$$\Omega M \to \Lambda M \to M$$

collapses, and thus that i_* is injective, and $i^!$ is surjective. In particular there is an element $U \in H_{2n-1}(\Lambda M)$ with $i^!(U) = W$. It follows from Equation 27 that $i^! U^{\bullet m} = W^{\times m}$ and thus that $U^{\bullet m} \neq 0$ for all m. By Equation 29, $i_*(W) = A \bullet U$. It follows from Equation 28, and the fact that i_* is injective that

$$i_*(A \bullet U) \bullet U^{\bullet m} = i_*(W \times i^! U^m) = i_*(W^{\times m}) \neq 0.$$

The Chas-Sullivan product is always trivial on the image of i_*, so $A \bullet A = 0$. Because the rank of $H_*(\Lambda(S^n); \mathbb{Z}_2)$ is at most one in each dimension, Equation 22 follows.

7. Continuity Lemma and Density Theorem

If M is a compact manifold, the space $\mathcal{G}_1$ is defined in the Open Mapping theorem 1.5 in the introduction. The length, average index, index, and nullity of closed geodesics have the following continuity properties:

(i) The map $\mathcal{G}_1 \to \mathbb{R}^2$ given by $(g, \gamma) \to (l_g(\gamma), \alpha_{\gamma,g})$ is continuous.
(ii) The map $\mathcal{G}_1 \to \mathbb{R}$ given by $(g, \gamma) \to \mathrm{ind}(\gamma)$ is *lower semi- continuous.*
(iii) The map $\mathcal{G}_1 \to \mathbb{R}$ given by $(g, \gamma) \to \mathrm{null}(\gamma)$ is *upper semi- continuous.*

In particular if $(g_q, \gamma_q) \to (g, \gamma)$, then $\liminf(\mathrm{ind}(\gamma_q)) \geq \mathrm{ind}(\gamma)$ and $\limsup(\mathrm{ind} + \mathrm{null}(\gamma_q)) \leq (\mathrm{ind} + \mathrm{null})(\gamma)$.

The *Continuity Lemma* 1.6 is a consequence of the *Resonance Theorem* 1.1 and the *Powers Lemma* 4.2: Fix a field G. For a fixed homology or cohomology class X, $\mathsf{cr}X$ depends continuously on the metric. By the Powers Lemma

$$\frac{cr(\omega^{\circledast m})}{m} \leq \frac{2(n-1)}{\overline{\alpha}_{g,G}} \leq \frac{cr(\Theta^{\bullet m})}{m}.$$

Since the left and right terms have the same limit as $m \to \infty$, the Theorem follows.

Proof of the Density Theorem 1.2. Fix a metric g and let $\varepsilon > 0$ be given. Take a nontrivial homology class $X \in H_d(\Lambda;\mathbb{Q})$ and approximate g by a sequence of bumpy metrics g_i. Let Λ_i^a be the set of closed curves γ with $F_i(\gamma) \leq a$, where $F_i = \sqrt{E_i}$ in the metric g_i. Given $\mu > 0$, for i sufficiently large, the d-dimensional class X will lie in the image of $H_d\left(\Lambda_i^{\mathsf{cr}(X)+\mu}\right)$, and will have nontrivial image in $H_d\left(\Lambda_i^{\mathsf{cr}(X)+\mu}, \Lambda_i^{\mathsf{cr}(X)-\mu}\right)$. Moreover by basic nondegenerate Morse theory, for each sufficiently large i, the critical level $\mathsf{cr}_i X$ of X in the metric g_i will be equal to $F_i(\gamma_i)$ for some (nondegenerate) closed geodesic γ_i with $\operatorname{ind}(\gamma_i) \leq d \leq \operatorname{ind}(\gamma_i) + 1$. Note also that

$$\lim_{i\to\infty} \mathsf{cr}_i(X) = \lim_{i\to\infty} F_i(\gamma_i) = \mathsf{cr}(X).$$

From Ascoli-Arzelà's theorem and the continuity of the map $g \mapsto \mathsf{cr}_g(X)$ we conclude that some subsequence of $\{\gamma_i\}$ will converge to a closed geodesic γ_X of length $\mathsf{cr}(X)$ with the property that

$$\operatorname{ind}(\gamma_X) \leq \operatorname{ind}(\gamma_i) \leq d \leq \operatorname{ind} + \operatorname{null}(\gamma_i) + 1 \leq \operatorname{ind} + \operatorname{null}(\gamma_X) + 1\,.$$

By the *Resonance Theorem* 1.1, if d is sufficiently large, $\frac{d}{\mathsf{cr}(X)}$ is arbitrarily close to $\overline{\alpha}_{g,\mathbb{Q}}$. Using the *Resonant Iterates Lemma* 10.1, if d is sufficiently large, we conclude $\overline{\alpha}_{\gamma_X} \in (\overline{\alpha}_{g,\mathbb{Q}} - \varepsilon, \overline{\alpha}_{g,\mathbb{Q}} + \varepsilon)$. Thus if d is sufficiently large, for each nontrivial homology class X there is a closed geodesic γ_X of length $\mathsf{cr}(X)$ with $\overline{\alpha}_{\gamma_X} \in (\overline{\alpha}_{g,\mathbb{Q}} - \varepsilon, \overline{\alpha}_{g,\mathbb{Q}} + \varepsilon)$.

We can assume that γ_X are isolated geodesics: It follows from (i) that if (for any $\varepsilon > 0$) there were a nonisolated closed geodesic γ with mean frequency $\overline{\alpha}_{\gamma,\mathbb{Q}} \in (\overline{\alpha}_{g,\mathbb{Q}} - \varepsilon, \overline{\alpha}_{g,\mathbb{Q}} + \varepsilon)$, then there would be infinitely many closed geodesics γ_i with mean frequency in $(\overline{\alpha}_{g,\mathbb{Q}} - \varepsilon, \overline{\alpha}_{g,\mathbb{Q}} + \varepsilon)$. Moreover the average index α_i would be bounded, and so the sum of the inverted average indices mentioned in the density theorem would in fact be infinite.

Thus we may (for purposes of proving the Density Theorem), assume that for sufficiently small ε all closed geodesics γ with mean frequency $\overline{\alpha}_\gamma \in (\overline{\alpha}_{g,\mathbb{Q}} - \varepsilon, \overline{\alpha}_{g,\mathbb{Q}} + \varepsilon)$ are isolated.

For each *prime* closed geodesic γ with $\overline{\alpha}_\gamma \in (\overline{\alpha}_{g,\mathbb{Q}} - \varepsilon, \overline{\alpha}_{g,\mathbb{Q}} + \varepsilon)$, the inverted index $\frac{1}{\overline{\alpha}_\gamma}$ is the number of critical levels per degree contributed by the iterates of γ. Let $N(d)$ be the number of distinct critical levels of rational homology classes of degree $\leq d$. It will be enough to show that

$$\lim_{d\to\infty} \frac{N(d)}{d} \geq \begin{cases} \frac{1}{n-1} & ; \quad n \text{ odd} \\ \frac{1}{2(n-1)} & ; \quad n \text{ even} \end{cases} . \tag{30}$$

We use the description of the homology of Λ given in Section 6. If n is even we use Equation 11, the *Duality Lemma* 3.1, and Equation 5 to conclude that the classes $B \bullet \Theta^{\bullet m}$ have distinct critical values. Since $\bullet\Theta$ has degree $2(n-1)$, the theorem is proved in this case.

Now consider the case where n is odd. Let $\sigma : H_k(\Lambda) \to H_{k+1-n}(\Lambda)$ by $\sigma(X) = A \bullet \Delta X$. Using Equation 23 we see that there is a nontrivial infinite string of rational classes

$$\ldots \underset{\sigma}{\longrightarrow} X_j \quad \underset{\sigma}{\longrightarrow} X_{j-1} \underset{\sigma}{\longrightarrow} \ldots \underset{\sigma}{\longrightarrow} X_0 = A\,.$$

Now for all nontrivial X of positive degree, $\mathsf{cr}(\Delta X) \leq \mathsf{cr}(X)$, $\mathsf{cr}(A \bullet X) \leq \mathsf{cr}(X)$, and in fact $\mathsf{cr}(A \bullet X) < \mathsf{cr}(X)$ unless the support of the image of X in the level homology maps *onto* M by the evaluation map. But this would violate the hypothesis that the geodesics in Σ_X are isolated. Therefore Equation 30 follows. The theorem follows for n odd since σ has degree $-(n-1)$. q.e.d.

Proof of Corollary 1.3. Again we follow [**18**]: Under the given curvature assumption the length of a closed geodesic is bounded from below by $\pi(1+\lambda^{-1})$. cf. [**19**, Thm.1], which together with the lower curvature bound shows that the average index satisfies $\alpha_\gamma > n-1$, cf. [**20**, Lem.2]. The statement now follows from the *Density Theorem* 1.2. q.e.d.

Proof of Corollary 1.4. Let $n, g_0, \mathcal{U}$, and $N \in \mathbb{N}$ be given. Pick $D \in \mathbb{N}$ with

$$D \geq N(2n-2).$$

Because the curvature is positive in a neighborhood of g_0, there is a neighborhood $\mathcal{W}_0$ of g_0 and $L \in \mathbb{R}$ so that for every $g \in \mathcal{W}$, all closed geodesics on (S^n, g) of length $> L$ have average index satisfying

$$\alpha_\gamma \geq D\,.$$

There is a neighborhood $\mathcal{W}$ of g_0 in the space of metrics so that for every $g \in \mathcal{W}$, all closed geodesics on (S^n, g) lie in $\mathcal{U}$ *or* have length $> L$. q.e.d.

8. Perturbation result

We can increase the mean frequency $\overline{\alpha} = \alpha/\ell$ of a closed geodesic γ by increasing the average index $\alpha = \alpha_\gamma$ or by decreasing the length

$\ell = \ell(\gamma)$. The Open Mapping Theorem 1.5 is an immediate consequence of the following:

Theorem 8.1. (Perturbation Theorem) *Given a geodesic γ of positive length on a Riemannian or Finsler manifold (M, g), and an open neighborhood U of a point p on γ, at least one of the following is true:*

(i) *There is a smooth family of metrics g^s, $s \geq 0$, with $g^0 = g$, and with $g^s = g$ outside U, so that $\gamma^s =: \gamma$ remains geodesic in the metric g^s, the average index of γ^s is increasing (resp. decreasing) for $s \geq 0$, and with the length of γ^s constant.*
(ii) *There is a smooth family of metrics $g^{s,t}$, $0 \leq s, t$, with $g^{0,0} = g$, with $g^{s,t} = g$ outside U, so that $\gamma^{s,t} =: \gamma$ remains geodesic in the metric $g^{s,t}$, and so that on an open set of the first quadrant containing the positive $s-$axis, the Poincaré map associated to $\gamma^{s,t}$ has no eigenvalue on the unit circle and length $(\gamma^{s,t})$ =length$(\gamma)-t$ (resp. length$(\gamma^{s,t})$ =length$(\gamma)+t$).*

Note that in the latter case the average index is constant. In either case the mean frequency is increasing (resp. decreasing).

Let $\mathcal{M}$ be the space of real $2(n-1)$ by $2(n-1)$ matrices, and let $\mathfrak{S} = \mathrm{Sp}\,(2(n-1), \mathbb{R}) = \{X \in \mathcal{M} : X^*JX = J\}$, where

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in \mathfrak{S}\,.$$

The Lie algebra is $\mathfrak{sp}(2(n-1)) = \{A \in \mathcal{M} : A^*J + JA = 0\}$. The *positive cone* in the Lie algebra is $\{A \in \mathfrak{sp}(2(n-1)) : JA > 0\}$. The positive cone is invariant under conjugation by an element of $\mathfrak{S}$. A curve $P(s)$ in $\mathfrak{S}$ is a *+-curve* if the left translate of each tangent vector to the identity lies in the +-cone, i.e.

$$JP^{-1}\frac{dP}{ds} > 0$$

for all s .

Lemma 8.2. *Given a metric g_0, and a closed geodesic γ, consider a 1-parameter perturbation g_s of the metric on M that*

1) *keeps γ geodesic of the same length,*
2) *does not alter parallel transport along γ,*
3) *does not decrease the sectional curvature of any plane containing γ' along γ,*
4) *increases the sectional curvature of every plane containing γ' along γ to first order in the perturbation parameter s, in a neighborhood of some point on γ .*

Then to first order the Poincare map $P(s)$ moves in a $+$-direction at $s=0$, that is

$$JP^{-1}\left.\frac{dP}{ds}\right|_{s=0}>0\,.$$

Proof. This follows from the same argument as Bott's proof of [**2**, Prop.3.1, p.204-205] with minor modification. Let γ have length ℓ and speed 1. Pick an orthonormal basis $\mathbf{U}_1, \ldots \mathbf{U}_{n-1}$ for the subspace of $T_{\gamma(0)}M$ orthogonal to $\gamma'(0)$, and use parallel transport to get a basis (with $\gamma'(t)$) $\mathbf{U}_1(t), \ldots \mathbf{U}_{n-1}(t)$ for $T_{\gamma(t)}M$. Let $X_s(t) \in \mathfrak{S}$ be the solution of

$$\frac{\partial X_s(t)}{\partial t}=A_s(t)X_s(t)$$

with $X_s(0)=I$ for all s, where A is the matrix

$$A_s=\begin{pmatrix} 0 & I \\ -R_s & 0 \end{pmatrix}$$

with R_s giving the sectional curvatures $\langle R(\gamma', \bullet)\gamma', \bullet\rangle$ in terms of the parallel translated basis. Suppose that parallel transport along γ to the point $t=\ell$ results in the orthogonal symplectic transformation Q on our basis, that is $\mathbf{U}_i(\ell)=\sum \mathbf{U}_j(0)Q_{ji}$. The Poincaré map $P(s)$ under our hypotheses is given by

$$P(s)=\widehat{Q}X_s(\ell)$$

where

$$\widehat{Q}=\begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix},$$

and $P(s)$ moves in a $+$-direction at $s=0$ if and only if $X_s(\ell)$ does, since

$$J(X_s(\ell))^{-1}\left.\frac{\partial X_s(\ell)}{\partial s}\right|_{s=0}=JP^{-1}\left.\frac{dP}{ds}\right|_{s=0}.$$

By the same argument as in Bott's paper, we have (see [**2**, 3.2]):

$$J\left\{X_s^{-1}\frac{\partial X_s}{\partial s}\right\}\Bigg|_{t=\ell;s=0}=\int_0^\ell X_0^* T(t)X_0(t)dt \tag{31}$$

where

$$T(t)=\begin{pmatrix} \tau(t) & 0 \\ 0 & 0 \end{pmatrix}$$

with

$$\tau(t)=\left.\frac{\partial R_s(t)}{\partial s}\right|_{s=0}\geq 0$$

and thus $\tau(t)>0$ by hypothesis for all t in some open set. A nonzero vector $v=(x,y)\in\mathbb{R}^{2(n-1)}$ defines the initial conditions of a nontrivial solution of the Jacobi equation

$$\frac{\partial x(t)}{\partial t} = y(t)$$
$$\frac{\partial y(t)}{\partial t} = -R_0(t)x(t)$$

and

$$\langle X_0^* T(t) X_0(t) v, v \rangle = \langle \tau(t) x(t), x(t) \rangle \, .$$

Since the zeroes of $x(t)$ are isolated, the quadratic form given by Equation 31 evaluated at v is positive. q.e.d.

Remark 8.3. In this section we state the results for Riemannian and Finsler metrics. The proofs will be carried out for simplicity only in the Riemannian case. Here we point out the changes in the proofs for the Finsler case: For the following results compare [**19**]. Assume that $f : TM \to \mathbb{R}^{\geq 0}$ is a Finsler metric on M. For a nowhere vanishing vector field V defined on an open subset of the manifold we can define a Riemannian metric g^V by

$$g^V(X, Y) = \frac{1}{2} \left. \frac{\partial^2}{\partial s \partial t} \right|_{s=t=0} f^2 \left(V + sX + tY \right) \, .$$

If $x = (x_1, x_2, \ldots, x_n)$ are coordinates on the manifold and $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ are the induced coordinates on the tangent bundle, then the metric coefficients are given by:

$$g_{ij}^V(x, y) = \frac{1}{2} \frac{\partial^2}{\partial y_i \partial y_j} f^2(x, y) \, .$$

A curve $x(t)$ is a geodesic if the Lagrange equations

$$\frac{d}{dt} \frac{\partial f^2}{\partial y_j}(x(t), \dot{x}(t)) - \frac{\partial f^2}{\partial x_j}(x(t), \dot{x}(t)) = 0 \qquad (32)$$

hold for $j = 1, \ldots, n$. For a geodesic γ we take a nowhere vanishing vector field V defined in a neighborhood of the geodesic extending the velocity vector field γ' along γ. Then γ is also a geodesic of the Riemannian metric g^V and the Jacobi fields as well as the parallel transport along γ with respect to the Finsler metric and with respect to this Riemannian metric coincide. Therefore also the linearized Poincaré mappings of γ with respect to the Finsler metric and the Riemannian metric coincide. Let σ be a plane in the tangent space $T_{\gamma(t)}M$ containing $\gamma'(t)$. Then the flag curvature $K(\gamma'; \sigma)$ of the flag $(\gamma'; \sigma)$ with respect to the Finsler metric f coincides with the sectional curvature $K[g^V](\sigma)$ of a two-plane σ with respect to the Riemannian metric g^V. If we extend the following proofs to the Finsler case we use the Riemannian metric g^V and its sectional curvature.

Lemma 8.4. *Given a metric $g = g^{(0)}$, a closed geodesic γ, and a neighborhood U of the point $p = \gamma(0)$ there is a one-parameter smooth family $\overline{g} = g^{(s)}, s \in [0, \beta)$ of perturbations of the metric $g = g^{(0)}$ supported in U such that the following properties hold for all $s \in [0, \beta)$:*

1) *γ is a closed geodesic of $\overline{g} = g^{(s)}$ of the same length.*
2) *Parallel transport along γ is unchanged.*
3) *Sectional curvature of any plane containing γ' does not decrease, i.e the sectional curvature $\overline{K}(\sigma) = K[g^{(s)}](\sigma)$ with respect to the Riemannian metric $\overline{g} = g^{(s)}, s > 0$ of any plane $\sigma \subset T_{\gamma(t)}M$ containing $\gamma'(t)$ satisfies: $K[g^{(s)}](\sigma) \geq K[g](\sigma)$.*
4) *There is $\eta > 0$ such that the following holds: For all planes σ containing $\gamma'(t)$ the sectional curvature $K[g^{(s)}](\sigma)$ of the plane σ with respect to the Riemannian metric $g^{(s)}$ satisfies:*
$$K[g^{(s)}](\sigma) \geq K[g](\sigma) + s$$
for all $t \in (-\eta, \eta)$ and $s \in [0, \beta)$.

Proof. We follow the lines of the Proof of [**13**, Prop.3.3.7]: Choose an orthonormal frame $E_0, E_1, \ldots, E_{n-1}$ in $T_pM = T_{\gamma(0)}M$ with $E_0 = \gamma'(0)$ and extend this frame to an orthonormal and parallel frame field $E_0(t), E_1(t), E_2(t), \ldots, E_{n-1}(t)$ (parallel with respect to $g = g^{(0)}$) with $E_0(t) = \gamma'(t)$ along $\gamma = \gamma(t)$. Let $(t, x) = (t; x_1, x_2, \ldots, x_{n-1}) = (x_0; x_1, \ldots, x_{n-1})$. Then the mapping
$$(t; x_1, x_2, \ldots, x_{n-1}) \mapsto \exp_{\gamma(t)} \left(\sum_{i=1}^{n-1} x_i E_i(t) \right)$$
defines *Fermi coordinates* along the geodesic γ in a neigbhorhood of $0 \in \mathbb{R}^n$ resp. in a neighborhood $U' \subset M$ of $p = \gamma(0)$. Here exp denotes the exponential mapping of the Riemannian metric g.

We denote the coordinate fields by $\partial_0 = \partial/\partial x_0 = \partial/\partial t; \partial_1 = \partial/\partial x_1, \ldots, \partial_{n-1} = \partial/\partial x_{n-1}$ hence along γ we have: $E_i(t) = \partial_i(t; 0)$. Then the following equations hold for the coefficients $g_{ij} = g(\partial_i, \partial_j)$ of the Riemannian metric g, the Christoffel symbols Γ_{ij}^k and the coefficients $R_{ijkl} = g(R(\partial_i, \partial_j)\partial_k, \partial_l)$:
$$g_{ik}(t; 0) = \delta_{ik}; g_{ik,l}(t; 0) = \frac{\partial g_{ik}(t; 0)}{\partial x_l} = 0$$
$$R_{0ik0}(t; 0) = -\frac{1}{2} g_{00,ik}(t; 0) = -\frac{1}{2} \frac{\partial^2 g_{00}(t; 0)}{\partial x_i \partial x_k}.$$
In particular the Christoffel symbols Γ_{ij}^k vanish along the geodesic $\gamma = \gamma(t) : \Gamma_{ij}^k(t; 0) = 0$ for all t since the coordinate fields $\partial_1, \ldots, \partial_{n-1}$ are parallel along γ. We introduce the function $r := r(x)$ by the equation

$r^2 = \sum_{1\le i\le n-1} x_i^2$ and define a small perturbation $\overline{g} = g^{(s)}$ of the Riemannian metric $g = g^{(0)}$ by:

$$g_{ij}^{(s)}(t;x) = \overline{g}_{ij}(t;x) = g_{ij}(t;x) - s\delta_{i0}\delta_{j0}a(t)a(r^2)r^2\,. \tag{33}$$

For some small $\eta > 0$ the non-negative function a satisfies: $a(t) = 0$ for $|t| \ge 2\eta$ and $a(t) = 1$ for $|t| \le \eta$. Then $\overline{g}_{ij}(t;0) = \delta_{ij}; \overline{g}_{ij,k}(t;0) = 0$ and the Christoffel symbols $\overline{\Gamma}_{ij}^k(t;0)$ along γ satisfy $\overline{\Gamma}_{ij}^k(t;0) = \Gamma_{ij}^k(t;0) = 0$, i.e. the coordinate fields $\partial_1, \ldots, \partial_{n-1}$ are parallel along γ with respect to $g^{(s)}$. Therefore the curve $\gamma = \gamma(t)$ is also a geodesic of the Riemannian metric $g^{(s)}$ and the length and the parallel transport along γ is unchanged. This proves the first two claims.

Then we obtain for the Riemannian curvature tensor of the Riemannian metric $g^{(s)}$ along the curve $\gamma = \gamma(t)$: $\overline{R}(t;0) = \overline{R}_{ijkl}(t;0)$ from Equation 33:

$$\overline{R}_{i00i}(t;0) = \overline{R}_{i00}^i(t;0) = \overline{\Gamma}_{00,i}^i(t;0) - \overline{\Gamma}_{i0,0}^i(t;0) \tag{34}$$

$$= \frac{1}{2}\left(2\overline{g}_{0i.0i}(t;0) - \overline{g}_{00,ii}(t;0) - \overline{g}_{ii,00}(t;0)\right) \tag{35}$$

$$= R_{i00i}(t;0) + sa(t)\,. \tag{36}$$

For the computation we use the standard formulas for the Christoffel symbols:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}\left(g_{li,j} + g_{jl,i} - g_{ij,l}\right)$$

and for the Riemann curvature tensor $R(\partial_i, \partial_j)\partial_k = R_{ijk}^l \partial_l$:

$$R_{ijk}^l = \Gamma_{jk,i}^l - \Gamma_{ik,j}^l + \Gamma_{jk}^r \Gamma_{ir}^l - \Gamma_{ik}^r \Gamma_{jr}^l$$

with $R_{ijkl} = R_{ijk}^m g_{ml}$. Here we use Einstein's sum convention. Then the sectional curvature $K[g^{(s)}]\,(\partial_0, \partial_i)\,(t;0) = \overline{K}\,(\partial_0, \partial_i)$ of a plane generated by $\partial_0 = \gamma'(t)$ and ∂_i is given by: $\overline{K}(\partial_0, \partial_i) = R_{i00i}(t;0)$. This implies the third and the fourth statement since $a(t) \ge 0$ for all t and $a(t) = 1$ for all $t \in (-\eta, \eta)$. q.e.d.

Remark 8.5. In the Finsler case the small perturbation of the Finsler metric $f = f^0$ can be defined by as follows:

$$\left(f^{(s)}\right)^2(t, x_1, \ldots, x_{n-1}, y_0, y_1, \ldots, y_{n-1}) =$$
$$f^2(t, x_1, \ldots, x_{n-1}, y_0, y_1, \ldots, y_{n-1}) - sy_0^2 a(t)a(r^2)r^2.$$

Then γ is also a geodesic of the Finsler metric $f^{(s)}$, as one can check with the Lagrange equation 32. We let $V = \partial_0$, and use the osculating Riemannian metric g_s^V of the Finsler metric $f^{(s)}$:

$$(g_s^V)_{ij} = (g_0^V)_{ij} - sa\delta_{i0}\delta_{j0}a(t)a(r^2)r^2\,. \tag{37}$$

Hence g_s^V is of the form of the metric $\overline{g}$ given in Equation 33 and we can proceed as in the above proof to obtain the result also for the Finsler case.

Lemma 8.6. *Given a metric $g = g^{(0)}$, a closed geodesic γ of length $L = L(\gamma)$ and a neighborhood U of the point $p = \gamma(0)$ there is a one-parameter smooth family $\overline{g} = g^{(s)}, s \in [0, \beta)$ of perturbations of the metric $g = g^{(0)}$ supported in U such that the following properties hold for all $s \in [0, \beta)$:*

1) *γ is up to parametrization a closed geodesic of $\overline{g} = g^{(s)}$ of length $L + s$.*
2) *Parallel transport along γ is unchanged.*

Let $a : \mathbb{R} \to \mathbb{R}^{\geq 0}$ be a smooth function such that $a(t) = 0$ for $|t| \geq 2\eta$, and $a(t) = 1$ for all t with $|t| \leq \eta$ for sufficiently small $\eta > 0$. Then we define a perturbation $\overline{g} = g^{(s)}, s \in [0, \beta]$:

$$g_{ij}^{(s)}(t; x) = \overline{g}_{ij}(t; x) = g_{ij}(t; x) + s\delta_{i0}\delta_{j0}a(t)a(r^2)\,. \tag{38}$$

Proof. Along γ we have:

$$\overline{g}_{ij}(t; 0) = \delta_{ij} + s\delta_{i0}\delta_{j0}a(t)$$

and therefore

$$\overline{\Gamma}_{ij}^{k}(t; 0) = \left\{\overline{g}_{ki,j}(t; 0) + \overline{g}_{jk,i}(t; 0) - \overline{g}_{ij,k}(t; 0)\right\} / \left(2\overline{g}_{kk}(t; 0)\right) = 0$$

for $(i, j, k) \neq (0, 0, 0)$. We conclude that γ is up to parametrization a geodesic for the Riemannian metric $g^{(s)}$ of length

$$L + s \int_{-2\eta}^{2\eta} \sqrt{a(t)}\, dt \geq L + 2\eta s$$

and the coordinate fields $\partial_1, \ldots, \partial_{n-1}$ are parallel along γ with respect to the Riemannian metric $g^{(s)}$. Hence the parallel transport is unchanged and by changing the parameter s we obtain the claim. q.e.d.

Remark 8.7. In the Finsler case we use the following perturbation:

$$\begin{gathered}\left(f^{(s)}\right)^2 (t, x_1, \ldots, x_{n-1}, y_0, y_1, \ldots, y_{n-1}) = \\ f^2 (t, x_1, \ldots, x_{n-1}, y_0, y_1, \ldots, y_{n-1}) + s y_0^2 a(t) a(r^2).\end{gathered}$$

Then γ is up to parametrization also a geodesic of the Finsler metric $f^{(s)}$ (which follows from Equation 32) of length $\geq L + 2\eta s$ and the parallel transport along the reparametrized closed geodesic is unchanged.

Lemma 8.8. *Fix M, $\gamma \in \Lambda M$, $\ell \in \mathbb{R}$, and let g^s, $0 \leq s \leq \delta$, be a smooth path in the space of metrics on M with the property that for all s, $\gamma_s =: \gamma$ is geodesic in the metric g^s with length ℓ. Assume that parallel translation along γ_s is independent of s, and let $P(s)$ be the Poincaré map (determined up to conjugation in the symplectic group)* *of γ_s . Suppose that $P(s)$ is a $+$-curve at $s = 0$. Then for some $\varepsilon > 0$ one of the following is true:*

(i) *the average index of γ_s is increasing for $s \in [0, \varepsilon)$, or*
(ii) *$P(s)$ has no eigenvalue on the unit circle for all $s \in (0, \varepsilon)$.*

Proof. This lemma is a consequence of Bott's results. Let γ be a closed geodesic of length ℓ in the metric g, and, for $\lambda \in \mathbb{R}$ let $X_\lambda(t) \in \mathfrak{S}$ be the solution of the eigenvalue equation

$$\frac{\partial X_\lambda(t)}{\partial t} = A_\lambda(t) X_\lambda(t)$$

with $X_\lambda(0) = I$ for all λ, where A_λ is the matrix

$$A_\lambda = \begin{pmatrix} 0 & I \\ -(R + \lambda I) & 0 \end{pmatrix}$$

with R giving the sectional curvature along γ as above. Bott proved that the average index α_γ of γ is given by

$$\alpha_\gamma = \frac{1}{2\pi} \int \Lambda(e^{i\theta}) d\theta$$

where $\Lambda : \{z : |z| = 1\} \to \mathbb{Z}$ is the intersection number of the $+$-curve

$$\lambda \to X_\lambda(\ell); \ -\infty < \lambda < 0$$

with the cycle

$$B_z : \det(X - z\widehat{Q}^*) = 0\,.$$

Note that the multiplicity of z as an eigenvalue of

$$P = \widehat{Q} X,$$

is the same as the nullity of $(X - z\widehat{Q}^*)$. It seems to the authors that Bott ignored the fact that Q may be nontrivial; however his theorem is general enough to accommodate this case also.

Bott showed that the intersection number of curves with B_z is well defined; in particular curves whose endpoints are not on B_z, but that are homotopic with endpoints fixed will have the same intersection number with B_z. Every intersection of B_z with a $+$-curve is transverse and carries the same orientation; the intersection number of a $+$-curve with B_z at a point X is given by the nullity of $X - z\widehat{Q}^*$.

Given $(\{g_s\}, \gamma)$, $0 \le s \le \delta$ as above with $P^{-1}\frac{dP}{ds}|_{s=0}$ in the $+$-cone, we can assume that $P(s)$ is a $+$-curve for $s \in [0, \delta]$. For each s, the curve

$$\lambda \to X_{\lambda,s}(\ell); \ -\infty < \lambda < 0$$

is easily seen to be homotopic to the union of

$$\lambda \to X_{\lambda,0}(\ell); \ -\infty < \lambda \le 0$$

with the path

$$\tau \to \widehat{Q}^* P(\tau), 0 < \tau < s\,. \tag{39}$$

Let α_s be the average index of γ_s. Since the path given in Equation 39 is a $+$-curve, the intersection number $\Lambda_s(z)$ is nondecreasing in s at each z, and α_s is nondecreasing in s.

Now let $s_0,\, s_1 \in [0,\delta]$ with $s_0 < s_1$. Assume the path

$$\tau \to \widehat{Q}^* P(\tau), s_0 < \tau < s_1$$

intersects B_z for some z on the unit circle (in other words that z is an eigenvalue of $P(\tau)$ for some $\tau \in (s_0, s_1)$). It follows that $\Lambda_{s_1}(z)$ is strictly greater than the sum of $\Lambda_{s_0}(z)$ and the nullity of $X_{0,s_0} - z\widehat{Q}^*$, and thus that the same is true in some neighborhood of z. Thus Λ_{s_1} is strictly greater than Λ_{s_0} on some open set containing z, and $\alpha_{s_1} > \alpha_{s_0}$. Note that the set $H \subset [0,\delta]$ of s for which $P(s)$ has no eigenvalue on the unit circle is open, and that if $s \in H$, then $\alpha_s \in \mathbb{Z}$. It follows that H is a union of a finite number of intervals. If 0 is the endpoint of one of these intervals, we have (i); if not, (ii) holds. q.e.d.

Proof of Theorem 8.1. Let a metric g^0, a geodesic γ of length $\ell > 0$ in the metric g^0 , and a neighborhood U of $\gamma(0)$ be given. By Lemma 8.2 and Lemma 8.4, we can find $\beta > 0$ and a one-parameter smooth family $g^s, s \in [0,\beta)$ of perturbations of g^0, supported in U, and so that

1) γ is a closed geodesic of g^s of length ℓ.
2) Parallel transport along γ is unchanged.
3) The path of Poincaré maps $\{P(s) : s \in [0,\beta)\}$ is a $+$-curve.

By Lemma 8.8, (possibly after picking a new but still positive β) either

(i) the average index is increasing for $s \in [0,\beta)$, or
(ii) $P(s)$ has no eigenvalue on the unit circle, for $s \in (0,\beta)$.

In the first case we are done. In the second case we apply Lemma 8.6 to the geodesic γ in the metric $g^{s,0} =: g^s$, to get a family $g^{s,t}$ of perturbations supported in U, with $\gamma^{s,t} =: \gamma$ geodesic of length $\ell \mp t$ in the $g^{s,t}$ metric (Note the construction of Lemma 8.4 is continuous in s). The average index is an integer and thus locally constant in a neighborhood of any point $(\overline{g},\overline{\gamma}) \in \mathcal{G}_1$ where the Poincaré map of $\overline{\gamma}$ has no eigenvalue on the unit circle. The theorem follows. q.e.d.

9. The Ellipsoid theorem

Using standard comparison arguments one obtains the following estimates for the *mean frequency* $\overline{\alpha}_\gamma$ of a closed geodesic:

Lemma 9.1. (cf. [**18**, Rem.4.3], [**19**, Lem.3], [**20**, Lem.1]) *Let $\gamma : S^1 \to M$ be a closed geodesic on an n-dimensional manifold M with a Finsler metric with flag curvature $K = K(\gamma')$.*

(a) *If $K(\gamma') \geq \delta^2$ for some $\delta > 0$ then $\overline{\alpha}_\gamma \geq \frac{(n-1)\delta}{\pi}$.*

(b) *If* $K(\gamma') \le \Delta^2$ *for some* $\Delta > 0$ *then* $\overline{\alpha}_\gamma \le \frac{(n-1)\Delta}{\pi}$.

For surfaces we show in Lemma 9.3 that one can improve these inequalities in case of non-constant flag curvature. On a surface we can compute the mean frequency of the closed geodesic $\gamma : \mathbb{R} \to M$ using conjugate points;

$$\overline{\alpha}_\gamma = \lim_{m\to\infty} \frac{N_m(\gamma)}{m}$$

where $N_m(\gamma)$ is the number of points $\gamma(t)$ conjugate to $\gamma(0)$ along $\gamma\,|[0,m]$. If t_k is the parameter of the k-th conjugate point $\gamma(t_k)$ to $\gamma(0)$ then

$$\overline{\alpha}_\gamma = \lim_{k\to\infty} \frac{k}{t_k}\,.$$

Lemma 9.2. *Let* $\gamma_j : \mathbb{R} \to M_j^2\,, j = 1,2$ *be two geodesics parametrized by arc length on a surface* $M_j = M_j^2$ *endowed with a Finsler metric* $F_j, j = 1,2$ *with positive flag curvature* $K_j(t) = K_j(\gamma_j'(t)), j = 1,2$. *We denote by* $t_j, j = 1,2$ *the parameter of the first conjugate point* $\gamma(t_j)$ *of* $\gamma_j(0)$ *along* γ_j.

If $K_1(t) \le K_2(t)$ *for all* t *then* $t_1 \ge t_2$ *and equality only holds if* $K_1(t) = K_2(t)$ *for all* $t \in [0,t_1]$.

Proof. Denote by

$$I\left[\gamma_j, T\right](y,z) = \int_0^T \left\{ y'(t)z'(t) - K_j(t)y(t)z(t) \right\}\, dt \tag{40}$$

the *index form* of the geodesic $\gamma_j : [0,T] \to M_j$. We consider the index form on the space of smooth functions $y, z : [0,T] \to \mathbb{R}$ with $y(0) = z(0) = y(T) = z(T) = 0$. Assume that $K_1(t) \le K_2(t)$ for all t.

Then t_j is the largest positive number T_j such that the index form $I\left[\gamma_j, T_j\right]$ is positive definite for $T < T_j$ and degenerate for $T = T_j$. Choose the Jacobi field y_1 of the Finsler metric F_1 with $y_1(0) = y_1(t_1) = 0$ and $y_1'(0) = 1$. Then

$$I\left[\gamma_2, t_1\right](y_1,y_1) = \int_0^{t_1} \left\{ y_1'(t)^2 - K_2(t)\, y_1^2(t) \right\}\, dt$$

$$\le \int_0^{t_1} \left\{ y_1'(t)^2 - K_1(t)\, y_1^2(t) \right\}\, dt = I\left[\gamma_1, t_1\right](y_1,y_1) = 0. \tag{41}$$

Hence $t_1 \ge t_2$ follows. If $t_1 = t_2$ we also have equality in Equation 41 which is only possible if $K_1(t) = K_2(t)$ for all $t \in [0,t_1]$ since $y_1(t) > 0$ for all $t \in (0,t_1)$. q.e.d.

Lemma 9.3. *Let* $\gamma_j : \mathbb{R} \to M_j = M_j^2, j = 1,2$ *be a closed geodesic parametrized by arc length on a surface* $M_j = M_j^2$ *with Finsler metric* $F_j, j = 1,2$ *and positive flag curvature* $K_j(t) = K_j(\gamma_j'(t)), j = 1,2$.

Let $\overline{\alpha}_j; j=1,2$ *be the* mean frequency *of* γ_j *with respect to the Finsler metric* F_j.

If $K_1(t) \le K_2(t)$ *for all* $t \in \mathbb{R}$ *and if* $\{t \in \mathbb{R}; K_1(t) = K_2(t)\} \subset \mathbb{R}$ *is discrete then* $\overline{\alpha}_1 < \overline{\alpha}_2$.

Proof. Let $L_j, j=1,2$ be the length of the closed geodesic γ_j, of the metric F_j, i.e. $\gamma_j(t+L_j) = \gamma(t)$ for all t. For any $s \in [0, L_j]$ denote by $s + t_1^{(j)}(s)$ the parameter of the first conjugate point $\gamma_j\left(s + t_1^{(j)}(s)\right)$ of $\gamma_j(s)$. We conclude from Lemma 9.2: $t_1^{(1)}(s) > t_1^{(2)}(s)$ for all $s \in [0, L_j]$. Since the function $s \in \mathbb{R} \mapsto t_1^{(j)}(s) \in (0,\infty)$ is continuous and periodic there is an $\rho < 1$ such that $t_1^{(1)}(s) \ge \rho^{-1}\, t_1^{(2)}(s)$ for all $s \in \mathbb{R}$. Let $t_k^{(j)}(s)$ be the k-th conjugate point of $\gamma_j(s)$ then we conclude from Lemma 9.2 $t_k^{(1)}(s) \ge \rho^{-1} t_k^{(2)}(s)$ and

$$\overline{\alpha}_1 = \lim_{k\to\infty} \frac{k}{t_k^{(1)}(s)} \le \rho \lim_{k\to\infty} \frac{k}{t_k^{(2)}(s)} \le \rho\overline{\alpha}_2.$$

Since $\rho < 1$ the claim follows. q.e.d.

Lemma 9.4. *If* $\gamma : \mathbb{R} \to M$ *is a closed geodesic on a Finsler manifold* (M,g) *with* $\gamma(t+1) = \gamma(t)$ *and* $(e_1(t), e_2(t), \ldots, e_n(t))$ *is a parallel field of orthonormal basis along* γ *with* $e_1(t) = \gamma'(t)$ *and* $e_i(t+1) = e_i(t)$ *for all* t *and if the sectional curvatures satisfy*

$$0 < \delta_i^2 \le K\left(\gamma'(t), e_i(t)\right) \le \Delta_i^2$$

with equality only at a discrete set of parameters t, *then the mean frequency satisfies*

$$\frac{1}{\pi}\sum_{i=2}^{n} \delta_i < \overline{\alpha}_\gamma < \frac{1}{\pi}\sum_{i=2}^{n} \Delta_i\,.$$

Proof. With respect to the parallel orthonormal basis field the index form splits as a sum of $(n-1)$ forms, which coincide with the index form of a closed geodesic $\gamma_i(t), i=2,\ldots,n$ on a surface with Gauß curvature $K(\gamma'(t), e_i(t)), i=2,\ldots,n$ along γ_i. Therefore the mean frequency of γ equals the sum of the mean frequencies of the closed geodesics $\gamma_i, i = 2,\ldots,n$. Then the statement follows from Lemma 9.1. q.e.d.

On the ellipsoid

$$M = M(a_0,a_1,a_2) := \left\{(x_0,x_1,x_2) \in \mathbb{R}^3\,;\; \frac{x_0^2}{a_0^2} + \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1\right\}$$

in $\mathbb{R}^3$ with three distinct principal axis the intersections with the coordinate planes are ellipses which are up to parametrization simple closed geodesics $\gamma_1, \gamma_2, \gamma_3$. If $a_0 = a_1 < a_2$ then γ_1 is a circle and the ellipsoid is invariant under rotation around the x_2-axis. Hence there is a one-dimensional family of simple closed geodesics. In particular the mean frequencies $\overline{\alpha}_2$ and $\overline{\alpha}_3$ coincide. If $a_0 < a_1 = a_2$ then γ_3 is a circle and the ellipsoid is invariant under rotation around the x_0-axis. In this case there is one-dimensional family of simple closed geodesics, in particular $\overline{\alpha}_1$ and $\overline{\alpha}_2$ coincide.

With the help of the preceding Lemmata we can show:

Proposition 9.5. *Let $M = M(a_0, a_1, a_2) \subset \mathbb{R}^3$ be the 2-dimensional ellipsoid with $0 < a_0 < a_1 < a_2$ endowed with the induced Riemannian metric. Denote by $\gamma_i, i = 1, 2, 3$ the simple closed geodesic resp. the ellipse which parametrizes the intersection of the ellipsoid with the $(x_0, x_1), (x_0, x_2), (x_1, x_2)$-coordinate plane. Then we obtain for the Gauß curvature $K(\gamma_i(t)), i = 1, 2$ along $\gamma_i(t)$:*

$$\frac{a_0^2}{a_1^2 a_2^2} \leq K(\gamma_1(t)) \leq \frac{a_1^2}{a_0^2 a_2^2} \leq K(\gamma_3(t)) \leq \frac{a_2^2}{a_0^2 a_1^2}$$

and

$$\frac{a_0^2}{a_1^2 a_2^2} \leq K(\gamma_2(t)) \leq \frac{a_2^2}{a_0^2 a_1^2} .$$

Therefore we obtain for the mean frequencies $\overline{\alpha}_i = \overline{\alpha}_{\gamma_i}, i = 1, 2, 3$:

$$\frac{a_0}{\pi a_1 a_2} < \overline{\alpha}_1 < \frac{a_1}{\pi a_0 a_2} < \overline{\alpha}_3 < \frac{a_2}{\pi a_0 a_1}$$

and

$$\frac{a_0}{\pi a_1 a_2} < \overline{\alpha}_2 < \frac{a_2}{\pi a_0 a_1} .$$

If $0 < a_0 = a_1 < a_2$ then γ_1 is a circle and the ellipsoid is invariant under rotation around the x_2-axis. Hence there is a one-parameter family of closed geodesics generated by rotation of γ_2 resp. γ_3. In particular for all these geodesics the mean frequency coincides and:

$$\frac{1}{\pi a_2} = \overline{\alpha}_1 < \overline{\alpha}_2 = \overline{\alpha}_3 < \frac{a_2}{\pi a_0^2} . \tag{42}$$

If $0 < a_0 < a_1 = a_2$ then γ_3 is a circle and the ellipsoid is invariant under rotation around the x_0-axis. Hence by rotation of γ_0 we obtain a one-parameter family of closed geodesics and:

$$\frac{a_0}{\pi a_1^2} < \overline{\alpha}_1 = \overline{\alpha}_2 < \overline{\alpha}_3 = \frac{1}{\pi a_0} . \tag{43}$$

Proof. The ellipsoid $M(a_0, a_1, a_2)$ can be parametrized as follows:

$$x_0 = a_0 \cos u \sin v \, ; \, x_1 = a_1 \sin u \sin v \, ; \, x_2 = a_2 \cos v \, .$$

Then the Gauß curvature is given by

$$K(u, v) = \frac{a_0^2 a_1^2 a_2^2}{\left\{a_0^2 a_1^2 \cos^2 v + a_2^2 \left(a_1^2 \cos^2 u + a_0^2 \sin^2 u\right) \sin^2 v\right\}^2} , \tag{44}$$

cf. [**8**, §13]. The Gauß curvature along the ellipse $\gamma_1(t) = (a_0 \cos t, a_1 \sin t, 0)$ is given by:

$$K(\gamma_1(t)) = \frac{1}{a_2^2} \frac{a_0^2 a_1^2}{\left(a_1^2 \cos^2 t + a_0^2 \sin^2 t\right)^2}, \tag{45}$$

hence

$$\frac{a_0^2}{a_1^2 a_2^2} \le K(\gamma_1(t)) \le \frac{a_1^2}{a_0^2 a_2^2}. \tag{46}$$

Then the statements follow from Lemma 9.1 and Lemma 9.3, the estimate for the mean frequencies $\overline{\alpha}_2, \overline{\alpha}_3$ of the ellipses γ_2, γ_3 work analogously. q.e.d.

With the help of this Lemma we can present the

Proof of Theorem 1.7*(a).* For smooth functions $a_i(s), i = 0,1,2, s \in [0,1]$ with $0 < a_0(s) \le a_1(s) \le a_2(s)$, and $a_0(0) < a_1(0) = a_2(0)$ and $a_0(1) = a_2(1) < a_2(1)$ we obtain a smooth family of ellipsoids

$M(a_0(s), a_1(s), a_2(s))$ and we denote by $\overline{\alpha}_i(s), i = 1,2,3, s \in [0,1]$ the mean frequency of the simple closed geodesic γ_i. Then Equation 42 and Equation 43 imply $\overline{\alpha}_1(0) = \overline{\alpha}_2(0) < \overline{\alpha}_3(0)$ and $\overline{\alpha}_1(1) < \overline{\alpha}_2(1) = \overline{\alpha}_3(1)$. This implies that there is an non-empty open subset $I \subset (0,1)$ such that $a_0(s) < a_1(s) < a_2(s)$ for all $s \in I$. q.e.d.

Theorem 1.7(b) is a consequence of the following

Proposition 9.6. *Let $n = 2m$, resp. $n = 2m-1, m \ge 2$, and choose $\mu > 1$. Let $M = E(\mu,\lambda) = M(a_0, a_1, \ldots, a_n) := \Big\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^n\,;\ \sum_{i=0}^n (x_i/a_i)^2 = 1\Big\}$ be the n-dimensional ellipsoid with pairwise distinct principal axis $a_{2i} = \mu^i, a_{2i+1} = \lambda\mu^i, i \le m$, for some $\lambda > 1$ endowed with the induced Riemannian metric. Denote by $\gamma_{i+1} = \gamma_{(2i,2i+1)}, 0 \le i \le m$ the simple closed geodesic resp. the ellipse which parametrizes the intersection of the ellipsoid with the (x_{2i}, x_{2i+1})-coordinate plane. Then for sufficiently small $\lambda > 1$ the mean frequencies $\overline{\alpha}_i$ of the ellipses $\gamma_i, i = 1,2,\ldots,m$ satisfy:*

$$\overline{\alpha}_1 < \overline{\alpha}_2 < \cdots < \overline{\alpha}_m.$$

Proof. We give the proof in the case $n = 2m$: If $e_0, e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^{n+1}$, let $V_{i,j}$ for $j \notin \{2i, 2i+1\}$ be the three dimensional subspace $V_{i,j} = \{x_{2i}e_{2i} + x_{2i+1}e_{2j+1} + x_j e_j\,;\ x_{2i}, x_{2i+1}, x_j \in \mathbb{R}\}$ of $\mathbb{R}^{n+1}$. This space is the fixed point set of the reflection $R : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $R(x_k) = -x_k, k \in \{2i, 2i+1, j\}$ and $R(x_k) = x_k$ otherwise. Therefore the two-dimensional ellipsoid $V_{i,j} \cap E(\mu,\lambda)$ is a totally geodesic submanifold. The Gauß curvature along the ellipse γ_i as a geodesic on this two-dimensional ellipsoid satisfies (cf. Equation 46):

$$\frac{1}{a_j^2}\frac{1}{\lambda^2} \leq K\left(\gamma_i(t)\right) \leq \frac{1}{a_j^2}\,\lambda^2\,;\; j \notin \{2i, 2i+1\} \tag{47}$$

with equality only at a discrete set of parameters.

From Lemma 9.4 we conclude for the mean frequency $\overline{\alpha}_i$ of γ_i :

$$\frac{1+\lambda}{\lambda} \sum_{0\leq k\leq m; k\neq i} \mu^{-k} < \overline{\alpha}_i < (1+\lambda)\,\lambda \sum_{0\leq k\leq m; k\neq i} \mu^{-k}\,. \tag{48}$$

It follows from this estimate that for $1 < \lambda < 1 + (\mu-1)^2\mu^{-m-2}$ the inequality

$$\overline{\alpha}_i < \overline{\alpha}_{i+1}$$

is satisfied for $i = 1, 2, \ldots, m$. q.e.d.

10. Appendix A

In this appendix we give proofs for the following estimates for the difference $\deg(X) - \overline{\alpha}\,\mathsf{cr}(X)$:

Lemma 10.1. (Resonant Iterates) *Let $S = \mathbb{O}(2)$ or $S\mathbb{O}(2)$. If $X \in H_*(\Lambda)$ lies hanging on a closed geodesic γ of length $L = \mathsf{cr}(X)$ in the sense that X is in the image a class in of $H_*(\Lambda^{<L} \cup S\cdot\gamma)$ whose image in $H_*(\Lambda^{<L} \cup S\cdot\gamma\,,\,\Lambda^{<L})$ is nontrivial (cf. Equation 2), then*

$$-(n-1) \leq \deg(X) - \overline{\alpha}(\gamma)\,\mathsf{cr}(X) \leq n\,. \tag{49}$$

Proof. By standard Morse theory estimates

$$\mathrm{ind}(\gamma) \leq \deg(X) \leq \mathrm{ind}(\gamma) + \mathrm{null}(\gamma) + 1 \tag{50}$$

and $\mathrm{null}(\gamma) \leq 2(n-1)$. We obtain the lemma using these and the estimate

$$L\overline{\alpha}(\gamma) - (n-1) \leq \mathrm{ind}(\gamma) \leq L\overline{\alpha}(\gamma) + (n-1) - \mathrm{null}(\gamma) \tag{51}$$

(cf.[**14**, Theorem 10.1.2]) q.e.d.

Lemma 10.2. *If the metric g carries only finitely many closed geodesics, then for $\deg(X)$ sufficiently large, each homology class X satisfies*

$$|\deg(X) - \overline{\alpha}\,\mathsf{cr}(X)| \leq n\,.$$

Proof. By the *Resonant Iterates Lemma* 10.1, for each X there is a closed geodesic γ of mean frequency $\overline{\alpha}(\gamma)$ so that the point $(\mathsf{cr}(X), \deg X)$ lies at a vertical distance at most n from the line

$$d = \overline{\alpha}(\gamma)\ell.$$

On the other hand we conclude from the *Resonance Theorem* 1.1, that the point $(\mathsf{cr}(X), \deg X)$ lies at a bounded distance from the line

$$d = \overline{\alpha}_g \ell.$$

where $\overline{\alpha}_g$ is the global mean frequency. These can both be true for $\deg X$ large only if the slopes are equal: $\overline{\alpha}(\gamma) = \overline{\alpha}_g$. Lemma 10.2 then follows from another application of the *Resonant Iterates Lemma* 10.1.

q.e.d.

11. Appendix B

In this appendix we show the vanishing of the string bracket for spheres and rational Coefficients, see [**7**, p.186-187]. Let M be compact and orientable of dimension n. As in [**4**], one may consider the $T = S^1$-equivariant homology $H_*^T(\Lambda)$ of the free loop space $\Lambda(M)$. Let $\mathrm{ET} \to \mathrm{BT}$ be the classifying space and universal bundle for $T = S^1$; and let $\pi : \Lambda \times \mathrm{ET} \to \Lambda_{\mathrm{T}} = \Lambda \times_{\mathrm{T}} \mathrm{ET}$ be the Borel construction.

There are maps back and forth

$$\begin{aligned} \pi_* &: H_k(\Lambda) \to H_k^T(\Lambda) \\ \pi^! &: H_k^T(\Lambda) \to H_{k+1}(\Lambda) \\ \pi^* &: H_T^k(\Lambda, \Lambda^0) \to H^k(\Lambda, \Lambda^0) \\ \pi_! &: H^k(\Lambda, \Lambda^0) \to H_T^{k-1}(\Lambda, \Lambda^0) \end{aligned}$$

We will use rational coefficients. The Chas-Sullivan *string bracket* product on equivariant homology is defined using the Chas Sullivan product $\bullet$: if $X, Y \in H_*^T(\Lambda)$, then

$$[X, Y] = (-1)^{|X|} \pi_* \left(\pi^!(X) \bullet \pi^!(Y) \right). \tag{52}$$

Here $|X| = i - n$ if $X \in H_i^T(\Lambda)$. The action of $T = S^1$ preserves the energy function, so the (homology) string bracket satisfies the same energy estimates 5 as the Chas Sullivan product.

Similarly the (cohomology) product $\circledast$ gives rise to a product in equivariant cohomology:

$$x \circledcirc y = (-1)^{|x|} \pi_! \left(\pi^*(x) \circledast \pi^*(y) \right). \tag{53}$$

satisfying energy estimates as Equations 5, with $|x| = i + n - 1$ in degree i.

Proposition 11.1. *For an n-dimensional sphere M and for rational coefficients the string product $[.,.]$ on equivariant homology, cf. Equation* 52*, and the product $\circledcirc$ on equivariant cohomology, cf. Equation* 53 *both vanish.*

Proof. First cohomology: By [**10**, 4.2, p.104] if n is even: $H_T^k(\Lambda, \Lambda^0; \mathbb{Q}) = 0$ unless k is odd; if n is odd, $H_T^k(\Lambda, \Lambda^0; \mathbb{Q}) = 0$ unless

k is even. It follows from this and the definition (keeping track of the degrees) that the string cohomology bracket is trivial. Next homology: By if n is odd, [**10**, 4.2, p.105] $H_k^T(\Lambda, \mathbb{Q}) = 0$ unless k is even and an argument similar to the above argument for cohomology shows that the string homology bracket is trivial. According [**11**, p.143], if n is even, and $X \in H_k^T(\Lambda; \mathbb{Q})$ is nontrivial, then k is odd *or* X is in the image of $H_k^T(\divideontimes; \mathbb{Q})$ for a basepoint $\divideontimes \in \Lambda^0$. But $\pi^!$ is trivial on the image of $H_*^T(\divideontimes; \mathbb{Q})$ (since $\pi^!$ increases the degree by 1, and $H_*^T(\divideontimes; \mathbb{Q}) \neq 0$ only in even degrees). Thus if X is of even degree, $[X, Y] = 0$ for any Y. The rest of the argument is the same as before. q.e.d.

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