Extension, Enhancement and Analogue of the Vasiliev Inequality

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瓦西列夫不等式的推广、加强与类似

摘要:

本文对瓦西列夫不等式进行了一系列的推广、加强与类比,并得到了一系列优美的类似结果.笔者通过平常的积累,搜集了一些关于瓦西列夫不等式的资料,并进行了深入的探讨.通过引进参数,笔者将其次数和系数进行一般化,利用三大不等式、Jensen 不等式、Chebyshev 不等式等不等式并借助 maple13 进行辅助证明,得到了一些有价值的引理(引理 5)和定理(定理 3、6、10、16、21、26、27、28、29 等).这些新的不等式在某些求函数最值,以及证明某些不等式时很有用.它们也可解决更多高次不等式的问题.最后,笔者借助杨路教授的 Bottema2009 软件,得到了 8 个猜想.最近，前 3 个猜想已告解决，现只剩下猜想 4-8 供有兴趣的读者探讨.

创新之处：将瓦西列夫不等式进行了一系列的推广、加强，并且得到了一些类似不等式.

闪光之处:引理 5 是我们的新结果,由此引理得到了一系列的加强命题.遗憾的是此引理还不是最佳结果,最好的结果是猜想 9.

Extension, Enhancement and Analogue of the Vasiliev Inequality

This treatise strengthens, extends and analogizes the Vasiliev Inequality and reaches a series of beautiful similar conclusions. In our research, we make an in-depth discussion on the information we collected in respect of the Vasiliev Inequality. By introducing parameters, we generalize the degrees and the coefficients. Then we use **the Three Famous Inequalities**, **Chebyshev's Inequality** and **Jensen's Inequality** etc, and maple13 to acquire several valuable lemmas(the Fifth Lemma) and Propositions (Proposition 3， 6,10,16,21,26,27,28,29,etc). These new inequalities are very helpful in acquiring the extremes of several functions and proving some inequalities. In addition, many high degree inequality problems can be solved by means of them. Finally, with the assistance of Professor Yang's Software Bottema 2009, we obtain 8 conjectures. Recently, Conjecture 1, Conjecture 2,Conjecture 3, have been proved. Only Conjecture 4 to 8 are left unproved to the reader with interest.

Innovation: By strengthening, extending and analogizing the Vasiliev Inequality, we reach a series of beautiful similar conclusions.

Highlight: Lemma 5 is our new result, from which we obtain a series of extensive inequalities. However, we have not yet reached the optimum result.Conjecture 9 is our best result so far.

Key words: The Vasilev Inequality, Extension, Enhancement, Similarity, Bottema2009.

Extension, Enhancement and Analogue of the Vasiliev Inequality

I. Background knowledge

Passage [1] introduces a group of inequalities published on a Russian magazine called "The Middle School's Mathematics". One of them is the Vasiliev Inequality:

Assume that a, b, c are positive, and satisfy $a+b+c=1$, then:

$$\frac{a^2+b}{b+c}+\frac{b^2+c}{c+a}+\frac{c^2+a}{a+b}\geq 2\,. \tag{1}$$

Passage [2] extends Formula (1) to the following inequality:

Assume that a, b, c are positive, and satisfy $a+b+c=1$, and $\lambda\geq 1$, then:

$$\frac{\lambda a^2+b}{b+c}+\frac{\lambda b^2+c}{c+a}+\frac{\lambda c^2+a}{a+b}\geq \frac{\lambda+3}{2}\,. \tag{2}$$

Passage [3] extends Formula (1) to the following inequality:

Assume that a, b, c are positive, and $-1\leq\lambda\leq 1$, then:

$$\frac{a^2+\lambda b}{b+c}+\frac{b^2+\lambda c}{c+a}+\frac{c^2+\lambda a}{a+b}\geq \frac{1+3\lambda}{2}\,. \tag{3}$$

Passage [4] extends Formula (1) to

If $a,b,c,\lambda_1,\lambda_2,\mu_1,\mu_2\in R_+, a+b+c=1, n\in N_+$, and $n\geq 2$, then:

$$\frac{\mu_1 a^2+\mu_2 b^n}{\lambda_1 b+\lambda_2 c}+\frac{\mu_1 b^2+\mu_2 c^n}{\lambda_1 c+\lambda_2 a}+\frac{\mu_1 c^2+\mu_2 a^n}{\lambda_1 a+\lambda_2 b}\geq \frac{3^{n-2}\cdot\mu_1+\mu_2}{(\lambda_1+\lambda_2)\cdot 3^{n-2}}\,. \tag{4}$$

and $$\frac{\mu_1 a^{n+1}+\mu_2 b^n}{\lambda_1 b+\lambda_2 c}+\frac{\mu_1 b^{n+1}+\mu_2 c^n}{\lambda_1 c+\lambda_2 a}+\frac{\mu_1 c^{n+1}+\mu_2 a^n}{\lambda_1 a+\lambda_2 b}\geq \frac{3\mu_2+\mu_1}{(\lambda_1+\lambda_2)\cdot 3^{n-1}}\,. \tag{5}$$

Inspired by this, we decided to make a further research and finally obtained a series of conclusions.

II. Five Lemmas

For a more comprehensible and effective demonstration, we firstly put forward these 5 lemmas.

Lemma1 [4]: If $a,b,c,x,y,z\in R_+, n\in N_+$ and $n\geq 2$, then:

$$\frac{a^n}{x}+\frac{b^n}{y}+\frac{c^n}{z}\geq\frac{(a+b+c)^n}{3^{n-2}(x+y+z)}\,. \tag{6}$$

Lemma2 If $a,b,c,d,e,f,g,h\in R_+$, $n\in N_+$ and $n\geq 2$, then:

$$\frac{a^n}{e}+\frac{b^n}{f}+\frac{c^n}{g}+\frac{d^n}{h}\geq\frac{(a+b+c+d)^n}{4^{n-2}(e+f+g+h)}\,. \tag{7}$$

Proof: When $n \geq 2$, according to **power mean inequality**, we can obtain the following inequality:

$$a^{\frac{n}{2}} + b^{\frac{n}{2}} + c^{\frac{n}{2}} + d^{\frac{n}{2}} \geq 4 \cdot (\frac{a+b+c+d}{4})^{\frac{n}{2}} .$$

Thus, according to **Cauchy's Inequality**, we can prove the following inequality:

$$(e+f+g+h) \cdot (\frac{a^n}{e} + \frac{b^n}{f} + \frac{c^n}{g} + \frac{d^n}{h})$$

$$= \left[(\sqrt{e})^2 + (\sqrt{f})^2 + (\sqrt{g})^2 + (\sqrt{h})^2\right] \cdot \left[\left(\frac{a^{\frac{n}{2}}}{\sqrt{e}}\right)^2 + \left(\frac{b^{\frac{n}{2}}}{\sqrt{f}}\right)^2 + \left(\frac{c^{\frac{n}{2}}}{\sqrt{g}}\right)^2 + \left(\frac{d^{\frac{n}{2}}}{\sqrt{h}}\right)^2\right]$$

$$\geq (a^{\frac{n}{2}} + b^{\frac{n}{2}} + c^{\frac{n}{2}} + d^{\frac{n}{2}})^2 \geq \left[4 \cdot (\frac{a+b+c+d}{4})^{\frac{n}{2}}\right]^2 = \frac{(a+b+c+d)^n}{4^{n-2}} .$$

So, when $a,b,c,d,e,f,g,h \in R_+$, and $n \in N_+, n \geq 2$, we can draw the following conclusion:

$$\frac{a^n}{e} + \frac{b^n}{f} + \frac{c^n}{g} + \frac{d^n}{h} \geq \frac{(a+b+c+d)^n}{4^{n-2}(e+f+g+h)} .$$

End.

Lemma3 If $a,b,c,d,e,g,h,i,j \in R_+$, $n \in N_+$, and $n \geq 2$, then:

$$\frac{a^n}{f} + \frac{b^n}{g} + \frac{c^n}{h} + \frac{d^n}{i} + \frac{e^n}{j} \geq \frac{(a+b+c+d+e)^n}{5^{n-2}(f+g+h+i+j)} . \quad (8)$$

Proof: When $n \geq 2$, according to **power mean inequality**, we can obtain the following inequality:

$$a^{\frac{n}{2}} + b^{\frac{n}{2}} + c^{\frac{n}{2}} + d^{\frac{n}{2}} + e^{\frac{n}{2}} \geq 5 \cdot (\frac{a+b+c+d+e}{5})^{\frac{n}{2}} .$$

Thus, according to **Cauchy's Inequality**, we can prove the following inequality:

$$(f+g+h+i+j) \cdot (\frac{a^n}{f} + \frac{b^n}{g} + \frac{c^n}{h} + \frac{d^n}{i} + \frac{e^n}{j})$$

$$= \left[(\sqrt{f})^2 + (\sqrt{g})^2 + (\sqrt{h})^2 + (\sqrt{i})^2 + (\sqrt{j})^2\right] \cdot \left[\left(\frac{a^{\frac{n}{2}}}{\sqrt{f}}\right)^2 + \left(\frac{b^{\frac{n}{2}}}{\sqrt{g}}\right)^2 + \left(\frac{c^{\frac{n}{2}}}{\sqrt{h}}\right)^2 + \left(\frac{d^{\frac{n}{2}}}{\sqrt{i}}\right)^2 + \left(\frac{e^{\frac{n}{2}}}{\sqrt{j}}\right)^2\right]$$

$$\geq (a^{\frac{n}{2}} + b^{\frac{n}{2}} + c^{\frac{n}{2}} + d^{\frac{n}{2}} + e^{\frac{n}{2}})^2 \geq \left[5(\frac{a+b+c+d+e}{5})^{\frac{n}{2}}\right]^2$$

$$= \frac{(a+b+c+d+e)^n}{5^{n-2}} .$$

So when $a,b,c,d,e,f,g,h,i,j\in R_+$, $n\in N_+$, and $n\geq 2$, we can draw the following conclusion:

$$\frac{a^n}{f}+\frac{b^n}{g}+\frac{c^n}{h}+\frac{d^n}{i}+\frac{e^n}{j}\geq\frac{(a+b+c+d+e)^n}{5^{n-2}(f+g+h+i+j)}.$$

End.

Lemma4 If $\alpha_1,\alpha_2,\ldots\ldots,\alpha_{n-1},\alpha_n\in R_+$, $\beta_1,\beta_2,\ldots\ldots,\beta_{n-1},\beta_n\in R_+$, $m,n\in N_+$, and $n\geq 3, m\geq 2$, then:

$$\sum_{i=1}^{n}\frac{\alpha_i^{\,m}}{\beta_i}\geq\frac{\left(\sum_{i=1}^{n}\alpha_i\right)^m}{n^{m-2}\left(\sum_{i=1}^{n}\beta_i\right)}. \qquad (9)$$

Proof: When $n\geq 2$, according to power mean inequality, we can obtain the following inequality:

$$\sum_{i=1}^{n}\alpha_i^{\frac{m}{2}}\geq n\left(\frac{\sum_{i=1}^{n}\alpha_i}{n}\right)^{\frac{m}{2}}.$$

Thus, according to Cauchy's Inequality, we can prove the following inequality:

$$\left(\sum_{i=1}^{n}\beta_i\right)\cdot\left(\sum_{i=1}^{n}\frac{\alpha_i^m}{\beta_i}\right)=\left[\sum_{i=1}^{n}\left(\sqrt{\beta_i}\right)^2\right]\cdot\left[\sum_{i=1}^{n}\left(\frac{\alpha_i^{\frac{m}{2}}}{\sqrt{\beta_i}}\right)^2\right]$$

$$\geq\left(\sum_{i=1}^{n}\alpha_i^{\frac{m}{2}}\right)^2\geq\left[n\left(\frac{\sum_{i=1}^{n}\alpha_i}{n}\right)^{\frac{m}{2}}\right]^2=\frac{\left(\sum_{i=1}^{n}\alpha_i\right)^m}{n^{m-2}}.$$

So when $\alpha_1,\alpha_2,\ldots\ldots,\alpha_{n-1},\alpha_n\in R_+$, $\beta_1,\beta_2,\ldots\ldots,\beta_{n-1},\beta_n\in R_+$, $m,n\in N_+$, and $n\geq 3, m\geq 2$, we can draw the following conclusion:

$$\sum_{i=1}^{n}\frac{\alpha_i^{\,m}}{\beta_i}\geq\frac{\left(\sum_{i=1}^{n}\alpha_i\right)^m}{n^{m-2}\left(\sum_{i=1}^{n}\beta_i\right)}.$$

End.

Lemma5 If $x,y,z\in R_+$, then:

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\geq 3+\frac{27}{8}\cdot\frac{(x-y)^2+(y-z)^2+(z-x)^2}{(x+y+z)^2}. \qquad (10)$$

Proof： Notice that $(x^2y+y^2z+z^2x)-(xy^2+yz^2+zx^2)=-(x-y)(y-z)(z-x)$, which means

$$2(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-3)=\frac{2(x^2z+y^2x+z^2x)-6xyz}{xyz}$$
$$=\frac{(x^2z+y^2x+z^2y+xz^2+yx^2+zy^2-6xyz)+(x^2z+y^2x+z^2y)-(xz^2+yx^2+zy^2)}{xyz}$$
$$=\frac{\sum x\sum yz-9xyz+(x-y)(y-z)(z-x)}{xyz}.$$
$$(x-y)^2+(y-z)^2+(z-x)^2=2(x^2+y^2+z^2-xy-yz-zx)=2[(\sum x)^2-3\sum yz],$$

From this we can discover that Formula (10) equals to:

$$\frac{\sum x\sum yz-9xyz+(x-y)(y-z)(z-x)}{xyz}\geq\frac{27}{2}\cdot\frac{(\sum x)^2-3\sum yz}{(x+y+z)^2}. \qquad (11)$$

Transform it into: $\begin{cases}y+z=a,\\ z+x=b,\\ x+y=c,\end{cases}$ so $\begin{cases}x=s-a,\\ y=s-b,\\ z=s-c,\end{cases}$. Therein a,b,c are the sides of triangle ABC, $s=\frac{1}{2}(a+b+c)$ is its half perimeter.

According to some common triangular identities:

$\sum x=\sum(s-a)=s,\sum yz=\sum(s-b)(s-c)=r(4R+r),$

$\prod x=\prod(s-a)=r^2s$,and $(x-y)(y-z)(z-x)=-(a-b)(b-c)(c-a)$, (R,r are respectively the radii of the circumcircles and the inscribed circle), so we can discover that Formula (11) equals to:

$$\frac{s\cdot r(4R+r)-9sr^2-(a-b)(b-c)(c-a)}{sr^2}\geq\frac{27}{2}\cdot\frac{s^2-3r(4R+r)}{s^2}$$
$$\Leftrightarrow r[4s^2(R-2r)-\frac{27}{2}r(s^2-12Rr-3r^2)]\geq s(a-b)(b-c)(c-a)$$
$$\Leftrightarrow r^2[4s^2(R-2r)-\frac{27}{2}r(s^2-12Rr-3r^2)]^2\geq s^2(a-b)^2(b-c)^2(c-a)^2 \qquad (12)$$

According to some common triangular identities [21]:

$(a-b)^2(b-c)^2(c-a)^2=4r^2[-s^4+(4R^2+20Rr-2r^2)s^2-r(4R+r)^3]$, so we can discover that Formula (12) equals to:

$$[8s^2(R-2r)-27r(s^2-12Rr-3r^2)]^2\geq 16s^2[-s^4+(4R^2+20Rr-2r^2)s^2-r(4R+r)^3]$$
$$\Leftrightarrow 16s^6-(1008Rr-1881r^2)s^4+(1024R^3r+5952R^2r^2-26376Rr^3-6950r^4)s^2$$
$$+104976R^2r^4+52488Rr^5+6561r^6\geq 0$$
$$\Leftrightarrow 16(s^2-16Rr+5r^2)^3+(-240Rr+1641r^2)s^4+(1024R^3r-6336R^2r^2$$
$$-18696Rr^3-8150r^4)s^2+65536R^3r^3+43536R^2r^4+71688Rr^5+4561r^6\geq 0 \qquad (13)$$

According to Gerretsens' Inequality, $s^2\geq 16Rr-5r^2$. Thus if we want to prove Formula (13), we only need to prove:

$$\Leftrightarrow f(s^2)=(-240Rr+1641r^2)s^4+(1024R^3r-6336R^2r^2-18696Rr^3-8150r^4)s^2$$
$$+65536R^3r^3+43536R^2r^4+71688Rr^5+4561r^6\geq 0. \qquad (14)$$

When $-240Rr+1641r^2\leq 0$, which means when $\frac{R}{r}\geq 6.8375$, $f(s^2)$ is a function or a quadratic function of opening down about s^2, and $16Rr-5r^2\leq s^2\leq 4R^2+4Rr+3r^2$, to prove $f(s^2)\geq 0$, only prove $f(16Rr-5r^2)\geq 0$ and $f(4R^2+4Rr+3r^2)\geq 0$.

$f(16Rr-5r^2)=16384R^4r^2-102400R^3r^3+234576R^2r^4-233792Rr^5+86336r^6$

$=16r^2(1024R^2-2304Rr+1349r^2)(R-2r)^2$

$=16r^2[1024(R-\dfrac{9}{8})^2+53r^2](R-2r)^2\geq 0$,

$f(4R^2+4Rr+3r^2)=256R^5r-2672R^4r^2+11392R^3r^3-22976R^2r^4+20224Rr^5-5120r^6$

$=16r^2(16R^3-103R^2r+236Rr^2-80r^3)(R-2r)^2$

$=16r^2[16R^2-71Rr+94r^2)(R-2r)+108r^3](R-2r)^2$

$=16r^2\{[16(R-\dfrac{71}{32}r)^2+\dfrac{91215}{1024}r^2](R-2r)+108r^3\}(R-2r)^2\geq 0$.

When $-240Rr+1641r^2>0$, that $2\leq\dfrac{R}{r}<6.8375$, $f(s^2)$ is a quadratic function of opening down about s^2, and $16Rr-5r^2\leq s^2\leq 4R^2+4Rr+3r^2$.

Notice the discriminant about $f(s^2)$

$\Delta_{s^2}=(1024R^3r-6336R^2r^2-18696Rr^3-8150r^4)^2$

$-4(-240Rr+1641r^2)(65536R^3r^3+43536R^2r^4+71688Rr^5+4561r^6)$

$=1048576R^6r^2-12976128R^5r^3+64770048R^4r^4-168159232R^3r^5$

$+235867392R^2r^6-161436672Rr^7+36484096r^8$

$=256r^2(R-2r)^2(4096R^4-34304R^3r+99408R^2r^2-122024Rr^3+35629r^4)$

At this time a quadratic function of the coefficient of term about $f(s^2)$

$1024R^3r-6336R^2r^2-18696Rr^3-8150r^4$

$=(R-2r)(1024R^3r-6336R^2r^2-18696Rr^3-8150r^4)$

$=(R-2r)r^4[1024(\dfrac{R}{r})^3-6336(\dfrac{R}{r})^2-18696\dfrac{R}{r}-8150]$.

Let $\dfrac{R}{r}=t\in[2,6.8375)$, so $g(t)=1024t^3-6336t^2-18696t-8150$,

$g'(t)=3072t^2-12672t-18696$.

Let $g'(t)=0$, we acquire the root of the function is

$t=\dfrac{33+\sqrt{2647}}{16}$=5.27806274⋯, or $t=\dfrac{33-\sqrt{2647}}{16}<0$ (omitted).

When $t\in[2,\dfrac{33+\sqrt{2647}}{16}]$, $g'(t)<0$; When $t\in(\dfrac{33+\sqrt{2647}}{16},6.8375)$, $g'(t)>0$.

$\therefore$ When $t=\dfrac{33+\sqrt{2647}}{16}$, $g(t)$ reaches the minimum $g(\dfrac{33+\sqrt{2647}}{16})$.

$\because g(2)=-62694<0$, $g(\dfrac{547}{80})=-\dfrac{26216541}{250}<0$,

$\therefore g(t)_{\max}=\max\{g(2),g(\dfrac{547}{80})\}<0$.

$\therefore g(t)<0$.

The axis of symmetry of quadratic function $f(s^2)$ is

$s^2=-\dfrac{1024R^3r-6336R^2r^2-18696Rr^3-8150r^4}{2(-240Rr+1641r^2)}>0$.

If $\Delta_{s^2} > 0$, which means $4096R^4 - 34304R^3r + 99408R^2r^2 - 122024Rr^3 + 35629r^4 > 0$. With the assistance of maple13, we discover that when $2 \le \frac{R}{r} < 4.10490422\cdots$, function $f(s^2) = 0$ has two roots; Therein the larger one is

$$\frac{-(1024R^3r - 6336R^2r^2 - 18696Rr^3 - 8150r^4) + \sqrt{\Delta_{s^2}}}{2(-240Rr + 1641r^2)}.$$

We can prove $\frac{-(1024R^3r - 6336R^2r^2 - 18696Rr^3 - 8150r^4) + \sqrt{\Delta_{s^2}}}{2(-240Rr + 1641r^2)} \le 16Rr - 5r^2$.

$\Leftrightarrow -(1024R^3r - 6336R^2r^2 - 18696Rr^3 - 8150r^4) + \sqrt{\Delta_{s^2}}$

$\le 2(-240Rr + 1641r^2)(16Rr - 5r^2)$

$\Leftrightarrow \sqrt{\Delta_{s^2}} \le -1024R^3r - 1344R^2r^2 + 73608Rr^3 - 8260r^4$

$= r^4[-1024(\frac{R}{r})^3 - 1344(\frac{R}{r})^2 + 73608(\frac{R}{r}) - 8260]$.

令 $\frac{R}{r} = t \in [2, 6.8375)$, 则 $h(t) = -1024t^3 - 1344t^2 + 73608t - 8260$,

$h'(t) = -3072t^2 - 2688t + 73608$.

let $h'(t) = 0$, so we can acquire the root $t = \frac{-7 + 3\sqrt{687}}{16}$=4.47700340⋯,

or $t = \frac{-7 - 3\sqrt{687}}{16} < 0$ (omitted).

When $t \in [2, \frac{-7 + 3\sqrt{687}}{16})$, $g'(t) > 0$; when $t \in (\frac{-7 + 3\sqrt{687}}{16}, 6.8375)$, $g'(t) < 0$.

$\therefore$ When $t = \frac{-7 + 3\sqrt{687}}{16}$, $g(t)$ reaches the maximum $g(\frac{-7 + 3\sqrt{687}}{16})$.

$\because g(2) = 125388 > 0$, $g(\frac{547}{80}) = \frac{26216541}{250} > 0$,

$\therefore g(t)_{\min} = \max\{g(2), g(\frac{547}{80})\} > 0$.

So $-1024R^3r - 1344R^2r^2 + 73608Rr^3 - 8260r^4 \ge 0$.

For this we only need to prove

$\Delta_{s^2} \le (-1024R^3r - 1344R^2r^2 + 73608Rr^3 - 8260r^4)^2$

$\Leftrightarrow 1048576R^6r^2 - 12976128R^5r^3 + 64770048R^4r^4 - 168159232R^3r^5$
$+ 235867392R^2r^6 - 161436672Rr^7 + 36484096r^8$
$\le (-1024R^3r - 1344R^2r^2 + 73608Rr^3 - 8260r^4)^2$

$\Leftrightarrow 7864320R^5r^3 - 153059328R^4r^4 + 398662656R^3r^5 + 1893865152R^2r^6$
$+156359952Rr^7 - 36481071r^8 \geq 0$

$\Leftrightarrow 3r(547r - 80R)(-32768R^4 + 413696R^3r + 1167552R^2r^2 - 22231r^4) \geq 0$

Notice that $2 \leq \frac{R}{r} < 4.10490422\cdots$, so $547r - 80R > 0$. For this we only need to prove $-32768R^4 + 413696R^3r + 1167552R^2r^2 - 22231r^4 \geq 0$, That is, to prove $-32768(\frac{R}{r})^4 + 413696(\frac{R}{r})^3 + 1167552(\frac{R}{r})^2 - 22231 \geq 0$.

Let $\frac{R}{r} = t \in [2, 6.8375)$, so $p(t) = -32768t^4 + 413696t^3 + 1167552t^2 - 22231 \geq 0$,

$p'(t) = -131072t^3 + 1241088t^2 + 2335104t = 128t(-1024t^2 + 9696t + 18243)$
$= 128t[1024t(8-t) + 1504t + 18243] > 0$.

So when $t \in [2, 6.8375)$, $p(t)$ is monotonely increasing, which means $p(t) \geq p(2) = 7433257 > 0$.

So $-32768R^4 + 413696R^3r + 1167552R^2r^2 - 22231r^4 \geq 0$ is tenable.

According to the graph, if we want to prove $f(s^2) \geq 0$, we only need to prove $f(16Rr - 5r^2) \geq 0$

Thus

$f(16Rr - 5r^2) = 16384R^4r^2 - 102400R^3r^3 + 234576R^2r^4 - 233792Rr^5 + 86336r^6$
$= 16r^2(1024R^2 - 2304Rr + 1349r^2)(R - 2r)^2$
$= 16r^2[1024(R - \frac{9}{8})^2 + 53r^2](R - 2r)^2 \geq 0$.

If $\Delta_{s^2} \leq 0$, which equals to

$4096R^4 - 34304R^3r + 99408R^2r^2 - 122024Rr^3 + 35629r^4 \leq 0$, with the assistance of maple13, we can discover that when $4.10490422\cdots \leq \frac{R}{r} < 6.8375$, $f(s^2) \geq 0$ is always tenable.

Generalizing all the identities above, we can draw a conclusion that Formula (14) is tenable, so Lemma 5 is also tenable.

III. Three-dimensional Inequalities

Theorem1: If $a, b, c, \lambda_1, \lambda_2, \mu_1, \mu_2 \in R_+, a + b + c = 1, n, m \in N_+$, and $n \geq 2, m \geq 2$, and $\lambda_1, \lambda_2 \geq 1, \mu_1, \mu_2 \geq 1$, then:

$$\frac{\mu_1 a^m + \mu_2 b^n}{\lambda_1 b + \lambda_2 c} + \frac{\mu_1 b^m + \mu_2 c^n}{\lambda_1 c + \lambda_2 a} + \frac{\mu_1 c^m + \mu_2 a^n}{\lambda_1 a + \lambda_2 b} \geq \frac{9\mu_1}{3^m(\lambda_1 + \lambda_2)} + \frac{9\mu_2}{3^n(\lambda_1 + \lambda_2)}. \quad (15)$$

With equality if and only if a= b= c= 1/3.

Proof $\because n, m \in N_+$, and $n \geq 2, m \geq 2$,

∴ According to the assumption and the Lemma 1:

$$\frac{\mu_1 a^m + \mu_2 b^n}{\lambda_1 b + \lambda_2 c} + \frac{\mu_1 b^m + \mu_2 c^n}{\lambda_1 c + \lambda_2 a} + \frac{\mu_1 c^m + \mu_2 a^n}{\lambda_1 a + \lambda_2 b}$$

$$= \mu_1 \cdot \sum \frac{a^m}{\lambda_1 b + \lambda_2 c} + \mu_2 \cdot \sum \frac{b^n}{\lambda_1 b + \lambda_2 c}$$

$$\geq \mu_1 \cdot \frac{(a+b+c)^m}{3^{m-2}(\lambda_1+\lambda_2)\cdot(a+b+c)} + \mu_2 \cdot \frac{(a+b+c)^n}{3^{n-2}(\lambda_1+\lambda_2)\cdot(a+b+c)}$$

$$= \mu_1 \cdot \frac{1}{3^{m-2}(\lambda_1+\lambda_2)} + \mu_2 \cdot \frac{1}{3^{n-2}(\lambda_1+\lambda_2)}$$

$$= \frac{\mu_1}{3^{m-2}(\lambda_1+\lambda_2)} + \frac{\mu_2}{3^{n-2}(\lambda_1+\lambda_2)} = \frac{9\mu_1}{3^m\cdot(\lambda_1+\lambda_2)} + \frac{9\mu_2}{3^n\cdot(\lambda_1+\lambda_2)}.$$

Demonstration finished

Theorem2: If $a,b,c,\lambda_1,\lambda_2,\mu_1,\mu_2,\mu_3 \in R_+, a+b+c=1, n,m,k \in N_+$, and

$n \geq 2, m \geq 2, k \geq 2,\ \lambda_1,\lambda_2 \geq 1, \mu_1,\mu_2,\mu_3 \geq 1$,then:

$$\sum \frac{\mu_1 a^m + \mu_2 b^n + \mu_3 c^k}{\lambda_1 b + \lambda_2 c} \geq \frac{9}{\lambda_1+\lambda_2}\left(\frac{\mu_1}{3^m} + \frac{\mu_2}{3^n} + \frac{\mu_3}{3^k}\right). \quad (16)$$

With equality if and only if a= b= c= 1/3.

Proof ∵ $n,m,k \in N_+$, and $n \geq 2, m \geq 2, k \geq 2$,

∴ According to the assumption and the Lemma 1:

$$\sum \frac{\mu_1 a^m + \mu_2 b^n + \mu_3 c^k}{\lambda_1 b + \lambda_2 c}$$

$$= \mu_1 \cdot \sum \frac{a^m}{\lambda_1 b + \lambda_2 c} + \mu_2 \cdot \sum \frac{b^n}{\lambda_1 b + \lambda_2 c} + \mu_3 \cdot \sum \frac{c^k}{\lambda_1 b + \lambda_2 c}$$

$$\geq \mu_1 \cdot \frac{(a+b+c)^m}{3^{m-2}(\lambda_1+\lambda_2)\cdot(a+b+c)} + \mu_2 \cdot \frac{(a+b+c)^n}{3^{n-2}(\lambda_1+\lambda_2)\cdot(a+b+c)} + \mu_3 \cdot \frac{(a+b+c)^k}{3^{k-2}(\lambda_1+\lambda_2)\cdot(a+b+c)}$$

$$= \frac{\mu_1}{3^{m-2}(\lambda_1+\lambda_2)} + \frac{\mu_2}{3^{n-2}(\lambda_1+\lambda_2)} + \frac{\mu_3}{3^{k-2}(\lambda_1+\lambda_2)} = \frac{9}{\lambda_1+\lambda_2}\left(\frac{\mu_1}{3^m} + \frac{\mu_2}{3^n} + \frac{\mu_3}{3^k}\right).$$

Demonstration finished

Annotation: In Formula (16), if we make $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1, \mu_3 = 0, m = 2, n = 1$,we can acquire the Vasiliev Inequality; if we make $\lambda_1 = \lambda_2 = \mu_2 = 1, \mu_3 = 0, \mu_1 = \lambda, m = 2, n = 1$, we can acquire formula (2).

Theorem3 If a, b, c are positive, and satisfy $a+b+c=1$, then:

$$\frac{a+b^2}{(b+c)^2}+\frac{b+c^2}{(c+a)^2}+\frac{c+a^2}{(a+b)^2}\geq 3 \cdot \qquad (17)$$

With equality if and only if a= b= c= 1/3.

Proof $\frac{a+b^2}{(b+c)^2}+\frac{b+c^2}{(c+a)^2}+\frac{c+a^2}{(a+b)^2}=\sum\frac{a}{(b+c)^2}+\sum\frac{b^2}{(b+c)^2}$.

We firstly prove $\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\geq\frac{9}{4}$. (18)

According to the assumptions and Cauchy's Inequality,

$$\left[\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\right]\cdot(a+b+c)\geq\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2.$$

$$\because \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}=\frac{a}{b+c}-\frac{1}{2}+\frac{b}{c+a}-\frac{1}{2}+\frac{c}{a+b}-\frac{1}{2}$$

$$=\frac{1}{2}\left(\frac{a-b}{b+c}-\frac{a-c}{b+c}\right)+\frac{1}{2}\left(\frac{b-c}{c+a}-\frac{b-a}{c+a}\right)+\frac{1}{2}\left(\frac{c-a}{a+b}-\frac{c-b}{a+b}\right)$$

$$=\frac{1}{2}\left[\frac{(a-b)^2}{(b+c)\cdot(c+a)}+\frac{(b-c)^2}{(c+a)\cdot(a+b)}+\frac{(c-a)^2}{(b+c)\cdot(a+b)}\right]\geq 0 \cdot$$

$$\therefore \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

$$\therefore \left[\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\right]\cdot(a+b+c)\geq\frac{9}{4},$$

$$\therefore \frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\geq\frac{9}{4}.$$

Then we prove $\frac{b^2}{(b+c)^2}+\frac{c^2}{(c+a)^2}+\frac{a^2}{(a+b)^2}\geq\frac{3}{4}$. (19)

$\Leftrightarrow$ 5（$a^4b^2+b^4c^2+c^4a^2$）+（$a^2b^4+b^2c^4+c^2a^4$）+2（$a^3b^3+b^3c^3+c^3a^3$）

+2 $abc(a^3+b^3+c^3)\geq 2abc(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)+18a^2b^2c^2$.

According to the Average Value Inequality，

$a^4b^2+b^4c^2+c^4a^2\geq 3a^2b^2c^2$， $a^2b^4+b^2c^4+c^2a^4\geq 3a^2b^2c^2$， so

$$5(a^4b^2+b^4c^2+c^4a^2)+(a^2b^4+b^2c^4+c^2a^4)\geq 18a^2b^2c^2. \qquad (20)$$

Notice that

$$a^3b^3+c^3a^3=a^3(b^3+c^3)=a^3(b^3+c^3-b^2c-bc^2)+a^3(b^2c+bc^2)$$

$$=a^3(b-c)(b^2-c^2)+a^3(b^2c+bc^2)=a^3(b-c)^2(b+c)+a^3(b^2c+bc^2)$$

$\geq a^3(b^2c+bc^2)$, so $a^3b^3+c^3a^3 \geq a^3(b^2c+bc^2)$,

For the same reason, $b^3c^3+b^3a^3 \geq b^3(c^2a+ca^2)$,

$b^3c^3+c^3a^3 \geq c^3(a^2b+ab^2)$,

If adding up the three formulas above, we can obtain the following inequality.

$$2(a^3b^3+b^3c^3+c^3a^3) \geq abc(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2). \qquad (21)$$

Notice that $b^3+c^3 = b^3+c^3-b^2c-bc^2+b^2c+bc^2$

$=(b-c)(b^2-c^2)+(b^2c+bc^2)=(b-c)^2(b+c)+(b^2c+bc^2) \geq b^2c+bc^2$, so

$b^3+c^3 \geq b^2c+bc^2$, so $abc(b^3+c^3) \geq abc(b^2c+bc^2)$,

For the same reason, $abc(c^3+a^3) \geq abc(c^2a+ca^2)$,

$abc(a^3+b^3) \geq abc(a^2b+ab^2)$,

If adding up the three formulas above, we can obtain the following inequality.

$$2abc(a^3+b^3+c^3) \geq abc(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2). \qquad (22)$$

If adding up Formula (20), Formula (21), and Formula (22), we can obtain Formula (17).

By adding up Formula (18) and Formula (19) we can prove the Theorem 3 to be tenable.

$$\therefore \frac{a+b^2}{(b+c)^2}+\frac{b+c^2}{(c+a)^2}+\frac{c+a^2}{(a+b)^2} \geq \frac{3+9}{4}=3.$$

Demonstration finished.

Theorem4: If a, b, c are positive, and satisfy $a+b+c=1$, then:

$$\frac{a^2+b^3}{(b+c)^3}+\frac{b^2+c^3}{(c+a)^3}+\frac{c^2+a^3}{(a+b)^3} \geq \frac{3}{2}. \qquad (23)$$

With equality if and only if a= b= c= 1/3.

Proof: $\frac{a^2+b^3}{(b+c)^3}+\frac{b^2+c^3}{(c+a)^3}+\frac{c^2+a^3}{(a+b)^3}=\sum\frac{a^2}{(b+c)^3}+\sum\frac{b^3}{(b+c)^3}$.

Step I: According to the assumptions and the Cauchy inequality,

$$\left[\frac{a^3}{(a+b)^3}+\frac{b^3}{(b+c)^3}+\frac{c^3}{(c+a)^3}\right]\cdot\left[a(a+b)^3+b(b+c)^3+c(c+a)^3\right] \geq \left(a^2+b^2+c^2\right)^2.$$

Next to prove $8\left(a^2+b^2+c^2\right)^2 \geq 3\left[a(a+b)^3+b(b+c)^3+c(c+a)^3\right]$.

Namely prove

$$5\left(a^4+b^4+c^4\right)+7\left(a^2b^2+b^2c^2+c^2a^2\right) \geq 9\left(a^3b+b^3c+c^3a\right)+3\left(ab^3+bc^3+ca^3\right).$$

$\because 4a^4+b^4+7a^2b^2-9a^3b-3ab^3 = a^4-2a^2b^2+b^4+3a^4-9a^3b+9a^2b^2-3ab^3$

$$=\left(a^2-b^2\right)^2+3a(a-b)^3=(a-b)^2\left(4a^2-ab+b^2\right)\geq 0.$$

$\therefore 4a^4+b^4+7a^2b^2\geq 9a^3b+3ab^3$,

For the same reason, $4b^4+c^4+7b^2c^2\geq 9b^3c+3bc^3$, $4c^4+a^4+7c^2a^2\geq 9c^3a+3ca^3$,

If adding up the three formulas above, we can obtain the following inequality.

$$5\left(a^4+b^4+c^4\right)+7\left(a^2b^2+b^2c^2+c^2a^2\right)\geq 9\left(a^3b+b^3c+c^3a\right)+3\left(ab^3+bc^3+ca^3\right).$$

Then we can prove $\frac{a^3}{(a+b)^3}+\frac{b^3}{(b+c)^3}+\frac{c^3}{(c+a)^3}\geq\frac{3}{8}$.

Step II: Method 1 According to the Cauchy Inequality:

$$\left[\frac{a^2}{(b+c)^3}+\frac{b^2}{(c+a)^3}+\frac{c^2}{(a+b)^3}\right]\cdot[(b+c)+(c+a)+(a+b)]$$

$$=\left[\left(\sqrt{\frac{a^2}{(b+c)^3}}\right)^2+\left(\sqrt{\frac{b^2}{(c+a)^3}}\right)^2+\left(\sqrt{\frac{c^2}{(a+b)^3}}\right)^2\right]\cdot\left[\left(\sqrt{b+c}\right)^2+\left(\sqrt{c+a}\right)^2+\left(\sqrt{a+b}\right)^2\right]$$

$$\geq\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2.$$

$$\therefore\left[\frac{a^2}{(b+c)^3}+\frac{b^2}{(c+a)^3}+\frac{c^2}{(a+b)^3}\right]\cdot[(b+c)+(c+a)+(a+b)]$$

$$\geq\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2\geq\left(\frac{3}{2}\right)^2=\frac{9}{4}.$$

And $\left[\frac{a^2}{(b+c)^3}+\frac{b^2}{(c+a)^3}+\frac{c^2}{(a+b)^3}\right]\cdot[(b+c)+(c+a)+(a+b)]$

$$=2\left[\frac{a^2}{(b+c)^3}+\frac{b^2}{(c+a)^3}+\frac{c^2}{(a+b)^3}\right].$$

$$\therefore\left[\frac{a^2}{(b+c)^3}+\frac{b^2}{(c+a)^3}+\frac{c^2}{(a+b)^3}\right]\geq\frac{9}{4\cdot 2}=\frac{9}{8}.$$

$$\therefore\frac{a^2+b^3}{(b+c)^3}+\frac{b^2+c^3}{(c+a)^3}+\frac{c^2+a^3}{(a+b)^3}=\sum\frac{a^2}{(b+c)^3}+\sum\frac{b^3}{(b+c)^3}\geq\frac{9}{8}+\frac{3}{8}=\frac{3}{2}.$$

Demonstration finished

Method 2 $a+b+c=1\Leftrightarrow b+c=1-a$, $\frac{a^2}{(b+c)^3}=\frac{a^2}{(1-a)^3}$.

Assume the function $f(x)=\frac{x^2}{(1-x)^3}$, and $x\in(0,1)$,

$$f'(x)=\frac{2x+x^2}{(1-x)^4},\ f''(x)=\frac{2(x^2+4x+1)}{(1-x)^5}>0,$$

∴ $f(x)$ is a lower convex function when $x\in(0,1)$

According to the Jensen Inequality

$$\frac{f(x_1)+f(x_2)+\ldots\ldots+f(x_n)}{n}\geq f(\frac{x_1+x_2+\ldots\ldots+x_n}{n}).$$

When $n=3$, and $a=x_1, b=x_2, c=x_3$, which means when $a+b+c=1$,

$$f(a)+f(b)+f(c)\geq 3\cdot f(\frac{1}{3}),$$

$$\frac{a^2}{(b+c)^3}+\frac{b^2}{(c+a)^3}+\frac{c^2}{(a+b)^3}\geq\frac{9}{8}.$$

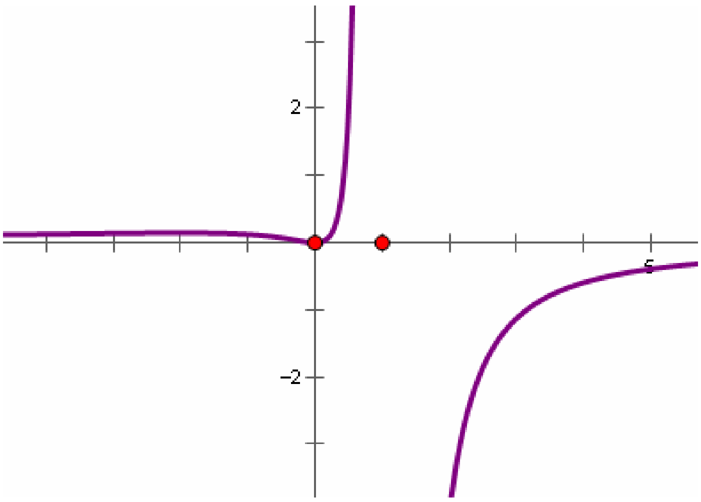
$$\therefore \frac{a^2+b^3}{(b+c)^3}+\frac{b^2+c^3}{(c+a)^3}+\frac{c^2+a^3}{(a+b)^3}=\sum\frac{a^2}{(b+c)^3}+\sum\frac{b^3}{(b+c)^3}\geq\frac{9}{8}+\frac{3}{8}=\frac{3}{2}.$$

Demonstration finished.

Method 3 Let $f(x)=\frac{x^2}{(1-x)^3}$, and $x\in(0,1)$, $f'(x)=\frac{2x+x^2}{(1-x)^4}$.

With the assistance of The Geometer's Sketchpad & reg, we found out that the second derivative of function is always positive when $x\in(0,1)$

The graph of the function's second derivative



According to the graph, we can prove $f'(\frac{1}{3})\cdot\left(x-\frac{1}{3}\right)\leq f(x)-f(\frac{1}{3})$

$$\Leftrightarrow \frac{2\cdot\frac{1}{3}+(\frac{1}{3})^2}{(1-\frac{1}{3})^4}\cdot\left(x-\frac{1}{3}\right)\leq\frac{x^2}{(1-x)^3}-\frac{(\frac{1}{3})^2}{(1-\frac{1}{3})^3}$$

$$\Leftrightarrow \frac{63}{16}\cdot\left(x-\frac{1}{3}\right)\le \frac{x^2}{(1-x)^3}-\frac{3}{8}$$

$$\Leftrightarrow \frac{(7x^2-18x+15)(3x-1)^2}{(1-x)^3}\ge 0$$

$$\Leftrightarrow \frac{[7(x-\frac{9}{7})^2+\frac{24}{7}](3x-1)^2}{(1-x)^3}\ge 0\,.$$

If we substitute x for a, b, and c respectively, we can obtain the following inequality:

$$\frac{a^2}{(b+c)^3}+\frac{b^2}{(c+a)^3}+\frac{c^2}{(a+b)^3}\ge\frac{9}{8}.$$

$$\therefore \frac{a^2+b^3}{(b+c)^3}+\frac{b^2+c^3}{(c+a)^3}+\frac{c^2+a^3}{(a+b)^3}=\sum\frac{a^2}{(b+c)^3}+\sum\frac{b^3}{(b+c)^3}\ge\frac{9}{8}+\frac{3}{8}=\frac{3}{2}.$$

Demonstration finished

Theorem5 If a, b, c are positive, and satisfy $a+b+c=1$, then:

$$\frac{a^n+b^{n+1}}{(b+c)^{n+1}}+\frac{b^n+c^{n+1}}{(c+a)^{n+1}}+\frac{c^n+a^{n+1}}{(a+b)^{n+1}}\ge\frac{6}{2^n}\,. \qquad (24)$$

With equality if and only if a= b= c= 1/3.

Proof: $\sum\frac{a^n+b^{n+1}}{(b+c)^{n+1}}=\sum\frac{a^n}{(b+c)^{n+1}}+\sum\frac{b^{n+1}}{(b+c)^{n+1}}\,.$

$$a+b+c=1\Leftrightarrow b+c=1-a\,,\frac{a^n}{(b+c)^{n+1}}=\frac{a^n}{(1-a)^{n+1}}\,,$$

Let $f(x)=\frac{x^n}{(1-x)^{n+1}}$, and $x\in(0,1)$, n$\in N_+$, $f'(x)=\frac{nx^{n-1}+x^n}{(1-x)^{n+2}}$,

$$f''(x)=\frac{2x^n+4nx^{n-1}+n(n-1)x^{n-2}}{(1-x)^{n+3}}>0\,,$$

So when $x\in(0,1)$, $f(x)$ is a lower convex function.

According to **Jensen's Inequality**,

$$\frac{f(x_1)+f(x_2)+\ldots\ldots+f(x_n)}{n}\ge f(\frac{x_1+x_2+\ldots\ldots+x_n}{n}).$$

When n=3, and $a=x_1, b=x_2, c=x_3$, which means when $a+b+c=1$,

$f(a)+f(b)+f(c)\ge 3\cdot f(\frac{1}{3})$, so $\frac{a^n}{(b+c)^{n+1}}+\frac{b^n}{(c+a)^{n+1}}+\frac{c^n}{(a+b)^{n+1}}\ge\frac{9}{2^{n+1}}\,.$

When n=2， $\frac{b^2}{(b+c)^2}+\frac{c^2}{(c+a)^2}+\frac{a^2}{(a+b)^2}\ge\frac{3}{4}=\frac{3}{2^2}$

This Inequality has already been proved in Theorem 3.

When $n\ge 3$， According to **power mean inequality**,

$$\frac{b^n}{(b+c)^n}+\frac{c^n}{(c+a)^n}+\frac{a^n}{(a+b)^n}=[\frac{b^2}{(b+c)^2}]^{\frac{n}{2}}+[\frac{c^2}{(c+a)^2}]^{\frac{n}{2}}+[\frac{a^2}{(a+b)^2}]^{\frac{n}{2}}$$

$$\ge 3\{\frac{1}{3}[\frac{b^2}{(b+c)^2}+\frac{c^2}{(c+a)^2}+\frac{a^2}{(a+b)^2}]\}^{\frac{n}{2}}$$

$$\ge 3(\frac{1}{3}\times\frac{3}{4})^{\frac{n}{2}}=3\left(\frac{1}{4}\right)^{\frac{n}{2}}=3(\frac{1}{2})^n=\frac{3}{2^n}.$$

So when $n\in N_+$， $n\ge 2$, $\frac{b^n}{(b+c)^n}+\frac{c^n}{(c+a)^n}+\frac{a^n}{(a+b)^n}\ge\frac{3}{2^n}$.

$$\sum\frac{a^n+b^{n+1}}{(b+c)^{n+1}}=\sum\frac{a^n}{(b+c)^{n+1}}+\sum\frac{b^{n+1}}{(b+c)^{n+1}}=\frac{3+9}{2^{n+1}}=\frac{6}{2^n}.$$

Demonstration finished.

Theorem6： If a,b,c are positive, $a+b+c=1$， and $m\ge 2, n\ge 2$, then:

$$\frac{a^m+b^{m+n}}{(b+c)^{m+n}}+\frac{b^m+c^{m+n}}{(c+a)^{m+n}}+\frac{c^m+a^{m+n}}{(a+b)^{m+n}}\ge\frac{3^{n+1}+3(a+b+c)^n}{2^{m+n}(a+b+c)^n}. \qquad (25)$$

With equality if and only if a= b= c= 1/3.

Proof： Assume that $a\ge b\ge c$, so $a+b\ge a+c\ge b+c$, $\frac{1}{b+c}\ge\frac{1}{c+a}\ge\frac{1}{a+b}$,

$a^m\ge b^m\ge c^m$, $\frac{1}{(b+c)^{m+n}}\ge\frac{1}{(c+a)^{m+n}}\ge\frac{1}{(a+b)^{m+n}}$,

According to **Chebyshev's Inequality**, **Cauchy's Inequality** and **power mean inequality**,

$$\frac{a^m}{(b+c)^{m+n}}+\frac{b^m}{(c+a)^{m+n}}+\frac{c^m}{(a+b)^{m+n}}$$

$$=a^m\cdot\frac{1}{(b+c)^{m+n}}+b^m\cdot\frac{1}{(c+a)^{m+n}}+c^m\cdot\frac{1}{(a+b)^{m+n}}$$

$$\ge\frac{1}{3}(a^m+b^m+c^m)[\frac{1}{(b+c)^{m+n}}+\frac{1}{(c+a)^{m+n}}+\frac{1}{(a+b)^{m+n}}]$$

$$\ge\frac{1}{3}\cdot 3(\frac{a+b+c}{3})^m\cdot 3[\frac{1}{3}(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b})]^{m+n}$$

$$\geq (\frac{a+b+c}{3})^m \cdot 3[\frac{1}{3}(\frac{9}{b+c+c+a+a+b})]^{m+n}$$

$$= \frac{3^{n+1}}{2^{m+n}(a+b+c)^n}.$$

According to the conclusions $\frac{b^n}{(b+c)^n}+\frac{c^n}{(c+a)^n}+\frac{a^n}{(a+b)^n} \geq \frac{3}{2^n}$ given by Theorem 5, we can easily prove the following inequality,

$$\frac{b^{m+n}}{(b+c)^{m+n}}+\frac{c^{m+n}}{(c+a)^{m+n}}+\frac{a^{m+n}}{(a+b)^{m+n}} \geq \frac{3}{2^{m+n}}$$

$$\therefore \frac{a^m+b^{m+n}}{(b+c)^{m+n}}+\frac{b^m+c^{m+n}}{(c+a)^{m+n}}+\frac{c^m+a^{m+n}}{(a+b)^{m+n}} \geq \frac{3^{n+1}+3(a+b+c)^n}{2^{m+n}(a+b+c)^n}.$$

Demonstration finished.

IV. Four-dimensional Inequalities

Theorem7: If $a,b,c,d,e \in R_+$ and $a+b+c+d=1$, then:

$$\frac{a^3+b^2}{b+c}+\frac{b^3+c^2}{c+d}+\frac{c^3+d^2}{d+a}+\frac{d^3+a^2}{a+b} \geq \frac{5}{8}. \quad (26)$$

With equality if and only if a= b= c= d= 1/4.

Proof: according to the assumption and the Lemma 2:

$$\sum\frac{a^3+b^2}{b+c}=\sum\frac{a^3}{b+c}+\sum\frac{b^2}{b+c}$$

$$\geq \frac{(a+b+c+d)^3}{4^{3-2}\cdot 2(a+b+c+d)}+\frac{(a+b+c+d)^2}{4^{2-2}\cdot 2(a+b+c+d)}=\frac{1}{4\cdot 2}+\frac{1}{2}=\frac{5}{8}.$$

Demonstration finished.

If we make a further extension, we can easily acquire some sequent conclusions:

Theorem8: If $a,b,c,d \in R^+, a+b+c+d=1, n \in N_+$, and $n \geq 2$, then:

$$\frac{a^{n+1}+b^n}{b+c}+\frac{b^{n+1}+c^n}{c+d}+\frac{c^{n+1}+d^n}{d+a}+\frac{d^{n+1}+a^n}{a+b} \geq \frac{10}{4^n}. \quad (27)$$

With equality if and only if a= b= c= d= 1/4.

Proof: according to the assumption and the Lemma 2:

$$\sum\frac{a^{n+1}+b^n}{b+c}=\sum\frac{a^{n+1}}{b+c}+\sum\frac{b^n}{b+c}$$

$$\geq \frac{(a+b+c+d)^{n+1}}{4^{n+1-2}\cdot 2(a+b+c+d)}+\frac{(a+b+c+d)^n}{4^{n-2}\cdot 2(a+b+c+d)}$$

$$= \frac{1}{4^{n-1}\cdot 2} + \frac{1}{4^{n-2}\cdot 2} = \frac{1+4}{4^{n-1}\cdot 2} = \frac{5}{4^{n-1}\cdot 2} = \frac{10}{4^{n-1}\cdot 4} = \frac{10}{4^{n}}.$$

Demonstration finished.

Annotation: In Formula (27), when n=2, it equals to Formula (26).

Theorem9: If $a,b,c,d,\lambda_1,\lambda_2,\mu_1,\mu_2 \in R_+$, $a+b+c+d=1$, $n \in N_+$,

and $n \geq 2, \lambda_1, \lambda_2, \mu_1, \mu_2 \geq 1$, then:

$$\frac{\mu_1 a^{n+1}+\mu_2 b^n}{\lambda_1 b+\lambda_2 c} + \frac{\mu_1 b^{n+1}+\mu_2 c^n}{\lambda_1 c+\lambda_2 d} + \frac{\mu_1 c^{n+1}+\mu_2 d^n}{\lambda_1 d+\lambda_2 a} + \frac{\mu_1 d^{n+1}+\mu_2 a^n}{\lambda_1 a+\lambda_2 b} \geq \frac{4\mu_2+\mu_1}{(\lambda_1+\lambda_2)\cdot 4^{n-1}} . \quad (28)$$

With equality if and only if a= b= c= d= 1/4.

Proof: according to the assumption and the Lemma 2:

$$\sum \frac{\mu_1 a^{n+1}+\mu_2 b^n}{\lambda_1 b+\lambda_2 c} = \mu_1 \sum \frac{a^{n+1}}{\lambda_1 b+\lambda_2 c} + \mu_2 \sum \frac{b^n}{\lambda_1 b+\lambda_2 c}$$

$$\geq \mu_1 \cdot \frac{(a+b+c+d)^{n+1}}{4^{n+1-2}\cdot(\lambda_1+\lambda_2)\cdot(a+b+c+d)} + \mu_2 \cdot \frac{(a+b+c+d)^n}{4^{n-2}\cdot(\lambda_1+\lambda_2)\cdot(a+b+c+d)}$$

$$= \mu_1 \cdot \frac{1}{4^{n-1}\cdot(\lambda_1+\lambda_2)} + \mu_2 \cdot \frac{4}{4^{n-1}\cdot(\lambda_1+\lambda_2)} = \frac{4\mu_2+\mu_1}{4^{n-1}\cdot(\lambda_1+\lambda_2)}.$$

Demonstration finished.

Theorem10: If $a,b,c,d,\lambda_1,\lambda_2,\mu_1,\mu_2 \in R_+, a+b+c+d=1$, $n,m \in N_+$,

and $n \geq 2$, $m \geq 2$, $\lambda_1, \lambda_2, \mu_1, \mu_2 \geq 1$, then:

$$\frac{\mu_1 a^m+\mu_2 b^n}{\lambda_1 b+\lambda_2 c} + \frac{\mu_1 b^m+\mu_2 c^n}{\lambda_1 c+\lambda_2 d} + \frac{\mu_1 c^m+\mu_2 d^n}{\lambda_1 d+\lambda_2 a} + \frac{\mu_1 d^m+\mu_2 a^n}{\lambda_1 a+\lambda_2 b} \geq \frac{16\mu_1}{4^m(\lambda_1+\lambda_2)} + \frac{16\mu_2}{4^n(\lambda_1+\lambda_2)}. \quad (29)$$

With equality if and only if a= b= c= d= 1/4.

Proof: according to the assumption and the Lemma 2:

$$\sum \frac{\mu_1 a^m+\mu_2 b^n}{\lambda_1 b+\lambda_2 c} = \mu_1 \sum \frac{a^m}{\lambda_1 b+\lambda_2 c} + \mu_2 \sum \frac{b^n}{\lambda_1 b+\lambda_2 c}$$

$$\geq \mu_1 \cdot \frac{(a+b+c+d)^m}{4^{m-2}\cdot(\lambda_1+\lambda_2)\cdot(a+b+c+d)} + \mu_2 \cdot \frac{(a+b+c+d)^n}{4^{n-2}\cdot(\lambda_1+\lambda_2)\cdot(a+b+c+d)}$$

$$= \mu_1 \cdot \frac{1}{4^{m-2}\cdot(\lambda_1+\lambda_2)} + \mu_2 \cdot \frac{1}{4^{n-2}\cdot(\lambda_1+\lambda_2)} = \frac{16\mu_1}{4^m(\lambda_1+\lambda_2)} + \frac{16\mu_2}{4^n(\lambda_1+\lambda_2)}.$$

Demonstration finished.

Theorem11: If $a,b,c,d \in R_+$, and $a+b+c+d=1$, then:

$$\frac{a^2+b^2+c^2}{d+b+c}+\frac{b^2+a^2+d^2}{c+a+d}+\frac{c^2+a^2+d^2}{b+a+d}+\frac{d^2+b^2+c^2}{a+b+c}\geq 1. \quad (30)$$

With equality if and only if a= b= c= d= 1/4.

Proof: according to the assumption and the Lemma 2:

$$\frac{a^2+b^2+c^2}{d+b+c}+\frac{b^2+a^2+d^2}{c+a+d}+\frac{c^2+a^2+d^2}{b+a+d}+\frac{d^2+b^2+c^2}{a+b+c}$$

$$=\sum\frac{a^2}{d+b+c}+\sum\frac{b^2}{d+b+c}+\sum\frac{c^2}{d+b+c}$$

$$\geq\frac{(a+b+c+d)^2}{4^{2-2}\cdot 3(a+b+c+d)}\cdot 3=1.$$

Demonstration finished.

If we homogenize the formula, we can get the following inequality:

$$\frac{a^2+b^2+c^2}{d+b+c}+\frac{b^2+a^2+d^2}{c+a+d}+\frac{c^2+a^2+d^2}{b+a+d}+\frac{d^2+b^2+c^2}{a+b+c}\geq a+b+c+d.$$

This is a more general inequality.

Theorem12: If $a,b,c,d\in R_+$, then:

$$\frac{a^2+b^2+c^2}{d+b+c}+\frac{b^2+a^2+d^2}{c+a+d}+\frac{c^2+a^2+d^2}{b+a+d}+\frac{d^2+b^2+c^2}{a+b+c}\geq a+b+c+d. \quad (31)$$

With equality if and only if a= b= c= d.

Theorem13 If $a,b,c,d\in R_+$, and $a+b+c+d=1$, then:

$$\frac{a^3+b^2+c^2}{d+b+c}+\frac{b^3+a^2+d^2}{c+a+d}+\frac{c^3+a^2+d^2}{b+a+d}+\frac{d^3+b^2+c^2}{a+b+c}\geq\frac{3}{4}. \quad (32)$$

With equality if and only if a= b= c= d= 1/4.

Proof: according to the assumption and the Lemma 2:

$$\frac{a^3+b^2+c^2}{d+b+c}+\frac{b^3+a^2+d^2}{c+a+d}+\frac{c^3+a^2+d^2}{b+a+d}+\frac{d^3+b^2+c^2}{a+b+c}$$

$$=\sum\frac{a^3}{d+b+c}+\sum\frac{b^2}{d+b+c}+\sum\frac{c^2}{d+b+c}$$

$$\geq\frac{(a+b+c+d)^2}{4^{2-2}\cdot 3(a+b+c+d)}\cdot 2+\frac{(a+b+c+d)^3}{4^{3-2}\cdot 3(a+b+c+d)}=\frac{2}{3}+\frac{1}{12}=\frac{3}{4}.$$

Demonstration finished.

Theorem14 If $a,b,c,d\in R_+, a+b+c+d=1, n\in N_+$, and $n\geq 2$, then:

$$\frac{a^3+b^n+c^n}{d+b+c}+\frac{b^3+a^n+d^n}{c+a+d}+\frac{c^3+a^n+d^n}{b+a+d}+\frac{d^3+b^n+c^n}{a+b+c}\geq\frac{32}{4^n\cdot 3}+\frac{1}{12}. \quad (33)$$

With equality if and only if a= b= c= d= 1/4.

Proof: according to the assumption and the Lemma 2:

$$\frac{a^3+b^n+c^n}{d+b+c}+\frac{b^3+a^n+d^n}{c+a+d}+\frac{c^3+a^n+d^n}{b+a+d}+\frac{d^3+b^n+c^n}{a+b+c}$$

$$=\sum\frac{a^3}{d+b+c}+\sum\frac{b^n}{d+b+c}+\sum\frac{c^n}{d+b+c}$$

$$\geq\frac{(a+b+c+d)^n}{4^{n-2}\cdot 3(a+b+c+d)}\cdot 2+\frac{(a+b+c+d)^3}{4^{3-2}\cdot 3(a+b+c+d)}=\frac{16}{4^n\cdot 3}\cdot 2+\frac{1}{12}=\frac{32}{4^n\cdot 3}+\frac{1}{12}.$$

Demonstration finished.

Theorem15 If $a,b,c,d\in R_+, a+b+c+d=1, n\in N_+$, and $n\geq 2$, then:

$$\frac{a^{n+1}+b^n+c^n}{d+b+c}+\frac{b^{n+1}+a^n+d^n}{c+a+d}+\frac{c^{n+1}+a^n+d^n}{b+a+d}+\frac{d^{n+1}+b^n+c^n}{a+b+c}\geq\frac{12}{4^n}. \qquad (34)$$

With equality if and only if a= b= c= d= 1/4.

Proof: according to the assumption and the Lemma 2:

$$\frac{a^{n+1}+b^n+c^n}{d+b+c}+\frac{b^{n+1}+a^n+d^n}{c+a+d}+\frac{c^{n+1}+a^n+d^n}{b+a+d}+\frac{d^{n+1}+b^n+c^n}{a+b+c}$$

$$=\sum\frac{a^{n+1}}{d+b+c}+\sum\frac{b^n}{d+b+c}+\sum\frac{c^n}{d+b+c}$$

$$\geq\frac{(a+b+c+d)^n}{4^{n-2}\cdot 3(a+b+c+d)}\cdot 2+\frac{(a+b+c+d)^{n+1}}{4^{n-1}\cdot 3(a+b+c+d)}=\frac{16}{4^n\cdot 3}\cdot 2+\frac{4}{4^n\cdot 3}=\frac{36}{4^n\cdot 3}=\frac{12}{4^n}.$$

Demonstration finished.

Theorem16 If $a,b,c,d\in R_+, a+b+c+d=1, n\in N_+, m,n\geq 2$, then:

$$\frac{a^m+b^n+c^n}{d+b+c}+\frac{b^m+a^n+d^n}{c+a+d}+\frac{c^m+a^n+d^n}{b+a+d}+\frac{d^m+b^n+c^n}{a+b+c}\geq\frac{32}{3\cdot 4^n}+\frac{16}{3\cdot 4^m}. \qquad (35)$$

With equality if and only if a= b= c= d= 1/4.

Proof: according to the assumption and the Lemma 2:

$$\frac{a^m+b^n+c^n}{d+b+c}+\frac{b^m+a^n+d^n}{c+a+d}+\frac{c^m+a^n+d^n}{b+a+d}+\frac{d^m+b^n+c^n}{a+b+c}$$

$$=\sum\frac{a^m}{d+b+c}+\sum\frac{b^n}{d+b+c}+\sum\frac{c^n}{d+b+c}$$

$$\geq\frac{(a+b+c+d)^n}{4^{n-2}\cdot 3(a+b+c+d)}\cdot 2+\frac{(a+b+c+d)^m}{4^{m-2}\cdot 3(a+b+c+d)}=\frac{16}{4^n\cdot 3}\cdot 2+\frac{16}{4^m\cdot 3}=\frac{32}{4^n\cdot 3}+\frac{16}{4^m\cdot 3}.$$

Demonstration finished.

V. Five-dimensional Inequalities

Theorem17 If $a,b,c,d,e \in R_+$ and $a+b+c+d+e=1$, then:

$$\frac{a^3+b^2}{b+c}+\frac{b^3+c^2}{c+d}+\frac{c^3+d^2}{d+e}+\frac{d^3+e^2}{e+a}+\frac{e^3+a^2}{a+b} \ge \frac{3}{5}. \qquad (36)$$

With equality if and only if a= b= c= d= e= 1/5.

Proof according to the assumption and the Lemma 3:

$$\sum \frac{a^3+b^2}{b+c}=\sum \frac{a^3}{b+c}+\sum \frac{b^2}{b+c}$$

$$\ge \frac{(a+b+c+d+e)^3}{5^{3-2}\cdot 2(a+b+c+d+e)}+\frac{(a+b+c+d+e)^2}{5^{2-2}\cdot 2(a+b+c+d+e)}=\frac{1}{5\cdot 2}+\frac{1}{2}=\frac{3}{5}.$$

Demonstration finished.

Theorem18: If $a,b,c,d,e \in R_+$ and $a+b+c+d+e=1$, then:

$$\frac{a^4+b^3}{b+c}+\frac{b^4+c^3}{c+d}+\frac{c^4+d^3}{d+e}+\frac{d^4+e^3}{e+a}+\frac{e^4+a^3}{a+b} \ge \frac{3}{25}. \qquad (37)$$

With equality if and only if a= b= c= d= e= 1/5.

Proof: according to the assumption and the Lemma 3:

$$\sum \frac{a^4+b^3}{b+c}=\sum \frac{a^4}{b+c}+\sum \frac{b^3}{b+c}$$

$$\ge \frac{(a+b+c+d+e)^4}{5^{4-2}\cdot 2(a+b+c+d+e)}+\frac{(a+b+c+d+e)^3}{5^{3-2}\cdot 2(a+b+c+d+e)}$$

$$=\frac{1}{50}+\frac{1}{10}=\frac{3}{25}.$$

Demonstration finished.

Theorem19: If $a,b,c,d,e \in R_+$ and $a+b+c+d+e=1$, and $n \ge 2$, then:

$$\frac{a^{n+1}+b^n}{b+c}+\frac{b^{n+1}+c^n}{c+d}+\frac{c^{n+1}+d^n}{d+e}+\frac{d^{n+1}+e^n}{e+a}+\frac{e^{n+1}+a^n}{a+b} \ge \frac{15}{5^n}. \qquad (38)$$

With equality if and only if a= b= c= d= e= 1/5.

Proof: according to the assumption and the Lemma 3:

$$\sum \frac{a^{n+1}+b^n}{b+c}=\sum \frac{a^{n+1}}{b+c}+\sum \frac{b^n}{b+c}$$

$$\ge \frac{(a+b+c+d+e)^{n+1}}{5^{n+1-2}\cdot 2(a+b+c+d+e)}+\frac{(a+b+c+d+e)^n}{5^{n-2}\cdot 2(a+b+c+d+e)}$$

$$=\frac{(a+b+c+d+e)^{n+1}}{5^{n+1-2}\cdot 2(a+b+c+d+e)}+\frac{(a+b+c+d+e)^{n}}{5^{n-2}\cdot 2(a+b+c+d+e)}$$

$$=\frac{1}{5^{n-1}\cdot 2}+\frac{1}{5^{n-2}\cdot 2}=\frac{15}{5^{n}}.$$

Demonstration finished.

Theorem20 If $a,b,c,d,e,n,m\in R_{+}$, $a+b+c+d+e=1$, and $n,m\geq 2$, then:

$$\frac{a^{m}+b^{n}}{b+c}+\frac{b^{m}+c^{n}}{c+d}+\frac{c^{m}+d^{n}}{d+e}+\frac{d^{m}+e^{n}}{e+a}+\frac{e^{m}+a^{n}}{a+b}\geq\frac{25}{2}\left(\frac{1}{5^{m}}+\frac{1}{5^{n}}\right). \quad (39)$$

With equality if and only if a= b= c= d= e= 1/5.

Proof according to the assumption and the Lemma 3:

$$\sum\frac{a^{m}+b^{n}}{b+c}=\sum\frac{a^{m}}{b+c}+\sum\frac{b^{n}}{b+c}$$

$$\geq\frac{(a+b+c+d+e)^{m}}{5^{m-2}\cdot 2(a+b+c+d+e)}+\frac{(a+b+c+d+e)^{n}}{5^{n-2}\cdot 2(a+b+c+d+e)}$$

$$=\frac{1}{2}\left(\frac{25}{5^{m}}+\frac{25}{5^{n}}\right)=\frac{25}{2}\left(\frac{1}{5^{m}}+\frac{1}{5^{n}}\right).$$

Demonstration finished.

Theorem21 If $a,b,c,d,e,\lambda_1,\lambda_2,\mu_1,\mu_2\in R_{+}, a+b+c+d+e=1, n,m\in N_{+}$,

and $n,m\geq 2$, $\lambda_1,\lambda_2,\mu_1,\mu_2\geq 1$, then

$$\frac{\mu_1 a^{m}+\mu_2 b^{n}}{\lambda_1 b+\lambda_2 c}+\frac{\mu_1 b^{m}+\mu_2 c^{n}}{\lambda_1 c+\lambda_2 d}+\frac{\mu_1 c^{m}+\mu_2 d^{n}}{\lambda_1 d+\lambda_2 e}+\frac{\mu_1 d^{m}+\mu_2 e^{n}}{\lambda_1 e+\lambda_2 a}$$

$$+\frac{\mu_1 e^{m}+\mu_2 a^{n}}{\lambda_1 a+\lambda_2 b}\geq\frac{25}{\lambda_1+\lambda_2}\cdot\left(\frac{\mu_1}{5^{m}}+\frac{\mu_2}{5^{n}}\right). \quad (40)$$

With equality if and only if a= b= c= d= e= 1/5.

Proof $\sum\frac{\mu_1 a^{m}+\mu_2 b^{n}}{\lambda_1 b+\lambda_2 c}=\mu_1\sum\frac{a^{m}}{\lambda_1 b+\lambda_2 c}+\mu_2\sum\frac{b^{n}}{\lambda_1 b+\lambda_2 c}$

$$\geq\mu_1\cdot\frac{(a+b+c+d+e)^{m}}{5^{m-2}\cdot(\lambda_1+\lambda_2)(a+b+c+d+e)}+\mu_2\cdot\frac{(a+b+c+d+e)^{n}}{5^{n-2}\cdot(\lambda_1+\lambda_2)(a+b+c+d+e)}$$

$$=\frac{25\mu_1}{5^{m}(\lambda_1+\lambda_2)}+\frac{25\mu_2}{5^{n}(\lambda_1+\lambda_2)}=\frac{25}{\lambda_1+\lambda_2}\cdot\left(\frac{\mu_1}{5^{m}}+\frac{\mu_2}{5^{n}}\right).$$

If we extend them to multidimensional inequalities, we can acquire the following theorems.

VI. Multi-dimensional Inequalities

Theorem22 If $a_1, a_2, \cdots, a_n \in R_+, a_1 + a_2 + \cdots + a_n = 1$ and note $a_{n+1} = a_1, a_{n+2} = a_2$, $n \geq 2$

then

$$\sum_{i=1}^{n} \frac{a_i^3 + a_{i+1}^2}{a_{i+1} + a_{i+2}} \geq \frac{n+1}{2n}. \tag{41}$$

With equality if and only if $a_1 = a_2 = \ldots = a_n = 1/n$.

Proof according to the assumption and the Lemma 4

$$\sum_{i=1}^{n} \frac{a_i^3 + a_{i+1}^2}{a_{i+1} + a_{i+2}} = \sum_{i=1}^{n} \frac{a_i^3}{a_{i+1} + a_{i+2}} + \sum_{i=1}^{n} \frac{a_{i+1}^2}{a_{i+1} + a_{i+2}}$$

$$\geq \frac{(\sum_{i=1}^{n} a_i)^3}{n^{3-2} \cdot 2\sum_{i=1}^{n} a_i} + \frac{(\sum_{i=1}^{n} a_i)^2}{n^{2-2} \cdot 2\sum_{i=1}^{n} a_i} = \frac{1}{2n} + \frac{1}{2} = \frac{1+n}{2n}.$$

Demonstration finished.

Theorem23 If $a_1, a_2, \cdots, a_n \in R_+, a_1 + a_2 + \cdots + a_n = 1$ and note $a_{n+1} = a_1, a_{n+2} = a_2$, $n \geq 2$

then

$$\sum_{i=1}^{n} \frac{a_i^4 + a_{i+1}^3}{a_{i+1} + a_{i+2}} \geq \frac{1+n}{2n^2}. \tag{42}$$

With equality if and only if $a_1 = a_2 = \ldots = a_n = 1/n$.

Proof: according to the assumption and the Lemma 4:

$$\sum_{i=1}^{n} \frac{a_i^4 + a_{i+1}^3}{a_{i+1} + a_{i+2}} = \sum_{i=1}^{n} \frac{a_i^4}{a_{i+1} + a_{i+2}} + \sum_{i=1}^{n} \frac{a_{i+1}^3}{a_{i+1} + a_{i+2}}$$

$$\geq \frac{\left(\sum_{i+1}^{n} a_i\right)^4}{n^{4-2} \cdot 2\sum_{i+1}^{n} a_i} + \frac{\left(\sum_{i+1}^{n} a_i\right)^3}{n^{3-2} \cdot 2\sum_{i+1}^{n} a_i} = \frac{1}{2n^2} + \frac{1}{2n} = \frac{1+n}{2n^2}.$$

Theorem24: If $a_1, a_2, \cdots, a_n \in R_+, m \in N_+$, and $a_1 + a_2 + \cdots + a_n = 1 \quad m \geq 2$,

note $a_{n+1} = a_1, a_{n+2} = a_2$,then:

$$\sum_{i=1}^{n} \frac{a_i^{m+1} + a_{i+1}^m}{a_{i+1} + a_{i+2}} \geq \frac{n+n^2}{2n^m}. \tag{43}$$

With equality if and only if $a_1 = a_2 = \ldots = a_n = 1/n$.

Proof: according to the assumption and the Lemma 4:

$$\sum_{i=1}^{n}\frac{a_i^{m+1}+a_{i+1}^m}{a_{i+1}+a_{i+2}}=\sum_{i=1}^{n}\frac{a_i^{m+1}}{a_{i+1}+a_{i+2}}+\sum_{i=1}^{n}\frac{a_{i+1}^m}{a_{i+1}+a_{i+2}}$$

$$\geq\frac{\left(\sum_{i+1}^{n}a_i\right)^{m+1}}{n^{m-1}\cdot 2\sum_{i+1}^{n}a_i}+\frac{\left(\sum_{i+1}^{n}a_i\right)^{m}}{n^{m-2}\cdot 2\sum_{i+1}^{n}a_i}=\frac{n}{2n^m}+\frac{n^2}{2n^m}=\frac{n+n^2}{2n^m}.$$

Demonstration finished.

Theorem25 If $a_1,a_2,\cdots,a_n\in R_+,m\in N_+$, and $a_1+a_2+\cdots+a_n=1$

$m\geq 2$,note $a_{n+1}=a_1,a_{n+2}=a_2,\lambda_1,\lambda_2,\mu_1,\mu_2\in R_+,\lambda_1,\lambda_2,\mu_1,\mu_2\geq 1$,then:

$$\sum_{i=1}^{n}\frac{\mu_1 a_i^{m+1}+\mu_2 a_{i+1}^m}{\lambda_1 a_{i+1}+\lambda_2 a_{i+2}}\geq\frac{\mu_1 n+\mu_2 n^2}{(\lambda_1+\lambda_2)n^m}. \qquad (44)$$

With equality if and only if $a_1=a_2=\ldots=a_n=1/n$.

Proof: according to the assumption and the Lemma 4:

$$\sum_{i=1}^{n}\frac{\mu_1 a_i^{m+1}+\mu_2 a_{i+1}^m}{\lambda_1 a_{i+1}+\lambda_2 a_{i+2}}=\mu_1\sum_{i=1}^{n}\frac{a_i^{m+1}}{\lambda_1 a_{i+1}+\lambda_2 a_{i+2}}+\mu_2\sum_{i=1}^{n}\frac{a_{i+1}^m}{\lambda_1 a_{i+1}+\lambda_2 a_{i+2}}$$

$$\geq\frac{\mu_1\left(\sum_{i+1}^{n}a_i\right)^{m+1}}{n^{m-1}\cdot(\lambda_1+\lambda_2)\sum_{i+1}^{n}a_i}+\frac{\mu_2\left(\sum_{i+1}^{n}a_i\right)^{m}}{n^{m-2}\cdot(\lambda_1+\lambda_2)\sum_{i+1}^{n}a_i}$$

$$=\frac{\mu_1 n}{(\lambda_1+\lambda_2)n^m}+\frac{\mu_2 n^2}{(\lambda_1+\lambda_2)n^m}=\frac{\mu_1 n+\mu_2 n^2}{(\lambda_1+\lambda_2)n^m}.$$

Demonstration finished.

Theorem26: If $a_1,a_2,\cdots,a_n\in R_+,m,k\in N_+$, and $a_1+a_2+\cdots+a_n=1$, and

$m,k\geq 2$,note $a_{n+1}=a_1,a_{n+2}=a_2,\lambda_1,\lambda_2,\mu_1,\mu_2\in R_+,\lambda_1,\lambda_2,\mu_1,\mu_2\geq 1$,then:

$$\sum_{i=1}^{n}\frac{\mu_1 a_i^{k}+\mu_2 a_{i+1}^m}{\lambda_1 a_{i+1}+\lambda_2 a_{i+2}}\geq\frac{n^2}{(\lambda_1+\lambda_2)}\cdot\left(\frac{\mu_1}{n^k}+\frac{\mu_2}{n^m}\right). \qquad (45)$$

With equality if and only if $a_1=a_2=\ldots=a_n=1/n$.

Proof: according to the assumption and the Lemma 4:

$$\sum_{i=1}^{n}\frac{\mu_1 a_i^{k}+\mu_2 a_{i+1}^m}{\lambda_1 a_{i+1}+\lambda_2 a_{i+2}}=\mu_1\sum_{i=1}^{n}\frac{a_i^{k}}{\lambda_1 a_{i+1}+\lambda_2 a_{i+2}}+\mu_2\sum_{i=1}^{n}\frac{a_{i+1}^m}{\lambda_1 a_{i+1}+\lambda_2 a_{i+2}}$$

$$\geq \frac{\mu_1\left(\sum_{i+1}^{n} a_i\right)^k}{n^{k-2}\cdot(\lambda_1+\lambda_2)\sum_{i+1}^{n} a_i}+\frac{\mu_2\left(\sum_{i+1}^{n} a_i\right)^m}{n^{m-2}\cdot(\lambda_1+\lambda_2)\sum_{i+1}^{n} a_i}$$

$$=\frac{\mu_1 n^2}{(\lambda_1+\lambda_2)n^k}+\frac{\mu_2 n^2}{(\lambda_1+\lambda_2)n^m}=\frac{n^2}{(\lambda_1+\lambda_2)}\cdot\left(\frac{\mu_1}{n^k}+\frac{\mu_2}{n^m}\right).$$

Demonstration finished.

VII. Enhancement

Passage [17] makes an enhancement on Formula (1) to get:

Assume that a, b, c are positive, and satisfy $a+b+c=1$, so:

$$\sum\frac{a^2+b}{b+c}\geq 2+\frac{3}{8}\sum(a-b)^2 . \tag{46}$$

Passage [5] makes an enhancement on Formula (46) to get:

$$\sum\frac{a^2+b}{b+c}\geq 2+\left(1-\frac{1}{\sqrt{16\sqrt{2}+13}}\right)\sum(a-b)^2 . \tag{47}$$

We notice that $1-\frac{1}{\sqrt{16\sqrt{2}+13}}$ =0.832464119…and Passage [5] also suggests that Formula (47) is not the best. Thus we out forward the following questions:

If $a,b,c\in R_+$, and $a+b+c=1$, seeking the maximum to set up the following permanent establishment of inequality,

Assume that a, b, c are positive, and satisfy $a+b+c=1$

$$\sum\frac{a^2+b}{b+c}\geq 2+\lambda\sum(a-b)^2 . \tag{48}$$

The study found that the (47) equation can be improved to:

Theorem 27 If $a,b,c\in R_+$,and $a+b+c=1$,then:

$$\sum\frac{a^2+b}{b+c}\geq 2+\frac{27}{32}\sum(a-b)^2 . \tag{49}$$

Proof: use $a+b, b+c, c+a$ to replace the x, y, z in lemma 5, then:

If $a,b,c\in R_+$, then:

$$\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\geq 3+\frac{27}{32}\cdot\frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a+b+c)^2} . \tag{50}$$

$$\sum\frac{a^2+b}{b+c}=\sum\frac{a[1-(b+c)]+b}{b+c}=\sum\frac{a+b}{b+c}-\sum a$$

$$\geq 3+\frac{27}{32}\cdot\frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a+b+c)^2}-1$$

$$=2+\frac{27}{32}[(a-b)^2+(b-c)^2+(c-a)^2].$$

Demonstration finished.

$\frac{27}{32}$=0.84375>0.832464119…,which means Formula (49) is more powerful than Formula (47).

With the assistance of Formula (46), Passage [18] prove the following inequality:

Assume that a, b, c are positive, and satisfy $a+b+c=1$, so:

$$\sum\frac{a^3+b}{b+c}\ge\frac{5}{3}. \quad (51)$$

What's more, we can extend Formula (51) to the following inequality,

Assume that a, b, c are positive, and satisfy $a+b+c=1$, so:

$$\sum\frac{a^3+b}{b+c}\ge\frac{5}{3}+\frac{1}{24}[(a-b)^2+(b-c)^2+(c-a)^2]. \quad (52)$$

With the assistance of Formula (49), we can make a further enhancement on Formula (52) to get

Theorem 28: If a, b, c are positive, and satisfy $a+b+c=1$, then:

$$\sum\frac{a^3+b}{b+c}\ge\frac{5}{3}+\frac{49}{96}\sum(a-b)^2. \quad (53)$$

Proof $\sum\frac{a^3+b}{b+c}=\sum\frac{a^2+b}{b+c}-(a^2+b^2+c^2)\ge 2+\frac{27}{32}\sum(a-b)^2-(a^2+b^2+c^2)$

$$=\frac{5}{3}+\frac{27}{32}\sum(a-b)^2+\frac{1}{3}(a+b+c)^2-(a^2+b^2+c^2)$$

$$=\frac{5}{3}+\frac{27}{32}\sum(a-b)^2-\frac{1}{3}\sum(a-b)^2$$

$$=\frac{5}{3}+\frac{49}{96}\sum(a-b)^2.$$

Demonstration finished.

We notice that $\frac{49}{96}$=0.510416666…>0.0416666666…=$\frac{1}{24}$. Thus, Formula (53) is more powerful than Formula (52).

Passage [5] and Passage [16] respectively use different method to prove Formula (54):

Assume that a, b, c are positive, and satisfy $a+b+c=1$, so:

$$\sum\frac{a^4+b}{b+c}\ge\frac{14}{9}. \quad (54)$$

With the assistance of Formula (49), we can make a further enhancement on Formula（54）to get

Theorem 29 If a, b, c are positive, and satisfy $a+b+c=1$, then:

$$\sum\frac{a^4+b}{b+c}\ge\frac{14}{9}+\frac{1}{96}\sum(a-b)^2. \quad (55)$$

Prove ： $\because\frac{a^4+b}{b+c}=\frac{a^3[1-(b+c)]+b}{b+c}=\frac{a^3+b}{b+c}-a^3$,

$$\therefore\sum\frac{a^4+b}{b+c}=\sum\frac{a^3+b}{b+c}-\sum a^3.$$

We notice that

$$\sum a^3-3abc=\frac{1}{2}(a+b+c)\sum(a-b)^2. \quad (56)$$

Combining Formula (56) and Formula (53), we can get

$$\therefore \sum \frac{a^4+b}{b+c} = \sum \frac{a^3+b}{b+c} - \sum a^3$$

$$\ge \frac{5}{3} + \frac{49}{96}\sum (a-b)^2 - 3abc - \frac{1}{2}\sum (a-b)^2$$

$$\ge \frac{5}{3} + (\frac{49}{96} - \frac{1}{2})\sum (a-b)^2 - 3\cdot(\frac{a+b+c}{3})^3$$

$$= \frac{5}{3} + \frac{1}{96}\sum (a-b)^2 - 3\cdot(\frac{1}{3})^3$$

$$= \frac{14}{9} + \frac{1}{96}\sum (a-b)^2$$

Demonstration finished.

We notice that $\frac{1}{96}$>0, which means Formula (56) is more powerful than Formula (55). So we can get another kind of enhancement and certification of Formula (53).

VIII. Unsolved Problems

Based on Formula (1) and Formula (51), Passage [18] puts forward several supposes.

Conjecture 1: If $a,b,c \in R_+$, and $a+b+c=1$, and $n \in N_+$, then:

$$\sum \frac{a^{n+1}+b}{b+c} \ge \frac{1+3^n}{2\cdot 3^{n-1}}. \qquad (57)$$

Passage [5] points out that when $n=1,2,3$, Formula (57) is tenable, while when $n \ge 6$, Formula(57) is incorrect. However, we still need to make a deeper research to find out whether when n=4 or n=5 Formula (57) is tenable.

With the assistance of program Bottema2009 made by Professor Lu Yang from the Chinese Academy of Science, we can easily find that when n=4 or n=5, Formula（57）is tenable.
But we are still unable to offer artificial certification, so we put forward the following two supposes.

Conjecture 2: If a, b, c are positive, and satisfy $a+b+c=1$, then:

$$\sum \frac{a^5+b}{b+c} \ge \frac{41}{27}. \qquad (58)$$

Conjecture 3: If a, b, c are positive, and satisfy $a+b+c=1$, then:

$$\sum \frac{a^6+b}{b+c} \ge \frac{122}{81}. \qquad (59)$$

With the assistance of program Bottema2009, we also found that the Inequality (49) can be enhanced to be Suppose 4.

Conjecture 4: If a, b, c are positive, and satisfy $a+b+c=1$,then:

$$\sum \frac{a^2+b}{b+c} \ge 2 + k\sum (a-b)^2. \qquad (60)$$

$k_{\max}$ =0.952078828…is the root of the equation:

$$192k^6 - 1152k^5 + 3296k^4 - 5264k^3 + 4940k^2 - 2004k + 23 = 0$$

Postscript: Recently passage [22] has proved Suppose 3. With the help of Suppose3, we can easily prove that Suppose 2 is tenable.

$$\sum\frac{a^5+b}{b+c}=\sum\frac{a^5(a+b+c)+b}{b+c}=\sum\frac{a^6+b}{b+c}+\sum a^5\ge\frac{122}{81}+3(\frac{1}{3}\sum a)^5$$

$$=\frac{122}{81}+3\times(\frac{1}{3})^5=\frac{122}{81}+\frac{1}{81}=\frac{41}{27}.$$

Demonstration finished.

Thus, Suppose 1 has been thoroughly solved. Now only Suppose 4 is left unsolved for readers who are interested in them to discuss.

Conjecture 5: If a, b, c are positive, and satisfy $a+b+c=1$, then:

$$\sum\frac{a^3+b}{b+c}\ge\frac{5}{3}+k\sum(a-b)^2. \qquad (61)$$

$k_{\max}$ =0.618745495…is the root of the equation

$$46656k^6-186624k^5+412128k^4-487728k^3+360180k^2-10548k-61931=0$$

Conjecture 6: If a, b, c are positive, and satisfy $a+b+c=1$,then:

$$\sum\frac{a^4+b}{b+c}\ge\frac{14}{9}+k\sum(a-b)^2\cdot \qquad (62)$$

$k_{\max}$ =0.334432193…is the root of the equation

$$11337408k^6-16376256k^5+26069040k^4-8440848k^3+6231384k^2$$
$$+2566804k-9238901=0$$

Conjecture 7: If a, b, c are positive, and satisfy $a+b+c=1$,then:

$$\sum\frac{a^5+b}{b+c}\ge\frac{41}{27}+k\sum(a-b)^2\cdot \qquad (63)$$

$k_{\max}$ =0.144954956…is the root of the equation

36714920303220705142379642 88k^{15} +2718490549611928136931134 6688k^{14} +602736608311206576087399137 28k^{13} -3737099105511443856469839 0528k^{12} -38625379039358090406761988 0960k^{11} -5493342109135928365223631 91296k^{10} +3508415602890263149873141 18656k^9 +2216456757427941779550577 164288k^8 +3418658793193231988534923 296768k^7 +281676433780987360549477348 1472k^6 +1313680169948284601618889 178368k^5 +325493187206561578334244994 560k^4 +4240419187833476797184532 0000k^3 -22078156795104082492082315 52k^2 -8281757950620158846749377 60k -22166100355575396002616160 7=0

Comjecture 8: If a, b, c are positive, and satisfy $a+b+c=1$, then:

$$\sum\frac{a^6+b}{b+c}\ge\frac{122}{81}+k\sum(a-b)^2\cdot \qquad (64)$$

$k_{\max}$ =0.02584295685…is the root of the equation

-1275458120676915 49+4482785340037365228k +1673156908872111373 2k^2 +29831803345471958208k^3 +1721509935025431499 2k^4 +4191716669843968k^5 +74732015308412849472k^6 +7948075077853747430 4k^7 +3082254302475769420 8k^8 +3794985346768570368k^9 =0

Particularly, when $k=\frac{1}{39}$ in the inequality (59), we can get:

$$\sum\frac{a^6+b}{b+c}\ge\frac{122}{81}+\frac{1}{39}\sum(a-b)^2\cdot \qquad (65)$$

Comjecture 9: If $x, y, z \in R_+$, then:

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geq 3+3\sqrt[3]{2} \cdot \frac{(x-y)^2+(y-z)^2+(z-x)^2}{(x+y+z)^2}. \tag{66}$$

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