THE COMPLETION OF THE MANIFOLD OF RIEMANNIAN METRICS

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Abstract

We give a description of the completion of the manifold of all smooth Riemannian metrics on a fixed smooth, closed, finite-dimensional, orientable manifold with respect to a natural metric called the L^2 metric. The primary motivation for studying this problem comes from Teichmüller theory, where similar considerations lead to a completion of the well-known Weil-Petersson metric. We give an application of the main theorem to the completions of Teichmüller space with respect to a class of metrics that generalize the Weil-Petersson metric.

Contents

1. Introduction

This is the second in a pair of papers studying the metric geometry of the Fréchet manifold $\mathcal{M}$ of all smooth Riemannian metrics on a smooth, closed, finite-dimensional, orientable manifold M. The manifold $\mathcal{M}$ carries a natural weak Riemannian metric called the L^2 metric, defined in the next section. In the first paper [5], we showed that the L^2 metric induces a metric space structure on $\mathcal{M}$ (a nontrivial statement for weak Riemannian metrics; see Section 2.1.3). In this paper, we will give the following description of the metric completion $\overline{\mathcal{M}}$ of $\mathcal{M}$ with respect to the L^2 metric. In the theorem, and the remainder of the paper, a positive-semidefinite symmetric $(0, 2)$-tensor based at $x \in M$ is called *nondegenerate* if it induces a positive-definite scalar product on $T_x M$, and *degenerate* otherwise.

Theorem. *Let $\mathcal{M}_f$ denote the space of all measurable, symmetric, finite-volume $(0, 2)$-tensor fields on M that induce a positive semidefinite scalar product on each tangent space of M. Then there is a natural identification*

$$\overline{\mathcal{M}} \cong \mathcal{M}_f / \sim.$$

Here, for $g_0, g_1 \in \mathcal{M}_f$, we say $g_0 \sim g_1$ if and only if the following statement holds for almost every $x \in M$:

If at least one of $g_0(x)$ or $g_1(x)$ is nondegenerate, then $g_0(x) = g_1(x)$.

Note that while $\mathcal{M}$ is a space of smooth objects, we must add in points corresponding to extremely singular objects in order to complete it. This is a reflection of the fact that the L^2 metric is a *weak* rather than a *strong* Riemannian metric. That is, the topology it induces on the tangent spaces—the L^2 topology—is weaker than the C^∞ topology coming from the manifold structure. In essence, the incompleteness of the tangent spaces then carries over to the manifold itself.

The manifold of Riemannian metrics—along with geometric structures on it—has been considered in several contexts. It originally arose in general relativity [6], and was subsequently studied by mathematicians [7, 9, 10]. In particular, the Riemannian geometry of the L^2 metric is well understood—its curvature, geodesics, and Jacobi fields

are explicitly known. The metric geometry of the L^2 metric, though, was not as clear up to this point, and this paper seeks to illuminate one aspect of that.

Our motivation for studying the completion of $\mathcal{M}$—besides the intrinsic interest to Riemannian geometers of investigating this important deformation space—came largely from Teichmüller theory. If the base manifold M is a closed Riemann surface of genus larger than one, the work of Fischer and Tromba [26] gives an identification of the Teichmüller space of M with $\mathcal{M}_{-1}/\mathcal{D}_0$, where $\mathcal{M}_{-1} \subset \mathcal{M}$ is the submanifold of hyperbolic metrics and $\mathcal{D}_0$ is the group of diffeomorphisms of M that are homotopic to the identity, acting on $\mathcal{M}_{-1}$ by pull-back. The L^2 metric restricted to $\mathcal{M}_{-1}$ descends to the Weil-Petersson metric on Teichmüller space, and its completion consists of adding in points corresponding to certain cusped hyperbolic metrics. The action of the mapping class group on Teichmüller space extends to this completion, and the quotient is homeomorphic to the Deligne-Mumford compactification of the moduli space of M. In Section 6, inspired by [11, 12], we generalize the Weil-Petersson metric and use the above theorem to formulate a condition on the completion of these generalized Weil-Petersson metrics.

The paper is organized as follows:

In Section 2, we recall the necessary background on the manifold of metrics, the L^2 metric, and completions of metric spaces. We also review some nonstandard geometric notions and fix notation and conventions for the paper.

In Section 3, we complete what we call amenable subsets of $\mathcal{M}$. They are defined in such a way that we can show that the completion of these subsets with respect to the L^2 *metric* on the subset is the same as with respect to the L^2 *norm* on $\mathcal{M}$ (this will be made precise below). This completion is the first step in a bootstrapping process of understanding the full completion.

In Section 4, we introduce a notion we call ω-*convergence* for Cauchy sequences in $\mathcal{M}$ that describes how a Cauchy sequence converges to an element of $\mathcal{M}_f/\sim$. It is a kind of pointwise a.e.-convergence—except on a subset where the sequence degenerates in a certain way, where no convergence can be demanded of Cauchy sequences. We then use the results of Section 3 to show that this convergence notion gives an injective map from the completion of $\mathcal{M}$ into $\mathcal{M}_f/\sim$. To do this, we need to show two things. First, we prove that every Cauchy sequence in $\mathcal{M}$ has an ω-convergent subsequence. Second, we show in two theorems that two Cauchy sequences are equivalent (in the sense of the completion of a metric space) if and only if they ω-subconverge to the same limit. This section comprises the most technically challenging portion of the paper. It also contains the following result, which is in our eyes one of the most unexpected and striking of the paper:

Proposition. *Suppose that $g_0, g_1 \in \mathcal{M}$, and let $E := \mathrm{carr}(g_1 - g_0) = \{x \in M \mid g_0(x) \neq g_1(x)\}$. Let d be the Riemannian distance function of the L^2 metric $(\cdot, \cdot)$. Then there exists a constant $C(n)$ depending only on $n := \dim M$ such that*

$$d(g_0, g_1) \leq C(n) \left(\sqrt{\mathrm{Vol}(E, g_0)} + \sqrt{\mathrm{Vol}(E, g_1)} \right).$$

In particular, we have

$$\mathrm{diam} \left(\{ \tilde{g} \in \mathcal{M} \mid \mathrm{Vol}(M, \tilde{g}) \leq \delta \} \right) \leq 2 C(n) \sqrt{\delta}.$$

The surprising thing about this proposition is that it says that two metrics can vary wildly, but as long as they do so on a set that has small volume with respect to each, they are close together in the L^2 metric.

In Section 5, we complete the proof of the main result. This is done by continuing the bootstrapping process begun in Section 3 to see that the map defined in Section 4 is in fact surjective. That is, we prove in stages that there are Cauchy sequences ω-converging to elements in ever larger subsets of $\mathcal{M}_f / \sim$.

In Section 6, we give the application to the geometry of Teichmüller space that was mentioned above.

Several different types of sequences and convergence notions enter into this work. For the reader's convenience, we have included an appendix which summarizes the relationships between these different concepts.

2. Preliminaries

2.1. The Manifold of Metrics. For the entirety of the paper, let M denote a fixed closed, orientable, n-dimensional C^∞ manifold. We fix an orientation on M, but all our results are independent of the choice of orientation.

The basic facts about the manifold of Riemannian metrics given in this section can be found in [**4**, §2.5]

We denote by S^2T^*M the vector bundle of symmetric $(0,2)$ tensors over M, and by $\mathcal{S}$ the Fréchet space of C^∞ sections of S^2T^*M. The space $\mathcal{M}$ of Riemannian metrics on M is an open subset of $\mathcal{S}$, and hence it is trivially a Fréchet manifold, with tangent space at each point canonically identified with $\mathcal{S}$. (For a detailed treatment of Fréchet manifolds, see, for example, [13]. For a more thorough treatment of the differential topology and geometry of $\mathcal{M}$, see [7].)

2.1.1. The L^2 Metric. $\mathcal{M}$ carries a natural Riemannian metric $(\cdot,\cdot)$, called the L^2 *metric*, induced by integration from the natural scalar product on S^2T^*M. Given any $g \in \mathcal{M}$ and $h, k \in \mathcal{S} \cong T_g\mathcal{M}$, we define

$$(h,k)_g := \int_M \mathrm{tr}_g(hk)\, d\mu_g.$$

Here, $\mathrm{tr}_g(hk)$ is given in local coordinates by

$$\mathrm{tr}(g^{-1}hg^{-1}k) = g^{ij}h_{il}g^{lm}k_{jm},$$

and μ_g denotes the volume form induced by g.

Remark 2.1. Alternatively, we may express $\mathrm{tr}_g(hk)$ without reference to coordinates as follows. For any $h \in S^2T_x^*M$, there is a unique $(1,1)$-tensor H (an endomorphism of T_xM) such that $h(X,Y) = g(H(X),Y)$ for all $X,Y \in T_xM$. Then $\mathrm{tr}_g(hk) = \mathrm{tr}(HK)$, where K is defined analogously to H and HK simply denotes the composition of endomorphisms.

Throughout the paper, we use the notation d for the distance function induced from $(\cdot,\cdot)$ by taking the infimum of the lengths of paths between two given points.

The L^2 metric is a weak Riemannian metric, which means that its induced topology on the tangent spaces of $\mathcal{M}$ is weaker than the manifold topology. This leads to some phenomena that are unfamiliar from the world of finite-dimensional Riemannian geometry, or even strong Riemannian metrics on Hilbert manifolds. For instance, the L^2 metric does not *a priori* induce a metric space structure on $\mathcal{M}$. In [5], we nevertheless showed directly that $(\mathcal{M},d)$ is a metric space, but other strange phenomena occur—for instance, the metric space topology of $(\mathcal{M},d)$ is weaker than the manifold topology of $\mathcal{M}$. Indeed, in Lemma 5.11 we will see the following: When considered as a subset of its completion, $\mathcal{M}$ contains no open d-ball around *any* point! For more information on weak Riemannian metrics, see [4, §2.4], [5, §3]

The basic Riemannian geometry of $(\mathcal{M}, (\cdot,\cdot))$ is relatively well understood. For example, it is known that the sectional curvature of $\mathcal{M}$ is nonpositive [9, Cor. 1.17], and the geodesics of $\mathcal{M}$ are known explicitly [9, Thm. 2.3], [10, Thm. 3.2].

We will also consider related structures restricted to a point $x \in M$. Let $\mathcal{S}_x := S^2T_x^*M$ denote the vector space of symmetric $(0,2)$-tensors at

x, and let $\mathcal{M}_x \subset \mathcal{S}_x$ denote the open subset of tensors inducing a positive definite scalar product on T_xM. Then $\mathcal{M}_x$ is an open submanifold of $\mathcal{S}_x$, and its tangent space at each point is canonically identified with $\mathcal{S}_x$. For each $g \in \mathcal{M}_x$, we define a scalar product $\langle \cdot, \cdot \rangle_g$ on $T_g\mathcal{M}_x \cong \mathcal{S}_x$ by setting, for all $h, k \in \mathcal{S}_x$,

$$\langle h, k \rangle_g := \mathrm{tr}_g(hk).$$

Then $\langle \cdot, \cdot \rangle$ defines a Riemannian metric on the finite-dimensional manifold $\mathcal{M}_x$.

For each $g \in \mathcal{M}$, we denote the L^2 norm induced by g on $\mathcal{S}$ with $\| \cdot \|_g$, that is, $\|h\|_g := \sqrt{(h, h)_g}$. For any $g_0, g_1 \in \mathcal{M}$, the norms $\| \cdot \|_{g_0}$ and $\| \cdot \|_{g_1}$ are equivalent [20, §IX.2]. (As Ebin pointed out [7, §4], this statement even holds if g_0 and g_1 are only required to be continuous.)

2.1.2. Geodesics. As noted above, the geodesic equation of $\mathcal{M}$ can be solved explicitly. We will not need the full expression for an arbitrary geodesic for our purposes, but rather only for very special geodesics.

We denote by $\mathcal{P} \subset C^\infty(M)$ the group of strictly positive smooth functions on M. This is a Fréchet Lie group that acts on $\mathcal{M}$ by pointwise multiplication. For any $g_0 \in \mathcal{M}$, the next proposition gives the geodesics of the orbit $\mathcal{P} \cdot g_0$.

Proposition 2.2 ([9, Prop. 2.1]). *The geodesic γ in $\mathcal{M}$ starting at $g_0 \in \mathcal{M}$ with initial tangent vector ρg_0, where $\rho \in C^\infty(M)$, is given by*

$$(2.1) \qquad\qquad \gamma(t) = \left(1 + n\frac{t}{4}\rho\right)^{4/n} g_0.$$

In particular, $\mathcal{P} \cdot g_0$ is a totally geodesic submanifold, and the exponential mapping $\exp_{g_0}$ is a diffeomorphism from the open set $U \cdot g_0 \subset T_{g_0}(\mathcal{P} \cdot g_0)$—where U is the set of functions ρ satisfying $\rho > -4/n$—onto $\mathcal{P} \cdot g_0$.

2.1.3. Metric Space Structures on $\mathcal{M}$. In [5], we proved the following theorem:

Theorem 2.3. $(\mathcal{M}, d)$, *where*

$$d(g_0, g_1) =$$

$$\inf \left\{ L(\gamma) = \int_0^1 \|\gamma'(t)\|_{\gamma(t)} \, dt \;\middle|\; \begin{array}{l} \gamma : [0,1] \to \mathcal{M} \text{ piecewise differ-} \\ \text{entiable, } \gamma(0) = g_0, \; \gamma(1) = g_1 \end{array} \right\},$$

for $g_0, g_1 \in \mathcal{M}$, is the distance function of the L^2 metric, is a metric space.

Convention 2.4. In the remainder of the paper, whenever $\mathcal{M}$ is (implicitly or otherwise) referred to as a metric space, the metric d is implied unless we explicitly state otherwise. For example, a "Cauchy sequence in $\mathcal{M}$" refers to a d-Cauchy sequence.

As mentioned above, the fact that the L^2 metric is a weak Riemannian metric means that general theorems imply only that d is a pseudometric. In fact, there are examples [**17**, **18**] of weak Riemannian metrics where the induced distance between any two points is always zero!

To prove Theorem 2.3, we defined a function on $\mathcal{M} \times \mathcal{M}$ that was manifestly a metric (in the sense of metric spaces) and showed that this metric bounded d from below in some way. We also showed that the function $\mathcal{M} \to \mathbb{R}$ sending a metric to the square root of its total volume is Lipschitz with respect to d. These results will play a role in what is to come, so we review them here, along with the relevant definitions.

First, we have the lemma on Lipschitz continuity of the square root of the volume function.

Lemma 2.5 ([**5**, Lemma 17]). *Let $g_0, g_1 \in \mathcal{M}$. Then for any measurable subset $Y \subseteq M$,*

$$\left| \sqrt{\mathrm{Vol}(Y, g_1)} - \sqrt{\mathrm{Vol}(Y, g_0)} \right| \leq \frac{\sqrt{n}}{4} d(g_0, g_1).$$

Next, we define the metric Θ_M on $\mathcal{M}$ that was mentioned above and state the lower bound it provides on d.

Definition 2.6. For each $x \in M$, consider $\mathcal{M}_x = \{\tilde{g} \in \mathcal{S}_x \mid \tilde{g} > 0\}$ (cf. Section 2.1.1). For any fixed $g \in \mathcal{M}$, define a Riemannian metric $\langle \cdot, \cdot \rangle^0$ on $\mathcal{M}_x$ by

$$\langle h, k \rangle_{\tilde{g}}^0 = \mathrm{tr}_{\tilde{g}}(hk) \det\left(g(x)^{-1}\tilde{g}\right) \qquad \forall h, k \in T_{\tilde{g}}\mathcal{M}_x \cong \mathcal{S}_x.$$

We denote by $L^{\langle \cdot, \cdot \rangle^0}$ the length of a path with respect to $\langle \cdot, \cdot \rangle^0$, and by θ_x^g the Riemannian distance function of $\langle \cdot, \cdot \rangle^0$.

Note that θ_x^g is automatically positive definite, since it is the distance function of a Riemannian metric on a finite-dimensional manifold. By integrating it in x, we can pass from a metric on $\mathcal{M}_x$ to a function on $\mathcal{M} \times \mathcal{M}$ as follows:

Lemma 2.7 ([**5**, Lemma 20, 21]). *For any measurable $Y \subseteq M$, define a function $\Theta_Y : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ by*

$$(2.2) \qquad \Theta_Y(g_0, g_1) = \int_Y \theta_x^g(g_0(x), g_1(x)) \, d\mu_g(x).$$

Then Θ_Y does not depend upon the choice of metric g used to define θ_x^g. (Indeed, even the integrand in (2.2) is independent of the choice of g.) Furthermore, Θ_Y is a pseudometric on $\mathcal{M}$, and Θ_M is a metric. Finally, if $Y_0 \subseteq Y_1$, then $\Theta_{Y_0}(g_0, g_1) \leq \Theta_{Y_1}(g_0, g_1)$ for all $g_0, g_1 \in \mathcal{M}$.

The lower bound on d is the following:

Proposition 2.8 ([**5**, Prop. 22]). *For any $Y \subseteq M$ and $g_0, g_1 \in \mathcal{M}$, we have the following inequality:*

$$\Theta_Y(g_0, g_1) \leq d(g_0, g_1) \left(\sqrt{n}\, d(g_0, g_1) + 2\sqrt{\mathrm{Vol}(M, g_0)} \right).$$

In particular, Θ_Y is a continuous pseudometric (w.r.t. d).

2.2. Completions of Metric Spaces. To fix notation and recall a few elementary points, we briefly review the completion of a metric space. We will simply state the definition and explore a couple of consequences of it, then give an alternative, equivalent viewpoint for path metric spaces.

The *precompletion* of (X, δ) is the set $\overline{(X, \delta)}^{\mathrm{pre}}$, usually just denoted by $\overline{X}^{\mathrm{pre}}$, consisting of all Cauchy sequences of X, together with the distance function

$$\delta(\{x_k\}, \{y_k\}) := \lim_{k \to \infty} \delta(x_k, y_k).$$

(We denote the distance function of the precompletion of a space using the same symbol as for the space itself; which distance function is meant will always be clear from the context.)

The *completion* of (X, δ) is a quotient space of $\overline{X}^{\mathrm{pre}}$, $\overline{X} := \overline{X}^{\mathrm{pre}}/\sim$, where $\sim$ is the equivalence relation defined by

$$(2.3) \qquad\qquad \{x_k\} \sim \{y_k\} \iff \delta(\{x_k\}, \{y_k\}) = 0.$$

Note that if $\{x_k\}$ is a Cauchy sequence in X and $\{x_{k_l}\}$ is a subsequence, then clearly $\{x_{k_l}\} \sim \{x_k\}$. Thus, given an element of the precompletion of X, we can always pass to a subsequence and still be talking about the same element of the completion.

Recall that a *path metric space* is a metric space for which the distance between any two points coincides with the infimum of the lengths of rectifiable curves joining the two points. (We will also call a rectifiable curve a *finite-length path*.)

The following theorem describes the completion of a path metric space in terms of finite-length paths. Its proof is straightforward.

Theorem 2.9 ([**4**, Thm. 2.2]). *Let (X, δ) be a path metric space. Then the following description of the completion of (X, δ) is equivalent to the definition given above, in the sense that there exists an isometry between the two completions which restricts to the identity on X.*

Define the precompletion $\overline{X}^{\mathrm{pre}}$ of X to be the set of rectifiable curves $\alpha : (0, 1] \to X$. It carries the pseudometric

$$(2.4) \qquad\qquad \delta(\alpha_0, \alpha_1) := \lim_{t \to 0} \delta(\alpha_0(t), \alpha_1(t)).$$

Then the completion of (X, δ) is $\overline{X} := \overline{X}^{\mathrm{pre}}/\sim$, where $\alpha_0 \sim \alpha_1 \iff \delta(\alpha_0, \alpha_1) = 0$.

2.3. Geometric Preliminaries. We now give some of the nonstandard geometric facts that we will need, in order to fix notation and recall the relevant notions.

2.3.1. Sections of the Endomorphism Bundle of M. We begin with a small convention:

Convention 2.10. When we refer to a section of a fiber bundle over M, we mean section only in the set-theoretic sense, unless otherwise specified. In particular, measurable, continuous, and smooth sections will be explicitly identified as such.

Given a section H of the endomorphism bundle of M, the determinant and trace of H are well-defined functions over M. Furthermore, if H is measurable, continuous, or smooth, then the determinant and trace will be so as well, since they are smooth functions from the space of $n \times n$ matrices into $\mathbb{R}$. If H is (pointwise) self-adjoint with respect to some Riemannian metric on M, i.e., there exists $g \in \mathcal{M}$ such that $g(H(\cdot), \cdot) = g(\cdot, H(\cdot))$, then the eigenvalues of H are real, and in particular the minimal and maximal eigenvalues are well-defined functions on M.

The following proposition allows us to characterize positive definite and positive semidefinite $(0, 2)$-tensors.

Proposition 2.11 ([**15**, Thm. 7.2.1]). *A symmetric $n \times n$ matrix T is positive definite (resp. positive semidefinite) if and only if all eigenvalues of T are positive (resp. nonnegative).*

In particular, if T is positive definite (resp. positive semidefinite), then $\det T > 0$ (resp. $\det T \geq 0$). If T is positive semidefinite but not positive definite, then $\det T = 0$.

We also need a result on the eigenvalues of a section of the endomorphism bundle.

Lemma 2.12 ([**4**, Lemma 2.11]). *Let h be any continuous, symmetric $(0, 2)$-tensor field. Suppose g is a Riemannian metric on M, and let H be the $(1, 1)$-tensor field obtained from h by raising an index using g (cf. Remark 2.1). (That is, locally $H^i_j = g^{ik} h_{kj}$, or $h(\cdot, \cdot) = g(H(\cdot), \cdot)$.) Then H is a continuous section of the endomorphism bundle $\mathrm{End}(M)$. Denote by $\lambda^H_{\min}(x)$ the smallest eigenvalue of $H(x)$. We have that*

1) *$\lambda^H_{\min}$ is a continuous function and*
2) *if h is positive definite, then $\min_{x \in M} \lambda^H_{\min}(x) > 0$.*

Furthermore, if $\lambda^H_{\max}(x)$ denotes the largest eigenvalue of $H(x)$, then $\lambda^H_{\max}$ is a continuous and hence bounded function.

Proof. Given a symmetric $n \times n$ matrix A, we have (with $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denoting the Euclidean scalar product on $\mathbb{R}^n$) $\lambda^A_{\max} = \sup_{\langle\!\langle v, v \rangle\!\rangle} \langle\!\langle v, Av \rangle\!\rangle$. From this, one can easily deduce that $\lambda^A_{\max}$ is a convex function from the space

of $n \times n$ symmetric matrices to $\mathbb{R}$. Furthermore, since $\lambda_{\min}^A = -\lambda_{\max}^{-A}$, $\lambda_{\min}^A$ is a concave function. Thus both mappings are continuous [**23**, Thm. 10.1]. The proof of the continuity of $\lambda_{\min}^H$ and $\lambda_{\max}^H$ then follows via a standard argument using compactness of the sphere bundle $SM \subset TM$.

The bound on the minimal eigenvalue follows from continuity and Proposition 2.11.   q.e.d.

2.3.2. Lebesgue Measure on Manifolds. The concept of Lebesgue measurability carries over from $\mathbb{R}^n$ to smooth (or even topological) manifolds by simply declaring a subset E to be Lebesgue measurable if $\phi(E \cap U)$ is Lebesgue measurable for each chart (U, ϕ) in a maximal atlas. Transition functions will also map nullsets to nullsets, so the notion of a nullset is well-defined.

With Lebesgue measurable sets well-defined, the concept of a measurable function or a measurable map between manifolds is also well-defined. We can also speak about measures on the σ-algebra of Lebesgue sets on the manifold. (In contrast to $\mathbb{R}^n$, there is no one canonical Lebesgue measure on a general manifold.) For example, any nonnegative n-form μ on M with measurable representative in any chart induces a measure on M. If μ is a smooth volume form, then in particular μ-nullsets are precisely the Lebesgue nullsets described above.

Convention 2.13. Unless we explicitly state otherwise, measurability of subsets of M, functions on M, and sections of fiber bundles over M, as well as the concept of a nullset in and measure on M, all refer to the concepts deriving from the Lebesgue σ-algebra and nullsets, as described in the preceding two paragraphs.

Additionally, if we write that a statement holds almost everywhere, we mean that it holds outside of a Lebesgue nullset.

It is not hard to see that the same relation between Lebesgue measurable sets and Borel measurable sets that holds on $\mathbb{R}^n$ [**24**, 11.11(d)] also holds on M. Namely, any Lebesgue measurable set E can be decomposed as $E = F \cup G$, where F is Borel measurable and G is a Lebesgue-nullset.

2.4. Notation and Conventions. Before we begin with the main body of the work, we will describe all nonstandard notation and conventions that will be used throughout the text.

The first thing we do is fix a reference metric, with respect to which all standard concepts will be defined.

Convention 2.14. For the remainder of the paper, we fix an element $g \in \mathcal{M}$. Whenever we refer to the L^p norm, L^p topology, L^p convergence etc., we mean that induced by g unless we explicitly state otherwise. (The L^p norm depends on the choice of g, but the L^p topology and the notion of L^p convergence do not.)

If we have a tensor $h \in \mathcal{S}$, we denote by the capital letter H the tensor obtained by raising an index with g, i.e., locally $H^i_j := g^{ik}h_{kj}$, or invariantly $h(\cdot, \cdot) = g(H(\cdot), \cdot)$. Given a point $x \in M$ and an element $a \in \mathcal{M}_x$, the capital letter A means the same—i.e., we assume some coordinates and write $A = g(x)^{-1}a$, though for readability we will generally omit x from the notation (or again, we may write invariantly $a(\cdot, \cdot) = g(x)(A(\cdot), \cdot)$). We use the same convention for accented characters—e.g., for $\tilde{h} \in \mathcal{S}$ and $\hat{a} \in \mathcal{S}_x$, by convention $\tilde{h}(\cdot, \cdot) = g(\tilde{H}(\cdot), \cdot)$ and $\hat{a}(\cdot, \cdot) = g(x)(\hat{A}(\cdot), \cdot)$—and for characters with sub-/superscripts.

Next, we'll fix an atlas of coordinates on M that is convenient to work with.

Definition 2.15. We call a finite atlas of coordinates $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ for M, where A is some index set, *amenable* if for each $\alpha \in A$, there exists a different coordinate chart (V_α, ψ_α) (which does not necessarily belong to $\{(U_\alpha, \phi_\alpha)\}$) such that

$$(2.5) \qquad \mathrm{cl}(U_\alpha) \subset V_\alpha \quad \text{and} \quad \phi_\alpha = \psi_\alpha|U_\alpha,$$

where $\mathrm{cl}(U_\alpha)$ denotes the closure of U_α in the topology of M.

Remark 2.16. Note that, by compactness of M, the set $\mathrm{cl}(U_\alpha)$ in the above definition is itself compact, and hence U_α is relatively compact as a subset of the domain of the larger chart (V_α, ϕ_α)—a fact which will be crucial throughout the paper.

Compactness of M implies that an amenable atlas for M always exists. Thus, we may make the following convention.

Convention 2.17. For the remainder of this paper, we work over a fixed amenable coordinate atlas $\{(U_\alpha, \phi_\alpha)\}$ for all computations and concepts that require local coordinates.

The next lemma heuristically says the following: in amenable coordinates, the coordinate representations of a smooth metric are somehow "uniformly positive definite". Additionally, the coefficients satisfy a uniform upper bound.

Lemma 2.18. *For any metric $\tilde{g} \in \mathcal{M}$, there exist constants $\delta(\tilde{g}) > 0$ and $C(\tilde{g}) < \infty$, depending only on $\tilde{g}$, with the property that for any α, any $x \in U_\alpha$, and $1 \leq i, j \leq n$,*

$$(2.6) \qquad |\tilde{g}_{ij}(x)| \leq C(\tilde{g}) \text{ and } \lambda^{\tilde{G}}_{\min}(x) \geq \delta(\tilde{g}),$$

where we of course mean the value of $\tilde{g}_{ij}(x)$ in the chart (U_α, ϕ_α).

Proof. Note that Definition 2.15 implies that $\phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ is a relatively compact subset of $\psi_\alpha(V_\alpha)$. Thus, the proof of the first inequality is immediate and the second is clear from Lemma 2.12. q.e.d.

Remark 2.19. The estimate $|\tilde{g}_{ij}(x)| \leq C(\tilde{g})$ also implies an upper bound in terms of $C(\tilde{g})$ on $\det \tilde{g}(x)$. This is clear from the fact that the determinant is a homogeneous polynomial in $\tilde{g}_{ij}(x)$ with $n!$ terms and coefficients ± 1.

The main point of using a fixed amenable coordinate atlas is the following: it gives us an easily understood and uniform—but nevertheless coordinate-dependent—notion of how "large" or "small" a metric is. The dependence of this notion on coordinates is perhaps somewhat dissatisfying at first glance, but it should be seen as merely an aid in our quest to prove statements that are, indeed, invariant in nature.

It is necessary to introduce somewhat more general objects than Riemannian metrics in this paper:

Definition 2.20. Let $\tilde{g}$ be a (set-theoretic) section of $S^2 T^* M$. Then $\tilde{g}$ is called a *(Riemannian) semimetric* if it induces a positive semidefinite scalar product on $T_x M$ for each $x \in M$.

We now make a couple of definitions on semimetrics and sequences of metrics.

Definition 2.21. Let $\tilde{g}$ be a semimetric on M (which we do not assume to be even measurable). We define the set

$$X_{\tilde{g}} := \{x \in M \mid \tilde{g}(x) \text{ is degenerate}\} \subset M,$$

which we call the *deflated set* of $\tilde{g}$. (Recall that by saying $\tilde{g}(x)$ is degenerate, we mean that it is not positive definite.)

We call $\tilde{g}$ *bounded* if there exists a constant C such that

$$|\tilde{g}_{ij}(x)| \leq C$$

for almost every $x \in M$ and all $1 \leq i, j \leq n$. Otherwise $\tilde{g}$ is called *unbounded*.

Definition 2.22. Let $\{g_k\} \subset \mathcal{M}$ be any sequence. We define the set

$$D_{\{g_k\}} := \{x \in M \mid \forall \delta > 0, \ \exists k \in \mathbb{N} \text{ s.t. } \det G_k(x) < \delta\},$$

which we call the *deflated set* of $\{g_k\}$.

Remark 2.23. We note here that:

1) Given any sequence $\{g_k\} \subset \mathcal{M}$, its deflated set is measurable. In fact, since $D_{\{g_k\}} = \cap_{m=1}^{\infty}(\cup_{k=1}^{\infty}\{x \in M \mid \det G_k(x) < \frac{1}{m}\})$, the deflated set is even Borel-measurable.
2) If $\tilde{g}$ is a measurable semimetric, then $X_{\tilde{g}}$ is measurable, since it is the set of points where $\det \tilde{G} = 0$.

Note that any measurable semimetric $\tilde{g}$ on M induces a measure on M that is absolutely continuous with respect to the fixed volume form μ_g.

A measurable Riemannian semimetric $\tilde{g}$ on M gives rise to an "L^2 scalar product" on measurable functions in the following way. For any two measurable functions ρ and σ on M, we define, as usual, $\mu_{\tilde{g}} := \sqrt{\det \tilde{g}}\, dx^1 \cdots dx^n$ and

$$(2.7) \qquad (\rho, \sigma)_{\tilde{g}} := \int_M \rho\sigma\,\mu_{\tilde{g}}.$$

(We denote this by the same symbol as the L^2 scalar product on $\mathcal{S}$; which is meant will always be clear from the context.) We put "L^2 scalar product" in quotation marks because unless we put specific conditions on ρ, σ, and $\tilde{g}$, (2.7) is not guaranteed to be finite or even well-defined. It suffices, for example, to require that ρ and σ be bounded and that the total volume $\mathrm{Vol}(M, \tilde{g}) = \int_M \mu_{\tilde{g}}$ of $\tilde{g}$ be finite.

3. Amenable Subsets

We begin the study of the completion of $\mathcal{M}$ in this section, by first completing very special subsets of $\mathcal{M}$ called *amenable subsets* (defined below). The main result of the section is that the completion of such a subset with respect to d coincides with the completion with respect to the L^2 norm on $\mathcal{S}$, the vector space in which $\mathcal{M}$ resides.

3.1. Amenable Subsets and their Basic Properties. For the following definition, recall that we work over an amenable atlas (cf. Definition 2.15).

Definition 3.1. We call a subset $\mathcal{U} \subset \mathcal{M}$ *amenable* if $\mathcal{U}$ is convex and we can find constants $C, \delta > 0$ such that for all $\tilde{g} \in \mathcal{U}$, $x \in M$ and $1 \leq i, j \leq n$,
$$\lambda_{\min}^{\tilde{G}}(x) \geq \delta$$
(where we recall that $\tilde{G} = g^{-1}\tilde{g}$, with g our fixed metric) and
$$|\tilde{g}_{ij}(x)| \leq C.$$

Remark 3.2. We make a couple of remarks about the definition:
1) The requirement that $\mathcal{U}$ is convex is technical, and is there to insure that we can consider simple, straight-line paths between points of $\mathcal{U}$ to estimate the distance between them.
2) Recall that the function sending a positive-semidefinite matrix to its minimal eigenvalue is concave (cf. the proof of Lemma 2.12). Also, the absolute value function on $\mathbb{R}$ is convex by the triangle inequality. Therefore, the two bounds given in Definition 3.1 are compatible with the requirement of convexity.

Definition 3.3. If $\mathcal{U} \subset \mathcal{M}$ is any subset, we denote by $\mathcal{U}^0$ the L^2-completion of $\mathcal{U}$ (that is, the completion of $\mathcal{U}$ with respect to $\|\cdot\|_g$).

Remark 3.4. Note that each element of the L^2 completion of $\mathcal{S}$ can be identified with an element of $L^2(S^2T^*M)$, i.e., an L^2-integrable measurable section of S^2T^*M, modulo an identification of sections that agree a.e. In particular, if $\mathcal{U}$ is an amenable subset, then $\mathcal{U}^0$ can be naturally identified with the set of measurable sections comprising the closure of $\mathcal{U}$ as a subset of $L^2(S^2T^*M)$.

Let us give a few examples of amenable subsets, which will be needed in later proofs.

Example 3.5.

1) The convex hull of any finite subset of elements of $\mathcal{M}$ is clearly amenable.
2) The convex hull of the union of a finite number of amenable subsets is amenable.
3) The "pointwise convex hull" of an amenable subset $\mathcal{U}$ is also amenable. What we mean by this is that the set

$$\mathcal{U}' := \{fg_0 + (1-f)g_1 \mid g_0, g_1 \in \mathcal{U}, \ f \in C^\infty(M), \ 0 \le f \le 1\}$$

is amenable, even with the same bounds C and δ as for $\mathcal{U}$. This follows immediately from the definition and Remark 3.2(2).

We now state another fact that will be needed in proofs below as a lemma, since it is not so immediately seen to be true.

Lemma 3.6. *Let $g_0 \in \mathcal{M}$ and $h \in \mathcal{S}$, and assume that $g_0 + h \in \mathcal{M}$. Then there exists an amenable subset $\mathcal{U}$ such that for any measurable function f with $0 \le f \le 1$, $g_0 + fh \in \mathcal{U}^0$, and such that if additionally f is smooth, then $g_0 + fh \in \mathcal{U}$.*

Proof. Let $\mathcal{V}$ be an amenable subset containing g_0 and $g_0 + h$, as in Example 3.5(1), and let $\mathcal{U}$ be the pointwise convex hull of $\mathcal{V}$, as in Example 3.5(3). For a fixed function f as in the hypotheses of the lemma, let $f_k \in C^\infty(M)$ be smooth functions converging to f in L^2 as $k \to \infty$, and such that $0 \le f_k \le 1$. (That $f \in L^2(M, g)$ follows from the fact that it is a measurable, bounded function and that $\mathrm{Vol}(M, g) < \infty$.) Note that $g_0 + f_k h = f_k(g_0 + h) + (1 - f_k)g_0 \in \mathcal{U}$ by pointwise convexity. But then $g_0 + fh$, as the L^2 limit of $g_0 + f_k h$, is contained in $\mathcal{U}^0$. If f was smooth in the first place, we may choose $f_k = f$ for all $k \in \mathbb{N}$, so $g_0 + fh \in \mathcal{U}$. Since $\mathcal{U}$ was chosen independently of f, the statement is proved.   q.e.d.

One useful property the metrics $\tilde{g}$ of an amenable subset have is that the Radon-Nikodym derivatives $(\mu_{\tilde{g}}/\mu_g)$, with respect to the reference volume form μ_g, are bounded away from zero and infinity independently of $\tilde{g}$.

Lemma 3.7. *Let $\mathcal{U}$ be an amenable subset. Then there exists a constant $K > 0$ such that for all $\tilde{g} \in \mathcal{U}$,*

$$(3.1) \qquad \frac{1}{K} \leq \left(\frac{\mu_{\tilde{g}}}{\mu_g}\right) \leq K$$

Proof. First, we note that

$$\left(\frac{\mu_{\tilde{g}}}{\mu_g}\right) = \sqrt{\det \tilde{G}} \quad \text{and} \quad \left(\frac{\mu_{\tilde{g}}}{\mu_g}\right)^{-1} = \left(\frac{\mu_g}{\mu_{\tilde{g}}}\right) = (\det \tilde{G})^{-1/2}.$$

So (3.1) is equivalent to upper bounds on both $\det \tilde{G}$ and $(\det \tilde{G})^{-1}$.

Now, if the eigenvalues of $\tilde{G}$ are $\lambda_1^{\tilde{G}}, \ldots, \lambda_n^{\tilde{G}}$, then

$$\det \tilde{G} = \lambda_1^{\tilde{G}} \cdots \lambda_n^{\tilde{G}} \geq \left(\lambda_{\min}^{\tilde{G}}\right)^n \geq \delta^n,$$

where δ is the constant guaranteed by the fact that $\tilde{g} \in \mathcal{U}$. This allows us to bound $(\det \tilde{G})^{-1}$ from above.

To bound $\det \tilde{G}$ from above, it is sufficient to bound the absolute value of the coefficients of $\tilde{G} = g^{-1}\tilde{g}$ from above. But bounds on the coefficients of $\tilde{g}$ are already assured by the fact that $\tilde{g} \in \mathcal{U}$, and bounds on the coefficients of g^{-1} are guaranteed by the fact that g^{-1} is a fixed, smooth cometric on M. So we are finished. q.e.d.

Amenable subsets guarantee good behavior of the norms on $\mathcal{S}$ that are defined by their members—namely, the norms are in some sense "uniformly equivalent". More precisely, we have:

Lemma 3.8. *Let $\mathcal{U} \subset \mathcal{M}$ be an amenable subset. Then there exist constants K and K', depending only on $\mathcal{U}$, such that for all $\tilde{g} \in \mathcal{U}$, all $h \in \mathcal{S}$, all $x \in M$, and all $k \in \mathcal{S}_x$,*

$$\frac{1}{K}\langle k, k \rangle_{g(x)} \leq \langle k, k \rangle_{\tilde{g}(x)} \leq K\langle k, k \rangle_{g(x)}$$

and

$$\frac{1}{K'}\|h\|_g \leq \|h\|_{\tilde{g}} \leq K'\|h\|_g.$$

Proof. The first statement is equivalent to the following. Let

$$T_{\tilde{g}} : (S^2 T^* M, \langle \cdot, \cdot \rangle_{\tilde{g}}) \to (S^2 T^* M, \langle \cdot, \cdot \rangle_g)$$

be the identity mapping on the level of sets, sending the bundle $S^2 T^* M$ with the Riemannian structure $\langle \cdot, \cdot \rangle_{\tilde{g}}$ to itself with the Riemannian structure $\langle \cdot, \cdot \rangle_g$. Let $N(T_{\tilde{g}})(x)$ be the operator norm of $T_{\tilde{g}}(x) : \mathcal{S}_x \to \mathcal{S}_x$, and let $N(T_{\tilde{g}}^{-1})(x)$ be defined similarly. Then the first statement of the lemma holds if and only if there exists a constant K such that

$$N(T_{\tilde{g}})(x)^2, N(T_{\tilde{g}}^{-1})(x)^2 \leq K.$$

But $N(T_{\tilde{g}})$ and $N(T_{\tilde{g}}^{-1})$ are continuous functions on the compact manifold M for fixed $\tilde{g}$. Secondly, we notice that both $N(T_{\tilde{g}})(x)$ and

$N(T_{\tilde{g}}^{-1})(x)$ depend only on the coordinate representations of $\tilde{g}(x)$ and $g(x)$. But $\tilde{g}(x)$ and $g(x)$ can only range over a compact subset of the space of positive definite $n \times n$ symmetric matrices, because $\tilde{g}$ ranges over an amenable subset. This implies the existence of K.

The second statement now follows from the first, as well as from the bounds on $(\mu_{\tilde{g}}/\mu_g)$ given by Lemma 3.7.   q.e.d.

Lemma 3.7 immediately implies that the function $\tilde{g} \mapsto \mathrm{Vol}(M, \tilde{g})$ is bounded when restricted to any amenable subset. Recalling the form of the estimate in Proposition 2.8 then shows the following lemma.

Lemma 3.9. *Let $\mathcal{U}$ be an amenable subset. Then there exists a constant V such that for any $g_0, g_1 \in \mathcal{U}$ and $Y \subset M$,*

$$\Theta_Y(g_0, g_1) \leq 2d(g_0, g_1)\left(\frac{\sqrt{n}}{2}d(g_0, g_1) + \sqrt{V}\right).$$

Specifically, this inequality holds with $V = \sup_{\tilde{g} \in \mathcal{U}} \mathrm{Vol}(M, \tilde{g})$, which is finite by the discussion preceding the lemma.

3.2. The Completion of $\mathcal{U}$ with Respect to d and $\|\cdot\|_g$. We are now ready to prove a result that, in particular, implies equivalence of the topologies defined by d and $\|\cdot\|_g$ on an amenable subset $\mathcal{U}$.

Theorem 3.10. *Consider the L^2 topology on $\mathcal{M}$ induced from the scalar product $(\cdot, \cdot)_g$ (where g is fixed). Let $\mathcal{U} \subset \mathcal{M}$ be any amenable subset.*

Then the L^2 topology on $\mathcal{U}$ coincides with the topology induced from the restriction of the Riemannian distance function d of $\mathcal{M}$ to $\mathcal{U}$.

Additionally, the following holds:

1) *There exists a constant K such that $d(g_0, g_1) \leq K \|g_1 - g_0\|_g$ for all $g_0, g_1 \in \mathcal{U}$.*
2) *For any $\epsilon > 0$, there exists $\delta > 0$ such that if $d(g_0, g_1) < \delta$, then $\|g_0 - g_1\|_g < \epsilon$.*

Proof. To prove (1), consider the linear path γ from g_0 to g_1. We then have

$$
\begin{aligned}
L(\gamma) &= \int_0^1 \left\|\gamma'(t)\right\|_{\gamma(t)} dt = \int_0^1 \|g_1 - g_0\|_{\gamma(t)} \, dt \\
&\leq \int_0^1 K' \|g_1 - g_0\|_g \, dt = K' \|g_1 - g_0\|_g,
\end{aligned}
$$

(3.2)

where K' is the constant guaranteed by Lemma 3.8. Since $d(g_0, g_1) \leq L(\gamma)$ and the constant K' depends only on the set $\mathcal{U}$, this inequality is shown.

We now move on to statement (2). To prove this, we will essentially show that the distance induced by the (fixed) scalar product $\langle \cdot, \cdot \rangle_g$ and is

bounded by the distance function θ_x^g, in a uniform way, when restricted to an amenable subset.

To make this precise, we first consider metrics in $\mathcal{U}$ as sections of the bundle $S_+^2 T^* M \to M$ of positive definite $(0,2)$-tensors on M, and define

$$I(\mathcal{U}) := \{\tilde{g}(M) \mid \tilde{g} \in \mathcal{U}\} \subset S_+^2 T^* M.$$

(Here, $\tilde{g}(M)$ denotes the image of $\tilde{g}$ as a map $M \to S_+^2 T^* M$.) By the definition of an amenable subset and the compactness of M, $I(\mathcal{U})$ is a precompact subset of $S_+^2 T^* M$.

For the moment, fix an arbitrary point $x \in M$ and an arbitrary element a_0 of the finite-dimensional manifold $\mathcal{M}_x$. Consider the Riemannian metric $\langle \cdot, \cdot \rangle^0$ on $\mathcal{M}_x$. Since $\langle \cdot, \cdot \rangle_a^0$ varies smoothly with a, we may, by restricting to a sufficiently small neighborhood of a_0 in $\mathcal{M}_x$, ensure that $\langle \cdot, \cdot \rangle_{a_0}^0$ approximates arbitrarily closely a constant (flat) metric—say, in the C^2 sense. It then follows that there exist $0 < C(x, a_0), \zeta(x, a_0) < \infty$ such that if $\theta_x^g(a_0, a_1) < \zeta(x, a_0)$, then

$$(3.3) \qquad \sqrt{\langle a_1 - a_0, a_1 - a_0 \rangle_{a_0}^0} < C(x, a_0) \theta_x^g(a_0, a_1),$$

since the left-hand side is just the distance from a_0 to a_1 in the metric induced by extending $\langle \cdot, \cdot \rangle_{a_0}^0$ constantly from the point a_0.

Since the Riemannian metric $\langle \cdot, \cdot \rangle^0$ on the fibers of $S_+^2 T^* M$ is smooth, the constants $C(x, a_0)$ and $\zeta(x, a_0)$ of (3.3) may be chosen uniformly with respect to x and a_0 if we restrict to $a_0 \in I(\mathcal{U})$, since then both x and a_0 vary over (pre)compact subsets. Let C and ζ denote such a uniform choice of these constants, with of course $0 < C, \zeta < \infty$.

On the other hand, if we define

$$(3.4) \quad D := \sup_{x \in M} \left\{ \sqrt{\langle a_1 - a_0, a_1 - a_0 \rangle_{g(x)}} \;\middle|\; a_0, a_1 \in I(\mathcal{U}) \cap S_+^2 T_x^* M \right\},$$

then the precompactness of $I(\mathcal{U})$ implies that $D < \infty$.

Now, let $g_0, g_1 \in \mathcal{U}$ be given. Of course, we have $g_0(x), g_1(x) \in I(\mathcal{U})$ for all $x \in M$. Thus, if $x \in M$ is such that $\theta_x^g(g_0(x), g_1(x)) < \zeta$, then we can use Lemmas 3.7 and 3.8 to see that there exist constants K_0 and K_1, depending only on $\mathcal{U}$, such that

$$
\begin{aligned}
\langle g_1(x) - g_0(x), g_1(x) - g_0(x) \rangle_{g(x)} & \\
&\leq K_0 \langle g_1(x) - g_0(x), g_1(x) - g_0(x) \rangle_{g_0(x)} \\
(3.5) \qquad &\leq K_1 \langle g_1(x) - g_0(x), g_1(x) - g_0(x) \rangle_{g_0(x)}^0 \\
&< C^2 K_1 \theta_x^g(g_0(x), g_1(x))^2 \\
&< C^2 K_1 \zeta \theta_x^g(g_0(x), g_1(x)).
\end{aligned}
$$

Here, C is the uniform constant mentioned above such that (3.3) holds.

On the other hand, if $x \in M$ is such that $\theta_x^g(g_0(x), g_1(x)) \geq \zeta$, then
(3.6)
$$\langle g_1(x) - g_0(x), g_1(x) - g_0(x) \rangle_{g(x)} \leq D^2 = \frac{D^2 \zeta}{\zeta} \leq \frac{D^2}{\zeta} \theta_x^g(g_0(x), g_1(x)).$$

Thus, we let $C' := \max\{C^2 K_1 \zeta, D^2/\zeta\}$; note that C' depends only on $\mathcal{U}$, not on g_0 g_1.

Finally, choose $\delta > 0$ such that

(3.7)
$$2\delta \left(\frac{\sqrt{n}}{2} \delta + \sqrt{V} \right) < \frac{\epsilon^2}{C'},$$

where V is the constant of Lemma 3.9. We claim that $d(g_0, g_1) < \delta$ implies $\|g_1 - g_0\|_g < \epsilon$. For in that case, Lemma 3.9 and (3.7) imply that $\Theta_M(g_0, g_1) < \epsilon^2/C'$, while (3.5) and (3.6) give

$$\|g_1 - g_0\|_g^2 = \int_M \langle g_1(x) - g_0(x), g_1(x) - g_0(x) \rangle_{g(x)} \, d\mu_g(x)$$

$$\leq C' \int_M \theta_x^g(g_0(x), g_1(x)) \, d\mu_g(x) < \epsilon^2.$$

Since C' and V, and hence δ, were independent of g_0 and g_1, this completes the proof.   q.e.d.

Theorem 3.10 will give us our first result regarding the completion of $\mathcal{M}$. First, though, we need to prove a statement about metric spaces.

Let's look back at Theorem 3.10 again. The first statement says that for any amenable subset $\mathcal{U}$ and any $g \in \mathcal{M}$, d is uniformly Lipschitz continuous with respect to $\|\cdot\|_g$ when viewed as a function on $\mathcal{U} \times \mathcal{U}$. The second statement says that $\|\cdot\|_g$ is uniformly continuous on $\mathcal{U} \times \mathcal{U}$ with respect to d. To put this knowledge to good use, we will need the following lemma:

Lemma 3.11. *Let X be a set, and let two metrics, d_1 and d_2, be defined on X. Denote by $\phi : (X, d_1) \to (X, d_2)$ the map which is the identity on the level of sets, i.e., ϕ simply maps $x \mapsto x$. Finally, denote by $\overline{X}^1$ and $\overline{X}^2$ the completions of X with respect to d_1 and d_2, respectively.*

If both ϕ and ϕ^{-1} are uniformly continuous, then there is a natural uniformly continuous homeomorphism between $\overline{X}^1$ and $\overline{X}^2$.

Proof. The proof follows in a straightforward manner from the definition of the completion of a metric space from Section 2.2, and the fact that a uniformly continuous function maps Cauchy sequences to Cauchy sequences. The natural homeomorphism is the unique continuous extension of ϕ to $\overline{X^1}$. This extension is uniformly continuous.
                                              q.e.d.

Now, Theorem 3.10 and Lemma 3.11 immediately imply

Theorem 3.12. *Let $\mathcal{U}$ be an amenable subset. Then we can identify $\overline{\mathcal{U}}$, the completion of $\mathcal{U}$ with respect to d, with $\mathcal{U}^0$, the L^2-completion of $\mathcal{U}$, in the sense of Lemma* 3.11. *We can make the natural homeomorphism $\overline{\mathcal{U}} \to \mathcal{U}^0$ into an isometry by placing a metric on $\mathcal{U}^0$ defined by*

$$d(g_0, g_1) = \lim_{k \to \infty} d(g_k^0, g_k^1),$$

where $\{g_k^0\}$ and $\{g_k^1\}$ are any sequences in $\mathcal{U}$ that L^2-converge to g_0 and g_1, respectively.

We have thus found a nice description of the completion of very special subsets of $\mathcal{M}$. As already discussed, our plan now is to start removing the nice properties that allowed us to understand amenable subsets so clearly, advancing through the completions of ever larger and more generally defined subsets of $\mathcal{M}$. Before that, though, we need to study general Cauchy sequences in $\mathcal{M}$ more closely in the next section.

4. Cauchy sequences and ω-convergence

In this chapter, we introduce and study a fundamental notion of convergence of our own invention for d-Cauchy sequences in $\mathcal{M}$. We call this ω-*convergence*, and its importance is made clear through two theorems we will prove, an existence and a uniqueness result. The existence result, proved in Section 4.1, says that every d-Cauchy sequence has a subsequence that ω-converges to a measurable semimetric, which we will then show has finite total volume. The uniqueness result, proved in Section 4.3, is that two ω-convergent Cauchy sequences in $\mathcal{M}$ are equivalent (in the sense of (2.3)) if and only if they have the same ω-limit. (Please note that my usage of the term "ω-limit" is a new coinage, and not related to the usage of this term in dynamical systems.)

These results allow us to identify an equivalence class of d-Cauchy sequences with the unique ω-limit that its representatives subconverge to, and thus give a geometric meaning to points of $\overline{\mathcal{M}}$.

4.1. Existence of the ω-Limit. We begin this section with an important estimate and some examples, followed by the definition of ω-convergence and some of its basic properties. After that, we start on the existence proof by showing a pointwise version, i.e., an analogous result on $\mathcal{M}_x$. Finally, we globalize this pointwise result to show the existence of an ω-convergent subsequence for any Cauchy sequence in $\mathcal{M}$.

4.1.1. Volume-Based Estimates on d and Examples. The following surprising proposition shows us that two metrics that differ only on a subset with small (intrinsic) volume are close with respect to d.

Proposition 4.1. *Suppose that $g_0, g_1 \in \mathcal{M}$, and let $E := \mathrm{carr}(g_1 - g_0) = \{x \in M \mid g_0(x) \neq g_1(x)\}$. Then there exists a constant $C(n)$ depending only on $n = \dim M$ such that*

$$(4.1) \qquad d(g_0, g_1) \leq C(n) \left(\sqrt{\mathrm{Vol}(E, g_0)} + \sqrt{\mathrm{Vol}(E, g_1)} \right).$$

In particular, we have

$$\mathrm{diam}\left(\{\tilde{g} \in \mathcal{M} \mid \mathrm{Vol}(M, \tilde{g}) \leq \delta\} \right) \leq 2C(n)\sqrt{\delta}.$$

Proof. The second statement follows immediately from the first, so we only prove the first.

The heuristic idea is the following. We want to construct a family of paths with three pieces, depending on a real parameter s, such that the metrics do not change on $M \setminus E$ as we travel along the paths. Therefore, we pretend that we can restrict all calculations to E. On E, the first piece of the path is the straight line from g_0 to sg_0 for some small positive number s. It is easy to compute a bound for the length of this path based on $\mathrm{Vol}(E, g_0)$. The second piece is the straight line from sg_0 to sg_1, which, as we will see, has length approaching zero for $s \to 0$. The last piece is the straight line from sg_1 to g_1, which again has length bounded from above by an expression involving $\mathrm{Vol}(E, g_1)$.

Our job is to now take this heuristic picture, which uses paths of L^2 metrics, and construct a family of paths of *smooth* metrics that captures the essential properties.

For each $k \in \mathbb{N}$ and $s \in (0, 1]$, we define three families of metrics as follows. Choose closed sets $F_k \subseteq E$ and open sets U_k containing E such that $\mathrm{Vol}(U_k, g_i) - \mathrm{Vol}(F_k, g_i) \leq 1/k$ for $i = 0, 1$. (This is possible because the Lebesgue measure is regular.) Let $f_{k,s} \in C^\infty(M)$ be functions with the following properties:

1) $f_{k,s}(x) = s$ if $x \in F_k$,
2) $f_{k,s}(x) = 1$ if $x \notin U_k$ and
3) $s \leq f_{k,s}(x) \leq 1$ for all $x \in M$.

Now, for $t \in [0, 1]$, define

$$\hat{\gamma}^{k,s}(t) := \hat{g}_t^{k,s} := ((1 - t) + t f_{k,s})g_0,$$

$$\bar{\gamma}^{k,s}(t) := \bar{g}_t^{k,s} := f_{k,s}((1 - t)g_0 + t g_1),$$

$$\tilde{\gamma}^{k,s}(t) := \tilde{g}_t^{k,s} := ((1 - t) + t f_{k,s})g_1.$$

We view $\hat{\gamma}^{k,s}$, $\bar{\gamma}^{k,s}$, and $\tilde{\gamma}^{k,s}$ as paths in t depending on the family parameter s. Furthermore, we define a concatenated path $\gamma^{k,s} := \hat{\gamma}^{k,s} * \bar{\gamma}^{k,s} * (\tilde{\gamma}^{k,s})^{-1}$, where of course the inverse means we run through the path backwards. It is easy to see that $\gamma^{k,s}(0) = g_0$ and $\gamma^{k,s}(1) = g_1$ for all s.

We now investigate the length of each piece of $\gamma^{k,s}$ separately, starting with that of $\hat{\gamma}^{k,s}$. Recalling that by Convention 2.14, $G_0 = g^{-1}g_0$, we

compute

$$L(\hat{\gamma}^{k,s}) = \int_0^1 \left(\int_M \mathrm{tr}_{((1-t)+tf_{k,s})g_0} \left(((f_{k,s}-1)g_0)^2 \right) \right.$$
$$\left. \cdot \sqrt{\det \left(((1-t)+tf_{k,s})G_0 \right)} \, \mu_g \right)^{1/2} dt$$

$$= \int_0^1 \left(\int_{U_k} ((1-t)+tf_{k,s})^{\frac{n}{2}-2} \right.$$
$$\left. \cdot \mathrm{tr}_{g_0} \left(((1-f_{k,s})g_0)^2 \right) \sqrt{\det G_0} \, \mu_g \right)^{1/2} dt.$$

since $\det(\lambda A) = \lambda^{n/2} \det A$ for any $n \times n$-matrix A and $\lambda \in \mathbb{R}$. Note that in the last line, we only integrate over U_k, which is justified by the fact that $1 - f_{k,s} = 0$ on $M \setminus U_k$. Since $s > 0$, it is easy to see that

$$(1 - f_{k,s})^2 \le (1-s)^2 < 1,$$

from which we can compute the estimate

$$L(\hat{\gamma}^{k,s}) < \int_0^1 \left(n \int_{U_k} ((1-t)+tf_{k,s})^{\frac{n}{2}-2} \, \mu_{g_0} \right)^{1/2} dt.$$

Now, to estimate this, we note that for $n \ge 4$, $\frac{n}{2} - 2 \ge 0$ and therefore $f_{k,s} \le 1$ implies that

$$(4.2) \qquad\qquad L(\hat{\gamma}^{k,s}) < \sqrt{n \, \mathrm{Vol}(U_k, g_0)}.$$

For $1 \le n \le 3$, $\frac{n}{2} - 2 < 0$ and therefore one can compute that $f_{k,s} \ge s > 0$ implies

$$((1-t)+tf_{k,s})^{\frac{n}{2}-2} \le (1-t)^{\frac{n}{2}-2}.$$

In this case, then,

$$(4.3) \qquad\qquad L(\hat{\gamma}^{k,s}) < \sqrt{n \, \mathrm{Vol}(U_k, g_0)} \int_0^1 (1-t)^{\frac{n}{4}-1} \, dt,$$

and the integral term is finite since $\frac{n}{4} - 1 > -1$. Furthermore, the value of this integral depends only on n. Putting together (4.2) and (4.3) therefore gives

$$L(\hat{\gamma}^{k,s}) \le C(n) \sqrt{\mathrm{Vol}(U_k, g_0)},$$

where $C(n)$ is a constant depending only on n.

In exact analogy, we can show that the same estimate holds with g_1 in place of g_0.

Next, we look at the second piece of $\gamma^{k,s}$. Here we have, using that $g_1 - g_0 \equiv 0$ on $M \setminus E$,

$$
\left\| (\bar{\gamma}^{k,s})'(t) \right\|^2_{\bar{\gamma}^{k,s}(t)} = \int_M \mathrm{tr}_{f_{k,s}((1-t)g_0+tg_1)} \left((f_{k,s}(g_1 - g_0))^2 \right)
$$
$$
\cdot \sqrt{\det \left(f_{k,s}((1-t)G_0 + tG_1) \right)} \, \mu_g
$$
$$
= \int_E f_{k,s}^{n/2} \, \mathrm{tr}_{(1-t)g_0+tg_1} \left((g_1 - g_0)^2 \right)
$$
$$
\cdot \sqrt{\det((1-t)G_0 + tG_1)} \, \mu_g.
$$

Since $f_{k,s}(x) = s$ if $x \in F_k$, and $f_{k,s}(x) \le 1$ for all $x \in M$, it follows from the above that

$$
\left\| (\bar{\gamma}^{k,s})'(t) \right\|^2_{\bar{\gamma}^{k,s}(t)}
$$
$$
\le s^{n/2} \int_{F_k} \mathrm{tr}_{(1-t)g_0+tg_1} \left((g_1 - g_0)^2 \right) \sqrt{\det((1-t)G_0 + tG_1)} \, \mu_g
$$
$$
+ \int_{E \setminus F_k} \mathrm{tr}_{(1-t)g_0+tg_1} \left((g_1 - g_0)^2 \right) \sqrt{\det((1-t)G_0 + tG_1)} \, \mu_g.
$$

Since $\mathrm{tr}_{(1-t)g_0+tg_1} \left((g_1 - g_0)^2 \right)$ and $\det((1-t)G_0 + tG_1)$ vary smoothly with x and t over the compact space $M \times [0,1]$, we have

$$
K := \max_{x \in M, t \in [0,1]} \mathrm{tr}_{(1-t)g_0+tg_1} \left((g_1 - g_0)^2 \right) \sqrt{\det((1-t)G_0 + tG_1)} < \infty.
$$

Thus, we obtain

$$
\left\| (\bar{\gamma}^{k,s})'(t) \right\|^2_{\bar{\gamma}^{k,s}(t)} \le s^{n/2} K \, \mathrm{Vol}(F_k, g) + K \, \mathrm{Vol}(E \setminus F_k, g).
$$

Since the right-hand side of this inequality is independent of t, $\mathrm{Vol}(F_k, g)$ is bounded independently of k, and by assumption $\mathrm{Vol}(E \setminus F_k, g) \to 0$ as $k \to \infty$, we have

$$
\lim_{k \to \infty} \lim_{s \to 0} L(\bar{\gamma}^{k,s}) = 0.
$$

Combining these considerations gives the desired estimate.   q.e.d.

As the following examples show, Proposition 4.1 implies that we cannot expect a Cauchy sequence in $\mathcal{M}$ to converge pointwise over subsets of M whose volume vanishes in the limit. Indeed, we cannot control its behavior at all.

Example 4.2. Consider the case where M is a two-dimensional torus, $M = T^2$. On the standard chart for T^2 ($[0,1] \times [0,1]$ with opposite

edges identified), define the following sequences of flat metrics:

$$g_k^1 := \begin{pmatrix} 1 & 0 \\ 0 & k^{-1} \end{pmatrix}, \qquad g_k^2 := \begin{pmatrix} k^{-1} & 0 \\ 0 & k^{-1} \end{pmatrix},$$

$$g_k^3 := \begin{pmatrix} e^k & 0 \\ 0 & e^{-2k} \end{pmatrix}, \qquad g_k^4 := \begin{pmatrix} |\cos k| & 0 \\ 0 & k^{-1} \end{pmatrix}.$$

Since $\mathrm{Vol}(T^2, g_k^i) \to 0$ for all $i = 1, 2, 3, 4$, Proposition 4.1 implies that each of these sequences is d-Cauchy, and all are equivalent. Yet in terms of (pointed) Gromov–Hausdorff convergence, if d_k^i denotes the distance-function on M induced by g_k^i, then $\{(M, d_k^1)\}$ converges to a circle (one dimension of T^2 collapses), $\{(M, d_k^2)\}$ converges to a point (both dimensions collapse), $\{(M, d_k^3)\}$ converges to the real line, and $\{(M, d_k^4)\}$ does not converge at all. (The sequences $\{g_k^1\}$ and $\{g_k^2\}$ are related to the notion of collapse introduced by Cheeger and Gromov [**3**, cf. Ex. 0.4]. However, in contrast to their definition of collapse as a sequence of metrics on M with injectivity radius uniformly converging to zero and with uniformly bounded curvature, one can easily modify the examples given here to find Cauchy sequences in $(\mathcal{M}, d)$ that collapse with unbounded curvature and/or only on a subset of M.)

4.1.2. ω-Convergence and its Basic Properties. In this subsection, we give a convergence notion suited to the completion of $\mathcal{M}$, in that it allows sequences to behave badly on sets that collapse in the limit.

First, though, recall that we define general measure-theoretic notions (e.g., the notion of something holding almost everywhere, or a.e.) using the fixed reference metric g (cf. Convention 2.14). Furthermore, we need one definition before that of ω-convergence.

Definition 4.3. We denote by $\mathcal{M}_m$ the set of all measurable semi-metrics on M. That is, $\mathcal{M}_m$ is the set of all sections of $S^2 T^* M$ that have measurable coefficients and that induce a positive semidefinite scalar product on $T_x M$ for each $x \in M$.

Define an equivalence relation "$\sim$" on $\mathcal{M}_m$ by $g_0 \sim g_1$ if and only if

1) their deflated sets X_{g_0} and X_{g_1} differ at most by a nullset, and
2) $g_0(x) = g_1(x)$ for almost every $x \in M \setminus (X_{g_0} \cup X_{g_1})$.

We denote the quotient space of $\mathcal{M}_m$ by

$$\widehat{\mathcal{M}_m} := \mathcal{M}_m / \sim.$$

Definition 4.4. Let $\{g_k\}$ be a sequence in $\mathcal{M}$, and let $[g_\infty] \in \widehat{\mathcal{M}_m}$. Recall that we denote the deflated set of the sequence $\{g_k\}$ by $D_{\{g_k\}}$ and the deflated set of an individual semimetric $\tilde{g}$ by $X_{\tilde{g}}$ (cf. Definitions 2.21 and 2.22). We say that $\{g_k\}$ ω-converges to $[g_\infty]$ if for every representative $\tilde{g}_\infty \in [g_\infty]$, the following holds:

1) $\{g_k\}$ is d-Cauchy,

2) $X_{\tilde{g}_\infty}$ and $D_{\{g_k\}}$ differ at most by a nullset,
3) $g_k(x) \to \tilde{g}_\infty(x)$ for almost every $x \in M \setminus D_{\{g_k\}}$, and
4) $\sum_{k=1}^\infty d(g_k, g_{k+1}) < \infty$.

We call $[g_\infty]$ the ω-*limit* of the sequence $\{g_k\}$ and write $g_k \xrightarrow{\omega} [g_\infty]$.
(The use of the definite article "the" for the ω-limit of a sequence will
be justified in Lemma 4.5 below.) More generally, if $\{g_k\}$ is a d-Cauchy
sequence containing a subsequence that ω-converges to $[g_\infty]$, then we
say that $\{g_k\}$ ω-*subconverges* to $[g_\infty]$, write $g_k \xrightarrow{\omega} g_\infty$, and call $[g_\infty]$ an
ω-limit of $\{g_k\}$.

Loosely speaking, the utility of Definition 4.4 is that, in a sense to
be made precise, an ω-convergent sequence in $\mathcal{M}$ will d-converge to
the same limit. The full identification of the completion of $\mathcal{M}$ with
(equivalence classes of) semimetrics that allows this is given in the main
result of this paper, Theorem 5.19.

Condition (1) in the definition is simply there for convenience, so we
don't have to repeatedly assume that a sequence is ω-convergent *and*
Cauchy. Condition (4) is technical and will aid us in proofs. Conceptu-
ally, it means that we can find paths α_k connecting g_k to g_{k+1} such that
the concatenated path $\alpha_1 * \alpha_2 * \cdots$ has finite length. If $\{g_k\}$ is d-Cauchy,
then this can always be achieved by passing to a subsequence. (We re-
mark here, however, that these two conditions are not independent. In
fact, (4) implies (1).)

Note that condition (3) implies that if $g_k \xrightarrow{\omega} [g_\infty]$, then for almost
all $x \in M \setminus D_{\{g_k\}}$, there exists some $\delta(x) > 0$ such that

$$\det G_k(x) \geq \delta(x)$$

for all $k \in \mathbb{N}$.

We now move on to proving some properties of ω-convergence. We
first state an entirely trivial consequence of Definitions 4.3 and 4.4.

Lemma 4.5. *Let $[g_\infty] \in \mathcal{M}$, and let $\{g_k\}$ be a sequence in $\mathcal{M}$. Sup-
pose that for one given representative $g_\infty \in [g_\infty]$, $\{g_k\}$ together with g_∞
satisfies conditions (1)–(4) of Definition 4.4. Then these conditions are
also satisfied for $\{g_k\}$ together with every other representative of $[g_\infty]$.*

*Therefore, if can we verify these conditions for one representative of
an equivalence class, this already implies $\{g_k\} \xrightarrow{\omega} [g_\infty]$.*

We can thus consistently say that $\{g_k\}$ ω-converges to an individual
semimetric $g_\infty \in \mathcal{M}_m$ if the two together satisfy conditions (1)–(4) of
Definition 4.4.

The next property of ω-convergence is obvious from property (2) of
Definition 4.4.

Lemma 4.6. *If $\{g_k^0\}$ and $\{g_k^1\}$ both ω-converge to the same element $[g_\infty] \in \widehat{\mathcal{M}_m}$, then $\{g_k^0\}$ and $\{g_k^1\}$ have the same deflated set, up to a nullset.*

Recall that the main goal of this section is to show that each Cauchy sequence in $\mathcal{M}$ has an ω-convergent subsequence. To do this, we will first prove a pointwise result in the following subsection.

4.1.3. (Riemannian) Metrics on $\mathcal{M}_x$ Revisited. In order to more closely study the metric Θ_M on $\mathcal{M}$, we now take a closer look at the Riemannian metrics $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle^0$ (see Section 2.1.1 and Definition 2.6, respectively) that we have defined on the finite-dimensional manifold $\mathcal{M}_x$. The relationship between the two is quite simple:

$$(4.4) \quad \langle h, k \rangle_{\tilde{g}}^0 = \langle h, k \rangle_{\tilde{g}} \det \tilde{G} \quad \text{for all } \tilde{g} \in \mathcal{M}_x \text{ and } h, k \in T_{\tilde{g}} \mathcal{M}_x \cong \mathcal{S}_x.$$

Thus, we will first study the simpler Riemannian metric $\langle \cdot, \cdot \rangle$ and find out what properties of $\langle \cdot, \cdot \rangle^0$ we can deduce in this way. Despite their close relationship, their qualitative properties are very different—in particular, $\mathcal{M}_x$ is complete with respect to $\langle \cdot, \cdot \rangle$. We will show this using a simplified version of the analogous computations for $(\cdot, \cdot)$ on $\mathcal{M}$ carried out in [**9**, Thm. 2.3].

Before we start, let's clear up some notation.

Definition 4.7. By d_x, we denote the distance function induced on $\mathcal{M}_x$ by $\langle \cdot, \cdot \rangle$. We denote the $\langle \cdot, \cdot \rangle$-length of a path γ in $\mathcal{M}_x$ by $L^{\langle \cdot, \cdot \rangle}(\gamma)$ and the $\langle \cdot, \cdot \rangle^0$-length by $L^{\langle \cdot, \cdot \rangle^0}(\gamma)$.

Now we compute the Christoffel symbols.

Proposition 4.8. *Let h and k be constant (that is, translation-invariant) vector fields on $\mathcal{M}_x$, and denote the Levi-Civita connection of $\langle \cdot, \cdot \rangle$ by ∇. Then the Christoffel symbols of $\langle \cdot, \cdot \rangle$ are given by*

$$\Gamma(h, k) = \nabla_h k|_{\tilde{g}} = -\frac{1}{2} \left(h\tilde{g}^{-1}k + k\tilde{g}^{-1}h \right).$$

Proof. All computations are done at the base point $\tilde{g}$, which we will omit from the notation for convenience. Let ℓ be any other constant vector field on $\mathcal{M}_x$. Using the Koszul formula, we can compute that

$$(4.5) \qquad 2\langle \nabla_h k, \ell \rangle = h\langle k, \ell \rangle + k\langle \ell, h \rangle - \ell\langle h, k \rangle.$$

Using the fact that the derivative of the map $\hat{g} \mapsto \hat{g}^{-1}$ at the point $\tilde{g}$ is given by $a \mapsto -\tilde{g}^{-1}a\tilde{g}^{-1}$, one can compute that

$$h\langle k, \ell \rangle = -\operatorname{tr}\left((\tilde{g}^{-1}h\tilde{g}^{-1}k)(\tilde{g}^{-1}\ell) \right) - \operatorname{tr}\left((\tilde{g}^{-1}k)(\tilde{g}^{-1}h\tilde{g}^{-1}\ell) \right).$$

Repeating the same computation for the other permutations and substituting the results into (4.5) then gives the result.   q.e.d.

Using this, it is a relatively simple matter to solve the geodesic equation of $\langle \cdot, \cdot \rangle$.

Proposition 4.9. *The geodesic γ in $(\mathcal{M}_x, \langle \cdot, \cdot \rangle)$ with initial data $\gamma(0) = g_0$, $\gamma'(0) = h$ is given by*

$$\gamma(t) = g_0 e^{tg_0^{-1}h}.$$

In particular, $(\mathcal{M}_x, d_x)$ is a complete metric space.

Proof. Let $a(t) := \gamma'(t)$. Since γ is a geodesic, we have

$$(4.6) \quad 0 = \nabla_{a(t)} a(t) = a'(t) + \Gamma(a(t), a(t)) = a'(t) - a(t)\gamma(t)^{-1}a(t)$$

by Proposition 4.8. (Note that Γ denotes the Christoffel symbols here, and *not* $g(x)^{-1}\gamma$.) Now, multiplying (4.6) on the left by $\gamma(t)^{-1}$ gives

$$(\gamma(t)^{-1}a(t))' = 0.$$

Thus $\gamma(t)^{-1}\gamma'(t)$ is constant, or $\log(\gamma(t))' = \gamma(t)^{-1}\gamma'(t) \equiv g_0^{-1}h$. The geodesic equation now follows, and since $g_0 e^{tg_0^{-1}h} \in \mathcal{M}_x$ for all $t \in \mathbb{R}$, the Hopf-Rinow theorem implies that $(\mathcal{M}_x, d_x)$ is complete.   q.e.d.

We now want to use Proposition 4.9 and (4.4) to characterize θ_x^g-Cauchy sequences in $\mathcal{M}_x$. First, though, we need a lemma that is a pointwise version of Lemma 2.5. The proof is completely analogous to that of Lemma 2.5, and so we omit it.

Lemma 4.10. *Let $a_0, a_1 \in \mathcal{M}_x$. Then*

$$\left| \sqrt{\det A_1} - \sqrt{\det A_0} \right| \leq \frac{\sqrt{n}}{2} \theta_x^g(a_0, a_1).$$

(Recall Convention 2.14 for the definitions of A_i.)

Proposition 4.11. *Let a_k be a θ_x^g-Cauchy sequence. Then either*

1) *$\det A_k \to 0$ as $k \to \infty$, or*
2) *there exist constants $C, \eta > 0$ such that $|(a_k)_{ij}| \leq C$ and $\det A_k \geq \eta$ for all $1 \leq i, j \leq n$ and $k \in \mathbb{N}$.*

Proof. Keeping Lemma 4.10 in mind, it is more convenient to work with the square root of the determinant. This is, of course, completely equivalent for our purposes.

Now, by Lemma 4.10, the map $a \mapsto \sqrt{\det A}$ is θ_x^g-Lipschitz. Since a_k is θ_x^g-Cauchy, $L := \lim_{k \to \infty} \sqrt{\det A_k}$ is well-defined.

If for every $\eta > 0$, there exists k such that $\sqrt{\det A_k} \leq \eta$, then clearly $L = 0$.

It remains to show that if there exist i and j such that for all $C > 0$, there is a k such that $|(a_k)_{ij}| > C$, then $\sqrt{\det A_k} \to 0$. We will assume that $L > 0$ and show a contradiction.

Let's say that we are given $b_0, b_1 \in \mathcal{M}_x$ with $\det B_0, \det B_1 \geq \delta$. Let

$$L_{-\delta} := \inf \left\{ L^{\langle \cdot, \cdot \rangle^0}(\beta) \,\middle|\, \begin{array}{l} \beta : [0,1] \to \mathcal{M}_x \text{ is a path from } b_0 \text{ to } b_1 \text{ with} \\ \det(g(x)^{-1}\beta(t)) \leq \delta/2 \text{ for some } t \in (0,1) \end{array} \right\},$$

$$L_{+\delta} := \inf \left\{ L^{\langle \cdot, \cdot \rangle^0}(\beta) \,\middle|\, \begin{array}{l} \beta : [0,1] \to \mathcal{M}_x \text{ is a path from } b_0 \text{ to } b_1 \text{ with} \\ \det(g(x)^{-1}\beta(t)) \geq \delta/2 \text{ for all } t \in [0,1] \end{array} \right\}.$$

It is easy to see that $\theta_x^g(b_0, b_1) = \min(L_{-\delta}, L_{+\delta})$. Now let β be a path as in the definition of $L_{-\delta}$, let $b_t := \beta(t)$, and assume $\tau \in (0,1)$ is such that $\det B_\tau \leq \delta/2$. Then using Lemma 4.10, we have

$$L^{\langle \cdot, \cdot \rangle^0}(\beta) \geq \frac{\sqrt{n}}{2} \left| \sqrt{\det B_0} - \sqrt{\det B_\tau} \right| + \frac{\sqrt{n}}{2} \left| \sqrt{\det B_1} - \sqrt{\det B_\tau} \right|$$

$$\geq \sqrt{n} \left(\sqrt{\delta} - \sqrt{\frac{\delta}{2}} \right) = \sqrt{n} \left(1 - \frac{1}{\sqrt{2}} \right) \sqrt{\delta}.$$

Therefore $L_{-\delta} \geq \sqrt{n}(1 - 1/\sqrt{2})\sqrt{\delta}$. Then, if β is a path as in the definition of $L_{+\delta}$, we have

$$L(\beta) = \int_0^1 \sqrt{\langle \beta'(t), \beta'(t) \rangle^0} \, dt = \int_0^1 \sqrt{\langle \beta'(t), \beta'(t) \rangle} \det(g(x)^{-1}\beta(t)) \, dt$$

$$\geq \sqrt{\frac{\delta}{2}} \int_0^1 \sqrt{\langle \beta'(t), \beta'(t) \rangle} \, dt \geq \sqrt{\frac{\delta}{2}} \, d_x(b_0, b_1).$$

This gives $L_{+\delta} \geq \sqrt{\delta/2} \, d_x(b_0, b_1)$. Putting this together, we get that

$$(4.7) \qquad \theta_x^g(b_0, b_1) \geq \min\{\sqrt{n}(1 - 1/\sqrt{2})\sqrt{\delta}, \sqrt{\delta/2} \, d_x(b_0, b_1)\}$$

whenever $\det B_0, \det B_1 \geq \delta$.

Now, let's apply the considerations of the last paragraph to the problem at hand. Let i and j be, as above, the indices for which $|(a_k)_{ij}|$ is unbounded, and choose a subsequence, which we again denote by a_k, such that $|(a_k)_{ij}| \geq k$ for all $k \in \mathbb{N}$. Passing to this subsequence does not change the limit $\lim_{k \to \infty} \sqrt{\det A_k}$.

Next, choose $K \in \mathbb{N}$ such that $k \geq K$ implies $\sqrt{\det A_k} \geq L/2$ and $k, l \geq K$ implies $\theta_x^g(a_k, a_l) \leq \frac{\sqrt{n}}{4}(1 - 1/\sqrt{2})L$. The latter assumption is possible since a_k is Cauchy. By (4.7), if $k \geq K$, we also have

$$\theta_x^g(a_K, a_k) \geq \min \left\{ \frac{\sqrt{n}}{2}(1 - 1/\sqrt{2})L, d_x(a_K, a_k)\frac{L}{2\sqrt{2}} \right\}.$$

But $\theta_x^g(a_K, a_k) \geq \frac{\sqrt{n}}{2}(1 - 1/\sqrt{2})L$ violates our assumptions on K. Furthermore, $d_x(a_K, a_k) \to \infty$ since $|(a_k)_{ij}| \to \infty$ and $(\mathcal{M}_x, \langle \cdot, \cdot \rangle)$ is complete. Therefore, if $\theta_x^g(a_K, a_k) \geq d_x(a_K, a_k)\frac{L}{2\sqrt{2}}$ for all k, then our assumptions on K are violated as well. Thus we have achieved the desired contradiction. $\hfill$ q.e.d.

Since for every pair of constants $C, \eta > 0$, the set of elements $\tilde{g}$ of $\mathcal{M}_x$ with $|\tilde{g}_{ij}| \leq C$ and $\det \tilde{G} \geq \eta$ for all $1 \leq i, j \leq n$ is compact, we immediately get the following corollary of Proposition 4.11:

Corollary 4.12. *Let $\{g_k\}$ be a θ_x^g-Cauchy sequence. Then either*

1) $\det G_k \to 0$ *as $k \to \infty$, or*
2) *there exists an element $g_\infty \in \mathcal{M}_x$ such that $g_k \to g_\infty$, with convergence in the manifold topology of $\mathcal{M}_x$.*

This is essentially a pointwise equivalent of ω-convergence. In the next subsection, we will globalize this result. Before we do that, though, we use this opportune moment to prove two last pointwise results, which will be useful in Section 4.3. The first is the pointwise analog of Proposition 4.1. Again, the proof is analogous to that of Proposition 4.1, so we omit it.

Proposition 4.13. *Let $\tilde{g}, \hat{g} \in \mathcal{M}_x$. Then there exists a constant $C'(n)$, depending only on n, such that*

$$\theta_x^g(\tilde{g}, \hat{g}) \leq C'(n) \left(\sqrt{\det \tilde{G}} + \sqrt{\det \hat{G}} \right).$$

The last pointwise result we need combines Corollary 4.12 and Proposition 4.13 to give a description of the completion of the metric space $(\mathcal{M}_x, \theta_x^g)$.

Theorem 4.14. *For any given $x \in M$, let $\mathrm{cl}(\mathcal{M}_x)$ denote the closure of $\mathcal{M}_x \subset \mathcal{S}_x$ with regard to the natural topology inherited by $\mathcal{S}_x$ from the manifold topology of M. Then $\mathrm{cl}(\mathcal{M}_x)$ consists of all positive semidefinite $(0, 2)$-tensors at x. Let us denote the boundary of $\mathcal{M}_x$, as a subspace of $\mathcal{S}_x$, by $\partial \mathcal{M}_x$.*

Then the completion of $(\mathcal{M}_x, \theta_x^g)$ is homeomorphic to the quotient of the space $\mathrm{cl}(\mathcal{M}_x)$ where $\partial \mathcal{M}_x$ has been identified to a single point. Under this identification, the distance function is given by

$$\theta_x^g([g_0], [g_1]) = \lim_{k \to \infty} \theta_x^g(g_k^0, g_k^1),$$

where $[g_0], [g_1] \in \mathrm{cl}(\mathcal{M}_x)/\partial \mathcal{M}_x$ and $\{g_k^0\}$ and $\{g_k^1\}$ are any sequences in $\mathcal{M}_x$ such that the sequences $\{[g_k^0]\}$ and $\{[g_k^1]\}$ in $\mathrm{cl}(\mathcal{M}_x)/\partial \mathcal{M}_x$ converge to $[g_0]$ and $[g_1]$, respectively.

Proof. Let $\{g_k\}$ be any sequence in $\mathcal{M}_x$. By Corollary 4.12, if $\{g_k\}$ is θ_x^g-Cauchy then either $g_k \to g_\infty \in \mathcal{M}_x$ (with convergence in the topology of $\mathcal{S}_x$), or $\det G_k \to 0$. In fact, by the equivalence of the topologies on $\mathcal{M}_x$ inherited from $\mathcal{S}_x$ and θ_x^g, one sees as well that if $g_k \to g_\infty \in \mathcal{M}_x$ in the topology of $\mathcal{S}_x$, then $\{g_k\}$ is θ_x^g-Cauchy. Furthermore, two θ_x^g-Cauchy sequences in $\mathcal{M}_x$ that converge to distinct elements of $\mathcal{M}_x$ are inequivalent as θ_x^g-Cauchy sequences.

By Proposition 4.13, all sequences with $\underline{\det G_k \to 0}$ are θ_x^g-Cauchy and equivalent, and so they are identified in $\overline{(\mathcal{M}_x, \theta_x^g)}$.

Thus, there is a natural bijection $\phi : \mathrm{cl}(\mathcal{M}_x)/\partial\mathcal{M}_x \to \overline{(\mathcal{M}_x, \theta_x^g)}$ identifying an element $[g_0] \in \mathrm{cl}(\mathcal{M}_x)/\partial\mathcal{M}_x$ with an equivalence class of θ_x^g-Cauchy sequences whose projections to $\mathrm{cl}(\mathcal{M}_x)/\partial\mathcal{M}_x$ have limit $[g_0]$. Continuity of ϕ is immediately implied by the preceding arguments.

To establish continuity of ϕ^{-1}, consider θ_x^g-Cauchy sequences $\{g_k^l\}$ and $\{g_k^\infty\}$, for $l \in \mathbb{N}$, such that the sequences $\{[g_k^l]\}$ and $\{[g_k^\infty]\}$ in $\mathrm{cl}(\mathcal{M}_x)/\partial\mathcal{M}_x$ have limits $[g^l]$ and $[g^\infty]$. (Note that $[g^l] = \phi^{-1}(\{g_k^l\})$ and $[g^\infty] = \phi^{-1}(\{g_k^\infty\})$.) Let $\tilde{g}^l \in [g^l]$ and $\tilde{g}^\infty \in [g^\infty]$ be any representatives. Suppose that we have $\theta_x^g(\{g_k^l\}, \{g_k^\infty\}) \to 0$ as $l \to \infty$; then

$$(4.8) \qquad \lim_{l \to \infty} \lim_{k \to \infty} \theta_x^g(g_k^l, g_k^\infty) = 0.$$

To complete the proof, we wish to show that $\lim_{l \to \infty}[g^l] = [g^\infty]$ in the topology of $\mathrm{cl}(\mathcal{M}_x)/\partial\mathcal{M}_x$.

First, we see by Lemma 4.10,

$$
\begin{aligned}
(4.9) \qquad 0 &= \lim_{l \to \infty} \lim_{k \to \infty} \theta_x^g(g_k^l, g_k^\infty) \geq \frac{\sqrt{n}}{2} \lim_{l \to \infty} \lim_{k \to \infty} \left| \sqrt{\det G_k^\infty} - \sqrt{\det G_k^l} \right| \\
&= \frac{\sqrt{n}}{2} \lim_{l \to \infty} \left| \sqrt{\det \tilde{G}^\infty} - \sqrt{\det \tilde{G}^l} \right|.
\end{aligned}
$$

(Note that $\det \tilde{G}^l$ and $\det \tilde{G}^\infty$ do not depend of the choice of representative for $[g^l]$ or $[g^\infty]$.) The last equality holds since $\lim_{k \to \infty} \det G_k^l = \det \tilde{G}^l$ follows from Lemma 4.10 if $\tilde{g}^l \in \mathcal{M}_x$, and otherwise it follows since $\lim_{k \to \infty} \det G_k^l = 0 = \det \tilde{G}^l$. An analogous argument holds for $\{g_k^\infty\}$ and $\tilde{g}^\infty$.

So suppose that $\tilde{g}^\infty \in \partial\mathcal{M}_x$. Then (4.9) implies that $\det \tilde{G}^l \to 0$, and in particular, $[g^l] \to [g^\infty]$ in the topology of $\mathrm{cl}(\mathcal{M}_x)/\partial\mathcal{M}_x$. On the other hand, if $\tilde{g}^\infty \in \mathcal{M}_x$, then (4.9) implies that $\tilde{g}^l \in \mathcal{M}_x$ for sufficiently large l, and so the following is well defined and follows from (4.8):

$$0 = \lim_{l \to \infty} \lim_{k \to \infty} \theta_x^g(g_k^l, g_k^\infty) = \lim_{l \to \infty} \theta_x^g(\tilde{g}^l, \tilde{g}^\infty).$$

But this in turn implies that $[g^l] \to [g^\infty]$ as $l \to \infty$ in the topology of $\mathrm{cl}(\mathcal{M}_x)/\partial\mathcal{M}_x$. This completes the proof.   q.e.d.

4.1.4. The Existence Proof. We now wish to globalize Corollary 4.12 to characterize d-Cauchy sequences, using Proposition 2.8 to reduce questions about d to questions about the simpler metric Θ_M.

Lemma 4.15. *Let $\{g_k\}$ be a Cauchy sequence in $\mathcal{M}$. Assume that*

$$(4.10) \qquad \sum_{k=1}^{\infty} d(g_k, g_{k+1}) < \infty.$$

Then the following holds:

$$(4.11) \qquad \sum_{k=1}^{\infty} \Theta_M(g_k, g_{k+1}) < \infty.$$

Furthermore, define functions Ω and Ω_N for each $N \in \mathbb{N}$ by

$$\Omega_N(x) := \sum_{k=1}^{N} \theta_x^g(g_k(x), g_{k+1}(x)), \quad \Omega(x) := \sum_{k=1}^{\infty} \theta_x^g(g_k(x), g_{k+1}(x)).$$

Then Ω is a.e. finite, $\Omega \in L^1(M, g)$ and $\Omega_N \xrightarrow{L^1} \Omega$. Furthermore, by definition, Ω_N converges to Ω pointwise.

Proof. The statement that $\Omega_N \to \Omega$ pointwise is clear. So we move on to the other statements.

Lemma 2.5 implies that $\sqrt{\mathrm{Vol}(M, g_k)}$ is a Cauchy sequence in $\mathbb{R}$. Therefore it is bounded, and we can find a constant V such that we have $\sqrt{\mathrm{Vol}(M, g_k)} \leq V$ for all k. Thus, by Proposition 2.8,

$$\Theta_M(g_k, g_{k+1}) \leq d(g_k, g_{k+1}) \left(\sqrt{n} d(g_k, g_{k+1}) + 2V \right).$$

But for large k, since $\{g_k\}$ is Cauchy, we must have $d(g_k, g_{k+1}) \leq 1$, so

$$\Theta_M(g_k, g_{k+1}) \leq (\sqrt{n} + 2V) d(g_k, g_{k+1}).$$

The estimate (4.11) is now immediate from (4.10).

We can then compute

$$\int_M \Omega \, \mu_g = \int_M \left(\sum_{k=1}^{\infty} \theta_x^g(g_k, g_{k+1}) \right) \mu_g = \sum_{k=1}^{\infty} \int_M \theta_x^g(g_k, g_{k+1}) \, \mu_g$$

$$= \sum_{k=1}^{\infty} \Theta_M(g_k, g_{k+1}) < \infty,$$

where exchanging the infinite sum and the integral is valid due to a theorem of Lebesgue [**14**, Thm. (12.21)]. Finiteness follows from the first part of the lemma. This proves that Ω is a.e. finite and $\Omega \in L^1(M, g)$. It remains to show that $\Omega_N \xrightarrow{L^1} \Omega$. But this is now immediate from a classical theorem of F. Riesz [**22**], which states that if $1 \leq p < \infty$, $f_i \to f$ a.e. and $\|f_i\|_p \to \|f\|_p$, then $f_i \xrightarrow{L^p} f$.   q.e.d.

Using this lemma, we can globalize Corollary 4.12.

Proposition 4.16. *Let $\{g_k\}$ be a Cauchy sequence in $\mathcal{M}$ such that*

$$\sum_{k=1}^{\infty} d(g_k, g_{k+1}) < \infty.$$

Then for almost every $x \in M$, $\{g_k(x)\}$ is θ_x^g-Cauchy and either:

1) $\det G_k(x) \to 0$ *as $k \to \infty$, or*

2) $g_k(x)$ *is a convergent sequence in* $\mathcal{M}_x$.
Furthermore, (1) *holds for almost every* $x \in D_{\{g_k\}}$, *and* (2) *holds for almost every* $x \in M \setminus D_{\{g_k\}}$.

Proof. By our assumption, all the conclusions of Lemma 4.15 hold. In particular, $\Omega_N \to \Omega$ pointwise and Ω is a.e. finite. Therefore, for almost every $x \in M$,

$$(4.12) \qquad \sum_{k=1}^{\infty} \theta_x^g(g_k(x), g_{k+1}(x)) = \Omega(x) < \infty.$$

From this, it is immediate that $\{g_k(x)\}$ is θ_x^g-Cauchy. The remaining results follow from Corollary 4.12.   q.e.d.

This proposition essentially delivers us the proof of the existence result.

Theorem 4.17. *For every Cauchy sequence* $\{g_k\}$ *in* $\mathcal{M}$, *there exists an element* $[g_\infty] \in \widehat{\mathcal{M}_m}$ *and a subsequence* $\{g_{k_l}\}$ *such that* $\{g_{k_l}\}$ ω-*converges to* $[g_\infty]$.
Explicitly, given the subsequence $\{g_{k_l}\}$, $[g_\infty]$ *is the unique equivalence class containing the element* $g_\infty \in \mathcal{M}_m$ *defined as follows. At points* $x \in M$ *where* $\{g_{k_l}(x)\}$ *is* θ_x^g-*Cauchy*,
   1) $g_\infty(x) := 0$ *for* $x \in D_{\{g_{k_l}\}}$ *and*
   2) $g_\infty(x) := \lim g_{k_l}(x)$ *for* $x \in M \setminus D_{\{g_{k_l}\}}$.
At points $x \in M$ *where* $\{g_{k_l}(x)\}$ *is not* θ_x^g-*Cauchy, we set* $g_\infty(x) := 0$.

Proof. Let $\{g_{k_l}\}$ be a subsequence of $\{g_k\}$ such that

$$\sum_{l=1}^{\infty} d(g_{k_l}, g_{k_l+1}) < \infty.$$

Then $\{g_{k_l}\}$ satisfies properties (1) and (4) of Definition 4.4, as well as the hypotheses of Corollary 4.16. Thus $\{g_{k_l}\}$ is a.e. θ_x^g-Cauchy, and so g_∞ is defined a.e. by the two conditions given above. From this, it is immediate that $\{g_{k_l}\}$ together with g_∞ also satisfies properties (2) and (3) of Definition 4.4. Thus, $\{g_{k_l}\}$ ω-converges to g_∞, and by Lemma 4.5 it therefore ω-converges to $[g_\infty]$—provided we can show that $g_\infty \in \mathcal{M}_m$. But g_∞ is clearly a semimetric, and it is the pointwise limit of the measurable semimetrics $\chi(M \setminus D_{\{g_{k_l}\}})g_{k_l}$ (one can easily construct the set $D_{\{g_{k_l}\}}$ as a countable intersection of open sets, so it is measurable). Therefore g_∞ itself is measurable.   q.e.d.

Knowing now that a Cauchy sequence in $\mathcal{M}$ ω-subconverges (see Definition 4.4), we go further into the properties of ω-convergence. Note that as of yet, we have not addressed whether all ω-convergent subsequences of a Cauchy sequence in $\mathcal{M}$ have the same ω-limit. Resolving this issue is postponed until Theorem 4.28 below.

4.2. ω-Convergence and the Concept of Volume. In this brief subsection, we wish to prove that the volumes of measurable subsets behave well under ω-convergence. To make this precise, we begin with a definition.

Definition 4.18. Let $\mathcal{M}_f \subset \mathcal{M}_m$ denote the set of all elements of $\mathcal{M}_m$ that have finite volume, where the volume form of $\tilde{g} \in \mathcal{M}_m$ is defined in the usual way by $\mu_{\tilde{g}}(x) = \sqrt{\det \tilde{g}(x)} dx^1 \wedge \cdots \wedge dx^n$ in coordinates. (Note that $\mu_{\tilde{g}}$ is a measure on the Lebesgue sets of M, as discussed in §2.3.2.)

We also define $\widehat{\mathcal{M}_f} := \mathcal{M}_f/\sim$, where "$\sim$" is the equivalence relation introduced in Definition 4.3.

For any $\tilde{g} \in \mathcal{M}_f$ and any measurable subset $Y \subseteq M$, we define $\mathrm{Vol}(Y, \tilde{g}) := \int_Y d\mu_{\tilde{g}}$.

In this subsection, we want to show that if $\{g_k\}$ ω-converges to $[g_\infty]$ and $Y \subseteq M$ is measurable, then for any representative $g_\infty \in [g_\infty]$,

$$(4.13) \qquad\qquad \mathrm{Vol}(Y, g_k) \to \mathrm{Vol}(Y, g_\infty).$$

To see that the above expression is well-defined, note that given any two representatives $g_\infty^0, g_\infty^1 \in [g_\infty]$, we have that $\mu_{g_\infty^0} = \mu_{g_\infty^1}$ as measures—it is clear from Definition 4.3 that $\mu_{g_\infty^0}$ and $\mu_{g_\infty^1}$ can differ at most on a nullset. Thus $\mathrm{Vol}(Y, g_\infty^0) = \mathrm{Vol}(Y, g_\infty^1)$.

The proof of (4.13) is achieved via the Lebesgue dominated convergence theorem with the help of the next two lemmas.

Lemma 4.19. *Let $\{g_k\}$ ω-converge to $g_\infty \in \mathcal{M}_m$. Then*

$$\left(\frac{\mu_{g_k}}{\mu_g} \right) \xrightarrow{\text{a.e.}} \left(\frac{\mu_{g_\infty}}{\mu_g} \right).$$

Proof. We first prove that for almost every $x \in D_{\{g_k\}}$,

$$\left(\frac{\mu_{g_k}}{\mu_g} \right) = \sqrt{\det G_k(x)} \to 0 = \sqrt{\det G_\infty} = \left(\frac{\mu_{g_\infty}}{\mu_g} \right)$$

as $k \to \infty$. By the definition of the deflated set, for every $x \in D_{\{g_k\}}$ and $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $\det G_k(x) < \epsilon$. But we also know from Proposition 4.16 and property (4) of Definition 4.4 that $\{g_k(x)\}$ is θ_x^g-Cauchy for almost every $x \in M$. Hence, by Lemma 4.10, $\left\{ \sqrt{\det G_k(x)} \right\}$ is a Cauchy sequence in $\mathbb{R}$ at such points. Therefore it has a limit, and this limit must be 0.

Now, for almost every $x \in M \setminus D_{\{g_k\}}$, $g_k(x) \to g_\infty(x)$. Since the determinant is a continuous map from the space of $n \times n$ matrices into $\mathbb{R}$, this immediately implies that $\det G_k(x) \to \det G_\infty(x)$ for almost every $x \in M \setminus D_{\{g_k\}}$. q.e.d.

Our next task is to find an L^1 function that dominates (μ_{g_k}/μ_g).

Lemma 4.20. *Let $\{g_k\}$ be a Cauchy sequence such that*

$$\sum_{k=1}^{\infty} d(g_k, g_{k+1}) < \infty,$$

and let Ω be the function of Lemma 4.15. Then

$$\left(\frac{\mu_{g_k}}{\mu_g}\right)(x) \leq \frac{\sqrt{n}}{2}\Omega(x) + \left(\frac{\mu_{g_1}}{\mu_g}\right)(x)$$

for almost every $x \in M$ and all $k \in \mathbb{N}$.

Proof. By Proposition 4.16, $\{g_k(x)\}$ is θ_x^g-Cauchy for almost every $x \in M$. Let $x \in M$ be a point where this holds. Then by Lemma 4.10, the triangle inequality, and the definitions of Ω_N and Ω, we have, for any fixed k,

$$\left|\sqrt{\det G_k} - \sqrt{\det G_1}\right| \leq \frac{\sqrt{n}}{2}\theta_x^g(g_k, g_1) \leq \frac{\sqrt{n}}{2}\sum_{m=1}^{k-1}\theta_x^g(g_m, g_{m+1})$$

$$= \frac{\sqrt{n}}{2}\Omega_{k-1}(x) \leq \frac{\sqrt{n}}{2}\Omega(x).$$

The result is now immediate. q.e.d.

Now, since μ_{g_1} is smooth, it has finite volume, and hence $(\mu_{g_1}/\mu_g) \in L^1(M, g)$. We have already seen in Lemma 4.15 that $\Omega \in L^1(M, g)$. Therefore Lemmas 4.19 and 4.20 allow us to apply the Lebesgue dominated convergence theorem to obtain:

Theorem 4.21. *Let $\{g_k\}$ ω-converge to $g_\infty \in \mathcal{M}_m$, and let $Y \subseteq M$ be any measurable subset. Then $\mathrm{Vol}(Y, g_k) \to \mathrm{Vol}(Y, g_\infty)$.*

An immediate corollary of this theorem and Lemma 2.5 is that the total volume of an ω-limit is finite.

Corollary 4.22. *If $g_\infty \in \mathcal{M}_m$ is an ω-limit of a sequence $\{g_k\}$ in $\mathcal{M}$, then $\mathrm{Vol}(M, g_\infty) < \infty$, i.e., $g_\infty \in \mathcal{M}_f$.*

Furthermore, as we might have suspected from the beginning, the volume of the deflated set $D_{\{g_k\}}$ of an ω-convergent sequence vanishes in the limit. This is because $\mathrm{Vol}(D_{\{g_k\}}, g_\infty) = 0$.

Corollary 4.23. *Let $\{g_k\}$ ω-converge to $g_\infty \in \mathcal{M}_f$. Then we have $\mathrm{Vol}(D_{\{g_k\}}, g_l) \to 0$ as $l \to \infty$. Furthermore, if $Y \subset M$ is such that $Y \setminus D_{\{g_k\}}$ has positive μ_g-measure, then $\mathrm{Vol}(Y, g_\infty) > 0$.*

Since we now know the volume of an ω-limit is finite, we can refine Theorem 4.17:

Theorem 4.24. *For every Cauchy sequence $\{g_k\}$ in $\mathcal{M}$, there exists an element $[g_\infty] \in \widehat{\mathcal{M}_f}$ such that some subsequence of $\{g_k\}$ ω-converges to $[g_\infty]$.*

4.3. Uniqueness of the ω-Limit. The goal of this section is to prove the uniqueness of the ω-limit in the sense mentioned in the introduction to the chapter: we will show that two ω-convergent Cauchy sequences in $\mathcal{M}$ are equivalent if and only if they have the same ω-limit. We prove each direction in a separate subsection.

4.3.1. First Uniqueness Result. We first prove the statement that if two ω-convergent Cauchy sequences are equivalent, then their ω-limits agree. To do so, we will extend the pseudometric Θ_Y (cf. Definition 2.7) to the precompletion of $\mathcal{M}$. For this, we need an easy lemma.

Lemma 4.25. *Let $Y \subseteq M$ be measurable. If $\{g_k\}$ is a d-Cauchy sequence, then it is also Θ_Y-Cauchy.*

Proof. As noted in the proof of Lemma 4.15, since $\{g_k\}$ is d-Cauchy, the sequence $\sqrt{\mathrm{Vol}(M, g_k)}$ in $\mathbb{R}$ is bounded, so Proposition 2.8 gives the result easily.   q.e.d.

Now we give the extension of Θ_Y mentioned above.

Proposition 4.26. *Let $Y \subseteq M$ be measurable. Then the pseudometric Θ_Y on $\mathcal{M}$ can be extended to a pseudometric on $\overline{\mathcal{M}}^{\mathrm{pre}}$, the precompletion of $\mathcal{M}$, via*

$$(4.14) \qquad \Theta_Y(\{g_k^0\}, \{g_k^1\}) := \lim_{k \to \infty} \Theta_Y(g_k^0, g_k^1).$$

This pseudometric is weaker than d in the sense that $d(\{g_k^0\}, \{g_k^1\}) = 0$ implies $\Theta_Y(\{g_k^0\}, \{g_k^1\}) = 0$ for any d-Cauchy sequences $\{g_k^0\}$ and $\{g_k^1\}$. More precisely, given two such sequences, if $\{g_{k_l}^i\}$ is any ω-convergent subsequence of $\{g_k^i\}$, $i \in \{0, 1\}$, then

$$(4.15) \quad \Theta_Y(\{g_{k_l}^0\}, \{g_{k_l}^1\}) \le$$
$$d(\{g_{k_l}^0\}, \{g_{k_l}^1\}) \left(\sqrt{n}\, d(\{g_{k_l}^0\}, \{g_{k_l}^1\}) + 2\sqrt{\mathrm{Vol}(M, g_0)} \right),$$

where g_0 is any element of $\mathcal{M}_f$ for which $g_{k_l}^0 \xrightarrow{\omega} [g_0]$.

Furthermore, let any $g_0, g_1 \in \mathcal{M}_f$ be given. If $\{g_k^0\}$ and $\{g_k^1\}$ are sequences in $\mathcal{M}$ that ω-converge to g_0 and g_1, respectively, then we have

$$(4.16) \qquad \Theta_Y(\{g_k^0\}, \{g_k^1\}) = \int_Y \theta_x^g(g_0(x), g_1(x))\, \mu_g(x).$$

Remark 4.27. The existence of the ω-convergent subsequences used in the Proposition has been proven, but it has not been proven that there is a unique ω-limit to which a d-Cauchy sequence subconverges (see Definition 4.4).

Proof of Proposition 4.26. The construction of a pseudometric on the precompletion of a metric space can be carried over to the case where we begin with a pseudometric space. Therefore, the limit in (4.14) is

well-defined due to the fact that $\{g_k^0\}$ and $\{g_k^1\}$ are Cauchy sequences with respect to Θ_Y, and (4.14) indeed defines a pseudometric.

The inequality (4.15) is proved via the following simple computation, which uses (4.14), Proposition 2.8, and Theorem 4.21:

$$
\begin{aligned}
\Theta_Y(\{g_{k_l}^0\}, \{g_{k_l}^1\}) &= \lim_{l\to\infty} \Theta_Y(g_{k_l}^0, g_{k_l}^1) \\
&\leq \lim_{l\to\infty} d(g_{k_l}^0, g_{k_l}^1)\left(\sqrt{n}\, d(g_{k_l}^0, g_{k_l}^1) + 2\sqrt{\mathrm{Vol}(M, g_{k_l}^0)}\right) \\
&= d(\{g_{k_l}^0\}, \{g_{k_l}^1\})\left(\sqrt{n}\, d(\{g_{k_l}^0\}, \{g_{k_l}^1\}) + 2\sqrt{\mathrm{Vol}(M, g_0)}\right).
\end{aligned}
$$

As for (4.16), note first that $\theta_x^g(g_0(x), g_1(x))$ is well-defined by Theorem 4.14, since g_0 and g_1 are positive-semidefinite tensors at each point $x \in M$. To prove (4.16), we will first use Fatou's Lemma to show that $\theta_x^g(g_0(x), g_1(x))$ is integrable. We will then use this to apply the Lebesgue dominated convergence theorem.

By Proposition 4.16, for almost every $x \in M$, $\{g_k^0(x)\}$ and $\{g_k^1(x)\}$ are θ_x^g-Cauchy. At such points, by definition,

$$(4.17) \qquad \theta_x^g(g_0(x), g_1(x)) = \lim_{k\to\infty} \theta_x^g(g_k^0(x), g_k^1(x)).$$

So defining

$$f_k(x) := \theta_x^g(g_k^0(x), g_k^1(x)), \quad f(x) := \theta_x^g(g_0(x), g_1(x)),$$

we have $f_k \to f$ a.e.

Now, note that

$$\Theta_Y(g_k^0, g_k^1) = \int_Y f_k(x)\, \mu_g(x).$$

We have already seen that $\lim_{k\to\infty} \Theta_Y(g_k^0, g_k^1)$ exists, so $\{\Theta_Y(g_k^0, g_k^1)\}$ is in particular a bounded sequence of real numbers. Thus Fatou's Lemma applies to the sequence $\{f_k\}$, and we obtain

$$\int_Y f(x)\, d\mu_g(x) \leq \liminf_{k\to\infty} \int_Y f_k(x)\, d\mu_g(x) = \liminf_{k\to\infty} \Theta_Y(g_k^0, g_k^1) < \infty,$$

where we have used Fatou's Lemma in the first inequality.

Now we wish to verify the assumptions of the Lebesgue dominated convergence theorem for f_k and f. We note that for each $l > k$, the triangle inequality gives

$$
\begin{aligned}
f_k(x) \leq \sum_{m=k}^{l-1} \theta_x^g(g_m^0(x), g_{m+1}^0(x)) \\
+ \theta_x^g(g_l^0(x), g_l^1(x)) + \sum_{m=k}^{l-1} \theta_x^g(g_m^1(x), g_{m+1}^1(x)).
\end{aligned}
$$

Starting the sum above at $m = 1$ instead of $m = k$ and taking the limit $l \to \infty$ gives, for almost every $x \in M$,

$$f_k(x) \le \sum_{m=1}^{\infty} \theta_x^g(g_m^0(x), g_{m+1}^0(x)) + f(x) + \sum_{m=1}^{\infty} \theta_x^g(g_m^1(x), g_{m+1}^1(x)),$$

where we have used (4.17). Now we claim that the right-hand side of the above inequality is L^1-integrable. We already showed f is integrable using Fatou's Lemma. As for the two infinite sums, they are each also integrable by Lemma 4.15 and ω-convergence of g_k^i, $i = 0, 1$ (specifically, property (4) of Definition 4.4). Thus each f_k is bounded a.e. by an L^1 function not depending on k.

Knowing all of this, we can apply the Lebesgue dominated convergence theorem to show

$$\Theta_Y(\{g_k^0\}, \{g_k^1\}) = \lim_{k \to \infty} \Theta_Y(g_k^0, g_k^1) = \lim_{k \to \infty} \int_Y f_k \, \mu_g$$

$$= \int_Y f \, \mu_g = \int_Y \theta_x^g(g_0(x), g_1(x)) \, \mu_g(x),$$

which completes the proof.   q.e.d.

With this proposition, proving the first uniqueness result becomes a relatively simple matter.

Theorem 4.28. *Let two ω-convergent sequences $\{g_k^0\}$ and $\{g_k^1\}$, with ω-limits $[g_0]$ and $[g_1]$, respectively, be given. If $\{g_k^0\}$ and $\{g_k^1\}$ are d-equivalent, i.e., if*

$$\lim_{k \to \infty} d(g_k^0, g_k^1) = 0,$$

then $[g_0] = [g_1]$.

Proof. Suppose the contrary; then for any representatives $g_0 \in [g_0]$ and $g_1 \in [g_1]$, one of two possibilities holds:

1) X_{g_0} and X_{g_1} differ by a set of positive μ_g-measure, or
2) $X_{g_0} = X_{g_1}$, up to a μ_g-nullset, but g_0 and g_1 differ on a set E with $E \cap (X_{g_0} \cup X_{g_1}) = \emptyset$ and $\mathrm{Vol}(E, g) > 0$, where g is our fixed metric.

We will show that neither of these possibilities can actually occur.

To rule out (1), let $X_i := D_{\{g_k^i\}}$ denote the deflated set of the sequence $\{g_k^i\}$ for $i = 0, 1$. Then X_i is measurable by Remark 2.23(1). We claim $X_0 = X_1$, up to a nullset. If this is not true, then by swapping the two sequences if necessary, we see that $Y := (X_0 \setminus X_1)$ has positive volume with respect to g_1 (cf. Corollary 4.23) and zero volume with respect to g_0. (Y is simply the set on which $\{g_k^0\}$ deflates and $\{g_k^1\}$ doesn't.) But

then by Lemma 2.5,

$$\liminf_{k\to\infty} d(g_k^0, g_k^1) \geq \lim_{k\to\infty} \frac{4}{\sqrt{n}} \left| \sqrt{\mathrm{Vol}(Y, g_k^1)} - \sqrt{\mathrm{Vol}(Y, g_k^0)} \right|$$

$$= \frac{4}{\sqrt{n}} \sqrt{\mathrm{Vol}(Y, g_1)} > 0,$$

where we have used Theorem 4.21. This contradicts the assumptions of the theorem, so in fact $X_0 = X_1$ up to a nullset. Since by property (2) of Definition 4.4, $X_{g_i} = D_{\{g_k^i\}}$ up to a nullset as well, (1) cannot hold.

So suppose that (2) holds. Note that on E, g_0 and g_1 are both positive definite. Note also that E has positive g-volume and that $\theta_x^g(g_0(x), g_1(x)) > 0$ for all $x \in E$. Thus, $\int_E \theta_x^g(g_0(x), g_1(x)) \, d\mu_g > 0$, and so we can conclude from Proposition 4.26 (specifically (4.16)), that $\Theta_E(\{g_k^0\}, \{g_k^1\}) > 0$. But then this and (4.15) also imply that

$$\lim_{k\to\infty} d(g_k^0, g_k^1) = d(\{g_k^0\}, \{g_k^1\}) > 0.$$

This contradicts the assumptions of the theorem, and so (2) cannot hold either. q.e.d.

Corollary 4.29. *Let $\{g_k\}$ be a d-Cauchy sequence in M. Then all ω-convergent subsequences of $\{g_k\}$ have the same ω-limit.*

Remark 4.30. In view of this corollary and Theorem 4.17, we can unambiguously define *the* ω-limit of a d-Cauchy sequence in $\mathcal{M}$ to be the ω-limit of any ω-convergent subsequence. Thus the element $[g_\infty] \in \widehat{\mathcal{M}_m}$ defined in Theorem 4.17 is determined uniquely by the sequence $\{g_k\}$, and in fact $[g_\infty] \in \widehat{\mathcal{M}_f}$.

Theorems 4.17 and 4.28 combine to show that there exists a well-defined map from $\overline{\mathcal{M}}$ to $\widehat{\mathcal{M}_f}$:

Definition 4.31. Denote by $\Omega : \overline{\mathcal{M}} \to \widehat{\mathcal{M}_f}$ the map sending an equivalence class of Cauchy sequences to the unique element of $\widehat{\mathcal{M}_f}$ that all of its representatives ω-subconverge to.

Our goal in the following subsection is to see that this map is injective, and in Section 5, we will show that it is surjective.

4.3.2. Second Uniqueness Result. Our goal in this subsection is to prove the following statement: up to d-equivalence, there is only one d-Cauchy sequence ω-converging to a given element of $\widehat{\mathcal{M}_f}$. That is, if we have two sequences $\{g_k^0\}, \{g_k^1\}$ that both ω-converge to the same $[g_\infty] \in \widehat{\mathcal{M}_f}$, then

$$d(\{g_k^0\}, \{g_k^1\}) = \lim_{k\to\infty} d(g_k^0, g_k^1) = 0.$$

We will first prove the above statement for sequences that remain within a given amenable subset $\mathcal{U}$, and will then use this to extend the

proof to arbitrary sequences. In what follows, we will often implicitly make use of the identification of elements of $\mathcal{U}^0$—the $\|\cdot\|_g$-completion of $\mathcal{U}$—with particular measurable sections in $L^2(S^2 T^* M)$, as explained in Remark 3.4.

Proposition 4.32. *Let $\mathcal{U}$ be an amenable subset, and let $\mathcal{U}^0$ be the L^2-completion of $\mathcal{U}$. If two sequences $\{g_k^0\}$ and $\{g_k^1\}$ in $\mathcal{U}$ both ω-converge to $[g_\infty] \in \widehat{\mathcal{M}_m}$, then $[g_\infty] \in \widehat{\mathcal{M}_f}$ and $\{g_k^0\}$ and $\{g_k^1\}$ are d-equivalent. That is,*

$$\lim_{k \to \infty} d(g_k^0, g_k^1) = 0.$$

Furthermore, up to differences on a nullset, $[g_\infty]$ only contains one representative, g_∞, and $\{g_k^0\}$ and $\{g_k^1\}$ both L^2-converge to g_∞. In particular, $g_\infty \in \mathcal{U}^0$.

Proof. Note that Definition 3.1 of an amenable subset implies that the deflated sets of $\{g_k^0\}$ and $\{g_k^1\}$ are empty. Therefore, all representatives of $[g_\infty]$ differ at most by a nullset, and property (3) of Definition 4.4 implies that $g_k^0, g_k^1 \xrightarrow{\text{a.e.}} g_\infty$.

Since all g_k^0 and g_k^1 satisfy the same bounds a.e. in each coordinate chart, it is easy to see that the set

$$\{|(g_k^l)_{ij}|^2 \mid 1 \le i, j \le n, \ k \in \mathbb{N}\}$$

is uniformly integrable in each coordinate chart for both $l = 0$ and $l = 1$. Therefore, using [**14**, (13.39)(a)] (this also follows quickly from the Vitali convergence theorem [**14**, (13.38)]), we see that $\{g_k^0\}$ and $\{g_k^1\}$ converge in L^2 to g_∞, proving the second statement. This also implies that $\lim_{k \to \infty} \|g_k^1 - g_k^0\|_g = 0$. But now, invoking Theorem 3.10 gives $\lim_{k \to \infty} d(g_k^0, g_k^1) = 0$.   q.e.d.

The next lemma establishes the strong correspondence between L^2- and ω-convergence within amenable subsets.

Lemma 4.33. *Let $\mathcal{U} \subset \mathcal{M}$ be amenable, and let $\tilde{g} \in \mathcal{U}^0$. Then for any sequence $\{g_k\}$ in $\mathcal{U}$ that L^2-converges to $\tilde{g}$, there exists a subsequence $\{g_{k_l}\}$ that ω-converges to $\tilde{g}$.*

In particular, for any element $\tilde{g} \in \mathcal{U}^0$, we can always find a sequence in $\mathcal{U}$ that both L^2- and ω-converges to $\tilde{g}$.

Proof. Let $\{g_k\}$ be any sequence L^2-converging to $\tilde{g} \in \mathcal{U}^0$. Then $\tilde{g}$ together with any subsequence of $\{g_k\}$ already satisfies properties (1) and (2) of Definition 4.4. This is clear from Theorem 3.10 and Definition 3.1 of an amenable subset. (Property (2) is empty here, as $\{g_k\}$ has empty deflated set by the definition of an amenable subset.) Since $\{g_k\}$ is d-Cauchy by Theorem 3.10, it is also easy to see that there is a subsequence $\{g_{k_m}\}$ of $\{g_k\}$ satisfying property (4) of ω-convergence.

To verify property (3), note that L^2-convergence of $\{g_{k_m}\}$ implies that there exists a subsequence $\{g_{k_l}\}$ of $\{g_{k_m}\}$ that converges to $\tilde{g}$ a.e. [**14**, (13.33), (11.26)]. q.e.d.

Given the results that we have so far, we can give an alternative description of the completion of an amenable set using ω-convergence instead of L^2-convergence.

Proposition 4.34. *Let $\mathcal{U} \subset \mathcal{M}$ be an amenable subset. Then ω-convergence implements the homeomorphism in Theorem 3.12. That is, there is a natural homeomorphism between $\overline{\mathcal{U}}$ and $\mathcal{U}^0$ given by identifying each equivalence class of Cauchy sequences $[\{g_k\}]$ with the unique element of $\mathcal{U}^0$ that they ω-subconverge to. Furthermore, $\mathcal{U}^0$ can be naturally identified with a subset of $\widehat{\mathcal{M}_f}$ as in Remark 3.4.*

Proof. The existence result (Theorem 4.17), the first uniqueness result (Theorem 4.28) and Proposition 4.32 together imply that for every equivalence class $[\{g_k\}]$ of d-Cauchy sequences in $\mathcal{U}$, there is a unique L^2 metric $g_\infty \in \mathcal{U}^0$ such that every representative of $[\{g_k\}]$ ω-subconverges to g_∞, and that the representatives of a different equivalence class cannot also ω-subconverge to g_∞. This gives us the map from $\overline{\mathcal{U}}$ to $\mathcal{U}^0$ and shows that it is injective. Furthermore, by Lemma 4.33, there is a sequence in $\mathcal{U}$ ω-subconverging to every element of $\mathcal{U}^0$. Thus, this map is also surjective. q.e.d.

With this identification, we can define a metric on $\mathcal{U}^0$ by declaring the bijection of the previous proposition to be an isometry. The result is the following:

Definition 4.35. Let $\mathcal{U}$ be an amenable subset. By $d_{\mathcal{U}}$, we denote the metric on the completion of $\mathcal{U}$, which we identify with the L^2-completion $\mathcal{U}^0$ via Proposition 4.34. Thus, for $g_0, g_1 \in \mathcal{U}^0$ and any sequences $g_k^0 \xrightarrow{\omega} g_0$, $g_k^1 \xrightarrow{\omega} g_1$, we have

$$(4.18) \qquad d_{\mathcal{U}}(g_0, g_1) = \lim_{k \to \infty} d(g_k^0, g_k^1).$$

Note that by the preceding results, to define $d_{\mathcal{U}}$ via (4.18) it suffices to assume that $\{g_k^0\}$ and $\{g_k^1\}$ L^2-converge to g_0 and g_1, respectively.

The next lemma, the proof of which is immediate, shows that the metric $d_{\mathcal{U}}$ is nicely compatible with the metric d.

Lemma 4.36. *Let $\mathcal{U} \subset \mathcal{M}$ be amenable, and suppose $g_0, g_1 \in \mathcal{U}$ and $g_2 \in \mathcal{U}^0$. Then*

1) $d(g_0, g_1) = d_{\mathcal{U}}(g_0, g_1)$, *and*
2) $d(g_0, g_1) \leq d_{\mathcal{U}}(g_0, g_2) + d_{\mathcal{U}}(g_2, g_1)$.

With a little bit of effort, we can use previous results to extend Proposition 4.1, a statement about $\mathcal{M}$, to the completion of an amenable subset. We first prove a very special case in a lemma, followed by the full result.

Lemma 4.37. *Let $\mathcal{U}$ be any amenable subset and $g^0, g^1 \in \mathcal{U}$. Let $C(n)$ be the constant of Proposition 4.1, and let $E \subseteq M$ be measurable. Then*

$$d_{\mathcal{U}}(g^0, \chi(M \setminus E)g^0 + \chi(E)g^1) \leq C(n) \left(\sqrt{\mathrm{Vol}(E, g^0)} + \sqrt{\mathrm{Vol}(E, g^1)} \right).$$

Proof. For each $k \in \mathbb{N}$, choose closed subsets F_k and open subsets U_k such that $F_k \subseteq E \subseteq U_k$ and $\mathrm{Vol}(U_k, g) - \mathrm{Vol}(F_k, g) \leq 1/k$. Furthermore, choose functions $f_k \in C^\infty(M)$ satisfying

1) $0 \leq f_k(x) \leq 1$ for all $x \in M$,
2) $f_k(x) = 1$ for $x \in F_k$ and
3) $f_k(x) = 0$ for $x \notin U_k$.

Then it is not hard to see that the sequence defined by

$$g_k := (1 - f_k)g^0 + f_k g^1$$

L^2-converges to $\chi(M \setminus E)g^0 + \chi(E)g^1$, so in particular

$$(4.19) \qquad d_{\mathcal{U}}(g^0, \chi(M \setminus E)g^0 + \chi(E)g^1) = \lim_{k \to \infty} d(g^0, g_k).$$

Furthermore, since g^0 and all g_k are smooth, Proposition 4.1 gives

$$(4.20) \qquad d(g^0, g_k) \leq C(n) \left(\sqrt{\mathrm{Vol}(U_k, g^0)} + \sqrt{\mathrm{Vol}(U_k, g_k)} \right).$$

By our assumptions on the sets U_k, it is clear that $\mathrm{Vol}(U_k, g^0) \to \mathrm{Vol}(E, g^0)$. So if we can show that $\mathrm{Vol}(U_k, g_k) \to \mathrm{Vol}(E, g^1)$, then (4.19) and (4.20) combine to give the desired result.

Now, because $g_k = g^1$ on F_k, we have

$$\mathrm{Vol}(U_k, g_k) = \int_{F_k} \mu_{g^1} + \int_{U_k \setminus F_k} \mu_{g_k}.$$

The first term converges to $\mathrm{Vol}(E, g^1)$ as $k \to \infty$ by the definition of F_k. We claim that the second term converges to zero. Note that since the bounds of Definition 3.1 are pointwise convex, we can enlarge $\mathcal{U}$ to an amenable subset containing g_k for each $k \in \mathbb{N}$. (By the definition, each g_k is contained in the "pointwise convex hull" of $\mathcal{U}$; see Example 3.5(3).) Therefore, by Lemma 3.7, there exists a constant K such that

$$\left(\frac{\mu_{g_k}}{\mu_g} \right) \leq K.$$

But using this, our claim is clear from the assumptions on U_k and F_k.
$$\text{q.e.d.}$$

At this point, we establish the following convention. If $\mathcal{U}$ is amenable, $g_0 \in \mathcal{U}^0$, and $Y \subseteq M$ is measurable, choose any $\tilde{g}_0 \in \mathcal{M}_f$ that represents g_0, and define $\mathrm{Vol}(Y, g_0) := \mathrm{Vol}(Y, \tilde{g}_0)$. Note that this definition is independent of the choice of representative $\tilde{g}_0$, since any other representative will differ from $\tilde{g}_0$ at most on a nullset.

Theorem 4.38. *Let $\mathcal{U}$ be any amenable subset with L^2-completion $\mathcal{U}^0$. Suppose that $g_0, g_1 \in \mathcal{U}^0$, let $\tilde{g}_0, \tilde{g}_1 \in \mathcal{M}_f$ represent g_0 and g_1, respectively, and let $E := \mathrm{carr}(\tilde{g}_1 - \tilde{g}_0) = \{x \in M \mid \tilde{g}_0(x) \neq \tilde{g}_1(x)\}$. Then there exists a constant $C(n)$ depending only on $n = \dim M$ such that*

$$d_{\mathcal{U}}(g_0, g_1) \leq C(n) \left(\sqrt{\mathrm{Vol}(E, g_0)} + \sqrt{\mathrm{Vol}(E, g_1)} \right).$$

In particular, we have

$$\mathrm{diam}_{\mathcal{U}} \left(\{ \tilde{g} \in \mathcal{U}^0 \mid \mathrm{Vol}(M, \tilde{g}) \leq \delta \} \right) \leq 2C(n)\sqrt{\delta}.$$

Proof. Using Lemma 4.33, choose any two sequences $\{g_k^0\}$ and $\{g_k^1\}$ in $\mathcal{U}$ that both L^2- and ω-converge to g_0 and g_1, respectively. By Definition 4.35, $d_{\mathcal{U}}(g_0, g_1) = \lim_{k \to \infty} d(g_k^0, g_k^1)$. We claim that

$$\lim_{k \to \infty} d(g_k^0, g_k^1) \leq C(n) \left(\sqrt{\mathrm{Vol}(E, g_0)} + \sqrt{\mathrm{Vol}(E, g_1)} \right),$$

which would complete the proof.

By the triangle inequality (2) of Lemma 4.36, we have
(4.21)
$$d(g_k^0, g_k^1) \leq d_{\mathcal{U}}(g_k^0, \chi(M \setminus E)g_k^0 + \chi(E)g_k^1) + d_{\mathcal{U}}(\chi(M \setminus E)g_k^0 + \chi(E)g_k^1, g_k^1).$$

By Lemma 4.37 and Theorem 4.21, we can conclude

$$\lim_{k \to \infty} d_{\mathcal{U}}(g_k^0, \chi(M \setminus E)g_k^0 + \chi(E)g_k^1) \leq$$
$$C(n) \left(\sqrt{\mathrm{Vol}(E, g_0)} + \sqrt{\mathrm{Vol}(E, g_1)} \right).$$

Therefore, if we can show that the second term of (4.21) converges to zero as $k \to \infty$, then we will have the desired result. But $\{g_k^0\}$ L^2-converges to g_0 and $\{g_k^1\}$ L^2-converges to g_1. Additionally, $\chi(M \setminus E)g_0 = \chi(M \setminus E)g_1$, and thus $\chi(M \setminus E)g_k^0 + \chi(E)g_k^1$ L^2-converges to g_1. This implies that the L^2 distance between g_k^1 and $\chi(M \setminus E)g_k^0 + \chi(E)g_k^1$ converges to zero as $k \to \infty$. Thus, continuity of the map $\mathcal{U}^0 \to \overline{\mathcal{U}}$ given in Proposition 4.34 implies that

$$\lim_{k \to \infty} d_{\mathcal{U}}(\chi(M \setminus E)g_k^0 + \chi(E)g_k^1, g_k^1) = 0,$$

as was to be shown. q.e.d.

Next, we need another technical result that will help us in extending the second uniqueness result from amenable subsets to all of $\mathcal{M}$.

Proposition 4.39. *Say $g_0 \in \mathcal{M}$ and $h \in \mathcal{S}$ with $g_0 + h \in \mathcal{M}$, and let $E \subseteq M$ be any open set. Define an L^2 tensor field $g_1 \in \mathcal{S}^0$ by $g_1 := g_0 + h^0$, where $h^0 := \chi(E)h$. (Here, we are using the identification of the L^2 completion $\mathcal{S}^0$ of $\mathcal{S}$ with $L^2(S^2 T^* M)$, as in Remark 3.4.) Finally, define a path γ of L^2 metrics by $\gamma(t) := g_t := g_0 + th^0$, $t \in [0,1]$.*

Then there exists an amenable subset $\mathcal{U}$ such that $g_t \in \mathcal{U}^0$ for all t, so in particular $d_{\mathcal{U}}(g_0, g_1)$ is well-defined. Furthermore,

$$(4.22) \qquad d_{\mathcal{U}}(g_0, g_1) \leq L(\gamma) := \int_0^1 \left\| h^0 \right\|_{g_t} dt.$$

Lastly, let $C, \delta > 0$ be such that the metrics g_t, $t \in [0,1]$, all satisfy the bounds

$$|(g_t)_{ij}(x)| \leq C \quad \text{and} \quad \lambda_{\min}^{G_t}(x) \geq \delta$$

for $1 \leq i, j \leq n$ and almost every $x \in M$. (That this is satisfied for some C and δ is guaranteed by $g_t \in \mathcal{U}^0$.) Then there is a constant $K = K(C, \delta)$ such that

$$d_{\mathcal{U}}(g_0, g_1) \leq K \left\| h^0 \right\|_g.$$

Proof. The existence of an amenable subset $\mathcal{U}$ such that $g_0 + fh \in \mathcal{U}^0$ for any measurable function f with $0 \leq f \leq 1$, and such that $g_0 + fh \in \mathcal{U}$ if additionally f is smooth, is given by Lemma 3.6. In particular, $g_t = g_0 + t\chi(E)h \in \mathcal{U}^0$ for all $t \in [0,1]$. So we turn to the proof of (4.22).

Let any $\epsilon > 0$ be given. By Theorem 3.10, we can choose $\delta > 0$ such that for any $\tilde{g}_0, \tilde{g}_1 \in \mathcal{U}$, $\|\tilde{g}_1 - \tilde{g}_0\|_g < \delta$ implies $d(\tilde{g}_0, \tilde{g}_1) < \epsilon$.

Next, for each $k \in \mathbb{N}$, we choose closed sets $F_k \subseteq E$ and open sets $U_k \supseteq E$ with the property that $\mathrm{Vol}(U_k, g) - \mathrm{Vol}(F_k, g) < 1/k$. Given this, let's even restrict ourselves to k large enough that

$$(4.23) \qquad \left\| \chi(U_k \setminus F_k) h \right\|_g < \min\{\delta, \epsilon\}.$$

We then choose $f_k \in C^\infty(M)$ satisfying

1) $f_k(x) = 1$ if $x \in F_k$,
2) $f_k(x) = 0$ if $x \notin U_k$ and
3) $0 \leq f_k(x) \leq 1$ for all $x \in M$,

The first consequence of our assumptions above is

$$(4.24) \qquad \left\| g_1 - (g_0 + f_k h) \right\|_g \leq \left\| \chi(U_k \setminus F_k) h \right\|_g < \delta.$$

The second inequality is (4.23), and the first inequality holds for two reasons. First, on both F_k and $M \setminus U_k$, $g_0 + f_k h = g_0 + \chi(F_k)h = g_1$. Second, on $U_k \setminus F_k$, $g_1 - (g_0 + f_k h) = (\chi(E) - f_k)h$, and by our third assumption on f_k, $0 \leq 1 - f_k \leq 1$, so $|\chi(E) - f_k| \leq 1$. Now, inequality (4.24) allows us to conclude, by our assumption on δ, that

$$(4.25) \qquad d_{\mathcal{U}}(g_0 + f_k h, g_1) < \epsilon.$$

Since by the triangle inequality

$$d_{\mathcal{U}}(g_0, g_1) \leq d_{\mathcal{U}}(g_0, g_0 + f_k h) + d_{\mathcal{U}}(g_0 + f_k h, g_1) < d_{\mathcal{U}}(g_0, g_0 + f_k h) + \epsilon,$$

we will now estimate $d_{\mathcal{U}}(g_0, g_0 + f_k h)$ to prove (4.22).

To do this, define a path γ^k in $\mathcal{M}$, for $t \in [0, 1]$, by $\gamma^k(t) := g_t^k := g_0 + t f_k h$. Then we have, as is easy to see,

$$(4.26) \qquad d(g_0, g_1^k) = d(g_0, g_0 + f_k h) \leq L(\gamma^k) = \int_0^1 \|f_k h\|_{g_t^k} \, dt$$

This is almost what we want, but we first have to replace $f_k h$ with $h^0 = \chi(E)h$. Also note that the L^2 norm in (4.26) is that of g_t^k. To put this in a form useful for proving (4.22), we therefore also have to replace g_t^k with g_t.

Using the facts that on F_k, $f_k h = h^0$ and $g_t^k = g_t$, as well as that $f_k = 0$ on $M \setminus U_k$, we can write

$$(4.27) \qquad \|f_k h\|_{g_t^k}^2 = \int_{F_k} \mathrm{tr}_{g_t} \left((h^0)^2 \right) \mu_{g_t} + \int_{U_k \setminus F_k} \mathrm{tr}_{g_t^k} \left((f_k h)^2 \right) \mu_{g_t^k}.$$

For the first term above, we clearly have

$$(4.28) \qquad \int_{F_k} \mathrm{tr}_{g_t} \left((h^0)^2 \right) \mu_{g_t} \leq \left\| h^0 \right\|_{g_t}^2 .$$

As for the second term, it can be rewritten and estimated by

$$\int_{U_k \setminus F_k} \mathrm{tr}_{g_t^k} \left((f_k h)^2 \right) \mu_{g_t^k} = \|\chi(U_k \setminus F_k) f_k h\|_{g_t^k}^2 \leq \|\chi(U_k \setminus F_k) h\|_{g_t^k}^2 ,$$

where the inequality follows from our third assumption on f_k above. Now, by the definition of g_t^k and our choice of $\mathcal{U}$ at the beginning of the proof, we have that $g_t^k \in \mathcal{U}$ for all $t \in [0, 1]$ and all $k \in \mathbb{N}$. Therefore, by Lemma 3.8, there exists a constant $K' = K'(\mathcal{U}, g_0, g_1)$—i.e., K' does not depend on k—such that

$$\|\chi(U_k \setminus F_k) h\|_{g_t^k} \leq K' \|\chi(U_k \setminus F_k) h\|_g .$$

But by (4.23), we have that $\|\chi(U_k \setminus F_k) h\|_g < \epsilon$. Combining this with (4.27) and (4.28), we therefore get

$$\|f_k h\|_{g_t^k}^2 \leq \left\| h^0 \right\|_{g_t}^2 + (K' \epsilon)^2 \leq \left(\left\| h^0 \right\|_{g_t} + K' \epsilon \right)^2 .$$

The above inequality, substituted into (4.26), gives

$$d(g_0, g_1^k) \leq \int_0^1 \left(\left\| h^0 \right\|_{g_t} + K' \epsilon \right) dt = L(\gamma) + K' \epsilon.$$

Using the above inequality and (4.25),

$$d_{\mathcal{U}}(g_0, g_1) \leq d(g_0, g_1^k) + d_{\mathcal{U}}(g_1^k, g_1) < L(\gamma) + (1 + K') \epsilon.$$

Since ϵ was arbitrary and K' is independent of k, we are finished with the proof of (4.22).

Finally, the third statement follows from the following estimate, which is proved in exactly the same way as Lemma 3.8:

$$\left\|h^0\right\|_{g_t} = \left(\int_E \mathrm{tr}_{g_t}\left(h^2\right)\mu_{g_t}\right)^{1/2} \leq K(C,\delta)\left\|h^0\right\|_g.$$

q.e.d.

With Theorem 4.38 and Proposition 4.39 as part of our toolbox, we are now ready to take on the proof of the second uniqueness result in its full generality.

So let two d-Cauchy sequences $\{g_k^0\}$ and $\{g_k^1\}$, as well as some $g_\infty \in \mathcal{M}_f$, be given. Suppose further that $\{g_k^0\}$ and $\{g_k^1\}$ both ω-converge to g_∞ as $k \to \infty$. We will prove that

$$(4.29) \qquad\qquad \lim_{k\to\infty} d(g_k^0, g_k^1) = 0.$$

The heuristic idea of our proof is very simple, which is belied by the rather technical nature of the rigorous proof. The point, though, is essentially that for all $l \in \mathbb{N}$, we break M up into two sets, E_l and $M \setminus E_l$. The set E_l has positive volume with respect to g_∞, but $\{g_k^0\}$ and $\{g_k^1\}$ L^2-converge to g_∞ on E_l, so the contribution of E_l to $d(g_k^0, g_k^1)$ vanishes in the limit $k \to \infty$. The set $M \setminus E_l$ contains the deflated sets of $\{g_k^0\}$ and $\{g_k^1\}$, so the sequences need not converge on $M \setminus E_l$. However, we choose things such that $\mathrm{Vol}(M \setminus E_l, g_\infty)$ vanishes in the limit $l \to \infty$, so that Proposition 4.1 implies that the contribution of $M \setminus E_l$ to $d(g_k^0, g_k^1)$ vanishes after taking the limits $k \to \infty$ and $l \to \infty$ in succession.

The rigorous proof is achieved in three basic steps, which we will describe after some brief preparation.

For each $l \in \mathbb{N}$, let

$$(4.30) \qquad E_l := \left\{ x \in M \;\middle|\; \begin{array}{l} \det g_k^i(x) > l^{-1}, \; \left|(g_k^i)_{rs}(x)\right| < l \\ \forall i = 0,1; \; k \in \mathbb{N}; \; 1 \leq r,s \leq n \end{array} \right\},$$

where these local notions are of course defined with respect to our fixed amenable atlas (cf. Convention 2.17), and the inequalities in the definition should hold in each chart containing the point x in question. Thus, E_l is a set over which the sequences g_k^i neither deflate nor become unbounded. We first note that for each $k \in \mathbb{N}$, there exists an amenable subset $\mathcal{U}_k$ such that the metrics

$$g_k^0, \; g_k^1 \text{ and } g_k^0 + \chi(E_l)(g_k^1 - g_k^0)$$

are contained in $\mathcal{U}_k^0$. Finding $\mathcal{U}_k$ such that $g_k^0 + \chi(E_l)(g_k^1 - g_k^0) \in \mathcal{U}_k^0$ for all $l \in \mathbb{N}$ is guaranteed by Lemma 3.6. (Note that, as follows from the lemma, $\mathcal{U}_k$ may be chosen independently of l.) Then $\mathcal{U}_k$ can be enlarged to contain g_k^0 and g_k^1 by Example 3.5(2).

The steps in our proof are the following. We will show first that

$$(4.31) \qquad \lim_{k \to \infty} d_{\mathcal{U}_k}(g_k^0, g_k^0 + \chi(E_l)(g_k^1 - g_k^0)) = 0$$

for all fixed $l \in \mathbb{N}$. Second,

$$(4.32) \quad \limsup_{k \to \infty} d_{\mathcal{U}_k}(g_k^0 + \chi(E_l)(g_k^1 - g_k^0), g_k^1) \leq 2C(n)\sqrt{\mathrm{Vol}(M \setminus E_l, g_\infty)}$$

for all fixed $l \in \mathbb{N}$ (where $C(n)$ is the constant from Theorem 4.38). And third,

$$(4.33) \qquad \lim_{l \to \infty} \mathrm{Vol}(E_l, g_\infty) = \mathrm{Vol}(M, g_\infty).$$

Since the triangle inequality of Lemma 4.36(2) implies that

$$d(g_k^0, g_k^1) \leq d_{\mathcal{U}_k}(g_k^0, g_k^0 + \chi(E_l)(g_k^1 - g_k^0)) + d_{\mathcal{U}_k}(g_k^0 + \chi(E_l)(g_k^1 - g_k^0), g_k^1)$$

for all $l \in \mathbb{N}$, taking the lim sup as $k \to \infty$ followed by the limit as $l \to \infty$ of both sides then gives (4.29).

We now prove each of (4.31), (4.32) and (4.33) in its own lemma.

Lemma 4.40. *For each $l \in \mathbb{N}$,*

$$\lim_{k \to \infty} d_{\mathcal{U}_k}(g_k^0, g_k^0 + \chi(E_l)(g_k^1 - g_k^0)) = 0.$$

Proof. Fix $l \in \mathbb{N}$. We know that

$$g_k^0, g_k^0 + \chi(E_l)(g_k^1 - g_k^0) \in \mathcal{U}_k^0,$$

where $\mathcal{U}_k$ is an amenable subset. (Recall that the choice of $\mathcal{U}_k$ was made independently of l.) Therefore, for each fixed $k \in \mathbb{N}$, Proposition 4.39 applies to give

$$(4.34) \qquad d_{\mathcal{U}_k}(g_k^0, g_k^0 + \chi(E_l)(g_k^1 - g_k^0)) \leq K_l \left\| \chi(E_l)(g_k^1 - g_k^0) \right\|_g,$$

where K_l is some constant depending only on l.

Now, recalling the definition (4.30) of E_l, we note that for all $1 \leq i, j \leq n$ and all $k \in \mathbb{N}$, we have $|(g_k^1)_{ij}(x) - (g_k^0)_{ij}(x)|^2 \leq 4l^2$ for $x \in E_l$, and hence the family of (local) functions

$$\{\chi(E_l)((g_k^1)_{ij} - (g_k^0)_{ij}) \mid 1 \leq i, j \leq n, \ k \in \mathbb{N}\}$$

is uniformly integrable. Furthermore, since property (3) of Definition 4.4 implies that $\chi(E_l)g_k^a \to \chi(E_l)g_\infty$ a.e. for $a = 0, 1$, we have that $\chi(E_l)(g_k^1 - g_k^0) \to 0$ a.e. Therefore, as in the proof of Proposition 4.32, we can use the Vitali convergence theorem to show that

$$\left\| \chi(E_l)(g_k^1 - g_k^0) \right\|_g \to 0$$

as $k \to \infty$. Together with (4.34), this implies the result immediately.
$$\text{q.e.d.}$$

Lemma 4.41. *For each $l \in \mathbb{N}$,*

$$\limsup_{k \to \infty} d_{\mathcal{U}_k}(g_k^0 + \chi(E_l)(g_k^1 - g_k^0), g_k^1) \leq 2C(n)\sqrt{\mathrm{Vol}(M \setminus E_l, g_\infty)}.$$

Proof. Fix $l \in \mathbb{N}$. First note that $g_k^1 = g_k^0 + \chi(E_l)(g_k^1 - g_k^0)$ on E_l. Therefore, by Theorem 4.38,

$$d_{\mathcal{U}_k}(g_k^0 + \chi(E_l)(g_k^1 - g_k^0), g_k^1) \leq$$
$$C(n) \left(\sqrt{\mathrm{Vol}(M \setminus E_l, g_k^0)} + \sqrt{\mathrm{Vol}(M \setminus E_l, g_k^1)} \right).$$

But now the result follows immediately from Theorem 4.21, since we have $\mathrm{Vol}(M \setminus E_l, g_i^k) \to \mathrm{Vol}(M \setminus E_l, g_\infty)$ for $i = 0, 1$.   q.e.d.

Lemma 4.42. *For each* $l \in \mathbb{N}$,

$$\lim_{l \to \infty} \mathrm{Vol}(E_l, g_\infty) = \mathrm{Vol}(M, g_\infty).$$

Proof. Fix $l \in \mathbb{N}$. Recall that $X_{g_\infty} \subseteq M$ denotes the deflated set of g_∞, i.e., the set where g_∞ is degenerate. Note that X_{g_∞} is measurable by Remark 2.23(2), and it has volume zero w.r.t. g_∞, since $\mu_{g_\infty} \equiv 0$ on X_{g_∞}. Therefore $\mathrm{Vol}(M, g_\infty) = \mathrm{Vol}(M \setminus X_{g_\infty}, g_\infty)$.

We note that by Proposition 4.16, $\chi(E_l)$ converges a.e. to $\chi(M \setminus X_{g_\infty})$ and that $\chi(E_l)(x) \leq 1$ for all $x \in M$. Since g_∞ has finite volume, the constant function 1 is integrable w.r.t. μ_{g_∞}, and therefore the Lebesgue dominated convergence theorem implies that

$$\lim_{l \to \infty} \mathrm{Vol}(E_l, g_\infty) = \lim_{l \to \infty} \int_M \chi(E_l) \, \mu_{g_\infty} = \int_M \chi(M \setminus X_{g_\infty}) \, \mu_{g_\infty}$$
$$= \mathrm{Vol}(M \setminus X_{g_\infty}, g_\infty).$$

q.e.d.

As already noted, Lemmas 4.40, 4.41 and 4.42 combine to give the desired result. We summarize what we have just proved in a theorem.

Theorem 4.43. *Let* $[g_\infty] \in \widehat{\mathcal{M}_f}$. *Suppose we have two sequences* $\{g_k^0\}$ *and* $\{g_k^1\}$ *with* $g_k^0, g_k^1 \xrightarrow{\omega} [g_\infty]$ *as* $k \to \infty$. *Then*

$$\lim_{k \to \infty} d(g_k^0, g_k^1) = 0,$$

that is, $\{g_k^0\}$ *and* $\{g_k^1\}$ *are equivalent in the precompletion* $\overline{\mathcal{M}}^{\mathrm{pre}}$ *of* $\mathcal{M}$.

Theorem 4.43 shows that the map Ω defined in Definition 4.31 is injective, and therefore identifies $\overline{\mathcal{M}}$ with a subset of $\widehat{\mathcal{M}_f}$. In the next section, we will prove that $\overline{\mathcal{M}}$ is actually identified with all of $\widehat{\mathcal{M}_f}$, i.e., Ω is surjective. We note here that for the purpose of studying $\overline{\mathcal{M}}$ we can now use Ω to drop the distinction between an ω-convergent sequence and the element of $\widehat{\mathcal{M}_f}$ that it converges to. We will employ this trick in what follows to simplify formulas and proofs.

5. The completion of $\mathcal{M}$

In this section, our previous efforts come to fruition and we are able to complete our description of $\overline{\mathcal{M}}$ by proving, in Section 5.3, that the map $\Omega : \overline{\mathcal{M}} \to \widehat{\mathcal{M}_f}$ defined in the previous chapter is a bijection.

Section 5.1 provides some necessary preparation for the surjectivity proof by going into more depth on the behavior of volume forms under ω-convergence. After this, Section 5.2 presents a partial result on the image of Ω. Namely, we show that all equivalence classes of measurable, bounded semimetrics (cf. Definition 2.21) are contained in $\Omega(\overline{\mathcal{M}})$. This marks the final preparation we need to prove the main result.

5.1. Measures induced by measurable semimetrics. For use in Section 5.3, we need to record a couple of properties of the measure $\mu_{\tilde{g}}$ induced by an element $\tilde{g} \in \mathcal{M}_f$.

The first property is continuity of the norms of continuous functions under ω-convergence. It follows immediately from Theorem 4.21 and the Portmanteau theorem [**25**, Thm. 8.1]:

Lemma 5.1. *Let $\tilde{g} \in \mathcal{M}_f$, and let $\rho \in C^0(M)$ be any continuous function. If the sequence $\{g_k\}$ ω-converges to $\tilde{g}$, then μ_{g_k} converges weakly to $\mu_{\tilde{g}}$, so in particular*

$$\lim_{k \to \infty} \|\rho\|_{g_k} = \|\rho\|_{\tilde{g}}.$$

The next fact we establish is that if $\tilde{g} \in \mathcal{M}_f$, i.e., $\tilde{g}$ is a measurable, finite-volume semimetric, then the set of C^∞ functions is dense in $L^p(M, \tilde{g})$ for $1 \le p < \infty$, just as in the case of a smooth volume form.

To prove this claim, we first state a fact about measures on $\mathbb{R}^n$. One can prove it almost identically to [**2**, Cor. 4.2.2], where the statement is made for Borel measures. To prove it for Lebesgue measures, one must simply approximate Lebesgue-measurable sets by Borel-measurable sets using the discussion of Section 2.3.2.

Theorem 5.2. *Let a nonnegative measure ν on the algebra of Lebesgue sets in $\mathbb{R}^n$ be bounded on bounded sets. Then the class $C_0^\infty(\mathbb{R}^n)$ of smooth functions with bounded support is dense in $L^p(\mathbb{R}^n, \nu)$, $1 \le p < \infty$.*

Now, since any $\tilde{g} \in \mathcal{M}_f$ has finite volume, its induced measure $\mu_{\tilde{g}}$ clearly satisfies the hypotheses of the theorem in any coordinate chart. Therefore, we have:

Corollary 5.3. *If $\tilde{g} \in \mathcal{M}_f$, then $C^\infty(M)$ is dense in $L^p(M, \tilde{g})$.*

5.2. Bounded semimetrics. In this section, we go one step further in our understanding of the injection $\Omega : \overline{\mathcal{M}} \to \widehat{\mathcal{M}_f}$ that was introduced in Definition 4.31. Specifically, we want to see that the image

$\Omega(\overline{\mathcal{M}})$ contains all equivalence classes of bounded, measurable semi-metrics (cf. Definition 2.21).

Our strategy for proving this is to first prove the fact for smooth semimetrics by showing that for any smooth semimetric g_0, there is a finite-length path $\gamma : (0,1] \to \mathcal{M}$ with $\lim_{t\to 0} \gamma(t) = g_0$ (where we take the limit in the C^∞ topology of $\mathcal{S}$). Then, if we simply let t_k be any monotonically decreasing sequence converging to zero, it is trivial to show $\gamma(t_k) \xrightarrow{\omega} g_0$ as $k \to \infty$. We then use this to handle the general, nonsmooth case.

5.2.1. Paths to the boundary. Before we get into the proofs, we put ourselves in the proper setting, for which we first need to introduce the notion of a quasi-amenable subset. These are defined by weakening the requirements for an amenable subset (cf. Definition 3.1), giving up the condition of being "uniformly inflated":

Definition 5.4. We call a subset $\mathcal{U} \subset \mathcal{M}$ *quasi-amenable* if $\mathcal{U}$ is convex and we can find a constant C such that for all $\tilde{g} \in \mathcal{U}$, $x \in M$ and $1 \le i, j \le n$,

$$(5.1) \qquad |\tilde{g}_{ij}(x)| \le C$$

in our given amenable atlas.

We also define $\partial\mathcal{M}$ to be the boundary of $\mathcal{M}$ as a topological subset of $\mathcal{S}$. Thus, it consists of all smooth semimetrics that somewhere fail to be positive definite.

Let $\mathcal{U}$ be any quasi-amenable subset, and denote by $\mathrm{cl}(\mathcal{U})$ the closure of $\mathcal{U}$ in the C^∞ topology of $\mathcal{S}$. Thus, $\mathrm{cl}(\mathcal{U})$ may contain some smooth semimetrics. One can easily see that any $g_0 \in \partial\mathcal{M}$ is contained in $\mathrm{cl}(\mathcal{U})$ for an appropriate quasi-amenable subset $\mathcal{U}$.

Now, suppose some $g_0 \in \mathrm{cl}(\mathcal{U}) \cap \partial\mathcal{M}$ is given, and let $g_1 \in \mathcal{U}$ have the property that $h := g_1 - g_0 \in \mathcal{M}$. Finding such a g_1 may require that we enlarge $\mathcal{U}$ to a different quasi-amenable subset. Since g_0 is positive semidefinite, it suffices to choose g_1 such that for each $x \in M$, $g_1(x)(X, X) > g_0(x)(X, X)$ for all $X \in T_x M$.

Exploiting the fact that $\mathcal{M}$ is an open positive cone in the linear space $\mathcal{S}$, we define the simplest path imaginable from g_0 to g_1:

$$(5.2) \qquad \gamma(t) := g_t := g_0 + th, \quad 0 \le t \le 1.$$

We claim that for $t \in (0,1]$, $g_t \in \mathrm{cl}(\mathcal{U}) \cap \mathcal{M}$, where $\mathrm{cl}(\mathcal{U})$ denotes the closure of $\mathcal{U}$ in the topology of $\mathcal{S}$, and that $\mathrm{cl}(\mathcal{U}) \cap \mathcal{M}$ is quasi-amenable. For the latter statement, note that if metrics in $\mathcal{U}$ satisfy the bounds (5.1), then so do all semimetrics in $\mathrm{cl}(\mathcal{U})$. For the former statement, note that the closure of a convex set in a topological vector space is convex, so $\mathrm{cl}(\mathcal{U})$ is convex. As $g_0 \in \mathrm{cl}(\mathcal{U})$, we have that for all $t \in [0,1]$, g_t is a convex combination of g_0 and g_1, and is therefore contained in

cl($\mathcal{U}$). Furthermore, g_t is smooth for all $t \in [0, 1]$, and positive definite for all $t > 0$.

Recall that the length of $\gamma(t)$ is given by

$$
(5.3) \quad
\begin{aligned}
L(\gamma) &= \int_0^1 \left(\int_M \mathrm{tr}_{\gamma(t)}(\gamma'(t)^2) \, \mu_{\gamma(t)} \right)^{1/2} dt \\
&= \int_0^1 \left(\int_M \mathrm{tr}_{g_t}(h^2) \sqrt{\det(g^{-1} g_t)} \, \mu_g \right)^{1/2} dt.
\end{aligned}
$$

To prove that γ is a finite-length path, we will therefore estimate the inner integrand. This estimate will follow from pointwise estimates combined with a compactness/continuity argument.

5.2.2. Pointwise estimates. Let $A = (a_{ij})$ and $B = (b_{ij})$ be real, symmetric $n \times n$ matrices, with $A_t := A + tB$ for $t \in (0, 1]$. We will assume that $B > 0$ and that $A \geq 0$. (In this scheme, A and B play the role of $g_0(x)$ and $h(x)$, respectively, at some point $x \in M$.) Furthermore, we fix an arbitrary matrix C that is invertible and symmetric (this plays the role of $g(x)$).

To get a pointwise estimate on $\mathrm{tr}_{g_t}(h^2) \sqrt{\det g^{-1} g_t}$, we need to estimate $\mathrm{tr}_{A_t}(B^2) \sqrt{\det(C^{-1} A_t)}$. We prove the desired estimate in two lemmas.

For any symmetric matrix D, let $\lambda_{\min}^D = \lambda_1^D \leq \cdots \leq \lambda_n^D = \lambda_{\max}^D$ be its eigenvalues numbered in increasing order.

Lemma 5.5.

$$
\begin{aligned}
\lambda_{\min}^{A_t} &\geq \lambda_{\min}^A + t\lambda_{\min}^B \\
\lambda_{\max}^{A_t} &\leq \lambda_{\max}^A + t\lambda_{\max}^B \leq \lambda_{\max}^A + \lambda_{\max}^B.
\end{aligned}
$$

Proof. Immediate from the concavity/convexity of the minimal/maximal eigenvalue (cf. the proof of Lemma 2.12).   q.e.d.

Lemma 5.6.

$$
\mathrm{tr}_{A_t}\left(B^2\right) \sqrt{\det C^{-1} A_t} \leq \frac{n \left(\lambda_{\max}^B\right)^2 \left(\lambda_{\max}^{A_t}\right)^{\frac{n-1}{2}}}{\sqrt{\det C} \left(\lambda_{\min}^B\right)^{3/2}} \frac{1}{t^{3/2}}.
$$

Proof. We focus on the trace term first.

Since B is a symmetric matrix, there exists a basis for which B is diagonal, so that $B = \mathrm{diag}(\lambda_1^B, \ldots, \lambda_n^B)$. In this basis, if we denote A_t^{-1} by (a_t^{ij}), then we have

$$
(5.4) \quad
\begin{aligned}
\mathrm{tr}\left(\left(A_t^{-1} B\right)^2\right) &= \sum_{ij} a_t^{ij} \lambda_j^B a_t^{ji} \lambda_i^B \leq \left(\lambda_{\max}^B\right)^2 \sum_{ij} \left(a_t^{ij}\right)^2 \\
&= \left(\lambda_{\max}^B\right)^2 \mathrm{tr}\left(A_t^{-2}\right),
\end{aligned}
$$

where we have used the symmetry of A_t^{-1}.

We note that the trace of the square of a matrix is given by the sum of the squares of its eigenvalues. Therefore,

$$(5.5) \qquad \operatorname{tr}\left(A_t^{-2}\right) = \sum_i \left(\lambda_i^{A_t}\right)^{-2} \leq n \left(\lambda_{\min}^{A_t}\right)^{-2}.$$

This takes care of the trace term.

For the determinant term, we clearly have $\det A_t \leq \lambda_{\min}^{A_t}(\lambda_{\max}^{A_t})^{n-1}$. Combining this, equations (5.4) and (5.5), the estimate of Lemma 5.5, and the fact that $\lambda_{\min}^{A} \geq 0$ (as $A \geq 0$) now implies the result.   q.e.d.

5.2.3. Finiteness of $L(\gamma)$. We want to use the pointwise estimate of Lemma 5.6 to prove the main result of the section.

It is clear that to pass from the pointwise result of Lemma 5.6 to a global result, we will have to estimate the maximum and minimum eigenvalues of h, as well as the maximum eigenvalue of g_t. We begin by noting that since we work over an amenable coordinate atlas (cf. Definition 2.15), all components of h, g and g_0 are bounded in absolute value. Therefore, so are their determinants. In particular, since $g > 0$ and $h > 0$, we can assume that $C \geq \det g \geq C_0$ and $C_1 \geq \det h \geq C_2$ over each chart of the amenable atlas for some constants $C_0, C_1, C_2 > 0$.

Lemma 5.7. *The quantities $\lambda_{\max}^h$ and $\lambda_{\max}^{g_t}$, as local functions on each coordinate chart, are uniformly bounded, say $\lambda_{\max}^h(x) \leq C_3$ and $\lambda_{\max}^{g_t}(x) \leq C_4$ for all x and t.*

Proof. Note that g_t lies in the quasi-amenable subset $\mathcal{U}$ for all $0 < t \leq 1$, so we have upper bounds (in absolute value) on the coefficients of g_t and h that are uniform in x and t. Thus, the bounds on their maximal eigenvalues follow straighforwardly from the min-max theorem [**21**, Thm. XIII.1].   q.e.d.

Lemma 5.8. *The quantity $\lambda_{\min}^h$, as a function over each coordinate chart, is uniformly bounded away from 0, say $\lambda_{\min}^h \geq C_5 > 0$.*

Proof. It is clear that $\det h(x) \leq \lambda_{\min}^h(x)\lambda_{\max}^h(x)^{n-1}$. So by Lemma 5.7 and the discussion that precedes it, $\lambda_{\min}^h(x) \geq \lambda_{\max}^h(x)^{1-n} \det h(x) \geq C_3^{1-n}C_2 =: C_5$.   q.e.d.

Theorem 5.9. *Define a path γ as in (5.2). Then*

$$L(\gamma) < \infty.$$

Proof. At each point $x \in M$, we use Lemma 5.6 to see

$$(5.6) \quad \operatorname{tr}_{g_t(x)}(h(x)^2)\sqrt{\det(g(x)^{-1}g_t(x))} \leq \frac{1}{\sqrt{C_0}}\frac{C_3^2}{C_5^{3/2}}C_4^{\frac{n-1}{2}}\frac{1}{t^{3/2}} =: \frac{C_6}{t^{3/2}}.$$

The result then follows from (5.3) and the integrability of $t^{-3/4}$.
                                          q.e.d.

5.2.4. Bounded, nonsmooth semimetrics. We now proceed to the proof that the equivalence class of any bounded semimetric, not just smooth ones, is contained in $\Omega(\overline{\mathcal{M}})$.

So far, we know from Proposition 4.34 that the equivalence class of any measurable metric that can be obtained as the L^2 limit of a sequence of metrics from an amenable subset belongs to $\Omega(\overline{\mathcal{M}})$. We also know from the preceding arguments that any smooth semimetric $\tilde{g}$ is in the image of Ω. Given the remarks at the end of Section 4, we can therefore unambiguously write things like $d(g_0, g_1)$—where g_0 and g_1 are known to belong to the image of Ω—in place of expressions involving sequences ω-converging to g_0 and g_1.

To begin proving the result on bounded, nonsmooth semimetrics, we want to prove a result about quasi-amenable subsets that is a generalization of Theorem 3.10—weakened so that it still applies for these more general subsets. First, though, we need to prove a couple of lemmas.

Lemma 5.10. *Let $\mathcal{U} \subset \mathcal{M}$ be quasi-amenable. Recall that we denote the closure of $\mathcal{U}$ in the C^∞ topology of $\mathcal{S}$ by $\mathrm{cl}(\mathcal{U})$, and we denote the boundary of $\mathcal{M}$ in the C^∞ topology of $\mathcal{S}$ by $\partial\mathcal{M}$. Then for each $\epsilon > 0$, there exists $\delta > 0$ such that $d(g_0, g_0 + \delta g) < \epsilon$ for all $g_0 \in \mathrm{cl}(\mathcal{U}) \cap \partial\mathcal{M}$.*

Proof. For any $g_0 \in \mathrm{cl}(\mathcal{U}) \cap \partial\mathcal{M}$, we consider the path γ given by $\gamma(t) := g_t := g_0 + th$, where $h := \delta g$, $\delta > 0$, and $t \in (0, 1]$. The proof consists of reexamining the estimates of Theorem 5.9 and showing that they only depend on upper bounds on the entries of g_0 (and g, but we get these automatically when we work over an amenable atlas), and that the bound on the length of g_t goes to zero as $\delta \to 0$.

So, recall the main estimate (5.6) of Theorem 5.9:

$$\mathrm{tr}_{g_t(x)}(h(x)^2)\sqrt{\det(g(x)^{-1}g_t(x))} \leq \frac{n\left(\lambda^h_{\max}(x)\right)^2\left(\lambda^{g_t}_{\max}(x)\right)^{\frac{n-1}{2}}}{\sqrt{\det g(x)}\left(\lambda^h_{\min}(x)\right)^{3/2}}\frac{1}{t^{3/2}}.$$

Since $\det g(x)$ is constant w.r.t. δ, we ignore this term. By Lemma 5.5,

$$\lambda^{g_t}_{\max}(x) \leq \lambda^{g_0}_{\max}(x) + \lambda^h_{\max}(x) = \lambda^{g_0}_{\max}(x) + \delta\lambda^g_{\max}(x).$$

Therefore, using the same arguments as in Lemma 5.7, we see that $\lambda^{g_t}_{\max}(x)$ is bounded from above, uniformly in x and t, by a constant that decreases as δ decreases. Furthermore, this constant does not depend on our choice of $g_0 \in \mathrm{cl}(\mathcal{U}) \cap \partial\mathcal{M}$, since the proof of Lemma 5.7 depended only on uniform upper bounds on the entries of g_0, and we are guaranteed the same upper bounds on all elements of $\mathrm{cl}(\mathcal{U}) \cap \partial\mathcal{M}$ since $\mathcal{U}$ is quasi-amenable.

We now focus our attention on the term

$$\frac{\left(\lambda^h_{\max}(x)\right)^2}{\left(\lambda^h_{\min}(x)\right)^{3/2}} = \frac{\left(\delta\lambda^g_{\max}(x)\right)^2}{\left(\delta\lambda^g_{\min}(x)\right)^{3/2}} = \frac{\left(\lambda^g_{\max}(x)\right)^2}{\left(\lambda^g_{\min}(x)\right)^{3/2}}\sqrt{\delta}.$$

This expression clearly goes to zero as $\delta \to 0$. Therefore, we have shown that an estimate of the form (5.6) holds with a constant C_6 that depends only on $\mathcal{U}$ and δ, and which approaches zero as $\delta \to 0$. The result now follows.   q.e.d.

The next lemma implies, in particular, that $\partial \mathcal{M}$ is *not closed* in the L^2 topology of $\mathcal{S}$, nor is it in the topology of d on $\Omega(\overline{\mathcal{M}})$. It also implies that around any point in $\mathcal{M}$, there exists no L^2- or d-open neighborhood.

Lemma 5.11. *Let $\mathcal{U} \in \mathcal{M}$ be any quasi-amenable subset. Then for all $\epsilon > 0$, there exists a function $\rho_\epsilon \in C^\infty(M)$ with the properties that for all $g_1 \in \mathcal{U}$,*

1) $\rho_\epsilon g_1 \in \partial \mathcal{M}$,
2) $0 \le \rho_\epsilon(x) \le 1$ *for all $x \in M$,*
3) $\|g_1 - \rho_\epsilon g_1\|_g < \epsilon$ *and*
4) $d(g_1, \rho_\epsilon g_1) < \epsilon$.

Proof. Let $x_0 \in M$ be any point, and for each $k \in \mathbb{N}$, choose a function $\rho_k \in C^\infty(M)$ satisfying

1) $\rho_k(x_0) = 0$,
2) $0 \le \rho_k(x) \le 1$ for all $x \in M$ and
3) $\rho_k \equiv 1$ outside an open set Z_k with $\mathrm{Vol}(Z_k, g) \le 1/k$.

Then clearly $\|g_1 - \rho_k g_1\|_g \to 0$ as $k \to \infty$, and this convergence is uniform in g_1 because of the upper bounds guaranteed by the fact that $g_1 \in \mathcal{U}$. Using arguments similar to those in the last lemma, we can also see that the length of the path γ^k given by $\gamma^k(t) := \rho_k g_1 + t(g_1 - \rho_k g_1)$ converges to zero as $k \to \infty$. Therefore, choosing k large enough gives the desired function.   q.e.d.

The next theorem is the desired analog of Theorem 3.10. Note that only one half of Theorem 3.10 holds in this case, and even this is proved only in a weaker form.

Theorem 5.12. *Let $\mathcal{U} \subset \mathcal{M}$ be quasi-amenable. Then for all $\epsilon > 0$, there exists $\delta > 0$ such that if $g_0, g_1 \in \mathrm{cl}(\mathcal{U})$ with $\|g_0 - g_1\|_g < \delta$, then $d(g_0, g_1) < \epsilon$.*

Proof. First, we enlarge $\mathcal{U}$ if necessary to include *all* metrics satisfying the bound given in Definition 5.4. This enlarged $\mathcal{U}$ is then clearly convex by the triangle inequality for the absolute value, and hence it is still a quasi-amenable subset.

Now, let $\epsilon > 0$ be given. We prove the statement first for $g_0, g_1 \in \mathrm{cl}(\mathcal{U}) \cap \partial \mathcal{M}$, then use this to prove the general case.

By Lemma 5.10, we can choose $\delta_1 > 0$ such that $d(\hat{g}, \hat{g} + \delta_1 g) < \epsilon/3$ for all $\hat{g} \in \mathrm{cl}(\mathcal{U}) \cap \partial \mathcal{M}$. We define an amenable subset of $\mathcal{M}$ by

$$\mathcal{U}' := \{\hat{g} + \delta_1 g \mid \hat{g} \in \mathcal{U}\}.$$

Lemma 5.5 implies that this set is indeed amenable. Now, by Theorem 3.10, there exists $\delta > 0$ such that if $\tilde{g}_0, \tilde{g}_1 \in \mathcal{U}'$ with $\|\tilde{g}_0 - \tilde{g}_1\|_g < \delta$, then $d(\tilde{g}_0, \tilde{g}_1) < \epsilon/3$. Let $g_0, g_1 \in \text{cl}(\mathcal{U}) \cap \partial\mathcal{M}$ be such that $\|g_0 - g_1\|_g < \delta$. If we define $\tilde{g}_i := g_i + \delta_1 g$ for $i = 1, 2$, then it is clear that $\|\tilde{g}_0 - \tilde{g}_1\|_g = \|g_0 - g_1\|_g < \delta$. Given this and the definition of δ_1, we have

$$d(g_0, g_1) \leq d(g_0, \tilde{g}_0) + d(\tilde{g}_0, \tilde{g}_1) + d(\tilde{g}_1, g_1) < \epsilon.$$

Now we prove the general case. Let $\epsilon > 0$ be given. By the special case we just proved, we can choose $\delta > 0$ such that if $\tilde{g}_0, \tilde{g}_1 \in \text{cl}(\mathcal{U}) \cap \partial\mathcal{M}$ with $\|\tilde{g}_0 - \tilde{g}_1\|_g < \delta$, then $d(\tilde{g}_0, \tilde{g}_1) < \epsilon/3$. Let $g_0, g_1 \in \mathcal{U}$ be any elements with $\|g_0 - g_1\|_g < \delta$. By Lemma 5.11 and our enlargement of $\mathcal{U}$, we can choose a function $\rho \in C^\infty(M)$ such that for $i = 0, 1$,

1) $\rho g_i \in \text{cl}(\mathcal{U}) \cap \partial\mathcal{M}$,
2) $0 \leq \rho(x) \leq 1$ for all $x \in M$, and
3) $d(g_i, \rho g_i) < \epsilon/3$.

(If $g_i \in \text{cl}(\mathcal{U}) \cap \partial\mathcal{M}$ for both $i = 1$ and 2, we might as well just choose $\rho \equiv 1$.) In particular, the second property of ρ implies that

$$\|\rho g_1 - \rho g_0\|_g \leq \|g_1 - g_0\|_g < \delta.$$

Then we immediately get

$$d(g_0, g_1) \leq d(g_0, \rho g_0) + d(\rho g_0, \rho g_1) + d(\rho g_1, g_1) < \epsilon.$$

This proves the general case and thus the theorem. q.e.d.

Using the relationship between d and $\|\cdot\|_g$ determined in Theorem 5.12, we can prove the following.

Proposition 5.13. *Let $[\tilde{g}] \in \widehat{\mathcal{M}_f}$ be an equivalence class of bounded, measurable semimetrics. Then for any representative $\tilde{g} \in [\tilde{g}]$, there exists a sequence $\{g_k\}$ in $\mathcal{M}$ that both L^2- and ω-converges to $\tilde{g}$. Thus $[\tilde{g}] \in \Omega(\overline{\mathcal{M}})$.*

Moreover, suppose $\tilde{g} \in \mathcal{U}^0$ for some quasi-amenable subset $\mathcal{U} \subset \mathcal{M}$. Then for any sequence $\{g_l\}$ in $\mathcal{U}$ that L^2-converges to $\tilde{g}$, $\{g_l\}$ is d-Cauchy and there exists a subsequence $\{g_k\}$ that also ω-converges to $\tilde{g}$.

Proof. It is clear that for every bounded representative $\tilde{g} \in [\tilde{g}]$, we can find a quasi-amenable subset $\mathcal{U} \subset \mathcal{M}$ such that $\tilde{g} \in \mathcal{U}^0$. Thus, there exists a sequence $\{g_l\}$ that L^2-converges to $\tilde{g}$. It is d-Cauchy by Theorem 5.12. We wish to show that it contains a subsequence $\{g_k\}$ that also ω-converges to $\tilde{g}$, so we still need to verify properties (2)–(4) of Definition 4.4.

By passing to a subsequence, we can assume that property (4) is satisfied for $\{g_l\}$. Property (3) is verified in the same way as in the proof of Lemma 4.33. That is, L^2-convergence of $\{g_l\}$ implies that there exists a subsequence $\{g_k\}$ of $\{g_l\}$ that converges to $\tilde{g}$ a.e. In particular, $g_k(x) \to \tilde{g}(x)$ for almost every $x \in M \setminus D_{\{g_k\}}$. Furthermore,

a.e.-convergence of $\{g_k\}$ to $\tilde{g}$ and continuity of the determinant function imply that $\det G_k(x) \to \det \tilde{G}(x)$ for almost every $x \in M$. Thus, $\det G_k(x) \to 0$ for almost every $x \in M$ with $\det \tilde{G}(x) = 0$, and so $X_{\tilde{g}} \subseteq D_{\{g_k\}}$, up to a nullset. On the other hand, if $x \in M \setminus X_{\tilde{g}}$, then $\det \tilde{G}(x) > 0$, so almost surely $\lim_{k\to\infty} \det G_k(x) > 0$ as well. This implies the reverse inclusion $D_{\{g_k\}} \subseteq X_{\tilde{g}}$ (up to a nullset), and so property (2) holds for $\{g_k\}$.   q.e.d.

Thus, like we did for more restricted types of metrics before, this proposition allows us to cease to distinguish between bounded semimetrics and sequences ω-converging to them.

5.3. Unbounded metrics and the proof of the main result. Up to this point, we have an injection $\Omega : \overline{\mathcal{M}} \to \widehat{\mathcal{M}_f}$, and we have determined that the image $\Omega(\overline{\mathcal{M}})$ contains all equivalence classes containing bounded semimetrics. In this section, we prove that Ω is surjective.

We first introduce a definition and explore a few consequences of it that will be used in the proof of surjectivity.

Definition 5.14. For $k \in \mathbb{N}$, let $g_k, g_0, \tilde{g} \in \mathcal{M}_f$ be given. For $\epsilon > 0$, define

$$E_k^\epsilon := \left\{ x \in M \ \Big| \ \sqrt{\langle g_0(x) - g_k(x), g_0 - g_k(x) \rangle_{g(x)}} \geq \epsilon \right\}.$$

We say that $\{g_k\}$ converges to g_0 in $\mu_{\tilde{g}}$-measure if, for all $\epsilon > 0$, $\mathrm{Vol}(E_k^\epsilon, \mu_{\tilde{g}}) \to 0$ as $k \to \infty$.

Lemma 5.15. *For $k \in \mathbb{N}$, let $g_k, g_0, \tilde{g} \in \mathcal{M}_f$ be given.*

1) *If $g_k \to g_0$ $\mu_{\tilde{g}}$-a.e., then $g_k \to g_0$ in $\mu_{\tilde{g}}$-measure.*
2) *If $g_k \to g_0$ in $\mu_{\tilde{g}}$-measure, then there exists a subsequence $\{g_{k_l}\}$ of $\{g_k\}$ that converges $\mu_{\tilde{g}}$-a.e. to g_0.*
3) *For $\hat{g}, \bar{g} \in \mathcal{M}_f$, define*

$$d_{\mathrm{meas}}^{\tilde{g}}(\hat{g}, \bar{g}) := \int_M \frac{\sqrt{\langle \bar{g}(x) - \hat{g}(x), \bar{g} - \hat{g}(x) \rangle_{g(x)}}}{1 + \sqrt{\langle \bar{g}(x) - \hat{g}(x), \bar{g} - \hat{g}(x) \rangle_{g(x)}}} \, d\mu_{\tilde{g}}.$$

Then $d_{\mathrm{meas}}^{\tilde{g}}$ metrizes the topology of convergence in measure in $\mathcal{M}_f$ (if we, as usual in this context, identify elements of $\mathcal{M}_f$ that agree a.e.).

Proof. All three of these statements are proved in exact analogy with the case of real- or complex-valued functions on a finite measure space; cf. (11.31), (11.26), and (12.47), respectively, of [**14**].   q.e.d.

With this technical result in hand, we may now move on to the surjectivity of Ω.

Theorem 5.16. *Let any $[\tilde{g}] \in \widehat{\mathcal{M}_f}$ be given. Then there exists a sequence $\{g_k\}$ in $\mathcal{M}$ such that*

$$g_k \xrightarrow{\omega} [\tilde{g}].$$

Proof. In view of Proposition 5.13, it remains only to prove this for the equivalence class of a measurable, unbounded semimetric $\tilde{g} \in \mathcal{M}_f$.

Given any element $\hat{g} \in \mathcal{M}_f$, we can define $\exp_{\hat{g}}$ on tensors of the form $\sigma \hat{g}$, where σ is any function, purely algebraically. We simply set

(5.7)
$$\exp_{\hat{g}}(\sigma \hat{g}) := \left(1 + \frac{n}{4}\sigma\right)^{4/n} \hat{g},$$

so that the expression coincides with the usual one if $\hat{g} \in \mathcal{M}$ and $\sigma \in C^\infty(M)$ with $\sigma > -\frac{4}{n}$ (cf. Proposition 2.2). If σ is additionally measurable, then $\exp_{\hat{g}}(\sigma \hat{g})$ will also be measurable.

Now, let $\tilde{g} \in \mathcal{M}_f$. Then we can find a measurable, positive function ξ on M such that $g_0 := \xi \tilde{g}$ is a bounded semimetric. A simple calculation using the finite volume of $\tilde{g}$ shows that $\rho := \xi^{-1} \in L^{n/2}(M, g_0)$.

Define the map ψ by $\psi(\sigma) := \exp_{g_0}(\sigma g_0)$, and let

(5.8)
$$\lambda := \frac{4}{n}\left(\rho^{n/4} - 1\right).$$

Then clearly $\psi(\lambda) = \rho g_0 = \tilde{g}$. The fact that $\rho \in L^{n/2}(M, g_0)$ implies that $\rho^{n/4} \in L^2(M, g_0)$, and the constant function 1 lies in $L^2(M, g_0)$ since $\mathrm{Vol}(M, g_0)$ is finite. Hence $\lambda \in L^2(M, g_0)$, and by Corollary 5.3 we can find a sequence $\{\lambda_k\}$ of smooth functions that converge in $L^2(M, g_0)$ to λ.

Since $\lambda_k \to \lambda$ in $L^2(M, g_0)$, as in the proof of Lemma 4.33 we can pass to a subsequence and assume that $\lambda_k \to \lambda$ pointwise a.e., where here, "almost everywhere" means with respect to μ_{g_0}. With respect to the fixed, smooth, strictly positive volume form μ_g, this actually means that $\lambda_k(x) \to \lambda(x)$ for almost every $x \in M \setminus X_{g_0}$, since X_{g_0} is a nullset with respect to μ_{g_0}. Note also that $X_{g_0} = X_{\tilde{g}}$, since we assumed that the function ξ is positive. Therefore $\lambda_k(x) \to \lambda(x)$ for almost every $x \in M \setminus X_{\tilde{g}}$.

Furthermore, since from (5.8) and positivity of ξ it is clear that $\lambda > -\frac{4}{n}$, we can choose the sequence $\{\lambda_k\}$ such that $\lambda_k > -\frac{4}{n}$ for all $k \in \mathbb{N}$. (Here we are also using the fact that g_0 has finite volume.) This implies, in particular, that $X_{\psi(\lambda_k)} = X_{g_0} = X_{\tilde{g}}$, which is easily seen from (5.7).

We make one last assumption on the sequence $\{\lambda_k\}$. Namely, by passing to a subsequence, we can assume that

(5.9)
$$\sum_{k=1}^{\infty} \|\lambda_{k+1} - \lambda_k\|_{g_0} < \infty.$$

Now, we claim that

(5.10)
$$d(\psi(\sigma), \psi(\tau)) \le \sqrt{n}\,\|\tau - \sigma\|_{g_0}$$

for all $\sigma, \tau \in C^\infty(M)$ with $\sigma, \tau > -\frac{4}{n}$. We delay the proof of this statement to Lemma 5.18 below and first finish the proof of the theorem.

We wish to construct a sequence that ω-converges to $\tilde{g}$ using the sequence $\{\psi(\lambda_k)\}$. We can't use $\{\psi(\lambda_k)\}$ directly, since it is a sequence in $\Omega(\overline{\mathcal{M}})$, not $\mathcal{M}$ itself. So we first verify the properties of ω-convergence for $\{\psi(\lambda_k)\}$ and then construct a sequence in $\mathcal{M}$ that approximates $\{\psi(\lambda_k)\}$ well enough that it still satisfies all the conditions for ω-convergence.

Since the sequence $\{\lambda_k\}$ is convergent in $L^2(M, g_0)$, it is also Cauchy in $L^2(M, g_0)$. Using the inequality (5.10), it is then immediate that $\{\psi(\lambda_k)\}$ is a Cauchy sequence in $(\Omega(\overline{\mathcal{M}}), d)$. This verifies property (1) of ω-convergence (cf. Definition 4.4).

We next verify property (3). Note that $X_{\tilde{g}} \subseteq D_{\{\psi(\lambda_k)\}}$, since we have already shown that $X_{\psi(\lambda_k)} = X_{\tilde{g}}$. (Keep in mind here the subtle point that $X_{\psi(\lambda_k)}$ is the deflated set of the *individual* semimetric $\psi(\lambda_k)$, while $D_{\{\psi(\lambda_k)\}}$ is the deflated set of the *sequence* $\{\psi(\lambda_k)\}$. Refer to Definitions 2.21 and 2.22 for details.) The inclusion implies that

$$M \setminus D_{\{\psi(\lambda_k)\}} \subseteq M \setminus X_{\tilde{g}},$$

so it suffices to show that $\psi(\lambda_k)(x) \to \tilde{g}(x)$ for almost every $x \in M \setminus X_{\tilde{g}}$. But this is clear from the fact, proved above, that $\lambda_k(x) \to \lambda(x)$ for almost every $x \in M \setminus X_{\tilde{g}}$.

To verify property (2), we claim that $D_{\{\psi(\lambda_k)\}} = X_{\tilde{g}}$, up to a nullset. In the previous paragraph, we already showed that $X_{\tilde{g}} \subseteq D_{\{\psi(\lambda_k)\}}$. The reverse inclusion holds, up to a nullset, using the same argument as in the proof of Proposition 5.13 and the fact that for almost every $x \in M \setminus X_{\tilde{g}}$, $\{\psi(\lambda_k)(x)\}$ converges to $\tilde{g}(x)$.

The last property to verify is (4). But this is immediate from (5.9) and (5.10).

So we have shown that $\{\psi(\lambda_k)\}$ satisfies the properties of ω-convergence, save that it is a sequence of measurable semimetrics, rather than a sequence of smooth metrics as required. To get a sequence in $\mathcal{M}$ that ω-converges to $\tilde{g}$, recall that each of the functions λ_k is smooth and therefore bounded, and also that g_0 is a bounded, measurable semimetric. Therefore, for each fixed $k \in \mathbb{N}$, $\psi(\lambda_k)$ is also a bounded, measurable semimetric, and so by Proposition 5.13 we can find a sequence $\{g_k^l\}$ in $\mathcal{M}$ that ω-converges to $\psi(\lambda_k)$ as $l \to \infty$. We will construct a sequence ω-converging to $\tilde{g}$ by a diagonal argument which we now describe.

First, let us construct a sequence $\{g_k^{l_k}\}$ satisfying property (4) of Definition 4.4. This follows by a standard argument using the fact that $\sum_k d(\psi(\lambda_k), \psi(\lambda_{k+1})) < \infty$, as well as that $\lim_{l \to \infty} d(g_k^l, \psi(\lambda_k)) = 0$ for all $k \in \mathbb{N}$. Indeed, let l_k be chosen large enough that, for all $l \geq l_k$,

$$d(g_k^l, \psi(\lambda_k)) < \frac{1}{2^k}.$$

Then since

$$d(g_k^{l_k}, g_{k+1}^{l_{k+1}}) \leq d(g_k^{l_k}, \psi(\lambda_k)) + d(\psi(\lambda_k), \psi(\lambda_{k+1})) + d(\psi(\lambda_{k+1}), g_{k+1}^{l_{k+1}}),$$

we have

$$\sum_k d(g_k^{l_k}, g_{k+1}^{l_{k+1}}) < \sum_k d(\psi(\lambda_k), \psi(\lambda_{k+1})) + \frac{3}{4} \sum_k \frac{1}{2^k} < \infty.$$

Note that property (1) of Definition 4.4 follows, as noted in the discussion following the definition.

We next modify $\{g_k^{l_k}\}$ to satisfy property (3) of Definition 4.4. To do this, recall that $\psi(\lambda_k) \to \tilde{g}$ $\mu_{\tilde{g}}$-a.e. Furthermore, one sees from Definition 4.4 that, for all k, $g_k^l \to \psi(\lambda_k)$ $\mu_{\psi(\lambda_k)}$-a.e. as $l \to \infty$. But, as already noted, $X_{\psi(\lambda_k)} = X_{\tilde{g}}$, so in fact $g_k^l \to \psi(\lambda_k)$ $\mu_{\tilde{g}}$-a.e. By Lemma 5.15, then, $d_{\text{meas}}^{\tilde{g}}(\psi(\lambda_k), \tilde{g}) \to 0$ as $k \to \infty$, and $d_{\text{meas}}^{\tilde{g}}(g_k^l, \psi(\lambda_k)) \to 0$ as $l \to \infty$ for all k. Thus, by making l_k larger if necessary, and since

$$d_{\text{meas}}^{\tilde{g}}(g_k^{l_k}, \tilde{g}) \leq d_{\text{meas}}^{\tilde{g}}(g_k^{l_k}, \psi(\lambda_k)) + d_{\text{meas}}^{\tilde{g}}(\psi(\lambda_k), \tilde{g}),$$

we may assume that, for all k and all $l \geq l_k$, $d_{\text{meas}}^{\tilde{g}}(g_k^l, \tilde{g}) < 1/k$, while still insuring that properties (1) and (4) of Definition 4.4 are satisfied. In particular, by Lemma 5.15(3), $g_k^{l_k} \to \tilde{g}$ in $\mu_{\tilde{g}}$-measure, and so Lemma 5.15(2) implies that we may pass to a subsequence to obtain that $g_k^{l_k} \to \tilde{g}$ $\mu_{\tilde{g}}$-a.e. Property (3) of Definition 4.4 now follows.

To modify $\{g_k^{l_k}\}$ so that it satisfies property (2) of Definition 4.4, note that by ω-convergence of g_k^l to $\psi(\lambda_k)$, $\det G_k^l \to 0$ for μ_g-almost every $x \in X_{\psi(\lambda_k)} = X_{\tilde{g}}$ and for all k. Consider $\mu := \mu_g | X_{\tilde{g}}$, the measure on $X_{\tilde{g}}$ obtained from μ_g by restriction, and let δ denote the standard metric of convergence in μ-measure for real-valued functions on $X_{\tilde{g}}$ (this is defined analogously to Lemma 5.15(3), cf. [**14**, (12.47)]). If $f_k^l := \det G_k^l | X_{\tilde{g}}$, then $\delta(f_k^l, 0) \to 0$ as $l \to \infty$ for all k. Thus, by making l_k larger if necessary, we may assume that $\delta(f_k^{l_k}, 0) < 1/k$ for all k, while still insuring that properties (1), (3) and (4) of Definition 4.4. In particular, by passing to a subsequence, we may assume that $f_k^{l_k} \to 0$ μ-a.e. [**14**, (11.26)]. This shows that $X_{\tilde{g}} \subseteq D_{\{g_k^{l_k}\}}$. The reverse inclusion $D_{\{g_k^{l_k}\}} \subseteq X_{\tilde{g}}$ then follows, as in the proof of Proposition 5.13, from the fact that $g_k^{l_k}(x) \to \tilde{g}(x)$ for almost every $x \in M \setminus X_{\tilde{g}}$. Thus property (2) of Definition 4.4 is also satisfied by $\{g_k^{l_k}\}$, and we have found the desired sequence.

It still remains to prove (5.10). The following two lemmas do this and thus complete the proof of the theorem.   q.e.d.

Lemma 5.17. *Let $\tilde{g} \in \mathcal{M}$. If $\sigma, \tau \in C^\infty(M)$ satisfy $\sigma, \tau > -\frac{4}{n}$, then*

$$d(\exp_{\tilde{g}}(\sigma\tilde{g}), \exp_{\tilde{g}}(\tau\tilde{g})) \leq \sqrt{n} \, \|\tau - \sigma\|_{\tilde{g}} \, .$$

Proof. Let $\hat{g} := \exp_{\tilde{g}}(\sigma\tilde{g})$. We first note that $\mathcal{P}\cdot\tilde{g} = \mathcal{P}\cdot\hat{g}$. Therefore, by Proposition 2.2, there is a neighborhood $V \subset C^\infty(M)$ such that $\exp_{\hat{g}} : V\cdot\hat{g} \to \mathcal{P}\cdot\hat{g}$ is a diffeomorphism.

Now

$$(5.11) \qquad d(\exp_{\tilde{g}}(\sigma\tilde{g}), \exp_{\tilde{g}}(\tau\tilde{g})) \leq \left\|\exp_{\hat{g}}^{-1}\exp_{\tilde{g}}(\tau\tilde{g})\right\|_{\hat{g}},$$

since the right-hand side is the length of a radial geodesic emanating from $\exp_{\tilde{g}}(\sigma\tilde{g})$ and ending at $\exp_{\tilde{g}}(\tau\tilde{g})$. Showing that the right-hand side of (5.11) is equal to $\sqrt{n}\,\|\tau - \sigma\|_{\tilde{g}}$ is a straightforward computation using (2.1). q.e.d.

Lemma 5.18. *Let g_0 and ψ be as in the proof of Theorem 5.16. If $\sigma, \tau \in C^\infty(M)$ satisfy $\sigma, \tau > -\frac{4}{n}$, then*

$$d(\psi(\sigma), \psi(\tau)) \leq \sqrt{n}\,\|\tau - \sigma\|_{g_0}.$$

Proof. Since g_0 is bounded, we can find a quasi-amenable subset $\mathcal{U}$ such that $g_0 \in \mathcal{U}^0$, i.e., such that g_0 belongs to the completion of $\mathcal{U}$ with respect to $\|\cdot\|_g$. Using Proposition 5.13, choose a sequence $\{g_k\}$ in $\mathcal{U}$ that both L^2- and ω-converges to g_0. For each $k \in \mathbb{N}$, define a map ψ_k by $\psi_k(\kappa) := \exp_{g_k}(\kappa g_k)$.

By the triangle inequality, we have
$$(5.12)$$
$$d(\psi(\sigma), \psi(\tau)) \leq d(\psi(\sigma), \psi_k(\sigma)) + d(\psi_k(\sigma), \psi_k(\tau)) + d(\psi_k(\tau), \psi(\tau))$$

for each k. But since $g_k \in \mathcal{M}$, Lemma 5.17 applies to give

$$(5.13) \qquad d(\psi_k(\sigma), \psi_k(\tau)) \leq \sqrt{n}\,\|\tau - \sigma\|_{g_k} \overset{k\to\infty}{\longrightarrow} \sqrt{n}\,\|\tau - \sigma\|_{g_0},$$

where the convergence follows from Lemma 5.1. By (5.12) and (5.13), if we can show that

$$(5.14) \qquad d(\psi(\sigma), \psi_k(\sigma)) \to 0 \quad \text{and} \quad d(\psi_k(\tau), \psi(\tau)) \to 0,$$

then we are finished. But it is not hard to show that $\psi_k(\sigma)$ L^2-converges to $\psi(\sigma)$, which then implies (5.14) by Proposition 5.13. q.e.d.

From the results of Section 4, we already know that the map $\Omega : \overline{\mathcal{M}} \to \widehat{\mathcal{M}_f}$ is an injection. Theorem 5.16 now states that this map is a surjection as well. Thus, we have already proved the main result of this paper, which we state again here in full detail.

Theorem 5.19. *There is a natural identification of $\overline{\mathcal{M}}$, the completion of $\mathcal{M}$ with respect to the L^2 metric, with $\widehat{\mathcal{M}_f}$, the set of measurable semimetrics with finite volume on M modulo the equivalence given in Definition 4.3.*

This identification is given by a bijection $\Omega : \overline{\mathcal{M}} \to \widehat{\mathcal{M}_f}$, where we map an equivalence class $[\{g_k\}]$ of d-Cauchy sequences to the unique

element of $\widehat{\mathcal{M}_f}$ that all of its members ω-subconverge to. This map is an isometry if we give $\widehat{\mathcal{M}_f}$ the metric $\bar{d}$ defined by

$$\bar{d}([g_0], [g_1]) := \lim_{k \to \infty} d(g_k^0, g_k^1)$$

where $\{g_k^0\}$ and $\{g_k^1\}$ are any sequences in $\mathcal{M}$ ω-subconverging to $[g_0]$ and $[g_1]$, respectively.

Here, we briefly note what geometric notions are well-defined for elements of $\widehat{\mathcal{M}_f}$. Given an equivalence class $[\tilde{g}] \in \widehat{\mathcal{M}_f}$, the metric space structure of different representatives may differ—e.g., if $M = T^2$, the torus, then the equivalence class of the zero metric also contains a geometric circle, where only one dimension has collapsed. (Note that here, as in Example 4.2, we are identifying a semimetric in $\mathcal{M}_f$ with the metric space associated to it by defining the distance between two points of M to be the infimum of the lengths of piecewise differentiable paths between them, and identifying points at distance zero from one another.) On the other hand, since representatives of a given equivalence class in $\widehat{\mathcal{M}_f}$ all have equal induced measures, things like L^p spaces of functions are well-defined for an equivalence class, as they are the same across all representatives.

To end this section, we remark that one might hope that Theorem 5.19 would give some information on the completion of the space $\mathcal{M}/\mathcal{D}$ of *Riemannian structures* with respect to the distance that the L^2 metric induces on it. (Here, $\mathcal{D}$ is the group of orientation-preserving diffeomorphisms of M acting by pull-back.) $\mathcal{M}/\mathcal{D}$ is the moduli space of Riemannian metrics, and hence is of great intrinsic interest to geometers. The problem here is that the proof of Theorem 5.19 does not indicate which degenerations of Cauchy sequences of metrics arise from "vertical" degenerations—that is, sequences $\{\varphi_n^* \tilde{g}\}$, where $\{\varphi_n\} \subset \mathcal{D}$ is a degenerating sequence of diffeomorphisms—and "horizontal" degenerations— that is, sequences of metrics that can be connected by horizontal paths. (See [**9**, §3] for a discussion of horizontal and vertical paths on $\mathcal{M}$.) Of course, only horizontal degenerations are relevant for the quotient. So there is some work remaining to do in order to understand the completion of $\mathcal{M}/\mathcal{D}$. We hope to investigate these questions in a future paper.

6. Application to Teichmüller Theory

In this section, we describe an application of our main theorem to the theory of Teichmüller space. Teichmüller space was historically defined in the context of complex manifolds, but the work of Fischer and Tromba translates this original picture into the context of Riemannian geometry, using the manifold of metrics [**26**]. We outline this construction of Teichmüller space in the first subsection, then define the much-studied

Weil-Petersson metric. In the second subsection, we prove a result on the completions of a class of metrics that generalize the Weil-Petersson metric.

6.1. The Weil-Petersson Metric on Teichmüller Space.

Convention 6.1. For the remainder of the paper, let our base manifold M be a connected, smooth, closed, oriented, two-dimensional manifold of genus $p \geq 2$.

Convention 6.2. In this chapter, we abandon Convention 2.14. That is, when we write g for a metric in $\mathcal{M}$, we no longer assume that this is fixed, but allow g to vary arbitrarily.

We have already noted that the group $\mathcal{P}$ acts on $\mathcal{M}$ by pointwise multiplication, and it turns out that the quotient $\mathcal{M}/\mathcal{P}$ is a smooth Fréchet manifold. Let $\mathcal{D}$ be the Fréchet Lie group of orientation-preserving diffeomorphisms of M, and let $\mathcal{D}_0 \subset \mathcal{D}$ be the subgroup of diffeomorphisms homotopic to the identity. Then both $\mathcal{D}$ and $\mathcal{D}_0$ act on $\mathcal{M}$ and $\mathcal{M}/\mathcal{P}$ by pull-back. Let $\mathcal{T}$ denote the Teichmüller space of M, $\mathcal{R}$ the Riemann moduli space of M, and $MCG := \mathcal{D}/\mathcal{D}_0$ the *mapping class group* of M. Then there are identifications

$$\mathcal{T} \cong {(\mathcal{M}/\mathcal{P})}\big/{\mathcal{D}_0}, \qquad \mathcal{R} \cong {\mathcal{T}}\big/{MCG} \cong {(\mathcal{M}/\mathcal{P})}\big/{\mathcal{D}},$$

where the first identification is a diffeomorphism. Note that Teichmüller space is finite-dimensional.

By the Poincaré uniformization theorem, there exists a unique hyperbolic metric (one with scalar curvature -1) in each conformal class $[g] \in \mathcal{M}/\mathcal{P}$. Furthermore, one can show that the subset $\mathcal{M}_{-1} \subset \mathcal{M}$ of hyperbolic metrics is a smooth Fréchet submanifold, and moreover that $\mathcal{M}_{-1}$ is the image of a smooth section of the principal $\mathcal{P}$-bundle $\mathcal{M} \to \mathcal{M}/\mathcal{P}$. It is easy to see that $\mathcal{M}_{-1}$ is $\mathcal{D}$-invariant, and therefore

$$\mathcal{T} \cong {\mathcal{M}_{-1}}\big/{\mathcal{D}_0}, \qquad \mathcal{R} \cong {\mathcal{M}_{-1}}\big/{\mathcal{D}},$$

where the first identification is again a diffeomorphism. We denote by $\pi : \mathcal{M}_{-1} \to \mathcal{M}_{-1}/\mathcal{D}_0$ the projection.

It is not hard to see that the L^2 metric $(\cdot, \cdot)$ on $\mathcal{M}$ is $\mathcal{D}$-invariant, so it descends to a MCG-invariant Riemannian metric, also denoted $(\cdot, \cdot)$, on the quotient $\mathcal{M}_{-1}/\mathcal{D}_0$. This metric is called the *Weil-Petersson metric*. (It differs from the usual definition of the Weil-Petersson metric by a constant scalar factor; cf. [**26**, §2.6].) With these definitions, we see that $(\mathcal{M}_{-1}, (\cdot, \cdot)) \to (\mathcal{M}_{-1}/\mathcal{D}_0, (\cdot, \cdot))$ is a weak Riemannian principal $\mathcal{D}_0$-bundle—that is, at each point $g \in \mathcal{M}_{-1}$, the differential $D\pi(g)$ of the projection is an isometry when restricted to the horizontal tangent space at g.

6.2. Generalized Weil-Petersson Metrics. We now wish to generalize the construction of the Weil-Petersson metric by selecting a different section of $\mathcal{M} \to \mathcal{M}/\mathcal{P}$. In fact, we will simultaneously consider every smooth section whose image $\mathcal{N}$ is $\mathcal{D}$-invariant, which we require so that we still have diffeomorphisms $\mathcal{T} \cong \mathcal{N}/\mathcal{D}_0$ and $\mathcal{R} \cong \mathcal{N}/\mathcal{D}$. This idea is directly inspired by [**11**] and [**12**], though our metrics on Teichmüller space differ from theirs.

Definition 6.3. We call the image of a smooth section of $\mathcal{M} \to \mathcal{M}/\mathcal{P}$ a *modular submanifold* if it is $\mathcal{D}$-invariant. Given a modular submanifold $\mathcal{N} \subset \mathcal{M}$, we call the quotients $\mathcal{N}/\mathcal{D}_0$ and $\mathcal{N}/\mathcal{D}$ the *$\mathcal{N}$-model of Teichmüller space* and the *$\mathcal{N}$-model of moduli space*, respectively.

For the remainder of the talk, let $\mathcal{N}$ be an arbitrary modular submanifold. It is not hard to see that $\mathcal{N} \cong \mathcal{M}_{-1}$ via a $\mathcal{D}$-equivariant diffeomorphism, so in fact we do have the desired diffeomorphisms $\mathcal{T} \cong \mathcal{N}/\mathcal{D}_0$ and $\mathcal{R} \cong \mathcal{N}/\mathcal{D}$.

Remark 6.4. Modular submanifolds other than $\mathcal{M}_{-1}$ indeed exist— for example, there are the Bergman and Arakelov metrics (see [**12**, §1] for details). We briefly describe the Bergman metric on a Riemann surface here. Recall that conformal structures (elements of $\mathcal{M}/\mathcal{P}$) are in one-to-one correspondence with complex structures on the surface, so we can work with these instead. Let c be a complex structure on M. Then the space of holomorphic one-forms on (M, c) has complex dimension p, the genus of M [**8**, Prop. III.2.7]. Let $\theta_1, \ldots, \theta_p$ be an L^2-orthonormal basis of this space. That is,

$$\frac{i}{2} \int_M \theta_j \wedge \overline{\theta_k} = \delta_{jk}.$$

The complex Bergman metric is defined by

$$g_B^{\mathbb{C}} := \frac{1}{p} \sum_{j=1}^p \theta_j \bar{\theta}_j.$$

Note that $g_B^{\mathbb{C}}$ is nondegenerate, since for any $x \in M$, there exists a holomorphic one-form on M that does not vanish at x [**8**, III.5.8]. Given a holomorphic local coordinate z on M, we can write $g^{\mathbb{C}}(z) = \lambda_B^2 dz d\bar{z}$ with λ_B real. The *Bergman metric* is the Riemannian metric associated to $g_B^{\mathbb{C}}$; if $z = x + iy$ for real local coordinates (x, y), then $g_B = \lambda_B^2(dx^2 + dy^2)$.

From this construction, it can be shown that the set $\mathcal{M}_B$ of all Bergman metrics on M is indeed a modular submanifold. This fact is essentially contained in [**11**, **12**], however, one can also argue as follows. $\mathcal{D}$-invariance of $\mathcal{M}_B$ is clear from the construction, as is the fact that $\mathcal{M}_B$ is the image of a set-theoretic section of $\mathcal{M} \to \mathcal{M}/\mathcal{P}$. Furthermore, Bers [**1**, Thm. I] constructs a smooth global section σ of $\mathcal{M}/\mathcal{P} \to \mathcal{T}$ and

shows that the holomorphic one-forms $\{w_j(\tau) \mid j = 1, \ldots, p\}$ on $\sigma(\tau)$ normalized by their periods are smooth functions of τ [**1**, Thm. III]. Since an L^2-orthonormal basis of holomorphic one-forms can be formed from $\{w_j(\tau)\}$ using the period matrix of $\sigma(\tau)$ [**11**, p. 303], and the period matrix of $\sigma(\tau)$ depends smoothly on τ [**1**, p. 210], σ induces a smooth injection $\tilde{\sigma} : \mathcal{T} \to \mathcal{M}_B$ with the property that $\mathcal{D}_0 \cdot \tilde{\sigma}(\mathcal{T}) = \mathcal{M}_B$. Since the $\mathcal{D}_0$-action is smooth, we have that $\mathcal{M}_B$ is smooth as well.

As in the case of the submanifold $\mathcal{M}_{-1}$, the L^2 metric on $\mathcal{M}$ projects to an MCG-invariant metric on $\mathcal{N}/\mathcal{D}_0$. We call this metric the *generalized Weil-Petersson metric* on the $\mathcal{N}$-model of Teichmüller space. As in the case of the bundle $\mathcal{M}_{-1} \to \mathcal{M}_{-1}/\mathcal{D}_0$, these metrics turn the bundle $\mathcal{N} \to \mathcal{N}/\mathcal{D}_0$ into a weak Riemannian principal $\mathcal{D}_0$-bundle.

Theorem 6.5. *For any C^1 path $\gamma : [0,1] \to \mathcal{N}/\mathcal{D}_0$ and any $g \in \pi_{\mathcal{N}}^{-1}(\gamma(0))$, there exists a unique horizontal lift $\tilde{\gamma} : [0,1] \to \mathcal{N}$ with $\tilde{\gamma}(0) = g$.*

Furthermore, $L(\tilde{\gamma}) = L(\gamma)$ and $\tilde{\gamma}$ has minimal length among the class of curves whose image projects to γ under $\pi_{\mathcal{N}}$.

Proof. The existence of horizontal lifts is not usually guaranteed on Fréchet manifolds, but since $\mathcal{N}/\mathcal{D}_0$ is finite-dimensional, the horizontal tangent space of $\mathcal{N}$ is finite-dimensional at each point. Therefore, integral curves of horizontal vector fields exist (cf. [**19**, Thm. 7.2 and Dfn. 5.6ff]).

An alternative proof, one which does not rely on the existence theory of solutions to ODEs in Fréchet spaces, is given in [**4**, Thm. 6.16].

Minimality of $\tilde{g}$ can be easily shown using the fact that $\mathcal{N} \to \mathcal{N}/\mathcal{D}_0$ is a weak Riemannian principal bundle.   q.e.d.

The next theorem applies the paper's main theorem to the completion of $\mathcal{N}/\mathcal{D}_0$ with respect to a generalized Weil-Petersson metric. In the following, we denote the distance function of $(\mathcal{N}, (\cdot, \cdot))$ by $d_{\mathcal{N}}$.

Theorem 6.6. *Let $\{[g_k]\}$ be a Cauchy sequence in the $\mathcal{N}$-model of Teichmüller space, $\mathcal{N}/\mathcal{D}_0$, with respect to the generalized Weil-Petersson metric. Then there exist representatives $\tilde{g}_k \in [g_k]$ and an element $[g_\infty] \in \widehat{\mathcal{M}_f}$ such that $\{\tilde{g}_k\}$ is a $d_{\mathcal{N}}$-Cauchy sequence that ω-subconverges to $[g_\infty]$.*

Furthermore, if $\{[g_k^0]\}$ and $\{[g_k^1]\}$ are equivalent Cauchy sequences in $\mathcal{N}/\mathcal{D}_0$, then there exist representatives $\tilde{g}_k^0 \in [g_k^0]$ and $\tilde{g}_k^1 \in [g_k^1]$, as well as an element $[g_\infty] \in \widehat{\mathcal{M}_f}$, such that $\{\tilde{g}_k^0\}$ and $\{\tilde{g}_k^1\}$ are $d_{\mathcal{N}}$-Cauchy sequences that both ω-subconverge to $[g_\infty]$.

Finally, if $\{[g_k^0]\}$ and $\{[g_k^1]\}$ are inequivalent Cauchy sequences in $\mathcal{N}/\mathcal{D}_0$, then there exists no choice of representatives $\tilde{g}_k^0 \in [g_k^0]$ and $\tilde{g}_k^1 \in [g_k^1]$ such that $\{\tilde{g}_k^0\}$ and $\{\tilde{g}_k^1\}$ ω-subconverge to the same element of $\widehat{\mathcal{M}_f}$.

Proof. The first claim would follow from Theorem 5.19 if we could show that there are representatives $\tilde{g}_k \in [g_k]$ such that $\{\tilde{g}_k\}$ is a $d_{\mathcal{N}}$-Cauchy sequence, since this implies that it is also a d-Cauchy sequence. So this is what we will show.

Let's denote the distance function induced by the generalized Weil-Petersson metric on $\mathcal{N}/\mathcal{D}_0$ by δ. For each $k \in \mathbb{N}$, let $\gamma_k : [0,1] \to \mathcal{N}/\mathcal{D}_0$ be any path from $[g_k]$ to $[g_{k+1}]$ such that

$$L(\gamma_k) \leq 2\delta([g_k],[g_{k+1}]).$$

For any $\tilde{g}_1 \in \pi_{\mathcal{N}}^{-1}([g_1])$, let $\tilde{\gamma}_1$ be the horizontal lift of γ_1 to $\mathcal{N}$ with $\tilde{\gamma}_1(0) = \tilde{g}_1$ which is guaranteed by Theorem 6.5. Then clearly $\tilde{g}_2 := \gamma_2(1) \in \pi_{\mathcal{N}}^{-1}([g_2])$. Furthermore,

$$d_{\mathcal{N}}(\tilde{g}_1, \tilde{g}_2) \leq L(\tilde{\gamma}_1) = L(\gamma_1) \leq 2\delta([g_1],[g_2]).$$

We repeat this process, i.e., let $\tilde{\gamma}_2$ be the unique horizontal lift of γ_2 with $\tilde{\gamma}_2(0) = \tilde{g}_2$, and set $\tilde{g}_3 := \tilde{\gamma}_2(1)$, etc. In this way, we get a sequence of representatives $\tilde{g}_k \in [g_k]$ such that for each $k \in \mathbb{N}$,

$$d_{\mathcal{N}}(\tilde{g}_k, \tilde{g}_{k+1}) \leq 2\delta([g_k],[g_{k+1}]).$$

Thus, since $\{[g_k]\}$ is a Cauchy sequence, $\{\tilde{g}_k\}$ is a $d_{\mathcal{N}}$-Cauchy sequence, as was to be shown.

The proof of the second statement is similar.

To prove the last statement, note that since $\{[g_k^0]\}$ and $\{[g_k^1]\}$ are inequivalent, we have

$$\liminf_{k\to\infty} \delta([g_k^0],[g_k^1]) > 0.$$

Thus by Theorem 6.5, no matter what representatives $\tilde{g}_k^0 \in [g_k^0]$ and $\tilde{g}_k^1 \in [g_k^1]$ we choose,

$$\liminf_{k\to\infty} d_{\mathcal{N}}(\tilde{g}_k^0, \tilde{g}_k^1) \geq \epsilon > 0.$$

So Theorem 5.19 implies the statement immediately.   q.e.d.

Theorem 6.6 generalizes what is known about the completion of the Weil-Petersson metric [**16**], which is completed by adding in certain cusped hyperbolic surfaces—which in particular can be viewed as elements of $\widehat{\mathcal{M}_f}$. However, they are only degenerate or singular along a set of disjoint simple closed geodesics—connecting with the complex-analytic viewpoint of degeneration—whereas elements of $\widehat{\mathcal{M}_f}$ can be degenerate or singular over an arbitrary subset of M. With more investigation and perhaps appropriate conditions on the modular submanifold $\mathcal{N}$, we expect that the statement of Theorem 6.6 can be considerably improved.

Despite the shortcomings of the above result, we see it as quite useful, as it gives relatively strong information about a new class of metrics on Teichmüller space—namely, that their completions can consist only of

finite-volume metrics. Furthermore, it illustrates the potential for applications of our main theorem and provides a starting point for further investigations.

Appendix. Relations between Various Notions of Convergence and Cauchy Sequences

In the following chart, we illustrate the relationships between the different notions of Cauchy and convergent sequences on $\mathcal{M}$. We let $\{g_k\}$ be a sequence in $\mathcal{M}$ and $\tilde{g} \in \mathcal{M}_f$. A double arrow ("$\Longrightarrow$") between two statements means that the one implies the other. A single arrow ("$\longrightarrow$") means that one statement implies the other, assuming the condition that is listed below the chart.
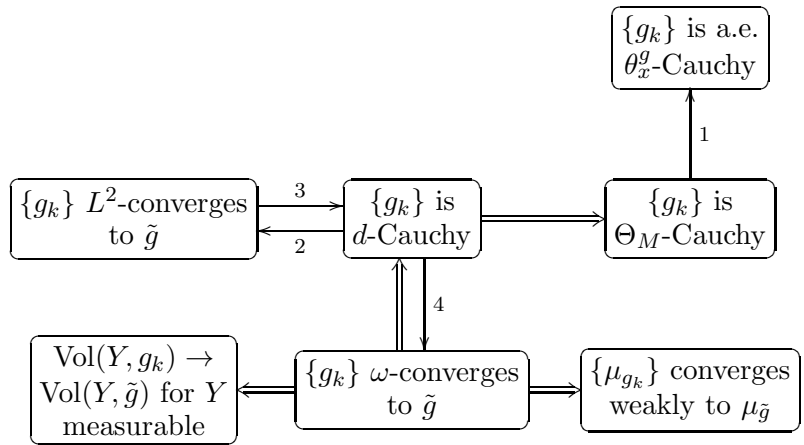
1) After passing to a subsequence
2) If there exists an amenable subset $\mathcal{U}$ such that $\{g_k\} \subset \mathcal{U}$, then there exists some $\tilde{g} \in \mathcal{U}^0$ such that the implication holds
3) If there exists a quasi-amenable subset $\mathcal{U}$ such that $\{g_k\} \subset \mathcal{U}$
4) After passing to a subsequence, there exists some $\tilde{g} \in \mathcal{M}_f$ such that the implication holds

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